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PROJECTIVE DIFFERENTIAL GEOMETRY OF GEODESICS, AND INTEGRABLE SYSTEMS

Abstract. This article is based on a lecture course on the projective differential geometry of geodesics, and integrable systems, held at Politecnico di Torino by two of the authors (GM and AV) in the academic years 2021 and 2022. The main objects of geodesic projective differential geometry are introduced, such as projective connections, metrizable projective structures and projective symmetries. Lie's classical problem on metrics with projective symmetries receives special attention, along with interrelations of projective differential geometry and integrable systems. C-projective geometry is briefly presented as a natural extension. The reader should be acquainted with classical differential geometry, but otherwise the article is self-contained. For the sake of tangibility, the text focuses on the 2-dimensional case.

1. Motivation

A parametrized curve in $\mathbb{R}^n$

$$\gamma : \mathbb{R} \supseteq I \ni t \to \gamma(t) \in \mathbb{R}^n,$$

where I is an interval of $\mathbb{R}$, can be interpreted as the solution of an equation of motion for a particle in $\mathbb{R}^n$, where the (external) parameter t is the time. Of course we could replace $\mathbb{R}^n$ by any other manifold, such as the n-sphere $\mathbb{S}^n$ or the n-torus $\mathbb{T}^n$, for instance. Straight lines are precisely the curves minimizing the distance between two distinct points of $\mathbb{R}^n$. For instance, consider the case $n = 2$. Then the curve

$$\gamma : \mathbb{R} \ni t \to \gamma(t) = (t, t) \in \mathbb{R}^2 \tag{1}$$

is a (parametrized) curve whose *trajectory* is the line described by the diagonal $y = x$. Now note that also the curve

$$\gamma : \mathbb{R} \ni t \to \gamma(t) = (t^3, t^3) \in \mathbb{R}^2 \tag{2}$$

has the same trajectory as [1]. What is the difference between curves [1] and [2]? The answer is that the curve [1] represents a *uniform rectilinear motion* of $\mathbb{R}^2$, called a *parametrized geodesic* of $\mathbb{R}^2$, whereas [2] does not, since, although the motion is rectilinear, is not uniform. In general, parametrized geodesics in $\mathbb{R}^n$ are represented by uniform rectilinear motions (i.e. zero acceleration) whereas in a generic manifold such as $\mathbb{S}^n$, $\mathbb{T}^n$, etc. (assumed to be immersed in $\mathbb{R}^{n+1}$), a parametrized geodesic can be seen as a particle moving freely on the considered manifold, which we can

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interpret as the *constraint*. This means that the acceleration is effectively determined only by the constraint ("spontaneous movement"). Interpreted mechanically within $\mathbb{R}^{n+1}$, the particle acceleration splits into its components parallel and orthogonal to the constrained tangent space. The constraint reaction ensures that the particle satisfies the constraint. This is elaborated upon in Section 2. In any case, curves [1] and [2] share the same trajectory, $y = x$, that we call *unparametrized geodesic*: in fact, any curve admitting $y = x$ as a trajectory admits a reparametrization of the external parameter such that it becomes a parametrized geodesic with respect to the new parameter. The main goal here is to study the geometry of unparametrized geodesics and their relation with the parametrized ones. In particular, we shall see when a system of unparametrized curves are the unparametrized geodesics of a metric. Metrics sharing the same unparametrized geodesics are called *projectively equivalent*, and our aim will be to study their symmetries (more precisely infinitesimal symmetries of unparametrized geodesics) and their first integrals.

2. Parametrized geodesics as spontaneous movements

Let S be an immersed surface in $\mathbb{R}^3$. Let

$$\gamma : \mathbb{R} \supseteq I \ni t \to \gamma(t) \in S \tag{3}$$

be a parametrized curve on S. A vector field V along γ is a mapping

$$V : \mathbb{R} \ni t \to V(t) \in T_{\gamma(t)}S \subset \mathbb{R}^3$$

where $V(t)$ is a tangent vector to the surface S in the point $\gamma(t)$. Note that, even if $V(t)$ is tangent to S, in general $\frac{dV}{dt}$ is not. With this in mind, we define the following differential operator

$$\frac{D}{dt} : V \to \frac{dV}{dt} - \left(\frac{dV}{dt} \cdot N\right) N \tag{4}$$

where

$$N : I \ni t \to N(t) \in \mathbb{R}^3$$

with $N(t)$ being a unit vector orthogonal to the surface S at the point $\gamma(t)$, and $\frac{dV}{dt} \cdot N$ being the standard scalar product of $\frac{dV}{dt}$ and N. In other words, $\frac{DV}{dt}$ is the orthogonal projection of $\frac{dV}{dt}$ onto the tangent space $T_{\gamma(t)}S$ of S at the point $\gamma(t)$.

The field V is called *parallel* (w.r.t. γ) if $\frac{DV}{dt} = 0$. The curve [3] is a (parametrized) geodesic of the surface $S \subset \mathbb{R}^3$ if $\dot{\gamma}$ is parallel (w.r.t. γ), i.e., if $\frac{D\dot{\gamma}}{dt} = \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$.

Due to [4], $\gamma(t)$ is a geodesic of S if and only if

$$\ddot{\gamma} - (\ddot{\gamma} \cdot N) N = 0 .$$

A physical interpretation is the following: a parametrized geodesic of S can be interpreted as the equation of motion of a particle, moving on S, on which only the acceleration coming from the constraint is present.

EXAMPLE 1. Let $S = \mathbb{S}^2$ be the 2-dimensional sphere in $\mathbb{R}^3$ with radius 1 and its center at the origin O. Let Q_1 and Q_2 be two points on $\mathbb{S}^2$ such that $\overrightarrow{OQ_1}$ are orthogonal to $\overrightarrow{OQ_2}$. By using the identification of R with $\overrightarrow{OR}$, we verify that a great circle of $\mathbb{S}^2$,

$$\gamma(t) = \cos(at)Q_1 + \sin(at)Q_2, \quad a \in \mathbb{R},$$

is a geodesic. We have that

$$\ddot{\gamma}(t) = -a^2\cos(at)Q_1 - a^2\sin(at)Q_2 = -a^2\gamma(t).$$

Then, if N is the operator that associates to $Q_1 \in S^2$ the unit vector $\overrightarrow{OQ_1}$, we have that

$$\frac{D\dot{\gamma}}{dt} = \ddot{\gamma} - (\ddot{\gamma}\cdot N)N = -a^2\gamma - (-a^2\gamma\cdot N)N = -a^2\gamma + a^2(\gamma\cdot N)N = 0$$

since $(\gamma\cdot N)N = \gamma$, as $\gamma(t)$ is parallel to $N(t)$ and satisfies $\|\gamma(t)\| = 1$.

From now on the Einstein convention on repeated indices will be used unless otherwise specified. Similarly, smoothness is assumed unless otherwise specified.

3. Parametrized vs. unparametrized geodesics

In coordinates, if

$$P : \mathbb{R}^2 \supseteq \Omega \ni (y^1, y^2) \to P(y^1, y^2) \in \mathbb{R}^3$$

is a parametrized regular surface, then

$$(P_{y^1}, P_{y^2}) := \left(\frac{\partial P}{\partial y^1}, \frac{\partial P}{\partial y^2}\right)$$

is a basis for its tangent space, so that we can write a vector field $V(t)$ along

$$\gamma(t) = P(y^1(t), y^2(t))$$

as follows

$$V(t) = V^i(t)P_{y^i}(y^1(t), y^2(t)) = V^1(t)P_{y^1}(y^1(t), y^2(t)) + V^2(t)P_{y^2}(y^1(t), y^2(t)).$$

Then

$$\frac{dV}{dt} = \frac{dV^i}{dt}P_{y^i} + V^i\frac{dP_{y^i}}{dt} = \frac{dV^i}{dt}P_{y^i} + V^iP_{y^iy^h}\dot{y}^h = \frac{dV^i}{dt}P_{y^i} + V^i\dot{y}^h(\Gamma^k_{ih}P_{y^k} + \Gamma_{ih}N) \tag{5}$$

for certain functions $\Gamma^k_{ih} = \Gamma^k_{ih}(y^1, y^2)$, $\Gamma_{ih} = \Gamma_{ih}(y^1, y^2)$, because in general $P_{y^iy^h}$ is not tangent to the surface. It has a tangent component, namely $\Gamma^k_{ih}P_{y^k}$, as well as a normal component, namely $\Gamma_{ih}N$.* Because of [5] we have

$$\frac{DV}{dt} = \left(\frac{dV^k}{dt} + \Gamma^k_{ih}V^i\dot{y}^h\right)P_{y^k}. \tag{6}$$

*The field N can be chosen equal to $\frac{P_{y^1}\times P_{y^2}}{\|P_{y^1}\times P_{y^2}\|}$.

In the particular case when

$$V(t) = \dot{\gamma}(t) = \dot{y}^i(t) P_{y^i}(y^1(t), y^2(t)),$$

we obtain, from [6], the equation of parametrized geodesics:

$$\ddot{y}^k + \Gamma^k_{ih} \dot{y}^i \dot{y}^h = 0. \tag{7}$$

It is easy to see that for a curve the property of being a (parametrized) geodesic is not invariant w.r.t. an arbitrary change of the parameter t: the curve remains a geodesic only if such a parameter change is affine, i.e. $t \to at + b$, $a, b \in \mathbb{R}$. Yet, the class of equations

$$\frac{D\dot{\gamma}}{dt} = f\dot{\gamma}, \quad \text{for some function} \quad f = f(t),$$

locally described by

$$\ddot{y}^k + \Gamma^k_{ih} \dot{y}^i \dot{y}^h = f\dot{y}^k, \quad f = f(t), \tag{8}$$

is invariant w.r.t. any change of the parameter t. Indeed any solution to [8] is a *reparametrized* geodesic.

By renaming the coordinates (y^1, y^2) as follows,

$$(x, y) = (y^1, y^2),$$

System [8] becomes

$$\begin{cases} \ddot{x} + \Gamma^1_{11}\dot{x}^2 + 2\Gamma^1_{12}\dot{x}\dot{y} + \Gamma^1_{22}\dot{y}^2 = f\dot{x} \\ \ddot{y} + \Gamma^2_{11}\dot{x}^2 + 2\Gamma^2_{12}\dot{x}\dot{y} + \Gamma^2_{22}\dot{y}^2 = f\dot{y} \end{cases} \tag{9}$$

A natural question arises: is it possible to eliminate the parameter t from System [9] in order to obtain a new differential equation which is parameter-free? The answer is positive. In fact, it suffices to require the condition for which a generic curve $y = y(x)$ is the trajectory of a (parametrized) geodesic. We have that

$$y_x = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}, \quad y_{xx} = \frac{d^2y}{dx^2} = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^3}.$$

Now, if we multiply the second equation of [9] by $\dot{x}$ and subtract from it the first equation of [9] multiplied by $\dot{y}$, we obtain

$$y_{xx} = -\Gamma^2_{11} + (\Gamma^1_{11} - 2\Gamma^2_{12})\, y_x + (2\Gamma^1_{12} - \Gamma^2_{22})\, y_x^2 + \Gamma^1_{22}\, y_x^3. \tag{10}$$

Note that the previous ODE is a polynomial of third degree in the first derivatives y_x, i.e., is of the type

$$y_{xx} = A_0(x, y) + A_1(x, y)\, y_x + A_2(x, y)\, y_x^2 + A_3(x, y)\, y_x^3. \tag{11}$$

Equation [11] is called the 2-*dimensional equation of unparametrized geodesics* or 2-*dimensional projective connection*. Accordingly, Equation [10] is called the 2-*dimensional projective connection associated to* ∇. Sometimes it is called a *a* 2-*dimensional system of paths*. Its importance can be seen from the following proposition.

PROPOSITION 6. *The class of ODEs [11] is invariant up to an arbitrary change of coordinates*

$$(x, y) \to \big(u(x, y), v(x, y)\big). \tag{12}$$

Proof. In fact, the transformation [12] induces the following transformations of first and second derivatives, more precisely [12] induces the transformation

$$(x, y, y_x, y_{xx}) \to (u, v, v_u, v_{uu})$$

where

$$v_u = \frac{v_x + v_y y_x}{u_x + u_y y_x}, \quad v_{uu} = \alpha \cdot y_{xx} + \beta \tag{13}$$

and

$$\begin{aligned} \alpha &= \frac{u_x v_y - u_y v_x}{(u_x + u_y y_x)^3}, \\ \beta &= \frac{v_{xx} + 2v_{xy}y_x + v_{yy}y_x^2}{(u_x + u_y y_x)^2} - \frac{(v_x + v_y y_x)(u_{xx} + 2u_{xy}y_x + u_{yy}y_x^2)}{(u_x + u_y y_x)^3}. \end{aligned}$$

By substituting the above quantities into the equation

$$v_{uu} = B_0(u, v) + B_1(u, v)v_u + B_2(u, v)v_u^2 + B_3(u, v)v_u^3 \tag{14}$$

we obtain an equation in (x, y, y_x, y_{xx}) of the form [11]. □

Proposition 6 shows that the class of 2-dimensional projective connections, i.e., ODEs of type [11], are closed under changes of the coordinates. For instance, an equation of type $y_{xx} = y_x^4$ cannot be mapped into an equation of type [14] by means of [12]. This leads to the following proposition.

PROPOSITION 7. *An ODE of the form*

$$y_{xx} = f(x, y, y_x)$$

represents an equation of unparametrized geodesics if and only if $\frac{\partial^4 f}{\partial y_x^4} = 0$.

REMARK 1. All that we have said so far can be easily generalized to arbitrary dimension. In fact, the equation of n-dimensional parametrized geodesics is [7] with k, i, h ranging from 1 to n. The same consideration holds for Equation [8]. By renaming the coordinates $(y^1, y^2, \ldots, y^n)$ as follows,

$$(y^1, y^2, \ldots, y^n) = (x, y^2, \ldots, y^n),$$

and eliminating the parameter from the (system of) equations we obtain $(k = 2, \ldots, n)$

$$\begin{aligned} y_{xx}^k = -\Gamma_{11}^k - \sum_{i=2}^{n} (2\Gamma_{1i}^k - \delta_i^k \Gamma_{11}^1) y_x^i \\ - \sum_{i,j=2}^{n} (\Gamma_{ij}^k - \delta_i^k \Gamma_{1j}^1 - \delta_j^k \Gamma_{1i}^1) y_x^i y_x^j + \sum_{i,j=2}^{n} \Gamma_{ij}^1 y_x^i y_x^j y_x^k, \end{aligned} \tag{15}$$

that is a system of ODEs of the following type ($k = 2, \dots, n$)

$$y^k_{xx} = f^k_{11} + \sum_{i=2}^{n} f^k_{1i} y^i_x + \sum_{i,j=2}^{n} f^k_{ij} y^i_x y^j_x + \sum_{i,j=2}^{n} f_{ij} y^i_x y^j_x y^k_x. \tag{16}$$

In analogy with [11], System [15] is called the *n-dimensional system of unparametrized geodesics* or *n-dimensional projective connection*. More specifically, System [15] is called the *n-dimensional projective connection associated to ∇*.

Since the paper, for the most part, is concerned with the 2-dimensional case, we will mainly consider Equations [10] and [11].

4. Notation

Let M be a smooth manifold. The algebra of smooth functions on M will be denoted by $C^\infty(M)$, the *tangent bundle* of M by TM and the *cotangent bundle* by T^*M. A *vector field* on M is a section of TM, whereas a *differential 1-form* is a section of T^*M. The $C^\infty(M)$-module of vector fields on M will be denoted by $\mathscr{X}(M)$ and the $C^\infty(M)$-module of differential 1-forms on M by $\Lambda^1(M)$.

A chart on M, in general, will be denoted by $(y^1, \dots, y^n)$, where $n = \dim(M)$. A local basis of TM (or T^*M) will be denoted by

$$\frac{\partial}{\partial y^i} \quad \text{or} \quad \partial_{y^i} \qquad (\text{respectively } dy^i),$$

so that a vector field X (resp. a differential 1-form ρ) on M, locally, reads

$$X = X^i \partial_{y^i}, \quad \rho = f_i dy^i, \qquad \text{where } X^i, f_i \in C^\infty(M).$$

A (p,q)-*tensor field* on M is a $C^\infty(M)$-multilinear map

$$\underbrace{\Lambda^1(M) \times \cdots \times \Lambda^1(M)}_{\text{p times}} \times \underbrace{\mathscr{X}(M) \times \cdots \times \mathscr{X}(M)}_{\text{q times}} \to C^\infty(M).$$

In particular, a *metric* g on M is a symmetric non-degenerate $(0,2)$-tensor field. We will not make assumptions on the signature of g: we let it arbitrary, unless otherwise specified.

A *differential k-form* is a totally skew-symmetric $(k,0)$-tensor field. We denote the $C^\infty(M)$-module of differential k-forms on M by $\Lambda^k(M)$. If $m = \dim(M)$, then any element of $\Lambda^m(M)$ is called a *volume form* on M. The *exterior product* of two differential forms is denoted by $\wedge : \Lambda^k(M) \times \Lambda^\ell(M) \to \Lambda^{k+\ell}(M)$.

If A_1, A_2 are $(0,1)$-tensor fields, then their *symmetric product* is given by

$$A_1 \odot A_2 := \frac{1}{2}(A_1 \otimes A_2 + A_2 \otimes A_1).$$

In the case of differential forms, we will often omit the symbol $\odot$, for instance,

$$dy^i dy^j := dy^i \odot dy^j = \frac{1}{2}(dy^i \otimes dy^j + dy^j \otimes dy^i).$$

We denote by S^kV the kth *symmetric tensor power* of V, which is defined analogously in the obvious way.

5. Connections, projective connections and projective symmetries

DEFINITION 7. *Let M be a smooth manifold. A* covariant derivative *(or* connection*) on M is a function $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, $(X, Y) \to \nabla_X Y$, such that the following properties hold $\forall X, Y, Z \in \mathcal{X}(M)$ and $\forall f \in C^\infty(M)$:*

- $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$;
- $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$;
- $\nabla_{fX} Y = f \nabla_X Y$;
- $\nabla_X (fY) = X(f)Y + f\nabla_X Y$.

If ∇ is *torsion-free*, i.e. if

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad \forall X, Y \in \mathcal{X}(M),$$

then we will call it an *affine connection*. The covariant derivative for tensor fields of type (k, l) will be introduced in Section 7.

Warning: Unless stated otherwise, we only consider affine connections from now on.

Let now $(y^1, \ldots, y^n)$ be a system of coordinates on an open set $U \subseteq M$ of the n-dimensional manifold M. We have, on this open set, that $X = X^i \partial_{y^i}$ and $Y = Y^j \partial_{y^j}$ where $\partial_{y^i} = \frac{\partial}{\partial y^i}$, such that

$$\begin{aligned} \nabla_X Y = \nabla_{X^i \partial_{y^i}} Y^j \partial_{y^j} &= X^i \nabla_{\partial_{y^i}} Y^j \partial_{y^j} \\ &= X^i \left(\frac{\partial Y^j}{\partial y^i} \partial_{y^j} + Y^j \nabla_{\partial_{y^i}} \partial_{y^j} \right) = X^i \left(\frac{\partial Y^j}{\partial y^i} \partial_{y^j} + Y^j \Gamma^k_{ij} \partial_{y^k} \right) \end{aligned}$$

for some functions $\Gamma^k_{ij} \in C^\infty(U)$ that we call the *Christoffel symbols* of the connection ∇. We then finally obtain that, locally,

$$\nabla_X Y = \left(X^i \frac{\partial Y^k}{\partial y^i} + X^i Y^j \Gamma^k_{ij} \right) \frac{\partial}{\partial y^k} = \left(X(Y^k) + \Gamma^k_{ij} X^i Y^j \right) \frac{\partial}{\partial y^k} \tag{17}$$

From [17] we infer, if $p \in M$,

$$(\nabla_X Y)_p = \left(X_p(Y^k) + \Gamma^k_{ij}(p) X^i(p) Y^j(p) \right) \left. \frac{\partial}{\partial y^k} \right|_p$$

which implies that, if $\gamma(t)$ is a curve such that $\dot{\gamma}(0) = X_p$,

$$(\nabla_X Y)_p = \left(\frac{d(Y^k \circ \gamma)}{dt}(0) + \Gamma^k_{ij}(p) X^i(p) Y^j(p) \right) \frac{\partial}{\partial y^k}\bigg|_p . \tag{18}$$

In other words, $(\nabla_X Y)_p$ depends only on X_p and on the values that Y takes on the curve γ. This allows to define $\nabla_v Y$ with $v \in T_p M$ in the following way:

$$\nabla_v Y := (\nabla_X Y)_p$$

where X is a vector field such that $X_p = v$. We can then define the covariant derivative of a vector field Y along a curve γ:

$$\nabla_{\dot{\gamma}(t)} Y = \nabla_{X_p} Y \tag{19}$$

where $X_p = \dot{\gamma}(t)$. Note that, in the light of the aforementioned, in order to compute [19] it suffices to know the values of Y along the curve γ. In other words, we may define $\nabla_{\dot{\gamma}(t)}$ on operators of the kind

$$V : t \to V(t) \in T_{\gamma(t)} M ,$$

which are called *vector fields along* γ. We denote by $\Gamma(\gamma)$ the set of vector field along γ. Of course a restriction of a vector field Y onto γ belongs to $\Gamma(\gamma)$. We define another operator,

$$\frac{D}{dt} : \Gamma(\gamma) \to \Gamma(\gamma), \tag{20}$$

by

$$\frac{DV}{dt}(s) := \nabla_{\dot{\gamma}(s)} V .$$

The operator [20] coincides with [4], which however has been introduced from a completely different perspective.

DEFINITION 8. *A curve* $\gamma : \mathbb{R} \supseteq I \to M$ *is called a* parametrized geodesic *if*

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \frac{D\dot{\gamma}}{dt} = 0 .$$

It is not difficult to confirm that, given the expressions [17]–[18], locally γ is a parametrized geodesic if and only if

$$\ddot{y}^k + \Gamma^k_{ih} \dot{y}^i \dot{y}^h = 0 . \tag{21}$$

This coincides with [7].

DEFINITION 9. *The set of connections sharing the same unparametrized geodesics of* ∇ *form an equivalence class that we denote by* $[\nabla]$, *called the* projective class *of* ∇ *or* projective structure *on* M. *Two connections in the same projective class are called* projectively equivalent. *A* projective connection *on a manifold* M *is a system of ODEs of type [16]. A* projective connection associated to a connection ∇ *is the system of ODEs [15].*

Given the above definitions, it is justified to say that projective connections realize projective structures in terms of a system of ODEs.

A natural question arises: when do two connections ∇ and $\hat{\nabla}$ on M define the same projective connection? Due to the definition of projective connections associated to affine connections, they are, essentially, particular systems of ODEs that are polynomial in y^k_x. By equating the coefficients of these polynomials we find the following condition

$$\hat{\nabla} - \nabla = \psi \otimes \mathrm{Id} + \mathrm{Id} \otimes \psi, \quad \psi \in \Lambda^1(M),$$

or, locally,

$$\hat{\Gamma}^k_{ij} - \Gamma^k_{ij} = \delta^k_i \psi_j + \delta^k_j \psi_i. \tag{22}$$

REMARK 2. If in [10] we perform the following substitutions

$$\Gamma^1_{11} \to \Gamma^1_{11} + 2\psi_1, \quad \Gamma^1_{12} \to \Gamma^1_{12} + \psi_2, \quad \Gamma^1_{22} \to \Gamma^1_{22}$$

$$\Gamma^2_{11} \to \Gamma^2_{11}, \qquad \Gamma^2_{12} \to \Gamma^2_{12} + \psi_1, \quad \Gamma^2_{22} \to \Gamma^2_{22} + 2\psi_2$$

the ODE does indeed not change, no matter the choice of ψ.

DEFINITION 10. *A* projective transformation *on a manifold M is a local diffeomorphism sending unparametrized geodesics into unparametrized geodesics. A* projective vector field *is a vector field whose local flow is made of projective transformations.*

6. Projective vector fields as infinitesimal symmetries of projective connections

As we know, Sophus Lie has shown that the set of projective vector fields has a Lie algebra structure [21]. Indeed, a projective vector field can be understood as a symmetry of the system of ODEs [16]. The set of the infinitesimal symmetries for a given system of ODEs form a Lie algebra, which has been called its *symmetry Lie algebra*. The reader interested in more details about this topic may like to consult [34], for instance. Indeed, a *finite symmetry* of System [16] is a local diffemorphism

$$(x, y^2, \ldots, y^n) \to (u(x, y^2, \ldots, y^n), \ldots v^n(x, y^2, \ldots, y^n))$$

sending solutions to solutions. whereas an infinitesimal symmetry is a vector field

$$X = a\partial_x + b^i \partial_{y^i}, \quad a = a(x, y^2, \ldots, y^n), \quad b^i = b^i(x, y^2, \ldots, y^n). \tag{23}$$

Its local flow in turn is made up of finite symmetries. Recalling Definition 10, we therefore infer that projective transformations and projective vector fields are finite and, respectively, infinitesimal symmetries of System [16].

We now ask ourselves how to characterize such symmetries through differential equations. The unknown functions of such PDEs have to be a and b^i of [23], whose

solution precisely gives the Lie algebra of symmetries of [16]. A convenient framework for this approach to projective symmetries and projective vector fields is that of jet spaces. As usual, let M be a smooth manifold of dimension n. Let $\alpha \subset M$ be an embedded 1-dimensional submanifold of M, i.e., α is an unparametrized curve. In appropriate local coordinates $(x, y^2, \dots, y^n)$ on M, α is described by $(x, y^2(x), \dots, y^n(x))$. The variable x will be called the *independent* variable, to distinguish it from the remaining coordinates y^i, that will be called the *dependent* ones. We say that the above chart is *admissible* for α or, equivalently, that α is (locally) admissible for the chart (x, y^i). It is easy to see that, given two unparametrized curves α and $\bar{\alpha}$ passing through the same point $p \in M$, there always exists a chart (through p) of M that is admissible for both. This ensures that the following definition is well posed.

DEFINITION 11. *Two unparametrized curves $\alpha = (x, y^i(x))$ and $\bar{\alpha} = (x, \bar{y}^i(x))$ through a point $p \in M$ are called* k–equivalent *at p if and only if*

$$\left.\frac{d^h y^i}{dx^h}\right|_p = \left.\frac{d^h \bar{y}^i}{dx^h}\right|_p, \quad \forall i \in \{2, \dots, n\}, \quad h \leq k.$$

The class of k-equivalent unparametrized curves at the point $p \in M$ is denoted by $[\alpha]^k_p$. The union

$$J^k(1, M) := \bigcup_{p \in M} \left\{ [\alpha]^k_p \mid \alpha \text{ is an unparametrized curve of } M \text{ passing through } p \right\}$$

of all these equivalence classes is the space of k–jets of unparametrized curves *of M.*

From now on, when there is no risk of confusion, we abbreviate

$$J^k := J^k(1, M).$$

It is easy to see that we have a tower of bundles

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & J^k & \longrightarrow & J^{k-1} & \longrightarrow & \dots & \longrightarrow & J^1 & \longrightarrow & J^0 = M \\ \dots & \longrightarrow & [\alpha]^k_p & \longrightarrow & [\alpha]^{k-1}_p & \longrightarrow & \dots & \longrightarrow & [\alpha]^1_p & \longrightarrow & p \end{array}$$

The space J^k has the natural structure of a smooth manifold: one way to see this is to extend the local coordinate system (x, y^i) on M to a coordinate system

$$\left(x, y^i, y^i_x, y^i_{xx}, \dots, y^i_k := y^i_{\underbrace{x \cdots x}_{\text{k times}}} \right)$$

on J^k, where each coordinate function $y^i_{\underbrace{x \cdots x}_{\text{h times}}}$, with $h \leq k$, is unambiguously defined by

$$y^i_h \left([\alpha]^h_p \right) = \left.\frac{d^h y^i}{dx^h}\right|_p.$$

The operator of *total derivation truncated to the order k* is given by

$$D_x^{(k)} := \partial_x + \sum_{i=2}^{n} \sum_{h=1}^{k} y_h^i \partial_{y_{h-1}^i}$$

and the *total derivation* operator $D_x^{(\infty)}$ is denoted by D_x. A local diffeomorphism

$$\phi : M \to M$$

can be interpreted as the change of local coordinates (dependent and independent variables) on M, where dependent and independent variables may be mixed. A local diffeomorphism ϕ induces a local diffeomorphism $\phi^{(k)} : J^k \to J^k$,

$$\phi^{(k)} : J^k \ni [\alpha]_p^k \to [\phi(\alpha)]_{\phi(p)}^k \in J^k, \tag{24}$$

called the *k-th prolongation* of ϕ. The following example details the construction in two dimensions.

EXAMPLE 2. Let $M = \mathbb{R}^2$ and consider a local diffeomorphism

$$\phi : (x, y) \to (u(x, y), v(x, y)).$$

The prolongation [24] then allows to relate all higher derivatives of v with respect to u in terms of derivatives of y with respect to x. For example, the prolongations $\phi^{(1)} : (x, y, y_x) \to (u, v, v_u)$ and $\phi^{(2)} : (x, y, y_x, y_{xx}) \to (u, v, v_u, v_{uu})$ are given by, respectively, the first and the second formula of [13], which explicitate the transformations of velocities and accelerations.

A vector field X on M can be prolonged to a vector field $X^{(k)}$ on J^k by prolonging its local flow. A straightforward computation shows that the prolongation to J^k of the vector field $X = a\partial_x + b^i \partial_{y^i}$ given by [23] is

$$X^{(k)} = \underbrace{a\partial_x + b_0^i \partial_{y^i}}_{=X} + b_1^i \partial_{y_1^i} + b_2^i \partial_{y_2^i} + \dots \tag{25}$$

where the functions b_h^i are recursively defined by

$$b_0^i = b^i, \qquad b_{h+1}^i = D_x(b_h^i) - y_{h+1}^i D_x(a).$$

The key point for computing infinitesimal symmetries of the system of ODEs [16] is to interpret such a system as an $(n-1)$-codimensional submanifold of J^2 since it is described by the vanishing of $(n-1)$ smooth functions on J^2. Let us denote such functions by F^k, $k = 2, \dots, n$. From this geometric viewpoint, an infinitesimal symmetry X of [16] is a vector field [23] such that $X^{(2)}$ is tangent to the aforementioned $(n-1)$-codimensional submanifold. Locally, this means

$$X^{(2)}(F^k) = 0 \quad \text{once restricted to the submanifold of } J^2 \text{ described by } \{F^2 = \dots = F^n = 0\}. \tag{26}$$

6.1. Projective vector fields of 2-dimensional projective connections

Let us consider the 2-dimensional case, i.e., let us find the conditions such that the vector field

$$X = a\partial_x + b\partial_y, \quad a = a(x,y), \quad b = b(x,y),$$

is an infinitesimal symmetry of [11] or, in other words, a projective vector field of the projective structure [11]. We note that, in this case, [11] can be interpreted as a hypersurface of J^2 described by

$$\left(x, y, y_x, A_0(x,y) + A_1(x,y)y_x + A_2(x,y)y_x^2 + A_3(x,y)y_x^3\right)$$

where (x, y, y_x) play the role of local parameters, i.e., [11] can be interpreted as the zero set of the function

$$y_{xx} - A_0(x,y) - A_1(x,y)y_x - A_2(x,y)y_x^2 - A_3(x,y)y_x^3. \tag{27}$$

The prolongation $X^{(2)}$ to J^2 of X is, in view of [25],

$$X^{(2)} = a\partial_x + b\partial_y + \big(D_x(b) - y_x D_x(a)\big)\partial_{y_x} + \big(D_x(D_x(b) - y_x D_x(a)) - y_{xx} D_x(a)\big)\partial_{y_{xx}}.$$

Then, by following Formula [26], we apply the vector field $X^{(2)}$ to the function [27] and subsequently replace y_{xx} by

$$A_0(x,y) + A_1(x,y)y_x + A_2(x,y)y_x^2 + A_3(x,y)y_x^3.$$

We obtain a polynomial of third degree in y_x. This polynomial vanishes if X is a projective symmetry, and this condition is described by the following system:

$$\left.\begin{aligned} b_{xx} - 2A_0a_x - A_1b_x + A_0b_y - A_{0x}a - A_{0y}b &= 0 \\ -a_{xx} + 2b_{xy} - A_1a_x - 3A_0a_y - 2A_2b_x - A_{1x}a - A_{1y}b &= 0 \\ -2a_{xy} + b_{yy} - 2A_1a_y - 3A_3b_x - A_2b_y - A_{2x}a - A_{2y}b &= 0 \\ -a_{yy} + A_3a_x - A_2a_y - 2A_3b_y - A_{3x}a - A_{3y}b &= 0 \end{aligned}\right\}. \tag{28}$$

Three examples will now be provided to illustrate the aforementioned theoretical considerations. We begin with the simplest case, the so-called *flat projective connection.*

EXAMPLE 3. Let us consider the most simple second order ODE, which simultaneously is the most simple conceivable projective connection,

$$y_{xx} = 0. \tag{29}$$

Substituting $A_0 = A_1 = A_2 = A_3 = 0$ into System [28] we obtain

$$\left.\begin{aligned} b_{xx} &= 0 \\ -a_{xx} + 2b_{xy} &= 0 \\ -2a_{xy} + b_{yy} &= 0 \\ a_{yy} &= 0 \end{aligned}\right\}$$

whose general solution depends on 8 parameters C_i $(i = 1,\dots,8)$:

$$a = C_1x^2 + C_3xy + C_7x + C_6y + C_8, \quad b = C_1xy + C_2x + C_3y^2 + C_4y + C_5.$$

Therefore, the Lie algebra of projective vector fields of [29] is generated by

$$\partial_x, \quad \partial_y, \quad x\partial_x, \quad x\partial_y, \quad y\partial_x, \quad y\partial_y, \quad x^2\partial_x + xy\partial_y, \quad xy\partial_x + y^2\partial_y.$$

The following projective connection is the one obtained from the round metric on $\mathbb{S}^2$. We are going to find that its associated symmetry Lie algebra is isomorphic to that in the previous example.

EXAMPLE 4. Let us consider the projective connection

$$y_{xx} = -2\cot(x)y_x - \cos(x)\sin(x)y_x^3. \tag{30}$$

Substituting $A_0 = 0$, $A_1 = -2\cot(x)$, $A_2 = 0$ and $A_3 = -\cos(x)\sin(x)$ into System [28], and solving it, we obtain that the Lie algebra of projective vector fields of [30] is generated by the following 8 vector fields:

$$\sin(y)\frac{\partial}{\partial x} + \cos(y)\cot(x)\frac{\partial}{\partial y}, \qquad -\cos(y)\frac{\partial}{\partial x} + \cot(x)\sin(y)\frac{\partial}{\partial y},$$
$$\sin(2x)\frac{\partial}{\partial x}, \qquad \cos(y)(\sin(x))^2\frac{\partial}{\partial x},$$
$$(\sin(x))^2\sin(y)\frac{\partial}{\partial x}, \qquad -\tfrac{1}{2}\cos(2y)\sin(2x)\frac{\partial}{\partial x} + \sin(2y)\frac{\partial}{\partial y},$$
$$\tfrac{1}{2}\sin(2x)\sin(2y)\frac{\partial}{\partial x} + \cos(2y)\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial y}.$$

We end our series of examples with two specific projective connections that we are going to encounter again later.

EXAMPLE 5. Consider the projective connection

$$y_{xx} = -e^{-2x}y_x^3. \tag{31}$$

Following the same steps as before, we obtain that it admits a 2-dimensional Lie algebra of projective vector fields generated by

$$\partial_y, \quad \partial_x + y\partial_y.$$

Now consider the projective connection

$$y_{xx} = \frac{1}{2}y_x - e^{-2x}y_x^3. \tag{32}$$

We find that it admits a 3-dimensional Lie algebra of projective vector fields generated by

$$\partial_y, \quad \partial_x + y\partial_y \quad \text{and} \quad y\partial_x + \frac{1}{2}y^2\partial_y.$$

Equations [31] and [32] cannot be mapped one into the other by a transformation [12] since they have a Lie algebra of projective vector fields of different dimension. Indeed, a necessary condition for two 2-dimensional projective connections to be linked by a transformation of type [12] is that the dimensions of their Lie algebra of projective vector fields coincide.

Keeping the thought at the end of Example 5 in mind, we observe that in the Examples 3 and 4 the symmetry Lie algebras have the same dimension. Indeed they are isomorphic and we may therefore ask, whether the projective connections [29] and [30] can be mapped one into the other by a transformation [12]. Indeed the following classical result [2, 22] holds:

THEOREM 9. *A projective connection [11] can be mapped to $v_{uu} = 0$ by a transformation of the form [12] if and only if it possesses an 8-dimensional Lie algebra of projective symmetries. Such a Lie algebra is isomorphic to* $\mathfrak{sl}(3,\mathbb{R})$.

This means that 2-dimensional metrics with an 8-dimensional Lie algebra of projective symmetries are projectively equivalent to the flat metric.

7. Levi-Civita connections and the metrizability of projective connections

A connection ∇ on a manifold M induces, via Leibniz' rule, a connection on the space of the tensor fields on M (which we also denote by ∇). In local coordinates, the formula

$$\nabla_{\partial_{y^i}} dy^k = -\Gamma^k_{ij} dy^j$$

introduces smooth functions Γ^k_{ij} called *Christoffel symbols*. In the presence of a (pseudo-) Riemannian metric, there exists a natural connection, the *Levi-Civita connection* of g. We recall its definition, keeping in mind that a metric g is, in particular, a $(0,2)$-tensor field on M.

DEFINITION 12. *Let g be a metric on M. A connection ∇ is called* Levi-Civita *connection if it is* torsion-free, *i.e.* $\nabla_X Y - \nabla_Y X = [X, Y]$, $\forall\, X, Y \in \mathfrak{X}(M)$, *and* metric, *i.e.* $\nabla_X(g) = 0$, $\forall\, X \in \mathfrak{X}(M)$.

As we know from introductory differential geometry, a *unique* Levi-Civita connection exists for a given metric g on M due to Koszul's formula. Locally, it is described by its Christoffel symbols, given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kh} \left(\frac{\partial g_{jh}}{\partial y^i} + \frac{\partial g_{ih}}{\partial y^j} - \frac{\partial g_{ij}}{\partial y^h} \right). \tag{33}$$

In projective differential geometry we face the following, natural problem: If we are provided with a projective structure, e.g. through a given projective connection, can we decide whether it is realized via a Levi-Civita connection? This is called the *metrizability problem* or *metrization problem*.

7.1. Metrizability

In this section we investigate the obstructions for the existence of a metric that realizes a given projective structure through its Levi-Civita connection. Clearly, if it exists, such a metric g is not unique: any constant multiple cg, $c \neq 0$, yields the same Levi-Civita connection and thus the same projective structure. As we shall see later, also the Levi-Civita connection is not necessarily unique. Recalling Definition 9, we extend the concept of projective equivalence to metrics.

DEFINITION 13. *Two metrics are called* projectively equivalent *if their Levi-Civita connections share the same unparametrized geodesics.*

Problem of metrizability of projective connections. We consider the following, natural question: Given a projective connection [16], is it possible to find a metric g such that the projective connection associated to its Levi-Civita connection is equal to [16]? In general, the answer is negative. For instance, the projective connection

$$y_{xx} = yy_x \tag{34}$$

is not metrizable, as we shall see in Example 7 below.

In order to quantify the property of a given projective structure to be metrizable, we aim to determine a system of PDEs characterizing the metrics g_{ij} such that [16] is equal to [15] with Γ^k_{ij} given by [33]. This problem has been solved in general, but for brevity and tangibility of the discussion, we will focus on the 2-dimensional case, in which the differential equations can be concisely written down. The system of PDEs is, due to [11],

$$\begin{cases} A_0 = \frac{1}{2}\frac{g_{12}g_{11_x} - 2g_{11}g_{12_x} + g_{11}g_{11_y}}{g_{11}g_{22} - g_{12}^2} \\ A_1 = \frac{1}{2}\frac{3g_{12}g_{11_y} - 2g_{11}g_{22_x} + g_{22}g_{11_x} - 2g_{12}g_{12_x}}{g_{11}g_{22} - g_{12}^2} \\ A_2 = \frac{1}{2}\frac{2g_{12}g_{12_y} - 3g_{12}g_{22_x} - g_{11}g_{22_y} + 2g_{22}g_{11_y}}{g_{11}g_{22} - g_{12}^2} \\ A_3 = \frac{1}{2}\frac{2g_{22}g_{12_y} - g_{22}g_{22_x} - g_{12}g_{22_y}}{g_{11}g_{22} - g_{12}^2} \end{cases} \tag{35}$$

System [35] is highly non-linear and finding the conditions on functions A_i such that there exists a solution (g_{11}, g_{12}, g_{22}), even by assuming extra hypotheses, turns out to be very hard. A key result concerning the metrizability of 2-dimensional projective structures is contained in the following proposition.

PROPOSITION 8. *The projective connection of a 2-dimensional metric*

$$g = g_{ij}dy^i dy^j$$

is [11] if and only if the entries of the matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} := \begin{pmatrix} \frac{g_{11}}{|g_{11}g_{22}-g_{12}^2|^{\frac{2}{3}}} & \frac{g_{12}}{|g_{11}g_{22}-g_{12}^2|^{\frac{2}{3}}} \\ \frac{g_{12}}{|g_{11}g_{22}-g_{12}^2|^{\frac{2}{3}}} & \frac{g_{22}}{|g_{11}g_{22}-g_{12}^2|^{\frac{2}{3}}} \end{pmatrix}$$

satisfy the following linear system *of PDEs:*

$$\begin{cases} a_{11,x} - \frac{2}{3}A_1 a_{11} + 2A_0 a_{12} = 0 \\ a_{11,y} + 2a_{12,x} - \frac{4}{3}A_2 a_{11} + \frac{2}{3}A_1 a_{12} + 2A_0 a_{22} = 0 \\ 2a_{12,y} + a_{22,x} - 2A_3 a_{11} - \frac{2}{3}A_2 a_{12} + \frac{4}{3}A_1 a_{22} = 0 \\ a_{22,y} - 2A_3 a_{12} + \frac{2}{3}A_2 a_{22} = 0 \end{cases} \tag{36}$$

We remark that the system [36] can be formulated in terms of a Lax pair. This means that there (locally) exist matrices $\mathbf{L}, \mathbf{M}$ (representing $(1,1)$-tensor fields) such that

$$\frac{d}{dt}\mathbf{L} = [\mathbf{M}, \mathbf{L}].$$

This will be explained in detail in Section 8.4.

EXAMPLE 6. The projective connections [29] and [30] are metrizable as they are the equations of unparametrized geodesics, respectively, of the Euclidean metric $dx^2 + dy^2$ (Gaussian curvature equal to 0) and of the standard metric on the sphere $dx^2 + \sin^2(x)dy^2$ (Gaussian curvature equal to 1). A similar result holds true also if we consider the hyperbolic metric (Gaussian curvature equal to -1). Of course, there could be more metrics admitting the same projective connection. For instance, $k_1 dx^2 + k_2 dy^2$ admits $y_{xx} = 0$ as projective connection for any $k_1, k_2 \in \mathbb{R}$ such that $k_1 k_2 \neq 0$.

EXAMPLE 7. The projective connection [34] is not metrizable. Indeed, by substituting $A_0 = 0$, $A_1 = y$, $A_2 = 0$ and $A_3 = 0$ into System [36], from the first and the last equation we obtain

$$a_{11} = f(y)e^{\frac{2}{3}xy}, \quad a_{22} = h(x). \tag{37}$$

By substituting [37] into the third equation of [36] we obtain

$$a_{12} = -\frac{1}{3}hy^2 - \frac{1}{2}h_x y + q(x). \tag{38}$$

Then, by substituting [37]–[38] into the second equation of System [36], we obtain an equation that vanishes if and only if $f(y) = h(x) = q(x) = 0$, i.e., only the trivial solution $a_{ij} = 0$ is possible.

We may think of [36] as a local expression, meaning that the a_{ij} can be simply interpreted as entries of a matrix. However, it must be noted that the a_{ij} do not

transform like the components of a tensor field, and therefore the matrix $a = (a_{ij})$ does not represent a $(0,2)$-tensor field. Indeed, changing coordinates (i.e., changing the chart by which we describe a local patch on our manifold), we have that, locally, the matrix g transforms as

$$g \to J^T g J$$

where J is the Jacobi matrix of the changing of coordinates. Then

$$a = |\det(g)|^{-\frac{2}{3}} g \to |\det(J^T g J)|^{-\frac{2}{3}} J^T g J = |\det(J)|^{-\frac{4}{3}} J^T a J,$$

i.e., a represents a section in the bundle $S^2(M) \otimes (\Lambda^2(M))^{-\frac{4}{3}}$.

REMARK 3. It should be mentioned that the situation is different in higher dimensions: First of all, we observe that in dimension 2, there are two different ways that allow us to linearize System [35]. One possible linearization is [36], but another is given as follows. Consider the matrix (σ^{ij}) given by

$$\sigma = |\det(g)|^{\frac{1}{3}} g^{-1}. \tag{39}$$

Rewriting [35] in terms of the new objects σ^{ij}, we obtain the linear system

$$\begin{cases} \sigma^{22}_{,x} - \frac{2}{3} A_1 \sigma^{22} - 2A_0 \sigma^{12} = 0 \\ \sigma^{22}_{,y} - 2\sigma^{12}_{,x} - \frac{4}{3} A_2 \sigma^{22} - \frac{2}{3} A_1 \sigma^{12} + 2A_0 \sigma^{11} = 0 \\ -2\sigma^{12}_{,y} + \sigma^{11}_{,x} - 2A_3 \sigma^{22} + \frac{2}{3} A_2 \sigma^{12} + \frac{4}{3} A_1 \sigma^{11} = 0 \\ \sigma^{11}_{,y} + 2A_3 \sigma^{12} + \frac{2}{3} A_2 \sigma^{11} = 0 \end{cases} \tag{40}$$

Next, we notice that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{g_{11}}{|g_{11}g_{22} - g_{12}^2|^{\frac{2}{3}}} & \frac{g_{12}}{|g_{11}g_{22} - g_{12}^2|^{\frac{2}{3}}} \\ \frac{g_{12}}{|g_{11}g_{22} - g_{12}^2|^{\frac{2}{3}}} & \frac{g_{22}}{|g_{11}g_{22} - g_{12}^2|^{\frac{2}{3}}} \end{pmatrix} = \begin{pmatrix} \sigma^{22} & -\sigma^{12} \\ -\sigma^{12} & \sigma^{11} \end{pmatrix}$$

and therefore $A = (a_{ij})$ is the comatrix of the matrix (σ^{ij}).

Note that the entries of the comatrix, in dimension $n = 2$, are linear in the components of the initial matrix. In higher dimension n, this is not true any longer and the entries of (σ^{ij}) become homogeneous polynomials of degree $n-1$ in the objects a_{ij}. Indeed, if the dimension is $n \geq 3$ the non-linear system analogous to [35] will linearize for new objects σ^{ij}

$$\sigma^{ij} = |\det(g)|^{\frac{1}{n+1}} g^{ij}, \tag{41}$$

defined via the inverse metric g^{ij} corresponding to g (see Proposition 9 below).

Metrizable projective structures in arbitrarily high dimensions are studied in [15], for example. For the two-dimensional case, we refer the reader to [8], where

a (local) geometric obstruction for metrizability is obtained. The metrizability of 3-dimensional path geometries is discussed in [13], including curvature obstructions for local metrizability.

PROPOSITION 9 ([15]). *A metric g on an n-dimensional manifold lies in the projective class of a given connection ∇ if and only if σ^{ij} defined by [41] is a solution of the linear system*

$$\nabla_a \sigma^{bc} - \frac{1}{n+1}(\delta^c_a \nabla_i \sigma^{ib} + \delta^b_a \nabla_i \sigma^{ic}) = 0 \tag{42}$$

with

$$\nabla_a \sigma^{bc} = \sigma^{bc}_{,a} + \Gamma^b_{ad}\sigma^{dc} + \Gamma^c_{ad}\sigma^{db} - \frac{2}{n+1}\Gamma^d_{da}\sigma^{bc},$$

where Γ^k_{ij} are the Christoffel symbols.

REMARK 4. It should be stressed that not all solutions to System [42] correspond to a metric via Formula [41]. Indeed, solutions $\sigma = (\sigma^{ij})$ whose rank is strictly less than n might exist. A solution σ corresponds to a metric g if and only if the solution σ is not degenerate, i.e., has rank equal to n.

7.2. Degree of mobility

Systems [36] and [40] are linearized systems of PDEs equivalent to [35]. As linear PDEs, the solutions (a_{ij}) and (σ^{ij}) to these equations form vector spaces. The dimensions of these two vector subspaces obviously coincide and we say that this dimension is the *degree of mobility* (also called *mobility degree* or simply *mobility*) of the underlying 2-dimensional projective structure $[\nabla]$. In general, by considering Proposition 9, we give the following definition, which holds in any dimension.

DEFINITION 14. *Let $[\nabla]$ be a projective structure on an n-dimensional manifold M. The dimension D of the space of solutions σ of [42] is called* degree of mobility *of $[\nabla]$. Moreover, if g is a metric, then D is called the* degree of mobility *of g iff D is the degree of mobility of the projective class $[\nabla^g]$ of the Levi-Civita metric ∇^g of g.*

As we said, the space of solutions of System [42] is a vector space. This means that if σ_1 and σ_2 are solutions (of maximal rank) to [42], also

$$c_1\sigma_1 + c_2\sigma_2 \overset{[41]}{=} c_1|\det(g_1)|^{\frac{1}{n+1}} g_1^{-1} + c_2|\det(g_2)|^{\frac{1}{n+1}} g_2^{-1}, \quad c_i \in \mathbb{R}, \tag{43}$$

is a solution to [42]. Thus, by computing the metric associated to [43] by inverting Formula [41], we obtain that

$$\frac{\left(c_1|\det(g_1)|^{\frac{1}{n+1}} g_1^{-1} + c_2|\det(g_2)|^{\frac{1}{n+1}} g_2^{-1}\right)^{-1}}{\det\left(c_1|\det(g_1)|^{\frac{1}{n+1}} g_1^{-1} + c_2|\det(g_2)|^{\frac{1}{n+1}} g_2^{-1}\right)} \tag{44}$$

are projectively equivalent metric for any $c_i \in \mathbb{R}$. For later reference we introduce the following notation:

$$\mathcal{M} = \{\sigma : \sigma \text{ solves [42]}\}. \tag{45}$$

Then the degree of mobility is $D = \dim(\mathcal{M})$. The degree of mobility is an important concept for studying metrics with projective vector fields. For example, the following proposition holds:

PROPOSITION 10. *If an n-dimensional metric g has degree of mobility one, i.e. $D = 1$, then all its projective vector fields are homothetic.*

Proof. Let $[\nabla^g]$ be the projective structure induced by the Levi-Civita connection ∇^g of g and assume v is a projective vector field for it. Then $\mathcal{M}$ (cf. [45]) is invariant under the action of v. In other words, if σ is a solution to System [42], then also $\mathcal{L}_v\sigma$ is a solution. We recall that $\mathcal{L}_v\sigma$ is the Lie derivative of σ along v,

$$\mathcal{L}_v\sigma = |\det(g)|^{\frac{1}{n+1}}\,\mathcal{L}_v g + \frac{1}{n+1}\,\mathrm{tr}_g(\mathcal{L}_v g)\,\sigma\,. \tag{46}$$

The claim is then confirmed by choosing adapted coordinates (x^k) such that $v = \partial_{x^1}$. Due to the projective invariance of System [42], the coefficients in [42] cannot depend on x^1. Differentiating [42] w.r.t. x^1, hence yields the claim that $\mathcal{L}_v\sigma$ is also a solution. □

We conclude by remarking on the upper bound of the degree of mobility.

THEOREM 10 ([20]). *Let g be an n-dimensional metric. Then the degree of mobility is bounded by $D_{max} = \binom{n+2}{2}$. The maximum is attained if and only if the metric g has constant sectional curvature.*

8. Metrizability and Integrability

Hamiltonian systems play an important role in mathematical physics, and particularly in classical mechanics, being concerned with the movement of "objects" (such as point particles) over time in a realistic context, even if using a simplified model. According to Newton's law of inertia, *if no force acts on the particle, it will move along a straight line, and with constant velocity.* In other words, it will move along a geodesic of $\mathbb{R}^3$.

A classical two-body problem from celestial mechanics is the (idealized) movement of the earth around the sun, determined by Newton's gravitational formula [32]. The planetary trajectories have been described by Kepler [17, 18] based on Copernicus's heliocentric model [11]. Kepler's laws imply several orbital invariants, i.e. functions $T^*M \to \mathbb{R}$ that remain constant along the planetary trajectories. It is a well-known fact, due to Liouville and Arnold [24, 25] that sufficiently many orbital invariants guarantee *integrability*, i.e. Hamiltonian systems can, in principle, be solved using algebraic operations and the integration of known functions ("quadrature").

8.1. Hamiltonian mechanics on manifolds

Let M be a smooth manifold. Its tangent bundle TM can be interpreted as the space of positions and velocities (at a specified point), and likewise its cotangent bundle T^*M can be viewed as the space of positions and so-called *momenta* ($N = T^*M$ is also called the the *phase space*). We recall that, on N, there is a natural Poisson bracket: If $\theta : N \to T^*N$ is the tautological 1-form, then the formula $\omega = -d\theta$ defines a symplectic structure on N. Using local Darboux coordinates $(q, p) = (q^i, p_i)$ on N, we have

$$\theta(q,p) = \sum p_i dq^i \quad \text{and} \quad \omega = -d\theta = \sum_i dq^i \wedge dp_i .$$

Introducing the Hamiltonian vector field $X_F \in TN$ of a function $F \in C^\infty(N)$, $X_F = \omega^{-1}(dF)$, the natural Poisson bracket is

$$\{-,-\} : C^\infty(N) \times C^\infty(N) \to \mathbb{R}$$

with $\{F_1, F_2\} = \omega(X_{F_1}, X_{F_2}) = X_{F_1}(dF_2) = -X_{F_2}(dF_1)$. Let g be a metric on M. Then the naturally defined function $H : N \to \mathbb{R}$,

$$H_g(q,p) = g_q^{-1}(p,p),$$

is called the *(free) Hamiltonian* associated to g, and often extended by a *potential* $V \in C^\infty(M)$

$$H(q,p) = H_g(q,p) + V(q).$$

Such Hamiltonians naturally arise in problems where *dissipative terms* can be neglected, i.e. if the effects of friction, air resistance, magnetic fields etc. can be ignored.

THEOREM 11. *Let $H_g : T^*M \to \mathbb{R}$ be the Hamiltonian associated to a metric g on a smooth manifold M. Then solutions $(q(t), p(t))$ of*

$$\dot{q} = -\frac{\partial H_g}{\partial p} \qquad \dot{p} = \frac{\partial H_g}{\partial q} \tag{47}$$

project to geodesics of g under π.

Indeed, all geodesics arise in this way. In this sense, solutions of *Hamilton's equations* [47] generalize geodesics.

EXAMPLE 8. Consider $\mathbb{R}^3$ with the flat metric $g = dx^2 + dy^2 + dz^2$ and Hamiltonian $H = p_x^2 + p_y^2 + p_z^2$. Hamilton's equations imply

$$\dot{x} = -2p_x \qquad \dot{y} = -2p_y \qquad \dot{z} = -2p_z \qquad \dot{p}_x = 0 \qquad \dot{p}_y = 0 \qquad \dot{p}_z = 0.$$

We infer $\ddot{x} = 0$, $\ddot{y} = 0$ and $\ddot{z} = 0$, and thus the orbits are straight lines with constant velocity (i.e. geodesics).

Next consider the Hamiltonian (parameterized by μ)

$$H = H_g + V = p_x^2 + p_y^2 + p_z^2 + \frac{\mu}{\sqrt{x^2+y^2+z^2}}.$$

Hamilton's equations then become, with $r = \sqrt{x^2+y^2+z^2}$,

$$\dot{x} = -2p_x \qquad \dot{y} = -2p_y \qquad \dot{z} = -2p_z \qquad \dot{p}_x = -\frac{\mu x}{r^3} \qquad \dot{p}_y = -\frac{\mu y}{r^3} \qquad \dot{p}_z = -\frac{\mu z}{r^3}.$$

We see that the solutions are not geodesics for g any longer unless $\mu = 0$.

8.2. Lagrange formalism

On a manifold M with metric g, the geodesic equation [21] can be obtained and studied in various ways, e.g. using *Lagrange's formalism* or the *Hamilton-Jacobi formalism*. In this section we will discuss the link to the Lagrangian viewpoint, working on TM instead of T^*M. Indeed [21] can be written as

$$\frac{d}{dt}\frac{\partial \mathrm{L}.}{\partial \dot{y}^k} - \frac{\partial \mathrm{L}.}{\partial y^k} = 0$$

with a Lagrangian function

$$\mathrm{L}: v \in TM \to g(v,v) \in \mathbb{R}.$$

The claim can be proven straightforwardly, though a little tediously, once we have specified the Lagrange function L. Locally the correct Lagrangian is given by

$$\mathrm{L} = g_{ij}\dot{y}^i\dot{y}^j.$$

With L at hand, the proof can be completed after substituting [33] into [21].

It remains to verify that this approach is equivalent to the Hamiltonian approach discussed earlier. This is achieved by means of the *Legendre transformation*

$$\mathfrak{L}: TM \ni v \to g(v,-) \in T^*M, \tag{48}$$

which can be interpreted as yielding the differential 1-form *dual* to a vector field v. The Hamiltonian is thus obtained as a mapping

$$H: T^*M \ni \theta \to \mathrm{L}\big(\mathfrak{L}^{-1}(\theta)\big). \tag{49}$$

Locally, the result of the Legendre transformation is

$$H = g^{ij}p_ip_j, \qquad \text{where } p_i = g_{ij}\dot{y}^j. \tag{50}$$

So H indeed coincides with the induced Hamiltonian obtained in the previous paragraph.

8.3. Killing tensor fields

Killing tensor fields are important objects on manifolds. They are defined as symmetric $(0,m)$-tensor fields such that

$$\nabla_X K(X,\dots,X) = 0, \qquad \forall\ X \in \mathfrak{X}(M). \tag{51}$$

We shall see in the next section, that Killing tensor fields are closely linked with orbital constants (integrals) of mechanical systems. In the present section we first want to investigate the interrelation between Killing tensor fields of type $(0, m)$ with $m \leq 2$ and metrizable projective connections. We start by introducing the *projective Killing tensor field* a_{ij}, which is invariant under projective transformations,

$$a_{ij} = |\det(g)|^{-\frac{2}{n+1}} K_{ij}. \tag{52}$$

While this construction is possible in any dimension, a particular phenomenon happens in dimension 2 specifically due to the fact that the entries of the comatrix in dimension 2 are linear in the entries of the initial matrix. We have already discussed this phenomenon, in different context, with System [36]. Using Equation [52] we can then compute the Killing tensor fields from a_{ij}. A direct computation confirms that the Killing equation [51], rewritten in terms of a_{ij}, coincides with [36]. For a 2-dimensional metric g this allows us to compute a Killing tensor field K from a metric $\hat{g} \sim g$ projectively equivalent to g via

$$K = \left|\frac{\det(g)}{\det(\hat{g})}\right|^{\frac{2}{3}} \hat{g} \tag{53}$$

We illustrate this by an example, which can be attributed to Liouville [24]. Let

$$g = (x-y)(dx^2+dy^2), \qquad \hat{g} = \left(\frac{1}{x}-\frac{1}{y}\right)\left(\frac{dx^2}{x}+\frac{dy^2}{y}\right).$$

Both have the same projective connection

$$y'' = -\frac{1}{2(x-y)}(1+y'+y'^2+y'^3)$$

and are therefore projectively equivalent. The solution

$$\sigma^{ij} = |\det(\hat{g})|^{\frac{1}{3}}\hat{g}^{ij}$$

can be expressed in coordinates (x, y) by

$$\sigma^{ij} = -|x-y|^{-\frac{1}{3}}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

and the corresponding Killing tensor field [53] is $K = (y-x)(ydx^2+xdy^2)$.

Killing vector fields. The simplest example of Killing tensor fields occurs if $m = 1$. Using the metric g, we can identify them with Killing vector fields, which simultaneously are projective vector fields. Indeed, if K is a $(0,1)$-Killing tensor field, and $g(X, Y) = K(Y)$ for all $Y \in \mathfrak{X}(M)$, then $\mathcal{L}_X g = 0$, by a direct computation. This result is presented in Example 9 below from a slightly different perspective. We mention, without proof, two further touch points of projective differential geometry and Killing vector fields. The first one has been observed by Gabriel Kœnigs in the 1890s.

THEOREM 12 ([14]). *If a 2-dimensional metric admits four linearly independent (0,2)-Killing tensor fields, then the metric admits a (non-zero) Killing vector field.*

The converse is only true if the dimension of the Killing algebra is precisely 1-dimensional. If, however, there exist two linearly independent Killing vector fields, then a 2-dimensional metric is already of constant Gaussian curvature.

The second result that deserves to be mentioned here is due to Knebelman [19]: the following theorem defines a mapping that in several contexts is very useful when working with 2-dimensional manifolds.

THEOREM 13. *Let $g, \hat{g}$ be projectively equivalent metrics and let ξ be a Killing vector field for g. Then*

$$\hat{\xi}^j = \left|\frac{\det(\hat{g})}{\det(g)}\right|^{\frac{1}{3}} (\hat{g}^{-1})^{jk} g_{ki} \xi^i \tag{54}$$

are components of a Killing vector field of $\hat{g}$.

Note that we can rewrite the map [54] introducing a (0,2)-Killing tensor field h as in [53],

$$\hat{\xi}^j = g^{ja} h_{ab} \xi^b .$$

We may view this in several ways. For example, given the metrics g and $\hat{g}$, or alternatively h, we can view it as a map from the space of Killing vector fields into the space of projective vector fields (note that this is a linear map as $|\det(\hat{g})|^{\frac{1}{3}} \hat{g}^{-1}$ is linear in a_{ij}). Alternatively, we could view it as a map from the metrization space $\mathscr{M}$ to the space of Killing vector fields.

8.4. Integrals of the geodesic equation

In the previous section we have discussed Killing tensor fields. When discussing classical mechanics the concept of *constants of the motion, integrals of the geodesic equation* and *integrals of the geodesic flow* is often used alternatively.

DEFINITION 15. *A first integral of the geodesic equation $\nabla_{\dot\gamma}\dot\gamma = \frac{D\dot\gamma}{dt} = 0$ is a function $f : TM \to \mathbb{R}$ such that $f(\dot\gamma(t))$ is constant for any (parametrized) geodesic γ.*

Locally, taking into account [21], f is a first integral if and only if

$$0 = \frac{d}{dt} f\big(\dot\gamma(t)\big) = \frac{d}{dt} f\big(t, y^k(t), \dot y^k(t)\big) = f_{y^k}\dot y^k + f_{\dot y^k}\ddot y^k = f_{y^k}\dot y^k - f_{\dot y^k}\Gamma^k_{ij}\dot y^i \dot y^j .$$

Dually, in presence of a metric g on M, we can relate the existence of first integrals of the geodesic equation with those of the Hamiltonian [50].

DEFINITION 16. *A first integral of a Hamiltonian function $H : T^*M \to \mathbb{R}$ is a*

function $F : T^*M \to \mathbb{R}$ such that

$$\{F, H\} = 0, \quad \textit{where} \quad \{F, H\} = \frac{\partial F}{\partial y^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial y^i}.$$

The bracket $\{\cdot,\cdot\}$ is called the Poisson bracket.

This definition is motivated by the following observation: on account of [47], we have that

$$\dot{F} = \frac{\partial F}{\partial y^i}\dot{y}^i + \frac{\partial F}{\partial p_i}\dot{p}_i = -\frac{\partial F}{\partial y^i}\frac{\partial H}{\partial p_i} + \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial y^i} = \{H, F\} = 0$$

Note that the Hamiltonian itself is also an integral of the motion. A direct computation yields the following proposition.

PROPOSITION 11. *Let g be a metric on M. Then a function $f : TM \to \mathbb{R}$ is an integral of the geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$ if and only if $F = f \circ \mathfrak{L}^{-1} : T^*M \to \mathbb{R}$ is a first integral of the Hamiltonian [49].*

First integrals polynomial in momenta. An important class of first integrals are those that are *homogeneous polynomials* with respect to the coordinates $\dot{y}^i$ of TM (or, equivalently, the coordinates p_i of T^*M). In particular, the existence of linear or quadratic first integrals has a direct geometrical interpretation, especially for $\dim M = 2$. Since any homogeneous polynomial I of degree k can be written as

$$I(\xi) = K(\xi, \ldots, \xi) \tag{55}$$

where K is a (totally) symmetric tensor field of type $(0, k)$, we can interpret first integrals that are homogeneous polynomials in the fibre coordinates as symmetric $(0, k)$-tensor fields. Locally, if $\xi = \dot{y}^i \partial_{y^i}$,

$$I(\xi) = K_{i_1 \ldots i_k} \dot{y}^{i_1} \cdots \dot{y}^{i_k}$$

where $K_{i_1 \ldots i_k} \in C^\infty(M)$. By a direct computation, one verifies that $K_{i_1 \ldots i_k}$ are components of a Killing tensor field.

LEMMA 5. *If $F = K^{j_1 \ldots j_m} p_{j_1} \cdots p_{j_m}$ is a first integral for $H = H_g$, then*

$$K_{i_1 \ldots i_m} = g_{i_1 j_1} \cdots g_{i_m j_m} K^{j_1 \ldots j_m}$$

are components of a Killing tensor field of g.

In particular, note that there is a 1-to-1 correspondence between Killing vector fields and first integrals linear in momenta. The example below discusses this aspect in detail.

EXAMPLE 9. Let g be a metric on M. Then the function $f = K_j \dot{y}^j$, $K_j \in C^\infty(M)$, is a first integral of the geodesic equation (or, equivalently, in view of Proposition 11, the function $K^j p_j$ is a first integral of the Hamiltonian H_g) if and only if $g^{hk} K_k \partial_h$ is a Killing vector field of g.

Proof. Locally, by using the formulas in the beginning of Section 7, we have that

$$\mathscr{L}_X g = \mathscr{L}_X(g_{ij} dy^i \otimes dy^j) = \left(X(g_{ij}) + g_{hj} X^h_{y^i} + g_{ih} X^h_{y^j}\right) dy^i \otimes dy^j .$$

Thus, $g^{hk} K_k \partial_h$ is a Killing vector field of the metric g if and only if

$$g^{hk} K_k \frac{\partial g_{ij}}{\partial y^h} + g_{hj}\left(\frac{\partial g^{hk}}{\partial y^i} K_k + g^{hk} \frac{\partial K_k}{\partial y^i}\right) + g_{hi}\left(\frac{\partial g^{hk}}{\partial y^j} K_k + g^{hk} \frac{\partial K_k}{\partial y^j}\right) = 0. \tag{56}$$

Taking into account the Levi-Civita conditions [33], a straightforward computation shows that [56] assumes the form

$$\frac{1}{2}\left(\frac{\partial K_j}{\partial y^i} + \frac{\partial K_i}{\partial y^j}\right) - K_k \Gamma^k_{ij} = 0, \tag{57}$$

that is the condition for which $f = K_j \dot{y}^j$ is a first integral. □

Note that, in the situation of Example 9, the differential 1-form $\rho = K_j dy^j$ is a Killing tensor field and $\mathfrak{L}^{-1}(\rho)$, where $\mathfrak{L}$ is defined by [48], is the Killing vector field of Example 9.

Lax pairs. In Section 7.1 we mentioned that the system [36] can be formulated in terms of a Lax pair, i.e. there (locally) exist matrices $\mathbf{L}, \mathbf{M}$ (representing $(1,1)$-tensor fields) such that

$$\frac{d}{dt}\mathbf{L} = [\mathbf{M}, \mathbf{L}].$$

Following [26], we introduce the matrices $\mathbf{L} = (L^a{}_b)$ and $\mathbf{M} = (M^a{}_b)$ by

$$L^a{}_b = F g^{ak} p_k p_b, \qquad M^a{}_b = 2 g^{mk} p_k \Gamma^a_{mb},$$

where $F = |\det(g)|^{2/3} a_{kl} g^{ki} g^{lj} p_i p_j$. The function F thus is a first integral of the natural Hamiltonian $H = g^{ij} p_i p_j$ induced by the metric, i.e. $\dot{F} = 0$. A direct computation using Hamilton's equations [47] for H indeed shows (commas denote partial derivatives with respect to a coordinate direction)

$$\begin{aligned}
\dot{L}^a{}_b &= F g^{ac}_{,k} \dot{q}^k p_c p_b + F g^{ac} \dot{p}_c p_b + F g^{ac} p_c \dot{p}_b \\
&= F g^{ac}_{,k} (-2 g^{jk} p_j) p_c p_b + F g^{ac} (g^{ij}_{,c} p_i p_j) p_b + F g^{ac} p_c (g^{ij}_{,b} p_i p_j) \\
&= 2 F g^{jk} g^{cm} \Gamma^a_{mk} p_j p_c p_b - 2 F g^{ac} p_c g^{im} \Gamma^j_{mb} p_i p_j \\
&= M^a{}_k L^k{}_b - L^a{}_j M^j{}_b = [M, L]^a{}_b
\end{aligned}$$

where we have used the formula

$$g^{ij}_{,c} = -\left(g^{im}\Gamma^{j}_{mc} + g^{jm}\Gamma^{i}_{mc}\right),$$

which follows from $(g^{ia}g_{aj})_{,c} = 0$ and Formula [33] for the Christoffel symbols.

First integrals quadratic in momenta. Within the context of metrizable projective structures, first integrals quadratic in momenta play a particular role due to the following theorem, which we already met in the case $n = 2$ in Section 8.3.

THEOREM 14 ([36, 40]). *Let g and $\hat{g}$ be metrics on an n-dimensional manifold M. If g and $\hat{g}$ are projectively equivalent, then*

$$I : TM \ni \xi \to I(\xi) = \left|\frac{\det(g)}{\det(\hat{g})}\right|^{\frac{2}{n+1}} \hat{g}(\xi,\xi) \tag{58}$$

is a quadratic integral of the geodesic equation of g. In other words,

$$\left|\frac{\det(g)}{\det(\hat{g})}\right|^{\frac{2}{n+1}} \hat{g} \tag{59}$$

is a Killing tensor field of the geodesic equation of g.
In dimension $n = 2$, more precisely, if $\hat{g}$ is projectively equivalent to g, it is of the form

$$\hat{g} = \left(\frac{\det(g)}{\det(h)}\right)^2 h,$$

where h is a quadratic integral of g with $\det(h) \neq 0$.

COROLLARY 4. *The degree of mobility of a 2-dimensional metric and the dimension of its space of quadratic integrals coincide.*

EXAMPLE 10. Let us consider the 2-dimensional metric

$$g = (4x^2 + y^2 + 1)(dx^2 + dy^2).$$

The vector space of the quadratic integrals of this metric is generated by the following quadratic functions on TM:

$$\begin{cases} \mathrm{L} = (4x^2 + y^2 + 1)(\dot{x}^2 + \dot{y}^2), \\ F_1 = (4x^2 + y^2 + 1)(y^2\dot{x}^2 - (4x^2 + 1)\dot{y}^2), \\ F_2 = (4x^2 + y^2 + 1)\big(xy^2(\dot{x}^2 + \dot{y}^2) + (4x^2 + y^2 + 1)\dot{y}(x\dot{y} - y\dot{x})\big). \end{cases}$$

Note that L is the Lagrangian. The above quadratic integrals can be given in terms of Killing tensor fields

$$\begin{cases} K_1 := g = (4x^2 + y^2 + 1)(dx^2 + dy^2), \\ K_2 = (4x^2 + y^2 + 1)(y^2dx^2 - (4x^2 + 1)dy^2), \\ K_3 = (4x^2 + y^2 + 1)\big(xy^2(dx^2 + dy^2) + (4x^2 + y^2 + 1)dy(xdy - ydx)\big), \end{cases}$$

or, equivalently, as integrals of the Hamiltonian $H = g^{ij}p_ip_j$, $(p,q) = (p_1,p_2)$ (cf. [50]):

$$\begin{cases} H = \frac{p^2+q^2}{4x^2+y^2+1}, \\ F_1 = \frac{y^2p^2-(4x^2+1)q^2}{4x^2+y^2+1}, \\ F_2 = \frac{-xy^2p^2+y(4x^2+y^2+1)pq-x(4x^2+2y^2+1)q^2}{4x^2+y^2+1}. \end{cases}$$

Then, by inverting Formula [59], we obtain the local description of all metrics that are projectively equivalent to g. We note that the metric g does not admit any (non-zero) Killing vector field, i.e., g does not possess linear integrals.

First integrals as "horizontalizations" in jet spaces. Another, slightly different viewpoint on first integrals is the following. The form $\rho = K_j dy^j$ can be "horizontalized", i.e., it can be restricted to the (truncated) total derivative $D_t = \partial_t + \dot{y}^k\partial_{y^k}$, so that $K_j\dot{y}^j dt$ is obtained, i.e., a volume form on $\mathbb{R}$ with coefficients depending on the first jet variables $(t, y^k, \dot{y}^k)$. The "volume" function $f = K_j\dot{y}^j$ is that appearing in Example 9. In other words, one can introduce a 'horizontal differential" d_h, which takes $f = K_j\dot{y}^j$ to

$$d_h(f) := \left(\frac{\partial K_j}{\partial y^i}\dot{y}^i\dot{y}^j + K_j\ddot{y}^j\right)dt = \left(\frac{1}{2}\left(\frac{\partial K_j}{\partial y^i} + \frac{\partial K_i}{\partial y^j}\right)\dot{y}^i\dot{y}^j + K_j\ddot{y}^j\right)dt.$$

If we restrict $d_h(f)$ to the geodesic equation [21] ("on shell" condition) and equate the result to zero, we obtain the conditions [57]. Similarly, we can do the same reasoning for a generic polynomial [55] of degree k. Essentially, one can say that a first integral of the geodesic equation [21] is a function on $J^0 = M$ whose horizontal differential vanishes on the equation. This reasoning can actually be extended to general systems of PDE by using the so called *$\mathscr{C}$-spectral sequences, variational sequences* and *variational bicomplexes*. For more insight about this topics, see [1,3, 42].

8.5. The theorems by Liouville-Arnold and Topalov-Matveev

By direct computation, one can show the following result, which is a special case of Emmy Noether's correspondence principle of symmetries and integrals of motion [33]. Let us consider the particular case of $(0,1)$-Killing tensor fields.

In view of Example 9, if $F = K^j(q)p_j$ is a first integral of the Hamiltonian H_g, then $\xi = K^j(q)\partial_j$ is a Killing vector field of g, and vice versa. Now assume we have a Hamiltonian $H(q,p)$ which does not depend on a particular coordinate q^j. By Hamilton's equations we thus have $\dot{p}_j = 0$ and so $F = p_j$ is a first integral. We infer that $\frac{\partial}{\partial q^j}$ is a Killing vector field. The converse is also true, and is a simple example of *Noether's theorem*: Symmetries of a mechanical system correspond to functions that are constant along trajectories of the system.

Theorem 14 provides, in arbitrary dimension $n \geq 2$, an integral of motion. In fact one can find more: the Topalov-Matveev theorem [40] even asserts integrability

for n-dimensional Riemannian manifolds. This theorem is our goal in this section, for which we need some preparations.

DEFINITION 17 (Liouville-Arnold). *Let H be a Hamiltonian on a manifold of dimension n. Let $F_k : T^*M \to \mathbb{R}$ $(1 \leq k \leq n)$ be n first integrals for H, i.e.*

$$\{H, F_k\} = 0,$$

and moreover

- $F_1 = H$
- $(F_k)_{1 \leq k \leq n}$ *are functionally independent*
- $\{F_i, F_j\} = 0$ *for each pair* (i, j).

Then the collection $(H, F_2, \ldots, F_n)$ is called an integrable system.

Sometimes, one says that H is integrable without specifying the functions F_k, meaning that such functions exist. Note that the functions F_k are not unique as we can go over to linear combinations of them. The following theorem is fundamental.

THEOREM 15 (Liouville-Arnold). *Let M be an n-dimensional manifold with integrable system $(H, F_2, \ldots, F_n)$. Then*

1. *The solution trajectories can be found using only methods of* quadrature, *i.e. algebraic operations and the integration of specific functions.*
2. *There are new coordinates of T^*M, called* action-angle variables Q, P, *such that*

$$\dot{Q} = -\frac{\partial H(Q,P)}{\partial P} \qquad \dot{P} = 0$$

3. *The level sets $L(c_1, \ldots, c_n) = \{\mu \in T^*M : F_i(\mu) = c_i\}$ are* lagrangian submanifolds, *i.e. the level sets L are maximal and satisfy $\omega|_{TL} = 0$. The phase-space is therefore locally of the form $\mathbb{R}^m \times \mathbb{T}^{2n-m}$.*

Let us look at dimension 2. A corollary of Theorem 14 is the following.

COROLLARY 5. *If $g \sim \hat{g}$ are 2-dimensional projectively equivalent metrics, then H_g and $H_{\hat{g}}$ are integrable Hamiltonians.*

Somewhat surprisingly, an analogous fact about the integrability of the Hamiltonian holds in higher dimensions as well. This is the statement of the following theorem.

THEOREM 16 (Topalov-Matveev [40]). *If $g \sim \hat{g}$ are n-dimensional projectively equivalent Riemannian metrics $(n \geq 2)$, then H_g and $H_{\hat{g}}$ are integrable Hamiltonians.*

Proof. We only give a rough overview of how the integrals are obtained. First, define the endomorphism $G: TM \to TM$, $G^i_j = g^{ia}\hat{g}_{aj}$. Its characteristic polynomial is

$$\chi(t) = \det(G - t\mathbb{1}) = (-1)^n t^n + \cdots + c_n$$

with $c_0 = (-1)^n$. Next, define

$$S_k = \left|\frac{\det(g)}{\det(\hat{g})}\right|^{\frac{k+2}{n+1}} \sum_{i=0}^{k} c_i G^{k-i+1}, \quad k \in \{0, \ldots, n-1\}. \tag{60}$$

With $g(\xi, -) = p$, the first integrals of H_g are thus obtained as

$$F_k(x, p) = g(S_k\xi, \xi). \tag{61}$$

Such integrals commute pairwise. □

It is worthwhile to mention two special cases of Formula (60). The first special case occurs if $k = 0$. We then have

$$S_0 = (-1)^n \left|\frac{\det(g)}{\det(\hat{g})}\right|^{\frac{2}{n+1}} G$$

and

$$F_0(x, p) = g\left((-1)^n \left|\frac{\det(g)}{\det(\hat{g})}\right|^{\frac{2}{n+1}} G(\xi), \xi\right) = (-1)^n \left|\frac{\det(g)}{\det(\hat{g})}\right|^{\frac{2}{n+1}} \hat{g}(\xi, \xi),$$

which coincides, possibly up to a sign, with Painlevé's classical first integral [58]. The other special case we would like to mention here occurs for $k = n-1$,

$$S_{n-1} = \left|\frac{\det(g)}{\det(\hat{g})}\right| \sum_{i=0}^{n-1} c_i G^{k-i+1} = -c_n \left|\frac{\det(g)}{\det(\hat{g})}\right| \mathbb{1} = \pm\mathbb{1}.$$

Here we have used that $c_n = \det(G) = \frac{\det(\hat{g})}{\det(g)}$. We conclude that the integral F_{n-1} associated to S_{n-1} is the Hamiltonian H_g, possibly up to a sign.

9. Classical problems posed by Sophus Lie

In his seminal 1882 work on projective symmetry algebras, Sophus Lie states the following classical problems on metrizable projective structures:

Lie's first problem [21]: Find all 2-dimensional metrics admitting a Lie algebra of projective vector fields of dimension greater or equal to 1.

Lie's second problem [21]: Find all 2-dimensional metrics admitting a Lie algebra of projective vector fields of dimension greater or equal to 2.

The above problems are subject to various interpretations. For instance, one can study both the local and global aspects of the problems. One could attempt to

find a local description of such metrics without caring if the obtained list is composed of pairwise non-isometric ones, or ask for a sharp list. We would like to mention that Lie's problems can be posed in higher dimensions, too. For instance, [29] studies the analog of Lie's problems in dimension 3.

Below, we shall consider the local version of Lie's problems by giving a strategy to obtain a list of pairwise non-isometric metrics satisfying the requirements of the above problems. Actually the solutions of the above two problems require different techniques. We first discuss how to solve Lie's second problem; we then proceed to metrics admitting exactly one projective vector field, thus solving also Lie's first problem. This corresponds to the chronological development, as Lie's second problem was indeed solved first [9].

9.1. The main theorem on Lie's second problem

We are now going to discuss Lie's second problem. To facilitate the exposition, we begin with the following, preliminary proposition suggested by Theorem 9 as well as Examples 3, 4 and 6. It concerns the maximal dimension of the Lie algebra of projective vector fields of 2-dimensional metrics.

PROPOSITION 12. *A 2-dimensional projective connection admitting an 8-dimensional Lie algebra of projective vector fields is metrizable. In particular, any 2-dimensional metric whose projective connection admits an 8-dimensional Lie algebra of projective vector fields is of constant Gaussian curvature, and vice versa.*

The purpose of the current section is to understand the idea of the proof of the following theorem.

THEOREM 17 ([9]). *Let g be a 2-dimensional metric on a manifold M and assume that the dimension of the Lie algebra of the projective vector fields is greater or equal to 2. Then, almost every point of M has a neighborhood U such that the restriction of g to U has constant curvature, or such that there exists a coordinate system (x, y) on U in which the metric g takes one of the following forms:*

1. *Metrics with a 2-dimensional Lie algebra of projective vector fields.*
   (a) $\varepsilon_1 e^{(b+2)x}dx^2 + \varepsilon_2 e^{bx}dy^2$, *where* $b \in \mathbb{R}\setminus\{-2,0,1\}$ *and* $\varepsilon_i \in \{-1,1\}$ *are constants,*
   (b) $a\left(\frac{e^{(b+2)x}dx^2}{(e^{bx}+\varepsilon_2)^2} + \varepsilon_1\frac{e^{bx}dy^2}{e^{bx}+\varepsilon_2}\right)$, *where* $a \in \mathbb{R}\setminus\{0\}$, $b \in \mathbb{R}\setminus\{-2,0,1\}$, *and* $\varepsilon_i \in \{-1,1\}$ *are constants, and*
   (c) $a\left(\frac{e^{2x}dx^2}{x^2} + \varepsilon\frac{dy^2}{x}\right)$, *where* $a \in \mathbb{R}\setminus\{0\}$, *and* $\varepsilon \in \{1,-1\}$ *are constants.*
2. *Metrics with a 3-dimensional Lie algebra of projective vector fields.*
   (a) $\varepsilon_1 e^{3x}dx^2 + \varepsilon_2 e^{x}dy^2$, *where* $\varepsilon_i \in \{-1,1\}$ *are constants,*

(b) $a\left(\frac{e^{3x}dx^2}{(e^x+\varepsilon_2)^2}+\varepsilon_1\frac{e^x dy^2}{(e^x+\varepsilon_2)}\right)$, *where* $a\in\mathbb{R}\setminus\{0\}$, $\varepsilon_i\in\{-1,1\}$ *are constants, and*

(c) $a\left(\frac{dx^2}{(2x^2+cx+\varepsilon_2)^2x}+\varepsilon_1\frac{xdy^2}{(2x^2+cx+\varepsilon_2)}\right)$, *where* $a>0$, $\varepsilon_i\in\{-1,1\}$, $c\in\mathbb{R}$ *are constants.*

No two distinct metrics from this list are isometric.

This theorem provides, in a neighborhood of almost every point, a solution to Lie's second problem. The proof we present here is based on the research article [9] and based on the following steps.

Strategy for the solution of Lie's second problem. To solve Lie's second problem we shall follow the following strategy, that can be summarized in the following steps:

(1) Understand which Lie algebras are admissible as Lie algebras of projective vector fields;

(2) Find all possible realizations of the above Lie algebras in terms of vector fields on an open subset of $\mathbb{R}^2$;

(3) Find all projective connections [11], up to a coordinate changing [12], admitting the vector fields of point 9.1 as projective vector fields;

(4) Once obtained the projective connections of point (3), understand which of them are metrizable by concretely finding all metrics realizing them.

Step 1

S. Lie proved [22] that the Lie algebras of 2-dimensional metrics, or, equivalently, the symmetry Lie algebras of the ODEs [11], can only be the following ones:

$$0,\quad \mathbb{R},\quad \mathfrak{s}:[X,Y]=X,\quad \mathfrak{sl}(2,\mathbb{R}),\quad \mathfrak{sl}(3,\mathbb{R}), \tag{62}$$

where $\mathfrak{s}$ is the 2-dimensional non-commutative Lie algebra. Since in this section we are interested in metrics admitting at least a 2-dimensional Lie algebra of projective vector fields, we shall concentrate on the last three ones. In view of Proposition 12, if a metric admits $\mathfrak{sl}(3,\mathbb{R})$ as Lie algebra of projective vector fields it is of constant curvature, being $\dim\mathfrak{sl}(3,\mathbb{R})=8$. Note that there is a dimensional gap in the list (62) as $\dim\mathfrak{sl}(2,\mathbb{R})=3$ whereas $\dim\mathfrak{sl}(3,\mathbb{R})=8$. Therefore, if a metric admits 4 independent projective vector fields, then it is of constant curvature. Furthermore, since in this section we are interested in metrics admitting ai least a 2-dimensional Lie algebra of projective vector fields, the first two Lie algebras of [62] will not ne considered here. Thus, we will consider only

$$\mathfrak{s}:[X,Y]=X \quad \text{and} \quad \mathfrak{sl}(2,\mathbb{R}).$$

Step 2

S. Lie [21, 23] found all possible realizations of Lie algebra $\mathfrak{s}:[X,Y]=X$ in terms of vector fields on an open subset of $\mathbb{R}^2$, assuming the orbits have a fixed constant

dimension. He showed that, for almost every point, there exist coordinates (x, y) such that vector fields X, Y satisfying $[X, Y] = X$, by excluding the trivial case $X = 0$, are described by

1. *Orbits of dimension* 1: $X = e^y\partial_y$, $Y = -\partial_y$;
2. *Orbits of dimension* 2: $X = \partial_y$, $Y = \partial_x + y\partial_y$.

Realizations of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ follows from the above ones by considering that there is a basis (X, Y, Z) of $\mathfrak{sl}(2,\mathbb{R})$ satisfy the relations

$$[X, Y] = X, \quad [X, Z] = Y, \quad [Y, Z] = Z. \tag{63}$$

For a reason that will be clear later, for the computation of Z we shall consider only X, Y coming from the case 2 (indeed, later on, we shall see that projective vector fields of the case 1 lead to metrics of constant curvature). Conditions [63] translate into a system of PDEs whose general solution is

$$Z = (y + C_1 e^x)\partial_x + \left(\frac{y^2}{2} + C_2 e^{2x}\right)\partial_y, \quad C_i \in \mathbb{R}. \tag{64}$$

Step 3

Projective connections admitting 1 of Step 2 as projective vector fields. In order to find the most general projective connection admitting (at least) $X = e^y\partial_y$ and $Y = -\partial_y$ as projective vector fields one can consider System [28]. In fact, since $Y = -\partial_y$, i.e., $a = 0, b = -1$ is a solution to such system, then $A_i = A_i(x)$. Then, by substituting $a = 0, b = e^y$ in the obtained system and by integrating it we obtain $A_0 = 0$, $A_1 = A_1(x)$, $A_2 = 1$ and $A_3 = 0$, i.e., the projective connection we were looking for is

$$y_{xx} = A_1(x)y_x + y_x^2. \tag{65}$$

Projective connections admitting the orbit of Step 2 as projective vector fields. By applying the same reasoning as above, one can find that the most general projective connection admitting (at least) $X = \partial_y$, $Y = \partial_x + y\partial_y$ as projective vector fields is

$$y_{xx} = K_0 e^x + K_1 y_x + K_2 e^{-x} y_x^2 + K_3 e^{-2x} y_x^3, \quad K_i \in \mathbb{R}. \tag{66}$$

Step 4

Metrizability of the projective connection.
(Here we refer to [65].)

One can prove that [65] admits an 8-dimensional Lie algebra of symmetries. This is actually not an easy task as one should study the solution space of [28] with $A_0 = 0$, $A_1 = A_1(x)$, $A_2 = 1$ and $A_3 = 0$. A possible alternative way is to see if there exists a map of type [12] transforming [65] into $v_{uu} = 0$. This problem has been solved

in full generality by Liouville in [24]: a projective connection [11] can be mapped into $v_{uu} = 0$ by [12] if and only if

$$2A_{1xy} - A_{2xx} - 3A_{0yy} - 6A_0A_{3x} - 3A_3A_{0x} + 3A_0A_{2y} + 3A_2A_{0y} + A_1A_{2x} - 2A_1A_{1y} = 0$$
$$2A_{2xy} - A_{1yy} - 3A_{3xx} + 6A_3A_{0y} + 3A_0A_{3y} - 3A_3A_{1x} - 3A_1A_{3x} - A_2A_{1y} + 2A_2A_{2x} = 0$$

and a simple computation shows that [65] is such a projective connection. We conclude, in view of Proposition 12, that [65] is a metrizable projective connection and the metrics admitting such a projective connection are of constant Gaussian curvature.

Metrizability of the projective connection.
(Here we refer to [66].)

We have to distinguish two cases:

Case 1: the projective connection [66] admits *exactly* a 2-dimensional Lie algebra of projective symmetries;

Case 2: the projective connection [66] admits *exactly* a 3-dimensional Lie algebra of projective symmetries.

As we already said, if a metric admits a Lie algebra of projective vector fields of dimension greater than 3, then it is of constant curvature.

Discussion of Case 1. Here we sketch the method for obtaining the metrics 1a-1c of Theorem 17.

Since ∂_y is a projective vector field, by substituting $a = 0$ and $b = 1$ into System [28], one obtains that functions A_i depend on x only, implying that the coefficients of the linear system [40] (equivalently [36]) do not depend on y. Therefore, if $\sigma = \sigma^{ij}$ is a solution to [40], also $\frac{\partial^k \sigma}{\partial y^k}$ is a solution ($\forall\, k \in \mathbb{N}$), implying that any linear combination $\sum_{k=0} c_k \frac{\partial^k \sigma}{\partial y^k}$ is a solution ($c_k \in \mathbb{R}$). In view of Corollary 4, the dimension of the space of solutions to System [40] coincides with the dimension of the space of the quadratic integrals. On the other hand, the maximum dimension of such space is 6 in view of a theorem by Koenigs [14] (he also proved that the dimension 6 is attained only in the case of constant Gaussian curvature). Thus, any seven solutions to System [40] must be dependent, i.e., there always exist seven constants $c_i \in \mathbb{R}$, with some $c_i \neq 0$, such that

$$\sum_{k=0}^{6} c_k \frac{\partial^k \sigma}{\partial y^k} = 0\,.$$

But we have more: it can be shown that a metric whose Lie algebra of projective vector fields is 2-dimensional has degree of mobility (cf. Section 7.2 for definitions) less than four (see page 453 of [9]). Then, in the case under consideration, four independent solutions to System [40] cannot exist, so there must exist constants

$(c_0, c_1, c_2, c_3) \neq (0,0,0,0)$ such that

$$c_0\sigma + c_1\frac{\partial\sigma}{\partial y} + c_2\frac{\partial^2\sigma}{\partial y^2} + c_3\frac{\partial^3\sigma}{\partial y^3} = 0. \tag{67}$$

Also, we have a further restriction: indeed (see page 454 of [9]) it is possible to assume

$$K_2 \neq 0 \text{ and } K_1 = K_3 = 0, \quad \text{or}$$
$$K_2 = 0, K_3 \neq 0 \text{ and if } K_0 = 0 \text{ then } K_1 \notin \{\tfrac{1}{2}, 2\}.$$

Conditions [67], together with the above restrictions on the constants K_i, lead to all solutions σ to System [40] and then to metrics 1a-1c of Theorem 17. The proof of the fact that these metrics are pairwise non-isometric is contained in Section 2.7 of [9].

Discussion of Case 2. The strategy for solving this case, in principle, could go closely to that employed for the previous one, but, in view of the presence of an extra projective vector field, some simplifications arise. We only sketch the main ideas used in [9] leading to metrics 2a-2c of Theorem 17. Indeed, in [9] it is proved that the only possibility for the projective connection [66] to admit an infinitesimal symmetry of type [64] and, in the same time, to admit *exactly* a 3-dimensional symmetry Lie algebra, is that $K_0 = K_2 = 0$, $K_1 = \frac{1}{2}$, $K_3 \neq 0$, $C_1 = C_2 = 0$. To sum up, we have to consider the projective connection

$$y_{xx} = \frac{1}{2}y_x + K_3 e^{-2x} y_x^3, \quad K_3 \neq 0$$

with the following vector fields as a basis of the projective Lie algebra:

$$X = \partial_y, \quad Y = \partial_x + y\partial_y, \quad Z = y\partial_x + \frac{y^2}{2}\partial_y. \tag{68}$$

Furthermore, 2-dimensional metrics admitting exactly a 3-dimensional Lie algebra of projective symmetries are *superintegrable*, i.e., the dimension of the space of its quadratic integrals is four (see Lemma 2 and Corollary 3 of [9]) and such metrics admit a (non-zero) Killing vector field (see Theorem 12). Without loss of generality, we may suppose that X or Y or Z of [68] is a Killing vector field for the metric g, thus obtaining three distinct cases. Detailed computations are contained in Section 2.6 of [9].

9.2. Lie's first problem: a sketch of the main ideas for its solution

Lie's first problem has been solved in [28, 30]: a description of 2-dimensional metrics admitting a 1-dimensional Lie algebra of projective vector fields is found in [30], pioneering ideas that have been reconsidered in [28]. In the latter work sharp (i.e., pairwise non-isometric) normal forms are obtained. Here we can only give a brief outline of the main ideas in these papers. We first give a short review of [30] and then of [28]. Roughly speaking, the main results in [30] are based on the following

observations. First, if v is a projective vector field, then $\mathscr{L}_v$ sends solutions of the system [40] into solutions of the same system. Recalling the definition of $\mathscr{M}$ in [45], $\mathscr{L}_v$ is an endomorphism:

$$\mathscr{L}_v : \mathscr{M} = \{\sigma : \sigma \text{ solves } [40]\} \to \mathscr{M}. \tag{69}$$

In actuality, reference [30] considers the space of solutions of [36] rather than $\mathscr{M}$ defined in [45]; as we have pointed out in Section 7.1, however, both approaches are equivalent in the 2-dimensional case.

We suppose $\dim \mathscr{M} > 1$, as otherwise all projective vector fields are homothetic due to Proposition 10. Moreover, we suppose that the metric does not admit (non-zero) Killing vector fields. Recalling Theorem 12 and Corollary 4, we infer that $\dim \mathscr{M} < 4$. We therefore assume $\dim \mathscr{M} \in \{2,3\}$ in the following. For simplicity, we focus on the case $\dim \mathscr{M} = 2$, based on which the case $\dim \mathscr{M} = 3$ can be treated similarly. Therefore, with the assumptions made, we may consider $\mathscr{L}_v$ to be an endomorphism of the 2-dimensional vector space $\mathscr{M}$. Multiplying v by a non-zero constant and choosing an appropriate basis of $\mathscr{M}$, the matrix describing $\mathscr{L}_v$ thus assumes one of the following Jordan forms (with $\lambda \in \mathbb{R}$):

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}. \tag{70}$$

Due to [69], we further infer

$$\mathscr{L}_v \begin{pmatrix} \sigma \\ \hat{\sigma} \end{pmatrix} = A^T \begin{pmatrix} \sigma \\ \hat{\sigma} \end{pmatrix} \tag{71}$$

where $\sigma, \hat{\sigma}$ is a basis such the matrix A representing $\mathscr{L}_v$ is given by one of the matrices of [70] and $\mathscr{L}_v\sigma$ and $\mathscr{L}_v\hat{\sigma}$ are given by [46].

PROPOSITION 13 ([2, 5, 12]). *Let g and $\hat{g}$ be projectively equivalent, non-proportional 2-dimensional metrics. Then, in a neighborhood of almost every point, there are coordinates (x, y) such that the metrics assume one of the following three normal forms:*

	Liouville	*Complex-Liouville*	*Jordan-Block*
g	$(f_1 - f_2)(dx^2 \pm dy^2)$	$(\overline{h(z)} - h(z))\,(d\overline{z}^2 - dz^2)$	$(1 + x f_2')dxdy$
$\hat{g}$	$\left(\frac{1}{f_1} - \frac{1}{f_2}\right)\left(\frac{dx^2}{f_1} \pm \frac{dy^2}{f_2}\right)$	$\left(\frac{1}{h(z)} - \frac{1}{h(z)}\right)\left(\frac{d\overline{z}^2}{h(z)} - \frac{dz^2}{h(z)}\right)$	$\frac{1+xf_2'}{f_2^4}\,(-2f_2\,dxdy$ $+(1+xf_2')\,dy^2)$

Here, $f_1 = f_1(x)$ and $f_2 = f_2(y)$ are functions of one variable only, and in the complex Liouville case we use coordinates $z = x + iy$, $\overline{z} = x - iy$.

Recalling [39],

$$\sigma = |\det(g)|^{\frac{1}{3}} g^{-1}, \quad \sigma = |\det(\hat{g})|^{\frac{1}{3}} \hat{g}^{-1},$$

and taking into account Proposition 13, System [71] translates into a system of PDEs with the metric coefficients of $g, \hat{g}$ and the components of v as unknown functions. We obtain nine such systems, corresponding to the possible cases of [70] and of Proposition 13. The systems can be integrated: details of computation can be found in [30].

Once g and $\hat{g}$ have been obtained, Formula [44] provides all metrics that are in the same projective class as g and $\hat{g}$:

$$\mathfrak{P} = \left\{ \frac{\left(c_1 |\det(g)|^{\frac{1}{n+1}} g^{-1} + c_2 |\det(\hat{g})|^{\frac{1}{n+1}} \hat{g}^{-1}\right)^{-1}}{\det\left(c_1 |\det(g)|^{\frac{1}{n+1}} g^{-1} + c_2 |\det(\hat{g})|^{\frac{1}{n+1}} \hat{g}^{-1}\right)}, \; c_1, c_2 \in \mathbb{R} \right\}. \tag{72}$$

We emphasize that within a given projective class $\mathfrak{P}$, two metrics parametrized by [44] could be isometric. The obvious question therefore is how to identify isometric metrics in $\mathfrak{P}$ and distinguish pairwise non-isometric ones. Below we sketch some ideas from [28, 30] that allow one to answer these questions.

As we have already pointed out, in the situation under consideration, the interesting cases are those where the degree of mobility is 2 or 3. Again, for simplicity, we focus on the case $\dim \mathcal{M} = 2$ as those with $\dim \mathcal{M} = 3$ can be obtained by extending the solutions obtained for $\dim \mathcal{M} = 2$.

Let ϕ_t be the local flow of the projective vector field v (which is unique up to a scalar factor). Of course, $\phi_t^*(g')$ is isometric to g' for any t, since ϕ_t is a local diffeomorphism. We conclude that they therefore must belong to the same projective class. We hence have a 1-parameter family of maps

$$\phi_t^* : \mathfrak{P} \ni g' \to \phi_t^*(g') \in \mathfrak{P}$$

whose 1-dimensional orbits consist of isometric metrics. This yields a considerable simplification as one is able to eliminate one parameter in [72]. However, it must be emphasized that further isometric correspondences do – in general – exist. Additional strategies are therefore needed to solve Lie's first problem. Strategies that have been used in the cited literature include, for example, an exploitation of invariant properties of the so-called *Benenti tensor*, a particular $(1,1)$-tensor field associated to a pair of projectively equivalent metrics $g, \hat{g}$:

$$\mathscr{B}_i^j := \left|\frac{\det(\hat{g})}{\det(g)}\right|^{\frac{1}{3}} (\hat{g}^{-1})^{jk} g_{ki}.$$

In fact, we have already encountered this tensor, albeit in different guise. For instance Theorem 13 can be rephrased as follows: if g and $\hat{g}$ are projectively equivalent, then their Benenti tensor transforms a Killing vector field of g into a Killing vector field of $\hat{g}$.

10. Outlook: C-projective geometry

Many of the concepts that have been introduced in the previous sections have analogies in the realm of (almost) complex manifolds. An extensive overview of the subject

can be found in [10], although the topic is active and further results have been published since. The goal of this section is to briefly introduce some of the concepts of c-projective geometry and entice the reader to further studies in the literature.

An almost complex manifold is a smooth manifold M equipped with an almost complex structure $J : TM \to TM$ with the property that $J^2 = -Id$. An almost complex manifold is a complex manifold if J is integrable in the sense that the eigendistributions of J are integrable. This is the case if and only if the *Nijenhuis tensor* (also called *Nijenhuis torsion*) of J vanishes. This is the praised result of Newlander and Nirenberg [31].

A complex manifold equipped with a (pseudo-)Riemannian metric g is called a Kähler manifold if and only if $g(X, Y) = g(JX, JY)$ for all vector fields X, Y and the metric and the complex structure are compatible in the sense that J is parallel w.r.t. the Levi-Civita connection ∇ of g, i.e., $\nabla_X J = 0$ for all vector fields X. An early result on the topic was published by S. Bochner in 1947 which demonstrates that a direct translation of projective geometry gives a very limited range of examples: in [4] he showed that, if two Kähler metrics on the same complex manifold are projectively equivalent, then their Levi-Civita connections coincide.

An extension to the notion of projective equivalence was obtained by the examination not only of geodesics but a larger class of curves, the so called *J-planar curves*, which were introduced by T. Otsuki and Y. Tashiro in 1957 [35]. They introduced the concept on Kähler manifolds but the existence of a metric is not a necessary requirement for all aspects of the topic and we shall thus give a more general definition, but for brevity we will start out with all connections being complex (i.e. preserving J).

DEFINITION 18. *Let M be a smooth manifold equipped with an (almost) complex structure J and a* complex connection ∇ *(i.e. $\nabla_X J = 0$ for all vector field X). A* J-planar curve γ *(parameterized by t) is a curve whose acceleration $\nabla_{\dot\gamma}\dot\gamma$ lies in the span of the velocity vector $\dot\gamma$ and $J\dot\gamma$, i.e. there exist functions $\alpha(t), \beta(t)$, such that*

$$\nabla_{\dot\gamma}\dot\gamma = \alpha(t)\dot\gamma + \beta(t)J\dot\gamma \tag{73}$$

is fulfilled.

The definition above is independent of parametrization; a direct computation shows that upon reparametrization, Equation [73] is still satisfied for modified $\alpha(t)$ and $\beta(t)$.

DEFINITION 19. *Let M be a smooth manifold endowed with an (almost) complex structure J and two complex connections $\nabla, \bar\nabla$. Then $\bar\nabla$ is called* c-projectively equivalent *to ∇ if all J-planar curves of ∇ are J-planar curves of $\bar\nabla$ and vice versa. Naturally, if g and $\bar g$ are Kähler metrics (of arbitrary signature) they are called c-projectively equivalent if their Levi-Civita connections are c-projectively equivalent.*

At this point it shall be noted that if we have two pairs (∇, J) and $(\bar\nabla, \bar J)$ on a smooth manifold with $\nabla J = 0$ and $\bar\nabla \bar J$ and all J-planar curves of ∇ are at the same time $\bar J$-planar curves of $\bar\nabla$ then $J = \pm\bar J$, an explicit proof can be found in [37].

Now that we have a notion of c-projective equivalence we can ask when two complex connections $\nabla, \bar{\nabla}$ are c-projectively equivalent. An answer is given by a formula analogous to [22]:

PROPOSITION 14 ([10], see §2.1 and its references). *Two complex connections on an (almost) complex manifold are c-projectively equivalent if and only if there exists a (real) 1-form ψ such that*

$$\bar{\nabla}_a X^c - \nabla_a X^c = \frac{1}{2}(\psi_a \delta^c_b + \psi_b \delta^c_a - J^d_a \psi_d J^c_b - J^c_a \psi_d J^d_b) X^b .$$

This further implies that c-projectively equivalent metrics have the same torsion.

DEFINITION 20. *A* c-projective transformation *is a local diffeomorphism of M that sends J-planar curves into J-planar curves. A* c-projective vector field *is a vector field on M whose local flow acts by c-projective transformations.*

The c-projective vector fields of a complex connection form a Lie algebra.

10.1. C-Projective connections

In the following we will look at another description of a c-projective structure in terms of ODEs, obtaining the *c-projective connection*, an analog of the projective connection [15]. Let (M, J, ∇) be a real $2n$-dimensional smooth manifold equipped with a complex structure J and a complex connection ∇. Let

$$(y^1, \dots, y^{2n})$$

be coordinates of M. The (system of) equations of J-planar curves [73] is

$$\begin{cases} \ddot{y}^k + \Gamma^k_{ij}\dot{y}^i\dot{y}^j = \alpha(t)\dot{y}^k + \beta(t) J^k_m \dot{y}^m, \quad k \geq 3, \\ \ddot{y}^2 + \Gamma^2_{ij}\dot{y}^i\dot{y}^j = \alpha(t)\dot{y}^2 + \beta(t) J^2_m \dot{y}^m, \\ \ddot{y}^1 + \Gamma^1_{ij}\dot{y}^i\dot{y}^j = \alpha(t)\dot{y}^1 + \beta(t) J^1_m \dot{y}^m. \end{cases} \tag{74}$$

Solving the last two equations of [74] for α, β, we obtain

$$\alpha = \frac{J^1_m \dot{y}^m(\ddot{y}^2 + \Gamma^2_{ij}\dot{y}^i\dot{y}^j) - J^2_m \dot{y}^m(\ddot{y}^1 + \Gamma^1_{ij}\dot{y}^i\dot{y}^j)}{\dot{y}^2 J^1_m \dot{y}^m - \dot{y}^1 J^2_m \dot{y}^m},$$

$$\beta = \frac{\dot{y}^2(\ddot{y}^1 + \Gamma^1_{ij}\dot{y}^i\dot{y}^j) - \dot{y}^1(\ddot{y}^2 + \Gamma^2_{ij}\dot{y}^i\dot{y}^j)}{\dot{y}^2 J^1_m \dot{y}^m - \dot{y}^1 J^2_m \dot{y}^m}.$$

By substituting α and β in the first set of Equations [74] we obtain, for $k \geq 3$,

$$\ddot{y}^k + \Gamma^k_{ij}\dot{y}^i\dot{y}^j = \frac{J^1_m\dot{y}^m(\ddot{y}^2 + \Gamma^2_{ij}\dot{y}^i\dot{y}^j) - J^2_m\dot{y}^m(\ddot{y}^1 + \Gamma^1_{ij}\dot{y}^i\dot{y}^j)}{\dot{y}^2 J^1_m\dot{y}^m - \dot{y}^1 J^2_m\dot{y}^m}\dot{y}^k + \frac{\dot{y}^2(\ddot{y}^1 + \Gamma^1_{ij}\dot{y}^i\dot{y}^j) - \dot{y}^1(\ddot{y}^2 + \Gamma^2_{ij}\dot{y}^i\dot{y}^j)}{\dot{y}^2 J^1_m\dot{y}^m - \dot{y}^1 J^2_m\dot{y}^m} J^k_m\dot{y}^m. \quad (76)$$

Upon renaming the coordinates $y^1, \ldots, y^{2n}$ as follows,

$$(y^1, \ldots, y^{2n}) = (x, y^2, \ldots, y^{2n}),$$

and taking into account that

$$\dot{y}^m = y^m_x\dot{x}, \quad \ddot{y}^m = y^m_{xx}\dot{x}^m + y^m_x\ddot{x}, \quad \text{for } m \geq 2,$$

System [76] is easily transformed (assuming that $\dot{x} \neq 0$) into

$$\begin{aligned}
&\Big((y^k_{xx}y^2_x - y^2_{xx}y^k_x) + (\Gamma^k_{11}y^2_x - \Gamma^2_{11}y^k_x) \\
&\qquad + 2y^j_x(\Gamma^k_{1j}y^2_x - \Gamma^2_{1j}y^k_x) + y^i_x y^j_x(\Gamma^k_{ij}y^2_x - \Gamma^2_{ij}y^k_x)\Big)(J^1_1 + J^1_h y^h_x) \\
&+ \Big(-y^k_{xx} + (\Gamma^1_{11}y^k_x - \Gamma^k_{11}) + 2y^j_x(\Gamma^1_{1j}y^k_x - \Gamma^k_{1j}) + y^i_x y^j_x(\Gamma^1_{ij}y^k_x - \Gamma^k_{ij})\Big)(J^2_1 + J^2_h y^h_x) \\
&+ \Big(y^2_{xx} + (\Gamma^2_{11} - y^2_x\Gamma^1_{11}) + 2y^j_x(\Gamma^2_{1j} - y^2_x\Gamma^1_{1j}) + y^i_x y^j_x(\Gamma^2_{ij} - y^2_x\Gamma^1_{ij})\Big)(J^k_1 + J^k_h y^h_x) = 0, \quad (77)
\end{aligned}$$

where $h, i, j \geq 2$, $k \geq 3$.

We call System [77] the *c-projective connection* associated to (M, J, ∇).

REMARK 5. System [77] describes the c-projective structure $[\nabla]$ in terms of ODEs. Unlike in the case of (real) projective connections, second order derivatives of System [77] cannot be expressed in terms of the first ones, i.e., System [77] is underdetermined.

REMARK 6. Analogously to the case of (real) projective connections (see Section 6), a c-projective vector field is an infinitesimal symmetry of System [77]. With small modifications, one can employ the methods described in Section 6 also in this case. The following example will clarify this aspect.

10.2. An example: c-projective vector fields of the Fubini-Study metric

In this example we compute the c-projective vector fields of the Fubini-Study metric of $\mathbb{CP}^2$ [16, 39]; in particular, we show that they form a 16-dimensional Lie algebra isomorphic to $\mathfrak{sl}(3, \mathbb{C})$. This is the maximum dimension that the Lie algebra of c-projective vector fields of a Kähler metric of complex dimension 2 can have. On the other hand, it can be proven [10] that metrics of complex dimension 2 admitting a

16-dimensional Lie algebra of c-projective vector fields are c-projective equivalent to the Fubini-Study metric. For this reason, the Fubini-Study metric takes the role of the "flat" model in the context of c-projective geometry. The Fubini-Study metric g_{FS} is given by

$$g_{\mathrm{FS}} = \frac{(1+\sum_{i=1}^{2} z_i \bar{z}_i)\sum_{j=1}^{2} dz_j d\bar{z}_j - \sum_{i,j=1}^{2} \bar{z}_j z_i dz_j d\bar{z}_i}{(1+\sum_{i=1}^{2} z_i \bar{z}_i)^2}, \tag{78}$$

where $z_1 = x + iy$ and $z_2 = z + it$. The complex structure J is

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Taking the variable x as the independent variable and (y, z, t) as the dependent ones, the c-projective connection [77] associated to the Fubini-Study metric is given by the following system of two ODEs:

$$\begin{cases} y_x^2 z_{xx} - y_x z_x y_{xx} + t_x y_{xx} + z_{xx} = 0, \\ -t_x y_x y_{xx} + y_x^2 t_{xx} - z_x y_{xx} + t_{xx} = 0. \end{cases} \tag{79}$$

C-projective vector fields of the Fubini-Study metric [78] are infinitesimal symmetries of System [79], and vice versa (cf. Remark 6). To compute infinitesimal symmetries

$$a\partial_x + b^1\partial_y + b^2\partial_z + b^3\partial_t$$

of [79], with a, b^i smooth functions of (x, y, z, t), one can follow closely the method outlined in Section 6. In this case, the jet space of order 2 to be considered is locally described by

$$(x, y, z, t, y_x, z_x, t_x, y_{xx}, z_{xx}, t_{xx}).$$

Having said that, the procedure for finding infinitesimal symmetries of [79] is practically the same as described in Section 6, taking into account that now the 2-codimensional submanifold of the jet space of order 2 is described, for $t_x y_x + z_x \neq 0$ (by solving [79] for y_{xx} and z_{xx}),

$$y_{xx} = \frac{t_{xx}(y_x^2+1)}{t_x y_x + zx}, \quad z_{xx} = \frac{t_{xx}(y_x z_x - t_x)}{t_x y_x + z_x}.$$

Unlike in the case of real projective connections, Condition [26] in this case leads to the vanishing of certain polynomials in t_{xx}, y_x, z_x, t_x. By equating to zero the coefficients of these polynomials, we obtain a system of PDEs in the unknown functions a, b^i whose solution depends on 16 arbitrary constants. More precisely, the c-projective algebra of the Fubini-Study metric is generated by the following 16 vector

fields:

$$
\begin{array}{llll}
(x^2-y^2)\partial_x+2xy\partial_y+(xz-yt)\partial_z+(xt-yz)\partial_t, & y\partial_x-x\partial_y, & x\partial_x+y\partial_y, & \partial_x,\\
-2xy\partial_x+(x^2-y^2)\partial_y-(xt+yz)\partial_z+(xz-yt)\partial_t, & y\partial_z-x\partial_t, & x\partial_z+y\partial_t, & \partial_y,\\
-(xt+yz)\partial_x+(xz-yt)\partial_y-2zt\partial_z+(z^2-t^2)\partial_t, & t\partial_z-z\partial_t, & z\partial_z+t\partial_t, & \partial_z,\\
(xz-yt)\partial_x+(xt+yz)\partial_y+(z^2-t^2)\partial_z+2zt\partial_t, & t\partial_x-z\partial_y, & z\partial_x+t\partial_y, & \partial_t.
\end{array}
$$

REMARK 7. The Fubini-Study metric is c-projectively equivalent to the Euclidean metric. In fact the c-projective connection of the Euclidean metric $dx^2+dy^2+dz^2+dt^2$ is [79].

10.3. Closing remarks

For brevity we close our elaboration and leave the interested reader with a few notes for further reading: A linearization of the metrizability problem (similar to that in Section 7 for the real projective case) is possible also in the c-projective setting. Analogs of Remark 3 and Proposition 9 can be found in Section 5 of [10] as well as in Section 1.3 of [7], in terms of the tensor field A. An analog of Theorem 16 regarding the integrals of the geodesic flow on Kähler manifolds was published in [41]. C-projectively equivalent Kähler metrics possess canonical integrals of the geodesic flow that are linear in momenta and the corresponding Killing vector fields are described for example in [10]. With these Killing tensor fields at hand, the commutativity of a family of differential operators with the Laplace-Beltrami operator has been proven by one of the authors in [37, 38]. Lie's problem can be posed analogously in the complex setting [6, 27], which is a topic of ongoing research.

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