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Bayesian Equilibria of Heterogeneous Non-Atomic Routing Games With Private Information

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***Abstract*— We study non-atomic Bayesian routing games whereby a transportation network is shared by two types of traffic: a coordinated fleet and a mass of selfish users. The links in the network are characterized by travel time functions that depend both on the aggregate flow on the link and on a random state of the world W that, in general, is not directly observable. Rather, we assume that both the fleet coordinator and the selfish users know the prior distribution of W, observe private messages that may differ across populations and are correlated with W and possibly with one another, and make routing decisions based on this partial information. Under the assumption that both the state of the world and the private message sets are finite, we prove the existence and uniqueness of a Bayesian equilibrium for the ensuing Bayesian routing game.**

***Index Terms*— Non-atomic routing games, private information, Bayesian equilibria, transportation networks.**

I. Introduction

UNDERSTANDING how traffic flows emerge from the interaction of many travelers is a central problem in transportation science. A widely adopted approach models route choice as a strategic interaction among users of the transportation network. When the number of travelers is large, transportation systems can be modeled by non-atomic routing games in which each link of the network is associated with a delay function, describing the dependence of the travel time on the traffic flow, and populations of agents independently select routes to minimize their individual travel times. The resulting equilibrium concept is the Wardrop equilibrium [19], defined as a flow where no traveler can reduce her travel time by unilaterally deviating to a different path. Under standard assumptions on the delay functions, the existence and uniqueness of Wardrop equilibria are well known [2], and their efficiency has been extensively analyzed [15], [16].

While classical models assume that all travelers behave independently and selfishly, modern transportation systems increasingly involve actors capable of coordinating the routing decisions of a subset of vehicles. In particular, delivery services and mobility providers operate fleets whose routes are centrally optimized according to fleet objectives, such as minimizing total travel time or improving service efficiency. In such settings, a coordinated fleet interacts with a large population of independent travelers who choose routes selfishly. This coexistence introduces a fundamental form of *heterogeneity* in routing behavior, as different groups of users may pursue different objectives and respond differently to congestion [4], [6], [7], [11], [13], [18]. At the same time, the growing diffusion of digital mobility platforms and navigation applications naturally leads to routing problems with *private information*, where the state of the world is uncertain, and agents make decisions based on probabilistic beliefs. Such problems can be modeled using *Bayesian routing games*, where travel times depend on an uncertain state of the world and agents select routes according to their beliefs about that state [1], [10], [12]. The role of information in routing decisions has been recently studied, particularly in connection with the design of information provision mechanisms [1], [5], [8], [17], [20].

In this paper, we study a heterogeneous non-atomic routing game on a transportation network with two interacting populations: a coordinated fleet and a selfish population. The fleet coordinator selects the routes of its vehicles in order to optimize the total travel time of the fleet, while the selfish population selects routes according to Wardrop equilibrium behavior, minimizing individual travel costs, given their beliefs about the state of the world. The network is subject to an uncertain state that affects the delay functions, capturing stochastic events such as incidents, weather conditions, or roadworks. In general, we assume that neither the fleet nor the selfish users directly observe the state of the world: instead, they receive private signals that are correlated with it and among themselves. Assuming a commonly known prior distribution, we prove existence and uniqueness of Bayesian equilibria for these routing games. The importance of uniqueness in real-world applications can be appreciated, for instance, in the context of information design. Indeed, when a

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social planner seeks to influence network users through information, uniqueness enhances robustness and predictability, because the planner can anticipate a single outcome in response to a given policy or information structure.

The paper is organized as follows. In Section II, we describe the transportation network model and the notion of Bayesian equilibria. In Section III, we formulate the Bayesian routing game on the transportation network, we prove the existence of Bayesian equilibria, and provide sufficient conditions for their uniqueness. In Section IV, we summarize our findings.

II. Model

A. Transportation Network

The *transportation network* is modeled by a finite directed multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \xi)$, where $\mathcal{V}$ is the set of nodes, $\mathcal{E}$ is the set of links, and $\sigma, \xi : \mathcal{E} \to \mathcal{V}$ are two functions that associate each link with its tail and head node, respectively. We assume that there are no self-loops, i.e., $\sigma(e) \neq \xi(e)$ for every e in $\mathcal{E}$. We assume that there is a unique origin and destination o, d in $\mathcal{V}$, with $o \neq d$ and d reachable from o, and let $\mathcal{P}$ denote the set of paths from o to d. A path between two nodes i and j in $\mathcal{V}$ is a finite sequence of links $(e_1, \dots, e_n)$ such that $\sigma(e_1) = i$, $\sigma(e_{l+1}) = \xi(e_l)$ for every $l = 1, \cdots, n-1$, and $\xi(e_n) = j$. The transportation network topology is then fully characterized by its node-link incidence matrix B in $\mathbb{R}^{\mathcal{V}\times\mathcal{E}}$, with entries $B_{ie} = 1$ if $\sigma(e) = i$, $B_{ie} = -1$ if $\xi(e) = i$, or $B_{ie} = 0$ otherwise. Moreover, we define the link-path incidence matrix A in $\{0,1\}^{\mathcal{E}\times\mathcal{P}}$, with entries $A_{ep} = 1$ if link e belongs to path p, or $A_{ep} = 0$ otherwise.

We consider two populations, called *coordinated fleet* and *selfish population*. For simplicity, we assume that both populations have unitary mass, and define the exogenous inflow $\nu = \delta^{(o)} - \delta^{(d)}$ in $\mathbb{R}^{\mathcal{V}}$, where $\delta^{(i)}$ is the vector with 1 in the i-th entry $\delta^{(i)}_i$ and 0 in all other positions. Hence, the set of network flows for both populations is

$$\mathcal{F} = \{f \in \mathbb{R}^{\mathcal{E}}_+ : Bf = \nu\}. \tag{1}$$

The *state of the world* is described by a random variable W defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and taking values in a nonempty finite set $\mathcal{W}$. We shall assume without loss of generality that $p_w := \mathbb{P}(W = w) > 0$ for every w in $\mathcal{W}$. Each link of the network is endowed with a delay function $\tau_e : \mathcal{W} \times \mathbb{R}_+ \to \mathbb{R}_+$ that depends on the state of the world as well as on the aggregate flow over the link. The delay functions $\tau_e(W, f_e)$ are assumed to be $\mathcal{C}^2$, non-decreasing, and convex, in the aggregate flow f_e.

B. Information

We assume that both the fleet coordinator and all selfish users know the prior distribution $(p_w)_{w\in\mathcal{W}}$ of the state of the world, but in general do not directly observe its realization W. Rather, they observe some private messages, to be denoted by M for the fleet coordinator and $\bar{M}_s$ for the selfish agent s in $\mathcal{S} := [0,1)$, that convey some partial information about the state of the world W, and make their routing decisions based on the observed messages.

Formally, let $\mathcal{M} = \{1, \cdots, |\mathcal{M}|\}$ and $\bar{\mathcal{M}} = \{1, \cdots, |\bar{\mathcal{M}}|\}$ be two non-empty finite sets where the private messages observed by the fleet coordinator and, respectively, the selfish users, take values, and consider two maps $\pi : \mathcal{W} \to \Delta_{\mathcal{M}}$ and $\bar{\pi} : \mathcal{W} \times \mathcal{M} \to \Delta_{\bar{\mathcal{M}}}$ taking values in the simplices of probability vectors over $\mathcal{M}$ and $\bar{\mathcal{M}}$, respectively, where $\Delta_{\mathcal{K}} = \{x \in \mathbb{R}^{\mathcal{K}}_+ : \mathbf{1}'x = 1\}$. Let also V and $\bar{V}$ be two random variables defined over the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, both uniformly distributed over the interval $[0,1)$, mutually independent, and independent from W. Then, let

$$M = \min\left\{m \in \mathcal{M} : V \leq \sum_{1\leq l\leq m} \pi_l(W)\right\}, \tag{2}$$

be the message observed by the fleet coordinator, and let

$$\bar{M}_s = \min\left\{\bar{m} \in \bar{\mathcal{M}} : s + \bar{V} - \lfloor s + \bar{V}\rfloor \leq \sum_{1\leq l\leq \bar{m}} \bar{\pi}_l(W, M)\right\} \tag{3}$$

be the message received by every selfish agent s in $\mathcal{S}$.

The following result states that selfish users s in $\mathcal{S}$ have identical conditional distribution on their received messages $\bar{M}_s$ given the pair (W, M), and that the (random) fraction of selfish users that receive a message $\bar{m}$ in $\bar{\mathcal{M}}$ is exactly $\bar{\pi}_{\bar{m}}(W, M)$.

Lemma 1:

(i) The triple $(W, M, \bar{M}_s)$ has the same joint probability distribution

$$\mathbb{P}(W = w, M = m, \bar{M}_s = \bar{m}) = p_w \pi_m(w) \bar{\pi}_{\bar{m}}(w, m), \tag{4}$$

for every selfish user s in $\mathcal{S}$;

(ii) the fraction of selfish users receiving message $\bar{m}$ is

$$\bar{\pi}_{\bar{m}}(W, M) = \mathbb{P}(\bar{M}_0 = \bar{m} | W, M),$$

for every $\bar{m}$ in $\bar{\mathcal{M}}$.

Proof: (i) On the one hand, (2) implies that the conditional distribution of the message M given the state of the world W is $\mathbb{P}(M = m | W = w) = \pi_m(w)$. On the other hand, (3) and the fact that, for every s in $\mathcal{S}$, the random variable $s + \bar{V} - \lfloor s + \bar{V}\rfloor$ is uniformly distributed on the interval $[0,1)$ and independent from W and M, imply that the conditional probability distribution of $\bar{M}_s$ given W and M is $\mathbb{P}(\bar{M}_s = \bar{m} | W = w, M = m) = \bar{\pi}_{\bar{m}}(w, m)$. This readily implies (4).

(ii) Let $H_{\bar{m}} = \sum_{l=1}^{\bar{m}} \bar{\pi}_l(M, W) - \bar{V}$ for all $\bar{m}$ in $\bar{\mathcal{M}}$. Then, from (3), the set of selfish users that receive message $\bar{m}$ is

$$\begin{aligned}
&\{s : \bar{M}_s = \bar{m}\} \\
&= \{s : H_{\bar{m}-1} + \bar{V} < s + \bar{V} - \lfloor s + \bar{V}\rfloor \leq H_{\bar{m}} + \bar{V}\} \\
&= [0,1) \cap \left((H_{\bar{m}-1}, H_{\bar{m}}] \cup (H_{\bar{m}-1} + 1, H_{\bar{m}} + 1]\right)
\end{aligned}$$

whose Lebesgue measure is $H_{\bar{m}} - H_{\bar{m}-1} = \bar{\pi}_{\bar{m}}(W, M)$. ∎

Thanks to Lemma 1, we can characterize the posterior beliefs on the state of the world after receiving the private messages for both the fleet coordinator and the selfish users. In particular, Lemma 1(i) implies that, every two selfish users s and $\tilde{s}$ in $\mathcal{S}$ that observe the same message $\bar{M}_s = \bar{M}_{\tilde{s}}$ have the same posterior distribution on the pair (W, M) of the state of the world and the fleet coordinator's message.

C. Bayesian Equilibria

In this section we describe the user behavior and characterize the notion of equilibrium. Towards this goal, we define

$$\mathcal{X} = \{x : \mathcal{M} \to \mathcal{F}\}, \quad X_e = x_e(M) \quad \forall e \in \mathcal{E}.$$

An element x in $\mathcal{X}$ is a map from the message that the fleet coordinator receives to the flow allocated on the network, while X_e is the random variable describing such flow as a function of M. For the selfish population, we define

$$\begin{aligned}\mathcal{Y} &= \{y : \bar{\mathcal{M}} \to \Delta_{\mathcal{P}}\},\\ Z_e^y &= \sum_{p\in\mathcal{P}} A_{ep} \sum_{\bar m\in\bar{\mathcal{M}}} y_p(\bar m)\bar\pi_{\bar m}(W,M)\\ &= \mathbf{E}[(Ay(\bar M_0))_e|W,M].\end{aligned}$$

An element $y \in \mathcal{Y}$ associates with each message $\bar m \in \bar{\mathcal{M}}$ a distribution over paths, where $y_p(\bar m)$ represents the fraction of selfish agents choosing path p upon receiving message $\bar m$. Given (W, M), the quantity Z_e^y denotes the induced flow of the selfish population on link e. Specifically, it is obtained by aggregating, over all messages and paths, the fraction of agents receiving message $\bar m$ multiplied by the fraction of those agents selecting path p, and then projecting the resulting path flow onto the link space through the incidence matrix A. The set of admissible Z is

$$\mathcal{Z} = \{Z : \mathcal{W}\times\mathcal{M} \to \mathbb{R}_+^{\mathcal{E}} : Z = Z^y \text{ for } y \in \mathcal{Y}\}.$$

The goal of the fleet coordinator is to minimize the expected total travel time spent on the network by the fleet, given the flow of the selfish population, defined by

$$\Phi(x,Z) = \mathbf{E}\left[\sum_e X_e\tau_e(W, X_e + Z_e)\right]. \tag{5}$$

On the other hand, each user of the selfish population aims to travel along a path from o to d that minimizes the expected travel time conditioned on the message received. This yields the following notion of equilibrium.

Definition 1: A pair (x, Z) is a *Bayesian equilibrium* if:

(i)
$$x \in \arg\min_{\tilde x\in\mathcal{X}} \Phi(\tilde x, Z)\,; \tag{6}$$

(ii) there exists y in $\mathcal{Y}$ such that $Z = Z^y$, and for every message $\bar m$ in $\bar{\mathcal{M}}$ and path p in $\mathcal{P}$, $y_p(\bar m) > 0$ implies

$$\mathbf{E}\left[\sum_{e\in\mathcal{E}}(A_{ep} - A_{eq})\tau_e\big(W, X_e + Z_e\big)\big|\bar M_s = \bar m\right] \le 0 \tag{7}$$

for every path q in $\mathcal{P}$ and selfish user s in $\mathcal{S}$.

In plain words, a pair (x, Z) is an equilibrium if x minimizes the expected total travel time of the fleet given Z, and if every path p that is used by some selfish agents that receive message $\bar m$ has minimal expected travel time according to the posterior belief after receiving message $\bar m$ and given x. In the next section we shall investigate existence and uniqueness of such Bayesian equilibria.

Remark 1: Bayesian equilibria can be viewed as Bayes correlated equilibria where the private messages act as a correlation device [3]. However, in this paper we restrict our analysis to a special class of Bayes correlated equilibria, since by construction the fraction of selfish agents that receive each message is a deterministic function of the state of the world. Another loss of generality comes from the fact that the set of actions of the fleet coordinator is continuous, while the set of messages $\mathcal{M}$ that he can receive is finite.

Example 1: Consider a network with two nodes and two parallel links, with delay functions $\tau_1(W, f_1) = 4f_1 + W$, $\tau_2(W, f_2) = 8f_2$. The state of the world W takes value in $\mathcal{W} = \{0, 1\}$, with $p_0 = 1/4$, $p_1 = 3/4$. Suppose that the coordinated fleet and the selfish agents can receive one and two messages respectively, i.e., $\mathcal{M} = \{m\}$, $\bar{\mathcal{M}} = \{1, 2\}$, with

$$\bar\pi_1(0) = \bar\pi_2(0) = 1/2, \quad \bar\pi_1(1) = 2/3, \quad \bar\pi_2(1) = 1/3.$$

Hence, by Bayes' theorem, the posterior distribution about the state of the world for every selfish user s are

$$\mathbb{P}(W = 0|\bar M_s = 1) = 1/5, \quad \mathbb{P}(W = 0|\bar M_s = 2) = 1/3,$$
$$\mathbb{P}(W = 1|\bar M_s = 1) = 4/5, \quad \mathbb{P}(W = 1|\bar M_s = 2) = 2/3.$$

We now compute the set of Bayesian equilibria (x, Z). Since the network has only two parallel links, we have $x_2 = 1-x_1$ and $Z_2 = 1-Z_1$. Moreover, since the coordinated fleet has only one message, x is a vector with entries $x_1 = \alpha$, $x_2 = 1-\alpha$. Instead, for the selfish population we define $y_1(1) = \beta$, $y_1(2) = \gamma$. Therefore,

$$\begin{aligned}W = 0 &\implies Z_1 = y_1(1)\bar\pi_1(0) + y_1(2)\bar\pi_2(0) = (\beta+\gamma)/2,\\ W = 1 &\implies Z_1 = y_1(1)\bar\pi_1(1) + y_1(2)\bar\pi_2(1) = (2\beta+\gamma)/3.\end{aligned}$$

We now apply Definition 1 to find the Bayesian equilibria (x, Z). For the coordinated fleet, $\Phi(x, Z)$ takes the form

$$\begin{aligned}&\frac14\left[\alpha\tau_1\left(0,\alpha+\frac{\beta+\gamma}{2}\right) + (1-\alpha)\tau_2\left(0,2-\alpha-\frac{\beta+\gamma}{2}\right)\right]\\ &+\frac34\left[\alpha\tau_1\left(1,\alpha+\frac{2\beta+\gamma}{3}\right) + (1-\alpha)\tau_2\left(1,2-\alpha-\frac{2\beta+\gamma}{3}\right)\right],\end{aligned}$$

whose unique stationary point is achieved in α satisfying

$$24\alpha + 15\beta/2 + 9\gamma/2 - 93/4 = 0. \tag{8}$$

We search for interior equilibria for the selfish population by assuming that (7) is satisfied as an equality. This yields

$$\begin{aligned}&\mathbf{E}[\tau_1\big(W,\alpha+Z_1\big) - \tau_2\big(W,1-\alpha+1-Z_1\big)\big|\bar{\mathcal{M}}_s = 1] = 0,\\ &\mathbf{E}[\tau_2\big(W,1-\alpha+1-Z_1\big) - \tau_1\big(W,\alpha+Z_1\big)\big|\bar{\mathcal{M}}_s = 2] = 0.\end{aligned}$$

Using the posterior distributions and the expression of the delay functions, this yields

$$\begin{aligned}12\alpha + \frac{38}{5}\beta + \frac{22}{5}\gamma - \frac{76}{5} &= 0\\ -12\alpha - \frac{22}{3}\beta - \frac{14}{3}\gamma + \frac{46}{3} &= 0.\end{aligned}$$

Solving this equation together with (8) we obtain $\alpha = 2/3, \beta = 5/12, \gamma = 11/12$, and hence (x, y) and the corresponding Bayesian equilibrium (x, Z^y). Note that this is the unique interior equilibrium, but other equilibria may exist. The presence of other equilibria will be ruled out by Theorem 1.

III. Main Results

In this section we establish conditions for the existence and uniqueness of Bayesian equilibria. Towards this goal, we first define an auxiliary two-player game, whose Nash equilibria are equivalent to the Bayesian equilibria.

A. Auxiliary 2-Player Game

We start with a technical result that is instrumental to show the equivalence between Bayesian equilibria and Nash equilibria of a 2-player game.

Lemma 2: Z satisfies Definition 1(ii) if and only if it is solution of

$$Z \in \arg\min_{\tilde Z\in\mathcal{Z}} \bar\Phi(\tilde Z, x), \tag{9}$$

where $\bar{\Phi} : \mathcal{Z} \times \mathcal{X} \to \mathbb{R}_+$ is defined by

$$\bar{\Phi}(Z, x) = \mathbf{E}\left[\sum_{e\in\mathcal{E}} \int_0^{X_e+Z_e} \tau_e(W, r)dr\right].$$

Moreover, $\bar{\Phi}(Z, x)$ is convex in Z.

Proof: $\bar{\Phi}(Z, x)$ is convex in Z because the delay functions are non-decreasing. Since $\bar{\mathcal{M}}$ is finite, we can write y in matrix form by identifying $y_{mp} \equiv y_p(m)$. With this notation, y is a matrix in $\mathbb{R}_+^{\bar{\mathcal{M}}\times\mathcal{P}}$ subject to $y\mathbf{1} = \mathbf{1}$. Now, since by construction every Z in $\mathcal{Z}$ admits y in $\mathcal{Y}$ such that $Z = Z^y$, and since Z^y is a linear function of y, we can define $\Psi(y, x) = \bar{\Phi}(Z^y, x)$ and note that $\Psi(y, x)$ is convex in y, since it is the composition of a convex function with a linear function. Moreover,

$$Z^y \in \arg\min_{\tilde{Z}\in\mathcal{Z}} \bar{\Phi}(\tilde{Z}, x) \iff y \in \arg\min_{\tilde{y}\in\mathcal{Y}} \Psi(y, x). \tag{10}$$

We now prove that y solves the right-most optimization problem in (10) if and only if Z^y satisfies Definition 1(ii), concluding the proof. Because of the convexity of $\Psi(y, x)$ in y, the KKT conditions of the right-most optimization problem in (10) are necessary and sufficient for optimality (see Section III-A of [2] for the derivation), that is, y is solution of the right-most problem in (10) if and only if

$$\frac{\partial\Psi}{\partial y_{\bar{m}p}} \le \frac{\partial\Psi}{\partial y_{\bar{m}q}}, \quad \forall q \in \mathcal{P}, \tag{11}$$

for every $\bar{m}$ in $\bar{\mathcal{M}}$, p in $\mathcal{P}$ such that $y_{\bar{m}p} > 0$. Explicit computation yields

$$\frac{\partial\Psi}{\partial y_{\bar{m}p}} = \mathbf{E}\left[\sum_{e\in\mathcal{E}} A_{ep}\bar{\pi}_{\bar{m}}(W, M)\tau_e(W, X_e + Z_e)\right].$$

By definition of $\bar{\pi}_{\bar{m}}(W, M)$, it follows that $\mathbb{P}(W = w, M = m, \bar{M} = \bar{m}) = \mathbb{P}(W = w, M = m)\bar{\pi}_{\bar{m}}(W, M)$. Hence, dividing such expression by $\mathbb{P}(\bar{M}_0 = \bar{m})$, we get

$$\frac{\partial\Psi}{\partial y_{\bar{m}p}} = \mathbf{E}\left[\sum_{e\in\mathcal{E}} A_{ep}\tau_e(W, X_e + Z_e)\middle|\bar{M} = \bar{m}\right] \cdot \mathbb{P}(\bar{M}_0 = \bar{m}).$$

The proof is concluded by plugging this expression into (11) and noticing that this is equivalent to (7). ∎

We now introduce an auxiliary two-player game.

Definition 2: Consider the game Γ with player set $\{C, S\}$, action sets $\{\mathcal{X}, \mathcal{Z}\}$ and utility functions $\{u(x, Z), \bar{u}(Z, x)\}$, with

$$u(x, Z) = -\Phi(x, Z), \quad \bar{u}(Z, x) = -\bar{\Phi}(Z, x).$$

Here, player C represents the coordinating decision maker choosing the action x in $\mathcal{X}$, while player S is a nominal player representing the collection of all selfish agents in the original game. The artificial aggregate action Z in $\mathcal{Z}$ is defined for technical reasons and it summarizes the collective response of the selfish agents. Hence, the two-player game captures the interaction between the coordinator and the overall behavior of the selfish population.

Definition 3: A *pure-strategy Nash equilibrium* of game Γ is an action distribution (x, Z) such that

$$x \in \arg\max_{\tilde{x}\in\mathcal{X}} u(\tilde{x}, Z), \quad Z \in \arg\max_{\tilde{Z}\in\mathcal{Z}} \bar{u}(\tilde{Z}, x). \tag{12}$$

Proposition 1: (x, Z) is a pure-strategy Nash equilibrium of Γ if and only if it is a Bayesian equilibrium per Definition 1.

Proof: By definition of U, x satisfies the left-most relation of (12) if and only if it satisfies (6). By Lemma 2, Z satisfies the right-most relation of (12) if and only if it satisfies Definition 1(ii). This concludes the proof. ∎

B. Existence of Bayesian Equilibria

The next result provides sufficient conditions on the delay functions for the existence of Bayesian equilibria. Note that our contribution concerns a rich setting combining heterogeneous populations with different objectives, private information and Bayesian routing strategies conditioned on private messages. In this setting, the equilibrium is not a standard Wardrop equilibrium, but rather a coupled Bayesian equilibrium involving both an optimization problem for the coordinator and Wardrop-type conditions for the selfish population under uncertainty. The existence result is therefore not an immediate consequence of classical Wardrop theory.

Proposition 2: Assume that the delay functions $\tau_e(w, f_e)$ are convex in f_e for every w in $\mathcal{W}$. Then, there always exists a Bayesian equilibrium.

Proof: We first prove the existence of Nash equilibria of Γ. To do so, we rely on [9], which ensures that a pure-strategy Nash equilibrium of the game Γ exists if: (i) the action sets are finite-dimensional, compact, and convex; (ii) the utility function of each player is continuous in both actions and concave in the action of the player itself.

Because $\mathcal{W}$, $\mathcal{M}$, and $\bar{\mathcal{M}}$ are finite sets, the action spaces $\mathcal{X}$ and $\mathcal{Z}$ are finite-dimensional, compact, and convex. Moreover, the utility functions $u(x, Z)$ and $\bar{u}(Z, x)$ are continuous in both actions, since the delay functions are continuous. We now show that $u(x, Z)$ is concave in x and that $\bar{u}(Z, x)$ is concave in Z. To prove the concavity of $u(x, Z)$, note that

$$u(x, Z) = -\sum_{w\in\mathcal{W}}\sum_{m\in\mathcal{M}} \mathbb{P}(W = w, M = m) \cdot \sum_{e\in\mathcal{E}} x_e(m)\tau_e(w, x_e(m) + Z_e),$$

hence

$$\frac{\partial u(x, Z)}{\partial x_e(m)\partial x_{\tilde{e}}(\tilde{m})} = 0, \quad \text{if } e \neq \tilde{e} \text{ or } m \neq \tilde{m}.$$

Therefore, the Hessian of U is diagonal. Direct computation yields that the diagonal elements are non-positive if the delay functions $\tau_e(W, f_w)$ are non-decreasing and convex in f_e. This proves the concavity of $u(x, Z)$ in x. The concavity of $\bar{u}(Z, x)$ in Z follows from $\bar{u}(Z, x) = -\bar{\Phi}(Z, x)$ and from convexity of $\bar{\Phi}(Z, x)$ in Z, established by Lemma 2.

This proves the existence of a Nash equilibrium of the two-player game Γ. The existence of a Bayesian equilibrium then follows from Proposition 1. ∎

C. Uniqueness

This section provides sufficient conditions for the uniqueness of Bayesian equilibria.

Theorem 1: Assume that the delay functions are twice continuously differentiable in the flow variable and satisfy

$$2\tau_e'(w, f_e + \bar{f}_e) > f_e\tau_e''(w, f_e + \bar{f}_e) \ge 0 \tag{13}$$

for every e in $\mathcal{E}$, w in $\mathcal{W}$, and $f, \bar{f}$ in $\mathcal{F}$, where τ_e' and τ_e'' denote, respectively, the first and second derivatives of τ_e with respect to the flow variable f_e. Then, there exists a unique Bayesian equilibrium.

Proof: We prove that the auxiliary game Γ admits a unique Nash equilibrium. Then, the uniqueness of the Bayesian equilibrium follows from Proposition 1.

Since Z depends on (W, M), and since W, M can take a finite number of values, we can express the elements Z in $\mathcal{Z}$ in matrix form. Let $Q = (W, M)$ take values in $\mathcal{Q} = \mathcal{W} \times \mathcal{M}$. Since the set of actions $\mathcal{Z}$ of the selfish population is the image of a linear function Z^y, for every y in $\mathcal{Y}$, standard arguments of linear algebra imply the existence of two matrices H, K such that

$$\mathcal{Z} = \{Z \in \mathbb{R}_+^{\mathcal{Q}\times E} : HZ = K\}.$$

For convenience, we also represent $\mathcal{X}$ in matrix form by

$$\mathcal{X} = \left\{x \in \mathbb{R}_+^{\mathcal{M}\times\mathcal{E}} : \sum\nolimits_e B_{ie}x_{me} = \nu_i,\ \forall i \in \mathcal{V},\ \forall m \in \mathcal{M}\right\}.$$

For $q = (w, m)$ in $\mathcal{Q}$, let $q_1 = w$, $q_2 = m$. It will also prove useful to embed $\mathcal{X}$ into the augmented set

$$\hat{\mathcal{X}} = \Big\{\hat{x} \in \mathbb{R}_+^{\mathcal{Q}\times\mathcal{E}} : \sum_e B_{ie}\hat{x}_{qe} = \nu_i,\ \forall i \in \mathcal{V},\ \forall q \in \mathcal{Q},$$
$$\hat{x}_{qe} = \hat{x}_{q'e}, \forall e \in \mathcal{E}, \forall q, q' \in \mathcal{Q} \text{ s.t. } q_2 = q_2'\Big\}.$$

In plain words, $\hat{\mathcal{X}}$ is a new artificial set of actions for the fleet coordinator, where the action is formally allowed to depend on the state of the world, although the constraints enforce dependence only through the message. Notice that $\mathcal{X}$ can be embedded into $\hat{\mathcal{X}}$, and as a consequence, there exists a bijection between $\mathcal{X}$ and $\hat{\mathcal{X}}$. We also parametrize the utility functions of the two players in order to let them depend on x in $\hat{\mathcal{X}}$, Z in $\mathcal{Z}$. Let $v, \bar{v} : \hat{\mathcal{X}} \times \mathcal{Z} \to \mathbb{R}$ be defined by

$$v(x, Z) = -\sum_{q\in\mathcal{Q}} \mathbb{P}(Q = q) \sum_{e\in\mathcal{E}} x_{qe}\tau_e(q_1, x_{qe} + Z_{qe}),$$
$$\bar{v}(x, Z) = -\sum_{q\in\mathcal{Q}} \mathbb{P}(Q = q) \sum_{e\in\mathcal{E}} \int_0^{x_{qe}+Z_{qe}} \tau_e(q_1, r)\mathrm{d}r.$$

We have defined a two-player game $\hat{\Gamma}$ with action sets $\hat{\mathcal{X}}, \mathcal{Z}$ and utility functions $v, \bar{v}$, whose Nash equilibria (x, Z) satisfy

$$x \in \arg\max_{\tilde{x}\in\hat{\mathcal{X}}} v(\tilde{x}, Z), \quad Z \in \arg\max_{\tilde{Z}\in\mathcal{Z}} \bar{v}(\tilde{Z}, x). \tag{14}$$

It is fundamental to observe that, by construction, there is a bijection between Nash equilibria of $\hat{\Gamma}$ and those of Γ, so that Γ admits a unique Nash equilibrium if and only if $\hat{\Gamma}$ admits a unique Nash equilibrium.

We now aim to prove the uniqueness of the Nash equilibrium of $\hat{\Gamma}$. The Lagrangian function of the left-most optimization problem in (14) is

$$L(x, Z, \gamma, \lambda) = v(x, Z) + \sum_{e,q} \gamma_{qe}x_{qe} + \sum_{i,q} \lambda_{iq}\left(\sum_e B_{ie}x_{qe} - \nu_i\right),$$

where $\gamma_{eq} \geq 0$ and λ_{iq} are Lagrangian multipliers associated to inequality constraint $x_{qe} \geq 0$ and equality constraints $\sum_e B_{ie}x_{qe} = \nu_i$ that an element x in $\hat{\mathcal{X}}$ must satisfy. The Lagrangian function of the right-most optimization in (14) is

$$\bar{L}(Z, x, \omega, \delta) = \bar{v}(Z, x) + \sum_{q,e} \omega_{qe}Z_{qe} + \sum_{q',e} \delta_{q'e}\left(\sum_q H_{q'q}Z_{qe} - K_{q'e}\right),$$

where $\omega_{qe} \geq 0$ and $\delta_{q'e}$ are Lagrangian multipliers associated to inequality constraint $Z_{qe} \geq 0$ and equality constraints $\sum_q H_{q'q}Z_{qe} = K_{q'e}$ of $\mathcal{Z}$, respectively.

Suppose now that there are two different Nash equilibria $(x, Z) \neq (\tilde{x}, \tilde{Z})$. For (x, Z) to be a Nash equilibrium, there must exist a set of Lagrangian multipliers $(\gamma, \lambda, \omega, \delta)$ such that

$$\frac{\partial L(x, Z, \gamma, \lambda)}{\partial x_{qe}} = 0, \quad \frac{\partial \bar{L}(Z, x, \omega, \delta)}{\partial z_{qe}} = 0,$$

for every e in $\mathcal{E}$, q in $\mathcal{Q}$, which readily implies

$$\frac{\partial v(x, Z)}{\partial x_{qe}} + \gamma_{qe} + \sum_i \lambda_{iq}B_{ie} = 0,$$
$$\frac{\partial \bar{v}(Z, x)}{\partial Z_{qe}} + \omega_{qe} + \sum_{q'} \delta_{q'e}H_{q'q} = 0. \tag{15}$$

Equivalently, for $(\tilde{x}, \tilde{Z})$ to be a Nash equilibrium, there must exist a set of Lagrangian multipliers $(\tilde{\gamma}, \tilde{\lambda}, \tilde{\omega}, \tilde{\delta})$ such that

$$\frac{\partial v(\tilde{x}, \tilde{Z})}{\partial \tilde{x}_{qe}} + \tilde{\gamma}_{qe} + \sum_i \tilde{\lambda}_{iq}B_{ie} = 0,$$
$$\frac{\partial \bar{v}(\tilde{Z}, \tilde{x})}{\partial \tilde{Z}_{qe}} + \tilde{\omega}_{qe} + \sum_{q'} \tilde{\delta}_{q'e}H_{q'q} = 0. \tag{16}$$

Moreover, the complementary slackness constraints, i.e.,

$$\omega_{qe}Z_{qe} = 0,\ \gamma_{qe}x_{qe} = 0,\ \tilde{\omega}_{qe}\tilde{Z}_{qe} = 0,\ \tilde{\gamma}_{qe}\tilde{x}_{qe} = 0, \tag{17}$$

must hold true for every e in $\mathcal{E}$, q in $\mathcal{Q}$. We now multiply the first equation of (15) by $(\tilde{x}_{qe} - x_{qe})$, the second equation of (15) by $(\tilde{Z}_{qe} - Z_{qe})$, the first equation of (16) by $(x_{qe} - \tilde{x}_{qe})$ and the second equation of (16) by $(Z_{qe} - \tilde{Z}_{qe})$. Then, summing all the resulting terms for every e, q and rearranging them we get that $\psi + \chi = 0$, where

$$\psi = \sum_{q,e} (\tilde{x}_{qe} - x_{qe})\left(\frac{\partial v(x, Z)}{\partial x_{qe}} - \frac{\partial v(\tilde{x}, \tilde{Z})}{\partial \tilde{x}_{qe}}\right) + \sum_{q,e} (\tilde{Z}_{qe} - Z_{qe})\left(\frac{\partial \bar{v}(Z, x)}{\partial Z_{qe}} - \frac{\partial \bar{v}(\tilde{Z}, \tilde{x})}{\partial \tilde{Z}_{qe}}\right),$$

$$\chi = \sum_{q,e} (\tilde{x}_{qe} - x_{qe})\left(\gamma_{qe} - \tilde{\gamma}_{qe} + \sum_i \left(\lambda_{iq}B_{ie} - \tilde{\lambda}_{iq}B_{ie}\right)\right) + \sum_{q,e} (\tilde{Z}_{qe} - Z_{qe}) \left(\omega_{qe} - \tilde{\omega}_{qe} + \sum_{q'} \left(\delta_{q'e}H_{q'q} - \tilde{\delta}_{q'e}H_{q'q}\right)\right).$$

We now analyze χ and ψ separately, starting from χ. First, note that, for x and $\tilde{x}$ to belong to $\hat{\mathcal{X}}$, and for Z and $\tilde{Z}$ to belong to $\mathcal{Z}$,

$$\sum_e B_{ie} x_{qe} = \sum_e B_{ie} \tilde{x}_{qe} = \nu_i, \quad \forall i, q.$$
$$\sum_q H_{q'q} Z_{qe} = \sum_q H_{q'q} \tilde{Z}_{qe} = K_{q'e}, \quad \forall q', e.$$

Using these two equations and (17), we get

$$\chi = \sum_{q,e} (\tilde{x}_{qe}\gamma_{qe} + x_{qe}\tilde{\gamma}_{qe}) + \sum_{q,e} (\tilde{Z}_{qe}\omega_{qe} + Z_{qe}\tilde{\omega}_{qe}) \geq 0,$$

where the last equivalence follows since both the Lagrangian multipliers γ, ω and the actions x, Z are non-negative.

We now prove that $\psi \geq 0$ and $\psi = 0$ if and only if the two equilibria coincide. Note that $(x, Z) = (\tilde{x}, \tilde{Z})$ implies $\psi = 0$. We are then left to prove that $(x, Z) \neq (\tilde{x}, \tilde{Z})$ implies $\psi > 0$. To this end, we define $\sigma(x, Z) := v(x, Z) + \bar{v}(Z, x)$ and its pseudogradient

$$g(x, Z) := \begin{bmatrix} \nabla_x v(x, Z) \\ \nabla_Z \bar{v}(Z, x) \end{bmatrix},$$

obtained by stacking the gradient of the utility function of both players with respect to their actions. As shown in [14, Section 3], if $(x, Z) \neq (\tilde{x}, \tilde{Z})$ and $\sigma(x, Z)$ is diagonally strictly concave for every (x, Z), then $\psi > 0$. The same reference shows that a sufficient condition for $\sigma(x, Z)$ to be diagonally strictly concave is that the Jacobian $J(x, Z)$ of the pseudogradient $g(x, Z)$ is negative definite for every x, Z. Direct computation yields that $J(x, Z)$ has a block diagonal structure, since $\partial^2 V/(\partial x_{qe}\partial x_{q'e'}) = \partial^2 V/(\partial x_{qe}\partial Z_{q'e'}) = \partial^2 \bar{V}/(\partial Z_{qe}\partial Z_{q'e'}) = \partial^2 \bar{V}/(\partial Z_{qe}\partial Z_{q'e'}) = 0$ if $(q, e) \neq (q', e')$. Moreover,

$$\frac{\partial^2 v(x, Z)}{\partial x_{qe}^2} = -\mathbb{P}(Q = q)(2\tau_e' + x_{qe}\tau_e'')$$
$$\frac{\partial^2 v(x, Z)}{\partial x_{qe}\partial Z_{qe}} = -\mathbb{P}(Q = q)(\tau_e' + x_{qe}\tau_e'')$$
$$\frac{\partial^2 \bar{v}(Z, x)}{\partial Z_{qe}\partial x_{qe}} = \frac{\partial^2 \bar{v}(Z, x)}{\partial Z_{qe}^2} = -\mathbb{P}(Q = q)\tau_e'.$$

We are now left to prove that, for every e in $\mathcal{E}$ and q in Q, the (e, q)-th block of J is negative definite, so that the entire Jacobian is negative definite. To do so, we sum the block with its transpose, obtaining

$$-\mathbb{P}(Q = q) \begin{pmatrix} 2(2\tau_e' + x_{qe}\tau_e'') & 2\tau_e' + x_{qe}\tau_e'' \\ 2\tau_e' + x_{qe}\tau_e'' & 2\tau_e' \end{pmatrix}.$$

Observe that (13) implies that the delay functions are strictly increasing and convex in the flow variable. Hence, the trace of the matrix is negative, while the determinant is positive if and only if

$$2\tau_e'(W, f_e + \bar{f}_e) > f_e\tau_e''(W, f_e + \bar{f}_e), \quad \forall q \in \mathcal{Q}, \ \forall e \in \mathcal{E}.$$

This proves the uniqueness of the Nash equilibrium of $\hat{\Gamma}$, and therefore of Γ. Proposition 1 then implies the uniqueness of the Bayesian equilibrium. ■

Remark 2: A sufficient condition similar to (13) was established by [13] for uniqueness of the equilibria in presence of two populations in a deterministic setting. It is also worth noting that the uniqueness of the Nash equilibrium of Γ, which is an intermediate step to proving Theorem 1, is established using an argument similar to that in [14, Theorem 2]. In that work, sufficient conditions for uniqueness are derived for the case in which the action sets are subject to inequality constraints, but without any equality constraints.

IV. Conclusion

We have established existence and uniqueness of Bayesian equilibria for non-atomic Bayesian routing games, whereby a coordinated fleet and a population of selfish users share a transportation network. We have assumed that the travel times on the links depend on both the aggregate flow on them and a random state of the world that is generally not directly observable by either the fleet coordinator or the selfish users, who make routing decisions based on private partial information obtained from messages that are correlated with the state of the world and among themselves. Future directions include extending the analysis to continuous sets of states of the world $\mathcal{W}$ and sets of messages $\mathcal{M}$, as well as studying information design problems, where a system planner designs private messaging policies to optimize the social welfare.

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