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Wave Control via Scattering: From Spatial to Space–Time Interfaces

**Theory, design, and applications of structured media and
advanced wave manipulation**

By

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Declaration

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* This dissertation is presented in partial fulfillment of the requirements for **Ph.D. degree** in the Graduate School of Politecnico di Torino (ScuDo).

I would like to dedicate this thesis to my loving parents

Acknowledgements

And I would like to acknowledge ...

Abstract

Wave scattering in structured and time-modulated media provides a versatile framework for controlling energy transport beyond the limits of conventional materials. By combining spatial design with temporal variation, it becomes possible to tailor how waves propagate, interact, and exchange energy across a wide range of scales. In structured media, geometric features and material heterogeneity govern dispersion, polarization, localization, and directionality. In time-modulated systems, variations in material properties introduce additional degrees of freedom, enabling phenomena such as frequency conversion, non-reciprocal behaviour, and dynamic energy redistribution. This thesis presents a broad investigation of wave control strategies that draw on both spatial structuring and temporal modulation, addressing practical design applications as well as more fundamental physical phenomena. In the first part, arrays of coupled resonators are employed to manipulate the propagation of surface waves induced by low frequency ground vibrations. Building on similar design principles in acoustics, a sound absorbing panel is then developed for room acoustic applications. The third considered example examines elastic wave mode conversion in corrugated beams, showing how a nominally homogeneous structure can, through geometric modification, couple longitudinal and bending waves. The final part of the thesis focuses on more fundamental aspects of mechanics related to the interaction of waves with space-time interfaces. The propagation of a continuous deformation within a material is considered, giving rise to a moving transition front between distinct deformation states. The interaction between this front and propagating waves is analyzed, highlighting the underlying mechanisms of energy exchange in dynamically evolving media. Overall, this work shows that wave control can be systematically achieved through the engineering of scattering across space and time. From practical implementations in vibration mitigation and acoustic absorption to geometry-driven mode conversion and wave interaction with space-time interfaces, the results demonstrate a unified framework for manipulating wave propagation

and energy transport. This approach provides both effective design strategies and a deeper understanding of wave–matter interactions in structured and dynamically evolving media.

Contents

List of Figures	xii
------------------------	------------

List of Tables	xxvi
-----------------------	-------------

1 Introduction	1
1.1 Introduction	1
1.1.1 Fundamental aspects of wave propagation	1
1.1.2 Wave propagation in spatially varying media	2
1.1.3 Wave propagation in time varying media	9
2 Design and Field Validation of a Locally Resonant Metamaterial Barrier for Low-Frequency Ground Vibration Mitigation	16
2.1 Introduction	16
2.2 Unit Cell Design	19
2.3 UC Numerical Analysis	19
2.3.1 Dispersion analysis of the single cell	20
2.3.2 Transmission Analysis of the UC	21
2.4 UC Experimental Laboratory Tests	22
2.5 Soil-Structure interaction Assessment	24
2.6 Design of Anti-Vibration Modules	28
2.7 Numerical Analysis of the Anti-Vibration Modules	29

2.8	Field Test	30
2.8.1	Test Site	30
2.8.2	Measurement Setup	32
2.8.3	Soil characterization	35
2.8.4	Results	36
2.8.5	Point Source Excitation	37
2.9	Conclusions	42
3	Coiled-up acoustic resonators for subwavelength acoustic absorption	43
3.1	Introduction	43
3.2	Unit cell design and analytical model	45
3.2.1	Geometrical configuration	45
3.2.2	Equivalent acoustic model	47
3.2.3	Absorption coefficient	47
3.2.4	Parametric analysis and tuning	48
3.2.5	Experimental characterization of the unit cell and numerical comparison	49
3.3	Rainbow metamaterial panel design and characterization	55
3.3.1	Panel design and fabrication	55
3.3.2	FE model of the AM Panel	56
3.3.3	Panel characterization	58
3.3.4	AM panel sound absorption results	59
3.4	Numerical evaluation of different labyrinthine sound absorption panel solutions	61
3.4.1	Results and discussion	62
3.5	Conclusions	63
4	Wave mode conversion in phononic crystals using undulated beams	67

4.1	Introduction	67
4.2	Derivation of dispersion relations using the PWE method	71
4.3	Computation of dispersion relations and experimental verification	75
4.3.1	Computation of dispersion relations	76
4.3.2	Experimental verification of band gaps	79
4.4	Wave conversion application	81
4.4.1	Numerical investigation using a constant undulation profile	81
4.4.2	Experimental validation using a constant undulation profile	85
4.4.3	Numerical investigation using a graded profile	88
4.5	Conclusions	88
5	Wave propagation in a continuously deforming medium	91
5.1	Introduction	91
5.2	Space-time interface through deformation	93
5.2.1	Continuously deforming beam	94
5.3	Waves crossing deformation induced space-time interface	98
5.3.1	Transverse wave propagation in the undeformed region	98
5.3.2	Transverse wave propagation in the deformed region with transport velocity	99
5.3.3	Space–time varying transition	100
5.3.4	Governing equations in the co-moving frame	101
5.3.5	Continuity conditions at the interface	102
5.3.6	Scattering formulation	104
5.3.7	Phase conservation law in the laboratory frame	106
5.4	Numerical model and experimental validation	108
5.4.1	Modeling space-time interface in a hyperelastic medium	108
5.4.2	Transverse wave propagation across a finite-thickness interface	111

Contents	xi
5.4.3 Characterization of wave transmission across the moving interface	117
5.5 Conclusions	121
References	124

List of Figures

- 1.1 Schematic illustration of wave scattering at an interface for (top) acoustic and (bottom) elastic media. (a) Acoustic case, where a longitudinal incident wave I impinging on an interface between two fluids characterized by (c_1, ρ_1) and (c_2, ρ_2) generates a reflected wave R and a transmitted wave T . The scattering process involves a single mode and is governed by impedance mismatch. (b) Time evolution along the x_2 coordinate of the acoustic wave. (d) Elastic case, where an incident longitudinal wave I^P interacting with an interface between two solids characterized by (c_1^P, c_1^S, ρ_1) and (c_2^P, c_2^S, ρ_2) generates both longitudinal and transverse reflected (R^P, R^S) and transmitted (T^P, T^S) waves due to mode conversion. (d) Time evolution along the x_2 coordinate of the elastic wave. 7
- 1.2 (a) Time and space evolution of a scattering mechanism of an acoustic wave-packet across a spatial interface separating two media with different wave velocities and densities. (b) Fourier transform of the scattered field: the scattered field components share the same frequency but different wave vectors. (c) Time and space evolution of the same scattering process across a time interface. (d) Fourier transform of the scattered field: the scattered field components share the same wave-vector but different frequencies. 10
- 1.3 (a) Space-time evolution of a wave packet scattered across a time-space interface moving at velocity $-V_{int}$. (b) Fourier transform of the scattered field: transmitted and reflected wave component have different frequencies and wave vectors. 13

2.1	Schematic of an incoming ground-borne wave (a) at a frequency different from the resonators (ω_0) and (b) close to the resonance frequency.	18
2.2	Structures of the UC: (a) Exploded 3D view (b) Section view with details of the connections between steel beams and concrete pyramids.	20
2.3	(Left) 3D view of the UC and schematic of its IBZ. (Centre) dispersion diagram of the periodic structure. (Right) eigenmodes of the UC corresponding to the f_{top} and f_{bot} eigenfrequencies, respectively.	21
2.4	(Left) Schematic of the input and output positions in the numerical model and (right) the numerically-calculated transmission spectrum together with the eigenmode at f_{bot} shown in the inset.	22
2.5	(Above) Schematic of the experimental setup. (Below, from left to right) output side view (accelerometer position in red square), and input side view (shaker).	23
2.6	Comparison of the transmission curves obtained numerically and experimentally.	24
2.7	(a) Lumped parameters model of the UC and (b) schematic of the soil half-space with embedded resonators on its surface. (c) Dispersion relation for the Rayleigh waves propagating on a half space embedded with resonators. The gray region corresponds to the sound cone and the dashed line to the bulk longitudinal (c_L) and shear velocity (c_S), and the surface Rayleigh wave velocity (c_R) for a homogeneous medium. The dashed horizontal line is corresponding to resonance frequency calculated from the lumped model approximation.	26
2.8	Variation of the relative band-gap width (BG) for the soil types considered in the parametric analysis. Softer soils are associated with larger band-gap widths, whereas stiffer soils exhibit a reduced attenuation range.	28
2.9	(From left to right) exploded, front-, and isometric- views of the anti-vibration module consisting of six UCs and a surrounding U-shaped concrete block. Main element dimensions are indicated.	30

2.10	(a) Left: numerical transmission spectra of the anti-vibration modules PhV-MQ-A (with embedded unit cells) and PhV-MQ-B (U-block only). The shaded region highlights the low-frequency range where PhV-MQ-A exhibits reduced transmission due to locally resonant effects. The dashed vertical lines indicate the resonant frequencies of the system. Right: schematic representation of the two configurations. (b)–(c) Acceleration magnitude distributions (m/s^2 , colour scale) of the resonant modes at 32.5 Hz and 27.5 Hz, respectively, showing the vibration localization within the embedded metamaterial units.	31
2.11	(a) Satellite view of Elze (Germany) and its main railway station. The dashed lines indicate the area selected for the installation of the anti-vibration modules. (b) Detailed view of the test site, highlighting the railway tracks, the installation trench (150 m long and 2.4 m deep), the adjacent road, and the available free area.	33
2.12	DYNPACT [®] excitation device used during the in-field measurements. The system consists of a 50 kg drop weight released from a height of 1 m to generate an impulsive load in the soil for mobility and insertion loss evaluation.	34
2.13	Schematic of the experimental setup.	35
2.14	Results of the MASW analysis: soil profile in terms of shear wave velocity and dynamic shear modulus.	36
2.15	Point Source. Comparison of the IL at different distances from the source for (a) the solution PhV-MQ-A and (b) the solution PhV-MQ-B. The red shaded region indicates the frequency range in which the two UC types differ in spectral behaviour	39
2.16	Schematic representation of the construction of a line source from point sources, to obtain the line source mobility functions at a desired location by integrating point transfer mobility functions.	40
2.17	Line Source. Comparison of the IL at different distances from the source for the solution PhV-MQ-A (a) and for the solution PhV-MQ-B (b). The red shaded region indicates the frequency range in which the two UC types differ in spectral behaviour	41

- 3.1 Schematic of (a) the sound reflection problem at a hard boundary. (b) Engineered resonant layer working at frequency different and (c) close to the resonance. The layer is designed to maximize absorption (A) at resonance. Undulated arrows indicate the absorbed energy from the layer 44
- 3.2 (a) Geometry of the three-dimensional labyrinthine unit cell (UC) based on a space-coiling architecture, characterized by lateral size a , thickness t , and internal channel width w . (b) Schematic representation of the effective acoustic propagation path inside the UC, illustrating the elongation of the wavelength due to the folded channel geometry. (c) Equivalent one-dimensional acoustic model of the UC, represented as a lossy narrow duct of effective length l_{eff} and cross-sectional area S_c , rigidly backed and coupled to the external acoustic domain through a dissipative aperture of area S_a 46
- 3.3 Parametric map of the analytically predicted maximum absorption coefficient of the labyrinthine unit cell as a function of the lateral edge size a and the aperture cross-sectional area S_a , for a fixed unit cell thickness $t = 9.6$ mm. The lateral edge size is varied in the range $16 \text{ mm} \leq a \leq 29 \text{ mm}$, while the aperture area spans approximately $20 \text{ mm}^2 \leq S_a \leq 90 \text{ mm}^2$. Contour lines indicate the resonance frequency corresponding to the absorption peak. The selected unit cell geometry, providing near-perfect absorption within the target frequency range, is highlighted. 48
- 3.4 (a) Analytically predicted normal-incidence absorption spectrum of the selected labyrinthine unit cell, showing a pronounced absorption peak associated with the fundamental resonance. (b) Representation of the squared reflectance in the complex frequency plane for the lossless case, where a complex-conjugate zero–pole pair is observed away from the real frequency axis. (c) Corresponding reflectance map including thermo-viscous losses, where the zero is shifted towards the real axis. Perfect absorption is achieved when the critical coupling condition is satisfied, corresponding to the alignment of the reflectance zero with the real frequency axis. 49

- 3.5 Experimental setup for the impedance tube measurements. (a) Complete Kundt's tube apparatus used for normal-incidence absorption measurements, equipped with a broadband sound source and two flush-mounted microphones. (b) Detail of the opened tube showing the labyrinthine unit cell mounted in the custom sample holder, which ensures accurate positioning and minimizes circumferential air gaps. 51
- 3.6 Finite element model of the impedance tube containing the labyrinthine unit cell. (a) Computational domain and mesh of the impedance tube, including the air regions, the unit cell cavity and the inlet aperture; the blue region indicates the location of the imposed plane-wave excitation, while the yellow regions denote the surfaces where the averaged acoustic pressure is sampled. (b) Acoustic pressure field distribution at the peak absorption frequency. (c) Spatial distribution of the dissipated acoustic energy, highlighting the dominant contribution of thermo-viscous losses in the aperture region. 52
- 3.7 Comparison between analytical, numerical and experimental absorption spectra of the labyrinthine unit cell under normal incidence conditions. The results refer to a unit cell with lateral size $a = 19.7$ mm and thickness $t = 9.6$ mm, showing good agreement in terms of resonance frequency and absorption level, with minor discrepancies attributed to additional thermo-viscous losses in the experimental setup. 53
- 3.8 Variation of the absorption peak frequency as a function of the geometrical parameters of the labyrinthine unit cell. (a) Resonance frequency versus unit cell thickness t (with fixed lateral size $a = 21.2$ mm). (b) Resonance frequency versus lateral edge size a (with fixed thickness $t = 12.6$ mm). Symbols represent experimental results, while analytical and numerical predictions are shown for comparison. Error bars correspond to the measured peak bandwidth. 54

- 3.9 Modular architecture of the rainbow labyrinthine metamaterial panel. (a) Two adjacent macrocells, each composed of unit cells with lateral sizes a_1 and a_2 and different thicknesses (T1–T6) to generate multiple resonance frequencies. (b) Top view schematic of the complete panel, obtained by arranging and rotating mirrored macrocells to ensure spatial homogeneity of the acoustic response (dimensions in mm). (c) Three-dimensional view of two assembled macrocells highlighting the variation in unit cell thickness across the module. 56
- 3.10 Finite element model of the rainbow labyrinthine metamaterial panel. (a) Rectangular acoustic waveguide used to simulate normal-incidence sound propagation toward the panel and the backing cavity, with periodic boundary conditions applied on the lateral surfaces. (b) Corresponding finite element mesh, showing local refinement in the metamaterial region to accurately resolve the narrow labyrinthine channels and thermo-viscous effects. 57
- 3.11 Experimental characterization of the rainbow labyrinthine metamaterial panel in a small-scale reverberation room. (a) Measurement configuration with the panel placed on the floor and the flat surface exposed to the diffuse acoustic field. (b) Random-incidence absorption coefficient as a function of frequency, compared with analytical predictions and finite element simulations, showing near-unity absorption within the target range of 800–1300 Hz. 61
- 3.12 Finite element model of the rainbow metamaterial macrocell coupled with a backing cavity. (a) Acoustic domain used to simulate normal-incidence sound propagation toward the labyrinthine panel and the rear cavity, terminated with a Perfectly Matched Layer (PML) to avoid artificial reflections. (b) Corresponding finite element mesh, showing local refinement in the metamaterial region and in the backing cavity to accurately capture thermo-viscous effects and pressure gradients. 64

3.13	Finite element simulation results of the normal-incidence absorption coefficient for different panel configurations: baseline rainbow meta-material panel (AM), panel with a 5 mm foam-filled backing cavity (AM + foam backing), and panel with both foam-filled labyrinthine cavities and foam-filled backing cavity (foam AM + foam backing). The addition of the backing cavity shifts the absorption peaks toward lower frequencies, while foam filling leads to a broader but less pronounced absorption response.	65
4.1	(a) Schematic of the wave conversion process for a longitudinal incident wave impinging to an undulated region and converting into a bending mode. (b) Schematic profile of the undulated region for wave conversion.	71
4.2	Longitudinal-flexural wave mode conversion device. a. The beam with an undulated profile is designed to convert incident longitudinal waves (blue arrow) to reflected flexural waves (red arrow). b. General profile $d(x)$ of the undulated region with lattice length a , constant thickness h , and undulation amplitude d_0 . c. Schematic representation of constant (top panel) and graded (bottom panel) undulated profiles, with n_c undulated unit cells and a maximum height d_0	73

- 4.3 Dispersion relations computed using the PWE and FE as a function of the normalized undulation height $d_0/a \in [0, 0.2]$ in the frequency range $\bar{\omega} \in [0, \pi]$. a. Dispersion curves computed with the PWE method with the out-of-plane polarization metric (p) ranging from 0 (purely longitudinal, blue) to 1 (purely transverse, red). b. Comparison of dispersion curves obtained using the PWE (black circles) and FE (colored lines) for increasing values of d_0/a . For $d_0/a = 0$ (left panel), highlighted branches are labeled as L (longitudinal), F (flexural), T (torsional), and S (shear), with corresponding wave modes shown at the edge of the first Brillouin zone ($ka/\pi = 1$). For $d_0/a = 0.01$ (middle panel), the longitudinal and flexural branches present locking (L–F) due to the coupling between in-plane ($p < 0.5$) and out-of-plane ($p > 0.5$) branches. A similar behavior is observed for the torsional and shear branches (T–S), only captured by the FE model. For $d_0/a = 0.10$ (right panel), a significant band gap is noticed between the coupled L–F (T–S) branches, with a polarization $p \approx 0.5$ at the local maximum of the coupled branch. c. Isofrequency line for $\bar{\omega} = 1$ (black line) indicated on the second dispersion branches of (a), with a maximum value (green circle) obtained at $d_0/a = 0.093$ (green dashed line). The blue and red circles correspond, respectively, to the longitudinal and flexural wave modes at a flat unit cell ($d_0/a = 0$). d. Normalized group velocity (c_g/c_L) computed along the isofrequency line $\bar{\omega} = 1$ (black line), indicating a zero group velocity ($c_g = 0$) at $d_0/a = 0.093$ 77

- 4.4 Time harmonic simulations. a. Simulated system with two flat regions (lengths $n_l \times a$ and $n_r \times a$) divided by an undulated region (length $n_c \times a$). A time harmonic longitudinal force (F_x) is applied at the left end of the beam. b. Manufactured test specimen shown close to the undulated region. c. Experimental setup showing the fabricated sample with an attached piezoelectric transducer resting on a soft support and the laser Doppler vibrometer used to perform the measurements. d. Simulation results for normalized transverse displacements along the beam axis and the frequency range $\bar{\omega} \in [0.5, 1.85]$. The vertical dashed black lines indicate the undulated region ($x/a \in [0, n_c]$) and the horizontal dashed white lines indicate the band gap region. The colorbar represents absolute vertical displacements ($|u_z|$). e. Experimental results corresponding to the numerical simulation in (d), with the colorbar adjusted for visualization. 80
- 4.5 Numerical simulation of a longitudinal transient pulse applied to a beam with an undulated region. a. A longitudinal force (F_x) is applied using a modulated sinusoidal pulse (inset). Surfaces S_1 and S_2 (in green and orange, respectively), located at the edges of the flat regions, are considered for the analysis of mechanical energy flux. Two PMLs (lengths $n_{\text{PML}} \times a$) are included at the outer edges of each flat region. b. Longitudinal (φ_l) and flexural elastic power (φ_f) computed at the surfaces S_1 (green) and S_2 (orange). The sign indicates forward- (> 0) or backward-propagating energy (< 0). The dashed vertical lines indicate the time instants $t/T = 2.5, 5.0$, and 7.5 . c. Absolute normalized displacements represented at the time instants indicated by the dashed lines in (b). The colorbar represents absolute displacements ($|u|$). The blue and red arrows represent longitudinal and flexural waves, respectively. The curved arrows indicate the occurrence of wave conversion. 83

- 4.6 Numerical simulation and experimental validation of longitudinal to flexural wave conversion using a beam with a constant undulation profile. a. Computed longitudinal (left panel, u_x) and transverse (right panel, u_z) displacements at the top surface of the beam. Vertical dashed lines indicate the undulated region. Blue and red arrows represent, respectively, longitudinal and flexural wavefronts. The colorbar indicates normalized displacements (u). b. 2D-FFTs corresponding to the longitudinal (u_x , top panel) and transverse (u_z , bottom panel) displacements shown in (a). Blue and red dashed lines indicate, respectively, longitudinal and flexural analytical dispersion branches at the flat region. The colorbar indicates normalized 2D-FFT values ($|\text{FFT}(u)|^2$). (c,d). Same as (a,b), obtained experimentally. 86
- 4.7 Numerical simulation of longitudinal to flexural wave conversion using a beam with a graded undulation profile. a. Computed longitudinal (left panel, u_x) and transverse (right panel, u_z) displacements at the top surface of the beam. Vertical dashed lines indicate the undulated region. Blue and red arrows represent, respectively, longitudinal and flexural wavefronts. The colorbar indicates normalized displacements (u). b. 2D-FFTs corresponding to the longitudinal (u_x , top panel) and transverse (u_z , bottom panel) displacements shown in (a). Blue and red dashed lines indicate, respectively, longitudinal and flexural analytical dispersion branches at the flat region. The colorbar indicates normalized 2D-FFT values ($|\text{FFT}(u)|^2$). 89

- 5.1 Schematic of (a) wave scattering at a spac-time varying interface in a deforming medium in 1D, where one boundary of the medium is displaced at an imposed velocity causing the propagation of a deformation front. The gray arrows behind the front represent the translation velocity of the particles of the medium. (b) Corresponding 2D spectra representation showing frequency and wave-number conversion linked by the phase conservation relation represented by the black dashed line. The brown dashed lines correspond to the dispersion relation of the deformed medium at rest, which due to the drift velocity of the medium are rotated by a Doppler shift $V_0 k$, where V_0 is the velocity of the moving boundary. 94
- 5.2 (a) Soft ramp function $f(t)$ (left) and its time derivatives (right) for different smoothing parameters τ , compared to the ideal discontinuous ramp (dashed). (b) Alternative parametrization in terms of characteristic velocity V_0 , showing the same quantities $f(t)$ and df/dt . (c) Corresponding spatial profiles of normalized displacement u/L , strain $\partial_x u$, and velocity $\partial_t u$ along the bar at different times, highlighting the propagation of a smooth deformation front. (d) Space–time maps of displacement, stretch $1 + \partial_x u$, and velocity $\partial_t u$, illustrating the transition layer traveling through the medium and the associated gradients induced by the smoothed ramp loading for a positive (first row) or negative (second row) value of V_0 , and $\lambda_0 = 0.05$ and $\lambda_0 = 0.5$, respectively. 97
- 5.3 Schematic of transverse wave propagation across a deformation-induced space–time interface in an slender beam. (a) laboratory-frame view, where a localized stretching region generates a moving transition with effective velocity c_L , leading to a conversion of the incident (blue) to a transmitted (red) wavepacket. The dashed line represents the space–time trajectory of the interface. (b) Dispersion relation of the system at different pulling/contraction velocities. The black dotted line indicates the reference dispersion relation (c) transmitted mode amplitude calculated from the scattering matrix for a pre-stretch $\lambda_0 = 1.05$ and $\lambda_0 = 1.5$ at positive and negative pulling velocities, respectively. 107

- 5.4 (a) Longitudinal stress vs engineering strain plot for the Mooney-Rivlin elastic energy potential and comparison with the linear approximation. Black arrows highlight the initial stretches chosen for the simulation. (b) Schematic of the two different simulations where a velocity controlled (top) pull and (bottom) contraction is used as boundary condition. Red arrows represent the velocity field of the particles past the interface and the orange ones the velocity of the interface. (c) and (d) Comparison between analytical and numerical solutions of the displacement field, stretch, and particle velocity in the current configuration. Panel (c) corresponds to a tensile (loading) case, while panel (d) represents a release (unloading) case. 111
- 5.5 (a) Schematic of the hyperelastic strip subjected to axial deformation and transverse excitation. A localized wavepacket $\delta W(t)$ is generated at the top boundary, while a time-dependent axial displacement $\delta U(t)$ induces a propagating transition layer with characteristic time τ and onset time t_0 . Comparison between numerical and experimental result for (b) space–time representation transverse displacement field, showing the interaction of the incident wavepacket with the moving deformation front and the resulting transmitted wave. The change in slope of the wavefronts indicates a modification of phase velocity across the interface and the black contour lines indicate the longitudinal component of the deformation gradient. (c) Frequency–wavenumber spectrum of the wave field, highlighting the shift between incident and transmitted components. The superimposed dispersion curves (solid blue and red lines) and the phase-matching condition (dashed white line) confirm the validity of the phase conservation law and the predicted frequency–wavenumber conversion induced by the space–time interface. 114

- 5.6 (a) Schematic of the hyperelastic strip subjected to axial deformation and transverse excitation. A localized wavepacket $\delta W(t)$ is generated at the top boundary, while the other end controlled by the boundary displacement function δU . Numerical result for (b) space–time representation of the transverse displacement field, showing the interaction of the incident wavepacket with the deformation front and the resulting transmitted and reflected wave. The inversion of propagation direction takes place due to a larger drift velocity compared to the wave-packet velocity. (c) Frequency–wavenumber spectrum of the wave field, highlighting the shift between incident and transmitted components. The superimposed dispersion curves (solid blue and red lines) and the phase-matching condition (dashed white line) confirm the validity of the phase conservation law and the predicted frequency–wavenumber conversion induced by the space–time interface. 116
- 5.7 Numerical characterization of transverse wave transmission across a deformation-induced space–time interface. (a) Space–time evolution of the transverse displacement field, showing the incident wavepacket interacting with the moving transition layer and generating a transmitted wave with modified slope, indicative of a change in phase velocity. (b) Temporal evolution of the maximum transverse displacement $\max(W)$, highlighting two main events: wavepacket generation and interface crossing. The amplitude variation ΔW calculated as a peak to peak difference between incident wave amplitude (blue dot) and transmitted wave amplitude (orange dot), quantifies the effect of the interface. The intermediate fluctuation is associated with the propagation and reflection of a faster longitudinal disturbance. (c) Detailed view of $\max(W)$ for different drift velocities V_0 , showing a systematic reduction of the transmitted amplitude and a secondary modulation occurring at $t \approx 2L/c_L$, corresponding to the round-trip time of the longitudinal wave. (d) Transmission coefficient T as a function of the drift velocity V_0 , comparing numerical results with the analytical prediction. The good agreement confirms the validity of the sharp-interface model, while small deviations are attributed to the finite thickness of the transition layer. 119

- 5.8 Numerical characterization of transverse wave transmission across a deformation-induced space–time interface. (a) Space–time evolution of the transverse displacement field, showing the incident wavepacket interacting with the moving transition layer and generating a transmitted wave with modified slope, indicative of a change in phase velocity. (b) Temporal evolution of the maximum transverse displacement $\max(W)$, highlighting two main events: wavepacket generation and interface crossing. The amplitude variation ΔW , calculated as a peak to peak difference between incident wave amplitude (blue dot) and transmitted wave amplitude (orange dot), quantifies the effect of the interface, while the intermediate fluctuation is associated with the propagation and reflection of a faster longitudinal disturbance. (c) Detailed view of $\max(W)$ for different drift velocities V_0 . (d) Transmission coefficient T as a function of the drift velocity V_0 , comparing numerical results with the analytical prediction. The good agreement confirms the validity of the sharp-interface model, while small deviations are attributed to the finite thickness of the transition layer. 121

List of Tables

2.1	Mechanical properties of the soil types considered in the parametric analysis.	27
3.1	Melamine foam material properties adopted in the finite element simulations.	62

Chapter 1

Introduction

1.1 Introduction

1.1.1 Fundamental aspects of wave propagation

Wave propagation is a fundamental physical phenomenon through which energy, momentum, and information are transferred across space and time without the permanent transport of matter. Waves arise in a wide variety of systems, including mechanical, acoustic, electromagnetic, and quantum media, and their behavior is governed by the interaction between inertia and restoring forces present in the medium.

In its simplest form, wave motion can be described by the linear wave equation

$$\nabla^2 U(\mathbf{x}, t) - \frac{1}{c^2} \frac{\partial^2 U(\mathbf{x}, t)}{\partial t^2} = 0, \quad (1.1)$$

where $U(\mathbf{x}, t)$ represents the wave field and c is the propagation velocity. Harmonic solutions of the form

$$U(\mathbf{x}, t) = A e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \quad (1.2)$$

describe monochromatic waves characterized by amplitude A , angular frequency ω , and wave vector $\mathbf{k}$. The wave vector determines the wavelength $\lambda = 2\pi/k$ and the propagation direction, while the angular frequency is related to the temporal

oscillation period $T = 2\pi/\omega$.

The fundamental relationship linking spatial and temporal wave characteristics is

$$c = \frac{\omega}{k} = f\lambda, \quad (1.3)$$

where f is the frequency. For non-dispersive media, the propagation velocity is independent of frequency and all spectral components travel at the same speed. In dispersive systems, however, the relationship between ω and k becomes nonlinear, leading to frequency-dependent propagation. This dependence is described by the dispersion relation $\omega = \omega(k)$, which governs many wave phenomena such as pulse spreading, localization, and band-gap formation.

Wave propagation in elastic and acoustic media is strongly influenced by the material properties of the host medium. An important aspect of wave physics is the distinction between phase velocity and group velocity. The phase velocity,

$$v_p = \frac{\omega}{k}, \quad (1.4)$$

describes the speed of individual wavefronts, whereas the group velocity,

$$v_g = \frac{\partial \omega}{\partial k}, \quad (1.5)$$

represents the velocity at which wave packets and energy propagate. In dispersive media these two quantities generally differ, leading to rich dynamical phenomena and complex energy transport mechanisms.

When waves encounter spatial or temporal variations of material properties, scattering processes occur. At a conventional spatial interface separating two media with different wave velocities, frequency remains conserved while the wave vector changes according to the boundary conditions.

1.1.2 Wave propagation in spatially varying media

Scattering phenomena arise when propagating waves encounter spatial variations in material properties, leading to a redistribution of energy through reflection, transmission, and, in more complex media, mode conversion. These processes are

fundamental in acoustics and elasticity and form the basis for understanding wave interactions with interfaces, inclusions, and structured materials.

The simplest configuration illustrating scattering is that of a planar interface separating two homogeneous media. In this case, an incident wave generates reflected and transmitted waves, whose amplitudes and directions are determined by the governing equations and boundary conditions. This canonical problem provides the foundation for defining reflection and transmission coefficients and for extending the analysis to more complex scattering scenarios.

In fluids, acoustic wave propagation is governed by linearized equations of motion. Combining momentum conservation and the constitutive relation yields the scalar wave equation for the pressure field $p(\mathbf{x}, t)$:

$$\nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0, \quad (1.6)$$

where $c = \sqrt{B/\rho}$ is the speed of sound, with ρ the density and B the bulk modulus. For time-harmonic waves, this reduces to the Helmholtz equation:

$$\nabla^2 p + k^2 p = 0, \quad k = \frac{\omega}{c}. \quad (1.7)$$

At an interface between two fluids (fig. 1.1a and 1.1b) characterized by (ρ_1, c_1) and (ρ_2, c_2) , the wavefield must satisfy [1]:

- continuity of pressure,

$$p_1 = p_2, \quad (1.8)$$

- continuity of normal particle velocity,

$$\frac{1}{\rho_1} \frac{\partial p_1}{\partial n} = \frac{1}{\rho_2} \frac{\partial p_2}{\partial n}. \quad (1.9)$$

For a plane wave at normal incidence, introducing the acoustic impedance $Z = \rho c$, the reflection and transmission coefficients are:

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad T = \frac{2Z_2}{Z_1 + Z_2}, \quad (1.10)$$

with corresponding energy coefficients:

$$|R|^2 + \left| \frac{Z_2}{Z_1} T \right|^2 = 1 \quad (1.11)$$

For oblique incidence, phase continuity along the interface leads to Snell's law:

$$\frac{\sin \theta_t}{c_2} = \frac{\sin \theta_i}{c_1}, \quad (1.12)$$

which determines the propagation directions of reflected and transmitted waves.

In elastic bodies, in the absence of body forces, the propagation of elastic waves is governed by the balance of linear momentum,

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma}, \quad (1.13)$$

where ρ is the mass density, $\mathbf{u}(\mathbf{x}, t)$ is the displacement field, and $\boldsymbol{\sigma}$ is the Cauchy stress tensor. For an isotropic linear elastic solid, the constitutive relation is given by

$$\boldsymbol{\sigma} = \lambda (\nabla \cdot \mathbf{u}) \mathbf{I} + \mu (\nabla \mathbf{u} + \nabla \mathbf{u}^T), \quad (1.14)$$

where λ and μ are the Lamé parameters and $\mathbf{I}$ is the identity tensor. Substituting this constitutive law into the momentum balance yields the elastodynamic (Navier) equation,

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u}, \quad (1.15)$$

which, using the vector identity

$$\nabla^2 \mathbf{u} = \nabla (\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}),$$

can be rewritten as

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}). \quad (1.16)$$

- longitudinal (P) waves with velocity

$$c_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad (1.17)$$

- transverse (S) waves with velocity

$$c_S = \sqrt{\frac{\mu}{\rho}}. \quad (1.18)$$

At an interface between two elastic media (fig. 1.1a and 1.1b), the wavefield must satisfy:

- continuity of displacement,

$$\mathbf{u}_1 = \mathbf{u}_2, \quad (1.19)$$

- continuity of normal stresses,

$$\boldsymbol{\sigma}_1 \cdot \mathbf{n} = \boldsymbol{\sigma}_2 \cdot \mathbf{n}, \quad (1.20)$$

where $\boldsymbol{\sigma}_i$, with $i = 1, 2$ is the stress tensor evaluated at medium 1 and 2, respectively. These conditions couple the normal and tangential components of motion, potentially giving rise to mode conversion. For example, an incident P-wave can generate reflected and transmitted P and S waves.

The directions of the scattered waves are determined by phase continuity along the interface, which imposes conservation of the tangential component of the wave-vector. This leads to a generalized Snell's law:

$$\frac{\sin \theta_I}{c_1^P} = \frac{\sin \theta_R^P}{c_1^P} = \frac{\sin \theta_R^S}{c_1^S} = \frac{\sin \theta_T^P}{c_2^P} = \frac{\sin \theta_T^S}{c_2^S}. \quad (1.21)$$

Thus, each polarization propagates at a different angle determined by its wave velocity.

The amplitudes of the reflected and transmitted waves are obtained by solving the boundary conditions, leading to the Zoeppritz equations [2], which can be written in matrix form:

$$\mathbf{M} \begin{pmatrix} R^P \\ R^S \\ T^P \\ T^S \end{pmatrix} = \mathbf{b}, \quad (1.22)$$

where $\mathbf{M}$ is a 4×4 matrix depending on densities, wave velocities, and propagation angles.

A commonly used explicit form is:

$$\begin{pmatrix} -\sin \theta_i & -\cos \theta_i & \sin \theta_t^P & \cos \theta_t^S \\ \cos \theta_i & -\sin \theta_i & \cos \theta_t^P & -\sin \theta_t^S \\ \sin 2\theta_i & \frac{c_1^P}{c_1^S} \cos \theta_t^S & \frac{\rho_2(c_2^S)^2 c_1^P}{\rho_1(c_1^S)^2 c_2^P} \sin 2\theta_t^P & \frac{\rho_2 c_2^S c_1^P}{\rho_1(c_1^S)^2} \cos 2\theta_t^S \\ -\cos 2\theta_i & \frac{c_1^S}{c_1^P} \sin 2\theta_t^S & \frac{\rho_2 c_2^P}{\rho_1 c_1^P} \cos 2\theta_t^P & -\frac{\rho_2 c_2^S}{\rho_1 c_1^P} c_2^S \sin 2\theta_t^S \end{pmatrix} \begin{pmatrix} R^P \\ R^S \\ T^P \\ T^S \end{pmatrix} = \begin{pmatrix} -\sin \theta_i \\ \cos \theta_i \\ \sin 2\theta_i \\ \cos 2\theta_i \end{pmatrix}. \quad (1.23)$$

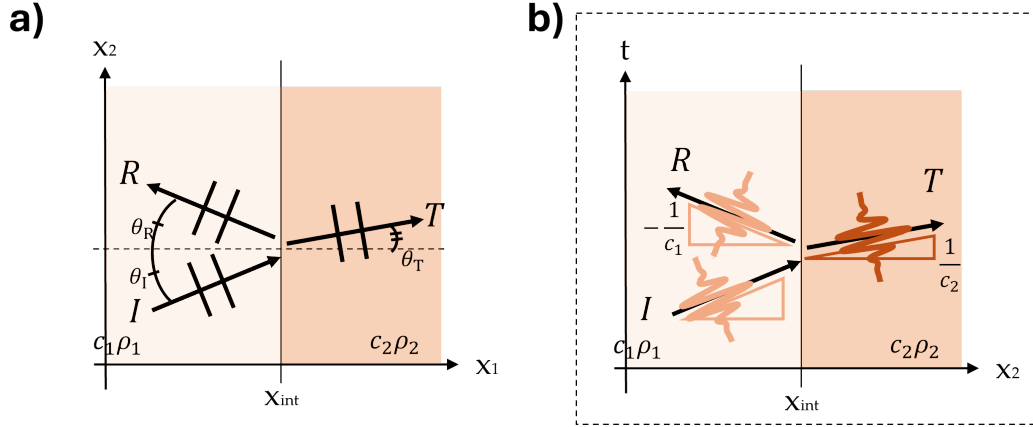
The energy transported by each mode leads to the generalized conservation relation:

$$|R^P|^2 + |R^S|^2 + |T^P|^2 + |T^S|^2 = 1. \quad (1.24)$$

Within this framework, scattering phenomena have been exploited as a design principle to control wave propagation. A prominent example of this approach is provided by phononic crystals, which are artificially engineered materials composed of a periodic arrangement of inclusions or material contrasts on the scale of the wavelength. This periodicity leads to repeated scattering and multiple reflections of waves as they propagate through the structure. The resulting interference effect, known as Bragg scattering, gives rise to characteristic dispersion relations with frequency ranges, known as band gaps, in which wave propagation is forbidden [3, 4].

In one of the earliest theoretical demonstrations of this concept, Kushwaha [5], and Sigalas [6] derived the dispersion relation for a periodic elastic and acoustic composite showing the presence of band gaps arising from the Bragg scattering phenomena.

Acoustics



Elasticity

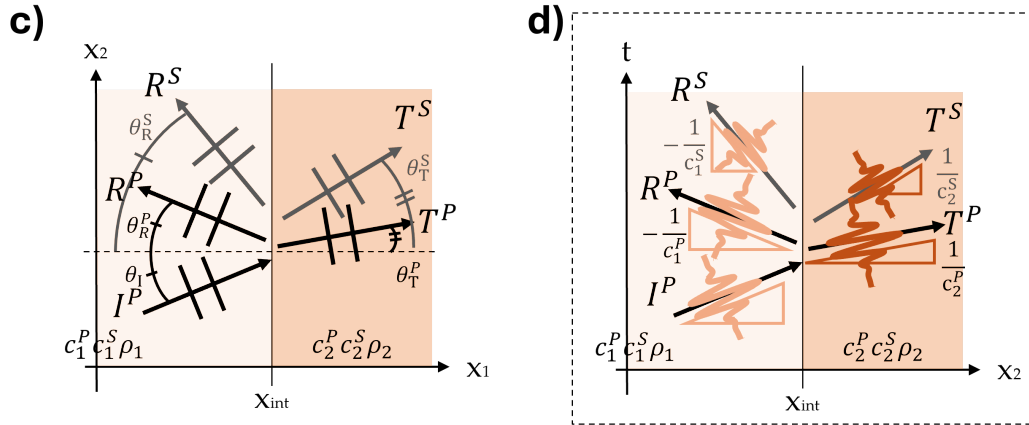


Fig. 1.1 Schematic illustration of wave scattering at an interface for (top) acoustic and (bottom) elastic media. (a) Acoustic case, where a longitudinal incident wave I impinging on an interface between two fluids characterized by (c_1, ρ_1) and (c_2, ρ_2) generates a reflected wave R and a transmitted wave T . The scattering process involves a single mode and is governed by impedance mismatch. (b) Time evolution along the x_2 coordinate of the acoustic wave. (d) Elastic case, where an incident longitudinal wave I^P interacting with an interface between two solids characterized by (c_1^P, c_1^S, ρ_1) and (c_2^P, c_2^S, ρ_2) generates both longitudinal and transverse reflected (R^P, R^S) and transmitted (T^P, T^S) waves due to mode conversion. (d) Time evolution along the x_2 coordinate of the elastic wave.

Several experimental validations of these theoretical predictions were then achieved in subsequent works [7–9], demonstrating sound attenuation in a two-dimensional array of rigid cylinders in air. In this case, the cylinders act as scattering centers, and the repeated reflections at their surfaces lead to a macroscopic reduction of transmission in certain frequency bands, confirming the role of multiple scattering in shaping wave propagation.

Beyond the formation of band gaps, scattering in phononic crystals has been exploited to achieve waveguiding and localization. Defect engineering, in which the periodicity is broken locally, allows the creation of states within the band gap where waves can propagate along prescribed paths [10, 11]. This mechanism relies on the suppression of scattering in the bulk of the crystal, while allowing controlled transmission along the defect, effectively guiding the wave through the structure [12–15].

In all these examples, the underlying mechanism remains the same: wave manipulation is achieved through the cumulative effect of multiple scattering events. By engineering the spatial distribution of scatterers, phononic crystals provide a powerful platform for controlling wave propagation through interference, enabling functionalities such as filtering, waveguiding, and localization.

Beyond scattering-based mechanisms in periodic media, wave control can also be achieved through resonance-driven effects. Locally resonant metamaterials represent a major advancement in this direction, enabling manipulation at scales much smaller than the wavelength. These systems are composed of resonant elements embedded in a host medium, enabling wave manipulation through strong, frequency-selective scattering. Unlike phononic crystals, where band gaps arise from Bragg scattering at wavelengths comparable to the lattice spacing, these systems exploit internal resonances to induce reflection and energy trapping at much larger wavelengths. The coupling between propagating waves and local resonators leads to hybridization effects, giving rise to deep subwavelength band gaps and effective medium behaviors such as negative dynamic mass density or bulk modulus. As a result, locally resonant metamaterials allow compact and efficient control of wave propagation, including attenuation, localization, and dispersion engineering. The foundational work by Liu *et al.* [16] introduced the concept of locally resonant sonic materials, demonstrating that subwavelength resonant inclusions embedded in a host medium can produce band gaps at frequencies well below those predicted by Bragg scattering.

This work established resonance-driven scattering as a powerful mechanism for wave attenuation. Subsequent studies further developed the theoretical framework by introducing effective medium descriptions. In particular, Liu [17] showed that such systems can exhibit negative effective mass density near resonance, providing a physical interpretation of the strong scattering and band gap formation. Similarly, Fang [18] demonstrated experimentally that locally resonant structures can exhibit negative effective bulk modulus, confirming that both constitutive parameters governing wave propagation can be engineered through resonance. Building on these concepts, Yang [19] proposed membrane-type acoustic metamaterials capable of achieving negative dynamic mass and strong low-frequency attenuation in compact structures. These designs highlighted the potential of locally resonant systems for practical applications, particularly in noise and vibration control. More generally, the field has been extensively reviewed and expanded in recent years. Hussein [20] provided a comprehensive overview of the dynamics of phononic materials, bridging the concepts of periodic scattering and local resonance. Cummer [21] and Ma [22] further emphasized the role of engineered scattering and resonance in achieving advanced functionalities, including wave redirection, cloaking, and broadband control. Together, these works demonstrate that locally resonant metamaterials exploit strong, frequency-selective scattering induced by internal resonances to achieve precise control over wave propagation, extending the capabilities of classical phononic crystals into the deep subwavelength regime.

1.1.3 Wave propagation in time varying media

A further level of control is introduced in time-varying media, where material properties evolve dynamically. Time-varying materials are systems in which constitutive parameters, such as density, bulk modulus, or elastic stiffness, are modulated over time. In contrast to conventional metamaterials, which rely solely on spatial structuring (fig. 1.2a, 1.2b) to control wave propagation through scattering and interference, time-varying media introduce an additional degree of freedom. As a result, these systems overcome fundamental limitations of static materials, such as reciprocity and fixed-frequency response, allowing for phenomena including frequency conversion, parametric amplification, and nonreciprocal wave propagation.

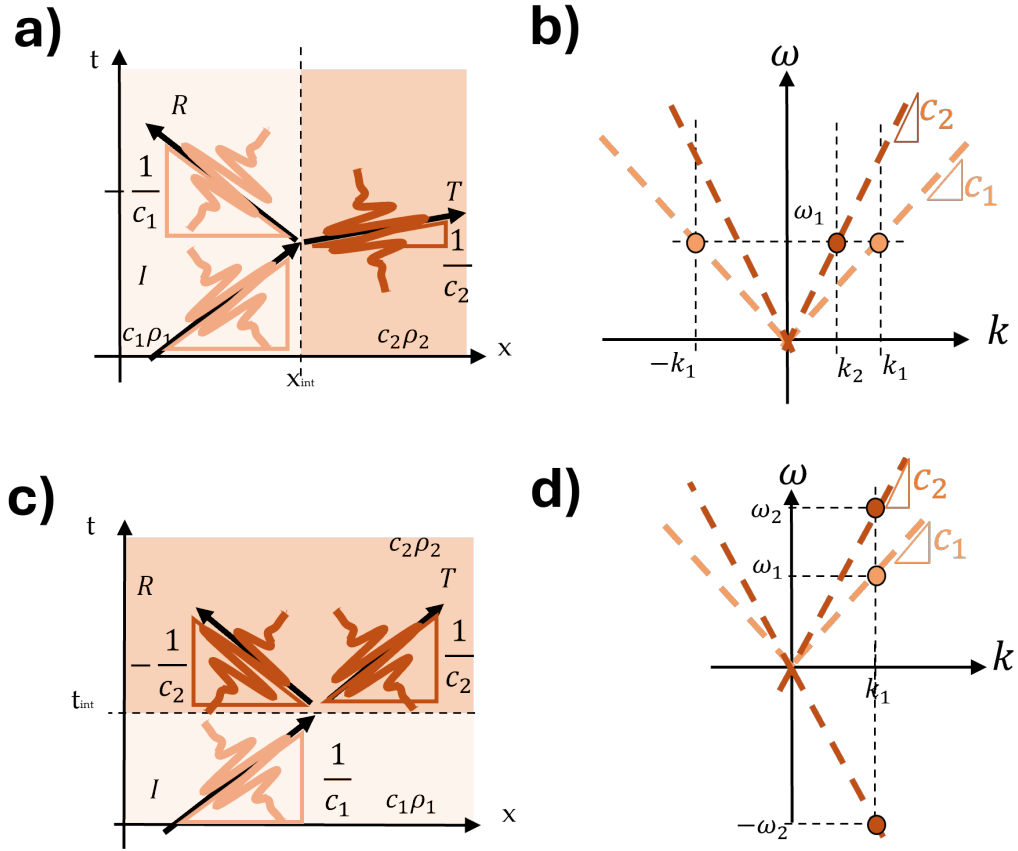


Fig. 1.2 (a) Time and space evolution of a scattering mechanism of an acoustic wave-packet across a spatial interface separating two media with different wave velocities and densities. (b) Fourier transform of the scattered field: the scattered field components share the same frequency but different wave vectors. (c) Time and space evolution of the same scattering process across a time interface. (d) Fourier transform of the scattered field: the scattered field components share the same wave-vector but different frequencies.

From a physical perspective, temporal modulation can be interpreted as a form of scattering in the time domain (fig.1.2c and 1.2d), where the modulation couples different wave states and redistributes energy across the frequency spectrum. Moreover, under spatial invariance, the phase-matching condition leads to the following frequency conversion relation:

$$\frac{\omega_1}{c_1} = \frac{\omega_2}{c_2}. \quad (1.25)$$

The concept of temporal modulation of material properties was first rigorously introduced by Morgenthaler [23], who investigated the propagation of electromagnetic waves in media with explicitly time-dependent constitutive parameters. In this

seminal work, it was shown that a temporal variation of the permittivity leads to a breakdown of frequency conservation, allowing for the generation of new frequency components and enabling energy exchange between waves. This study established the fundamental role of time-translation symmetry in wave propagation and highlighted how its breaking gives rise to inherently nonstationary wave phenomena.

Building on these early insights, temporal modulation has evolved into a powerful paradigm for wave control. In particular, recent works [24] have explored both theoretically and experimentally photonic systems, where temporal modulation can be implemented through ultrafast changes in the refractive index. In this context, temporal modulation has been used to reproduce time-varying counterpart of space-varying structures. In [25] Pacheco-Peña demonstrated how the refractive index of a medium can be modulated in time to obtain a temporal antireflection coating. In addition, Tirole and co-workers [26] have experimentally demonstrated the temporal analogue of Young's double-slit experiment. By applying a double temporal gating to a light beam reflected from a time-modulated mirror, they observed interference patterns emerging in the frequency domain. These studies provide clear theoretical and experimental evidence of how temporal variations can be used to tailor wave propagation in complex photonic environments.

More recently, the development of actively controlled metamaterials and metasurfaces has enabled practical implementations of space-time modulated systems. These studies bridge the gap between theoretical predictions and experimental realizations, highlighting the versatility of time-modulated platforms for advanced wave manipulation.

The first theoretical and experimental demonstrations of these concepts in acoustics were provided by the work of Bacot [27], who experimentally demonstrated the breaking of time-reversal symmetry in water waves through temporal modulation. More recently, Kim [28] and Wang [29] provided experimental evidence of non-reciprocal elastic wave propagation in dynamically modulated periodic structures, confirming the practical implementation of these concepts in solid mechanics. Comprehensive reviews by Cummer [21] and Galiffi [24] have further highlighted the potential of time-varying media for advanced wave control. These works demonstrate that time modulation extends the concept of scattering beyond the spatial domain, enabling unprecedented control over wave propagation through the combined manip-

ulation of space, time, and frequency.

In contrast to media that vary purely in space or purely in time—where wave manipulation relies on modulations that remain invariant in the complementary domain—space–time varying media combine both degrees of freedom, introducing modulations that evolve simultaneously in space and time. As a result, scattering processes are no longer confined to either the spatial or temporal domain, but instead involve a coupled redistribution of energy and momentum across both. This leads to interactions between wave components with different frequencies and wavelengths, giving rise to enriched dispersion relations and a broader range of phenomena.

These concepts have been significantly advanced with the advent of metamaterials and metasurfaces, where subwavelength structuring enables the realization of media with prescribed space–time-dependent effective properties. Such platforms provide unprecedented control over wave propagation, allowing for effects such as non-reciprocity, asymmetric mode conversion, and tunable dispersion [30, 31]. In particular, by imposing controlled variations of stiffness, density, or refractive index in both space and time, it becomes possible to break time-reversal symmetry and achieve nonreciprocal wave propagation without relying on nonlinearities or external bias fields [32, 33]. These ideas were originally developed in time-modulated circuits and electromagnetic systems [34, 35], and have since been extended to optics [36, 37].

From this perspective, space–time varying media can be viewed as a natural generalization of scattering-based wave control (Fig. 1.3a,b), where the redistribution of energy occurs jointly in space and time.

In this framework, phase-matching conditions are no longer purely spatial or temporal, but instead involve a coupling between frequency and wavevector through the velocity of the modulation. As demonstrated both numerically and experimentally [38, 39], this condition can be expressed as

$$\frac{\omega_2 - \omega_1}{k_2 - k_1} = -V_{int}, \quad (1.26)$$

highlighting the direct role of the interface velocity (V_{int}) in governing wave conversion processes. As a consequence, space–time modulation enables phenomena

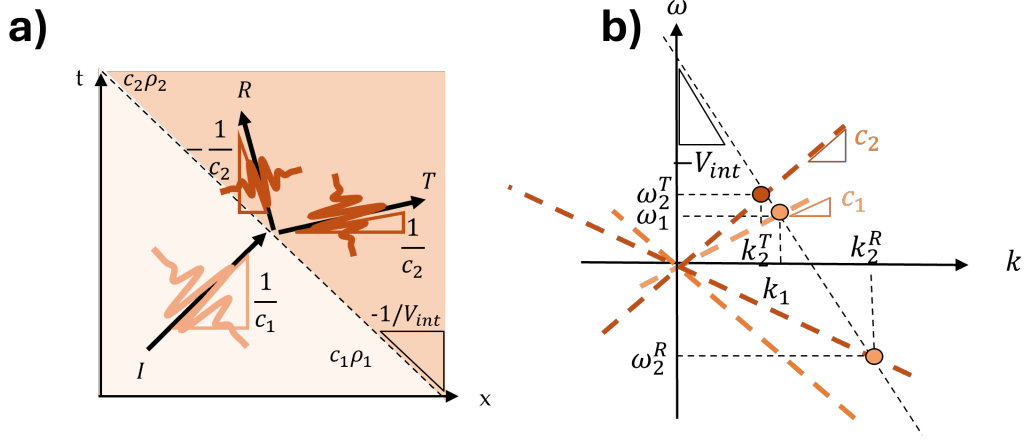


Fig. 1.3 (a) Space-time evolution of a wave packet scattered across a time-space interface moving at velocity $-V_{int}$. (b) Fourier transform of the scattered field: transmitted and reflected wave component have different frequencies and wave vectors.

such as frequency conversion, parametric amplification, and nonreciprocal wave propagation, extending the capabilities of conventional materials.

In acoustics, these principles have been exploited experimentally to break reciprocity and achieve directional wave control. Fleury [40, 41] demonstrated nonreciprocal sound propagation in a time-modulated acoustic circulator. In elasticity, both theoretical and experimental studies have shown that space–time modulation leads to nonreciprocal dispersion relations and asymmetric wave propagation. In particular, Trainiti [42, 43] introduced a framework for wave propagation in spatiotemporal periodic structures, which was later extended by Nassar [44] in the context of elastic metamaterials.

These works establish space–time varying media as a powerful platform for wave control, where scattering processes extend beyond purely spatial interactions to include temporal dynamics, enabling simultaneous manipulation of wave direction, frequency, and energy flow.

Within this broad framework, this thesis investigates three complementary approaches to wave manipulation. The first part focuses on structured metamaterials inspired by locally resonant systems, with an emphasis on practical applications such as the mitigation of ground-borne vibrations and acoustic absorption. In particular, the second chapter extends these concepts across scales, from subwavelength systems to large-scale implementations such as resonant trench barriers for Rayleigh wave

mitigation. A distinctive aspect of this work is the experimental validation of the proposed metamaterial concepts at field scale, a relatively rare achievement in the literature, where most studies remain confined to numerical simulations or laboratory-scale demonstrations. The third chapter addresses the design of a sound-absorbing metamaterial panel based on arrays of coiled resonators, achieving near-perfect absorption within a subwavelength thickness. The proposed design is further validated through reverberation-room measurements, providing one of the few demonstrations of metamaterial acoustic performance under realistic diffuse-field conditions.

The second part of this thesis focuses on the fundamental physics of wave scattering phenomena in elastic solids. Chapter 4 investigates mode conversion mechanisms in elastic structures, demonstrating how incident longitudinal waves can be partially transformed into flexural waves through geometric coupling. While most existing approaches rely on material heterogeneity, resonant inclusions, or multi-component structures to achieve mode conversion, this work shows that the phenomenon can arise solely from geometric modifications. In particular, it is demonstrated that an undulated region embedded within an otherwise homogeneous beam is sufficient to induce efficient mode conversion, providing a simple and effective strategy for controlling elastic wave propagation without introducing material discontinuities. Chapter 5 is motivated by recent experimental observations of deformation-induced space–time modulation in elastic media [39]. In contrast to most studies, where space–time interfaces are generated through externally prescribed variations of the material properties, this chapter investigates a fundamentally different mechanism in which the interface emerges naturally from the deformation of the medium itself. External forces modify the local strain state, thereby inducing a propagating space–time boundary that strongly affects wave propagation. While the acousto-elastic effect is traditionally studied under static or quasi-static deformation conditions, the present work extends the analysis to dynamically evolving deformation fields and their interaction with elastic waves. A theoretical framework describing the coupled evolution of deformation and wave motion is developed and validated through numerical simulations and experimental measurements. The results reveal how deformation can act as an active mechanism for wave manipulation, providing a new perspective on the generation of space–time interfaces in solids.

In this perspective, wave manipulation emerges as a direct consequence of controlled variations in the material properties. Structured and space–time varying

media thus provide complementary strategies to guide, scatter, and transform waves, defining a unified design space for advanced materials capable of controlling wave propagation across acoustic, elastic, and seismic domains.

Chapter 2

Design and Field Validation of a Locally Resonant Metamaterial Barrier for Low-Frequency Ground Vibration Mitigation

2.1 Introduction

In recent years, increasing attention has been devoted to the mitigation of seismic waves and ground-borne vibrations due to their detrimental impact on both infrastructure and human comfort. In particular, railway-induced vibrations represent a major issue in urban and suburban environments, as they propagate primarily in the form of Rayleigh waves along the soil surface. The mechanisms of generation and propagation of such waves, together with conventional mitigation strategies, are extensively discussed in the literature [45]. Among traditional countermeasures, trench barriers have been widely investigated. For instance, the effectiveness of stiff concrete wave barriers has been numerically and experimentally evaluated in [46], while the use of buried steel sheet piles as vibration barriers has been analyzed in [47].

More recently, the advent of metamaterials has opened new possibilities for vibration mitigation. Phononic crystals have been increasingly investigated as a solution for mitigating ground-borne vibrations by exploiting frequency band gaps that

inhibit elastic wave propagation in periodic media. Early theoretical and numerical works established the application of Floquet–Bloch theory to periodic structures for vibration attenuation [48, 49], while studies on periodic pile barriers demonstrated their effectiveness in reducing surface wave transmission in soils [50]. More recent research has focused on locally resonant designs capable of targeting low-frequency ranges relevant for railway and seismic applications, overcoming the limitations of Bragg scattering [51]. Numerical and experimental studies have confirmed that buried phononic crystal barriers and resonant inclusions can significantly attenuate Rayleigh waves and reduce vibration levels in practical scenarios [52, 53]. Applications to railway infrastructure, such as phononic crystal-enhanced tracks and engineered subsurface structures, further demonstrate the potential of these systems for real-world vibration mitigation [54, 55].

An alternative mechanism for band gap formation relies on local resonance. Locally resonant metamaterials can generate band gaps at frequencies corresponding to wavelengths much larger than the characteristic size of the unit cell [56]. The underlying mechanism for this design principle is to couple the soil to a resonant mass whose frequency matches the frequency range targeted by the application. This coupling produces a frequency selective behaviour of the medium which becomes transparent to incoming ground-borne vibrations outside the targeted frequency range (fig. 2.1a), and purely reflective for vibrations, within this range (fig. 2.1b). At resonance, the interaction between the incident wave and the local resonators opens a band gap. Within this frequency range, propagating solutions are replaced by evanescent fields that decay exponentially inside the medium. Consequently, the wave energy remains localized near the interface and is largely reflected back into the medium of origin. This feature makes them especially attractive for low-frequency vibration mitigation, where conventional Bragg-based solutions would require impractically large structures. In some configurations, Bragg scattering and local resonance effects may coexist or interact, leading to enhanced or hybrid band gap phenomena [57]. Additional coupling mechanisms such as mode veering, locking, and multiple Bragg wave coupling have also been reported [58–60].

The application of metamaterials to seismic and railway vibration mitigation has been explored in several studies. Numerical investigations of buried periodic inclusions and locally resonant barriers have demonstrated promising attenuation capabilities [61, 62]. Engineered metabarriers composed of resonant elements embedded in the soil have been proposed and experimentally validated at reduced scale

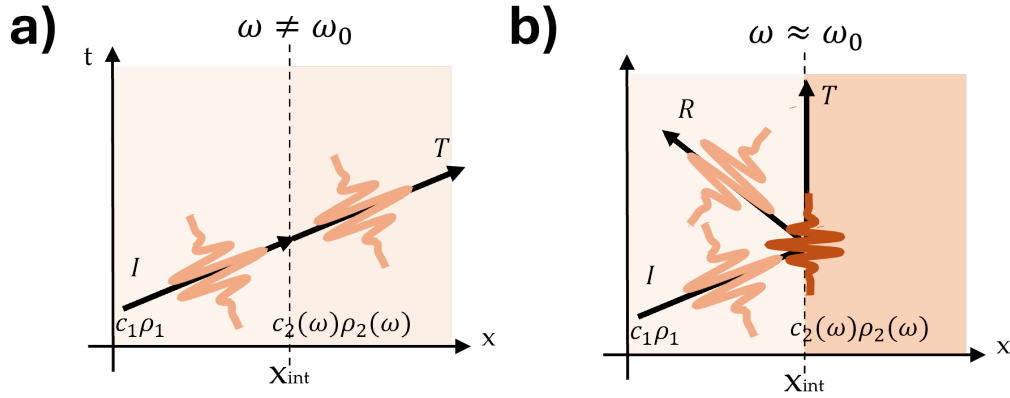


Fig. 2.1 Schematic of an incoming ground-borne wave (a) at a frequency different from the resonators (ω_0) and (b) close to the resonance frequency.

[63, 64]. Other configurations, such as clamped resonant pillars and metasurfaces interacting with Rayleigh waves, have shown the potential for ultra-low-frequency stop bands and surface wave manipulation [65–67]. Interestingly, even natural systems such as forests have been shown to act as seismic metamaterials through local resonance mechanisms [68].

Building upon previous developments in three-dimensional elastic metamaterials optimized for low-frequency vibration attenuation [69, 70], here we propose a modular metamaterial-based barrier specifically designed for the mitigation of railway-induced vibrations. The unit cell is engineered to maximize participating mass and minimize stiffness in order to achieve a resonance frequency around 30 Hz, which is particularly relevant for railway excitation spectra.

The objective of this chapter of my thesis is to describe the complete process of design, numerical validation, laboratory characterization, and full-scale field testing of the proposed metamaterial-based panel. After introducing the unit cell design, dispersion and transmission analyses are presented and compared with experimental laboratory results. The unit cells are then integrated into reinforced-concrete modules suitable for in-ground installation, whose dynamic behavior is numerically assessed. Finally, an in-situ experimental campaign performed at a railway site validates the effectiveness of the proposed solution in mitigating low-frequency ground-borne vibrations.

2.2 Unit Cell Design

The first objective of this chapter of my thesis is the design and the dynamic characterization of a Metamaterial Unit Cell (MM UC) working at subwavelength range, which could be used as fundamental unit for a barrier for railway-induced vibration mitigation in soil. The proposed UC structure is based on previous studies [70, 69] and chosen in order to both maximize the participating mass and minimize the stiffness as well as to tune the vibration damping properties in the low frequency range. Due to their design of thin beams and large masses, the modes featuring larger modal mass of the system (pyramids) are confined in the low passbands, while the ones featuring smaller modal mass (beams) are confined in the high passband, in what is known as mode separation [70, 69]. A 3D view of the proposed UC is shown in figure 2.2a. The main dimensions of the UC, made of 6 pyramids disposed symmetrically, are illustrated in figure 2.2b in a mid cross section view, where the main side of the cubic cell is assumed to be $a_{cell} = 0.36$ m, while the pyramid side is $l_{pyr} = 0.30$ m.

The materials chosen for the fabrication of the UCs are steel and concrete. The presence of fibres in the concrete mix increases its overall tensile, shear, and compressive strength, and is required to avoid using rebars as reinforcement in the pyramids, given the limited available volume. Although this is not strictly necessary for the operating phase, where no payloads are supported, it can be important for the installation phase, where impacts can occur.

Each pyramid is connected to the neighbouring ones through an L-shaped steel beams. Each beam presents hooks on both ends in order to strengthen the attachment the bond between beams and pyramidal elements. These are made of stainless steel AISI 630, have an 8 mm diameter circular section and are curved at 90° (with a total bending radius equal to 4.5 times the diameter of the beam).

2.3 UC Numerical Analysis

The dynamic behaviour of the chosen unit cell is first analysed numerically. Dispersion analysis is performed, investigating the potential behaviour for an infinite array of unit cells. Then, a transmission analysis of the single cell is carried out.

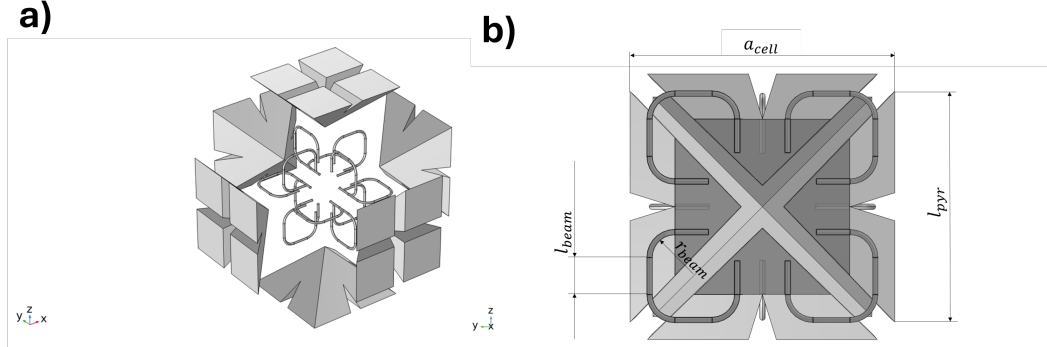


Fig. 2.2 Structures of the UC: (a) Exploded 3D view (b) Section view with details of the connections between steel beams and concrete pyramids.

2.3.1 Dispersion analysis of the single cell

Simulations are performed using the FE Method, in particular by means of the Structural Mechanics Module of COMSOL Multiphysics. The pyramids are modelled by means of solid tetrahedral elements, with the element size adapted to the geometrical details of the UC. The concrete material parameters are: density $\rho_c = 2500 \text{ kg/m}^3$, Young's Modulus $E_c = 30 \text{ GPa}$ and Poisson's ratio $\nu_c = 0.2$. On the other hand, the bars that connect the pyramids are modeled using beam elements following Timoshenko theory (simulations are also performed with solid elements instead of beam elements, but give no significant differences in results). The material parameters are: density $\rho_s = 7850 \text{ kg/m}^3$, Young's Modulus $E_s = 210 \text{ GPa}$ and Poisson's ratio $\nu_s = 0.3$. The phononic band structure of the structure in figure 2.3 is calculated implementing Bloch boundary conditions on the edges of the UC to account for the periodicity of the system [71]:

$$\vec{u}(\vec{r} + \vec{R}) = \vec{u}(\vec{r})e^{-i\vec{k}_F \cdot \vec{R}}, \quad (2.1)$$

where $\vec{u}$ represents the displacement vector, $\vec{R}$ is the lattice vector and $\vec{k}_F$ is the Bloch vector. Band structures are obtained by letting the wavevector $\vec{k}_F = (\vec{k}_{Fx}, \vec{k}_{Fy}, \vec{k}_{Fz})$ span selected points along the path of the Irreducible Brillouin Zone (IBZ) (Fig. 2). The main directions of symmetry in the reciprocal space $\Gamma X - XM - M\Gamma - \Gamma R - RX - XM - MR$ identify the 3D IBZ.

The bandgap opening mode at $f_{bot} = 32.7 \text{ Hz}$ involves a large percentage of the total mass of the system, i.e., the mobilization of the concrete pyramids. The

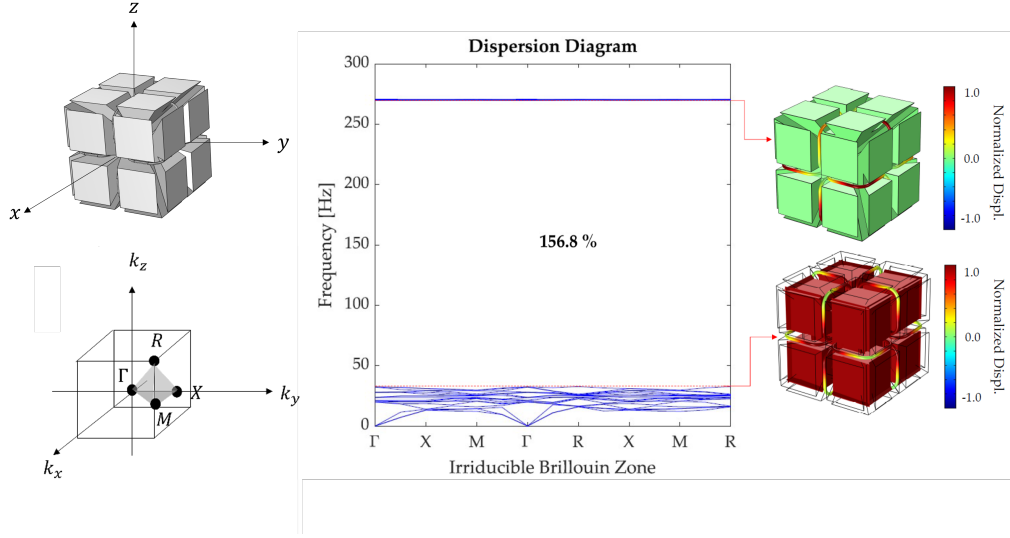


Fig. 2.3 (Left) 3D view of the UC and schematic of its IBZ. (Centre) dispersion diagram of the periodic structure. (Right) eigenmodes of the UC corresponding to the f_{top} and f_{bot} eigenfrequencies, respectively.

bandgap closing mode, at $f_{top} = 270.0$ Hz is characterized by the vibration of the steel beams instead of the concrete pyramids. The relative band gap width (in %) can be calculated as follows:

$$bandgap\% = 2 \cdot \frac{f_{top} - f_{bot}}{f_{top} + f_{bot}} = 156.8\%. \quad (2.2)$$

2.3.2 Transmission Analysis of the UC

The FE model of the system is used to numerically evaluate its mechanical transmission, calculated as:

$$T = 20 \cdot \log_{10} \left(\frac{A_{out}}{A_{in}} \right) \quad (2.3)$$

where A_{in} and A_{out} are the accelerations at the input and output areas, respectively. Input and output areas are chosen on opposite faces, as shown in fig. 2.4; the input area is excited with a harmonic force with amplitude equal to 1 N and the resulting acceleration is detected at the output area A_{out} , carrying out a sweep frequency

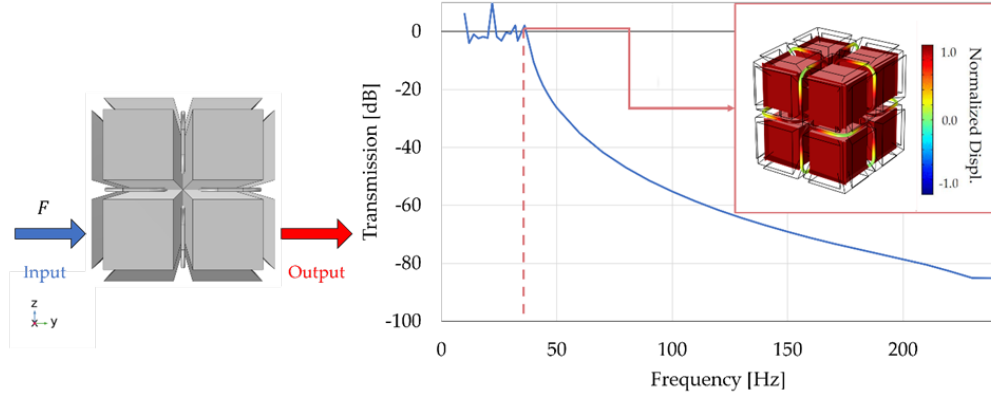


Fig. 2.4 (Left) Schematic of the input and output positions in the numerical model and (right) the numerically-calculated transmission spectrum together with the eigenmode at f_{bot} shown in the inset.

analysis. Results are consistent with band gap analysis, showing significant vibration attenuation between 35–250 Hz, which is the frequency range targeted in this study.

2.4 UC Experimental Laboratory Tests

The mechanical transmission of the system is experimentally evaluated by means of a setup comprising an arbitrary waveform generator Agilent 33220A, able to perform frequency sweeps; a NI-9234 sound and vibration recording module and a NI cDAQ-9171 data acquisition module; a permanent magnet shaker LDS V406, generally used for modal and structural health monitoring (e.g. bridge testing); two PCB Piezotronics 353B15 (sensitivity of 10 mV/g, resonant frequency of 70 kHz) accelerometers. The range of frequencies in which the actuator has a flat response is from 5 Hz to 9 kHz, with a nominal peak sine rating of 98 N, and with a movable mass of 0.20 kg. The actuator head is placed at the centre of the UC (fig. 2.5) in order to excite one of the lateral faces of the cube. The two accelerometers are positioned at two opposite faces, so that the input and output accelerations can be measured. The whole experimental setup is shown in Figure 4. The experiment is performed by using a sine sweep actuation and measuring the corresponding output with the accelerometer. The transmission is computed as in Equation 3.

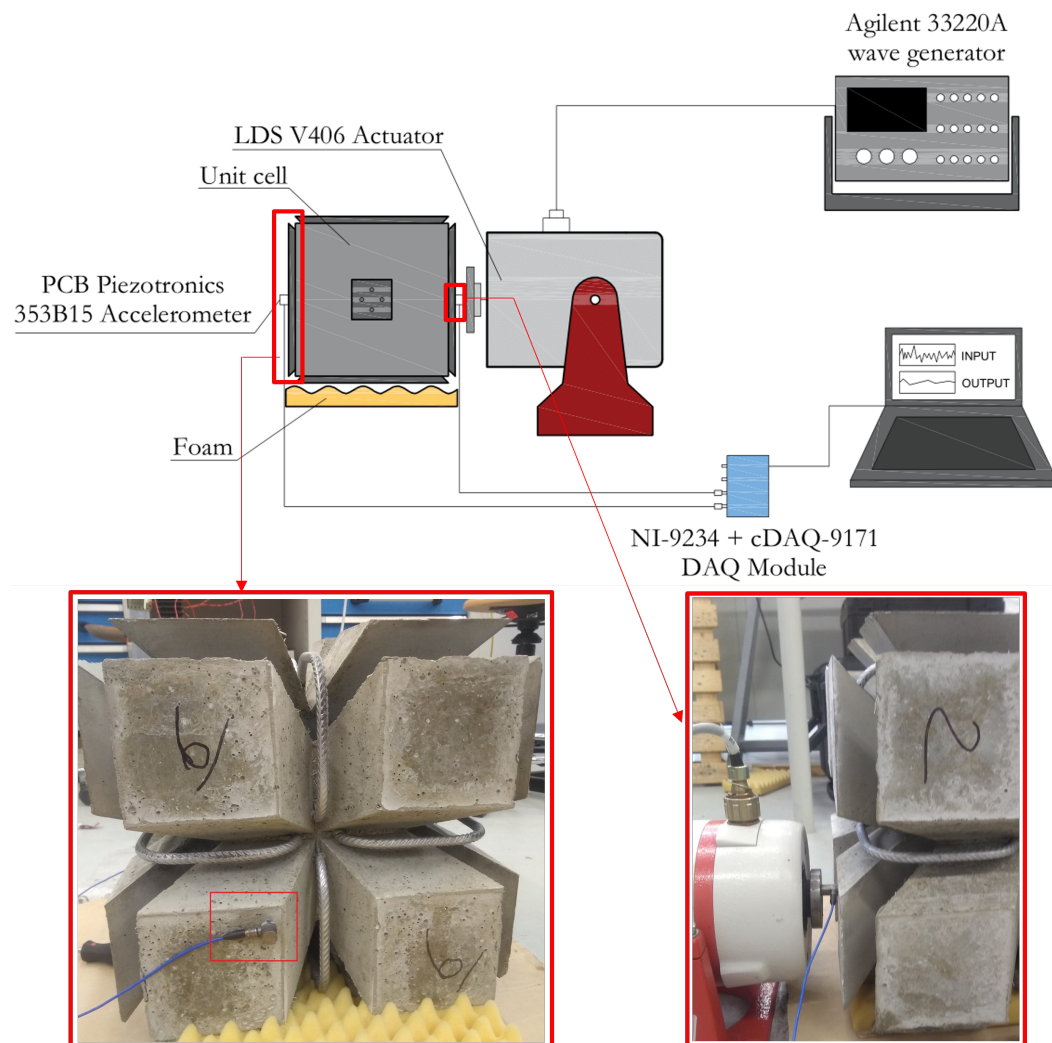


Fig. 2.5 (Above) Schematic of the experimental setup. (Below, from left to right) output side view (accelerometer position in red square), and input side view (shaker).

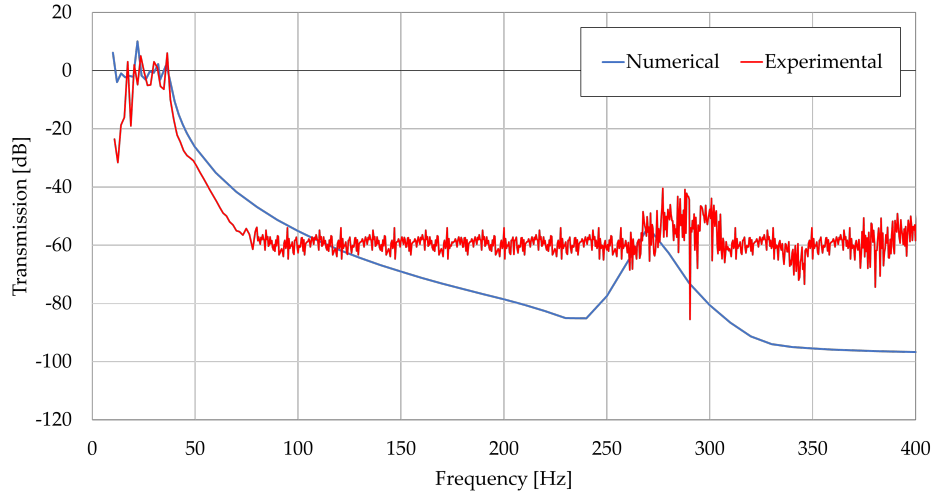


Fig. 2.6 Comparison of the transmission curves obtained numerically and experimentally.

Figure 2.9 shows an excellent agreement between experimental and numerical transmission curves. In particular, the clear drop in transmission occurs after 32.5 Hz, which corresponds to the UC mode involving pyramidal masses and the steel beams acting as flexural elements, as described in Section 3.1. Due to the sensitivity of the measurement instrument, the experimentally measured transmission appears constant above 100 Hz, as the acceleration amplitude is below the signal measurement noise.

2.5 Soil-Structure interaction Assessment

We first evaluate the quasistatic properties of the UC. The equivalent (fig.2.7a) stiffness k_{eff} can be estimated introducing a mono-atomic mass spring model of the UC, which is discussed in detail in the appendix ???. Thus, in our case $k_{eff} = 308$ N/mm. The equivalent mass is the mass of the double pyramid, i.e. $m_{eff} = 24.5$ kg. With this approximation, the corresponding BG opening frequency corresponds to $f = \frac{1}{\pi} \sqrt{k_{eff}/m_{eff}} = 35$ Hz, which is compatible with the numerical prediction (with a discrepancy of only 7.7 %).

Within this approximation, we can study the propagation of surface Rayleigh waves as depicted in fig.2.7b. The soil is modeled as an elastic half space, with density $\rho_s = 1300$ kg/m³, and longitudinal and shear wave velocity equal to $c_L = 227$ m/s and $c_S = 121$ m/s, respectively. As demonstrated in [63], the dynamic of the resonator

excited by the vertical soil motion w can be studied as:

$$m_{eff}(\ddot{Z} - \ddot{w}_0) + k_{eff}Z = 0, \quad (2.4)$$

where $w_0 = w(z=0)$. By using an effective medium approach, we can model the interaction of the resonant mass on the surface as an exchange of normal stress expressed as:

$$\sigma_{zz} = \frac{N}{A} = \frac{k_{eff}}{A}Z, \quad (2.5)$$

where $A = l_{pyr}^2$ is unit-cell area. The dispersion relation of Rayleigh waves is obtained by introducing the scalar (dilatational) and vector (shear) potentials in a homogeneous, isotropic half-space and assuming harmonic, depth-decaying fields, $\Phi = A e^{kz\sqrt{1-\omega^2/(k^2c_L^2)}} e^{i(\omega t-kx)}$ and $\Psi = B e^{kz\sqrt{1-\omega^2/(k^2c_S^2)}} e^{i(\omega t-kx)}$. The in-plane and out-of-plane displacements follow from the standard kinematic relations $u = \partial_x\Phi - \partial_z\Psi$ and $w = \partial_z\Phi + \partial_x\Psi$. The effect of the resonators is incorporated by replacing the classical stress-free boundary conditions at $z=0$ with effective tractions representing the resonator forces uniformly distributed over the unit-cell area A ; for the 1-dof case this yields $\sigma_{zz,res} = (K_v/A) \omega^2/(\omega_v^2 - \omega^2) w_0$, with $\omega_v = \sqrt{k_{eff}/m} = 2\pi f_{res}$. By expressing σ_{zz} in terms of Φ and Ψ through isotropic linear elastic constitutive relations and enforcing $\sigma_{zz}(0) = \sigma_{zz,res}$, one obtains a homogeneous linear system in the wave amplitudes; non-trivial solutions require the vanishing of its determinant, which delivers the closed-form coupled Rayleigh-resonator dispersion equation $\mathcal{D}(k, \omega) = 0$, (fig.2.7c) featuring the classical Rayleigh term plus additional frequency-dependent terms associated with the vertical and horizontal resonances and their coupling induced by the elliptical Rayleigh motion.

$$\begin{aligned} \mathcal{D}(k, \omega) = \left(\frac{\omega^2}{\omega_v^2} - 1 \right) & \left[\left(2 - \frac{\omega^2}{k^2 c_S^2} \right)^2 - 4 \sqrt{1 - \frac{\omega^2}{k^2 c_L^2}} \sqrt{1 - \frac{\omega^2}{k^2 c_S^2}} \right] - \\ & + \frac{m_{eff} \omega^4}{A c_S^4 \rho k^3} \sqrt{1 - \frac{\omega^2}{k^2 c_L^2}} = 0. \end{aligned} \quad (2.6)$$

In addition, it is of interest to calculate BG width arising from the soil-resonator. In fact, as shown by Palermo [63], the normalized width of the BG is proportional to soil properties as follows:

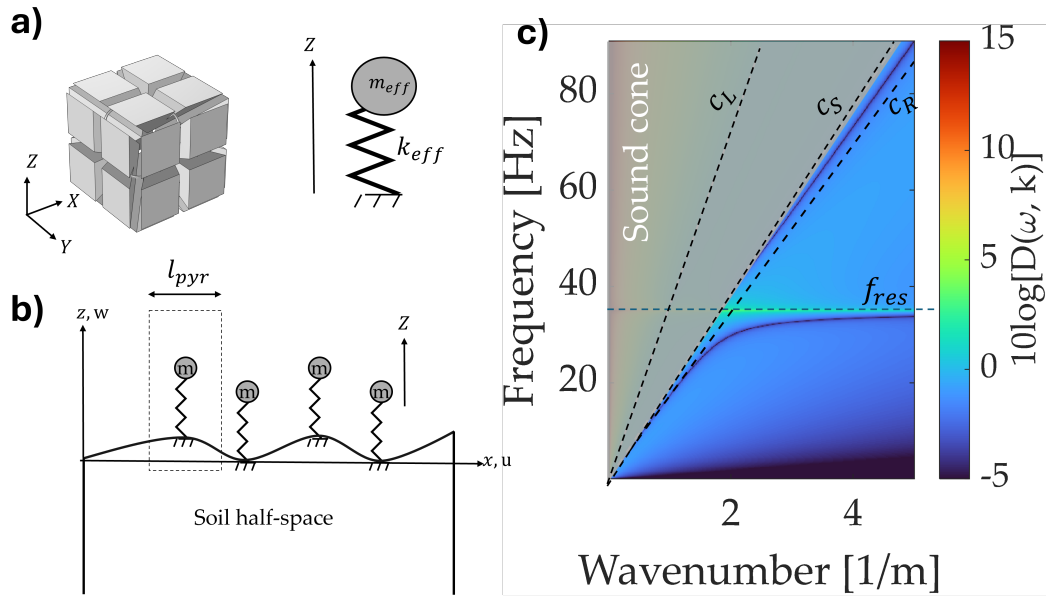


Fig. 2.7 (a) Lumped parameters model of the UC and (b) schematic of the soil half-space with embedded resonators on its surface. (c) Dispersion relation for the Rayleigh waves propagating on a half space embedded with resonators. The gray region corresponds to the sound cone and the dashed line to the bulk longitudinal (c_L) and shear velocity (c_S), and the surface Rayleigh wave velocity (c_R) for a homogeneous medium. The dashed horizontal line is corresponding to resonance frequency calculated from the lumped model approximation.

$$BG(\%) = \frac{2(\Gamma + \sqrt{(\Gamma^2 + 1)} - 1)}{\Gamma + \sqrt{(\Gamma^2 + 1)} + 1} \cdot 100, \Gamma = \frac{m_{eff}\omega_v}{2A\rho c_S} \sqrt{1 - \frac{c_S^2}{c_L^2}}. \quad (2.7)$$

To assess the influence of the supporting medium on the band-gap characteristics, a parametric study was carried out considering a range of representative soil types. The corresponding material properties are summarized in Table 2.1.:

Table 2.1 Mechanical properties of the soil types considered in the parametric analysis.

Soil Type	Density, ρ (kg/m³)	Longitudinal Velocity, c_L (m/s)	Shear Velocity, c_S (m/s)
Soft Clay	1700	1450	120
Stiff Clay	1900	1650	300
Loose Sand	1800	500	180
Dense Sand	2000	1800	450
Gravel	2200	2500	800

As shown in Fig. 2.8, the relative band-gap width decreases as the soil stiffness increases. The largest band gaps are obtained for soft clay, whereas progressively smaller values are observed for stiffer soils such as dense sand and gravel. This trend can be attributed to the increasing wave velocities (i.e. stiffness) of the supporting medium, which reduce the contrast between the dynamic properties of the soil and those of the resonator system.

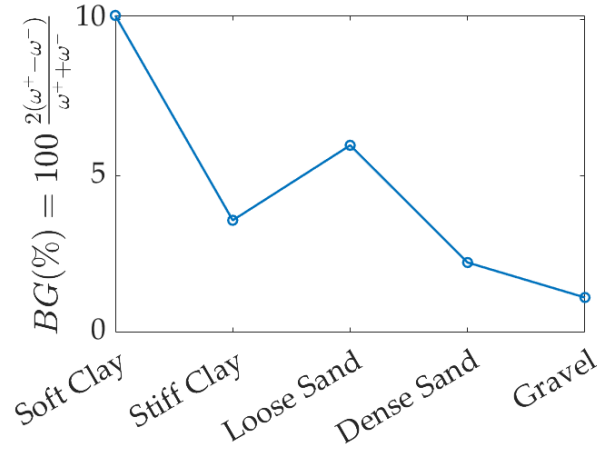


Fig. 2.8 Variation of the relative band-gap width (BG) for the soil types considered in the parametric analysis. Softer soils are associated with larger band-gap widths, whereas stiffer soils exhibit a reduced attenuation range.

However, it should be emphasized that Eq. (2.7) is derived assuming an infinite periodic array of identical resonators coupled to a homogeneous, isotropic, linear-elastic half-space. Consequently, the predicted band-gap width represents the intrinsic dispersion characteristics of the idealized soil–resonator system rather than the attenuation achievable in a practical installation. Finite arrays, soil heterogeneity, material damping, imperfect soil–resonator coupling, and construction tolerances may shift or reduce the effective band gap. Nevertheless, the analytical formulation provides valuable design guidelines by quantifying how the supporting soil influences the expected attenuation range and identifying soft soils as the most favorable conditions for maximizing the band-gap width.

2.6 Design of Anti-Vibration Modules

The UC presented in previous sections is now used for a field test for the mitigation of railway induced vibrations. There are some problematic aspects when directly burying the UC in the ground, which may decrease its efficiency: for instance, a non uniform contact with the soil and a potential alteration of the dynamic behaviour of the cell because the soil can fill the gaps between the components (i.e., between the pyramid cement blocks) and modify the modes and frequencies of the resonators in an uncontrolled way. Moreover, direct application of the UC in the ground may lead

to difficulties in installation and durability issues, since it would also assume a load bearing, as well as a vibration damping function.

To avoid these problems, an outer module which encloses the UCs is designed, ensuring a load-bearing function and uniform contact with the soil. The module is depicted in figure 2.9, and consists of a U-shaped reinforced-concrete block enclosing a $3 \times 2 \times 1$ array of UCs. The resulting antivibration module is named “PhV-MQ-A” in the following, while the sole U-block is referred to as “PhV-MQ-B”. In the manufacturing phase, the UCs of the PhV-MQ-A are attached to the walls of the block through a thin film of epoxy resin applied on the external faces of the pyramids and the internal ones of the U-block. This ensures coupling and vibration transmission at the interfaces between elements. The resulting design of the anti-vibration module therefore allows:

- to protect the cells from the soil to preserve both their dynamic behaviour and durability;
- a more uniform pressure distribution across their flat lateral surfaces
- an easier transport and field installation.

The presence of the U-shaped block modifies the behaviour of the connected UCs, and numerical analysis is necessary to determine the vibration characteristics of the whole module.

2.7 Numerical Analysis of the Anti-Vibration Modules

Numerical analysis is performed on PhV-MQ-A and PhV-MQ-B structures to investigate the effect of the casing on the structure. A FE modal analysis is performed by introducing a harmonic unit input force $F = Ae^{i\omega t}$, with $A = 1$ N applied on one side of the U-block along its normal direction. On the same surface and on the opposite one, the average of the normal acceleration amplitude is calculated, and the transmission spectrum is then computed by using equation 3 both for PhV-MQ-A and PhV-MQ-B, as shown in Fig. 2.10a. The two spectra display two minima in transmission in the low frequency range (5–45 Hz) for PhV-MQ-A that are absent

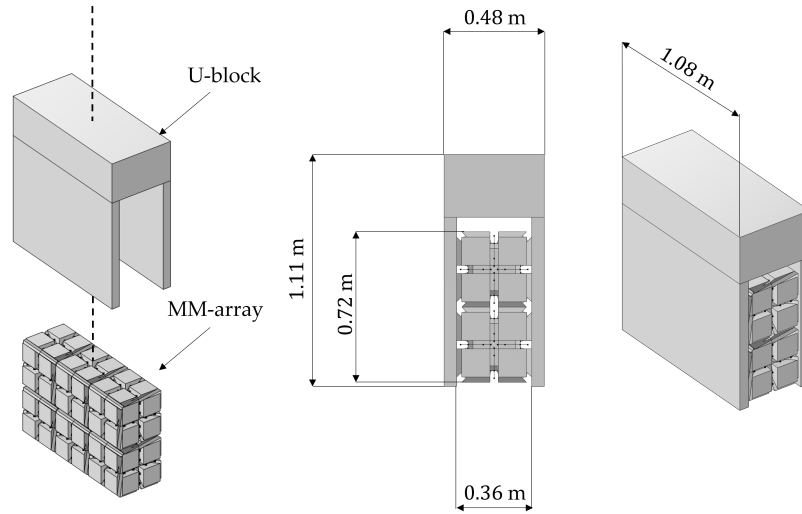


Fig. 2.9 (From left to right) exploded, front-, and isometric- views of the anti-vibration module consisting of six UCs and a surrounding U-shaped concrete block. Main element dimensions are indicated.

in PhV-MQ-B. These are associated with the “locally resonant” behavior of the structure. The lowest minimum is attained at $f = 32.5$ Hz, which corresponds to the eigenfrequency of the MM. The other occurs at $f = 27.5$ Hz, which corresponds to another resonant mode of the MM. The largest transmission peak for PhV-MQ-A is attained at 45 Hz, which corresponds instead to a resonance mode in which the largest acceleration is concentrated in the U-block of the structure. Figures 12.10b and c show that for both modal shapes, the two concrete walls of the U-block are rigidly coupled, and the system thus acts as a stiff barrier. Well beyond the resonance frequency, the walls are decoupled, and the system thus acts as a soft trench. At the resonance frequency however, the UCs vibrate and consequently absorb the vibration energy. This resonance is seen to take place at about 30 Hz. Additionally, it can be pointed out that the PhV-MQ-B (without UCs) does not display a resonance peak.

2.8 Field Test

2.8.1 Test Site

Finally, the modules were tested in real conditions to validate their efficiency in mitigating railway-induced vibrations. The testing site was located in Elze (see

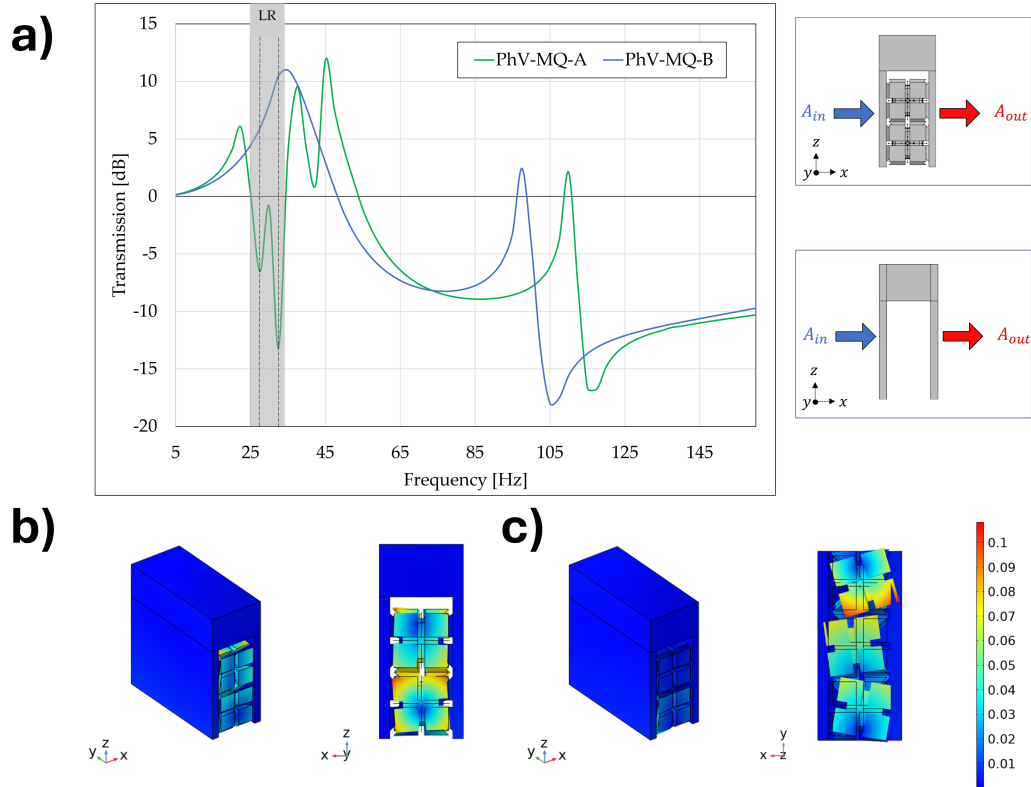


Fig. 2.10 (a) Left: numerical transmission spectra of the anti-vibration modules PhV-MQ-A (with embedded unit cells) and PhV-MQ-B (U-block only). The shaded region highlights the low-frequency range where PhV-MQ-A exhibits reduced transmission due to locally resonant effects. The dashed vertical lines indicate the resonant frequencies of the system. Right: schematic representation of the two configurations. (b)–(c) Acceleration magnitude distributions (m/s², colour scale) of the resonant modes at 32.5 Hz and 27.5 Hz, respectively, showing the vibration localization within the embedded metamaterial units.

Figure 2.11a), in Germany. The Elze train station of the Deutsche Bahn infrastructure is particularly suitable because of the presence of scarcely used track sections surrounded by sufficient space to allow for the installation and the testing in relevant environment (Figure 2.11b). The in-field tests were performed in two sessions, one for each of the two setups, which comprised 40 m-long trench barriers where modules PhV-MQ-A and then -B were installed. In particular, the installation of the solution PhV-MQ-B was aimed to separate the effect of the UCs from that of the concrete U-shaped super-structure.

2.8.2 Measurement Setup

Measurements were carried out by Baudynamik Heiland & Mistler GmbH [72] on the 11th–12th May (baseline prior to the installation of the modules) and on the 18th June 2020 (after installation).

The goal was to first determine the transfer function of the soil, i.e., soil mobility, with and without the anti-vibration solution. Mobility measurements were performed by means of an artificial (drop weight) excitation. The comparison between mobility measurements performed before and after the installation of the anti-vibration elements allowed to obtain the Insertion Loss (IL) provided by the anti-vibration modules. The mobility is computed from the measurement of the vibration velocity arising from a known input force. A drop weight excitation was used at several impact points on the track while the vibration response was recorded at several points beyond the anti-vibration modules. The location of the impact source and the output sensors varied, so that we will refer to a transfer mobility, which is dependent on the distance, the soil conditions and the type of excitation (point or line source).

A DYNPACT[®] excitation source was used, i.e., a device that drops a 50 kg weight from a height of 1 m (see Figure 9). This induces a pulse-shaped force in the ground that propagates to the load cell location, whose amplitude depends on subsoil stiffness. The proper coupling of the load cell with the subsoil is obtained by a load-distributing steel plate on a sand pad of 3 cm thickness. Before starting the measurements, several impacts were applied to avoid local compacting effects of the upper soil layers. The excitation was performed on the track bed between the rail profiles. A schematic of the impact and measuring point locations is shown in Fig. 2.13. Vibration measurements were performed with geophones, which record the

a)



b)

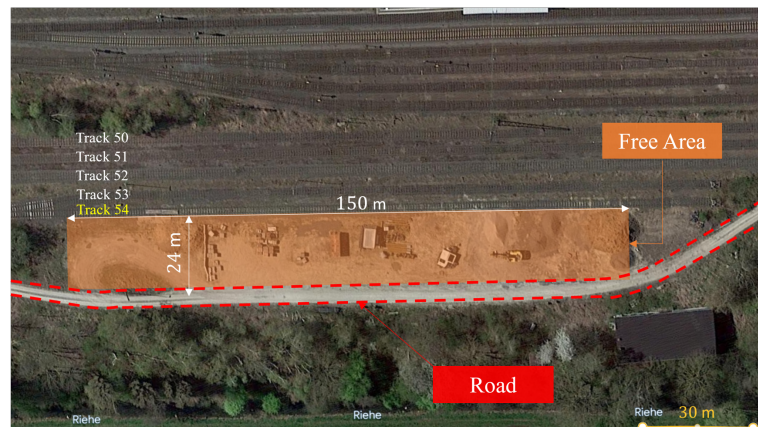


Fig. 2.11 (a) Satellite view of Elze (Germany) and its main railway station. The dashed lines indicate the area selected for the installation of the anti-vibration modules. (b) Detailed view of the test site, highlighting the railway tracks, the installation trench (150 m long and 2.4 m deep), the adjacent road, and the available free area.



Fig. 2.12 DYNPACT® excitation device used during the in-field measurements. The system consists of a 50 kg drop weight released from a height of 1 m to generate an impulsive load in the soil for mobility and insertion loss evaluation.

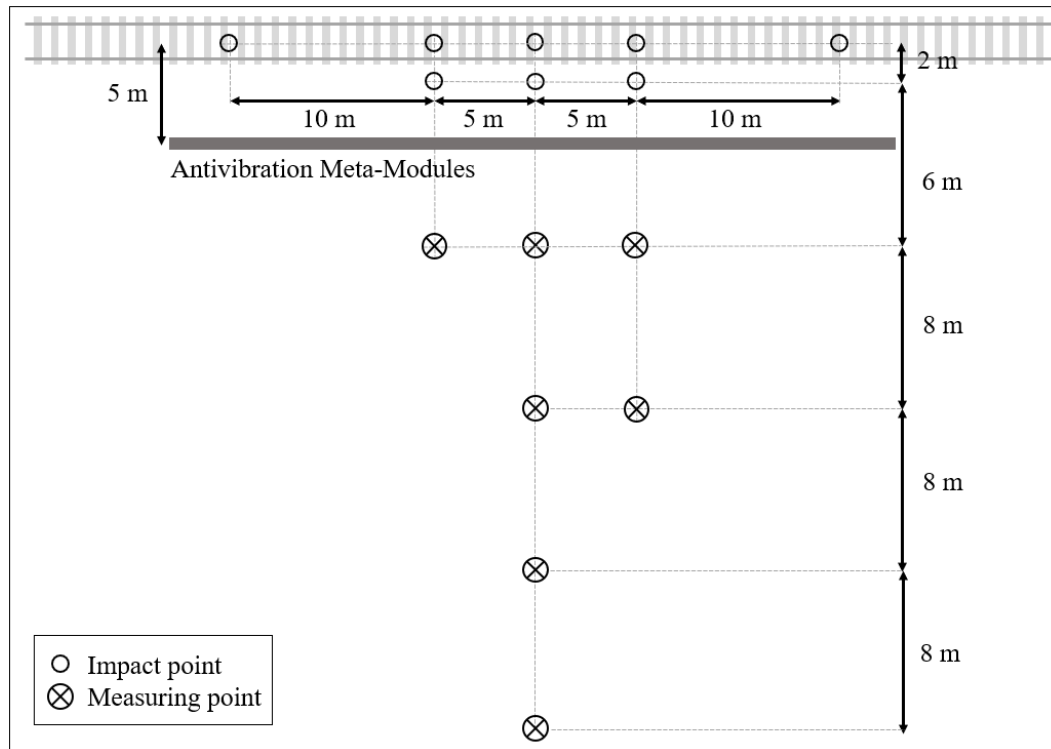


Fig. 2.13 Schematic of the experimental setup.

vibration velocity at the installation location. These were placed at 8, 16, 24, and 32 m from the center of the tracks, and kept in the same position for the measurements before and after antivibration meta-module installation, through the use of a sensor bed made of rapid hardening cement.

2.8.3 Soil characterization

In order to obtain information about the soil dynamic properties, additional sets of measurements were performed. First, a measuring chain with horizontal geophones in the radial direction, i.e. perpendicular to the tracks, were installed, while vibrations in the soil were induced in the vertical direction by a hammer impact. Secondly, a similar measuring chain was set up with horizontal geophones oriented in the transversal direction, i.e. parallel to the track, while the soil excitation was performed in the horizontal direction, perpendicular to the measurement chain line, with a hammer impact. The recorded data on surface vibration waves were analysed with the Multichannel Analysis of Surface Waves (MASW) method. This method is based

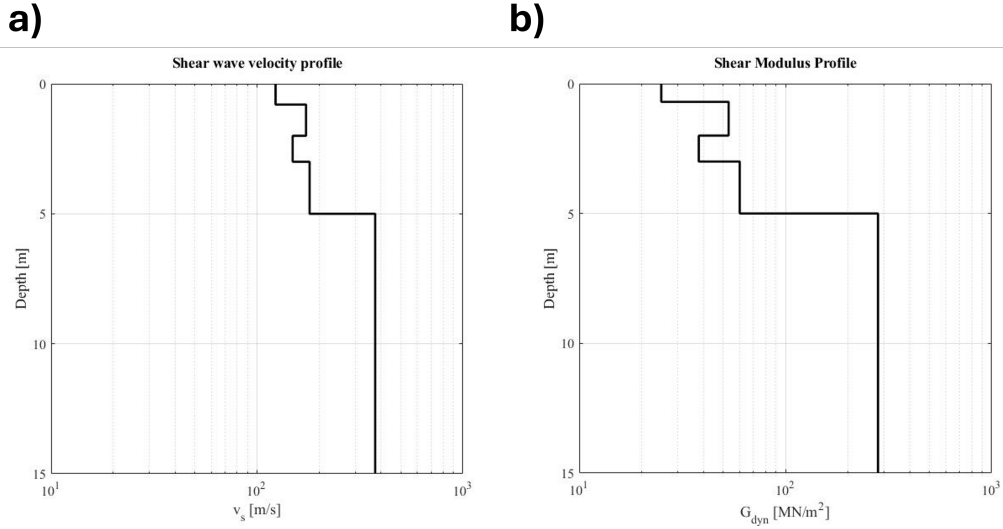


Fig. 2.14 Results of the MASW analysis: soil profile in terms of shear wave velocity and dynamic shear modulus.

on the fact that the waves produced by the hammer impact are modified according to the properties of the soil medium during their propagation. By measuring these dispersion properties at different distances from the vibration source, the different layers of the soil (soil profile) can iteratively be reconstructed [73, 74].

Each layer is described by two parameters: the Shear Wave Velocity c_s and the Dynamic Shear Modulus G_{dyn} : results are illustrated in Figure 2.14. The site characterization indicates that the shear-wave velocity profile (Fig. 2.14a) increases with depth by approximately 30%, ranging from about 100 to 130m/s. According to the classification reported in Table 2.1, the subsurface can therefore be approximated as a soft clay deposit. As shown previously, this soil type is associated with the largest relative band-gap width among the cases considered (fig. 2.8).

2.8.4 Results

In the following section, the main steps in the evaluation of the mobility transfer functions and of the IL are presented. Two excitation conditions are discussed: a point source and a line source excitation.

2.8.5 Point Source Excitation

Starting from the measurement setup previously described, the transfer functions are calculated for any combination of impact and measuring points as follows:

$$TF^{mnp}(f) = \frac{S_{A,m}^p(f) \cdot S_{E,n}(f)^*}{S_{E,n}(f) \cdot S_{E,n}(f)^*}, \quad (2.8)$$

where, $S_{A,m}^p(f)$ is the Fourier Transform of the vibration velocity measured at a distance p from the barrier and sampled by the m -th geophone. $S_{E,n}(f)$ is the Fourier Transform of the vibration velocity measured at the n -th impact point. The symbol $*$ denotes the complex-conjugate. To average the effect of distance and soil heterogeneities, the TF is used to compute a transfer mobility function as follows:

$$TM^p(f) = \frac{1}{M} \cdot \frac{1}{N} \sum_{m=1}^M \sum_{n=1}^N |TF^{mnp}(f)|^2, \quad (2.9)$$

where M is the number of geophones located at a distance p and N are the impact points, respectively.

For the computation of spectra, a force amplitude measurement trigger was set to $F_{trig} = 15 \text{ kN}$ with a pre-trigger time set to 0.1 s . Rectangle windowing was performed both on the excitation and response signals, with a window length of 4096 points and a sampling of 2560 Hz. The narrow-band transfer mobility functions are thus obtained with a frequency resolution of $\Delta f = 1/1.6 \text{ s} = 0.625 \text{ Hz}$, and then transformed into equivalent functions related to the 1/3 octave band frequencies. Finally, the IL at different distances from the source is calculated as:

$$IL(f) = 10 \log_{10} \left[\frac{TM_0^p(f)}{TM^p(f)} \right], \quad (2.10)$$

where $TM_0^l(f)$ is the transfer mobility function computed at a distance p from the input source before the installation.

Figs. 2.15a and 2.15b show a comparison of the IL spectra measured at 8, 16, 24, and 32 m from the point source excitation for the PhV-MQ-A and PhV-MQ-B structures, respectively. For both modules, the IL is close to zero up to the 20 Hz frequency band (grey region), due to the small depth of the barrier [47]. Within the frequency bands from 20 to 50 Hz (red region) the modules present different

behaviours. For the PhV-MQ-A, a maximum around 10 dB is attained at 31.5 Hz and preserved along the whole distance. This implies that the energy propagating within this frequency range is not transmitted across the barrier, but reflected and absorbed through the mechanism of local resonance. The PhV-MQ-B instead, presents two peaks around 8 dB at 25 and 40 Hz bands that rapidly decay below 2 dB with distance. More quantitatively, the PhV-MQ-A solution displays an increment in IL in the 31.5 Hz frequency band due to UC resonance of 6 dB with respect to the PhV-MQ-B solution at 8m from the source. The increment reaches up to 8.5 dB at 32 m from the excitation source. The overall vibration mitigation provided by the PhV-MQ-A solution with respect to the baseline (before installation) is observed in the frequency range 20-50 Hz (red transparency region) with peak values of 10 dB.

Thus, due to its small depth, the U-shaped module (PhV-MQ-B) is only capable of influencing the vibration propagation near the stiff barrier, but its effect strongly decays with distance. This is compatible with the so-called "wave healing" phenomenon which tends to reduce the scattering effect with the distance on both amplitude and phase from weak scatterers. Moreover, previous studies have demonstrated that in stratified soils, Rayleigh waves redirected to the soil tend to re-surface because of the increasing depth dependent velocity (mirage effect) [75, 76]. Instead, the U-shaped barrier with embedded metamaterial resonators (PhV-MQ-A) is effective in attenuating vibrations at all considered distances in the designed frequency range, demonstrating a considerable advantage. Experimental results are also consistent with full-field wave propagation simulations in the soil, reported in the Supplementary Material. Although at least from a qualitative point of view, experimental results confirm numerical predictions, the latter provide a forcibly simplified picture of the wave propagation problem, since it is difficult to choose realistic soil-structure interaction conditions as well as a correct model for soil viscoelasticity. However, we emphasize that the objective of this chapter of my thesis is not to compare experimental results to numerical simulations, but to provide a direct in-field experimental verification of the designed barriers.

Line Source Excitation

The mobility functions computed in the previous section for each point source can be conveniently transformed by integration into mobility functions for a line source

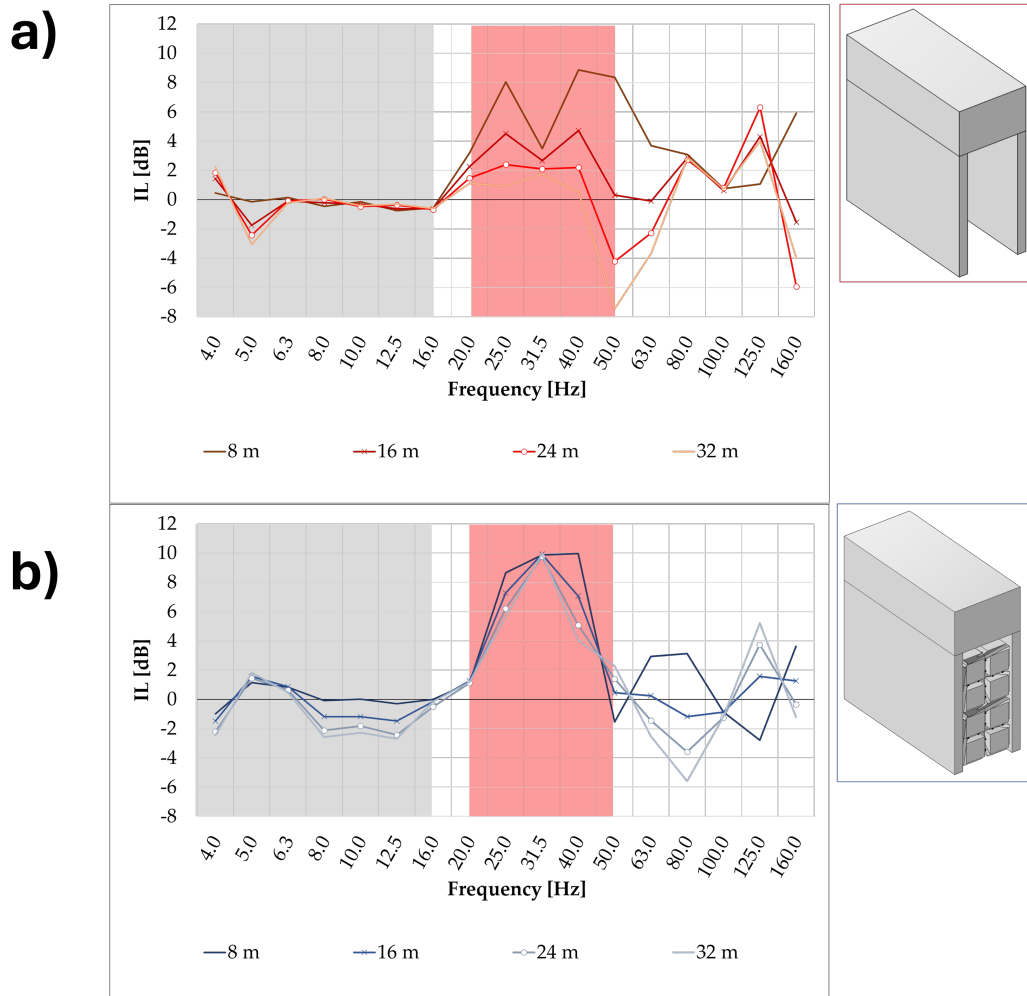


Fig. 2.15 Point Source. Comparison of the IL at different distances from the source for (a) the solution PhV-MQ-A and (b) the solution PhV-MQ-B. The red shaded region indicates the frequency range in which the two UC types differ in spectral behaviour

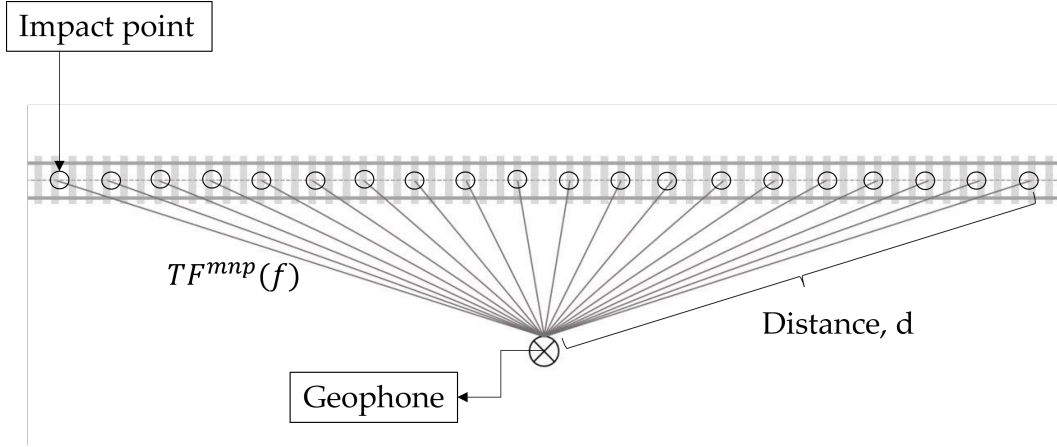


Fig. 2.16 Schematic representation of the construction of a line source from point sources, to obtain the line source mobility functions at a desired location by integrating point transfer mobility functions.

excitation, which is more representative of the real working conditions, namely the train excitation.

First of all, for each frequency band of the spectrum, a regression analysis is performed on the results in terms of transfer mobility functions for point sources. This regression analysis is based on the assumption of an exponential decay of the transfer mobility with distance d :

$$TM^l(f, d) = TM^p(f) \cdot \left(\frac{d_0}{d} \right)^{n_p(f)}, \quad (2.11)$$

where TM^l is the line transfer mobility, $d_0 = 8$ m and $n_p(f)$ quantifies the vibration decay with distance, which is computed as a regression value which fits the mobility decay with distance at a specific frequency band, as described in [?]. The line source transfer mobility function can then be computed at desired locations, integrating the transfer mobility functions generated for several point sources obtained at the previous step with the corresponding influence length (fig. 2.16). It is important to underline that a line source has a shorter distance decay coefficient than a point source. Results indicate that the following relation holds:

$$n_L = n_P + 0.5. \quad (2.12)$$

where n_L quantifies the line mobility decay. As has been described for the case

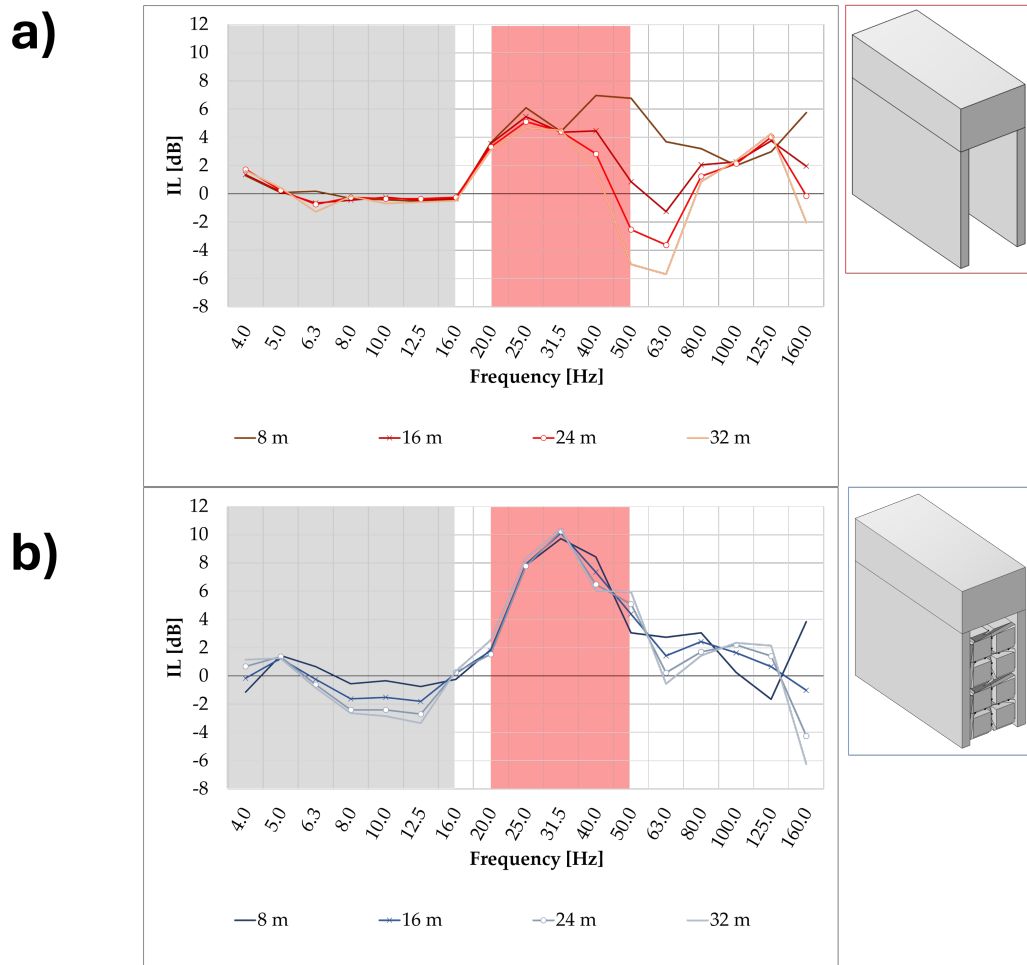


Fig. 2.17 Line Source. Comparison of the IL at different distances from the source for the solution PhV-MQ-A (a) and for the solution PhV-MQ-B (b). The red shaded region indicates the frequency range in which the two UC types differ in spectral behaviour

of point source, the IL can be obtained from the mobility transfer functions at the desired locations as the difference between the transfer mobility before and after the installation of the proposed solutions. The comparison of ILs at 8 m, 16 m, 24 m and 32 m from the line excitation source between the PhV-MQ-A and PhV-MQ-B is shown in Fig. 2.16a and 2.16, respectively. A similar interpretation of the results for a line source can be made as in the case of a point excitation. Here too, the resonance effect of the UCs leads up to a 6 dB of IL gain in the frequency bands 20 – 40 Hz, compared to the PhV-MQ-B. The overall vibration insulation effect of PhV-MQ-A is stable with distance and reaches up to 10 dB of mitigation.

2.9 Conclusions

In this Chapter, the process of design and validation in relevant environment of a MM-based modular panel for vibration mitigation in soils was performed. The MM UC was optimized to improve vibration insulation in the desired frequency range (30 Hz), exploiting its low frequency resonance. Its efficiency was firstly evaluated in numerical and experimental laboratory tests. A concrete U-block encasement was developed to protect the MM structure from the surrounding soil, in order to allow its vibration at resonance frequency. The design of the structure was then validated by means of FEM analysis. Then, a field experimental campaign for its validation was performed: the MM-based modular panel was installed and tested at a railway station site under an artificial excitation to investigate its effect of mitigation of railway-induced vibrations. An IL of up to 10 dB was obtained in the low frequency range (20-50 Hz), in accordance with numerical simulations, and presented in addition a stable insulation measured up to 32 m from the track. Further improvements in the working conditions can probably be obtained with better consolidation and settlement of the soil in the trench around the barrier to improve contact conditions and homogenise the soil properties in the vicinity of the barrier. Moreover, the application of a geotextile within the trench could prevent the soil from spreading inside the cavities of the metabarrier modifying the resonance modes and possibly leading to performance degradation. This work demonstrates the feasibility of using metamaterial solutions for the reduction of low frequency groundborne vibrations such as those emitted by rail traffic. This can constitute an attractive solution, given the non-invasive installation below surface, removed from the rails, the limited size (1 m depth) and realization using conventional building materials.

Chapter 3

Coiled-up acoustic resonators for subwavelength acoustic absorption

3.1 Introduction

Sound reflection is a fundamental acoustic phenomenon arising when an incident wave interacts with ideal purely reflective (hard) boundary or a medium with a relatively larger impedance, generating a reflected field governed by boundary conditions (fig. 3.1a) and typically described via a complex reflection coefficient [77, 78]. Within the broader framework of wave scattering, reflection redistributes acoustic energy in space, ranging from specular to diffuse regimes depending on surface geometry and material properties [79, 80]. These processes are central in room acoustics, where early reflections influence clarity and spatial perception, and late reflections contribute to reverberation [81]. Accurate modeling of reflection is therefore essential in applications such as concert hall design, recording studios, and auralization, employing both geometrical and wave-based methods [82, 83].

To address these problems, acoustic metamaterials (AMs) provide a natural framework to exploit wave scattering through tailored micro- and meso-scale architectures. These structures allow the realization of effective material properties not found in natural media, offering enhanced control over sound propagation and reflection through the generation of friction, dissipating the wave's energy and thus reducing the intensity of the scattered, hence reflected field [84]. This concept has been implemented in a variety of resonant architectures, including porous–metamaterial

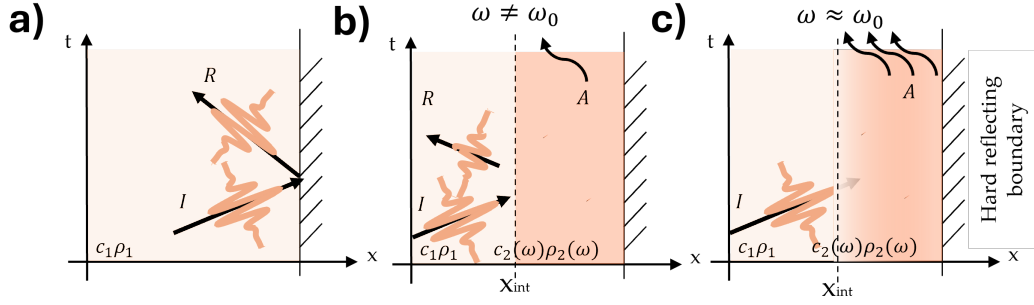


Fig. 3.1 Schematic of (a) the sound reflection problem at a hard boundary. (b) Engineered resonant layer working at frequency different and (c) close to the resonance. The layer is designed to maximize absorption (A) at resonance. Undulated arrows indicate the absorbed energy from the layer

hybrids, Helmholtz-type resonators, and membrane-based absorbers [85–90]. Perfect or near-perfect absorption in AMs can be achieved under critical coupling conditions, whereby intrinsic dissipative mechanisms balance radiative energy leakage at the resonance frequency of the AM by design[91] (figs. 3.1b and 3.1c). This capability is especially relevant for frequency-specific and especially low-frequency noise mitigation, where the conventional aim is to lower sound reflection within a certain frequency band through acoustic absorption which is constrained by the mass–thickness trade-off [92, 93]. Such limitations are critical in applications subject to stringent geometric and weight constraints, including aerospace and automotive engineering [94]. However, these systems are typically characterized by narrow operational bandwidths, limiting their effectiveness in broadband noise control.

A viable route to overcome this limitation consists of combining resonant elements with distinct characteristic frequencies, giving rise to the so-called acoustic “rainbow trapping” effect [87, 95–103]. In this context, labyrinthine or space-coiling metamaterials have attracted considerable interest, as they exploit wave propagation along tortuous subwavelength channels to achieve large effective refractive indices and reduced phase velocities [104]. Through appropriate geometric design, such structures enable tailored dispersion properties, broadband impedance matching, and enhanced sound absorption [105–107].

Despite extensive theoretical and numerical studies, experimental investigations of large-scale labyrinthine metamaterial panels operating under realistic, diffuse-field conditions remain scarce. The present work addresses this gap by proposing

and experimentally validating a modular, lightweight sound-absorbing panel based on a rainbow arrangement of three-dimensional labyrinthine unit cells, exploiting controlled spatial symmetry breaking to achieve efficient broadband absorption in the mid-to-low frequency range.

3.2 Unit cell design and analytical model

The sound-absorbing performance of the proposed metamaterial panel is based on the response of individual labyrinthine unit cells (UCs), which act as subwavelength resonant acoustic elements. The building block of this geometry is a finite size waveguide closed on one end. This geometry is characterized by a set of resonances proportional to a wavelength proportional to the waveguide's length according to the formula [108] $f_{res} = \frac{nc}{4L}$, where n is an odd integer number, and c is fluid's sound velocity. Due to inverse proportionality between the resonator's length and its resonance frequency, the use of such systems for low frequency applications requires large resonators' sizes. Therefore, to limit the encumbrance of this systems, the channel of this waveguide is coiled-up [109, 103, 110]. Each UC consists of a rigid cavity connected to the external acoustic domain through a narrow aperture, forming a coiled propagation path that effectively elongates the acoustic path within a compact volume. This space-coiling mechanism enables the realization of low-frequency resonances in structures with thicknesses much smaller than the corresponding free-space wavelength.

3.2.1 Geometrical configuration

The UC geometry considered in this chapter of my thesis (fig. 3.2a) is based on a three-dimensional labyrinthine structure inspired by space-filling curves. The internal channel (fig. 3.2b) follows a folded path of rectangular cross section $w \times t$, enclosed within a square cavity of lateral size a and thickness t . The UC is rigidly backed, so that acoustic energy enters the structure exclusively through the front aperture.

The key geometrical parameters governing the acoustic response of the UC are the lateral size a , the thickness t , and the aperture cross-sectional area $S_a = w^2$. By varying these parameters, the effective propagation length of the acoustic wave inside

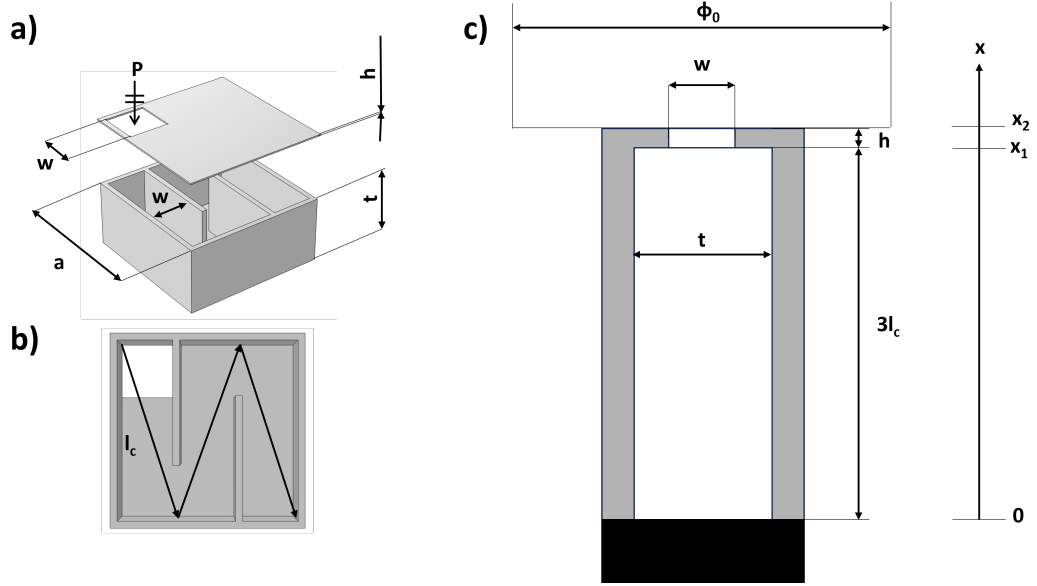


Fig. 3.2 (a) Geometry of the three-dimensional labyrinthine unit cell (UC) based on a space-coiling architecture, characterized by lateral size a , thickness t , and internal channel width w . (b) Schematic representation of the effective acoustic propagation path inside the UC, illustrating the elongation of the wavelength due to the folded channel geometry. (c) Equivalent one-dimensional acoustic model of the UC, represented as a lossy narrow duct of effective length l_{eff} and cross-sectional area S_c , rigidly backed and coupled to the external acoustic domain through a dissipative aperture of area S_a .

the cavity can be tuned, allowing direct control over the resonance frequency while preserving subwavelength overall dimensions.

3.2.2 Equivalent acoustic model

Under the assumption that the UC walls behave as acoustically rigid boundaries, the labyrinthine cavity can be modeled as an equivalent narrow duct terminated by a rigid backing (fig. 3.2c). Due to the small transverse dimensions of the internal channels, thermo-viscous losses play a significant role and must be explicitly accounted for. Following classical treatments of sound propagation in narrow ducts [111, 107, 106], the effective wavenumber and impedance inside the cavity are expressed in terms of complex, frequency-dependent density and compressibility. Their complete derivation is provided in the appendix ??.

The UC can therefore be represented by an equivalent one-dimensional model consisting of a lossy duct of effective length l_{eff} , closed at one end and coupled to the external acoustic domain through a dissipative aperture. The surface impedance Z_s of the UC is obtained by combining the impedance of the cavity and the aperture, accounting for impedance transformations due to cross-sectional changes.

3.2.3 Absorption coefficient

The normal-incidence sound absorption coefficient α of the UC is computed from the surface impedance as

$$\alpha(\omega) = 1 - \left| \frac{Z_s(\omega) - Z_0}{Z_s(\omega) + Z_0} \right|^2, \quad (3.1)$$

where $Z_0 = \rho_0 c_0$ is the characteristic impedance of air, with ρ_0 the air density and c_0 the speed of sound.

Perfect absorption occurs when the surface impedance matches the characteristic impedance of air, corresponding to a critical coupling condition in which intrinsic thermo-viscous losses exactly balance radiative leakage [91]. This condition can be achieved by appropriately tuning the UC geometrical parameters, in particular the aperture size and the effective cavity length.

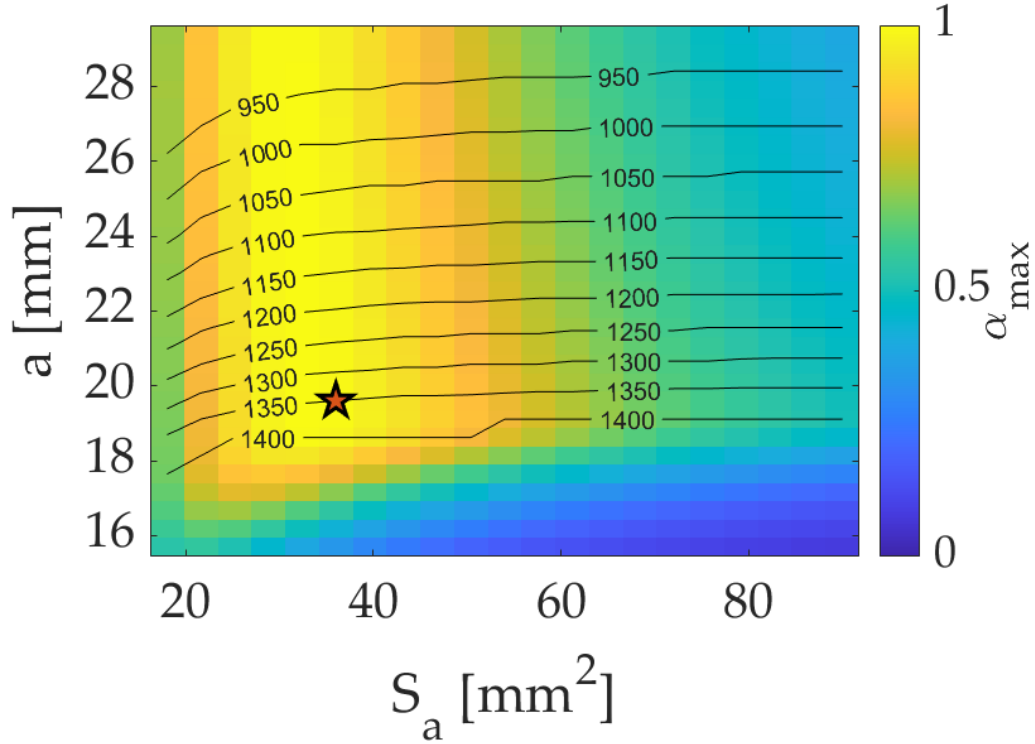


Fig. 3.3 Parametric map of the analytically predicted maximum absorption coefficient of the labyrinthine unit cell as a function of the lateral edge size a and the aperture cross-sectional area S_a , for a fixed unit cell thickness $t = 9.6$ mm. The lateral edge size is varied in the range $16 \text{ mm} \leq a \leq 29 \text{ mm}$, while the aperture area spans approximately $20 \text{ mm}^2 \leq S_a \leq 90 \text{ mm}^2$. Contour lines indicate the resonance frequency corresponding to the absorption peak. The selected unit cell geometry, providing near-perfect absorption within the target frequency range, is highlighted.

3.2.4 Parametric analysis and tuning

A parametric study based on the analytical formulation is carried out to quantify the influence of the UC geometry on its absorption performance. In particular, the lateral edge size a is varied in the range $16 \text{ mm} \leq a \leq 29 \text{ mm}$ by isotropically scaling the UC in the plane, while the aperture cross-sectional area S_a is varied from approximately 20 mm^2 to 90 mm^2 . The UC thickness is kept fixed at $t = 9.6$ mm in order to ensure subwavelength operation within the investigated frequency range. The resulting absorption map, shown in Fig.3.3, highlights the combined influence of these parameters on the maximum absorption coefficient and resonance frequency. Based on this parametric analysis, a representative UC configuration is selected that

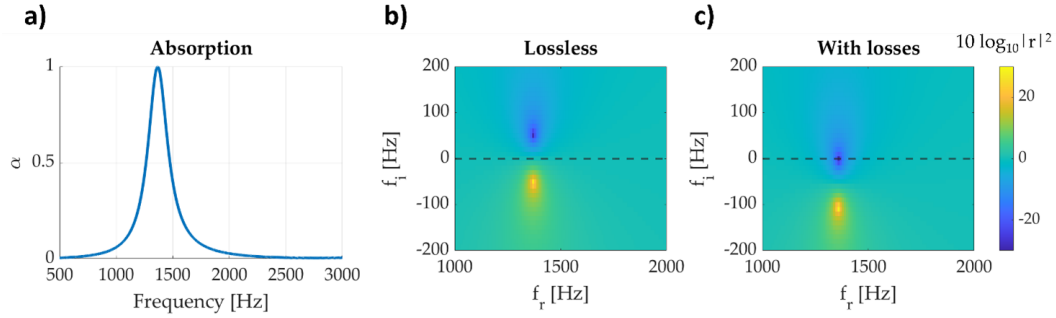


Fig. 3.4 (a) Analytically predicted normal-incidence absorption spectrum of the selected labyrinthine unit cell, showing a pronounced absorption peak associated with the fundamental resonance. (b) Representation of the squared reflectance in the complex frequency plane for the lossless case, where a complex-conjugate zero–pole pair is observed away from the real frequency axis. (c) Corresponding reflectance map including thermo-viscous losses, where the zero is shifted towards the real axis. Perfect absorption is achieved when the critical coupling condition is satisfied, corresponding to the alignment of the reflectance zero with the real frequency axis.

exhibits near-perfect absorption within the targeted frequency band. The analytically predicted absorption spectrum of this configuration is reported in Fig. 3.4(a), where a pronounced absorption peak associated with the fundamental resonance of the structure is observed. To further interpret the absorption mechanism, the reflectance is analysed in the complex frequency plane following the approach introduced in [91]. As illustrated in Fig. 3.4(b,c), the lossless system exhibits a complex-conjugate zero–pole pair, while the inclusion of thermo-viscous losses shifts the zero towards the real frequency axis. Perfect absorption is achieved when the critical coupling condition is met, corresponding to the alignment of the reflectance zero with the real frequency axis.

3.2.5 Experimental characterization of the unit cell and numerical comparison

The experimental characterization of the labyrinthine unit cell (UC) was performed to validate the analytical and numerical predictions of its sound absorption performance under normal incidence conditions. Measurements were carried out using an impedance tube setup, which allows accurate and repeatable evaluation of the absorption coefficient over a broad frequency range.

The tested UCs consist of a square labyrinthine cavity embedded within a rigid cylindrical body, where internal solid walls define a first-iteration Wunderlich curve geometry, as shown in Fig.3.2. The samples were fabricated via additive manufacturing to ensure high geometric fidelity. Polylactic acid (PLA) samples were produced using a Dremel Digilab FDM printer, while additional samples were manufactured in VeroYellow using a Stratasys J750 system. The mechanical stiffness of both polymers is sufficiently high to justify the rigid-wall assumption adopted in the analytical and numerical models, and no appreciable differences in acoustic response were observed between the two materials.

Normal-incidence absorption measurements were conducted using a Kundt's tube with an inner diameter of 35 mm, corresponding to a cut-off frequency of approximately 5.6 kHz. The experimental setup, shown in Fig. 3.5, follows the two-microphone transfer-function method in accordance with ISO 10534-2 and ASTM E1050 standards. Two flush-mounted 1/4" microphones were positioned at a fixed distance of 29 mm along the tube axis, enabling the determination of both the amplitude and phase of the acoustic pressure field. A broadband white-noise signal generated by a 2" loudspeaker was used as the excitation source. The absorption coefficient $\alpha(f)$ was measured over the frequency range 100–5000 Hz with a spectral resolution of 2 Hz. A custom-designed sample holder with a sliding rigid piston (Fig. 3.5b) ensured precise positioning of the samples and minimized circumferential air gaps, whose influence on impedance tube measurements is known to be non-negligible. To assess measurement repeatability, each UC configuration was tested using three nominally identical samples. Ambient temperature and atmospheric pressure were continuously monitored and accounted for in post-processing.

In parallel with the experiments, a finite element (FE) model reproducing the impedance tube configuration was developed in COMSOL Multiphysics (Fig. 3.6a). The model includes the air domain of the tube, the UC cavity, and the inlet aperture. Thermo-viscous losses within the labyrinthine cavity were modeled using the *narrow region acoustics* formulation, whereas losses at the aperture were described using a poro-acoustic model based on the Johnson–Champoux–Allard framework. A plane-wave excitation with unit pressure amplitude was imposed at the inlet, and the average pressure was sampled at two locations corresponding to the microphone positions in the experimental setup, allowing the direct numerical computation of the absorption coefficient via the transfer-function method. Figures 3.6b and c illustrate the numerically computed acoustic pressure field and dissipated energy distribution

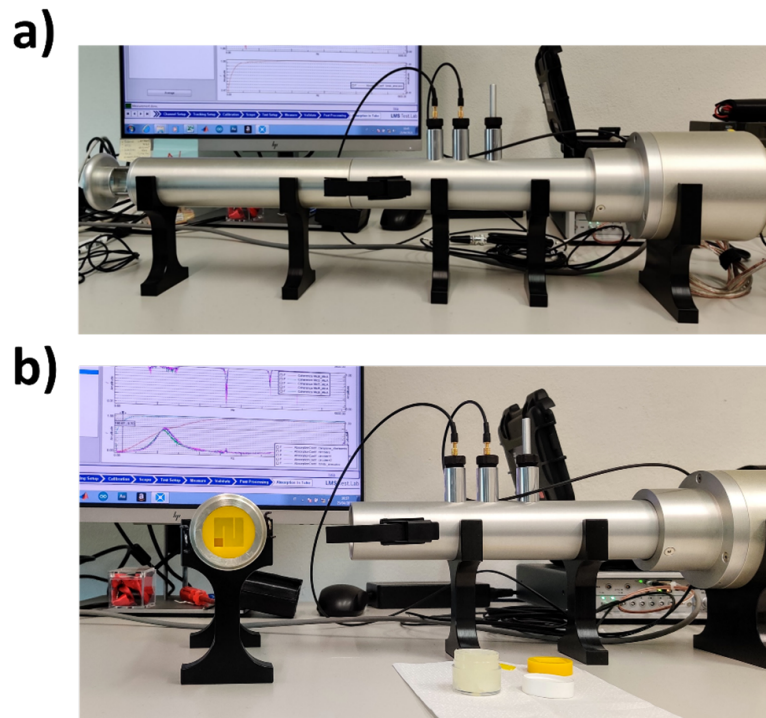


Fig. 3.5 Experimental setup for the impedance tube measurements. (a) Complete Kundt's tube apparatus used for normal-incidence absorption measurements, equipped with a broadband sound source and two flush-mounted microphones. (b) Detail of the opened tube showing the labyrinthine unit cell mounted in the custom sample holder, which ensures accurate positioning and minimizes circumferential air gaps.

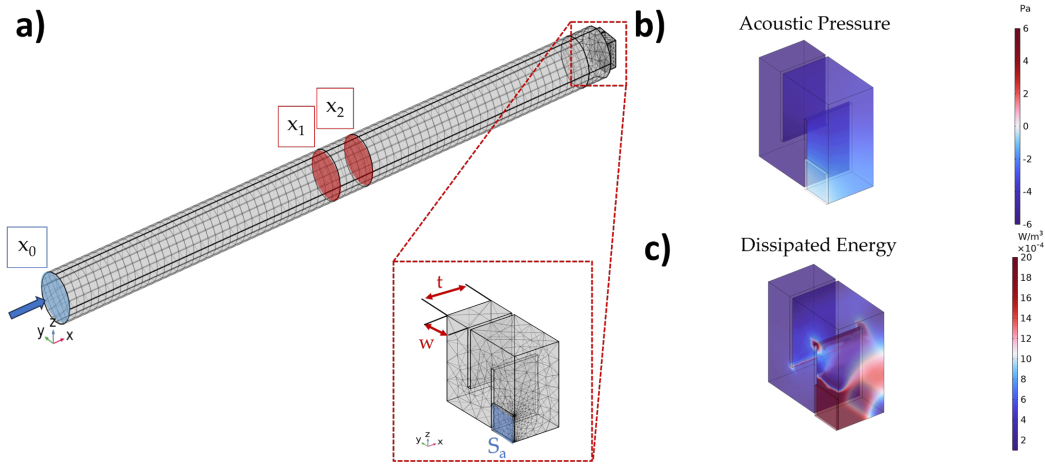


Fig. 3.6 Finite element model of the impedance tube containing the labyrinthine unit cell. (a) Computational domain and mesh of the impedance tube, including the air regions, the unit cell cavity and the inlet aperture; the blue region indicates the location of the imposed plane-wave excitation, while the yellow regions denote the surfaces where the averaged acoustic pressure is sampled. (b) Acoustic pressure field distribution at the peak absorption frequency. (c) Spatial distribution of the dissipated acoustic energy, highlighting the dominant contribution of thermo-viscous losses in the aperture region.

at the peak absorption frequency, respectively. Strong pressure amplification within the cavity is observed, together with a concentration of energy dissipation near the aperture region, indicating that thermo-viscous losses at the entrance play a dominant role in the absorption mechanism.

A representative comparison between analytical, numerical, and experimental absorption spectra is reported in Fig. 3.7. For a UC with lateral size $a = 19.7$ mm and thickness $t = 9.6$ mm, the experimental results show a maximum absorption coefficient of $\alpha \approx 0.95$ at a frequency of approximately 1480 Hz, with a bandwidth of about 390 Hz. The numerical model accurately reproduces both the peak position and bandwidth, while the analytical model slightly underestimates the absorption away from resonance. This discrepancy is mainly attributed to additional thermo-viscous losses occurring along the impedance tube walls, which are inherently included in the experimental and numerical configurations but neglected in the simplified analytical formulation. In order to quantify the dependence of the resonance frequency on the geometrical parameters of the unit cell, additional samples were fabricated and experimentally characterized, as summarized in Fig. 3.8. Two parametric campaigns were conducted: in the first (fig. 3.8a), the UC thickness t was varied within the

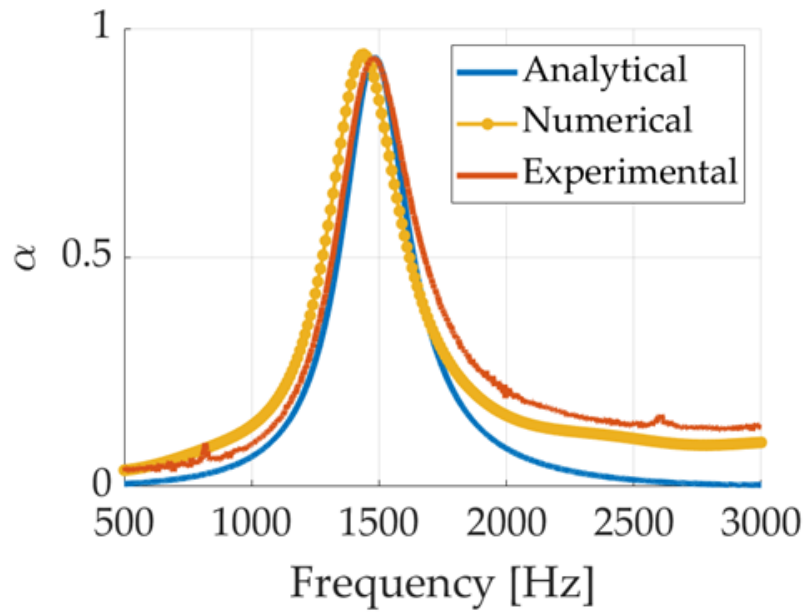


Fig. 3.7 Comparison between analytical, numerical and experimental absorption spectra of the labyrinthine unit cell under normal incidence conditions. The results refer to a unit cell with lateral size $a = 19.7$ mm and thickness $t = 9.6$ mm, showing good agreement in terms of resonance frequency and absorption level, with minor discrepancies attributed to additional thermo-viscous losses in the experimental setup.

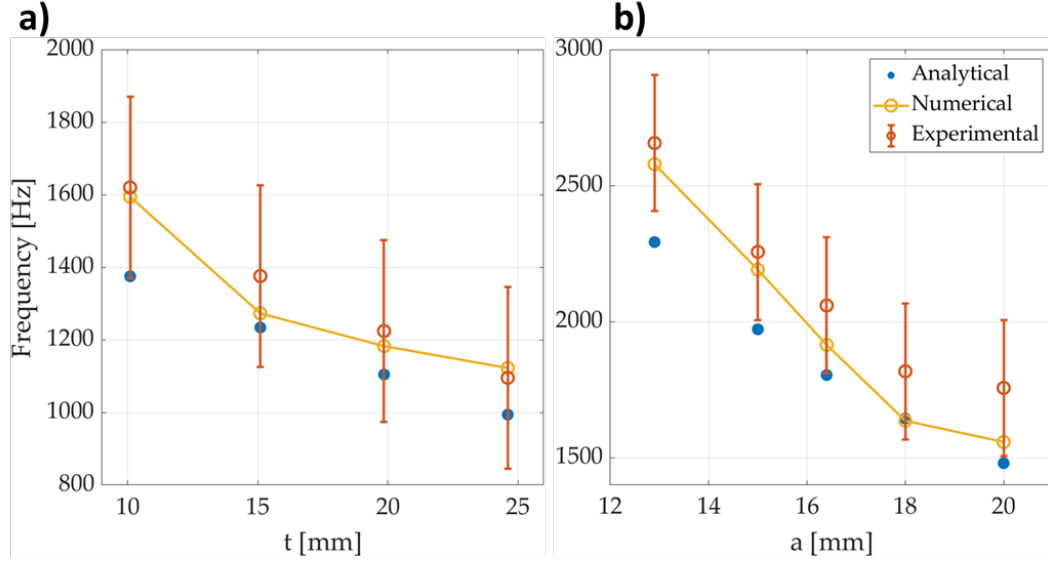


Fig. 3.8 Variation of the absorption peak frequency as a function of the geometrical parameters of the labyrinthine unit cell. (a) Resonance frequency versus unit cell thickness t (with fixed lateral size $a = 21.2$ mm). (b) Resonance frequency versus lateral edge size a (with fixed thickness $t = 12.6$ mm). Symbols represent experimental results, while analytical and numerical predictions are shown for comparison. Error bars correspond to the measured peak bandwidth.

range 10.0–25.0 mm while keeping the lateral edge size fixed ($a = 21.2$ mm); in the second (fig. 3.8b), the lateral size a was varied within the range 13.0–20.0 mm while maintaining constant the thickness ($t = 12.6$ mm). All samples were tested using the same impedance tube setup and measurement protocol described above to ensure consistency.

The experimentally measured peak absorption frequencies, reported in Fig. 3.8, show an approximately linear dependence on both geometric parameters. In particular, the resonance frequency decreases with increasing thickness with a slope of approximately $df_r/dt \approx -42.7 \text{ Hz mm}^{-1}$, while a stronger sensitivity is observed with respect to the lateral edge size, with $df_r/da \approx -145.3 \text{ Hz mm}^{-1}$. The experimental error bars correspond to the measured peak bandwidth. The results confirm that the lateral dimension provides coarse tuning over a wider frequency interval, whereas the thickness enables finer adjustment of the target resonance frequency. This behavior is consistent with both analytical and numerical predictions and constitutes the basis for the subsequent design of the rainbow metamaterial panel.

3.3 Rainbow metamaterial panel design and characterization

3.3.1 Panel design and fabrication

Following the experimental validation of the tuning capability of the individual labyrinthine unit cells (UCs), a broadband sound-absorbing panel was designed by combining multiple resonators with different geometrical parameters. The design strategy relies on the so-called *rainbow trapping* concept, whereby UCs characterized by distinct resonance frequencies are spatially arranged so that the superposition of their individual absorption peaks generates an extended effective absorption bandwidth.

Based on the impedance tube characterization, the target absorption range was selected between 800 and 1300 Hz. To cover this frequency interval, two UC lateral edge sizes were considered, namely $a_1 = 19.5$ mm and $a_2 = 26.0$ mm, combined with different thickness values. In particular, for the smaller unit cell (a_1) the thickness was varied within $t_1 = 10$ – 15 mm, whereas for the larger unit cell (a_2) thickness values in the range $t_2 = 23$ – 25 mm were adopted. This choice enables controlled positioning of multiple resonance peaks within the desired frequency band.

The UCs were arranged into modular elements of size 156×156 mm², referred to as *macrocells*, as illustrated in Fig. 3.9a. Each macrocell contains a quasi-periodic distribution of UCs with different thicknesses (T1–T6), ensuring that multiple resonance frequencies coexist within the same module. The internal wall thickness of each UC was scaled proportionally to its lateral size in order to preserve structural integrity while maintaining consistent acoustic behavior.

The complete panel consists of 20 macrocells arranged in the plane to cover a total surface of 628×784 mm², as shown schematically in Fig. 3.9b. Adjacent macrocells are rotated by 90° and mirrored in order to enhance spatial homogeneity of the acoustic response and reduce directional effects. A three-dimensional view of the assembled structure is reported in Fig. 3.9c. Under the assumption of weak coupling between adjacent UCs, the overall surface impedance of the panel can be approximated as the parallel combination of the impedances of the individual resonators, leading to a broadband absorption response resulting from the superposition of multiple subwavelength resonances.

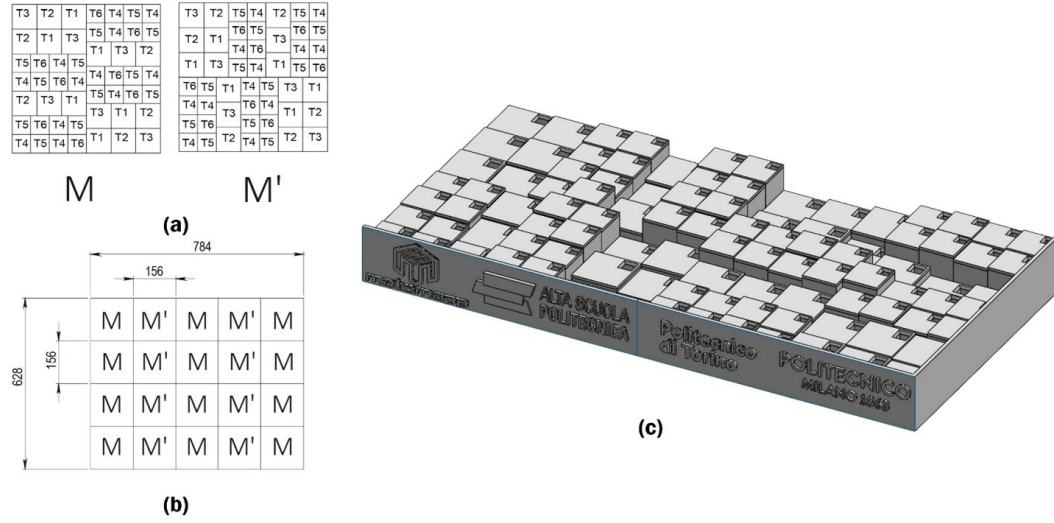


Fig. 3.9 Modular architecture of the rainbow labyrinthine metamaterial panel. (a) Two adjacent macrocells, each composed of unit cells with lateral sizes a_1 and a_2 and different thicknesses (T1–T6) to generate multiple resonance frequencies. (b) Top view schematic of the complete panel, obtained by arranging and rotating mirrored macrocells to ensure spatial homogeneity of the acoustic response (dimensions in mm). (c) Three-dimensional view of two assembled macrocells highlighting the variation in unit cell thickness across the module.

The panel was fabricated in polyamide by means of selective laser sintering (SLS), which enables the realization of complex hollow geometries without the need for supporting structures. Since powder removal requires open cavities during fabrication, each UC was provided with an additional auxiliary opening, which was subsequently sealed during testing to ensure acoustic tightness.

3.3.2 FE model of the AM Panel

In order to predict the acoustic response of the complete rainbow metamaterial (AM) panel and to support the interpretation of the experimental results, a fully acoustic finite element (FE) model was developed using COMSOL Multiphysics. The computational domain is shown in Fig. 3.10a and consists of a rectangular acoustic waveguide in which a plane wave is normally incident on the front surface of the labyrinthine AM panel. The corresponding finite element mesh, with local refinement in the metamaterial region, is reported in Fig. 3.10b. Periodic Bloch–Floquet boundary conditions were applied to the lateral surfaces of the acoustic waveguide (Fig. 3.10a) in order to reproduce an infinitely periodic arrangement

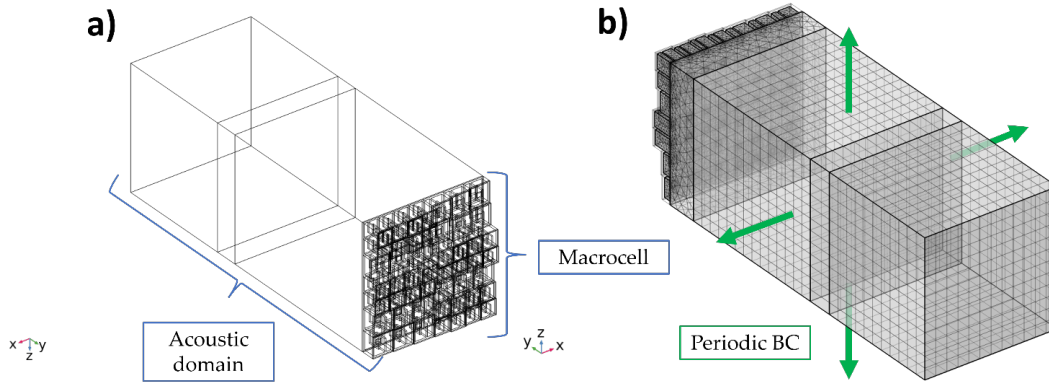


Fig. 3.10 Finite element model of the rainbow labyrinthine metamaterial panel. (a) Rectangular acoustic waveguide used to simulate normal-incidence sound propagation toward the panel and the backing cavity, with periodic boundary conditions applied on the lateral surfaces. (b) Corresponding finite element mesh, showing local refinement in the metamaterial region to accurately resolve the narrow labyrinthine channels and thermo-viscous effects.

of the macrocells in the plane of the panel. This approach enables modeling a representative portion of the structure while significantly reducing the computational cost. A plane-wave radiation condition with unit pressure amplitude was imposed at the inlet boundary, while a radiation condition was applied at the outlet to prevent artificial reflections.

The acoustic domains were discretized using approximately 5.4×10^5 finite elements, as illustrated in Fig. 3.10b. The mesh size was selected according to the smallest wavelength $\lambda_{\min}$ in the frequency range of interest, with a maximum element size $\Delta_{\max} = \lambda_{\min}/10$ and a minimum element size $\Delta_{\min} = \lambda_{\min}/30$, ensuring adequate resolution of both propagating waves in the waveguide and localized field gradients within the labyrinthine cavities.

Thermo-viscous losses within the narrow channels of the unit cells were accounted for using the *narrow region acoustics* formulation. This approach allows modelling viscous and thermal boundary layer effects without solving the full thermo-viscous equations throughout the entire domain. The narrow region condition was applied to the unit cell cavities and to the front and rear caps of the panel (see Fig. 3.10a). Losses were computed assuming rectangular duct cross-sections, using the channel width w and thickness t as geometric parameters.

All structural boundaries were considered acoustically rigid, as preliminary comparisons between fully coupled vibro-acoustic simulations and purely acoustic

models showed negligible differences in the investigated frequency range. The absorption coefficient was derived from the simulated surface impedance by evaluating the complex pressure and particle velocity fields at the panel interface.

3.3.3 Panel characterization

The acoustic performance of the complete rainbow metamaterial (AM) panel was experimentally evaluated in a small-scale reverberation room under diffuse field conditions. This methodology enables the determination of the random-incidence absorption coefficient, providing a more realistic assessment of the panel performance compared to normal-incidence impedance tube measurements.

The measurements were conducted in the small-scale reverberation chamber at Politecnico di Torino, whose performance has been previously validated for sound absorption measurements in the frequency range 400–5000 Hz. The mean reverberation time of the empty room in the 100–5000 Hz range is approximately 0.95 s, with a Schroeder frequency of about 1152 Hz. In order to enhance sound field diffusivity, the chamber is equipped with eight ceiling-mounted diffusers, covering approximately 13.5% of the total room surface. These features ensure sufficiently diffuse field conditions within the frequency band of interest.

The experimental configuration is shown in Fig. 3.11a. The panel was placed on the floor of the chamber with its flat surface exposed to the acoustic field, allowing sound waves to enter the labyrinthine unit cells through the front apertures while the rear surface remained sealed.

The measurement procedure followed the ISO 354 standard. The instrumentation chain consisted of six 1/4" BSWA Tech MPA451 microphones coupled with ICP104 preamplifiers (BSWA Technology Co., Ltd., Beijing, China), two ITA High-Frequency Dodecahedron loudspeakers with dedicated ITA power amplifiers (ITA-RWTH, Aachen, Germany), and a Roland OctaCapture UA-1010 multi-channel sound card (Roland Corporation, Japan). Signal generation, acquisition, and post-processing were performed using a MATLAB-based routine combined with the ITA-Toolbox.

A total of 12 source–microphone combinations were used, corresponding to six microphone positions and two source locations, ensuring adequate spatial averaging of the sound field. Reverberation times were measured in two configurations: with

and without the specimen. The integrated impulse response method was employed for the evaluation of decay curves.

Environmental conditions were continuously monitored throughout the experiments. The room temperature was maintained above 15°C, and the relative humidity was kept within the range 30–90%, in compliance with the validity requirements of ISO 354. These parameters were used to determine the sound speed and the air attenuation coefficients required for the calculation of the absorption area.

The equivalent absorption area A_T (in m^2) was calculated according to ISO 354 as

$$A_T = 55.3V \left(\frac{1}{c_2 t_2} - \frac{1}{c_1 t_1} \right) - 4V(m_2 - m_1), \quad (3.2)$$

where t_1 and t_2 are the reverberation times of the chamber without and with the specimen, respectively; V is the volume of the empty reverberation room; c_1 and c_2 are the sound velocities under the two environmental conditions; and m_1 and m_2 are the power attenuation coefficients accounting for air absorption.

The random-incidence absorption coefficient was then obtained as

$$\alpha = \frac{A_T}{S}, \quad (3.3)$$

where S is the surface area of the tested panel.

3.3.4 AM panel sound absorption results

The acoustic performance of the rainbow labyrinthine metamaterial panel was experimentally evaluated under diffuse-field conditions in the small-scale reverberation room. The panel features a flat upper surface equipped with apertures providing access to the labyrinthine unit cells, while the lower surface is fully sealed. The measurement configuration is shown in Fig. 3.11a, where the flat surface faces the incident diffuse sound field.

The experimentally measured random-incidence absorption spectrum is reported in Fig. 3.11b in third-octave bands over the frequency range 250–5000 Hz. The results demonstrate that the effective absorption coefficient approaches unity within

the target frequency interval of 800–1300 Hz. In particular, absorption values close to $\alpha \approx 1$ are achieved in this band, confirming the effectiveness of the rainbow design strategy based on the superposition of multiple subwavelength resonances.

For comparison, an analytical prediction (blue curve in Fig. 3.11b) was obtained by computing the global surface impedance of the panel assuming that the unit cells operate independently and in parallel. Under this assumption, the total surface impedance is derived from the parallel combination of the individual UC impedances, following the modelling approach described in [112]. Despite the fact that both the analytical model and the finite element (FE) simulations assume normal-incidence excitation, whereas the experimental measurements are performed under diffuse-field conditions, good agreement is observed within the main absorption band (800–1300 Hz).

The FE simulations under normal incidence (also shown in Fig. 3.11b) exhibit excellent consistency with the analytical model and closely reproduce the experimentally observed broadband absorption behaviour. Minor discrepancies are observed at higher frequencies (above approximately 3 kHz), which can be attributed to increased sensitivity to non-normal incidence effects and to the finite lateral dimensions of the panel.

Edge effects associated with the finite sample size were not explicitly included in the numerical model. However, according to the validated performance of the small-scale reverberation room [113], such effects become significant mainly below 400 Hz for the considered specimen dimensions. Furthermore, the impact of finite-size effects is generally more pronounced for thick and highly absorbing materials [114, 115]. Since the primary frequency range of interest (800–1300 Hz) lies well above this lower limit, the measured results can be considered representative of the intrinsic acoustic performance of the rainbow metamaterial panel.

Overall, the experimental characterization confirms that the proposed structure achieves near-ideal absorption within the targeted mid-frequency band while maintaining reduced thickness and modular architecture, thereby validating the effectiveness of the labyrinthine rainbow design approach.

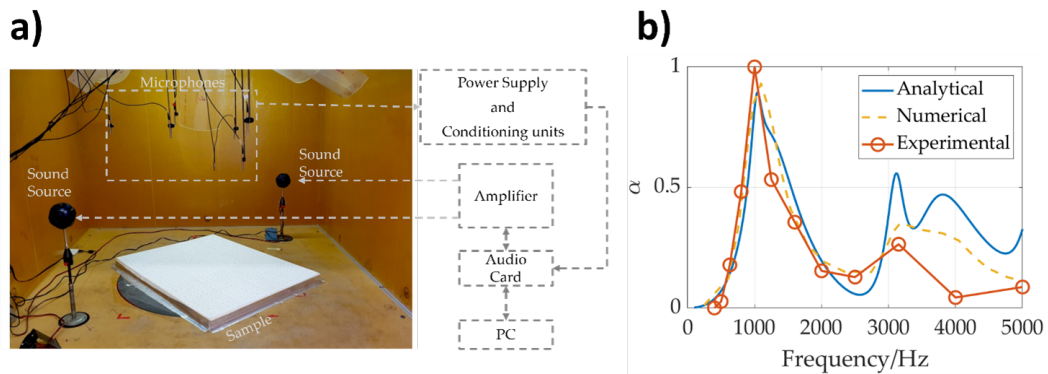


Fig. 3.11 Experimental characterization of the rainbow labyrinthine metamaterial panel in a small-scale reverberation room. (a) Measurement configuration with the panel placed on the floor and the flat surface exposed to the diffuse acoustic field. (b) Random-incidence absorption coefficient as a function of frequency, compared with analytical predictions and finite element simulations, showing near-unity absorption within the target range of 800–1300 Hz.

3.4 Numerical evaluation of different labyrinthine sound absorption panel solutions

Macrocell with backing cavity and porous filling

Following the promising agreement between analytical predictions, finite element (FE) simulations and reverberation room measurements, a further numerical investigation was conducted to explore possible enhancements of the performance of the rainbow metamaterial (AM) panel. The objective of this study was to assess the acoustic performance of alternative configurations that combine the labyrinthine structure with conventional porous absorbing materials.

Three additional configurations were considered:

- (i) addition of a melamine foam filling inside the labyrinthine unit cell (UC) cavities;
- (ii) introduction of a rigid rear backing cavity applied to the uneven bottom surface of the panel (with open apertures), optionally filled with melamine foam;
- (iii) combination of both foam-filled UCs and foam-filled backing cavity.

The numerical analysis was performed using COMSOL Multiphysics, focusing on a single macrocell considered representative of the global panel behaviour. Thermo-viscous losses were included using the same modelling approach described in Section 4.2.

In the configuration including a backing cavity, the AM panel was coupled to an additional acoustic domain of depth equal to four times the macrocell side length, as illustrated in Fig. 3.12a. Non-reflective Perfectly Matched Layer (PML) boundary conditions were applied at the rear boundary of the acoustic domain to prevent artificial reflections. The corresponding finite element mesh is shown in Fig. 3.12b, with local refinement in the metamaterial region.

All structural boundaries were assumed acoustically rigid, as preliminary vibro-acoustic simulations indicated negligible structural coupling effects in the investigated frequency range.

The porous material used in the simulations was melamine foam, modeled as an equivalent fluid according to a rigid-frame porous formulation. The material parameters adopted in the FE simulations are reported in Table 3.1.

Table 3.1 Melamine foam material properties adopted in the finite element simulations.

Parameter	Value
Porosity, Φ	0.995
Flow resistivity, σ (Pa·s/m ²)	10500
Thermal characteristic length, Λ' (μ m)	470
Viscous characteristic length, Λ (μ m)	240
Tortuosity factor, α_∞	1.0059

3.4.1 Results and discussion

The calculated normal-incidence absorption coefficients for the different configurations are reported in Fig. 3.13. The baseline configuration (rainbow AM panel without additional porous material) is shown as the reference case.

The introduction of a 5 mm thick foam-filled backing cavity produces a noticeable shift of the absorption peaks toward lower frequencies. In particular, the lowest dominant peak shifts from approximately 1 kHz to about 700 Hz, improving low-frequency attenuation. This effect can be attributed to the additional compliance

introduced by the backing cavity, which effectively modifies the acoustic impedance and lowers the global resonance frequency of the system.

Conversely, filling the internal AM cavities with melamine foam results in a more uniform absorption coefficient across the entire investigated frequency range. However, this configuration reduces the peak absorption values at the original metamaterial resonance frequencies. The additional distributed losses introduced by the porous medium broaden the response but partially damp the sharp subwavelength resonances that characterize the labyrinthine design.

The combined configuration (foam-filled AM cavities and foam-filled backing cavity) further smooths the absorption spectrum, producing a broadband attenuation profile at the expense of reduced peak values. Such a solution may be advantageous in applications that require distributed noise reduction over a wide frequency interval rather than high-efficiency absorption within a specific target band.

Overall, the numerical results demonstrate that hybrid solutions that combine labyrinthine metamaterials with conventional porous absorbers provide additional flexibility in tailoring the acoustic response. Depending on the application requirements, the system can be tuned either for strong peak absorption in a predefined frequency band or for broader, more uniform attenuation across an extended frequency range.

3.5 Conclusions

In this Chapter, a comprehensive design, modeling and experimental validation procedure for a broadband sound-absorbing panel based on labyrinthine acoustic metamaterials has been presented. The proposed approach relies on the controlled tuning of subwavelength unit cells (UCs) and their combination according to a rainbow strategy, enabling efficient mid-frequency sound attenuation within a reduced structural thickness.

The analytical modeling of the labyrinthine UC, based on thermo-viscous loss mechanisms and impedance matching considerations, was first validated through impedance tube measurements and finite element (FE) simulations. Experimental results confirmed that the resonance frequency can be tuned in a predictable manner by varying both the unit cell thickness and its lateral size. In particular, an approximately

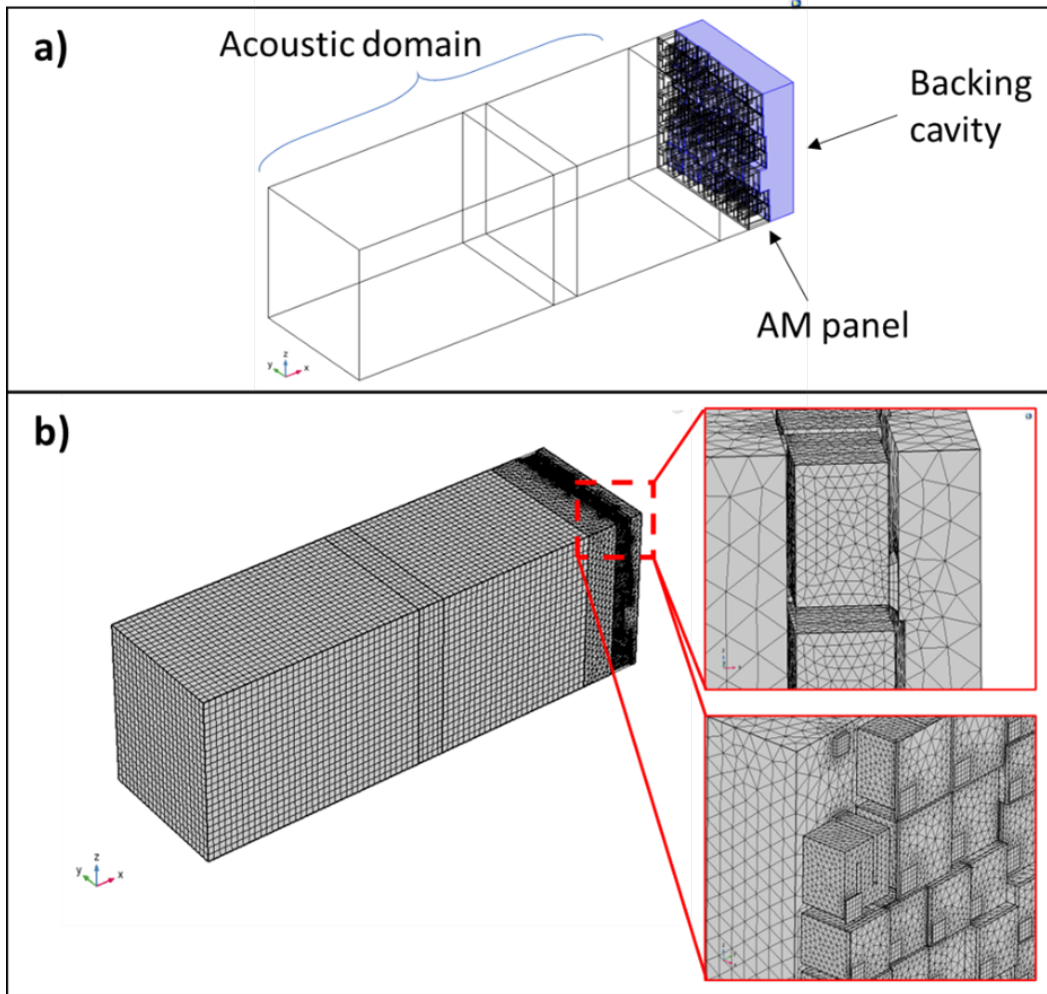


Fig. 3.12 Finite element model of the rainbow metamaterial macrocell coupled with a backing cavity. (a) Acoustic domain used to simulate normal-incidence sound propagation toward the labyrinthine panel and the rear cavity, terminated with a Perfectly Matched Layer (PML) to avoid artificial reflections. (b) Corresponding finite element mesh, showing local refinement in the metamaterial region and in the backing cavity to accurately capture thermo-viscous effects and pressure gradients.

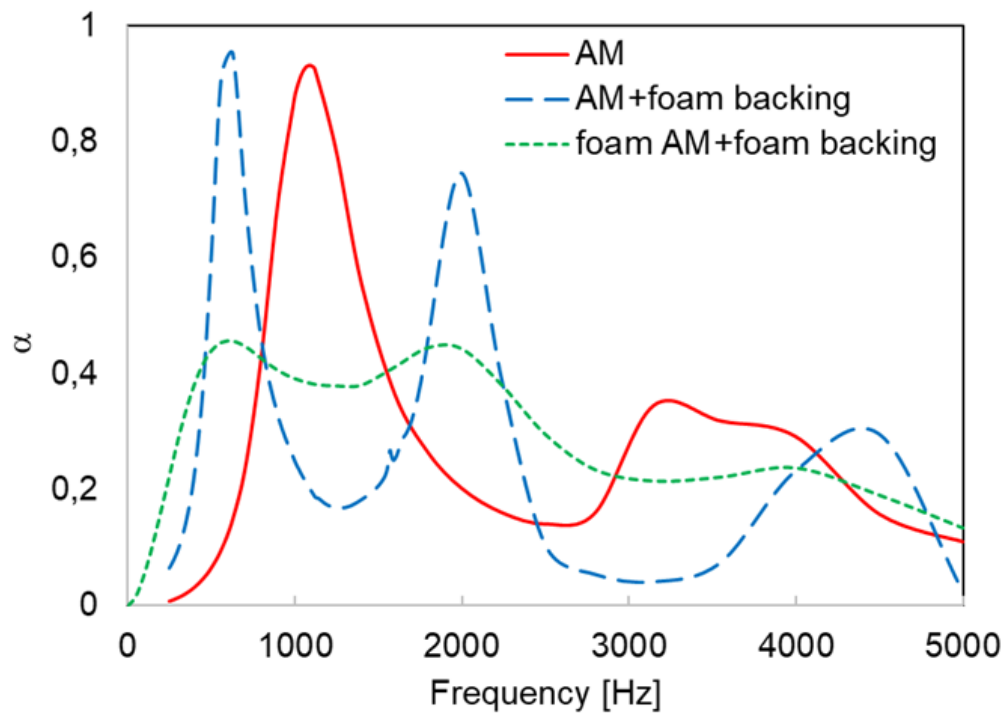


Fig. 3.13 Finite element simulation results of the normal-incidence absorption coefficient for different panel configurations: baseline rainbow metamaterial panel (AM), panel with a 5 mm foam-filled backing cavity (AM + foam backing), and panel with both foam-filled labyrinthine cavities and foam-filled backing cavity (foam AM + foam backing). The addition of the backing cavity shifts the absorption peaks toward lower frequencies, while foam filling leads to a broader but less pronounced absorption response.

linear dependence of the resonance frequency on these geometrical parameters was observed, enabling both coarse and fine tuning of the target operating frequency.

Building upon these results, a modular rainbow panel was designed by combining UCs with different geometrical configurations in a quasi-periodic arrangement. The superposition of multiple subwavelength resonances produced a broadband absorption response in the frequency range 800–1300 Hz. The complete panel, fabricated using selective laser sintering, was experimentally characterized in a small-scale reverberation room under diffuse-field conditions. The measured random-incidence absorption coefficient approached unity within the target frequency band, demonstrating excellent agreement with analytical impedance-based predictions and FE simulations.

Further numerical investigations explored hybrid configurations combining the labyrinthine metamaterial with conventional porous materials and backing cavities. The results showed that the addition of a foam-filled backing cavity shifts the absorption peaks toward lower frequencies, enhancing low-frequency performance, whereas foam filling within the unit cells broadens the response at the expense of peak absorption. These findings highlight the flexibility of the proposed design framework and its adaptability to different application requirements.

Overall, the presented study provides experimental proof of concept for an efficient, modular, and lightweight sound-absorbing solution based on labyrinthine metamaterials. The reduced thickness, tunability, and compatibility with additive manufacturing techniques make the proposed structure a promising candidate for practical noise mitigation applications in architectural acoustics, automotive engineering, and aerospace systems. The validated modeling framework further enables systematic optimization of future metamaterial-based absorbers tailored to specific frequency targets and design constraints.

Chapter 4

Wave mode conversion in phononic crystals using undulated beams

4.1 Introduction

Wave mode conversion is a fundamental phenomenon in applications such as acoustic and ultrasonic sensing, yielding enhanced sensitivity by transforming one wave mode into another that is more easily detectable (e.g., longitudinal into shear waves) [116, 117]. This feature is particularly useful in non-destructive testing [118], where mode conversion improves defect detection [119, 120], and also allows for specific interaction with various materials, yielding structural defect discrimination through selective detection [121].

Phononic crystals and elastic metamaterials have proven to be advantageous platform for manipulating elastic and acoustic waves due to their tailorable properties and versatility [122–125], with applications in sensing devices [126–128], energy harvesting [129, 130], one-way transmission [131], polarization control [132], topological waveguiding [133], and non-destructive testing [134, 135]. Although elastic metamaterials are typically based on locally resonant structures [16], phononic crystals are usually designed through spatially periodic variations of material or geometric properties. As a consequence, phononic crystals present functionalities associated with Bragg scattering [136, 137], leading to phenomena that occur at wavelengths comparable to the lattice spacing, thus limiting their operation in the subwavelength regime. Beyond applications based on static quantities [138, 139],

mechanical metamaterials are mostly notable for their recent development in dynamic (i.e., frequency-dependent) applications. Examples include acoustic metamaterials able to perform edge detection in the subwavelength range [140], anisotropic graded-index metamaterials that achieve high signal-to-noise ratio for weak acoustic signal detection [141], and cochlea-inspired metamaterials that enable the spatial discrimination of flexural waves based on their frequency content [142].

Elastic metamaterials have been widely employed in the design of wave mode conversion devices. For instance, Colombi et al. [143] proposed a system composed of a graded array of subwavelength resonators ("metawedge") on an elastic substrate that can either (i) convert incident surface Rayleigh waves into bulk shear waves without reflections or (ii) reflect Rayleigh waves, creating a system with asymmetric transmission properties. This concept was later studied by Chaplain et al. [144] to consider both longitudinal and flexural motion of the resonators and exploit Umklapp conversion. Tian et al. [145] proposed a metamaterial consisting of a thin plate substrate decorated with resonating stubs to convert ultrasonic Lamb waves (A_0 and S_0 modes) into purely longitudinal shear horizontal waves (SH_0 mode), demonstrating complete conversion with a single type of unit cell while maintaining the propagation direction. In another work, Wang et al. [146] utilized an ultra-thin metamaterial plate (thickness two orders smaller than the wavelength) endowed with asymmetrical resonators whose anisotropy increases the coupling between the longitudinal/transverse motion to achieve nearly perfect mode conversion (above 95%) between longitudinal and transverse waves. Specifically in the context of longitudinal-flexural wave mode conversion, Cao et al. [147] proposed a waveguide composed of periodic resonators whose Fano resonance hybridizes flexural and longitudinal waves, leading to highly localized bound modes [148] with a theoretically infinite quality factor and perfect mode conversion from flexural to longitudinal waves. In all of these cases, the desired functionalities are achieved with the use of locally resonant structures that may be challenging to manufacture, due to size and geometrical constraints.

Metamaterials also provide excellent candidates to design wave conversion waveguides. In [149], De Ponti et al. designed a graded array of asymmetrically distributed pairs of resonators that yields mode conversion between incident flexural waves and reflected torsional waves. A similar concept was then applied to obtain selective mode conversion between torsional, flexural in-plane, and flexural out-of-plane waves, yielding group velocity inversion for a single wave type [150]. Finally, Lemkalli et al. [151] have demonstrated that longitudinal waves can be partially

converted into torsional waves using a chiral unit cell whose edges are connected through tetragonal beams. Therefore, introducing geometrical asymmetry in the design of the periodic unit cell seems to be a fundamental ingredient for the design of passive wave conversion waveguides.

Focusing on wave mode conversion using phononic crystals, a common strategy is to design unit cells whose effective medium properties present an anisotropic behavior, yielding coupling between distinct wave modes. For instance, an anisotropic unit cell whose effective properties yield full conversion between longitudinal and shear waves in the ultrasonic regime was proposed in [152]. Full mode conversion was also achieved between longitudinal and shear modes in the ultrasonic regime in [153], where a transmodal Fabry-Pérot resonance mechanism is presented and investigated, and derive the conditions for broadband perfect mode conversion using an architected medium with tailored effective properties. A similar concept was used to achieve longitudinal-torsional wave mode conversion employing a structured tubular waveguide in [154]. However, designing phononic crystals that operate in the low-frequency regime and do not rely on locally resonant features remains an open challenge for wave mode conversion applications. In particular, single-phase structures may enable designs that are amenable to fabrication, allowing versatile and low-cost wave conversion devices.

Phononic crystals in one- and two-dimensional structures with undulated profiles have been shown to enable superior control over flexural waves. Pelat et al. demonstrated this concept using undulated profiles to manipulate the first Bragg band gap in beams [155], obtaining dispersion diagrams through the plane wave expansion (PWE) method. They also demonstrated that by using beams with asymmetric profiles (i.e., whose neutral line does not vary in transverse direction along the longitudinal coordinate), it is possible for longitudinal excitations to produce flexural motion and vice versa. This behavior, however, was not correlated with the particular states of polarization of each propagating branch in the dispersion diagrams. The concept of flexural wave control using undulated profiles was later generalized to two-dimensional media using thin plate models (i.e., with negligible shear strain and rotational inertia) with variable thickness to obtain low-frequency flexural band gaps with optimized widths by Dal Poggetto and Arruda [156]. Beyond purely flexural waves, folding-uncoupled origami metamaterial were also used to obtain band gaps based on coupled longitudinal-flexural behavior [157]. Likewise, Trainiti et al. [158] investigated the occurrence of band gaps in undulated thin beams and plates with

asymmetric profiles in the low-frequency range, observing that the introduction of slight changes in the beam curvature enables the coupling between longitudinal and flexural motion. In these works, however, the authors focused on the nucleation of band gaps without investigating the evolution of the polarization states, whether flexural or longitudinal, in the computed dispersion relations. Understanding the wave propagation behavior in waveguides according to the variation of wavenumber allows the identification of (i) coupling phenomena responsible for the nucleation of band gaps and (ii) the possibility of exploiting these systems to create wave conversion devices.

Another particularity of interest concerning wave propagation in periodic media is the existence of sub-Bragg mechanisms (i.e., occurring below the first-order Bragg condition) observed in photonic crystals [159]. Extending this concept to phononic crystals may allow subwavelength Bragg-like phenomena, with low-frequency wave manipulation capabilities [160, 161]. In this chapter of my thesis, I propose the design of beams with undulated profiles to perform mode conversion between longitudinal and flexural waves below the first-order Bragg condition. This capability is achieved through a single-phase monolithic structure that can be easily manufactured, and enables operation at various frequencies through the tuning of a single geometrical parameter (fig. 1.1a, 4.1b). Unlike conventional conversion mechanisms presented in the literature, our proposed structure also inverts the propagation direction, enabling wave conversion through reflection. The corresponding dispersion curves are calculated considering non-negligible shear strain and rotational inertia, which improves their accuracy. Our proposed wave conversion device is first justified analytically, then demonstrated numerically and validated experimentally.

This chapter is divided as follows. Section 4.2 presents a PWE formulation for the computation of dispersion diagrams in undulated beams considering normal-flexural coupling, shear strain, and rotational inertia. In Section 4.3 the proposed formulation is utilized to design a periodic unit cell with longitudinal-flexural coupling at a target operating frequency. The numerical and experimental validation of the proposed wave conversion devices are then presented in Section 4.4. Conclusions and outlook for future work are outlined in Section 4.5.

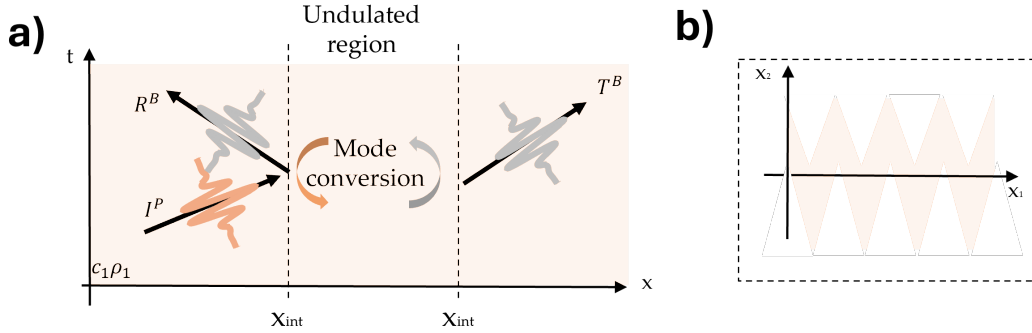


Fig. 4.1 (a) Schematic of the wave conversion process for a longitudinal incident wave impinging to an undulated region and converting into a bending mode. (b) Schematic profile of the undulated region for wave conversion.

4.2 Derivation of dispersion relations using the PWE method

Consider a beam whose centerline lies in the x -direction and presents an undulated profile in the z -direction, as illustrated in Fig. 4.2a. Although a simple sinusoidal function is considered for illustration purposes, the following derivation holds for any periodic shape that can be expressed as a Fourier series. The equations of motion relating the longitudinal (u_x) and transverse (u_z) displacements and the cross-section rotation (ψ_y) with respect to its initial configuration in a beam with an undulated profile, considering both flexural and shear motion, are given by [162]

$$\begin{aligned} \frac{\partial N}{\partial x} + \kappa Q &= \rho A \frac{\partial^2 u_x}{\partial t^2}, \\ \frac{\partial Q}{\partial x} - \kappa N &= \rho A \frac{\partial^2 u_z}{\partial t^2}, \\ \frac{\partial M}{\partial x} + Q &= \rho I \frac{\partial^2 \psi_y}{\partial t^2}, \end{aligned} \quad (4.1)$$

where x is the longitudinal beam coordinate, t is the time coordinate, $N = N(x)$, $Q = Q(x)$, and $M = M(x)$ are, respectively, the internal longitudinal force, shear force, and flexural moment resultants, $\kappa = \kappa(x)$ is the beam curvature, $\rho = \rho(x)$ is the specific mass density of the beam material, A and $I = I(x)$ are, respectively, the cross-sectional area and second moment of inertia of the beam with respect to its centerline. The internal forces ($N(x)$ and $Q(x)$) and moment resultants ($M(x)$) are

respectively given by [162]

$$\begin{aligned} N(x) &= AE \left(\frac{\partial u_x}{\partial x} + \kappa u_z \right), \\ M(x) &= EI \frac{\partial \psi_y}{\partial x}, \\ Q(x) &= \alpha A \mu \left(\frac{\partial u_z}{\partial x} - \kappa u_x - \psi_y \right), \end{aligned} \quad (4.2)$$

where E and μ are, respectively, the Young's modulus and shear modulus of the beam material, and α is a shear correction factor [163]. Combining Eqs. (4.1) and (4.2) and assuming constant material properties and cross-section, we obtain

$$\begin{aligned} &AE \frac{\partial^2 u_x}{\partial x^2} - \alpha A \mu \kappa^2 u_x + A(E + \alpha \mu) \kappa \frac{\partial u_z}{\partial x} \\ &+ AE \frac{\partial \kappa}{\partial x} u_z - \alpha A \mu \kappa \psi_y = \rho A \frac{\partial^2 u_x}{\partial t^2}, \end{aligned} \quad (4.3a)$$

$$\begin{aligned} &-A(E + \alpha \mu) \kappa \frac{\partial u_x}{\partial x} - \alpha A \mu \frac{\partial \kappa}{\partial x} u_x + \alpha A \mu \frac{\partial^2 u_z}{\partial x^2} \\ &-AE \kappa^2 u_z - \alpha A \mu \frac{\partial \psi_y}{\partial x} = \rho A \frac{\partial^2 u_z}{\partial t^2}, \end{aligned} \quad (4.3b)$$

$$\begin{aligned} &-\alpha A \mu \kappa u_x + \alpha A \mu \frac{\partial u_z}{\partial x} + EI \frac{\partial^2 \psi_y}{\partial x^2} \\ &-\alpha A \mu \psi_y = \rho I \frac{\partial^2 \psi_y}{\partial t^2}. \end{aligned} \quad (4.3c)$$

The computation of the dispersion relations $k = k(\omega)$ between the wavenumber k and the propagating frequencies ω in a beam with a periodic curvature can be efficiently computed using the PWE method. Following the methodology described in [164, 165], we start by expressing the beam periodic curvature in the form of a Fourier series as

$$\kappa(x) = \sum_{G_n} \hat{\kappa}(G_n) e^{iG_n x}, \quad (4.4)$$

where $G_n = n \frac{2\pi}{a}$, $n \in \mathcal{Z}$, represents the reciprocal lattice vectors for a unit cell of length a , $i = \sqrt{-1}$ is the imaginary unit number, and $\hat{\kappa}(G_n)$ are its Fourier series coefficients, which can be computed through the inverse relation

$$\hat{\kappa}(G_n) = \frac{1}{a} \int_0^a \kappa(x) e^{-iG_n x} dx. \quad (4.5)$$

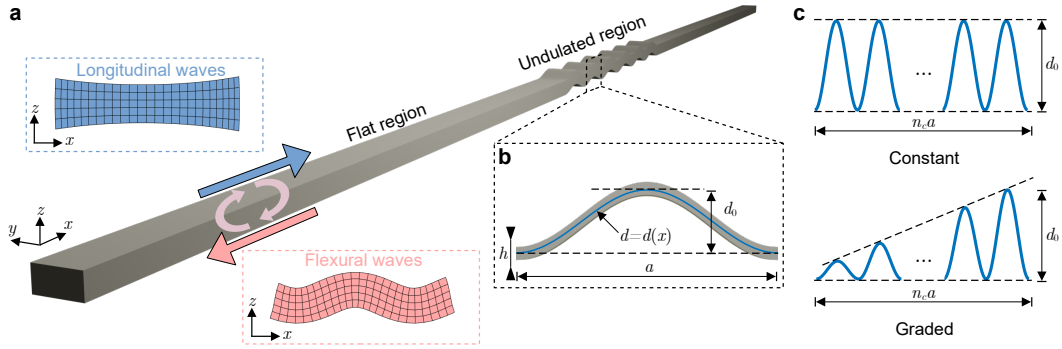


Fig. 4.2 Longitudinal-flexural wave mode conversion device. a. The beam with an undulated profile is designed to convert incident longitudinal waves (blue arrow) to reflected flexural waves (red arrow). b. General profile $d(x)$ of the undulated region with lattice length a , constant thickness h , and undulation amplitude d_0 . c. Schematic representation of constant (top panel) and graded (bottom panel) undulated profiles, with n_c undulated unit cells and a maximum height d_0 .

Similarly, the periodic wavelength-dependent displacements and rotation (u_x , u_z , ψ_y) can be represented by time-harmonic Bloch waves [166] expressed as

$$u(x, t) = e^{-i\omega t} \sum_{G_n} \hat{u}(G_n) e^{i(k+G_n)x}, \quad (4.6)$$

where $\hat{u} = \{\hat{u}_x, \hat{u}_z, \hat{\psi}_y\}$ are the Fourier series coefficients corresponding to the spatial periodic function that composes the Bloch wave solution of u_x , u_z , or ψ_y .

Equations (4.4) and (4.6) can be substituted into Eqs. (4.1) and (4.2), to yield the infinite system of linear equations given by

$$\begin{aligned} & \sum_{G_j} \left[E(k+G_j)^2 \delta_{i,j} + \alpha\mu \widehat{\kappa^2}(G_i - G_j) \right] \hat{u}_x(G_j) + \\ & - \sum_{G_j} \left[(E + \alpha\mu) \hat{\kappa}(G_i - G_j) i(k+G_j) + E \widehat{\kappa'}(G_i - G_j) \right] \hat{u}_z(G_j) + \\ & + \sum_{G_j} \left[\alpha\mu \hat{\kappa}(G_i - G_j) \right] \hat{\psi}_y(G_j) = \omega^2 \rho \hat{u}_x(G_i), \end{aligned} \quad (4.7a)$$

$$\begin{aligned} & \sum_{G_j} \left[(E + \alpha\mu) \hat{\kappa}(G_i - G_j) i(k+G_j) + \alpha\mu \widehat{\kappa'}(G_i - G_j) \right] \hat{u}_x(G_j) + \\ & + \sum_{G_j} \left[\alpha\mu (k+G_j)^2 \delta_{i,j} + E \widehat{\kappa^2}(G_i - G_j) \right] \hat{u}_z(G_j) + \\ & + \sum_{G_j} \left[\alpha\mu i(k+G_j) \delta_{i,j} \right] \hat{\psi}_y(G_j) = \omega^2 \rho \hat{u}_z(G_i), \end{aligned} \quad (4.7b)$$

$$\begin{aligned} & \sum_{G_j} \left[\alpha\mu \hat{\kappa}(G_i - G_j) \right] \hat{u}_x(G_j) - \sum_{G_j} \left[\alpha\mu i(k+G_j) \delta_{i,j} \right] \hat{u}_z(G_j) + \\ & + \sum_{G_j} \left[\frac{EI}{A} (k+G_j)^2 \delta_{i,j} + \alpha\mu \delta_{i,j} \right] \hat{\psi}_y(G_j) = \omega^2 \rho \frac{I}{A} \hat{\psi}_y(G_i), \end{aligned} \quad (4.7c)$$

where $\widehat{\kappa'}$ and $\widehat{\kappa^2}$ are, respectively, the Fourier series coefficients that represent the functions $\frac{\partial \kappa(x)}{\partial x}$ and $\kappa^2(x)$, G_i (G_j) corresponds to the i -th (j -th) reciprocal lattice vector, and $\delta_{i,j}$ is the Kronecker delta (i.e, $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ for $i \neq j$). For a given beam curvature κ , this system of equations can be used to obtain the propagating frequencies ω for each wavenumber k of interest, thus yielding a dispersion relation.

Consider now a beam profile $d(x)$ corresponding to a single undulation described by

$$d(x) = \frac{d_0}{2} \left(1 - \cos \left(\frac{2\pi}{a} x \right) \right), x \in [0, a], \quad (4.8)$$

where d_0 is the profile height for a fixed beam thickness h , as shown in Fig. 4.2b. The corresponding beam curvature is calculated as

$$\kappa(x) = \frac{\partial^2 d(x)}{\partial x^2} = d_0 \left(\frac{\pi}{a} \right)^2 \left(e^{i \frac{2\pi}{a} x} + e^{-i \frac{2\pi}{a} x} \right), \quad (4.9)$$

which can be immediately represented using a Fourier series in the form

$$\widehat{\kappa}(G_n) = d_0 \left(\frac{\pi}{a} \right)^2 (\delta_{n,1} + \delta_{n,-1}). \quad (4.10)$$

Analogously, the Fourier series coefficients for the terms $\frac{\partial \kappa(x)}{\partial x}$ and $\kappa^2(x)$ can be represented, respectively, as

$$\widehat{\kappa'}(G_n) = 2i d_0 \left(\frac{\pi}{a} \right)^3 \left(\delta_{n,1} - \delta_{n,-1} \right), \quad (4.11a)$$

$$\widehat{\kappa^2}(G_n) = d_0^2 \left(\frac{\pi}{a} \right)^4 \left(\delta_{n,2} + 2\delta_{n,0} + \delta_{n,-2} \right). \quad (4.11b)$$

Substituting Eqs. (4.10)–(4.11) back into Eq. (4.7) and truncating the number of plane waves by restricting the reciprocal lattice vector G_n to $n \in [-N, N]$, $N \in \mathcal{N}$, corresponding to $1 + 2N$ plane waves, yields the eigenproblem

$$\begin{bmatrix} \mathbf{D}_{xx} & \mathbf{D}_{xz} & \mathbf{D}_{xy} \\ \mathbf{D}_{zx} & \mathbf{D}_{zz} & \mathbf{D}_{zy} \\ \mathbf{D}_{yx} & \mathbf{D}_{yz} & \mathbf{D}_{yy} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}_x \\ \hat{\mathbf{u}}_z \\ \hat{\psi}_y \end{Bmatrix} = \omega^2 \begin{Bmatrix} \hat{\mathbf{u}}_x \\ \hat{\mathbf{u}}_z \\ \hat{\psi}_y \end{Bmatrix}, \quad (4.12)$$

where the matrix terms are described in Appendix ?? and the eigenvectors representing the Fourier series terms for the displacement and rotation components given by $\hat{\mathbf{u}} = \left\{ \hat{u}(G_{-N}) \cdots \hat{u}(G_{-1}) \hat{u}(G_0) \hat{u}(G_1) \cdots \hat{u}(G_N) \right\}^T$, for $u = \{u_x, u_z, \psi_y\}$.

The formulation above allows to compute dispersion relations corresponding to a periodic medium with a single undulation height, i.e., a constant (non-graded) profile. A corresponding undulated region is schematically illustrated in Fig. 4.2b (top panel), using n_c unit cells of length a with an undulation height d_0 . Also shown in Fig. 4.2c (bottom panel) is the alternative case of a graded profile, where a linear variation of the height d_0 is applied over n_c unit cells. This is later analyzed in Section 4.4.

4.3 Computation of dispersion relations and experimental verification

In this section, we apply the derived semi-analytical method to obtain the dispersion relations considering a homogeneous isotropic medium made of Aluminum, with Young's modulus $E = 70$ GPa, Poisson's ration 0.33, and specific mass density $\rho = 2700$ kg/m³. The cross section of the beam is rectangular, with width $b = 4$ mm and height $h = 2$ mm. The unit cell length is fixed as $a = 10$ mm, which was chosen

due to fabrication constraints for the total length of the test specimen (300 mm). Then, we utilize an experimental set-up to verify the existence of the computed band gaps.

4.3.1 Computation of dispersion relations

We initially use the PWE formulation given by Eq. (4.12) to compute the dispersion relations corresponding to the unit cell with increasing values of undulation d_0 . For these computations, we utilize 11 plane waves, which present good convergence due to the simple harmonic expression of the chosen undulation profile. The computed dispersion relations (frequency ω vs. wavenumber k) are presented in Fig. 4.3a for increasing values of $d_0/a \in [0, 0.2]$, where we also include in colour scale a polarization metric given by $p(\hat{\mathbf{u}}_x, \hat{\mathbf{u}}_z) = \sum_G |\hat{u}_z|^2 / (\sum_G |\hat{u}_x|^2 + |\hat{u}_z|^2)$, thus indicating $p = 0$ for purely longitudinal ($u_z = 0$) and $p = 1$ for purely transverse ($u_x = 0$) motion. We also utilize the normalized frequency $\bar{\omega} = \omega a / c_L$, where $c_L = \sqrt{E/\rho}$ is the (quasi-)longitudinal wave speed in the beam material, restricting the results to $\bar{\omega} \in [0, \pi]$. This frequency normalization is useful to represent the ratio between the unit cell length a and the longitudinal wavelength λ_L , obtained by recalling that $k_L = \omega / c_L = 2\pi / \lambda_L$, where k_L is the longitudinal wavenumber, which leads to the relation $\bar{\omega} / 2\pi = a / \lambda_L$.

We then numerically validate the dispersion relations obtained using the PWE method with the ones obtained using a finite element (FE)-based solution enforcing periodic boundary conditions [167]. For the FE computations, we use hexahedral linear solid elements [168]. These comparisons are shown in Fig. 4.3b using continuous lines and open circles for the results obtained using the FE and PWE formulations, respectively. For the FE-based results, we also show a polarization metric given by $p(u_x, u_y, u_z) = \int_V u_z^2 dV / \int_V (u_x^2 + u_y^2 + u_z^2) dV$, where V is the unit cell volume.

Considering the dispersion relations for the baseline case (without undulation, i.e., $d_0 = 0$), we notice (left panel in Fig. 4.3b) the occurrence of distinct branches corresponding to the flexural (F), in-plane shear (S), torsional (T), and longitudinal (L), behavior. The wave modes corresponding to each branch are indicated at the edge of the first Brillouin zone ($ka/\pi = 1$), using a red circle, blue diamond, red diamond, and blue circle, respectively, for the F, S, T, and L branches. Since the dynamic equations of the coupled normal and transverse behavior are two-dimensional (xz -

plane), the PWE method is only able to capture the flexural and longitudinal motions (i.e., branches whose wave modes are marked with circles).

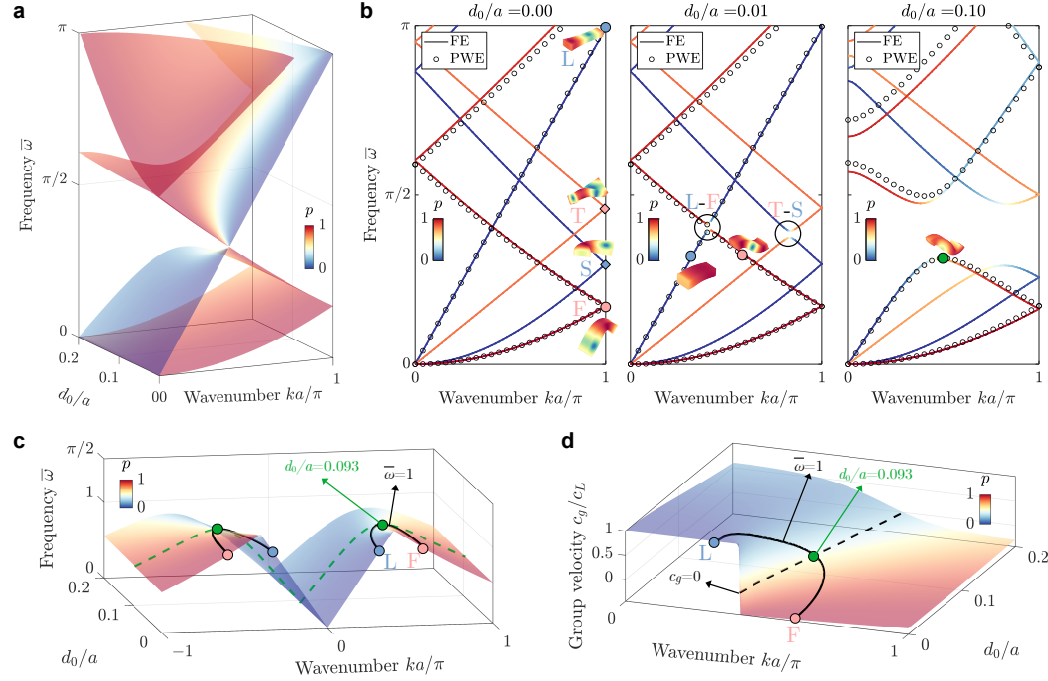


Fig. 4.3 Dispersion relations computed using the PWE and FE as a function of the normalized undulation height $d_0/a \in [0, 0.2]$ in the frequency range $\bar{\omega} \in [0, \pi]$. a. Dispersion curves computed with the PWE method with the out-of-plane polarization metric (p) ranging from 0 (purely longitudinal, blue) to 1 (purely transverse, red). b. Comparison of dispersion curves obtained using the PWE (black circles) and FE (colored lines) for increasing values of d_0/a . For $d_0/a = 0$ (left panel), highlighted branches are labeled as L (longitudinal), F (flexural), T (torsional), and S (shear), with corresponding wave modes shown at the edge of the first Brillouin zone ($ka/\pi = 1$). For $d_0/a = 0.01$ (middle panel), the longitudinal and flexural branches present locking (L–F) due to the coupling between in-plane ($p < 0.5$) and out-of-plane ($p > 0.5$) branches. A similar behavior is observed for the torsional and shear branches (T–S), only captured by the FE model. For $d_0/a = 0.10$ (right panel), a significant band gap is noticed between the coupled L–F (T–S) branches, with a polarization $p \approx 0.5$ at the local maximum of the coupled branch. c. Isofrequency line for $\bar{\omega} = 1$ (black line) indicated on the second dispersion branches of (a), with a maximum value (green circle) obtained at $d_0/a = 0.093$ (green dashed line). The blue and red circles correspond, respectively, to the longitudinal and flexural wave modes at a flat unit cell ($d_0/a = 0$). d. Normalized group velocity (c_g/c_L) computed along the isofrequency line $\bar{\omega} = 1$ (black line), indicating a zero group velocity ($c_g = 0$) at $d_0/a = 0.093$.

Next, we present the results obtained for $d_0/a = 0.01$ in Fig. 4.3b (middle panel). In this case, it is possible to notice a mode locking effect [169] (weak coupling between waves with same phase velocity and group velocities of opposite sign)

between the longitudinal and flexural (L–F) branches, which transition from the longitudinal to the flexural motion, for the lower branch (and vice-versa, for the upper branch). Indicative wave modes are also shown for $ka/\pi = 0.32$ and $ka/\pi = 0.60$ for the longitudinal (blue circle) and flexural wave modes (red circle), respectively. A similar locking effect is also noticed between the torsional and shear branches, which however are not represented by the PWE formulation. In this case, mode locking occurs below $\bar{\omega} = \pi/2$ ($a/\lambda_L < 1/4$), characterizing a sub-Bragg operation. We also show the dispersion relations obtained for $d_0/a = 0.10$, which present excellent agreement with the FE-based solution for $\bar{\omega} < \pi/2$ and relatively good agreement for $\bar{\omega} > \pi/2$. We also indicate a highly polarized longitudinal-flexural wave mode at $ka/\pi = 0.60$ (green circle). The results yielded by the PWE and FE methods present excellent agreement, with small divergences due to the limitations of the analytical equations used for the PWE formulation, which consider smooth curvatures (i.e., $d_0 \ll a$).

To obtain the undulation height (d_0/a) necessary to convert longitudinal to flexural waves at a given frequency, we intersect the second propagative branches shown in Fig. 4.3a with the isofrequency line corresponding to the desired operating frequency. Consider the illustrative case of $\bar{\omega} = 1$, shown in Fig. 4.3c (black line) combined with the second propagative branch in the first Brillouin zone ($ka/\pi \in [-1, 1]$). At this frequency, the longitudinal and flexural modes propagating in the flat structure ($d_0/a = 0$) are represented, respectively, using the blue and red circles (for both positive and negative wavenumbers). The point of maximum of the isofrequency curve (as projected onto the $k \times \omega$ plane) is located at $d_0/a = 0.093$, whose second propagative branch is highlighted as a green dashed curve and with a maximum indicated using a green circle. It is also interesting to recall that distinct points present varying values of group velocity along the isofrequency line, which are computed and shown in Fig. 4.3d, normalized with respect to the longitudinal wave velocity. The group velocity starts at $c_g/c_L = 1$, corresponding to purely longitudinal waves (blue circle), crosses the zero group velocity line $c_g/c_L = 0$ at $d_0/a = 0.093$ (green circle), and returns to $c_g/c_L = -0.41$ at $d_0/a = 0$ (red circle), corresponding to negative-propagating flexural waves. We note that the point of zero group velocity indicates the undulation height necessary for mode conversion. Additionally, by employing a graded undulated profile (see Fig. 4.2c, bottom panel), it is possible to achieve a smooth transition between longitudinal waves ($c_g > 0$) and the zero group velocity point ($c_g = 0$), which then leads to the reflection as flexural waves ($c_g < 0$).

However, it is important to note that such perfect conversion would require a graded structure with an infinite length to ensure a continuous transition between the group velocities of each contiguous unit cell, which is not realizable in practice.

Finally, we note that the discussed conversion mechanism concerns the conversion from longitudinal to flexural waves, which is straightforward due to their positive group velocity. In the case of conversion from flexural to longitudinal waves, additional modifications would be required to the structure in order to first excite flexural waves with a negative group velocity and then proceed to the conversion. This type of effect is thus not discussed in detail here.

4.3.2 Experimental verification of band gaps

To validate experimentally the computed dispersion relations, we verify the existence of a band gap in a beam having an undulated region with a profile described by $d_0/a = 0.093$, corresponding to a band gap in the frequency range $\bar{\omega} \in [1.0, 1.5]$. We note that the ratio between the unit cell length and the longitudinal wavelength in this case is given by $a/\lambda_L = 1/2\pi \approx 0.16$. Thus, considering a structure composed of 6 unit cells still allows to maintain subwavelength conditions ($6 \times a/\lambda_L \approx 0.95 < 1$). We then consider the structure represented in Fig. 4.4a, composed of an undulated region with n_c unit cells dividing two flat regions (i.e., $d_0/a = 0$) with lengths $n_l \times a$ and $n_r \times a$. Test specimens were then fabricated by metal powder sintering using Aluminum 5000 alloys (Al-Mg, EN AW 5056A) with a length of 300 mm, due to fabrication limitations. The flat and undulated regions are defined by the values $n_l = 14$, $n_c = 6$, and $n_r = 10$ unit cells. An example of a fabricated specimen is shown in Fig. 4.4b, zoomed at the undulated region. For the experimental validation, we attached a piezoelectric transducer to one side of the test specimen to excite longitudinal waves. A light and strongly damping material ("putty") was then applied to the opposite end of the specimen. This extremity rests on a layer of low-density foam fixed on a mechanical vise. This setup was chosen to prevent significant local stiffness increases at the specimen edges and to avoid non-linear contact phenomena, reducing transmitted vibrations and attenuating oscillations at the boundaries. The support is then mounted on a linear stage to measure transverse displacements along a 160 mm length using a data acquisition system by using a fixed laser Doppler vibrometer. The setup is shown in Fig. 4.4c.

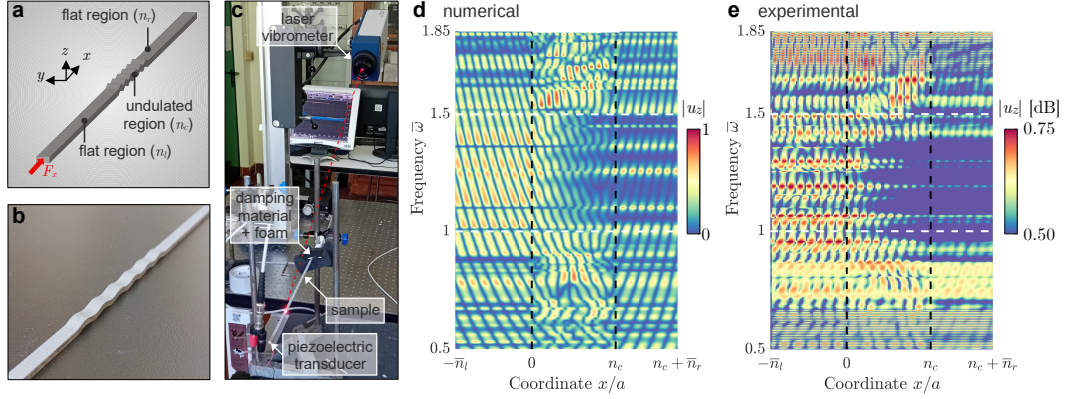


Fig. 4.4 Time harmonic simulations. a. Simulated system with two flat regions (lengths $n_l \times a$ and $n_r \times a$) divided by an undulated region (length $n_c \times a$). A time harmonic longitudinal force (F_x) is applied at the left end of the beam. b. Manufactured test specimen shown close to the undulated region. c. Experimental setup showing the fabricated sample with an attached piezoelectric transducer resting on a soft support and the laser Doppler vibrometer used to perform the measurements. d. Simulation results for normalized transverse displacements along the beam axis and the frequency range $\bar{\omega} \in [0.5, 1.85]$. The vertical dashed black lines indicate the undulated region ($x/a \in [0, n_c]$) and the horizontal dashed white lines indicate the band gap region. The colorbar represents absolute vertical displacements ($|u_z|$). e. Experimental results corresponding to the numerical simulation in (d), with the colorbar adjusted for visualization.

To verify the existence of band gaps, a harmonic signal is applied using a longitudinal force at the left edge (equal for all FE nodes), performing a sweep in the frequency range $\bar{\omega} \in [0.5, 1.85]$ (corresponding to $f \in [40, 150]$ kHz) and computing displacements in the transverse direction at the top surface of the beam using free boundary conditions at the other faces. We note that these boundary conditions are an approximation with respect to the experimental setup due to the small stiffness introduced by the foam and mechanical vise assembly. Results from FE simulations are indicated in Fig. 4.4d, with normalized displacements shown in absolute values in the range $[-\bar{n}_l, n_c + \bar{n}_r]$, with $\bar{n}_l = 5.5a$ and $\bar{n}_r = 4.5a$, chosen in accordance with experimental restrictions. Within the band gap (marked with horizontal dashed white lines), standing waves are formed within the flat regions (to the left, $x/a < 0$, and to the right, $x/a > n_c$, of the undulated region), while a significant attenuation is observed within the undulated region ($x/a \in [0, n_c]$, marked using vertical dashed black lines) from left to right.

Experimental results are shown in Fig. 4.4e, using a logarithmic scale for the resulting fast Fourier transforms (FFT) of the measured transverse displacements,

restricting their values to the $[0.50, 0.75]$ range to facilitate visualization. In this case, although the band gap is slightly shifted downwards in frequency, at around $\bar{\omega} \in [0.9, 1.4]$ (i.e., an approximate 10% decrease). The observed minor discrepancy in band gap frequencies is probably due to issues related to specimen fabrication and to the experimental setup: on the one hand, there is some uncertainty on material properties due to the induced porosity in the sintering process [170, 171]; on the other hand, experimental boundary conditions do not exactly reproduce ideal ones used in simulations (i.e., use of a mechanical vise to fix the putty material and foam assembly). Nonetheless, experimental results are still in very good agreement with numerical data. We also notice a particularly good correlation between numerical and experimental data in the undulated region ($x \in [0, n_c]$) outside the band gap region, i.e., for $\bar{\omega} < 1.0$ and $\bar{\omega} > 1.5$.

Finally, we note that in the usual conversion mechanisms presented in literature, both types of waves present the same propagation direction. In our proposed structures, however, longitudinal and flexural waves present an inversion of the propagation direction (i.e., opposing group velocity signs), enabling wave conversion through reflection. Time-harmonic simulations do not allow for a clear appreciation of the wave conversion mechanism due to reflections of waves at the boundaries. For this reason, in the next section we verify the wave conversion mechanisms using time-transient simulations and experiments.

4.4 Wave conversion application

4.4.1 Numerical investigation using a constant undulation profile

We now proceed to verify numerically the longitudinal-flexural wave conversion functionality of the structure presented in the last section. For the numerical time-transient simulations, two perfectly matched layers (PMLs), with a length of $n_{\text{PML}} \times a$ each, are added to both ends of the structure (purple region in Fig. 4.5a) to minimize reflections. We note here that the PMLs act as ideal absorbing boundaries, which do not accurately represent the experimental boundary conditions (see Fig. 4.4c)), but instead, allow the calculation of ideal transmission and reflection coefficients for the undulated structures. We consider the FE implementation as described in [172] with $n_{\text{PML}} = 15$. Additionally, we consider two surfaces of interest, labeled as S_1

and S_2 (green and orange surfaces, respectively, represented in Fig. 4.5a), which pass through the geometrical centers of the elements adjacent to each PML region, thus presenting a negligible distance to the PMLs (≈ 0.25 mm) with respect to the wavelength. These surfaces are used to investigate the displacements and mechanical energy flux, in which case the evaluation of internal stresses at Gauss points are required to minimize numerical errors associated with the FE implementation [168]. We then apply a longitudinal force following a signal with central frequency $\bar{\omega} = 1$ (fundamental period $T = 1/f = 12.34$ μ s) and normalized amplitude considering 5 pulses modulated by a Hanning window (inset of Fig. 4.5a). The time-transient simulations are performed considering a time-step incremental analysis using the Extended Newmark Method [172], with a total time duration of $t/T = 20$ and an incremental time step $\Delta_t = 0.5$ μ s ($T/\Delta_t \approx 25$).

In order to quantify the efficiency of wave conversion, we analyze it from a quantitative point of view by computing the corresponding elastic energy flux across surfaces S_1 and S_2 . The ratio of reflection or transmission between longitudinal and flexural waves are obtained using

$$\begin{aligned} R_f &= -\frac{\int \phi_f^-(S_1) dt}{\int \phi_l^+(S_1) dt}, \\ R_l &= -\frac{\int \phi_l^-(S_1) dt}{\int \phi_l^+(S_1) dt}, \\ T_l &= \frac{\int \phi_l^+(S_2) dt}{\int \phi_l^+(S_1) dt}, \end{aligned} \quad (4.13)$$

where $\phi_l(S_i)$ ($\phi_f(S_i)$) is the instantaneous longitudinal (flexural) power (surface integral of the elastic energy flux) at the surface S_i , $i = \{1, 2\}$, and the $+$ ($-$) superscript denotes energy flux in the positive (negative) direction. The derivation of these fluxes is given in Appendix ??.

These quantities are computed for surfaces S_1 and S_2 and shown (normalized) in Fig. 4.5b. We notice that $\phi_l(S_1)$ presents an inversion in sign due to its reflection at the interface between the flat and undulated regions. The transmitted power of longitudinal waves S_2 and reflected power of flexural waves S_1 , indicated, respectively, as $\phi_l(S_2)$ and $\phi_f(S_1)$ are also shown. The reflection (flexural and longitudinal waves) and transmission (longitudinal waves) coefficients are respectively computed as $R_f = 0.56$, $R_l = 0.09$, and $T_l = 0.33$, thus demonstrating that most of reflected

energy is converted in the form of flexural waves. Also, a non-negligible small energy fraction is reflected in the form of longitudinal waves due to the impedance mismatch between the undulated and the flat regions. We also note that, although the value of R_f does not seem particularly high, the ratio of peak absolute values between transverse and longitudinal displacements at S_1 is equal to 0.92 for the beam centerline (see Appendix??), which indicates that incident longitudinal waves can be used to generate flexural waves with a similar amplitude.

The corresponding computed displacements are illustrated in Fig. 4.5c for increasing time instants $t/T = 2.5, 5.0$, and 7.5 , with absolute normalized values represented in the colour scale. It becomes clear that an incident longitudinal wave is mostly reflected as flexural waves and marginally transmitted as longitudinal waves.

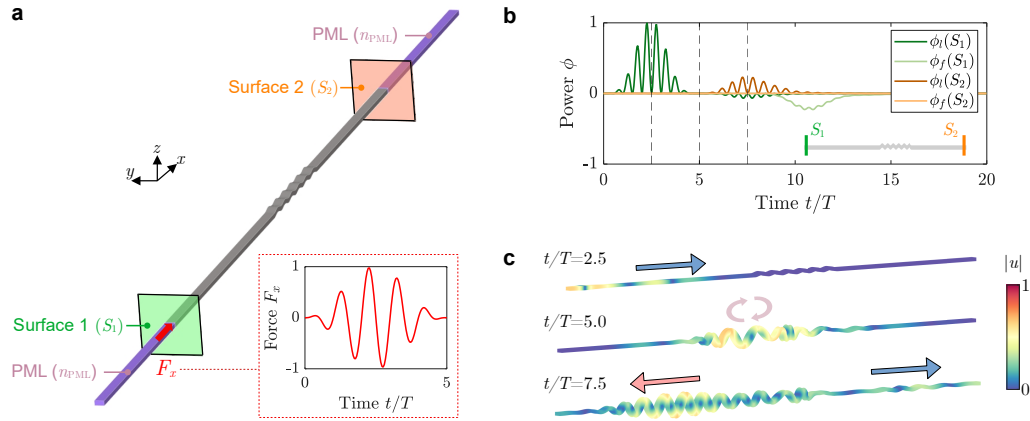


Fig. 4.5 Numerical simulation of a longitudinal transient pulse applied to a beam with an undulated region. a. A longitudinal force (F_x) is applied using a modulated sinusoidal pulse (inset). Surfaces S_1 and S_2 (in green and orange, respectively), located at the edges of the flat regions, are considered for the analysis of mechanical energy flux. Two PMLs (lengths $n_{\text{PML}} \times a$) are included at the outer edges of each flat region. b. Longitudinal (ϕ_l) and flexural elastic power (ϕ_f) computed at the surfaces S_1 (green) and S_2 (orange). The sign indicates forward- (> 0) or backward-propagating energy (< 0). The dashed vertical lines indicate the time instants $t/T = 2.5, 5.0$, and 7.5 . c. Absolute normalized displacements represented at the time instants indicated by the dashed lines in (b). The colorbar represents absolute displacements ($|u|$). The blue and red arrows represent longitudinal and flexural waves, respectively. The curved arrows indicate the occurrence of wave conversion.

For the sake of comparison, we perform the same analyses for the operating frequency $\bar{\omega} = 0.85$. The ratio $d_0/a = 0.18$ is obtained as the point of maximum for the curve intersecting the surface corresponding to the second branch at this frequency. In this case, the ratio between the unit cell length and the longitudinal

wavelength is given by $a/\lambda_L = 0.85/2\pi \approx 0.14$, which can accommodate $n_c = 7$ unit cells and still remain in the subwavelength operating range ($7 \times a/\lambda_L \approx 0.95 < 1$). We also set $n_l = 13$, keeping $n_r = 10$. We omit the graphical representation of these results for the sake of brevity. Instead, we only present here the transmission and reflection coefficients, which are obtained as $R_f = 0.59$, $R_l = 0.26$, and $T_l = 0.10$. Thus, there is an improvement in the flexural reflection coefficient due to the increase in both the d_0/a ratio and the number of unit cells. There is also an increase in the reflection coefficient and a decrease in the transmission coefficient of flexural waves, both owing to the larger impedance mismatch between the flat and undulated regions.

We note that the literature presents works which use (i) locally-resonant based systems to achieve the conversion between Lamb and shear [145], longitudinal and shear [146], flexural and torsional [149], torsional and flexural [150], or longitudinal and torsional waves [151], or (ii) structured media that do not rely on local resonances to achieve the conversion between longitudinal and shear [152, 153] or torsional modes [154]. We also note that an example of perfect mode conversion from flexural to longitudinal waves can be found in literature [147], but not the opposite (longitudinal to flexural). Thus, to the best of our knowledge, the coupling between longitudinal and flexural waves in periodic waveguides is mentioned in literature [158, 155], but its applicability for wave conversion purposes is not yet exploited. For this reason, a comparison between reflection and transmission coefficients is not direct, since different phenomena are quantified. Furthermore, our proposed system is based on a monolithic design, which facilitates its fabrication, not involving the use of locally resonant structures or intricate designs which could hinder its fabrication.

It is important to note that, although rigorous, the previous analyses perform numerical integrations of stresses and velocities on the beam cross-section, which is not experimentally feasible. From a practical point of view, we are mostly interested in measurable quantities, such as displacements and velocities. For this reason, we proceed to validate our findings experimentally by analyzing the time history of longitudinal and transverse displacements on the top surface of the beam.

4.4.2 Experimental validation using a constant undulation profile

Fig. 4.6a shows the computed normalized longitudinal and transverse displacements using normalized spatial (x/a) and time coordinates (t/T), with dashed black lines indicating the undulated region ($x/a \in [0, n_c]$). The longitudinal displacements (u_x , left panel) indicate an impinging longitudinal wave (bottom left blue arrow) being partially reflected at the interface between the flat and the undulated regions (top left blue arrow). In addition, part of the energy is converted as reflected flexural waves (red arrow), while another part is transmitted as longitudinal waves (right blue arrow). Although these refer to longitudinal displacements, the distinction between longitudinal and flexural waves can be made due to the differences in propagation velocities. Likewise, the transverse displacements (u_z , right panel) indicate a wavefront associated with reflected flexural waves (red arrow on the left), while the transmitted flexural waves (red arrow on the right) are barely noticeable. The transverse displacements associated with the Poisson's effect (i.e., due to longitudinal displacements) are negligible and cannot be distinguished.

We also perform two-dimensional fast Fourier transforms (2D-FFTs) on these data, obtaining a representation in the wavenumber-frequency space ($k \times \omega$). The numerical data are zero-padded (total of 2^{11} points) to improve the FFT resolution. The obtained surfaces are represented in Fig. 4.6b with the normalized coordinates ka/π and $\bar{\omega}$, using normalized absolute FFT values. We also plot, for reference, the analytical dispersion curves for longitudinal and flexural waves, respectively, using dashed blue and red curves. The observations concerning the time histories are confirmed by the 2D-FFT, where it is possible to notice the existence of incident ($k > 0$) longitudinal waves at $\bar{\omega} = 1$ (top panel, blue curve), which are reflected ($k < 0$) as longitudinal (blue curve) and flexural waves (red curve). Likewise, the 2D-FFT of transverse displacements (bottom panel) indicate that the reflected flexural waves ($k < 0$, red curve) are dominant with respect to transmitted flexural waves ($k > 0$).

An analogous analysis is then performed considering data obtained experimentally using the setup presented in Fig. 4.4c. We observe that although the putty material is attached to the specimen in the experimental setup, its attenuation is significantly smaller than that provided by numerical PMLs, resulting in more significant edge reflections. To this end, the vibrometer performs two sets of measurements over a 200 mm length, using (i) a grazing angle to obtain longitudinal displacements

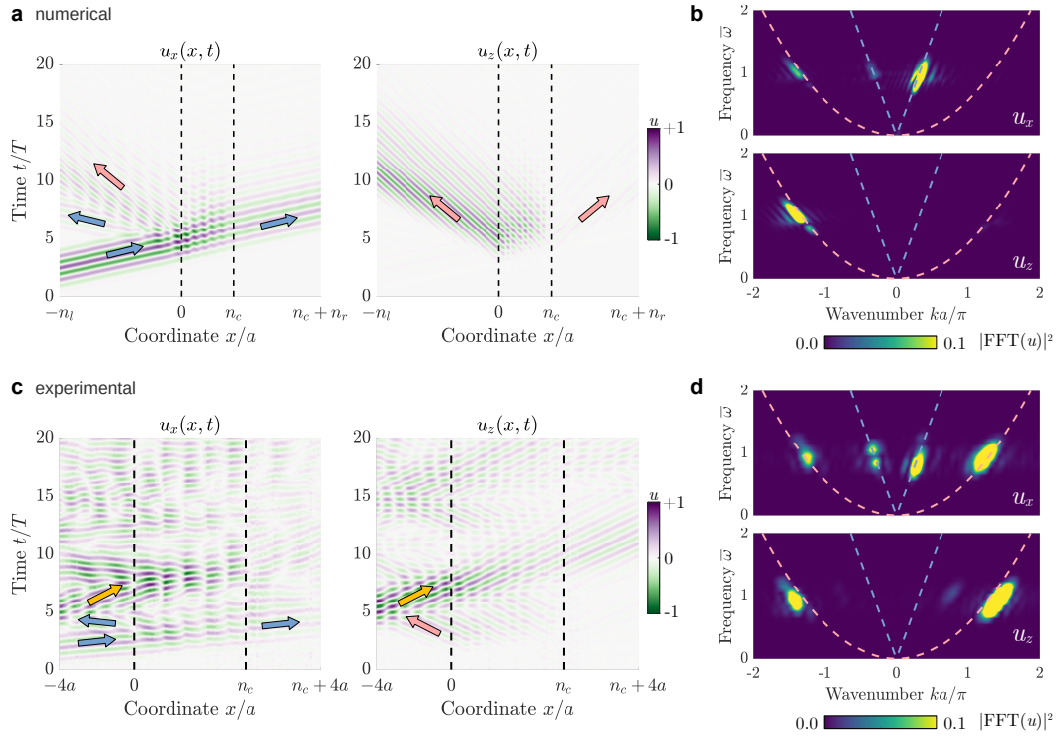


Fig. 4.6 Numerical simulation and experimental validation of longitudinal to flexural wave conversion using a beam with a constant undulation profile. a. Computed longitudinal (left panel, u_x) and transverse (right panel, u_z) displacements at the top surface of the beam. Vertical dashed lines indicate the undulated region. Blue and red arrows represent, respectively, longitudinal and flexural wavefronts. The colorbar indicates normalized displacements (u). b. 2D-FFTs corresponding to the longitudinal (u_x , top panel) and transverse (u_z , bottom panel) displacements shown in (a). Blue and red dashed lines indicate, respectively, longitudinal and flexural analytical dispersion branches at the flat region. The colorbar indicates normalized 2D-FFT values ($|\text{FFT}(u)|^2$). (c,d). Same as (a,b), obtained experimentally.

(u_x) and (ii) a normal angle to obtain transverse displacements (u_z). Fig. 4.6c presents these results (u_x , left panel, and u_z , right panel) for the time interval $t/T \in [0, 20]$. We note that due to positioning errors associated with the measured area (which is smaller than the beam length), we only represent a region corresponding to the obtained data which is symmetric with respect to the undulations (i.e., centered at $x/a = n_c/2$). The measured longitudinal displacements indicate the expected incident (bottom left blue arrow), transmitted (right blue arrow), and reflected longitudinal wavefronts (top left blue arrow). We also observe an additional flexural incident wavefront (yellow arrow), which can be distinguished due to its group velocity, which is smaller than the one associated with longitudinal waves, as revealed

by the inclination of the arrow. The existence of this incident flexural wave is caused by the spurious excitation of a flexural mode caused by the piezoelectric transducer attached longitudinally to the test specimen. Nonetheless, the wave conversion phenomenon can still be observed when analyzing the transverse displacements, where it is possible to notice reflected flexural waves (left red arrow). In this case too, the incident flexural wave is observed (yellow arrow), which possesses the same group velocity as the wave indicated by the red arrow with an opposing propagation direction. For $t/T > 10$, additional negative-going flexural waves are observed due to the subsequent incident longitudinal waves that impinge upon the undulated region after reflecting at the left edge of the test specimen.

The representation of this temporal data in the reciprocal space is obtained by performing a 2D-FFT on the displacements time history with an additional Hanning window in the time domain $t/T \in [0, 10]$ to minimize the effect of additional reflections. The resulting surfaces are also zero-padded to improve the FFT resolution. These results are shown in Fig. 4.6d. Concerning longitudinal displacements (u_x , top panel), the main differences with respect to the numerical simulation (see Fig. 4.6b) are increases (i) in the occurrence of backward-propagating waves ($k < 0$) caused by reflections at the right edge of the specimen and (ii) in the components associated with forward-propagating flexural waves (red dashed curve, $k > 0$) caused by the spurious mode excitation. In both cases, these effects are not observed in the numerical simulation due to the perfect longitudinal excitation and the presence of PMLs to reduce internal reflections. On the other hand, the main difference between numerical and experimental results (u_z , bottom panel) is the occurrence of forward-propagating flexural waves (red dashed curve, $k > 0$) caused by the spurious mode excitation. Also, due to the reduced time window considered for the 2D-FFT, the backward-propagating flexural waves (red dashed curve, $k < 0$) are associated with the conversion between longitudinal and flexural waves, thus demonstrating the wave conversion mechanism. A slight frequency shift is also observed, of the order of 5% to 10%, which is in agreement with the shift observed in Fig. 4.4e.

We conclude by noting that the use of a constant undulation profile yields significant reflection of longitudinal waves. Therefore, we next evaluate the use of a graded undulation profile as an alternative to reduce the reflection of longitudinal waves.

4.4.3 Numerical investigation using a graded profile

The same numerical analysis is now performed considering a graded undulated profile, where d_0/a presents a linear variation up to 0.093, distributed along 6 unit cells in the undulated region. The displacement history for longitudinal and transverse directions is presented in Fig. 4.7a. Concerning the longitudinal displacements (u_x , left panel), it is interesting to notice that no reflected longitudinal waves (blue arrows) are distinguishable. On the other hand, the time history of transverse displacements (u_z , right panel) indicates that the peak displacement values are achieved for larger time instants with respect to the beam with a constant undulated profile (when compared to the constant profile, see Fig. 4.6a, right panel), which is explained by the gradual modulation of group velocity inside the undulated region. These results are confirmed by their 2D-FFTs, shown in Fig. 4.7b, which show that the reflection of longitudinal waves is barely visible (u_x , top panel). The ratio between the peak values for the transverse and longitudinal displacements is computed as 0.37, indicating a decrease with respect to the constant undulation profile.

Finally, we compute the reflection and transmission coefficients, omitting the curves corresponding to mechanical power for the sake of brevity. The reflection (flexural and longitudinal waves) and transmission (longitudinal waves) coefficients are computed as $R_f = 0.41$, $R_l \approx 0$, and $T_l = 0.55$. We notice that the reflection coefficient for longitudinal waves is in good agreement with the 2D-FFT results. These results also demonstrate that the graded profile presents a decrease in performance with respect to flexural wave conversion when a short undulation length is considered, which is compensated, however, by the decrease in reflection of longitudinal waves. This design option thus provides a viable means of producing significant levels of flexural waves through the incidence of longitudinal waves with negligible reflection of longitudinal waves.

4.5 Conclusions

In this Chapter, we proposed the use of phononic crystals for longitudinal-flexural wave conversion applications using beams with periodic and graded undulated profiles. The design of these systems is performed by analyzing the dispersion relation using a semi-analytical method (PWE) based on the dynamic equations relating

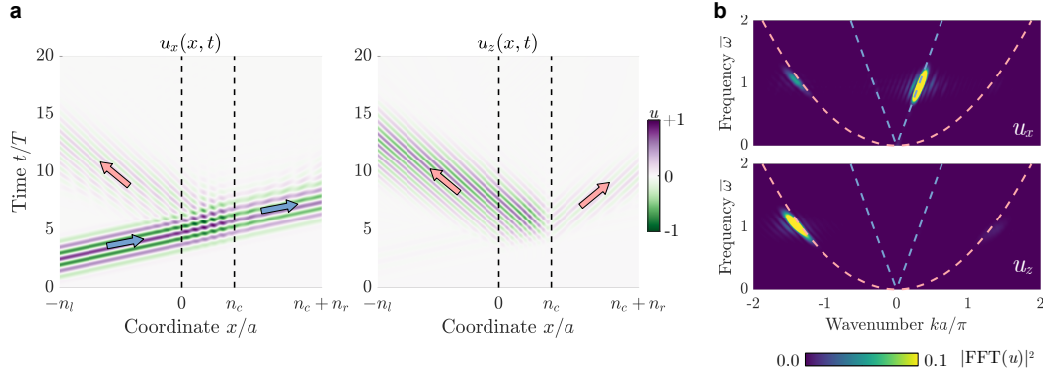


Fig. 4.7 Numerical simulation of longitudinal to flexural wave conversion using a beam with a graded undulation profile. a. Computed longitudinal (left panel, u_x) and transverse (right panel, u_z) displacements at the top surface of the beam. Vertical dashed lines indicate the undulated region. Blue and red arrows represent, respectively, longitudinal and flexural wavefronts. The colorbar indicates normalized displacements (u). b. 2D-FFTs corresponding to the longitudinal (u_x , top panel) and transverse (u_z , bottom panel) displacements shown in (a). Blue and red dashed lines indicate, respectively, longitudinal and flexural analytical dispersion branches at the flat region. The colorbar indicates normalized 2D-FFT values ($|\text{FFT}(u)|^2$).

longitudinal and transverse motion, also considering the effects of shear strain and rotational inertia. These dispersion relations present an excellent agreement with equivalent FE-based solutions. The results demonstrate that the curvature of the beam profile (i.e., undulation) can be rationally designed to obtain maximum coupling between the otherwise decoupled longitudinal and flexural branches of the dispersion diagram. We extend previous studies done on similar systems, which also achieved sub-Bragg band gaps, to include an investigation on the evolution of the polarization states involving flexural and longitudinal behavior. This allows to verify that the induced mode coupling promotes a shift from longitudinal to flexural behavior in the same branch through wave mode locking that occurs in a sub-Bragg regime.

This phenomenon can be exploited to design waveguides whose length is similar to the wavelength of longitudinal waves in the flat region of the material. For a given operating frequency, a finite sample was proposed and its computed band gap experimentally demonstrated using a test specimen fabricated in Aluminum. We then investigated the wave conversion capabilities of this waveguide with an undulated region of constant undulation profile using a longitudinal force applied as a modulated pulse and showing that most of the energy is reflected as flexural

waves. Also, a small portion of the longitudinal wave energy is reflected due to the impedance mismatch between the flat and undulated regions. This behavior was numerically and experimentally demonstrated, confirming the wave conversion mechanism in the wavenumber-frequency space, also demonstrating that a graded profile is not strictly necessary to achieve wave mode conversion. Finally, a beam with a graded undulated region is proposed to minimize the reflection of longitudinal waves, which is practically reduced to zero. As a consequence of the smaller number of unit cells with the target undulation value, the conversion to flexural waves is also diminished, although remaining significant.

Interesting directions for future investigations are the functional behavior of corrugated surfaces in biological systems such as shells and their bio-inspired applications [173]. Likewise, the application of undulated surfaces to achieve wave conversion for large scale vibration protection applications and the creation of meta-surfaces for topological waveguiding are additional promising directions.

Chapter 5

Wave propagation in a continuously deforming medium

5.1 Introduction

The propagation of mechanical waves in elastic media is inherently influenced by the mechanical state of the material through which they travel. This coupling is most prominently captured by the Acousto-Elastic (AE) effect, where the velocity of small-amplitude waves depends on the presence of finite stresses or deformations. The origin of this phenomenon lies in the nonlinear elastic response of solids, first systematically described by Murnaghan [174], and experimentally highlighted by Hughes and Kelly [175], who demonstrated that wave speeds vary measurably with applied stress.

A rigorous theoretical framework for this effect was later established through the theory of incremental deformations developed by Biot [176], which describes the propagation of small disturbances superimposed on a finitely deformed configuration. Within this approach, the effective elastic moduli governing wave motion explicitly depend on the underlying deformation, leading to a direct dependence of wave characteristics on the mechanical state. Early extensions of this framework to waveguides and structural elements were provided by Thurston and Brugger [177], while a comprehensive treatment within nonlinear elasticity is given by Ogden [178]. Together, these works establish the AE effect as a fundamental mechanism linking wave propagation and material deformation.

In recent years, the AE effect has gained renewed attention in the context of soft materials, where large and controllable deformations can be easily achieved. Experimental studies have shown that soft elastomers provide an ideal platform for observing and exploiting this coupling. In particular, Delory *et al.* [179] demonstrated that guided elastic waves in soft solids can be strongly tuned through pre-stretch, with significant modifications of the dispersion relations and modal structure. This behavior was further explored in [180], where highly stretched plates were shown to exhibit pronounced changes in wave propagation characteristics, offering a direct and controllable realization of AE effects. Complementary approaches based on ultrasonic measurements have also been used to quantify nonlinear elastic constants in soft solids, highlighting the practical relevance of acoustoelasticity for material characterization [181].

Despite these advances, the key point of the AE effect relies on a simplifying assumption: the deformation is static, or varies slowly enough that the medium can be treated as effectively time-invariant during wave propagation. Under this hypothesis, the deformation acts as a fixed background that modifies the local wave speed, but does not introduce any explicit time dependence in the governing equations. As further development of the AE effect, Dynamic Acoustoelastic Testing (DAET) is, instead, a non-destructive, vibro-acoustic method which uses a low-frequency, high-amplitude "pump" wave to generate dynamic strain, and a high-frequency, low-amplitude "probe" wave to measure changes in wave speed [182–184]. This technique is sensitive to material softening, damage, and hysteresis. However, many systems of interest—particularly in soft matter—exhibit deformations that evolve rapidly in both space and time, often in the form of localized fronts or propagating transitions. For instance, the viscoelastic response of a soft strip, undergoing large deformation, has been shown to involve complex transient dynamics and spatially non-uniform strain fields [185].

When the deformation becomes explicitly time-dependent, the separation between wave dynamics and material evolution breaks down. The medium can no longer be regarded as stationary; instead, it behaves as a space–time varying system in which the propagation of waves is continuously influenced by the evolving configuration. This viewpoint is made explicit in recent experiments by Delory *et al.* [186], where elastic wave packets interacting with a moving interface were shown to undergo frequency conversion and asymmetric transmission, key signatures of space–time modulation. In this context, the deformation itself acts as a mechanism

for dynamically modulating the effective material properties experienced by the wave.

These observations reveal a fundamental limitation of the classical AE framework: while it successfully captures the dependence of wave propagation on a prescribed deformation, it does not account for the dynamics of the deformation itself. The objective of this chapter is therefore to move beyond the experimental observation of space–time scattering in deformable media and establish a predictive theoretical framework for wave propagation in systems undergoing finite, space– and time-dependent deformations. In particular, we consider situations in which the deformation evolves in the form of sharp transitions or propagating fronts (Fig. 5.1a and 5.1b), giving rise to effective space–time modulations of the medium properties. By explicitly incorporating the deformation kinematics and mechanics into the governing equations, the proposed model captures not only the interaction of waves with a moving interface, but also the dynamics responsible for the generation and evolution of the interface itself. This enables the description of how evolving configurations influence wave velocity, transmission, reflection, frequency conversion, and energy transport. More broadly, the framework provides a unified connection between nonlinear elasticity, wave propagation, and space–time varying media, extending previous experimentally driven studies toward a general continuum-mechanics-based description with predictive numerical and experimental validation, particularly suited to soft elastomeric systems.

In the following, we show that deformation-induced interfaces naturally realize this class of space–time scattering problems. In particular, the propagation of a deformation front in an elastic medium generates a moving boundary across which both the mechanical state and the wave propagation properties change, leading to coupled frequency–wavenumber transformations.

5.2 Space-time interface through deformation

The analytical framework developed in this chapter is formulated for a slender elastic beam. First, we consider the longitudinal deformation of the beam and derive the displacement field associated with the imposed boundary motion. This analysis allows us to describe how a continuous axial deformation generates a propagating

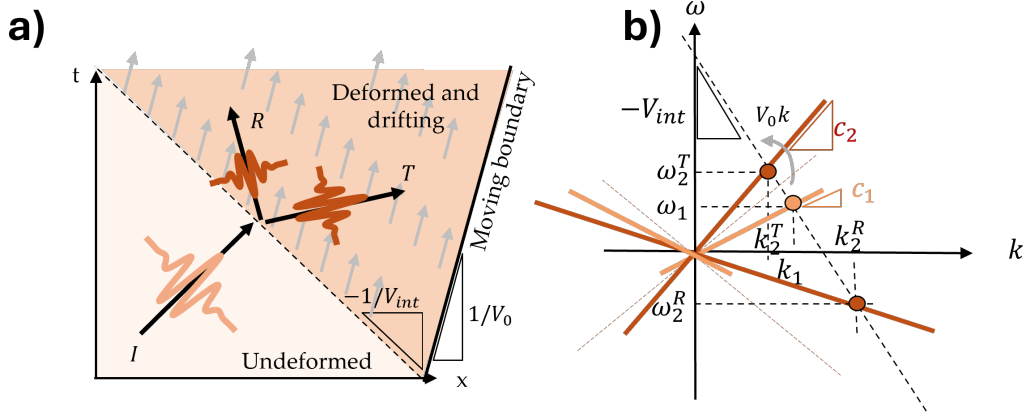


Fig. 5.1 Schematic of (a) wave scattering at a space-time varying interface in a deforming medium in 1D, where one boundary of the medium is displaced at an imposed velocity causing the propagation of a deformation front. The gray arrows behind the front represent the translation velocity of the particles of the medium. (b) Corresponding 2D spectra representation showing frequency and wave-number conversion linked by the phase conservation relation represented by the black dashed line. The brown dashed lines correspond to the dispersion relation of the deformed medium at rest, which due to the drift velocity of the medium are rotated by a Doppler shift $V_0 k$, where V_0 is the velocity of the moving boundary.

transition layer, which can be interpreted as a deformation-induced space–time interface separating regions with different mechanical Parameters.

5.2.1 Continuously deforming beam

The generation of a space time interface separating differently stretched regions of the material can be achieved through continuous deformation of the body. Let us now consider a simple case of a semi-infinite 1D bar in the linear elastic approximation. The equation of motion for the system, without any external applied load and neglecting the effect of body forces, is:

$$\frac{\partial^2 U}{\partial t^2} = c_L^2 \frac{\partial^2 U}{\partial x^2}, \quad (5.1)$$

where $U = U(x, t)$ is the longitudinal displacement and c_L is the longitudinal wave velocity. If the system is initially unperturbed, the initial conditions of the

problem are:

$$U(x, 0) = \lambda_0 x \quad (5.2)$$

$$\frac{\partial U}{\partial t}(x, 0) = 0. \quad (5.3)$$

where λ_0 is an initial pre-stretch of the medium. The system is perturbed at its boundary by a displacement of the form:

$$U(L, t) = f_L(t) = V_0 \tau \ln \left[1 + e^{(t-t_0)/\tau} \right], \quad (5.4)$$

where

$$V_0 = \lim_{t \rightarrow \infty} \frac{\partial U}{\partial t} \Big|_{X=L} = \lim_{t \rightarrow \infty} \frac{V_0}{1 + e^{(t-t_0)/\tau}} \quad (5.5)$$

is the boundary velocity and $\frac{\partial U}{\partial t}$ is the logistic function. By convention, we indicate an elongation when $V_0 > 0$ and a contraction when $V_0 < 0$. τ is a characteristic time. In addition, this function is a continuous approximation of a ramp function (fig.5.2a,b) together with its first and second derivatives, which correspond to velocity and acceleration, respectively. This boundary value problem can be solved through the use of the Laplace transform, which yields the solution:

$$U(x, t) = V_0 \tau \ln \left[1 + e^{\frac{1}{\tau} \left(t - t_0 - \frac{L-x}{c_L} \right)} \right] H \left(t - t_0 - \frac{L-x}{c_L} \right) + (\lambda_0 - 1)x. \quad (5.6)$$

where H is the Heaviside function. The displacement field defines a traveling transition layer governed by the characteristic variable

$$\xi = t - t_0 - \frac{L-x}{c_L}. \quad (5.7)$$

The condition $\xi = 0$ identifies the moving interface

$$x_f(t) = L - c_L(t - t_0), \quad (5.8)$$

which propagates along the medium with velocity c_L (fig.5.2c). The presence of the Heaviside function enforces causality, such that the displacement is nonzero only for $\xi > 0$. As a result, the domain is partitioned into two regions (fig. 5.2d): an undeformed region ahead of the front ($\xi < 0$) and a deformed region behind it ($\xi > 0$). The axial strain (fig.5.2b) is given by

$$\varepsilon(x,t) = \frac{\partial U}{\partial x} = \frac{V_0}{c_L} \frac{1}{1 + e^{-\xi/\tau}} H(\xi) + \lambda_0 - 1, \quad (5.9)$$

showing that the interface separates an unstretched region from a uniformly stretched region with asymptotic value $\varepsilon = V_0/c_L$. Similarly, the particle velocity (fig. 5.2c) is obtained as

$$u(x,t) = \frac{\partial U}{\partial t} = \frac{V_0}{1 + e^{-\xi/\tau}} H(\xi), \quad (5.10)$$

which corresponds to a smooth sigmoidal transition between two constant states. In particular, one has $u = 0$ ahead of the front and $u \rightarrow V_0$ behind it. Therefore, the solution describes a propagating kinematic interface connecting two homogeneous mechanical states: a quiescent, undeformed region and a uniformly moving, stretched region. The transition between these states is not discontinuous but occurs over a finite thickness controlled by the parameter τ , which regularizes the front and yields a smooth profile. This structure can thus be interpreted as a traveling interface separating two regions with distinct kinematic properties.

The longitudinal deformation field obtained above induces spatial and temporal variations of the geometric and mechanical properties of the beam, such as cross-sectional area, inertia, density, and axial stress. These variations directly affect the propagation of transverse waves, effectively transforming the beam into a space–time varying medium.

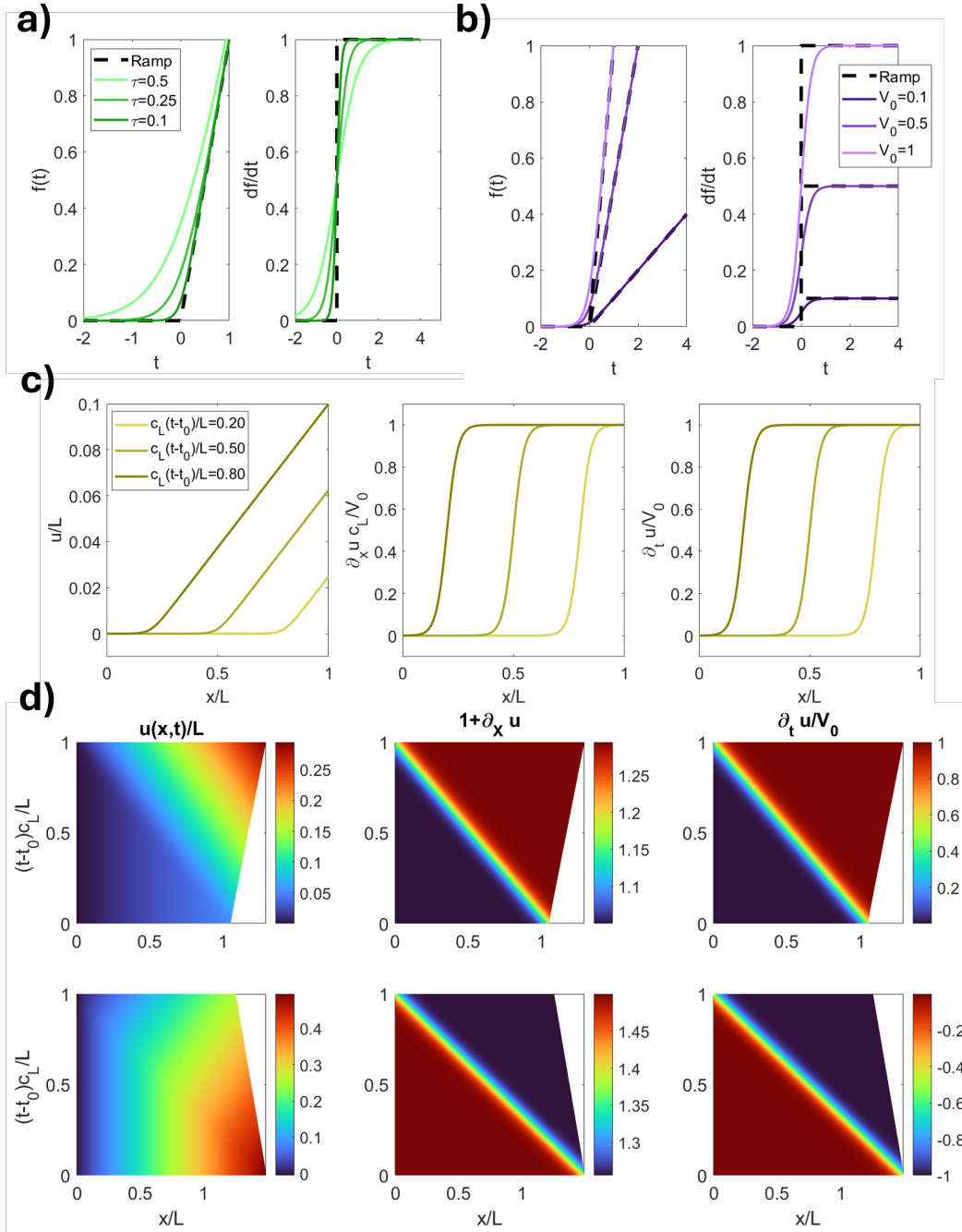


Fig. 5.2 (a) Soft ramp function $f(t)$ (left) and its time derivatives (right) for different smoothing parameters τ , compared to the ideal discontinuous ramp (dashed). (b) Alternative parametrization in terms of characteristic velocity V_0 , showing the same quantities $f(t)$ and df/dt . (c) Corresponding spatial profiles of normalized displacement u/L , strain $\partial_x u$, and velocity $\partial_t u$ along the bar at different times, highlighting the propagation of a smooth deformation front. (d) Space-time maps of displacement, stretch $1 + \partial_x u$, and velocity $\partial_t u$, illustrating the transition layer traveling through the medium and the associated gradients induced by the smoothed ramp loading for a positive (first row) or negative (second row) value of V_0 , and $\lambda_0 = 0.05$ and $\lambda_0 = 0.5$, respectively.

5.3 Waves crossing deformation induced space-time interface

We now investigate the propagation of transverse waves along the same beam, modeled within the Euler–Bernoulli approximation. The deformation field obtained from the longitudinal analysis is used to define the effective properties of the medium, leading to a space–time varying system in which transverse waves interact with a moving interface. This approach enables us to first characterize the kinematic mechanism responsible for the formation of the interface and then analyze its impact on wave propagation within a consistent beam framework.

5.3.1 Transverse wave propagation in the undeformed region

In the undeformed part of the strip (fig. 5.3a), the medium is at rest and is characterized by constant properties, corresponding to the region ahead of the deformation front. :

$$A_1, \quad I_1, \quad \rho_1, \quad \sigma_1,$$

where A_1 is the cross-sectional area, I_1 is the second moment of area, ρ_1 is the mass density, and σ_1 is the axial prestress.

Under the Euler–Bernoulli assumption, the transverse displacement $W(x, t)$ satisfies [187]:

$$\rho_1 A_1 \frac{\partial^2 W}{\partial t^2} - \sigma_1 A_1 \frac{\partial^2 W}{\partial x^2} + EI_1 \frac{\partial^4 W}{\partial x^4} = 0. \quad (5.11)$$

Seeking harmonic solutions of the form

$$W(x, t) = \hat{W} e^{i(kx - \omega t)}, \quad (5.12)$$

and substituting into Eq. (5.11), one obtains

$$-\rho_1 A_1 \omega^2 + \sigma_1 A_1 k^2 + EI_1 k^4 = 0, \quad (5.13)$$

from which the dispersion relation follows:

$$\omega = \pm \sqrt{\frac{EI_1}{\rho_1 A_1} k^4 + \frac{\sigma_1}{\rho_1} k^2}. \quad (5.14)$$

Since the medium is stationary, reciprocity holds:

$$\omega(k) = \omega(-k). \quad (5.15)$$

5.3.2 Transverse wave propagation in the deformed region with transport velocity

In the stretched region, the beam is not only prestressed but also transported with axial drift velocity V_0 (fig. 5.3a). The corresponding effective properties are denoted by

$$A_2, \quad I_2, \quad \rho_2, \quad \sigma_2.$$

These effective properties result from the longitudinal deformation field and may differ from those of the undeformed region due to stretching-induced changes in geometry and stress. Because the medium is moving, the time derivatives must be replaced by the material derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V_0 \frac{\partial}{\partial x}. \quad (5.16)$$

The governing equation [188, 189] then becomes:

$$\rho_2 A_2 \frac{D^2 W}{Dt^2} - \sigma_2 A_2 \frac{\partial^2 W}{\partial x^2} + EI_2 \frac{\partial^4 W}{\partial x^4} = 0. \quad (5.17)$$

Assuming again a plane-wave solution

$$W(x, t) = \hat{W} e^{i(kx - \omega t)}, \quad (5.18)$$

we note that

$$\frac{D}{Dt} \mapsto -i(\omega - V_0 k), \quad (5.19)$$

so that Eq. (5.17) yields

$$\rho_2 A_2 (\omega - V_0 k)^2 = \sigma_2 A_2 k^2 + E I_2 k^4. \quad (5.20)$$

Therefore, the dispersion relation in the drifting medium is

$$\omega = V_0 k \pm \sqrt{\frac{E I_2}{\rho_2 A_2} k^4 + \frac{\sigma_2}{\rho_2} k^2}. \quad (5.21)$$

The additional term $V_0 k$ represents a convective Doppler-like shift induced by the transport of the medium. As a consequence,

$$\omega(k) \neq \omega(-k), \quad (5.22)$$

and the system becomes non-reciprocal. The effect of deformation is illustrated in Fig. 5.3b for a pre-stretched beam of thickness h , which is either further stretched or relaxed at different driving velocities. The dispersion relations are computed for $|V_0|$ ranging from 0 up to the limiting value c_L , beyond which material failure would occur. As $V_0 \rightarrow c_L$, the dispersion branch associated with backward-propagating waves progressively flattens toward $\omega \approx 0$, indicating suppression of energy transport in that direction. This asymmetry reflects the breaking of time-reversal symmetry induced by the motion of the medium, and is a hallmark of space–time modulated systems.

In the relaxation case, the limiting velocity is set to $V_0 = -c_L(\lambda_0 - 1)$, corresponding to a configuration where the drifting region is unstretched and therefore stress-free. As this limit is approached, the internal stress $\sigma \rightarrow 0$, and the dominant propagating modes transition to bending waves, shifted by a convective term $V_0 k$. Furthermore, when $|V_0| > d\omega/dk$, a sign inversion of the temporal frequency ω occurs while k remains positive.

5.3.3 Space–time varying transition

Once an effective description of the medium before and after deformation is established, an appropriate reference frame is introduced to derive the continuity conditions governing the interface dynamics and to recast the system as a scatter-

ing problem. Since the transition between the statically deformed region and the deformed drifting region is not stationary in the laboratory frame, to analyze the scattering problem, it is convenient to adopt a reference frame in which the deformation front is stationary. Its material properties vary in both space and time according to:

$$\begin{aligned} A(x, t) &= A(x + c_L t), & I(x, t) &= I(x + c_L t), & \rho(x, t) &= \rho(x + c_L t), \\ \sigma(x, t) &= \sigma(x + c_L t), & V_0(x, t) &= V_0(x + c_L t), \end{aligned}$$

where c_L is the velocity of the moving transition.

To remove the explicit time dependence, it is convenient to introduce the co-moving coordinates

$$\xi = x + c_L t, \quad \tau = t. \quad (5.23)$$

In these variables, the material parameters depend only on ξ :

$$\begin{aligned} A(x, t) &= A(\xi), & I(x, t) &= I(\xi), & \rho(x, t) &= \rho(\xi), \\ \sigma(x, t) &= \sigma(\xi), & V_0(x, t) &= V_0(\xi). \end{aligned}$$

The problem is thus transformed into a stationary scattering problem in the (ξ, τ) frame.

5.3.4 Governing equations in the co-moving frame

In the co-moving frame, each side of the interface can be approximated as homogeneous.

For medium i ($i = 1, 2$), the governing equation reads

$$\rho_i A_i \frac{D^2 W}{D\tau^2} - \sigma_i A_i \frac{\partial^2 W}{\partial \xi^2} + EI_i \frac{\partial^4 W}{\partial \xi^4} = 0, \quad (5.24)$$

where the material derivative in the co-moving frame is

$$\frac{D}{D\tau} = \frac{\partial}{\partial \tau} + V_i \frac{\partial}{\partial \xi}. \quad (5.25)$$

Here V_i is the effective transport velocity in the co-moving frame:

$$V_1 = c_L, \quad V_2 = V_0 + c_L. \quad (5.26)$$

In this frame, the undeformed region appears to drift with velocity c_L , while the deformed region acquires an additional transport velocity V_0 , resulting in an effective relative motion across the interface. We look for solutions of the form

$$W_i(\xi, \tau) = \hat{W}_i e^{i(k_i \xi - \omega' \tau)}, \quad (5.27)$$

where ω' is conserved across the interface because the problem is stationary in the co-moving frame.

Substitution of Eq. (5.27) into Eq. (5.24) gives

$$\rho_i A_i (\omega' - V_i k_i)^2 = \sigma_i A_i k_i^2 + E I_i k_i^4, \quad (5.28)$$

or equivalently

$$\omega' = V_i k_i \pm \sqrt{\frac{E I_i}{\rho_i A_i} k_i^4 + \frac{\sigma_i}{\rho_i} k_i^2}. \quad (5.29)$$

Since ω' is the same on both sides of the interface, the scattered wavenumbers are obtained by solving Eq. (5.29) in each medium.

5.3.5 Continuity conditions at the interface

Let $\xi = \xi_{\text{int}}$ denote the interface position in the co-moving frame. The Euler–Bernoulli model requires four continuity conditions.

First, continuity of transverse displacement:

$$[W]_1^2 = 0. \quad (5.30)$$

Second, continuity of rotation:

$$\left[\frac{\partial W}{\partial \xi} \right]_1^2 = 0. \quad (5.31)$$

Third, continuity of the bending moment:

$$\left[-EI \frac{\partial^2 W}{\partial \xi^2} \right]_1^2 = 0. \quad (5.32)$$

To derive the last jump condition, we consider a vanishing control volume:

$$\mathcal{C}_\delta = [\xi_{\text{int}} - \delta, \xi_{\text{int}} + \delta] \quad (5.33)$$

around the stationary interface in the co-moving frame. Within this control volume (fig.5.3b) the transverse velocity of material points in the medium i is

$$\frac{DW}{D\tau} = \frac{\partial W}{\partial \tau} + V_i \frac{\partial W}{\partial \xi},$$

so that the transverse momentum density is

$$p_i = \rho_i A_i \frac{DW}{D\tau}.$$

The internal transverse force resultant in an Euler–Bernoulli beam under axial prestress is

$$Q_i = -EI_i \frac{\partial^3 W}{\partial \xi^3} + \sigma_i A_i \frac{\partial W}{\partial \xi}. \quad (5.34)$$

Applying the linear impulse–momentum theorem to $\mathcal{C}_\delta$, accounting for the flux of transverse momentum carried by the material crossing the control surfaces with speed V_i [190], yields

$$Q(\xi_{\text{int}} - \delta, \tau) - Q(\xi_{\text{int}} + \delta, \tau) = \frac{d}{d\tau} \int_{\mathcal{C}_\delta} \rho_i A_i \frac{\partial W}{\partial \tau} d\xi + \left[\rho_i A_i V_i \frac{DW}{D\tau} \right]_{\xi_{\text{int}} - \delta}^{\xi_{\text{int}} + \delta}.$$

In the limit $\delta \rightarrow 0$, the integral term vanishes, and the jump condition is obtained:

$$\left[Q - \rho A V \frac{DW}{D\tau} \right]_1^2 = 0,$$

that is,

$$\left[-EI \frac{\partial^3 W}{\partial \xi^3} + \sigma A \frac{\partial W}{\partial \xi} - \rho A V \frac{DW}{D\tau} \right]_1^2 = 0. \quad (5.35)$$

This shows that, in the co-moving frame, continuity applies not to the internal shear force alone, but to the generalized transverse traction corrected by the convective flux of transverse momentum.

5.3.6 Scattering formulation

In each medium, Eq. (5.28) admits four roots for k_i , corresponding in general to forward and backward propagating or evanescent contributions. The total field is written as a superposition of these modes:

$$W_m(\xi, \tau) = \sum_{n=1}^4 a_{mn} e^{i(k_{mn}\xi - \omega'\tau)}. \quad (5.36)$$

Applying the four continuity conditions (5.30)-(5.35) at $\xi = \xi_{\text{int}}$ yields a linear system relating incoming and outgoing modal amplitudes.

The interface conditions can be written in matrix form as

$$\mathbf{I}\mathbf{W}_\mathbf{I} = \mathbf{T}\mathbf{W}_\mathbf{T}, \quad (5.37)$$

where

$$\mathbf{W}_\mathbf{I} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{W}_\mathbf{T} = \begin{bmatrix} a_{T1} \\ a_{T4} \\ a_{T4} \\ a_{T4} \end{bmatrix}, \quad (5.38)$$

and

$$\mathbf{I} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ ik_{I,1} & ik_{I,2} & ik_{I,3} & ik_{I,4} \\ -EI_1 k_{I,1}^2 & -EI_1 k_{I,2}^2 & -EI_1 k_{I,3}^2 & -EI_1 k_{I,4}^2 \\ \Phi_1(k_{I,1}) & \Phi_1(k_{I,2}) & \Phi_1(k_{I,3}) & \Phi_1(k_{I,4}) \end{bmatrix}, \quad (5.39)$$

$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ ik_{T,1} & ik_{T,2} & ik_{T,3} & ik_{T,4} \\ -EI_2k_{T,1}^2 & -EI_2k_{T,2}^2 & -EI_2k_{T,3}^2 & -EI_2k_{T,4}^2 \\ \Phi_2(k_{T,1}) & \Phi_2(k_{T,2}) & \Phi_2(k_{T,3}) & \Phi_2(k_{T,4}) \end{bmatrix}, \quad (5.40)$$

with

$$\Phi_1(k) = iEk^3I_1 + i(N_1 - \rho_1A_1c_L^2)k + i\rho_1A_1\omega_1c_L, \quad (5.41)$$

$$\Phi_2(k) = iEk^3I_2 + i[N_2 - \rho_2A_2(c_L + V_0)^2]k + i\rho_2A_2\omega_1(c_L + V_0). \quad (5.42)$$

Finally,

$$\mathbf{W}_T = \mathbf{T}^{-1}\mathbf{I}\mathbf{W}_I = \mathbf{S}\mathbf{W}_I. \quad (5.43)$$

where $\mathbf{W}_I$ is the vector of incident amplitudes, $\mathbf{W}_T$ is the vector of transmitted/reflected amplitudes, and $\mathbf{S}$ is the scattering matrix.

The curves reported in Fig. 5.3c are obtained by varying the drift velocity V_0 while keeping the material parameters (E, ρ, A, I, σ) fixed to the values of the undeformed and deformed configurations introduced previously. For each value of V_0 , the dispersion relations in the two media are solved, the admissible modes are identified based on their propagation direction (or decay for complex roots), and the corresponding amplitudes are computed from the scattering matrix.

The blue and red curves correspond to forward- and backward-propagating modes, respectively, while the magenta and green curves correspond to the modes associated with the complex-wavenumber (evanescent) mode, which do not contribute to propagating wave fields. Hence, the latter remains negligible over the explored range of V_0 , indicating that the scattering process is dominated by propagating contributions, with only weak localization effects at the interface.

In addition, when $V_0 > 0$, the coefficients related to the forward and back propagating modes are both positive, while when $V_0 < 0$ they have opposite sign, due to phase inversion.

5.3.7 Phase conservation law in the laboratory frame

Returning to the laboratory frame, the conservation of the phase at the moving interface implies

$$k_1x - \omega_1t = k_2x - \omega_2t \quad \text{at } x = -c_Lt, \quad (5.44)$$

which leads to

$$\omega_2 - \omega_1 = -c_L(k_2 - k_1), \quad (5.45)$$

or

$$\frac{\omega_2 - \omega_1}{k_2 - k_1} = -c_L. \quad (5.46)$$

This relation expresses the conservation of phase across the moving interface and determines the frequency conversion induced by the space–time variation of the medium.

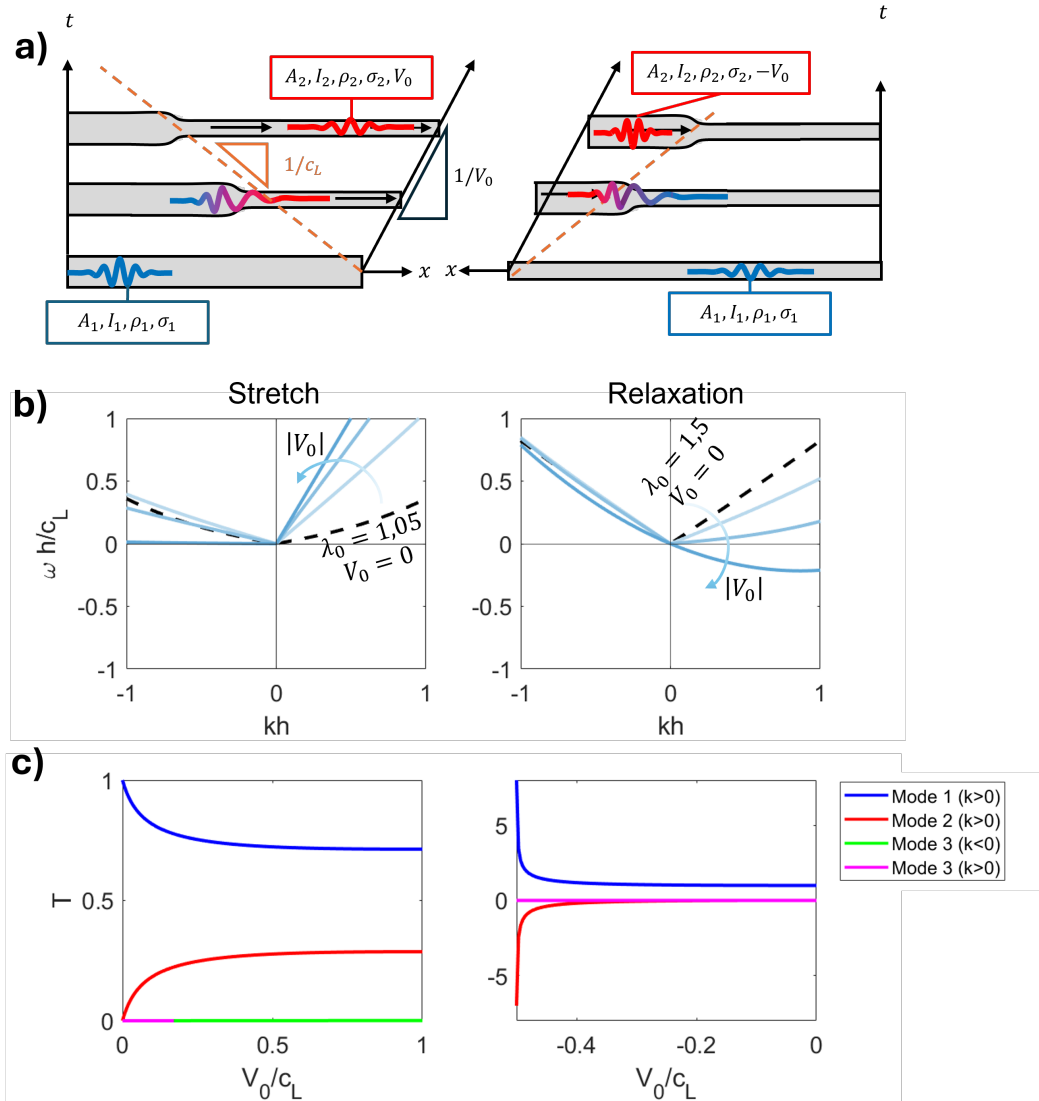


Fig. 5.3 Schematic of transverse wave propagation across a deformation-induced space-time interface in a slender beam. (a) laboratory-frame view, where a localized stretching region generates a moving transition with effective velocity c_L , leading to a conversion of the incident (blue) to a transmitted (red) wavepacket. The dashed line represents the space-time trajectory of the interface. (b) Dispersion relation of the system at different pulling/contraction velocities. The black dotted line indicates the reference dispersion relation (c) transmitted mode amplitude calculated from the scattering matrix for a pre-stretch $\lambda_0 = 1.05$ and $\lambda_0 = 1.5$ at positive and negative pulling velocities, respectively.

5.4 Numerical model and experimental validation

5.4.1 Modeling space-time interface in a hyperelastic medium

Motivated by previous experiments on hyperelastic elastomers [39], we adopt a finite-deformation model to capture the nonlinear coupling between deformation and wave propagation. The mechanical response of the material is described using a two-parameter Mooney–Rivlin hyperelastic constitutive model under the assumption of near incompressibility. In this framework, the strain energy density function is decomposed into isochoric and volumetric contributions as

$$W = W_{\text{iso}}(\bar{I}_1, \bar{I}_2) + W_{\text{vol}}(J), \quad (5.47)$$

where the isochoric part is given by

$$W_{\text{iso}} = (1 - \alpha)C_1(\bar{I}_1 - 3) + \alpha C_2(\bar{I}_2 - 3), \quad (5.48)$$

and the volumetric contribution penalizes deviations from incompressibility through

$$W_{\text{vol}} = \frac{K}{2}(J - 1)^2. \quad (5.49)$$

The material parameters are related to the linear elastic constants by

$$C_1 = C_2 = \frac{E}{2(1 + \nu)}, \quad K = \frac{E}{3(1 - 2\nu)}, \quad (5.50)$$

where $E = 67$ kPa is the Young's modulus and $\nu = 0.495$ is the Poisson ratio, chosen to enforce near incompressibility. The parameter $\alpha = 0.15$ is an empirical coefficient.

The invariants $\bar{I}_1$ and $\bar{I}_2$ are defined in terms of the isochoric part of the deformation tensor

$$\bar{\mathbf{F}} = J^{-1/3} \mathbf{F}, \quad (5.51)$$

as

$$\bar{I}_1 = \text{tr}(\bar{\mathbf{C}}), \quad \bar{I}_2 = \frac{1}{2} [(\text{tr}(\bar{\mathbf{C}}))^2 - \text{tr}(\bar{\mathbf{C}}^2)], \quad (5.52)$$

where $\bar{\mathbf{C}} = \bar{\mathbf{F}}^T \bar{\mathbf{F}}$. The deformation gradient $\mathbf{F} = \mathbf{I} + \partial \mathbf{U} / \partial \mathbf{X}$ maps the fiber deformation of the material from the material $\mathbf{X}$ to the current configuration $\mathbf{x}$, while

$\mathbf{U} = (U, V, W)$ is the displacement vector. $J = \det(\mathbf{F})$ represents the local volume ratio.

Figure 5.4a shows the dependence of the axial Cauchy stress on the engineering strain $\varepsilon = \lambda - 1$ over the range $-0.5 \leq \varepsilon \leq 0.8$. The nonlinear response is compared with the linear elastic prediction given by Hooke's law. A good agreement is observed within the approximate range $-0.2 \lesssim \varepsilon \lesssim 0.5$, while significant deviations appear outside this interval, highlighting the importance of the hyperelastic model for large deformations.

To numerically validate the analytical model, a simulation of an EcoflexTM strip is performed, as schematized in fig. 5.4b. The strip has length $L = 80$ cm and cross-sectional dimensions $W = 3$ cm and $h = 0.1$ cm.

The dynamics of the system is governed by the equation of motion

$$\rho_0 \frac{\partial^2 \mathbf{U}}{\partial t^2} = \nabla \cdot (\mathbf{FS}), \quad (5.53)$$

where $\rho_0 = 1000 \text{ kg/m}^3$ is the density in the reference configuration, and $\mathbf{S} = \mathbf{S}_d + \mathbf{F}^{-1} \mathbf{P}$ is the second Piola–Kirchhoff stress tensor.

The viscous contribution is modeled as

$$\mathbf{S}_d = \eta j \mathbf{C}^{-1}, \quad (5.54)$$

with $\eta = 0.05 \text{ Pa}\cdot\text{s}$, while the elastic contribution is defined through the first Piola–Kirchhoff stress tensor

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}. \quad (5.55)$$

The initial conditions are

$$U(X, 0) = (\lambda_i - 1)X, \quad (5.56)$$

$$\frac{\partial U}{\partial t}(X, 0) = 0, \quad (5.57)$$

where $\lambda_1 = 1.05$ and $\lambda_2 = 1.5$ represent an initial pre-stretch for the pulling and relaxation model, respectively.

The boundary conditions are prescribed as:

$$U(0, t) = 0, \quad (5.58)$$

$$U(L, t) = V_i \tau \ln \left(1 + e^{\frac{t-t_i}{\tau}} \right) + (\lambda_i - 1)L, \quad (5.59)$$

with $V_2 = 3$ m/s and $V_2 = -1$ m/s, $\tau = 5$ ms, and $t_1 = 180$ ms and $t_2 = 90$ ms. The solution was obtained using the FE method in the material domain and compared with the analytical prediction. In the comparison in the pulling configuration shown in fig. 5.4c, due to the small deviation from the linear regime, the velocity of the interface is the same both in the analytical and numerical solution. Whereas the broadening of the transition region across the characteristic $X + c_L t$ is due to the effect of the dissipation, not included in the analytical solution. In the comparison in the relaxation configuration shown in fig. 5.4d, due to the larger deviation from the linear regime, the velocity of the interface is slightly larger than the analytical one as we can appreciate from the different time of arrival of it to the opposite end of the beam.

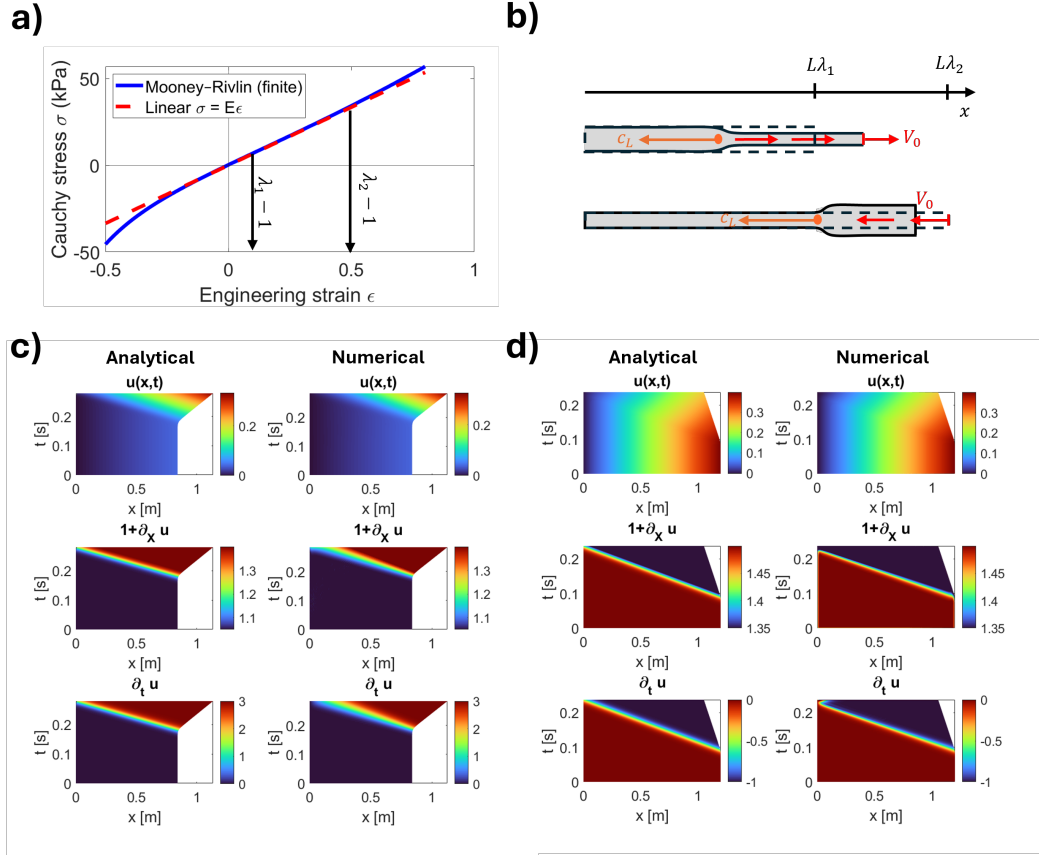


Fig. 5.4 (a) Longitudinal stress vs engineering strain plot for the Mooney-Rivlin elastic energy potential and comparison with the linear approximation. Black arrows highlight the initial stretches chosen for the simulation. (b) Schematic of the two different simulations where a velocity controlled (top) pull and (bottom) contraction is used as boundary condition. Red arrows represent the velocity field of the particles past the interface and the orange ones the velocity of the interface. (c) and (d) Comparison between analytical and numerical solutions of the displacement field, stretch, and particle velocity in the current configuration. Panel (c) corresponds to a tensile (loading) case, while panel (d) represents a release (unloading) case.

5.4.2 Transverse wave propagation across a finite-thickness interface

To overcome the limitations of the analytical model, in particular the assumption of an infinitely sharp interface, and include hyperelastic material characteristics, we perform finite element simulations of transverse wave propagation in a continuously deforming compliant (e.g. rubber) strip. The numerical framework allows

us to explicitly resolve the finite-thickness transition layer generated by the axial deformation and to assess its impact on wave scattering.

The simulations are carried out using a three-dimensional hyperelastic model based on the Mooney–Rivlin constitutive law introduced previously. The numerical procedure follows a two-step approach (fig. 5.4a): first, a static pre-strain is imposed through displacement-controlled boundary conditions to obtain a deformed configuration; second, a transient dynamic simulation is performed in which both axial motion and transverse excitation are applied.

In the static step, the strip of initial length $L_0 = 80$ cm and thickness t_0 is extended to a prescribed stretch λ_0 , leading to a deformed configuration characterized by modified geometry and stress distribution. This step establishes the initial condition for the dynamic problem and ensures consistency with the deformation-induced variation of the material properties.

Pulling configuration

In the dynamic step, as shown in fig. 5.5a, time-dependent boundary conditions are applied to generate the propagating transition layer and the transverse wavepacket. In particular, the axial displacement at one end is prescribed using the same smooth ramp function adopted in the analytical model, producing a moving deformation front with finite thickness. Simultaneously, a localized transverse excitation is imposed, generating a wave-packet with central frequency $f_0 = 60$ Hz and 20% bandwidth. This wave propagates along the strip and interacts with the moving interface.

The resulting wave field is shown in fig. 5.5b, where the evolution of the transverse displacement is represented in space–time coordinates. The incident wavepacket, initially propagating in the undeformed region, encounters the transition layer and undergoes scattering, producing a transmitted wave in the deformed drifting medium. The interaction is characterized by a clear change in slope of the wavefronts in the space–time diagram, indicating a modification of both phase velocity and wavelength.

In agreement with experiments in [39], only the forward propagating mode survives after the scattering with the interface, while the backward propagating one is suppressed. This phenomenon is probably due to the strain gradient across the interfaces which minimizes the reflection component. In addition, a second

contribution could also be due to the viscoelastic damping of the material attenuating the backward propagating component.

By studying the FFT spectrum, shown in fig. 5.5c, the transmitted wavepacket exhibits a frequency–wavenumber shift close to the phase conservation law derived in Eq. (5.46), confirming that the essential features of the space–time interface are preserved even in the absence of a sharp discontinuity. Specifically, the finite size of the transition layer introduces here additional phase shifts that cannot be captured by our model. In addition, the transition layer induces a gradual redistribution of energy, leading to a spreading of the spectral content and to partial mode conversion that is not captured by the analytical formulation.

These results demonstrate that the Euler–Bernoulli-based sharp-interface model provides an accurate description of the scattering process in the limit of thin transition layers, while the finite element simulations capture the more general regime in which the interface must be treated as a distributed space–time inhomogeneity. The numerical approach therefore validates the analytical predictions while extending their applicability to physically realistic configurations.

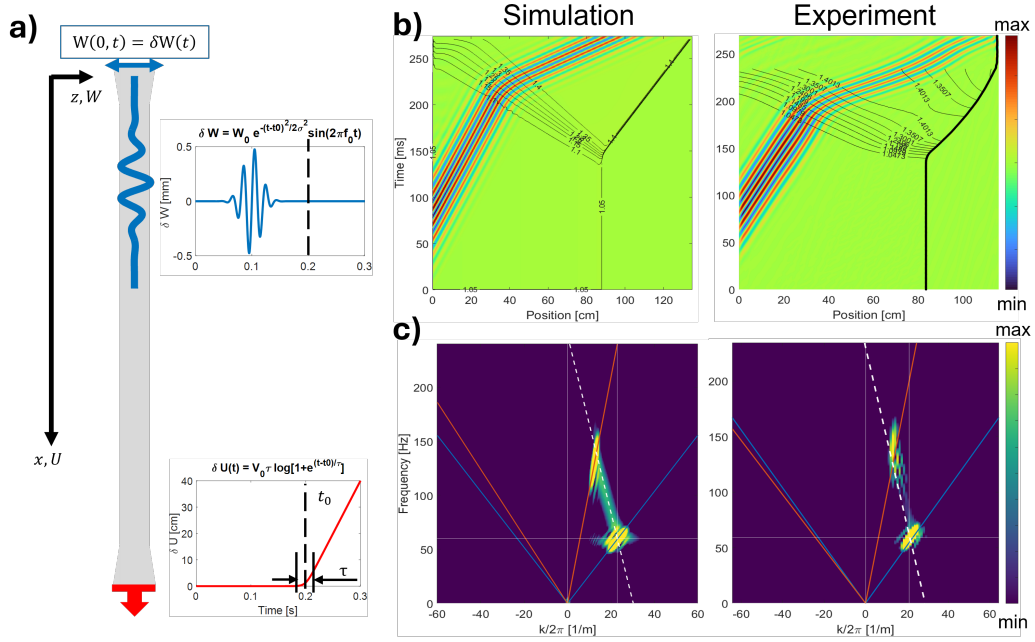


Fig. 5.5 (a) Schematic of the hyperelastic strip subjected to axial deformation and transverse excitation. A localized wavepacket $\delta W(t)$ is generated at the top boundary, while a time-dependent axial displacement $\delta U(t)$ induces a propagating transition layer with characteristic time τ and onset time t_0 . Comparison between numerical and experimental result for (b) space–time representation transverse displacement field, showing the interaction of the incident wavepacket with the moving deformation front and the resulting transmitted wave. The change in slope of the wavefronts indicates a modification of phase velocity across the interface and the black contour lines indicate the longitudinal component of the deformation gradient. (c) Frequency–wavenumber spectrum of the wave field, highlighting the shift between incident and transmitted components. The superimposed dispersion curves (solid blue and red lines) and the phase-matching condition (dashed white line) confirm the validity of the phase conservation law and the predicted frequency–wavenumber conversion induced by the space–time interface.

Relaxation configuration

As shown in fig. 5.5a, starting from a considerably pre-stretched configuration with $\lambda_0 = 1.6$, the transient of the model is studied applying time-dependent boundary conditions to generate the transverse wave-packet, with central frequency $f_0 = 50$ Hz and 30% bandwidth. Due to experimental constraints, the deformation gradient could not be controlled by a fixed deformation velocity. Therefore, the effect could be investigated by means of numerical simulation.

The resulting wave field is shown in fig.5.5b , where the evolution of the transverse displacement is represented in space–time coordinates. In this case, when the incident wave-packet encounters the transition layer and undergoes scattering, produces a transmitted wave that inverts its direction of propagation as the drift velocity of the medium is larger than the one of the wave-packet. In this case also a reflected wave packet is generated as predicted by the theory.

From the analysis of the FFT spectrum, shown in fig.5.5c, the transmitted wave-packet exhibits a negative frequency, as predicted analytically. This is interpreted as an inversion of the temporal phase of the wave packet due to the inversion of its direction of propagation. Again we notice how the phase conservation law still holds for both incident and transmitted frequencies and wave-numbers. Additionally, the finite size of the transition layer induces a gradual redistribution of energy, leading to a spreading of the spectral content and to partial mode conversion that is not captured by the analytical formulation.

These results demonstrate that the problem of relaxation triggers exotic effects in wave conversion, giving the possibility to access new phenomena as temporal phase inversion. Both analytical and FE models provide an accurate representation of these dynamics in describing the scattering process and in capturing the transient evolution of the system, respectively.

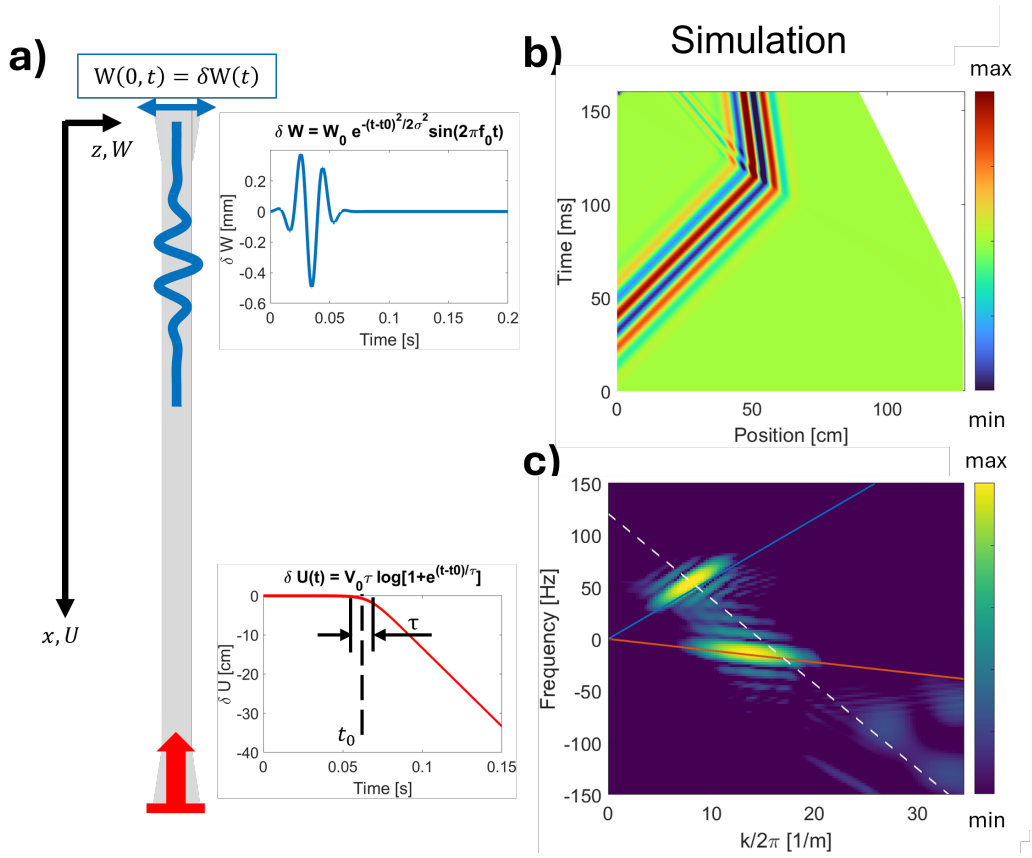


Fig. 5.6 (a) Schematic of the hyperelastic strip subjected to axial deformation and transverse excitation. A localized wavepacket $\delta W(t)$ is generated at the top boundary, while the other end controlled by the boundary displacement function δU . Numerical result for (b) space–time representation of the transverse displacement field, showing the interaction of the incident wavepacket with the deformation front and the resulting transmitted and reflected wave. The inversion of propagation direction takes place due to a larger drift velocity compared to the wave-packet velocity. (c) Frequency–wavenumber spectrum of the wave field, highlighting the shift between incident and transmitted components. The superimposed dispersion curves (solid blue and red lines) and the phase-matching condition (dashed white line) confirm the validity of the phase conservation law and the predicted frequency–wavenumber conversion induced by the space–time interface.

5.4.3 Characterization of wave transmission across the moving interface

Pulling configuration

Figure 5.7a shows the space–time evolution of the transverse displacement field. The incident wavepacket propagates in the undeformed region and encounters the moving transition layer, where a clear modification of the wavefront slope is observed. This change reflects the variation of phase velocity induced by the deformation and the onset of non-reciprocal propagation in the drifting medium. The interaction is localized around the transition region, confirming that the interface acts as the primary scattering mechanism despite its finite thickness.

To quantify the effect of the interaction, fig. 5.7b reports the temporal evolution of the maximum transverse displacement $\max(W)$ at a fixed observation point. Two distinct regimes can be identified: an initial increase corresponding to the generation and propagation of the incident wavepacket, followed by a second variation associated with the crossing of the interface. The difference between these two levels, denoted as ΔW , provides a direct measure of the amplitude change induced by the space–time interface.

In addition, a secondary fluctuation of the transverse amplitude can be observed between the wave-packet generation and interface crossing time instants identified in Fig. 5.7b. This feature originates from the simultaneous generation of a longitudinal disturbance associated with the imposed axial motion. Due to its higher propagation speed, the longitudinal wavepacket travels ahead of the transverse excitation and reaches the boundary of the medium in a shorter time. It is then reflected and propagates backward, eventually interacting with the transverse wavepacket.

This interaction does not lead to mode conversion in the classical sense, but rather to a transient modulation of the transverse displacement field induced by the coupling between axial and transverse kinematics. As a result, a weak interference pattern appears, manifested as a small fluctuation in the envelope of the transverse amplitude.

This interpretation is supported by the characteristic time scale of the phenomenon: the minimum of $\max(W)$ occurs at $t \approx 2L/c_L$, which corresponds to the round-trip time of a longitudinal wave traveling across the domain. This confirms

that the observed fluctuation is directly linked to the propagation and reflection of the longitudinal disturbance.

A more detailed view of this process is shown in fig. 5.7c, where the evolution of $\max(W)$ is reported for different values of the drift velocity V_0 . The curves reveal a systematic reduction of the transmitted amplitude as V_0 increases, indicating that the moving interface progressively attenuates the wavepacket. This behavior is consistent with the redistribution of energy among the available modes predicted by the analytical model.

Finally, fig. 5.7d summarizes the transmission coefficient T as a function of the drift velocity V_0 , comparing numerical results with the analytical prediction derived from the scattering formulation. A very good agreement is observed over the entire range of V_0 , confirming the validity of the sharp-interface approximation for predicting global transmission properties. The slight discrepancy at intermediate velocities can be attributed to the finite thickness of the transition layer, which introduces weak deviations from the idealized discontinuous model.

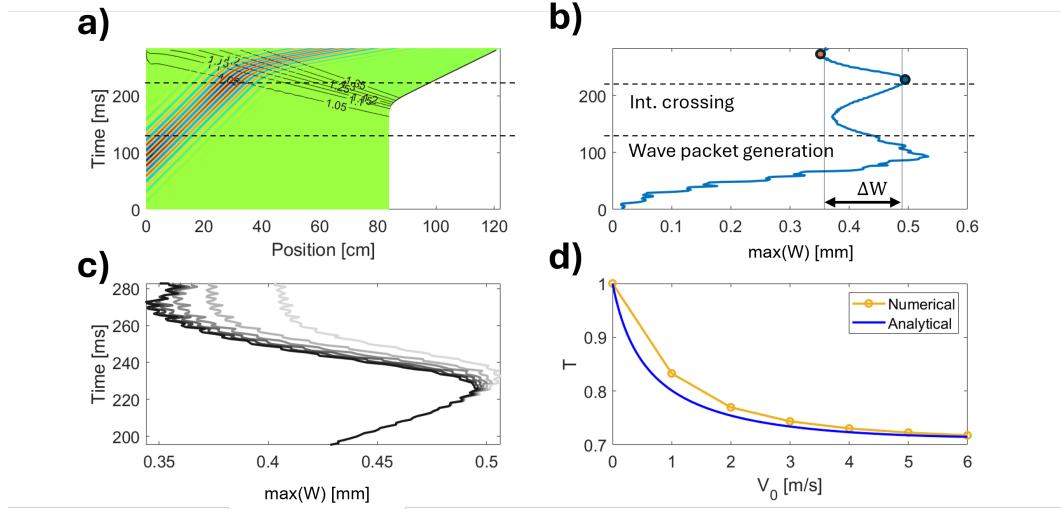


Fig. 5.7 Numerical characterization of transverse wave transmission across a deformation-induced space-time interface. (a) Space-time evolution of the transverse displacement field, showing the incident wavepacket interacting with the moving transition layer and generating a transmitted wave with modified slope, indicative of a change in phase velocity. (b) Temporal evolution of the maximum transverse displacement $\max(W)$, highlighting two main events: wavepacket generation and interface crossing. The amplitude variation ΔW calculated as a peak to peak difference between incident wave amplitude (blue dot) and transmitted wave amplitude (orange dot), quantifies the effect of the interface. The intermediate fluctuation is associated with the propagation and reflection of a faster longitudinal disturbance. (c) Detailed view of $\max(W)$ for different drift velocities V_0 , showing a systematic reduction of the transmitted amplitude and a secondary modulation occurring at $t \approx 2L/c_L$, corresponding to the round-trip time of the longitudinal wave. (d) Transmission coefficient T as a function of the drift velocity V_0 , comparing numerical results with the analytical prediction. The good agreement confirms the validity of the sharp-interface model, while small deviations are attributed to the finite thickness of the transition layer.

Relaxation configuration

Contrary to what happens in the pulling configuration, when a wave-packet crosses a decreasing deformation gradient, it decreases its linear momentum. Figure 5.8a shows the space-time evolution of the transverse displacement field propagating in a pre-stretched medium with $\lambda_0 = 1.6$. Here, the encounter with the interface produces a deceleration of the wave-packet's phase velocity, induced by the deformation.

To quantify the effect of the interaction, fig. 5.8b reports the temporal evolution of the maximum transverse displacement $\max(W)$ at all times. Two distinct regimes can be identified: an initial increase corresponding to the generation and propagation of

the incident wavepacket, followed by a second variation associated with the crossing of the interface. The difference between these two levels, denoted as ΔW , provides a direct measure of the amplitude change induced by the space–time interface.

The absence of the secondary fluctuation of the transverse amplitude can be explained due to the fact that the large initial stretch of the medium reduces the coupling between the longitudinal and transverse mode. Therefore the longitudinal wave excited during the wave-packet generation produces a negligible effect.

Fig. 5.8c shows the evolution of $\max(W)$ is reported for different values of the relaxation velocity V_0 . The curves reveal a systematic increase of the transmitted amplitude as V_0 decreases, indicating that the moving interface progressively amplifies the wavepacket. This behavior is consistent with the redistribution of energy among the available modes predicted by the analytical model.

Finally, fig. 5.8d summarizes the transmission coefficient T as a function of the drift velocity V_0 , comparing numerical results with the analytical prediction derived from the scattering formulation. A very good agreement is observed over the entire range of V_0 , confirming the validity of the sharp-interface approximation for predicting global transmission properties.

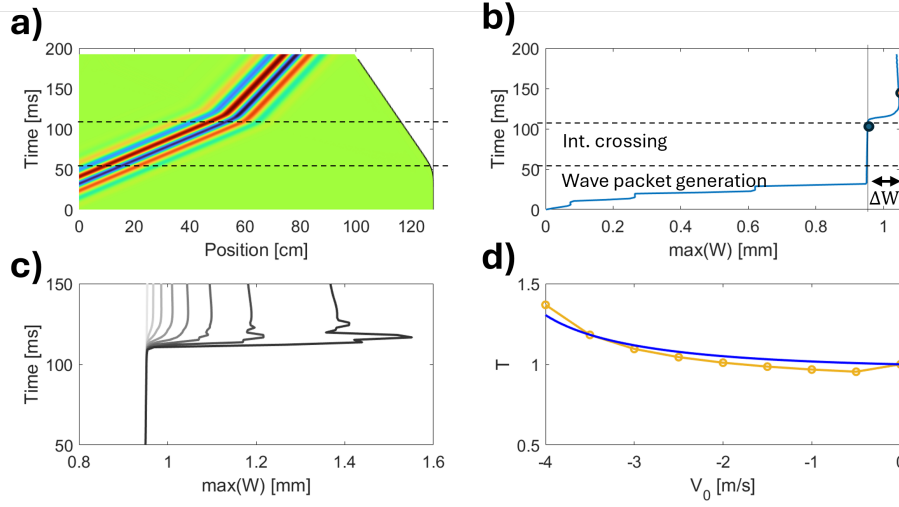


Fig. 5.8 Numerical characterization of transverse wave transmission across a deformation-induced space-time interface. (a) Space-time evolution of the transverse displacement field, showing the incident wavepacket interacting with the moving transition layer and generating a transmitted wave with modified slope, indicative of a change in phase velocity. (b) Temporal evolution of the maximum transverse displacement $\max(W)$, highlighting two main events: wavepacket generation and interface crossing. The amplitude variation ΔW , calculated as a peak to peak difference between incident wave amplitude (blue dot) and transmitted wave amplitude (orange dot), quantifies the effect of the interface, while the intermediate fluctuation is associated with the propagation and reflection of a faster longitudinal disturbance. (c) Detailed view of $\max(W)$ for different drift velocities V_0 . (d) Transmission coefficient T as a function of the drift velocity V_0 , comparing numerical results with the analytical prediction. The good agreement confirms the validity of the sharp-interface model, while small deviations are attributed to the finite thickness of the transition layer.

Overall, these results demonstrate that the deformation-induced interface governs both the kinematic and energetic aspects of wave propagation, and that the analytical framework captures the dominant features of the scattering process, while the numerical model resolves the additional effects associated with the continuous nature of the transition.

5.5 Conclusions

In this chapter, we analyze the three basic features of the time-space interface scattering configurations considered in this thesis, all of which occur in the supersonic

deformation regime where the deformation front moves faster than the transverse wave speed.

First, the deformation dynamics of a pulled and subsequently relaxing elastic beam has been modeled both analytically and numerically. The analysis shows that applying an axial stretch/relaxation at one end of the beam generates a deformation front that propagates along the beam at longitudinal wave velocity. The analytical solution, based on infinitesimal strain theory, provides the space-time evolution of the deformation gradient, while numerical simulations, based on finite strain theory, capture the geometric non-linearity of the deformation. This dynamic background medium is the essential prerequisite for studying wave interactions.

Second, building upon the derived deformation field, the interaction of transverse waves (bending or shear waves) with the moving longitudinal deformation gradient has been investigated. A comprehensive approach, that combines analytical methods, numerical simulations, and experimental validation, has been adopted. Analytically, a modified wave equation with space-time-dependent coefficients is derived and solved using the continuity condition at the moving deformation front. Numerically, time-domain FE simulations confirm the phenomena of phase conservation and amplitude modulation. Experimentally, using a silicon rubber strip (nearly compressible hyperelastic material), the predictions are validated.

Third, a systematic parametric study has been carried out over a range of deformation velocities. Both numerical simulations and dedicated experiments on silicon rubber specimens provide consistent results: the transmission coefficient decays/increases as the deformation velocity increases/decreases, respectively.

As further development, the next step is to recast wave propagation in deforming media within a variational framework. A Lagrangian formulation would allow to treat the medium and the wave field on equal footing and, more importantly, provide a systematic way to identify conserved quantities—if any are present—when space–time symmetries are broken. The central issue is straightforward: what is being conserved when the background itself evolves? While classical energy and momentum conservation generally fail, Noether’s theorem may still reveal generalized or adiabatic invariants, offering both physical insight and practical predictive tools.

This perspective naturally connects to cosmological analogies. Wave propagation in a deforming medium shares formal similarities with electromagnetic waves in

an expanding universe, where a time-dependent background induces frequency redshift and energy dilution [191–193]. In our case, deformation-driven modulation leads to frequency conversion and asymmetric mode coupling. By interpreting the deformation as an effective metric, one could recast the governing equations in a form akin to wave dynamics in curved spacetime, enabling the transfer of concepts such as particle production and horizon-like effects.

Although the analogy is not exact, it provides useful insight. Coupling this theoretical framework with experiments would allow the design of controlled deformation protocols that mimic cosmological scenarios. This would not only deepen the fundamental understanding of space–time varying media, but also position deformable metamaterials as accessible analog platforms for complex wave phenomena.

References

- [1] L. Rayleigh. *The Theory of Sound*. Macmillan, 1877.
- [2] K. Zoeppritz. Vii b. über reflexion und durchgang seismischer wellen durch unstetigkeitsflächen. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1919:66–84, 1919.
- [3] V. Laude. *Phononic Crystals*. De Gruyter, 2015.
- [4] N. Jiménez, O. Umnova, and J.-P. Groby. Acoustic waves in periodic structures, metamaterials, and porous media. *Ch. the Transfer Matrix Method in Acoustics, Springer International Publishing, Cham*, pages 103–164, 2021.
- [5] M. S. Kushwaha. Acoustic band structure of periodic elastic composites. *Phys. Rev. Lett.*, 71:2022, 1993.
- [6] M. M. Sigalas and E. N. Economou. Band structure of elastic waves in two-dimensional systems. *Solid State Communications*, 86(3):141–143, 1993.
- [7] R. Martínez-Sala, J. Sancho, J. V. Sánchez, V. Gómez, J. Llinares, and F. Meseguer. Sound attenuation by sculpture. *Nature*, 378:241, 1995.
- [8] D. Caballero, J. Sánchez-Dehesa, R. Martínez-Sala, C. Rubio, J. V. Sánchez-Pérez, L. Sanchis, and F. Meseguer. Suzuki phase in two-dimensional sonic crystals. *Physical Review B*, 64(6):064303, 2001.
- [9] M. M. Sigalas and E. N. Economou. Attenuation of multiple-scattered sound. *EPL (Europhysics Letters)*, 36(4):241–246, 1996.
- [10] M. M. Sigalas and E. N. Economou. Defect states in acoustic band gaps. *Journal of Sound and Vibration*, 158(2):377–382, 1996.
- [11] A. Khelif, A. Choujaa, S. Benchabane, B. Djafari-Rouhani, and V. Laude. Trapping and guiding of acoustic waves by defect modes in a full-band-gap ultrasonic crystal. *Physical Review B*, 68(21):214301, 2003.
- [12] A. Khelif, B. Djafari-Rouhani, J. O. Vasseur, and P. A. Deymier. Guiding and bending of acoustic waves in highly confined phononic crystal waveguides. *Applied Physics Letters*, 84(22):4400–4402, 2004.

- [13] Y. Pennec, B. Djafari-Rouhani, J. O. Vasseur, A. Khelif, and P. A. Deymier. Two-dimensional phononic crystals: Examples and applications. *Surface Science Reports*, 65(8):229–291, 2010.
- [14] Z. Liu, X. Zhang, Y. Mao, Y. Y. Zhu, Z. Yang, C. T. Chan, and P. Sheng. Three-dimensional phononic crystals with a full band gap. *Physical Review B*, 62(4):2446–2449, 2000.
- [15] M. Maldovan. Sound and heat revolutions in phononics. *Nature*, 503:209–217, 2013.
- [16] Z. Liu. Locally resonant sonic materials. *Science*, 289:1734, 2000.
- [17] Z. Liu, C. T. Chan, and P. Sheng. Negative effective mass density in acoustic metamaterials. *Physical Review B*, 71(1):014103, 2005.
- [18] N. Fang, D. Xi, J. Xu, M. Ambati, W. Srituravanich, C. Sun, and X. Zhang. Ultrasonic metamaterials with negative modulus. *Nature Materials*, 5:452–456, 2006.
- [19] Z. Yang, H. M. Dai, N. H. Chan, G. C. Ma, and P. Sheng. Membrane-type acoustic metamaterial with negative dynamic mass. *Physical Review Letters*, 101(20):204301, 2008.
- [20] M. I. Hussein. Dynamics of phononic materials and structures. *Appl. Mech. Rev.*, 66, 2014.
- [21] S. A. Cummer, J. Christensen, and A. Alù. Controlling sound with acoustic metamaterials. *Nature Reviews Materials*, 1:16001, 2016.
- [22] G. Ma and P. Sheng. Acoustic metamaterials: From local resonances to broad horizons. *Sci. Adv.*, 2:e1501595, 2016.
- [23] F. R. Morgenthaler. Velocity modulation of electromagnetic waves. *IRE Transactions on Microwave Theory and Techniques*, 6(2):167–172, 1958.
- [24] E. Galiffi, R. Tirole, S. Yin, H. Li, S. Vezzoli, P. A. Huidobro, M. G. Silveirinha, R. Sapienza, A. Alù, and J. B. Pendry. Photonics of time-varying media. *Advanced Photonics*, 4(1):014002–014002, 2022.
- [25] V. Pacheco-Peña and N. Engheta. Antireflection temporal coatings. *Optica*, 7(4):323–331, 2020.
- [26] R. Tirole, S. Vezzoli, E. Galiffi, I. Robertson, D. Maurice, B. Tilmann, S. A. Maier, J. B. Pendry, and R. Sapienza. Double-slit time diffraction at optical frequencies. *Nature Physics*, 19(7):999–1002, 2023.
- [27] V. Bacot, M. Labousse, A. Eddi, M. Fink, and E. Fort. Experimental evidence of nonreciprocal wave propagation in time-modulated media. *Nature Physics*, 12:972–977, 2016.

- [28] B. L. Kim, C. Chong, and C. Daraio. Temporal refraction in an acoustic phononic lattice. *Physical Review Letters*, 133(7):077201, 2024.
- [29] S. Wang, N. Shao, H. Chen, J. Chen, H. Qian, Q. Wu, H. Duan, A. Alú, and G. Huang. Experimental realization of temporal refraction and reflection in elastic beams. *Nature Communications*, 16(1):9520, 2025.
- [30] C. Caloz and Z.-L. Deck-Léger. Spacetime metamaterials—part i: General concepts. *IEEE Transactions on Antennas and Propagation*, 68:1569–1582, 2020.
- [31] Y. E. A. Liu. Reconfigurable intelligent surfaces: principles and opportunities. *IEEE Communications Surveys & Tutorials*, 23:1546–1577, 2021.
- [32] D. L. Sounas and A. Alù. Non-reciprocal photonics based on time modulation. *Nature Photonics*, 11:774–783, 2017.
- [33] H. E. A. Nassar. Nonreciprocity in acoustic and elastic materials. *Nature Reviews Materials*, 5:667–685, 2020.
- [34] E. S. Cassedy and A. A. Oliner. Dispersion relations in time-space periodic media. *Proceedings of the IEEE*, 51:1342–1359, 1963.
- [35] E. S. Cassedy. Dispersion relations in time-space periodic media, part ii. *Proceedings of the IEEE*, 55:1154–1168, 1967.
- [36] J. B. E. A. Pendry. Gain mechanism in time-dependent media. *Optica*, 8:636–637, 2021.
- [37] E. E. A. Galiffi. Photonics of time-varying media. *Advanced Photonics*, 4:014002, 2022.
- [38] Y. Ye, C. Liu, A. Marzani, E. Riva, X. Wang, and A. Palermo. Nonreciprocal scattering of elastic waves at time interfaces induced by spatiotemporal modulation. *Mechanical Systems and Signal Processing*, 2025.
- [39] A. Delory, C. Prada, M. Lanoy, A. Eddi, M. Fink, and F. Lemoult. Elastic wave packets crossing a space-time interface. *Physical Review Letters*, 2024.
- [40] R. Fleury, D. L. Sounas, and A. Alù. Sound isolation and giant linear nonreciprocity in a compact acoustic circulator. *Science*, 2014.
- [41] R. Fleury, D. L. Sounas, and A. Alù. Subwavelength ultrasonic circulator based on spatiotemporal modulation. *Physical Review B*, 91:174306, 2015.
- [42] G. Trainiti and M. Ruzzene. Non-reciprocal elastic wave propagation in spatiotemporal periodic structures. *New Journal of Physics*, 2016.
- [43] G. E. A. Trainiti. Time-periodic stiffness modulation in elastic metamaterials. *Physical Review Letters*, 122:124301, 2019.

- [44] H. Nassar, H. Chen, A. N. Norris, and G. L. Huang. Nonreciprocal wave propagation in modulated elastic metamaterials. *Proceedings of the Royal Society A*, 2017.
- [45] S. Ouakka, O. Verlinden, and G. Kouroussis. Railway ground vibration and mitigation measures: benchmarking of best practices. *Railway Engineering Science*, 30(1):1–22, 2022.
- [46] P. Coulier, V. Cuellar, G. Degrande, and G. Lombaert. Experimental and numerical evaluation of the effectiveness of a stiff wave barrier in the soil. *Soil Dynamics and Earthquake Engineering*, 77(1):238–253, 2015.
- [47] A. Dijckmans, A. Ekblad, A. Smekal, G. Degrande, and G. Lombaert. Efficacy of a sheet pile wall as a wave barrier for railway induced ground vibration. *Soil Dynamics and Earthquake Engineering*, 84(1):55–69, 2016.
- [48] M. I. Hussein, M. J. Leamy, and M. Ruzzene. Dynamics of phononic materials and structures: Historical origins, recent progress, and future outlook. *Applied Mechanics Reviews*, 66(4):040802, 2014.
- [49] M. Collet, M. Ouisse, M. Ruzzene, and M. Ichchou. Floquet–bloch decomposition for the computation of dispersion of two-dimensional periodic structures. *International Journal for Numerical Methods in Engineering*, 88(12):1252–1274, 2011.
- [50] G. Lombaert, G. Degrande, J. Kogut, and S. Francois. The experimental validation of a numerical model for the prediction of railway induced vibrations. *Journal of Sound and Vibration*, 297(3–5):512–535, 2006.
- [51] M. Oudich, M. Senesi, and B. Assouar. Tailoring structure-borne sound through bandgap engineering: From phononic crystals to metamaterials. *arXiv preprint arXiv:2207.05234*, 2022.
- [52] S. Kaçın et al. Experimental investigation of seismic wave attenuation using phononic crystal barriers. *Archive of Applied Mechanics*, 92:1231–1245, 2022.
- [53] J. Wang. A seismic shielding structure based on phononic crystals. *Revista de la Construcción*, 19(3):272–280, 2020.
- [54] X. Sheng et al. Phononic crystal-enhanced floating slab track for railway vibration mitigation. *KSCE Journal of Civil Engineering*, 2025.
- [55] M. Jalal, J. Wu, and A. Al-Mansour. Enhanced low-frequency vibration isolation via double-resonator phononic crystal. *Scientific Reports*, 15:XXXX, 2025.
- [56] Z. Liu, X. Zhang, Z. Mao, Y. Zhu, Z. Yang, C. Chan, and P. Sheng. Locally resonant sonic materials. *Science*, 289(1):1734–1736, 2000.

- [57] A. O. Krushynska, M. Miniaci, F. Bosia, and N. M. Pugno. Coupling local resonance with bragg band gaps in single-phase mechanical metamaterials. *Extreme Mechanics Letters*, 12:30–36, 2017.
- [58] H. M. van Driel and W. L. Vos. Multiple bragg wave coupling in photonic band-gap crystals. *Physical Review B*, 62(15):9872, 2000.
- [59] B. R. Mace and E. Manconi. Wave motion and dispersion phenomena: Veering, locking and strong coupling effects. *The Journal of the Acoustical Society of America*, 131(2):1015–1028, 2012.
- [60] E. Manconi and B. Mace. Veering and strong coupling effects in structural dynamics. *Journal of Vibration and Acoustics*, 139(2):021009, 2017.
- [61] A. Castanheira-Pinto, P. Alves-Costa, L. Godinho, and P. Amado-Mendes. On the application of continuous buried periodic inclusions on the filtering of traffic vibrations: A numerical study. *Soil Dynamics and Earthquake Engineering*, 113:391–405, 2018.
- [62] L. Lei, L. Miao, C. Li, X. Liang, and J. Wang. Locally resonant periodic wave barriers for vibration isolation in subway engineering. *KSCE Journal of Civil Engineering*, 25:1239–1251, 2021.
- [63] A. Palermo, S. Krödel, A. Marzani, and C. Daraio. Engineered metabarrier as shield from seismic surface waves. *Scientific Reports*, 2016.
- [64] M. Miniaci, A. Krushynska, F. Bosia, and N. M. Pugno. Large scale mechanical metamaterials as seismic shields. *New Journal of Physics*, 18(8):083041, 2016.
- [65] Y. Achaoui, T. Antonakakis, S. Brûlé, R. Craster, S. Enoch, and S. Guenneau. Clamped seismic metamaterials: ultra-low frequency stop bands. *New Journal of Physics*, 19(6):063022, 2017.
- [66] D. Colquitt, A. Colombi, R. Craster, P. Roux, and S. Guenneau. Seismic metasurfaces: Sub-wavelength resonators and rayleigh wave interaction. *Journal of the Mechanics and Physics of Solids*, 99:379–393, 2017.
- [67] A. Colombi, D. Colquitt, P. Roux, S. Guenneau, and R. V. Craster. A seismic metamaterial: The resonant metawedge. *Scientific Reports*, 6:27717, 2016.
- [68] A. Colombi, P. Roux, S. Guenneau, P. Gueguen, and R. V. Craster. Forests as a natural seismic metamaterial: Rayleigh wave bandgaps induced by local resonances. *Scientific Reports*, 2016.
- [69] L. D’Alessandro, E. Belloni, R. Ardito, F. Braghin, and A. Corigliano. Mechanical low-frequency filter via modes separation in 3d periodic structures. *Applied Physics Letters*, 2017.

- [70] L. D'Alessandro, R. Ardito, F. Braghin, and A. Corigliano. Low frequency 3d ultra-wide vibration attenuation via elastic metamaterial. *Scientific Reports*, 2019.
- [71] V. Laude. *Phononic Crystals: Artificial Crystals for Sonic, Acoustic, and Elastic Waves*. De Gruyter, 2015.
- [72] P. D. Villamil, M. Mistler, and F. Korda. Elze test area: Measurement report and evaluation of the insertion loss resulting from the installation of metamaterial elements. Technical Report 80-10634-01-D1, Baudynamik Heiland & Mistler GmbH, 2020.
- [73] G. Dal Moro. *Onde di superficie in geofisica applicata. Acquisizione e analisi di dati secondo tecniche MASW e HVSr*. Flaccovio Dario, 2014.
- [74] J.-S. L'Hereux and M. Long. Relationship between shear-wave velocity and geotechnical parameters for norwegian clays. *Journal of Geotechnical and Geoenvironmental Engineering*, 143(6):1, 2017.
- [75] R. Zaccherini, A. Palermo, A. Marzani, A. Colombi, V. Dertimanis, and E. Chatzi. Mitigation of rayleigh-like waves in granular media via multilayer resonant metabarriers. *Applied Physics Letters*, 117(25), 2020.
- [76] A. Quagliata, M. Ahearn, E. Boeker, C. Roof, L. Meister, and H. Singleton. Transit noise and vibration impact assessment manual. Technical report, 2018.
- [77] P. M. Morse and K. U. Ingard. *Theoretical Acoustics*. McGraw-Hill, 1968.
- [78] A. D. Pierce. *Acoustics: An Introduction to Its Physical Principles and Applications*. Acoustical Society of America, 1989.
- [79] H. Kuttruff. *Room Acoustics*. CRC Press, 5 edition, 2009.
- [80] T. J. Cox and P. D'Antonio. *Acoustic Absorbers and Diffusers: Theory, Design and Application*. Taylor & Francis, 2004.
- [81] M. Barron. *Auditorium Acoustics and Architectural Design*. Spon Press, 2 edition, 2010.
- [82] M. Vorländer. *Auralization: Fundamentals of Acoustics, Modelling, Simulation, Algorithms and Acoustic Virtual Reality*. Springer, 2008.
- [83] J. B. Allen and D. A. Berkley. Image method for efficiently simulating small-room acoustics. *Journal of the Acoustical Society of America*, 65(4):943–950, 1979.
- [84] X. Zhang, Z. Qu, and H. Wang. Engineering acoustic metamaterials for sound absorption: from uniform to gradient structures. *iScience*, 23:101110, 2020.
- [85] M. Yang and P. Sheng. Sound absorption structures: from porous media to acoustic metamaterials. *Annual Review of Materials Research*, 47:83–114, 2017.

- [86] C. Dong, Z. Liu, R. Pierce, X. Liu, and X. Yi. Sound absorption performance of a micro perforated sandwich panel with honeycomb-hierarchical pore structure core. *Applied Acoustics*, 203:109200, 2023.
- [87] L. Zhao and S. Zhou. Compact acoustic rainbow trapping in a bioinspired spiral array of graded locally resonant metamaterials. *Sensors*, 19:788, 2019.
- [88] M. Yang, G. Ma, Z. Yang, and P. Sheng. Coupled membranes with doubly negative mass density and bulk modulus. *Physical Review Letters*, 110:134301, 2013.
- [89] G. Ma, M. Yang, S. Xiao, Z. Yang, and P. Sheng. Acoustic metasurface with hybrid resonances. *Nature Materials*, 13:873–878, 2014.
- [90] N. Jiménez, W. Huang, V. Romero-García, V. Pagneux, and J.-P. Groby. Ultra-thin metamaterial for perfect and quasi-omnidirectional sound absorption. *Applied Physics Letters*, 109:121902, 2016.
- [91] V. Romero-García, G. Theocharis, O. Richoux, and V. Pagneux. Use of complex frequency plane to design broadband and sub-wavelength absorbers. *Journal of the Acoustical Society of America*, 139:3395–3403, 2016.
- [92] S. Kumar and H. Lee. The present and future role of acoustic metamaterials for architectural and urban noise mitigations. *Acoustics*, 1:590–607, 2019.
- [93] J. Li, X. Wen, and P. Sheng. Acoustic metamaterials. *Journal of Applied Physics*, 129:171103, 2021.
- [94] G. Palma, H. Mao, L. Burghignoli, P. Göransson, and U. Iemma. Acoustic metamaterials in aeronautics. *Applied Sciences*, 8:971, 2018.
- [95] V. Romero-García, G. Theocharis, O. Richoux, and V. Pagneux. Perfect and broadband acoustic absorption by critically coupled subwavelength resonators. *Scientific Reports*, 6:19519, 2016.
- [96] K. L. Tsakmakidis, A. D. Boardman, and O. Hess. Trapped rainbow storage of light in metamaterials. *Nature*, 450:397–401, 2007.
- [97] J. Zhu, J. Christensen, J. Jung, L. Martin-Moreno, X. Yin, L. Fok, X. Zhang, and F. J. Garcia-Vidal. Acoustic rainbow trapping. *Scientific Reports*, 3:1728, 2013.
- [98] J. Christensen, M. Willatzen, V. R. Velasco, and M.-H. Lu. Parity-time synthetic phononic media. *Physical Review Letters*, 116(20):207601, 2016.
- [99] N. Jiménez, V. Romero-García, V. Pagneux, and J.-P. Groby. Rainbow-trapping absorbers: Broadband, perfect and asymmetric sound absorption by subwavelength panels for transmission problems. *Scientific reports*, 7(1):13595, 2017.

- [100] Y. Peng et al. Gradient acoustic metasurfaces for broadband sound absorption. *Applied Physics Letters*, 114:091904, 2019.
- [101] M. Yang and P. Sheng. Subwavelength total acoustic absorption with degenerate resonators. *Scientific Reports*, 5:10980, 2015.
- [102] G. Ma and P. Sheng. Acoustic metasurface with hybrid resonances. *Science Advances*, 2(2):e1501595, 2016.
- [103] Z. Liang and J. Li. Extreme acoustic metamaterial by coiling up space. *Physical Review Letters*, 108(11):114301, 2012.
- [104] Z. Liang and J. Li. Extreme acoustic metamaterial by coiling up space. *Physical Review Letters*, 108:114301, 2012.
- [105] Z. Liang, T. Feng, S. Lok, F. Liu, K. B. Ng, C. H. Chan, J. Wang, S. Han, S. Lee, and J. Li. Space-coiling metamaterials with double negativity and conical dispersion. *Scientific Reports*, 3:1614, 2013.
- [106] M. R. Stinson. The propagation of plane sound waves in narrow and wide circular tubes. *Journal of the Acoustical Society of America*, 89:550–558, 1991.
- [107] C. Zwikker and C. W. Kosten. *Sound Absorbing Materials*. Elsevier, 1949.
- [108] H. Kuttruff. *Acoustics: An Introduction*. CRC Press, 2007.
- [109] Y. Xie, B.-I. Popa, L. Zigoneanu, and S. A. Cummer. Tapered labyrinthine acoustic metamaterials for broadband impedance matching. *Applied Physics Letters*, 103(20):201906, 2013.
- [110] C. Shen, Y. Xie, B.-I. Popa, L. Zigoneanu, and S. A. Cummer. Broadband acoustic metamaterial with space-coiling channels for sound absorption. *Applied Physics Letters*, 105(10):104101, 2014.
- [111] G. Kirchhoff. Ueber den einfluss der wärmeleitung in einem gase auf die schallbewegung. *Annalen der Physik*, 210:177–193, 1868.
- [112] A. Magnani, C. Marescotti, and F. Pompoli. Acoustic absorption modeling of single and multiple coiled-up resonators. *Applied Acoustics*, 186:108504, 2022.
- [113] L. Shtrepi and A. Prato. Towards a sustainable approach for sound absorption assessment of building materials: Validation of small-scale reverberation room measurements. *Applied Acoustics*, 165:107304, 2020.
- [114] A. De Bruijn. On the scattering of a plane wave by porous sound-absorbing strip. *Journal of the Acoustical Society of America*, 123(5):3141, 2008.

- [115] A. Duval, J.-F. Rondeau, L. Dejeager, F. Sgard, and N. Atalla. Diffuse field absorption coefficient simulation of porous materials in small reverberant rooms: Finite size and diffusivity issues. In *10ème Congrès Français d'Acoustique*, 2010.
- [116] B. A. Auld. *Acoustic Fields and Waves in Solids*. A Wiley-Interscience publication. Wiley, 1973.
- [117] K. F. Graff. *Wave Motion in Elastic Solids*. Courier Corporation, 2012.
- [118] B. Wang, S. Zhong, T.-L. Lee, K. S. Fancey, and J. Mi. Non-destructive testing and evaluation of composite materials/structures: A state-of-the-art review. *Advances in mechanical engineering*, 12(4):1687814020913761, 2020.
- [119] J. L. Rose. *Ultrasonic guided waves in solid media*. Cambridge university press, 2014.
- [120] C. Yeung and C. T. Ng. Time-domain spectral finite element method for analysis of torsional guided waves scattering and mode conversion by cracks in pipes. *Mechanical Systems and Signal Processing*, 128:305–317, 2019.
- [121] C. Prada, D. Clorennec, and D. Royer. Local vibration of an elastic plate and zero-group velocity lamb modes. *The Journal of the Acoustical Society of America*, 124(1):203–212, 2008.
- [122] A. Khelif and A. Adibi. Phononic crystals. *Berlin, Germany, Springer*, 10:978–1, 2015.
- [123] V. Laude. *Phononic Crystals*. De Gruyter, 2020.
- [124] R. Craster, S. Guenneau, M. Kadic, and M. Wegener. Mechanical metamaterials. *Reports on Progress in Physics*, 86(9):094501, 2023.
- [125] A. O. Krushynska, D. Torrent, A. M. Aragón, R. Ardito, O. R. Bilal, B. Bonello, F. Bosia, Y. Chen, J. Christensen, A. Colombi, et al. Emerging topics in nanophononics and elastic, acoustic, and mechanical metamaterials: an overview. *Nanophotonics*, 12(4):659–686, 2023.
- [126] P. Jiao, J. Mueller, J. R. Raney, X. Zheng, and A. H. Alavi. Mechanical metamaterials and beyond. *Nature communications*, 14(1):6004, 2023.
- [127] S. Pyo and K. Park. Mechanical metamaterials for sensor and actuator applications. *International Journal of Precision Engineering and Manufacturing-Green Technology*, 11(1):291–320, 2024.
- [128] M. Miniaci, A. S. Gliozzi, B. Morvan, A. Krushynska, F. Bosia, M. Scalerandi, and N. M. Pugno. Proof of concept for an ultrasensitive technique to detect and localize sources of elastic nonlinearity using phononic crystals. *Physical Review Letters*, 118(21):214301, 2019.

- [129] J. M. De Ponti, A. Colombi, R. Ardito, F. Braghin, A. Corigliano, and R. V. Craster. Graded elastic metasurface for enhanced energy harvesting. *New Journal of Physics*, 22(1):013013, 2020.
- [130] G. Lee, S.-J. Lee, J. Rho, and M. Kim. Acoustic and mechanical metamaterials for energy harvesting and self-powered sensing applications. *Materials Today Energy*, page 101387, 2023.
- [131] A. S. Gliozzi, M. Miniaci, A. O. Krushynska, B. Morvan, M. Scalerandi, N. M. Pugno, and F. Bosia. Proof of concept of a frequency-preserving and time-invariant metamaterial-based nonlinear acoustic diode. *Scientific Reports*, 9:9560, 2019.
- [132] J. M. De Ponti, E. Riva, F. Braghin, and R. Ardito. Elastic three-dimensional metaframe for selective wave filtering and polarization control. *Applied Physics Letters*, 119(21), 2021.
- [133] V. Laude. Principles and properties of phononic crystal waveguides. *Appl Materials*, 9(8), 2021.
- [134] A. Sukhovich, B. Merheb, K. Muralidharan, J. O. Vasseur, Y. Pennec, P. A. Deymier, and J. H. Page. Experimental and theoretical evidence for subwavelength imaging in phononic crystals. *Physical review letters*, 102(15):154301, 2009.
- [135] G. R. Sherwood, D. Chronopoulos, A. Marini, and F. Ciampa. 3D-printed phononic crystal waveguide transducers for nonlinear ultrasonic damage detection. *NDT & E International*, 121:102456, 2021.
- [136] J. H. Page, A. Sukhovich, S. Yang, M. L. Cowan, F. Van Der Biest, A. Tourin, M. Fink, Z. Liu, C. T. Chan, and P. Sheng. Phononic crystals. *physica status solidi (b)*, 241(15):3454–3462, 2004.
- [137] M.-H. Lu, L. Feng, and Y.-F. Chen. Phononic crystals and acoustic metamaterials. *Materials today*, 12(12):34–42, 2009.
- [138] S. L. Zhang, Y.-C. Lai, X. He, R. Liu, Y. Zi, and Z. L. Wang. Auxetic foam-based contact-mode triboelectric nanogenerator with highly sensitive self-powered strain sensing capabilities to monitor human body movement. *Advanced functional materials*, 27(25):1606695, 2017.
- [139] A. Mohammadi, Y. Tan, P. Choong, and D. Oetomo. Flexible mechanical metamaterials enabling soft tactile sensors with multiple sensitivities at multiple force sensing ranges. *Scientific reports*, 11(1):24125, 2021.
- [140] M. Molerón and C. Daraio. Acoustic metamaterial for subwavelength edge detection. *Nature communications*, 6(1):8037, 2015.
- [141] Y. Chen, H. Liu, M. Reilly, H. Bae, and M. Yu. Enhanced acoustic sensing through wave compression and pressure amplification in anisotropic metamaterials. *Nature communications*, 5(1):5247, 2014.

- [142] V. F. Dal Poggetto, F. Bosia, D. Urban, P. H. Beoletto, J. Torgersen, N. M. Pugno, and A. S. Gliozzi. Cochlea-inspired tonotopic resonators. *Materials & Design*, 227:111712, 2023.
- [143] A. Colombi. Seismic metamaterials. *Sci. Rep.*, 2016.
- [144] G. J. Chaplain, J. M. De Ponti, A. Colombi, R. Fuentes-Dominguez, P. Dryburg, D. Pieris, R. J. Smith, A. Clare, M. Clark, and R. V. Craster. Tailored elastic surface to body wave Umklapp conversion. *Nature communications*, 11(1):3267, 2020.
- [145] Y. Tian, Y. Song, Y. Shen, and Z. Yu. A metamaterial ultrasound mode convertor for complete transformation of Lamb waves into shear horizontal waves. *Ultrasonics*, 119:106627, 2022.
- [146] Q. Wang, W. Yu, H. Chang, D. Qiujiào, F. Liu, Z. Liu, and P. Peng. Total conversion between the longitudinal and transverse waves by ultrathin elastic metamaterials with anisotropic resonances. *Applied Physics Express*, 15(11):117001, 2022.
- [147] L. Cao, Y. Zhu, Y. Xu, S.-W. Fan, Z. Yang, and B. Assouar. Elastic bound state in the continuum with perfect mode conversion. *Journal of the Mechanics and Physics of Solids*, 154:104502, 2021.
- [148] V. F. Dal Poggetto, R. K. Pal, N. M. Pugno, and M. Miniaci. Topological bound modes in phononic lattices with nonlocal interactions. *International Journal of Mechanical Sciences*, 281:109503, 2024.
- [149] J. M. De Ponti, L. Iorio, E. Riva, R. Ardito, F. Braghin, and A. Corigliano. Selective mode conversion and rainbow trapping via graded elastic waveguides. *Physical Review Applied*, 16(3):034028, 2021.
- [150] L. Iorio, J. M. De Ponti, F. Maspero, and R. Ardito. Roton-like dispersion via polarisation change for elastic wave energy control in graded delay-lines. *Journal of Sound and Vibration*, 572:118167, 2024.
- [151] B. Lemkalli, M. Kadic, Y. El Badri, S. Guenneau, A. Mir, and Y. Achaoui. Longitudinal-twist wave converter based on chiral metamaterials. *New Journal of Physics*, 26(7):073026, 2024.
- [152] X. Yang, M. Kweun, and Y. Y. Kim. Monolayer metamaterial for full mode-converting transmission of elastic waves. *Applied Physics Letters*, 115(7), 2019.
- [153] J. M. Kweun, H. J. Lee, J. H. Oh, H. M. Seung, and Y. Y. Kim. Transmodal Fabry-Pérot resonance: theory and realization with elastic metamaterials. *Physical review letters*, 118(20):205901, 2017.
- [154] S. Yao, Y. Chai, X. Yang, and Y. Li. Elastic twisting metamaterial for perfect longitudinal-torsional wave mode conversion. *Applied Mathematics and Mechanics*, 44(4):515–524, 2023.

- [155] A. Pelat, T. Gallot, and F. Gautier. On the control of the first Bragg band gap in periodic continuously corrugated beam for flexural vibration. *Journal of Sound and Vibration*, 446:249–262, 2019.
- [156] V. F. Dal Poggetto and J. R. de França Arruda. Widening wave band gaps of periodic plates via shape optimization using spatial Fourier coefficients. *Mechanical Systems and Signal Processing*, 147:107098, 2021.
- [157] Z.-L. Xu, S.-F. Xu, and K.-C. Chuang. Coupled flexural-longitudinal waves in an origami metamaterial with uncoupled creases. *Physics Letters A*, 396:127232, 2021.
- [158] G. Trainiti, J. J. Rimoli, and M. Ruzzene. Wave propagation in periodically undulated beams and plates. *International Journal of Solids and Structures*, 75:260–276, 2015.
- [159] S. R. Huisman, R. V. Nair, A. Hartsuiker, L. A. Woldering, A. P. Mosk, and W. L. Vos. Observation of sub-Bragg diffraction of waves in crystals. *Physical review letters*, 108(8):083901, 2012.
- [160] M. O. Gzal, L. A. Bergman, K. H. Matlack, and A. F. Vakakis. Sub-Bragg phenomena in multilayered vibroacoustic phononic metamaterials. *arXiv preprint arXiv:2410.01178*, 2024.
- [161] M. O. Gzal, J. R. Tempelman, K. H. Matlack, and A. F. Vakakis. Subwavelength topological interface modes in a multilayered vibroacoustic metamaterial. *Frontiers in Acoustics*, 2:1489401, 2024.
- [162] P. Chidamparam and A. W. Leissa. Vibrations of planar curved beams, rings, and arches. *Applied Mechanics Reviews*, 46(9):467–483, 1993.
- [163] S. Timoshenko and J. M. Gere. *Mechanics of Materials*. Van Nostrand Reinhold Company, 1972.
- [164] V. F. Dal Poggetto and A. L. Serpa. Flexural wave band gaps in a ternary periodic metamaterial plate using the plane wave expansion method. *Journal of Sound and Vibration*, 495:115909, 2021.
- [165] V. F. Dal Poggetto, E. J. P. Miranda Jr, J. M. C. Dos Santos, and N. M. Pugno. Wave attenuation in viscoelastic hierarchical plates. *International Journal of Mechanical Sciences*, 236:107763, 2022.
- [166] F. Bloch. Über die Quantenmechanik der Elektronen in Kristallgittern. *Zeitschrift für Physik*, 52(7-8):555–600, 1929.
- [167] B. R. Mace and E. Manconi. Modelling wave propagation in two-dimensional structures using finite element analysis. *Journal of Sound and Vibration*, 318(4-5):884–902, 2008.
- [168] R. D. Cook. *Concepts and applications of finite element analysis*. John Wiley & Sons, 2007.

- [169] B. R. Mace and E. Manconi. Wave motion and dispersion phenomena: Veering, locking and strong coupling effects. *The Journal of the Acoustical Society of America*, 131(2):1015–1028, 2012.
- [170] Y. Pronina, M. Narykova, and M. Kachanov. Relating stiffness changes in porous materials to the evolution of pore space. *Mechanics of Materials*, 202:105236, 2025.
- [171] R. A. Hardin and C. Beckermann. Effect of porosity on the stiffness of cast steel. *Metallurgical and Materials Transactions A*, 38(12):2992–3006, 2007.
- [172] A. Fathi, B. Poursartip, and L. F. Kallivokas. Time-domain hybrid formulations for wave simulations in three-dimensional PML-truncated heterogeneous media. *International Journal for Numerical Methods in Engineering*, 101(3):165–198, 2015.
- [173] Y. Liu, V. F. Dal Poggetto, A. S. Gliozzi, N. M. Pugno, F. Bosia, and M. Tortello. Seashell-inspired polarization-sensitive tonotopic metasensor. *APL Materials*, 12(4), 2024.
- [174] F. D. Murnaghan. Finite deformations of an elastic solid. *American Journal of Mathematics*, 59(2):235–260, 1937.
- [175] D. S. Hughes and J. L. Kelly. The second-order elastic deformation of solids. *Physical Review*, 92(5):1145–1149, 1953.
- [176] M. A. Biot. *Mechanics of Incremental Deformations*. Wiley, 1965.
- [177] R. N. Thurston and K. Brugger. Elastic waves in rods and plates under initial stress. *The Journal of the Acoustical Society of America*, 36(3):548–557, 1964.
- [178] R. W. Ogden. *Non-Linear Elastic Deformations*. Dover, 1997.
- [179] A. Delory, F. Lemoult, M. Lanoy, A. Eddi, and C. Prada. Soft elastomers: A playground for guided waves. *The Journal of the Acoustical Society of America*, 151(5):3343–3358, 2022.
- [180] A. Delory, F. Lemoult, A. Eddi, and C. Prada. Guided elastic waves in a highly-stretched soft plate. *Extreme Mechanics Letters*, 61:102018, 2023.
- [181] J. L. Gennisson, M. Renier, S. Catheline, C. Barrière, J. Bercoff, M. Tanter, and M. Fink. Acoustoelasticity in soft solids: assessment of the nonlinear shear modulus. *The Journal of the Acoustical Society of America*, 122(6):3216–3219, 2007.
- [182] J. Rivière, G. Renaud, R. A. Guyer, and P. A. Johnson. Pump and probe waves in dynamic acousto-elasticity: Comprehensive description and comparison with nonlinear elastic theories. *Journal of applied physics*, 114(5), 2013.

- [183] M. Scalerandi, C. Mechri, M. Bentahar, A. Di Bella, A. S. Gliozzi, and M. Tortello. Role of slow dynamics in fast dynamics ultrasonic measurements. *Communications in Nonlinear Science and Numerical Simulation*, 91:105452, 2020.
- [184] J. A. Ten Cate and T. J. Shankland. Slow dynamics in the nonlinear elastic response of berea sandstone. *Geophysical Research Letters*, 23(21):3019–3022, 1996.
- [185] A. Delory, D. A. Kiefer, M. Lanoy, A. Eddi, C. Prada, and F. Lemoult. Viscoelastic dynamics of a soft strip subject to a large deformation. *Soft Matter*, 20(9):1983–1995, 2024.
- [186] A. Delory, C. Prada, M. Lanoy, A. Eddi, M. Fink, and F. Lemoult. Elastic wave packets crossing a space-time interface. *Physical Review Letters*, 133(26):267201, 2024.
- [187] J. F. Doyle. Wave propagation in structures. In *Wave Propagation in Structures: An FFT-Based Spectral Analysis Methodology*, pages 126–156. Springer, 1989.
- [188] W. D. Zhu and J. Ni. Energetics and stability of translating media with an arbitrarily varying length. *J. Vib. Acoust.*, 122(3):295–304, 2000.
- [189] L.-Q. Chen and X.-D. Yang. Steady-state response of axially moving viscoelastic beams with pulsating speed: comparison of two nonlinear models. *International Journal of Solids and Structures*, 42(1):37–50, 2005.
- [190] L.-Q. Shui, Z.-F. Yue, Y.-S. Liu, Q.-C. Liu, and J.-J. Guo. One-dimensional linear elastic waves at moving property interface. *Wave Motion*, 51(7):1179–1192, 2014.
- [191] E. Hubble. A relation between distance and radial velocity among extragalactic nebulae. *Proceedings of the National Academy of Sciences*, 15:168–173, 1929.
- [192] J.-P. Uzan. The fundamental constants and their variation: observational and theoretical status. *General Relativity and Gravitation*, 39:307–342, 2007.
- [193] B. P. E. A. Abbott. Observation of gravitational waves from a binary black hole merger. *Physical Review Letters*, 116:061102, 2016.