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Subspace Geometry of Pareto-Optimal ISAC

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Abstract—Integrated sensing and communications (ISAC) unifies radar-style sensing and wireless data transmission on a shared spectral and hardware platform. While its empirical motivations are well documented, rigorous design requires a geometric understanding of the tradeoff between sensing performance and communication reliability. This paper develops a theoretical ISAC framework based on a dual-functional MIMO transmitter and focuses on the geometry of covariance (second-order) design. A unified baseband signal model is introduced together with canonical metrics for sensing and communication expressed through Fisher information and achievable rates. Using information-theoretic and estimation-theoretic arguments, we characterize the Pareto-optimal tradeoff surface and show that optimal transmit covariances concentrate on low-dimensional informative subspaces. The resulting geometric interpretation provides a clear view of ISAC tradeoffs and complements existing RadCom studies [1]–[4].

Index Terms—Integrated sensing and communications, Dual-functional radar-communications, MIMO, Beamforming, Fisher information, Cramér–Rao bound, Pareto optimality

I. INTRODUCTION

The recent evolution of wireless systems has shifted from purely data-centric networks toward platforms that also perceive the physical environment. Integrated sensing and communications (ISAC)—also known as joint radar-communications (JRC)—pursues a single radio stack that *i*) illuminates the environment to estimate parameters such as range, Doppler, angle, or reflection coefficients, and *ii*) simultaneously conveys data to intended receivers using the same time–frequency–spatial resources. This integration is motivated by spectrum scarcity, hardware reuse, and emerging 6G use cases where communication and situational awareness must co-exist [1]–[4].

A core obstacle to systematic ISAC design is that sensing and communications evaluate “good” waveforms differently. Communications favor signals that maximize mutual information (achievable rate) under channel and power constraints, whereas sensing favors signals that optimize detection or estimation performance, often quantified via Fisher information, ambiguity functions, or beampattern metrics. This contrast is quantitative and naturally leads to a constrained multi-objective optimization problem combining information-theoretic and statistical criteria. This paper makes that coupling explicit and derives its analytical consequences.

A. Contributions and Scope

We develop a self-contained mathematical framework for ISAC that exposes the sensing–communication tradeoff at the level of transmit covariance and beamforming geometry.

The main contribution is a unified formalism yielding provable statements on the structure of Pareto-optimal designs, including informative-subspace characterizations, rank properties of optimal covariances, and continuous interpolation between sensing-dominant and communication-dominant operating points. Related ISAC/JRC work spans signal models and joint waveform/beamforming algorithms [1]–[3], surveys emphasizing architectural convergence [4], and information-theoretic rate–estimation tradeoffs [5], [6]. In contrast, we focus on the geometric structure of Pareto-optimal transmit covariances, deriving informative-subspace and rank characterizations that clarify the tradeoff geometry and complement Cramér–Rao bound (CRB)-driven numerical designs [7], [8].

II. SYSTEM MODEL: DUAL-FUNCTIONAL MIMO ISAC

We consider a downlink integrated sensing and communications (ISAC) system where a multi-antenna transmitter simultaneously serves communication users and senses the environment.

A. Transmit Signal and Power Constraint

Consider a transmitter with M antennas operating over a coherent processing interval of T complex baseband samples. The transmitted MIMO signal is collected into the matrix

$$\mathbf{X} = [\mathbf{x}[1], \dots, \mathbf{x}[T]] \in \mathbb{C}^{M \times T}, \quad (1)$$

where $\mathbf{x}[t] \in \mathbb{C}^M$ is the transmit vector at time t . The following average power constraint is assumed:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E} \|\mathbf{x}[t]\|_2^2 = \frac{1}{T} \mathbb{E} \|\mathbf{X}\|_F^2 \leq P. \quad (2)$$

A central design variable is the transmit covariance. We assume that the vectors $\mathbf{x}[t]$ are independent and identically distributed across time. Under a block-fading channel model, where the channel remains constant over a coherence interval of T symbols, we therefore focus on the spatial transmit covariance

$$\mathbf{Q} \triangleq \mathbb{E}[\mathbf{x}[t]\mathbf{x}[t]^H] \succeq \mathbf{0}, \quad \text{tr}(\mathbf{Q}) \leq P. \quad (3)$$

B. Communication Downlink

Let K single-antenna users observe

$$y_k[t] = \mathbf{h}_k^H \mathbf{x}[t] + n_k[t], \quad k = 1, \dots, K, \quad (4)$$

where $\mathbf{h}_k \in \mathbb{C}^M$ is the (known) downlink channel and $n_k[t] \sim \mathcal{CN}(0, \sigma_k^2)$ is proper complex Gaussian noise. Assuming Gaussian signaling with covariance $\mathbf{Q}$ and single-user decoding, we use the interference-free per-user benchmark

$$R_k(\mathbf{Q}) = \log_2 \left(1 + \frac{\mathbf{h}_k^H \mathbf{Q} \mathbf{h}_k}{\sigma_k^2} \right), \quad (5)$$

and typical system objectives include the weighted sum-rate $\sum_k w_k R_k(\mathbf{Q})$ or max-min fairness functionals. These rate-based metrics are used as canonical communication objectives in an idealized no-inter-user-interference case [2], [3].

C. Sensing Channel (Monostatic, Narrowband)

We adopt a monostatic narrowband sensing model in which the transmitter also acts as a receiver (full duplex with ideal self-interference cancellation). The received sensing return over T samples is [1], [3]

$$\mathbf{z}[t] = \alpha \mathbf{a}(\theta) \mathbf{a}(\theta)^H \mathbf{x}[t] + \mathbf{w}[t], \quad t = 1, \dots, T, \quad (6)$$

where $\alpha \in \mathbb{C}$ is a deterministic unknown complex reflection coefficient, θ is an unknown angle, $\mathbf{a}(\theta) \in \mathbb{C}^M$ is the *unit-norm* array steering vector, and $\mathbf{w}[t] \sim \mathcal{CN}(\mathbf{0}, \sigma_s^2 \mathbf{I})$ is sensing noise. Models of this type are standard in communication-centric radar sensing formulations and provide a bridge to waveform/covariance design [1], [3].

III. SENSING METRIC: FISHER INFORMATION AND CRB

Focusing on the estimation of θ in (6), we first treat α as deterministic and known; when α is unknown, it is handled as a nuisance parameter via Schur-complement elimination [9] (see (15)). In particular, let $\mathbf{a} \triangleq \mathbf{a}(\theta)$ and $\mathbf{b} \triangleq \partial \mathbf{a}(\theta) / \partial \theta$ (we henceforth drop the explicit dependence on θ for the sake of notation compactness). Under the covariance model (3), the sensing signal-to-noise structure involves quadratic forms in $\mathbf{Q}$.

Remark III.1 (Canonical gauge for steering vectors). *Let $\mathbf{a}(\theta) \in \mathbb{C}^M$ be a differentiable steering vector with $\|\mathbf{a}\|_2 = 1$. Differentiating $\mathbf{a}^H \mathbf{a} = 1$ yields $\Re\{\mathbf{a}^H \mathbf{b}\} = 0$. Furthermore, $\mathbf{a}$ is defined up to a θ -dependent phase factor, which can be chosen so that $\mathbf{a}^H \mathbf{b} = 0$ holds identically. This gauge choice is without loss of generality and leaves all second-order statistics and Fisher-information expressions unchanged.*

Proposition III.2. *Assume $\mathbf{x}[t] \sim \mathcal{CN}(\mathbf{0}, \mathbf{Q})$ i.i.d. across t and consider (6) with known α and unknown θ . Assume the gauge $\mathbf{a}^H \mathbf{b} = 0$, recall that $\mathbf{a}$ is unit-norm, and let $\mathbf{b} = \|\mathbf{b}\|_2 \bar{\mathbf{b}}$. Then, adopting the average Fisher information for θ*

$$\bar{\mathcal{I}}_\theta(\mathbf{Q}) \triangleq \mathbb{E}[\mathcal{I}_\theta(\mathbf{X})] = \frac{2T|\alpha|^2 \|\mathbf{b}\|_2^2}{\sigma_s^2} (\mathbf{a}^H \mathbf{Q} \mathbf{a} + \bar{\mathbf{b}}^H \mathbf{Q} \bar{\mathbf{b}}) \quad (7)$$

as sensing metric, where $\mathbf{X} = (\mathbf{x}[1], \dots, \mathbf{x}[T])$ is the transmitted sequence, any conditionally unbiased estimator $\hat{\theta}$ obeys the following Jensen-relaxed bound based on average FIM:

$$\text{Var}(\hat{\theta}) = \mathbb{E}[\text{Var}(\hat{\theta}|\mathbf{X})] \geq \mathbb{E}\left[\frac{1}{\mathcal{I}_\theta(\mathbf{X})}\right] \geq \frac{1}{\bar{\mathcal{I}}_\theta(\mathbf{Q})}. \quad (8)$$

Proof. For proper complex Gaussian observations $\mathbf{z}[t]$ with mean $\boldsymbol{\mu}_t = \alpha \mathbf{a} \mathbf{a}^H \mathbf{x}[t]$ and covariance $\sigma_s^2 \mathbf{I}$, the FIM equals $\frac{2}{\sigma_s^2} \sum_{t=1}^T \|\partial \boldsymbol{\mu}_t / \partial \theta\|_2^2$ [9]. From $\mathbf{a}^H \mathbf{b} = 0$ and $\mathbb{E}[\mathbf{x}[t] \mathbf{x}[t]^H] = \mathbf{Q}$ we get (7) and (8) derives from estimator unbiasedness, law of total variance, and Jensen's inequality. $\square$

Eq. (7) shows the key structural point: sensing quality is improved by allocating transmit covariance to increase both $\mathbf{a}^H \mathbf{Q} \mathbf{a}$ and $\bar{\mathbf{b}}^H \mathbf{Q} \bar{\mathbf{b}}$.

Communication quality, by (5), is improved by allocating covariance to directions favored by user channels $\mathbf{h}_k$. These directions generally differ, creating the ISAC tradeoff emphasized in the ISAC/JRC literature [2]–[4].

A. Rank-3 Covariance Tradeoff

To understand the tradeoff in a minimal setting, consider a single user ($K = 1$) with channel $\mathbf{h}$ and a single sensing parameter. Define

$$R(\mathbf{Q}) \triangleq \log_2 \left(1 + \frac{\mathbf{h}^H \mathbf{Q} \mathbf{h}}{\sigma_1^2} \right) \quad (\text{Commun. objective}) \quad (9)$$

$$\Phi(\mathbf{Q}) \triangleq \frac{1}{\bar{\mathcal{I}}_\theta(\mathbf{Q})} \quad (\text{Sensing loss}) \quad (10)$$

Now, we can use Pareto optimality to characterize the sensing–communication tradeoff. The covariance $\mathbf{Q}$ is *Pareto-optimal* if any strict improvement in one objective implies strict degradation in the other. The set of Pareto-optimal points forms the *Pareto frontier*, which captures the fundamental tradeoff between sensing and communication performance under the shared constraints.

Theorem III.3 (Rank-3 Pareto Optimality). *Assume $K = 1$ and let $\mathbf{x}[t] \sim \mathcal{CN}(\mathbf{0}, \mathbf{Q})$ i.i.d. across t with $\mathbf{Q} \succeq \mathbf{0}$ and $\text{tr}(\mathbf{Q}) \leq P$. Let the communication objective be $R(\mathbf{Q})$ in (9) and let the sensing loss be any function $\Phi(\mathbf{Q}) = \varphi(u, v)$ of $u = \mathbf{a}^H \mathbf{Q} \mathbf{a}$, $v = \bar{\mathbf{b}}^H \mathbf{Q} \bar{\mathbf{b}}$, where $\varphi(u, v)$ is non-increasing in each argument (e.g., $\Phi(\mathbf{Q}) = 1/\bar{\mathcal{I}}_\theta(\mathbf{Q})$ with $\bar{\mathcal{I}}_\theta(\mathbf{Q})$ in (7)).*

Then, any Pareto-optimal covariance $\mathbf{Q}^$ for the bi-objective problem*

$$\max_{\mathbf{Q} \succeq \mathbf{0}, \text{tr}(\mathbf{Q}) \leq P} R(\mathbf{Q}) \quad \text{and} \quad \min_{\mathbf{Q} \succeq \mathbf{0}, \text{tr}(\mathbf{Q}) \leq P} \Phi(\mathbf{Q})$$

satisfies

$$\text{range}(\mathbf{Q}^*) \subseteq \text{span}\{\mathbf{h}, \mathbf{a}, \bar{\mathbf{b}}\}, \quad (11)$$

so that $\text{rank}(\mathbf{Q}^*) \leq 3$. Moreover, the supported Pareto-optimal points are obtained by the solution of the following family of maximization problems, where $\lambda_a, \lambda_b \geq 0$ weight power allocation toward the informative directions $\mathbf{a}$ and $\bar{\mathbf{b}}$:

$$\begin{aligned} \max_{\mathbf{Q} \succeq \mathbf{0}} \quad & R(\mathbf{Q}) + \lambda_a \mathbf{a}^H \mathbf{Q} \mathbf{a} + \lambda_b \bar{\mathbf{b}}^H \mathbf{Q} \bar{\mathbf{b}} \\ \text{s.t.} \quad & \text{tr}(\mathbf{Q}) \leq P. \end{aligned} \quad (12)$$

Any optimizer of (12) is Pareto-optimal and satisfies (11).

Proof. Let $\mathcal{U} \triangleq \text{span}\{\mathbf{h}, \mathbf{a}, \bar{\mathbf{b}}\}$ and decompose any feasible $\mathbf{Q}$ as $\mathbf{Q} = \mathbf{Q}_\parallel + \mathbf{Q}_\perp$ where $\mathbf{Q}_\parallel = \mathbf{P}_\mathcal{U} \mathbf{Q} \mathbf{P}_\mathcal{U}$ and $\mathbf{Q}_\perp = \mathbf{P}_{\mathcal{U}^\perp} \mathbf{Q} \mathbf{P}_{\mathcal{U}^\perp}$, with $\mathbf{P}_\mathcal{U}$ the orthogonal projector onto $\mathcal{U}$.¹

¹If $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is an orthonormal base of $\mathcal{U}$, then $\mathbf{P}_\mathcal{U} = \sum_{i=1}^r \mathbf{u}_i \mathbf{u}_i^H$.

Since $\mathbf{h}, \mathbf{a}, \bar{\mathbf{b}} \in \mathcal{U}$, we have $\mathbf{h}^H \mathbf{Q} \mathbf{h} = \mathbf{h}^H \mathbf{Q}_{\parallel} \mathbf{h}$, $\mathbf{a}^H \mathbf{Q} \mathbf{a} = \mathbf{a}^H \mathbf{Q}_{\parallel} \mathbf{a}$, and $\bar{\mathbf{b}}^H \mathbf{Q} \bar{\mathbf{b}} = \bar{\mathbf{b}}^H \mathbf{Q}_{\parallel} \bar{\mathbf{b}}$. Hence both $R(\mathbf{Q})$ and $\Phi(\mathbf{Q})$ depend only on $\mathbf{Q}_{\parallel}$.

Moreover, $\text{tr}(\mathbf{Q}_{\parallel}) < \text{tr}(\mathbf{Q})$ if $\mathbf{Q}_{\perp} \neq \mathbf{0}$. In this case, let $\hat{\mathbf{Q}} = \frac{\text{tr}(\mathbf{Q})}{\text{tr}(\mathbf{Q}_{\parallel})} \mathbf{Q}_{\parallel}$, so that $\hat{\mathbf{Q}} \succeq \mathbf{0}$ and $\text{tr}(\hat{\mathbf{Q}}) = \text{tr}(\mathbf{Q}) \leq P$. Also, the inequality $\mathbf{h}^H \hat{\mathbf{Q}} \mathbf{h} > \mathbf{h}^H \mathbf{Q}_{\parallel} \mathbf{h}$ holds, so that $R(\hat{\mathbf{Q}}) > R(\mathbf{Q})$, while $\Phi(\hat{\mathbf{Q}}) \leq \Phi(\mathbf{Q})$, since $u = \mathbf{a}^H \mathbf{Q} \mathbf{a}$ and $v = \bar{\mathbf{b}}^H \mathbf{Q} \bar{\mathbf{b}}$ both increase under the scaling and $\varphi(u, v)$ is non-increasing in each argument.

Thus any Pareto-optimal $\mathbf{Q}^*$ must have $\mathbf{Q}_{\perp} = \mathbf{0}$, which is equivalent to (11). Finally, any optimizer of the weighted-sum scalarization (12) is Pareto-optimal and every supported Pareto-optimal point is attained by (12) for some $(\lambda_a, \lambda_b) \succeq \mathbf{0}$ under standard regularity conditions. $\square$

The theorem yields a neat geometric interpretation: in a single-user, single-parameter regime, any Pareto-optimal design concentrates its transmit covariance on an at most three-dimensional subspace spanned by the communication channel $\mathbf{h}$ and the two sensing-informative directions $\mathbf{a}$ and $\bar{\mathbf{b}}$. This subspace geometry also motivates the multi-user and multi-parameter setting, where Pareto-optimal covariances are likewise confined to informative subspaces, although the practical multi-user case requires explicit precoding/interference modeling [2]–[4].

Remark III.4. *If the communication receiver is colocated with the sensing target, the downlink channel $\mathbf{h}$ is aligned with the steering vector $\mathbf{a}$. In this case, the informative subspace in Theorem III.3 reduces to $\text{span}\{\mathbf{h}, \bar{\mathbf{b}}\}$ and the rank bound tightens to 2.*

IV. GENERAL LINEAR SENSING WITH NUISANCE

The single-target model can be generalized to an analytically tractable linear sensing channel capturing multiple scatterers and nuisance parameters. The received signal is

$$\mathbf{Z} = \mathbf{G}(\boldsymbol{\theta}) \mathbf{X} + \mathbf{W}, \quad (13)$$

where $\mathbf{G}(\boldsymbol{\theta}) \in \mathbb{C}^{M \times M}$ is a sensing operator depending on unknown parameters $\boldsymbol{\theta} \in \mathbb{R}^d$ (e.g., angles, delays, Doppler shifts, reflectivities)², and $\mathbf{W}$ has i.i.d. $\mathcal{CN}(0, \sigma_s^2)$ entries. This abstraction subsumes monostatic and multistatic narrowband models as well as linearized wideband models after matched filtering [1], [3].

Assuming $\mathbf{x}[t] \sim \mathcal{CN}(\mathbf{0}, \mathbf{Q})$ i.i.d., the log-likelihood is Gaussian with parameter-dependent mean. Now, defining the partial derivatives $\mathbf{G}_i \triangleq \partial \mathbf{G}(\boldsymbol{\theta}) / \partial \theta_i$, the Fisher information matrix (FIM) for $\boldsymbol{\theta}$ is [9]

$$\mathbf{J}_{\boldsymbol{\theta}}(\mathbf{Q}) = \frac{2T}{\sigma_s^2} \left[\Re \{ \text{tr}(\mathbf{G}_i \mathbf{Q} \mathbf{G}_j^H) \} \right]_{i,j=1}^d. \quad (14)$$

²For example, a monostatic model with L point scatterers at angles $\boldsymbol{\theta} = (\theta_1, \dots, \theta_L)$ and reflection coefficients $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_L)$ can be written as $\mathbf{G}(\boldsymbol{\theta}) = \sum_{\ell=1}^L \alpha_{\ell} \mathbf{a}(\theta_{\ell}) \mathbf{a}(\theta_{\ell})^H$. In this case, the Fisher information depends on quadratic forms of the transmit covariance projected onto the span of $\{\mathbf{a}(\theta_{\ell}), \partial \mathbf{a}(\theta_{\ell}) / \partial \theta_{\ell}\}_{\ell=1}^L$.

In the presence of nuisance parameters (e.g., unknown complex gains), the effective information for $\boldsymbol{\theta}_I$ follows from Schur-complement elimination [9], yielding the CRB

$$\text{CRB}(\boldsymbol{\theta}_I; \mathbf{Q}) = (\mathbf{J}_{II} - \mathbf{J}_{IN} \mathbf{J}_{NN}^{-1} \mathbf{J}_{NI})^{-1}, \quad (15)$$

provided $\mathbf{J}_{NN}$ is nonsingular.

A. Communication Model and Rate Region

On the communication side, we retain the downlink model of (4). Under Gaussian signaling with covariance $\mathbf{Q}$, the achievable rate region under the same interference-free single-user-decoding abstraction is characterized by

$$R_k(\mathbf{Q}) = \log_2 \left(1 + \frac{\mathbf{h}_k^H \mathbf{Q} \mathbf{h}_k}{\sigma_k^2} \right), \quad (16)$$

and the weighted sum-rate $\sum_k w_k R_k(\mathbf{Q})$, which is concave in $\mathbf{Q}$ for nonnegative weights w_k [10]. More general multiuser precoding/decoding and broadcast-channel capacity can be incorporated, but (16) is used here only as a tractable benchmark for ISAC tradeoffs [2], [3].

B. Multi-Objective Optimization for ISAC

We formalize ISAC design as a bi-objective optimization over the transmit covariance:

$$\max_{\mathbf{Q} \succeq \mathbf{0}} \left(\sum_k w_k R_k(\mathbf{Q}), -\Phi(\mathbf{Q}) \right) \quad \text{s.t.} \quad \text{tr}(\mathbf{Q}) \leq P, \quad (17)$$

where $\Phi(\mathbf{Q})$ is a sensing loss, such as $\text{tr}(\mathbf{J}_{\boldsymbol{\theta}}^{-1}(\mathbf{Q}))$. Standard problem scalarization approaches include weighted sums and constrained formulations:

$$\max_{\mathbf{Q} \succeq \mathbf{0}} \sum_k w_k R_k(\mathbf{Q}) \quad \text{s.t.} \quad \Phi(\mathbf{Q}) \leq \Phi_{\max}, \text{tr}(\mathbf{Q}) \leq P. \quad (18)$$

Theorem IV.1. *Assume $\Phi(\mathbf{Q})$ is convex on the feasible set, and $\sum_k w_k R_k(\mathbf{Q})$ is concave. Then the feasible set of (18) is convex, and every solution $\mathbf{Q}^*$ such that $\Phi(\mathbf{Q}^*) = \Phi_{\max}$ is Pareto-optimal for (17).*

Proof. Concavity of the objective and convexity of the constraints imply convexity of the feasible set. By standard multi-objective optimization theory, any optimum of a scalarized problem with a binding constraint corresponds to a Pareto-optimal point of the original bi-objective problem [11]. $\square$

C. Structural Properties of Optimal Covariance

We next establish geometric structure for optimal $\mathbf{Q}$ in representative regimes.

Theorem IV.2 (Informative Subspace). *Let*

$$\begin{aligned} \mathcal{U}_c &\triangleq \text{span}\{\mathbf{h}_1, \dots, \mathbf{h}_K\}, \\ \mathcal{U}_s &\triangleq \text{span}\left(\bigcup_{i=1}^d \text{range}(\mathbf{G}_i^H)\right). \end{aligned}$$

Define $\mathcal{U} \triangleq \mathcal{U}_c + \mathcal{U}_s$. Then, any Pareto-optimal covariance $\mathbf{Q}^$ for (17) satisfies*

$$\text{range}(\mathbf{Q}^*) \subseteq \mathcal{U}. \quad (19)$$

Proof sketch. Decompose $\mathbf{Q} = \mathbf{Q}_{\parallel} + \mathbf{Q}_{\perp}$ with $\mathbf{Q}_{\perp}$ supported on $\mathcal{U}^{\perp}$. Both $\sum_k R_k(\mathbf{Q})$ and $\mathbf{J}_{\theta}(\mathbf{Q})$ depend only on projections onto $\mathcal{U}$. Reallocating power from $\mathbf{Q}_{\perp}$ to $\mathbf{Q}_{\parallel}$ preserves feasibility and does not decrease either objective, with strict improvement unless $\mathbf{Q}_{\perp} = \mathbf{0}$. Hence Pareto-optimal $\mathbf{Q}$ must be supported on $\mathcal{U}$ [2], [3]. $\square$

V. KKT ANALYSIS AND RANK PROPERTIES

Here, we study the ISAC design problem (18) via Karush–Kuhn–Tucker (KKT) conditions [11] to obtain sharper structural results. Natural logarithms are used in this section for notational convenience. Consider

$$\min_{\mathbf{Q} \succeq \mathbf{0}} - \sum_{k=1}^K w_k \ln \left(1 + \frac{\mathbf{h}_k^H \mathbf{Q} \mathbf{h}_k}{\sigma_k^2} \right) \text{ s.t. } \begin{cases} \Phi(\mathbf{Q}) \leq \Phi_{\max} \\ \text{tr}(\mathbf{Q}) \leq P \end{cases}, \quad (20)$$

where $\Phi(\mathbf{Q})$ is a convex sensing loss such as $\text{tr}(\mathbf{J}_{\theta}^{-1}(\mathbf{Q}))$. Restricting optimization to the identifiable set $\{\mathbf{Q} : \mathbf{J}_{\theta}(\mathbf{Q}) \succ \mathbf{0}\}$, convexity holds since $\mathbf{J}_{\theta}(\mathbf{Q})$ is affine in $\mathbf{Q}$ and $\text{tr}(\mathbf{X}^{-1})$ is convex on $\mathbf{X} \succ \mathbf{0}$.

Let $\lambda \geq 0$ and $\mu \geq 0$ denote the Lagrange multipliers associated with the sensing and power constraints, respectively. The Lagrangian is

$$\begin{aligned} \mathcal{L}(\mathbf{Q}, \lambda, \mu) = & - \sum_k w_k \ln \left(1 + \frac{\mathbf{h}_k^H \mathbf{Q} \mathbf{h}_k}{\sigma_k^2} \right) \\ & + \lambda (\Phi(\mathbf{Q}) - \Phi_{\max}) + \mu (\text{tr}(\mathbf{Q}) - P). \end{aligned} \quad (21)$$

Theorem V.1 (Stationarity Condition). *At an optimum $\mathbf{Q}^*$, there exist $\lambda^*, \mu^* \geq 0$ and a dual slack matrix $\mathbf{Z}^* \succeq \mathbf{0}$ such that*

$$\mathbf{Z}^* = \mu^* \mathbf{I} + \lambda^* \nabla \Phi(\mathbf{Q}^*) - \sum_k \frac{w_k \mathbf{h}_k \mathbf{h}_k^H}{\sigma_k^2 + \mathbf{h}_k^H \mathbf{Q}^* \mathbf{h}_k} \succeq \mathbf{0}, \quad (22)$$

with complementary slackness

$$\lambda^* (\Phi(\mathbf{Q}^*) - \Phi_{\max}) = 0, \quad \mu^* (\text{tr}(\mathbf{Q}^*) - P) = 0,$$

and

$$\text{tr}(\mathbf{Z}^* \mathbf{Q}^*) = 0 \quad (\text{equivalently, } \mathbf{Z}^* \mathbf{Q}^* = \mathbf{0}).$$

Proof. The result follows from the KKT conditions for semidefinite programs [11]. The gradient of the negative weighted sum-rate is $\sum_k \frac{w_k \mathbf{h}_k \mathbf{h}_k^H}{\sigma_k^2 + \mathbf{h}_k^H \mathbf{Q}^* \mathbf{h}_k}$, while $\nabla \Phi(\mathbf{Q}^*)$ exists by convexity/differentiability. Dual feasibility gives $\mathbf{Z}^* \succeq \mathbf{0}$ and complementarity yields $\text{tr}(\mathbf{Z}^* \mathbf{Q}^*) = 0$. $\square$

A. Single-User, Single-Parameter

When $K = 1$ and θ is scalar, the informative subspace is at most three-dimensional, spanned by $\mathbf{h}$, $\mathbf{a}$, and $\bar{\mathbf{b}}$ as in (12). This regime provides analytic intuition and is frequently used as a benchmark in ISAC studies [1], [2].

B. Orthogonal Directions

If $\mathbf{h}_k^H \mathbf{b} = 0$ for all k , sensing and communication subspaces are orthogonal. Then the optimal covariance decomposes as

$$\mathbf{Q}^* = \mathbf{Q}_c^* + \mathbf{Q}_s^*, \quad (23)$$

where $\mathbf{Q}_c^*$ and $\mathbf{Q}_s^*$ solve independent communication-only and sensing-only problems with split power budgets. This decoupling represents an extreme but clarifying case and is consistent with intuitive “resource partitioning” baselines discussed in ISAC surveys [3], [4].

C. Communication-Only and Sensing-Only

Setting $\lambda^* = 0$ yields the standard communication-only covariance optimization (a concave maximization over $\mathbf{Q}$), while setting $w_k = 0$ yields a sensing-only design aligned with dominant sensing-information eigenvectors [1], [10]. ISAC solutions interpolate between these extremes as the tradeoff parameter varies, forming a Pareto frontier [2], [3].

D. Asymptotic and Design Implications

Two asymptotic effects clarify the Pareto geometry and are relevant to emerging 6G systems with large antenna arrays. As $M \rightarrow \infty$ with fixed (K, d) , user-channel and sensing-informative directions become nearly orthogonal under i.i.d. Rayleigh fading, increasing spatial degrees of freedom and easing the sensing–communication conflict.

At high SNR, achievable rates grow only logarithmically under uniform scaling of the transmit covariance, whereas CRB-type sensing errors decay inversely with $\bar{\mathbf{b}}^H \mathbf{Q} \bar{\mathbf{b}}$, yielding diminishing returns along the Pareto frontier.

From a 6G design perspective, this implies that: *i*) joint covariance and beamforming design is generally necessary; *ii*) low-rank signaling often suffices due to the informative-subspace structure; and *iii*) large arrays and flexible beam management enable near-decoupling of sensing and communication in favorable geometries.

VI. NUMERICAL EXAMPLES

This section presents numerical examples illustrating the sensing–communication tradeoff predicted by the proposed covariance-based framework. All results are obtained by Monte Carlo (MC) averaging over independent random channel realizations. Unless otherwise stated, the communication noise variance is normalized to $\sigma_1^2 = 1$ and the transmit power is $P = 1$. A key modeling point throughout this section is that the steering vector $\mathbf{a}(\theta)$ is unit-norm and differentiable in θ . As illustrated in Remark III.1, the *gauge* condition must be satisfied:

$$\mathbf{a}^H \bar{\mathbf{b}} = 0. \quad (24)$$

A. Single-User, Single-Parameter Tradeoff

We consider a transmitter with $M = 8$ antennas serving a single communication user with channel $\mathbf{h} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$. The transmit covariance is restricted to rank one, $\mathbf{Q} = \mathbf{s} \mathbf{s}^H$, with $\|\mathbf{s}\|_2^2 = P$. We parameterize the beamformer as

$$\mathbf{s} = \sqrt{P} (\cos \varphi \bar{\mathbf{h}} + \sin \varphi \bar{\mathbf{h}}_{\perp}), \quad \varphi \in [0, \frac{\pi}{2}], \quad (25)$$

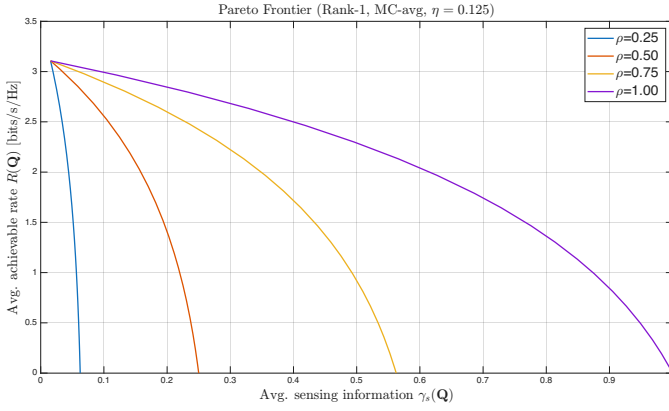


Fig. 1. Pareto frontier between the MC-averaged achievable rate $R(\mathbf{Q})$ and the sensing metric $\gamma_s(\mathbf{Q}) = \mathbf{a}^H \mathbf{Q} \mathbf{a} + \bar{\mathbf{b}}^H \mathbf{Q} \bar{\mathbf{b}}$ for a rank-one transmit covariance. Curves correspond to different values of $\rho > \eta$, with the gauge $\mathbf{a}^H \bar{\mathbf{b}} = 0$ enforced.

where $\bar{\mathbf{h}} = \mathbf{h}/\|\mathbf{h}\|_2$ and $\bar{\mathbf{h}}_\perp$ is a unit-norm vector orthogonal to $\bar{\mathbf{h}}$. The sensing steering direction is chosen to be *partially* aligned with the communication channel,

$$\mathbf{a} = \eta \bar{\mathbf{h}} + \sqrt{1 - \eta^2} \bar{\mathbf{h}}_\perp. \quad 0 < \eta < 1, \quad (26)$$

The parameter $\eta = |\mathbf{a}^H \bar{\mathbf{h}}|$ quantifies the degree of alignment between sensing and communication.

The unit-norm steering-derivative direction $\bar{\mathbf{b}}$ is constructed in the orthogonal complement of $\mathbf{a}$, so that (24) holds identically. Its alignment with the sweep direction $\bar{\mathbf{h}}_\perp$ is controlled by

$$\rho \triangleq |\bar{\mathbf{b}}^H \bar{\mathbf{h}}_\perp| \in [0, 1]. \quad (27)$$

As sensing metric, we use a quantity proportional to the averaged Fisher information,

$$\gamma_s(\mathbf{Q}) = \mathbf{a}^H \mathbf{Q} \mathbf{a} + \bar{\mathbf{b}}^H \mathbf{Q} \bar{\mathbf{b}}. \quad (28)$$

With the above construction, this reduces to

$$\gamma_s(\mathbf{Q}) = P(\eta^2 \cos^2 \varphi + \rho^2 \sin^2 \varphi). \quad (29)$$

According to (25), $\mathbf{h}^H \mathbf{Q} \mathbf{h} \propto \cos^2 \varphi$, hence $R(\mathbf{Q})$ is maximized at $\varphi = 0$. If $\rho \leq \eta$, $\gamma_s(\mathbf{Q})$ is also maximized at $\varphi = 0$, yielding a degenerate Pareto set. If $\rho > \eta$, the objectives are monotone in opposite directions, and varying φ traces a non-degenerate sensing–communication tradeoff. Fig. 1 shows the resulting Pareto frontiers. All frontiers intersect at $\varphi = 0$ since then $\gamma_s(\mathbf{Q})$ is independent of ρ . At $\varphi = \pi/2$, $R(\mathbf{Q}) = 0$.

B. Multi-User Tradeoff

We next consider the same benchmark downlink abstraction with $K = 3$ single-antenna users whose channels $\{\mathbf{h}_k\}_{k=1}^K$ are i.i.d. $\mathcal{CN}(\mathbf{0}, \mathbf{I})$. Communication performance is measured by the weighted sum-rate $\sum_{k=1}^K w_k R_k(\mathbf{Q})$ with uniform weights $w_k = 1/K$. The transmit covariance is again restricted to rank one and

$$\mathbf{s} = \sqrt{P}(\cos \varphi \mathbf{u}_c + \sin \varphi e^{j\psi} \mathbf{u}_\perp), \quad (30)$$

for $\varphi \in [0, \frac{\pi}{2}]$, $\psi \in [0, 2\pi)$, where $\mathbf{u}_c$ is the composite communication direction proportional to $\sum_{k=1}^K w_k \bar{\mathbf{h}}_k$ and normalized

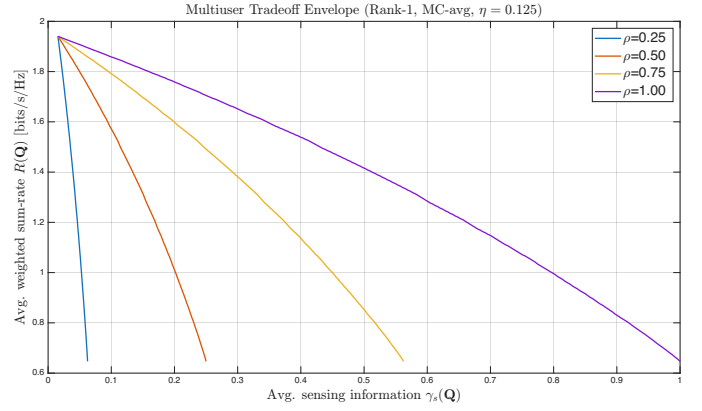


Fig. 2. Sensing–communication tradeoff envelope between the MC-averaged weighted sum-rate and the sensing metric $\gamma_s(\mathbf{Q})$ for the multiuser downlink. Non-degenerate tradeoffs arise only when $\rho > \eta$, i.e., when the sensing-informative direction is sufficiently distinct from the communication-favored subspace.

so that $\|\mathbf{u}_c\|_2 = 1$, and $\mathbf{u}_\perp$ is a unit-norm vector orthogonal to $\mathbf{u}_c$.

The sensing steering vector $\mathbf{a}$ is constructed with the same partial alignment parameter η , and the derivative direction $\bar{\mathbf{b}}$ is chosen orthogonal to $\mathbf{a}$ with $\rho = |\bar{\mathbf{b}}^H \mathbf{u}_\perp|$. The sensing metric again satisfies (29) and is independent of ψ .

For each $\rho > \eta$, the achievable set is evaluated over a dense (φ, ψ) grid and summarized by the upper envelope

$$R^*(\gamma) = \max_{\mathbf{s}: \gamma_s(\mathbf{Q}) \geq \gamma} \sum_{k=1}^K w_k R_k(\mathbf{Q}).$$

Fig. 2 shows the resulting sensing–communication tradeoff envelopes. Again, all envelopes intersect at $\varphi = 0$ since then $\mathbf{s} = \sqrt{P} \mathbf{u}_c$ and $\gamma_s(\mathbf{Q}) = P\eta^2$, so both the weighted sum-rate and $\gamma_s(\mathbf{Q})$ are independent of ρ . At $\varphi = \pi/2$, the corresponding weighted sum-rate is also independent of ρ .

VII. CONCLUSIONS

We presented a covariance-based analytical framework for ISAC that, within the adopted benchmark model, reveals the sensing–communication tradeoff through the geometry of Pareto-optimal low-dimensional informative subspaces shaped by the alignment between communication channels and sensing directions.

This subspace viewpoint explains why low-rank signaling and beamforming are often sufficient despite the high dimensionality of joint waveform design. For single-parameter sensing with a single user, Pareto-optimal designs lie in an at most three-dimensional subspace spanned by the communication channel and sensing-informative directions, reducing to two dimensions when the target and the receiver are colocated.

These results complement existing algorithmic and CRB-driven approaches by identifying the spatial degrees of freedom governing ISAC performance and emphasizing the fundamental role of subspace geometry, rather than full-dimensional waveform optimization.

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