Global-Local Modelling with Node Dependent Kinematics and Arbitrary Displacement Fields for Plates

Materials Research Proceedings 69 (2026) 1235-1239 https://doi.org/10.21741/9781644904251-213

Global-local Modelling with Node Dependent Kinematics and Arbitrary Displacement Fields for Plates

Daniele SCANO[1,a *], Erasmo CARRERA[1,b] and Enrico ZAPPINO[1,c]

[1]MUL2 Lab, Department of Mechanical and Aerospace Engineering, Corso Duca degli Abruzzi 24, 10129, Torino, Italy

[a]daniele.scano@polito.it, [b]erasmo.carrera@polito.it, [c]enrico.zappino@polito.it

Keywords: Finite Element Method, Higher-Order Theories, Kinematic Models, Unified Formulation, Node-Dependent Kinematics

Abstract. This paper introduces enhanced two-dimensional finite elements based on a node dependent kinematics method, which allows each node to adopt a unique thickness expansion. A key innovation of the proposed approach is the ability to independently select models for each of the three displacement variables, employing both Taylor-based and Legendre-based expansions for the cross-section. The formulation is developed within the framework of the Carrera Unified Formulation, which separates the three-dimensional displacement field into in-plane and cross-sectional components. The method is validated for a free-edge problem by comparison with results from the literature and highly refined benchmark solutions. The findings show that the optimal computational model for balancing accuracy and efficiency is problem specific.

Introduction

Laminated composite structures are extensively used in the aerospace and automotive industries for their lightweight and high-performance properties. However, their material inhomogeneity can cause complex behaviours, such as free-edge effects (i.e., stress concentrations at layer interfaces near discontinuities) that often serve as initiation sites for damage [1]. Refined theories, including those by Carrera based on Reissner's Mixed Variational Theorem and the Principle of Virtual Displacements (PVD) within the CUF framework [2, 3, 4], have improved upon classical models. Most existing studies use the Equivalent Single Layer (ESL) approach, which simplifies modelling by using variables independent of the number of layers. For more detailed and accurate stress analysis, the Layer-Wise (LW) approach is preferred, assigning separate variables to each layer.

While refined models improve accuracy, they also increase computational costs, especially when involving local phenomena like delamination [5]. To address this, hybrid modelling strategies have been developed, applying high-fidelity models only in critical regions and lower-fidelity models elsewhere, reducing computational load with minimal loss of accuracy. However, implementing such global–local approaches is complex, and numerous methods for integrating different structural theories have been proposed, as reviewed by Noor [6]. This work presents enhanced 2D finite elements based on the Node-Dependent Kinematics (NDK) method [7], which allows each node to adopt different section expansions and enables independent modelling of displacement variables without special coupling. As usual, the Lagrange (LW) and Taylor (ESL) polynomials are adopted as structural theories. The first step is the development of a unified method that isolates contributions of individual terms and avoids reformulating equations for each new theory, thereby balancing accuracy and computational efficiency. Building on Carrera's work [8, 9], the approach combines ESL and LW models within a consistent framework. The second step exploits the hierarchical nature of CUF to incorporate thickness-wise descriptions, assigning different structural theories to each node. Originally applied to plates and recently extended to plates [10], the NDK method provides flexibility in combining low- to high-order theories in 2D problems.

Materials Research Proceedings 69 (2026) 1235-1239 https://doi.org/10.21741/9781644904251-213

Structure of the work. This paper is organized as follows. Section *Theoretical formulation* discusses the principles of the Unified Formulation and the mathematical details of the Node-Dependent Kinematics (NDK) approach, including the governing equations. Section *Numerical results* presents the outcomes for out-of-plane stresses. Finally, Section *Conclusions* summarizes the key findings of the work.

Theoretical formulation

In classical elasticity, the stress tensor, $\boldsymbol{\sigma}_k$, and the strain tensor, $\boldsymbol{\epsilon}_k$, can be conveniently expressed in vector form as:

$$\boldsymbol{\sigma}^k = \left\{\sigma_{xx}^k\ \sigma_{yy}^k\ \sigma_{zz}^k\ \sigma_{yz}^k\ \sigma_{xz}^k\ \sigma_{xy}^k\ \right\}^T,\quad \boldsymbol{\epsilon}^k = \left\{\epsilon_{xx}^k\ \epsilon_{yy}^k\ \epsilon_{zz}^k\ \epsilon_{yz}^k\ \epsilon_{xz}^k\ \epsilon_{xy}^k\ \right\}^T \tag{1}$$

The geometric relationships between strains and displacements are defined as:

$$\boldsymbol{\epsilon}^k = \mathbf{D}\mathbf{u}^k \tag{2}$$

where $\mathbf{D}$ is the matrix of differential operators, valid under the assumption of small displacements and small rotations [3]. For linear elastic orthotropic materials, the constitutive relation is given by

$$\boldsymbol{\sigma}^k = \mathbf{C}^k\boldsymbol{\epsilon}^k \tag{3}$$

where $\mathbf{C}^k$ denotes the matrix of material coefficients (see [11] for details).

Unified formulation. In this method, the concept of the Unified formulation and the FE method are used together. Furthermore, every FE node possesses its unique set of structural theories, and each displacement component can use a different theory. These concepts can be summarized mathematically as follows for the real system:

$$\begin{aligned} u_x^k(x,y,z) &= N_i(x,y)F_{u_x\tau}^{ki}(z)q_{x_{\tau i}}^k,\ \ \tau = 1,\dots,M_{u_x}; i = 1,\dots,N_n \\ u_y^k(x,y,z) &= N_i(x,y)F_{u_y\tau}^{ki}(z)q_{y_{\tau i}}^k,\ \ \tau = 1,\dots,M_{u_y}; i = 1,\dots,N_n \\ u_z^k(x,y,z) &= N_i(x,y)F_{u_z\tau}^{ki}(z)q_{z_{\tau i}}^k,\ \ \tau = 1,\dots,M_{u_z}; i = 1,\dots,N_n \end{aligned} \tag{4}$$

where $F_{u_x\tau}^{ki}$, $F_{u_y\tau}^{ki}$, and $F_{u_z\tau}^{ki}$ denote the expansion functions associated with the generalized displacements $q_{x\,\tau i}^k$, $q_{y\,\tau i}^k$, and $q_{z\,\tau i}^k$, respectively. The index τ indicates summation, while M_{u_x}, M_{u_y}, and M_{u_z} represent the number of expansion terms for each displacement component. In this paper, seven-node Lagrange (LE7) expansions and Taylor functions are used. Moreover, N_i denotes the shape functions, with the repeated subscript i implying summation, and N_n the number of shape functions per element. For the numerical assessments carried out in this study, the classical nine-node Lagrangian element (Q9) is employed [11]. In the case of the virtual system, the indices i and τ are replaced by j and s, respectively.

Governing equations. By using the Principle of the Virtual Displacements:

$$\begin{aligned} &\delta L_{int} = \delta L_{ext} = \\ &\int_{V^k}\left(\delta\epsilon_{xx}^k\sigma_{xx}^k + \delta\epsilon_{yy}^k\sigma_{yy}^k + \delta\epsilon_{zz}^k\sigma_{zz}^k + \delta\epsilon_{yz}^k\sigma_{yz}^k + \delta\epsilon_{xz}^k\sigma_{xz}^k + \delta\epsilon_{zz}^k\sigma_{zz}^k\right)\mathrm{d}V^k = \\ &\delta u_x^k P_{u_x}^k + \delta u_y^k P_{u_y}^k + \delta u_z^k P_{u_z}^k \end{aligned} \tag{5}$$

It is possible to obtain three governing equations for each virtual displacement:

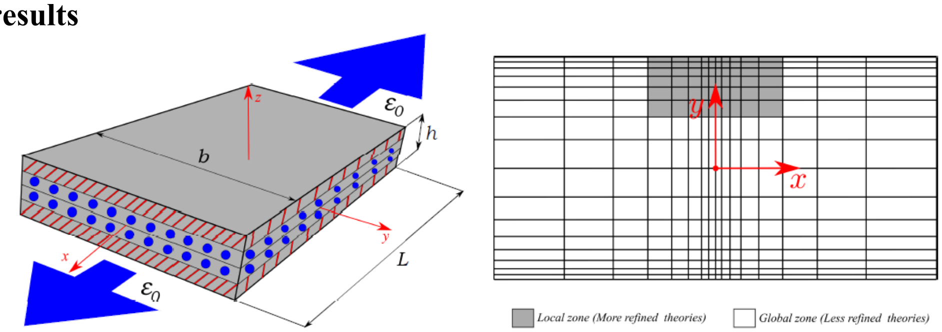
$$
\begin{aligned}
\delta q^k_{x_{sj}}&: K^k_{u_x u_x s\tau ji} q^k_{x_{\tau i}} + K^k_{u_x u_y s\tau ji} q^k_{y_{\tau i}} + K^k_{u_x u_z s\tau ji} q^k_{z_{\tau i}} = P^k_{u_x sj}\\
\delta q^k_{y_{sj}}&: K^k_{u_y u_x s\tau ji} q^k_{x_{\tau i}} + K^k_{u_y u_y s\tau ji} q^k_{y_{\tau i}} + K^k_{u_y u_z s\tau ji} q^k_{z_{\tau i}} = P^k_{u_y sj}\\
\delta q^k_{z_{sj}}&: K^k_{u_z u_x s\tau ji} q^k_{x_{\tau i}} + K^k_{u_z u_y s\tau ji} q^k_{y_{\tau i}} + K^k_{u_z u_z s\tau ji} q^k_{z_{\tau i}} = P^k_{u_z sj}
\end{aligned} \tag{6}
$$

The fundamental nuclei for the load vector are written in the followings:

$$
\begin{aligned}
P^k_{u_x sj} &= N_j F^{kj}_{u_x \tau} P^k_{u_x}\\
P^k_{u_y sj} &= N_j F^{kj}_{u_y \tau} P^k_{u_y}\\
P^k_{u_z sj} &= N_j F^{kj}_{u_z \tau} P^k_{u_z}
\end{aligned} \tag{7}
$$

Assembly. This work employs an innovative formulation of the Fundamental Nucleus (FN) that represents a significant departure from conventional CUF implementations. The key advancement lies in the treatment of FN components $K^k_{u_m u_l s\tau ji}$ and $P^k_{u_m sj}$ as single scalar elements rather than the traditional 3×3 matrix structure. The explicit expressions of $K^k_{u_m u_l s\tau ji}$ are not explicitly given for the sake of brevity. This streamlined approach requires only nine independent scalar parameters for complete stiffness characterization (and three for the load array), offering substantial computational advantages. The new formulation provides important theoretical and practical benefits compared to standard CUF implementations [2, 3]. Most notably, it eliminates the constraint requiring all variables to share the same expansion theory, a limitation that often forces unnecessary computational overhead in conventional approaches. See also [8,9]. The assembly across all displacement components, FE nodes, and elements results in the problem: $\mathbf{KU} = \mathbf{P}$.

Numerical results



(a) *Geometrical properties* (b) *NDK mesh*

Figure 1: Four-Layer composite plate. The study case is taken from [12] and [4].

The study investigates the free-edge behaviour of a symmetric composite plate (see Fig. 1a) with rectangular surface measuring L = 40 [mm], b = 20 [mm], and a thickness of h = 5 [mm]. The laminate has a stacking sequence of [45∘/−45∘]s, and its material properties are E_1 = 137.9 [GPa], $E_2 = E_3$ = 14.5 [GPa], $\nu_{12} = \nu_{13} = \nu_{23}$ = 0.21, and $G_{12} = G_{13} = \nu_{23}$ = 5.9[GPa]. The analysis, based on the works of Robbins and Reddy [12] and Scano et al. [4], applies a uniform axial strain ϵ_0 along the x-axis by imposing end displacements u at $x = \pm$L/2. This loading condition induces a constant strain field in the central region of the plate, provided that the plate length L is sufficiently large compared to the edge perturbed zones. In this problem, a finer mesh is required near the free edge; therefore, a non-uniformly distributed 16 × 16 finite element mesh is adopted. Figure 1b illustrates the mesh configuration, highlighting the local region with 208 FE nodes and the global region with 881 FE nodes, where more refined and less refined theories are employed, respectively. Shear, σ_{xz}, and transverse, σ_{zz}, stresses are calculated along the thickness of the plate near the free-edge (at y/b = 0.998) in mid-plate (at x = 0). The results are reported in a scaled form as follows

$$\bar{\sigma}_{xz}=\frac{\sigma_{xz}}{\epsilon_0},\quad \bar{\sigma}_{zz}=\frac{\sigma_{zz}}{\epsilon_0} \tag{9}$$

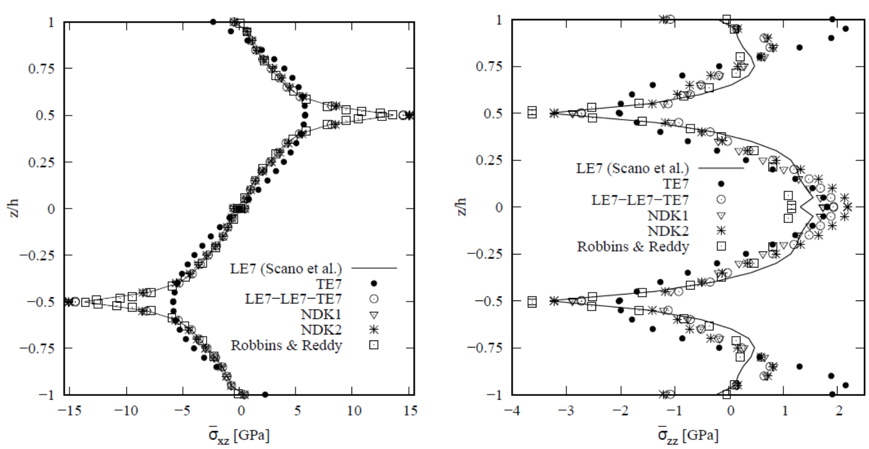
Table 1 presents the characteristics of the Unified Formulation models, including the adopted theories, the number of Degrees of Freedom (DOF), the corresponding DOF reduction, and the naming convention. Shear and transverse stresses along the thickness are studied, as displayed in Figs. 2a and 2b, respectively.

Table 1: Unified formulation models used for the angle-ply [45∘/−45∘]s laminate.

Global Model	Local Model	DOF	DOF Reduction [%]	Theory Definition
LE7-LE7-LE7	LE7-LE7-LE7	81675	—[a]	LE7 (Scano et al.)
TE7-TE7-TE7	TE7-TE7-TE7	26136	68.00	TE7
LE7-LE7-TE7	LE7-LE7-TE7	63162	22.67	LE7-LE7-TE7
TE2-TE2-TE2	LE7-LE7-TE7	19993	75.52	NDK1
TE1-TE1-TE2	LE7-LE7-TE7	18231	77.68	NDK2

(a): Taken as the reference solution.

The analysis reveals that lower-order models are sufficient for accurately representing the majority of the plate. However, when evaluating out-of-plane stresses, higher-order LW models are required to obtain accurate results for the u_x and u_y displacement components. Additionally, the two proposed NDK models demonstrate high efficiency, showing strong agreement with the reference results while achieving a degree-of-freedom reduction of over 75%, which is greater than that achieved with the TE7 model.



(a) *Shear stresses,* σ_{xz} (b) *Transverse stresses,* σ_{zz}

*Figure 2: Out-of-plane stresses along **z** of the angle-ply [45/-45]s laminate.*

Conclusions

The present work introduces an improved method for the analysis of two-dimensional structures within the framework of the CUF. The method is obtained by reformulating the traditional CUF approach and reorganizing the assembly of the FE matrices. The proposed framework relies on two main features: (1) Different expansions can be adopted for each displacement variable; in particular, Taylor- and Legendre-based expansions are employed. (2) The NDK approach is incorporated, allowing the adoption of a different model at each finite element node.

The paper demonstrates that the proposed method effectively joins different types of expansions and polynomial orders, allowing a significant reduction in the degrees of freedom through an accurate selection of expansions for each finite element node. It also enables the application of local kinematic refinements without altering the finite element mesh. Future work will focus on extending this approach to shell formulations and multifield analyses.

References

[1] J.N. Reddy, D.H. Robbins, Jr, Theories and computational models for composite laminates, Appl. Mech. Rev. 47(6) (1994) 147–169. https://doi.org/10.1115/1.3111076

[2] E. Carrera, Developments, ideas, and evaluations based upon Reissner's Mixed Variational Theorem in the modeling of multilayered plates and shells, Appl. Mech. Rev. 54(4) (2001) 301–329. https://doi.org/10.1115/1.1385512

[3] E. Carrera, M. Cinefra, M. Petrolo, E. Zappino, Finite Element Analysis of Structures Through Unified Formulation, Wiley, Amsterdam, 2014. https://doi.org/10.1002/9781118536643

[4] D. Scano, E. Carrera, M. Petrolo, Use of the 3D Equilibrium Equations in the Free-Edge Analyses for Laminated Structures with the Variable Kinematics Approach, Aerotec. Missili Spaz. 103 (2024) 179-195. https://doi.org/10.1007/s42496-023-00177-2

[5] A. Airoldi, A. Baldi, P. Bettini, G. Sala, Efficient modelling of forces and local strain evolution during delamination of composite laminates, Compos. B Eng. 72 (2015) 137–149. https://doi.org/10.1016/j.compositesb.2014.12.002

[6] A.K. Noor, Global-local methodologies and their application to nonlinear analysis, Finite Elem. Anal. Des. 2(4) (1986) 333–346. https://doi.org/10.1016/0168-874X(86)90020-X

[7] E. Zappino, G. Li, A. Pagani, E. Carrera, Global-local analysis of laminated plates by node-dependent kinematic finite elements with variable ESL/LW capabilities, Compos. Struct. 172 (2017) 1–14. https://doi.org/10.1016/j.compstruct.2017.03.057

[8] E. Carrera, D. Scano, Beam finite element models with node dependent kinematics and arbitrary displacement fields over the cross-section, Mech. Adv. Mater. Struct. (2025) 1-16. https://doi.org/10.1080/15376494.2024.2434198

[9] E. Carrera, D. Scano, E. Zappino, Plate finite elements with arbitrary displacement fields along the thickness, Finite Elem. Anal. Des. 244 (2025) 104296. https://doi.org/10.1016/j.finel.2024.104296

[10] D. Scano, E. Carrera, E. Zappino, Shell finite elements with arbitrary displacement fields along the thickness, Int. J. Solids Struct. 321 (2025) 113544. https://doi.org/10.1016/j.ijsolstr.2025.113544

[11] K.J. Bathe, Finite Element Procedure. Prentice hall, Upper Saddle River, New Jersey, USA, 1996.

[12] D.H. Robbins Jr., J.R. Reddy, Modelling of thick composites using a layerwise laminate theory, Int. J. Numer. Methods Eng. 36 (1993), 655–677. https://doi.org/10.1002/nme.1620360407