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Original

Optimal Activation in Asynchronous Opinion Dynamics / Proskurnikov, A., Raineri, R.. - (2026), pp. 3712-3717. (2026 European Control Conference (ECC) Reykjavik (Isl) 07-10 July 2026).

Availability:

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Publisher:

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Optimal Activation in Asynchronous Opinion Dynamics

Anton Proskurnikov, Roberta Raineri

Abstract—We study randomized asynchronous opinion dynamics in the Friedkin-Johnsen model, where agents update according to random activation. We establish a direct convergence proof with an explicit pathwise convergence rate and analyze its dependence on the activation probability vector. We then consider the problem of maximizing this rate. Although convexity of the objective remains open, the problem admits a reformulation with linear-bilinear constraints, which can be handled efficiently via sequential linear programming. In the single-agent update setting, the optimal activation distribution induces a novel centrality measure quantifying each agent’s importance for rapid equilibration. Numerical experiments show that this measure differs from classical centralities, while remaining correlated with influence-based ones.

I. INTRODUCTION

Many processes on networks are inherently asynchronous: nodes lack a common clock, activate at heterogeneous times, and communicate over unreliable, delayed, or intermittent channels. In opinion dynamics, the assumption of synchronous updating is particularly restrictive, since individuals revise their opinions only when triggered by new information, social interactions, or external events. These triggers occur irregularly and affect different agents at different times, reflecting heterogeneity in activity levels, attention, and exposure across online and offline environments [1]; this is also consistent with behavioral theories of persuasion, such as the Elaboration Likelihood Model [2].

While discrete-opinion models are often formulated asynchronously, with one agent or group updating at a time [3], the systems-and-control literature has mainly focused on synchronous models of real-valued opinion dynamics [4]. An exception is the asynchronous French–DeGroot model, which falls within the broader theory of random products of stochastic matrices and asynchronous consensus [5]–[10]. For more expressive opinion formation models that capture disagreement and clustering, there is, to our knowledge, no unified theory of asynchronous effects; existing results are largely model-specific and rely on different techniques. These models typically employ randomized activation policies of two main types: node activation, where agents wake up according to stochastic clocks and update using neighborhood information, and edge activation (gossip), where randomly selected links trigger pairwise interactions. For

instance, asynchronous variants of bounded-confidence models, most notably the Deffuant-Weisbuch model [11], have been studied using tools ranging from probability theory [9], [12]–[14] to game-theoretic methods [15], [16].

Unlike many theoretical opinion-dynamics models, the Friedkin–Johnsen (FJ) model has received empirical support in studies of real social groups [17]–[20] and remains a foundational framework for modeling interpersonal influence. Yet its asynchronous variants remain relatively unexplored. In gossip-based versions, the opinion trajectory generally fails to converge pointwise, although its expectation converges exponentially to the same equilibrium as in the synchronous model [21]–[23]. Similar behavior is observed in DeGroot-type models with stubborn agents [24]. In these settings, one can typically establish convergence only in distribution or for Cesàro averages [21], [25]. By contrast, agent-randomized activation yields convergent behavior under suitable assumptions. This setting has recently been analyzed within a broader framework of asynchronous proximal and Nash-seeking dynamics [26].

In this spirit, we revisit the asynchronous FJ model, provide a simpler convergence analysis under weaker assumptions. We also consider the problem of optimizing the convergence rate over the activation probabilities. This question is motivated by online social media, where activation probabilities can model users’ exposure to content and their propensity to engage. Because recommender systems and feed-ranking algorithms shape both exposure and engagement [27], the optimization can be viewed as an idealized design problem: which users should be prompted more frequently to accelerate convergence to equilibrium?

Contributions: First, we give a convergence result for the asynchronous FJ model that relaxes assumptions from [26] – notably, strictly positive self-confidence weights and the restriction to single-agent updates – and yields not only almost-sure convergence but an explicit pathwise rate. Second, we formulate the convergence rate maximization as an optimization problem and show that it admits a reformulation with linear objective and mixed linear-bilinear constraints, amenable to sequential linear programming. Third, in the single-agent setting, the optimal distribution p^* defines a new centrality measure, quantifying how frequently each agent should update for fastest equilibration. Experiments on random and real graphs show that this measure, termed wake-up centrality, is distinct from standard topological centralities while correlating with harmonic [28] and Friedkin’s influence [4] centralities, particularly for unweighted graphs.

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II. ASYNCHRONOUS OPINION FORMATION

Notation. Henceforth, inequalities between vectors and matrices are understood componentwise. For $\mathbf{p} \in \mathbb{R}^n$, we write $[\mathbf{p}] = \text{diag}(p_1, \dots, p_n)$. A nonnegative matrix is *substochastic* if its row sums do not exceed 1 and *stochastic* if the row sums equal 1. We denote the spectral radius of M by $\rho(M)$ and the probability simplex by $\Delta_n \doteq \{\mathbf{p} \geq \mathbf{0} : \mathbf{p}^\top \mathbf{1} = 1\}$.

FJ model. In synchronous form, the FJ model is given by

$$x(k+1) = Ax(k) + b, \quad (1)$$

where $x(k) \in \mathbb{R}^n$ collects the agents' opinions, A is a substochastic matrix, and $b \in \mathbb{R}^n$ is constant. In the original FJ model [17], $A = \Lambda W$ and $b = (I - \Lambda)s$, where W is a row-stochastic influence matrix, Λ is a diagonal susceptibility matrix, and s is the vector of innate opinions. We focus on the case $\rho(A) < 1$, for which every trajectory of the system (1) converges to the unique equilibrium

$$x^* \doteq (I - A)^{-1}b. \quad (2)$$

Asynchronous Update Rules

We consider two asynchronous variants of (1), differing in whether a single agent or a group updates at each step.

Single-Agent Update. At each time step, one agent $i \in \{1, \dots, n\}$ is selected at random according to a probability distribution $\mathbf{p} \in \Delta_n$ and updates its opinion according to the FJ rule, while all other agents keep their opinions unchanged. We assume that the selections are i.i.d. over time.

Algorithm 1: Asynchronous Single-Agent Activation

```

Input:  $A \in \mathbb{R}^{n \times n}$ ,  $b \in \mathbb{R}^n$ ,  $x(0)$ ,  $p \in \Delta_n$ 
for  $k = 0, 1, 2, \dots$  do
    sample  $i \sim p$ ;
     $x_i(k+1) \leftarrow \sum_{m=1}^n a_{im}x_m(k) + b_i$ ;
    for  $j \neq i$  do
         $x_j(k+1) \leftarrow x_j(k)$ 
    end
end

```

Group update. Multiple agents may update simultaneously, as in committees or group discussions. This is captured by a hypergraph $\mathcal{H} = (V, \mathcal{E})$, where each hyperedge $E \in \mathcal{E} \subseteq 2^V$ represents a group of agents that updates together. Each activated agent updates using its full neighborhood in A , not only the opinions of the co-activated agents, and the activation structure is independent of the influence structure. We assume that at each time step the activated hyperedge is selected independently from the same distribution; that is, the activation process is i.i.d. over time.

Here $A_{E,E}$ denotes the submatrix of A with rows/columns indexed by E ; similarly for $A_{E,\bar{E}}$, and x_E, b_E are subvectors corresponding to the subset of nodes E . Notice that the synchronous FJ model corresponds to the situation where $\mathcal{E}$ contains only one hyperedge $\{1, 2, \dots, n\}$.

Algorithm 2: Hypergraph-Activated Block Update

```

Input:  $A \in \mathbb{R}^{n \times n}$ ,  $b \in \mathbb{R}^n$ , initial  $x(0) \in \mathbb{R}^n$ 
Input: Hypergraph  $\mathcal{H} = (V, \mathcal{E})$  with  $V = \{1, \dots, n\}$ ,
        hyperedges  $\mathcal{E} \subseteq 2^V$ 
Input: Selection probabilities  $\{q_E\}_{E \in \mathcal{E}}$  with  $q_E \geq 0$ 
        and  $\sum_{E \in \mathcal{E}} q_E = 1$ 
for  $k = 0, 1, 2, \dots$  do
    Sample a hyperedge  $E \sim \{q_E\}_{E \in \mathcal{E}}$ ;
    Let  $\bar{E} \leftarrow V \setminus E$ ;
    // Block update; others hold
     $x_E(k+1) \leftarrow A_{E,E}x_E(k) + A_{E,\bar{E}}x_{\bar{E}}(k) + b_E$ ;
     $x_{\bar{E}}(k+1) \leftarrow x_{\bar{E}}(k)$ ;
end

```

The considered dynamics are related to asynchronous proximal methods [26] and can be interpreted as an asynchronous Gauss–Seidel iteration for $x = Ax + b$. Such methods are typically analyzed under symmetry of A , where the condition $\rho(A) < 1$ means that $I - A$ is positive definite, and the iteration reduces to randomized coordinate descent on a quadratic objective [29], [30]. The linear-positive structure of the FJ model instead yields a direct convergence proof without symmetry assumptions.

Expected Value Dynamics and Pathwise Convergence

The vector of activation probabilities. To proceed, let $p \in [0, 1]^n$ denote the vector of activation probabilities, where p_i is the probability that agent i is activated at a given step. In Algorithm 1, one has $p \in \Delta_n$. In Algorithm 2, instead, $p_i = \sum_{E: i \in E} q_E$, where the sum is over all hyperedges E containing agent i . In this case, p need not belong to Δ_n , since overlapping hyperedges may yield $\sum_i p_i > 1$.

In our subsequent results, we use extensively the matrix

$$M(p) \doteq I - [p] + [p]A = I - [p](I - A), \quad p \in [0, 1]^n,$$

and its spectral radius denoted by

$$R(p) \doteq \rho(M(p)).$$

Clearly, if A is substochastic, then so is $M(p)$, and therefore $R(p) \leq 1$ for all $p \in [0, 1]^n$. Moreover, if $p_i = 0$ for some i , then the i -th row of $M(p)$ equals $\mathbf{e}_i^\top$, so that $\mathbf{e}_i^\top M(p) = \mathbf{e}_i^\top$. Hence 1 is an eigenvalue of $M(p)$, and so $R(p) = 1$. The following proposition follows from the Perron–Frobenius theorem [31], and its proof is omitted.

Proposition 1. *For a substochastic matrix A with $\rho(A) < 1$ and positive probabilities $p_i > 0$ for all i , we have $R(p) < 1$.*

The role of $M(p)$ is clarified by the following.

Proposition 2. *With i.i.d. activation, the expected opinion vector $\mathbb{E}[x(k)]$ evolves in accordance with*

$$\mathbb{E}(x(k+1)) = M(p)\mathbb{E}(x(k)) + [p]b. \quad (3)$$

If $p_i > 0$ for all i , then $\mathbb{E}[x(k)] \rightarrow x^$ as $k \rightarrow \infty$, where x^* is given by (2). Moreover,*

$$|\mathbb{E}(x(k)) - x^*| \leq C R(p)^k,$$

where $C > 0$ depends on A , p , and the initial condition.

Proof sketch. Since the number of possible activation sequences over time steps $0, \dots, k$ is finite, the expectations $\mathbb{E}(x(k))$ exist. Let $\mathbb{I}_i(k) \in \{0, 1\}$ indicate that agent i updates at step k . Since $\mathbb{I}_i(k)$ is independent of $\mathbf{x}(k)$ and $\mathbb{E}(\mathbb{I}_i(k)) = p_i$, the equation (3) is obtained by taking the expectation in

$$x_i(k+1) = (1 - \mathbb{I}_i(k))x_i(k) + \mathbb{I}_i(k)(Ax(k) + b)_i. \quad (4)$$

The remaining claims follow from Proposition 1, since (3) is a linear system with state matrix $M(p)$ and has the same equilibrium x^* as the synchronous FJ model (1). $\square$

In fact, individual sample paths converge at the same rate, as the following result shows. For simplicity, we henceforth restrict to irreducible matrices A .

Theorem 1. *Let A be Schur stable, irreducible, and substochastic. Assume that all activation probabilities satisfy $p_i > 0$ for $i = 1, \dots, n$ in Algorithm 1 or Algorithm 2. Then, for any initial condition $x(0)$, almost surely,*

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \log |x(k) - x^*| \leq \log R(p) < 0. \quad (5)$$

In particular, $|x(k) - x^*| = O(R(p)^k)$ almost surely.

The proof, based on super-martingale convergence argument, is deferred to the extended version of this paper.

III. OPTIMIZATION OF ASYNCHRONOUS CONVERGENCE

We now study the problem of minimizing the function $R(p)$ – the convergence rate of the asynchronous FJ model. The first observation is that the synchronous spectral radius $\rho(A)$ provides a lower bound for $R(p)$.

Proposition 3. *Let A be substochastic with $\rho(A) < 1$. Then $R(p) \geq \rho(A)$, with strict inequality when $p_i < 1$ for all i .*

Proof sketch. It suffices to consider the irreducible case, since the general case follows by block-triangular decomposition. Let $v > 0$ be the Perron-Frobenius eigenvector with $Av = \rho(A)v$. Then it is easily checked that $M(p)v \geq \rho(A)v$, where the inequality is strict if $(I - [p])v > 0$. The statement is now immediate from the Collatz-Wielandt formula. $\square$

A similar argument based on the Collatz–Wielandt formula, in fact, allows to show a stronger fact that R is non-increasing: if $q \geq p \geq 0$, then $R(p) \geq R(q)$, with strict inequality whenever $q_i > p_i$ for some i . Proposition 3 corresponds to the case where q is the vector of ones.

While it is unsurprising that a single asynchronous update is slower than a synchronous one, even an epoch of n asynchronous updates under uniform selection, where each agent is activated once in expectation, does not match the effect of one synchronous step.

Proposition 4. *For a substochastic A with $n > 1$, $\rho(A) < 1$ and $p_{\text{unif}} = n^{-1}\mathbf{1}$, one has $R(p_{\text{unif}})^n > \rho(A)$.*

Proof. Notice that $M(p_{\text{unif}}) = n^{-1}((n-1)I + A)$, whence $R(p) = 1 + n^{-1}(\rho(A) - 1)$. The Bernoulli inequality $(1 +$

$x)^n > 1 + nx$ that is applicable for $x > -1$, $x \neq 0$ and $n \geq 2$, entails thus that $R(p)^n > 1 + \rho(A) - 1 = \rho(A)$. $\square$

Remark 1. *Assuming that the assumptions of the latter proposition hold, one has*

$$R(p_{\text{unif}})^n = \left(1 + \frac{\rho(A) - 1}{n}\right)^n \xrightarrow{n \rightarrow \infty} e^{\rho(A) - 1} > \rho(A).$$

However, as the simulations below illustrate, a suitably chosen non-uniform activation policy can make n asynchronous steps substantially faster than one synchronous update. This motivates the problem of choosing the activation probabilities so as to maximize the convergence rate, or equivalently, to minimize $R(p)$.

Problem 1 (Optimal activation probabilities). *For A being substochastic with $\rho(A) < 1$, find*

$$\min_{p \in \mathcal{P}} R(p), \quad (6)$$

where the set $\mathcal{P} \subseteq [0, 1]^n$ depends on the update scheme:

- **Single-agent update** (Algorithm 1):

$$\mathcal{P} = \Delta_n = \{p : p^\top \mathbf{1} = 1, p \geq 0\}.$$

- **Group update** (Algorithm 2):

$$\mathcal{P} = \left\{ p : p_i = \sum_{E: i \in E} q_E \quad \forall i, \quad \text{where} \right. \\ \left. \sum_{E \in \mathcal{E}} q_E = 1, q_E \geq 0 \quad \forall E \in \mathcal{E} \right\}.$$

Despite the simple form of $M(p)$, the function $R(p)$ remains difficult to characterize. In particular, its convexity is open in general, although numerical evidence suggests that $R(p)$ is often strictly convex for small-size matrices (see, e.g., Fig. 1 for a 3×3 illustration). This would imply uniqueness of the solution to (6). Convexity can nevertheless be established in some special cases, such as 2×2 matrices and matrices A that are diagonally symmetrizable, i.e., $D^{-1}AD$ is symmetric for some diagonal matrix $D > 0$.

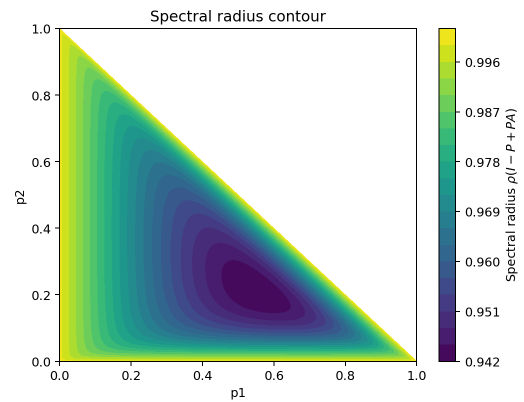


Fig. 1: The function $R(p)$ on the matrix $A = \begin{bmatrix} 0.16 & 0.58 & 0.25 \\ 0.08 & 0.34 & 0.26 \\ 0.16 & 0.30 & 0.29 \end{bmatrix}$.

Optimal Probabilities via Bilinear Programming

While the convexity of $R(p)$ remains conjectural, the optimization problem can be reformulated as follows: minimize a linear function subject to linear equality constraints and bilinear inequality constraints. This reformulation is well-suited to standard nonlinear programming methods, including sequential linear programming (SLP) and sequential quadratic programming (SQP). For brevity, we present the reformulation for the single-agent case (Algorithm 1), where $\mathcal{P} = \Delta_n$; the group update case follows similarly.

The first step applies a classical result from Perron-Frobenius theory [31]: $R(p) = \rho(M(p))$ is the minimal λ admitting a vector $x \neq 0$ satisfying $M(p)x \leq \lambda x$ and $x \geq 0$. By homogeneity, we may normalize x to satisfy $\sum_i x_i = 1$.

Problem 2. *Given a substochastic Schur stable matrix A , find the activation probability vector p that solves*

$$\begin{aligned} \min_{\lambda, x, p} \quad & \lambda \\ \text{s.t.} \quad & M(p)x \leq \lambda x \\ & x \geq 0, \quad x^\top \mathbf{1} = 1 \\ & p \geq 0, \quad p^\top \mathbf{1} = 1. \end{aligned} \quad (7)$$

In our experiments with large-scale graphs (next section), we solve this problem via SLP using the Gurobi solver [32].

IV. ACTIVATION PROBABILITIES AS CENTRALITY

The optimal activation probability vector p^* solving (6) naturally defines a centrality measure on the network: the larger p_i^* is, the more important agent i is for driving the system rapidly toward equilibrium. We refer to this measure as the *wake-up centrality* of the network. It is natural to compare the wake-up centrality with existing centralities commonly used in social-network analysis [23], [33] to clarify whether the new centrality captures a genuinely different notion of node importance, or whether it largely reproduces more classical structural or influence-based rankings such as betweenness, eigenvector centrality or PageRank.

Centralities on Weighted and Unweighted Graphs

To make the centrality measure concrete and comparable with classical alternatives, we now specialize the framework to the standard FJ model on weighted graphs [34]. Consider a social graph $\mathcal{G}$ with adjacency matrix W , where $W_{ij} \geq 0$ denotes the influence weight from agent j to agent i . Under single-agent activation, agent i updates according to

$$x_i(k+1) = \frac{\sum_{j \in \mathcal{V}} W_{ij} x_j(k) + \tilde{s}_i}{1 + \sum_{j \in \mathcal{V}} W_{ij}}, \quad (8)$$

where $\tilde{s}_i$ is agent i 's innate opinion, while all other agents keep their opinions unchanged. This model corresponds to

$$A = [d]W, \quad s = [d]\tilde{s},$$

where d is the vector with elements $d_i = 1/(1 + \sum_j w_{ij})$.

We compare the wake-up centrality with standard benchmark centrality measures, including degree, betweenness, closeness, eigenvector centrality, PageRank, Friedkin's influence centrality [4], [33], and harmonic centrality [28].

Influence centrality quantifies each agent's contribution to the average equilibrium opinion:

$$c_{\text{inf}}^\top s = \frac{\mathbf{1}^\top x^*}{n}, \quad (9)$$

Harmonic centrality measures the accessibility of a node by averaging the reciprocal distances to all other nodes:

$$c_{\text{harm}}(i) = \frac{1}{n-1} \sum_{j \neq i} \frac{1}{d(i, j)}, \quad (10)$$

where $d(i, j)$ is the shortest-path distance from i to j ; in weighted networks, distances are computed using minimum-weight paths [35]. The relationships among these measures have been examined through numerical simulations.

Simulations: Unweighted Network: First, we study the wake-up centrality on unweighted connected networks. We consider a synthetic model – Erdős-Rényi (ER) random networks and a real-world network of social contacts in a village in rural Malawi, available on Sociopatterns [36]. In each case, we compare wake-up centrality with the classical measures introduced earlier.

As shown in Fig.2 for an unweighted random ER network with 100 nodes, the wake-up centrality exhibits high positive correlation with harmonic and influence measures. However, despite the correlation between these measures, the distributions of network nodes ranks are non identical as exhibited by the cumulative weights distribution in Fig.2-(b). Similar correlations between the centrality measures are found for the Malawi contact network Fig.2-(c).

Simulations: Weighted Network: We now extend the analysis to weighted connected graphs, validating on a synthetic weighted ER graph (weights distributed uniformly on $[0, 1]$) and on an inter-provincial interaction network for Italy, where edge weights capture daily mobility flows between provinces estimated from official census data [37]. For the synthetic graph (Fig. 3(a)), wake-up centrality remains highly correlated with influence centrality, comparably to the unweighted case, while its correlation with the harmonic measure is markedly reduced and weak correlations with the other measures appear. For the real-world network (Fig. 3(b)), no dominant correlation emerges and all measures appear only weakly correlated.

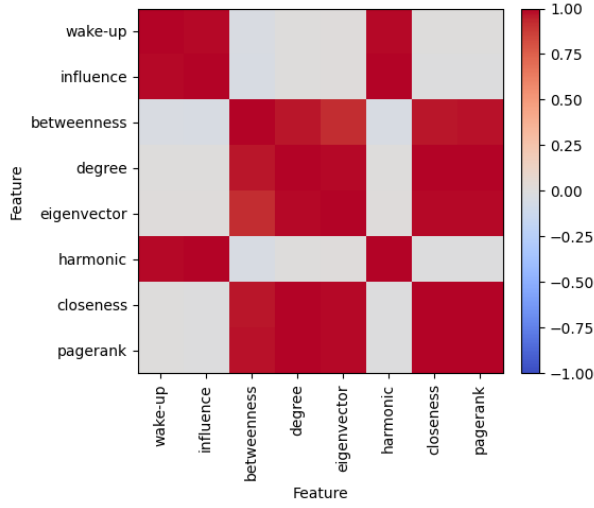
Despite the low correlation, wake-up and influence centralities yield nearly identical convergence rates, whereas harmonic centrality is far from optimal. While the synchronous convergence rate is $\rho(A) = 0.1080$, the asynchronous rates after $n = 107$ steps under the three activation policies are

$$R(p^*)^n = \mathbf{0.3887}, \quad R(p_{\text{inf}})^n = 0.3993, \quad R(p_{\text{harm}})^n = 0.9984.$$

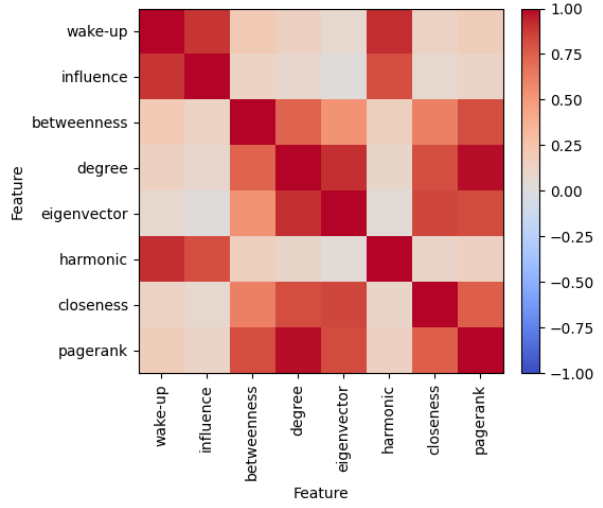
Wake-up centrality thus achieves the fastest asynchronous convergence, closely followed by influence centrality, while harmonic centrality performs poorly.

V. CONCLUSIONS AND FUTURE WORK

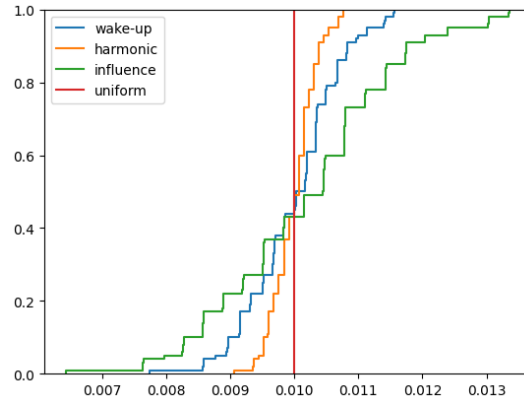
We analyzed an asynchronous variant of the Friedkin-Johnsen model under single-agent and group-based updates, deriving analytical properties of the spectral radius



(a) Correlations, ER

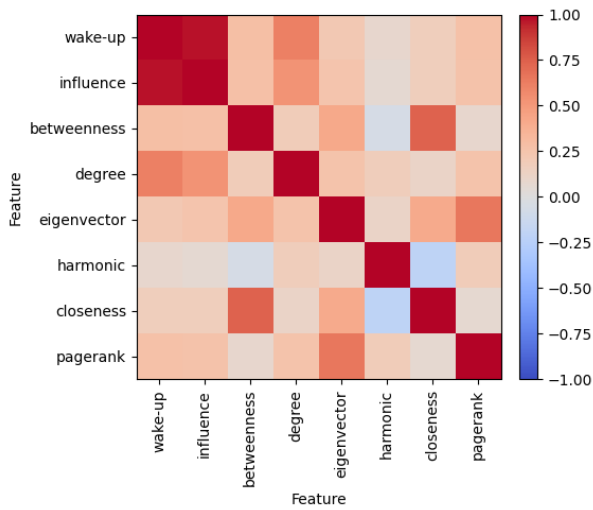


(b) Correlations, Malawi village

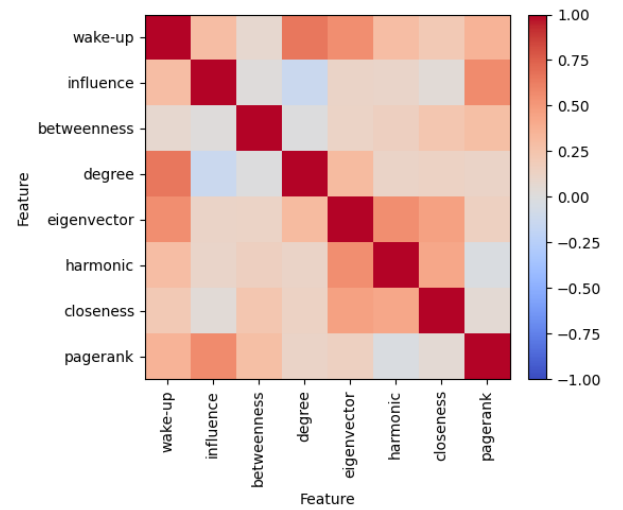


(c) Cumulative distribution, ER

Fig. 2: Comparison of unweighted centrality measures: (a) correlation matrix and (c) cumulative weight distribution for a 100-node ER random graph; (b) correlation matrix for the Malawi contact network



(a) ER graph



(b) Italian provinces network

Fig. 3: Comparison of correlation matrices for (a) a weighted random ER graph with 100 nodes, and (b) flow graph of Italian provinces.

governing convergence. To optimize convergence, we formulated a bilinear optimization problem whose solution defines

a new centrality measure, which correlates with influence and harmonic centralities while ensuring faster convergence. Numerical experiments on synthetic and real, unweighted and weighted networks support our findings. Several directions remain open. A key question is the convexity of $R(p)$, which would unlock convex optimization techniques and is especially relevant for group-based activation. The framework can also be extended to FJ dynamics on hypergraphs, to other dynamical processes (e.g., randomized fixed-point iterations for linear equations), and to settings with temporal correlations in activation, such as Markov update policies.

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