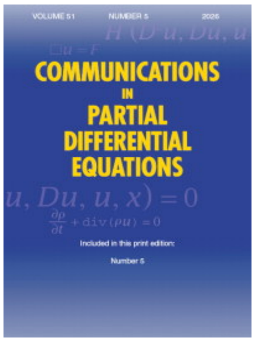

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Phase space analysis of higher-order dispersive equations with point interactions

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ABSTRACT
We investigate nonlinear, higher-order dispersive equations with measure (or even less regular) potentials and initial data with low regularity. Our approach is of distributional nature and relies on the phase space analysis (*via* Gabor wave packets) of the corresponding fundamental solution—in fact, locating the modulation/amalgam space regularity of such generalized Fresnel-type oscillatory functions is a problem of independent interest in harmonic analysis.

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1. Introduction

This note concerns the wave packet analysis of higher-order dispersive equations with measures (or even more singular distributions) as potentials. In the first place, let us consider time-independent potentials and linear problems of the form

$$\begin{cases} \frac{1}{2\pi i}\partial_t u = \mu(D)u + V(x)u \\ u(0,x) = f(x), \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{1}$$

where:

- $\mu(D) = \mathcal{F}^{-1}\mu\mathcal{F}$ is the Fourier multiplier with symbol $\mu\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$, which is a smooth, positively homogeneous function of degree $m \geq 2$ (in fact sufficiently large, depending on the regularity of V; see below), satisfying a non-degeneracy condition on the unit sphere (namely, $\det|\nabla^2\mu(x)| \neq 0$ if $|x| = 1$) to ensure the dispersive nature of the propagator $U_\mu(t) = e^{2\pi i t\mu(D)}$.
- The potential V is a temperate distribution that locally belongs to $\mathcal{F}L^\infty(\mathbb{R}^d)$ and is mildly localized—to be concrete, consider the case where V is a finite, complex Borel measure on $\mathbb{R}^d$.

A well-established approach to study (single or multi-source) point interaction models for the Schrödinger equation (viz., $\mu(x) = |x|^2$ and $u, f \in L^2(\mathbb{R}^d)$) largely relies on tools

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from the spectral theory of operators on Hilbert spaces, including for instance self-adjoint extensions and form methods—see for instance [1–7] and the references therein. For higher-order equations as above, it seems more convenient to look at (1) from a different angle, through the lens of distribution theory. Given that $V \in (\mathcal{F}L^\infty(\mathbb{R}^d))_{\mathrm{loc}}$, it is then natural to investigate the existence of solutions locally in $\mathcal{F}L^1(\mathbb{R}^d)$, so that the perturbation term Vu is a well-defined distribution in $(\mathcal{F}L^\infty(\mathbb{R}^d))_{\mathrm{loc}}$.

Standard techniques from the theory of dispersive PDE, combining Duhamel's integral representation of (1) with a suitable fixed-point theorem (cf [8].), motivate a preliminary analysis of the unperturbed equation

$$\begin{cases} \frac{1}{2\pi i}\partial_t u = \mu(D)u \\ u(0,x) = f(x), \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{2}$$

It is not difficult to see that the solution $u(t)$ is given by $u(t) = E(t,\cdot) * f$ for $t > 0$, where $E \in C^\infty((0,+\infty); \mathcal{S}'(\mathbb{R}^d))$ is the fundamental solution of (2), hence satisfying $E(t,\cdot) \to \delta$ in $\mathcal{S}'(\mathbb{R}^d)$ as $t \to 0^+$. More in detail, we have

$$E(t,x) = t^{-d/m} E_\mu(t^{-1/m}x),$$

where $E_\mu = \mathcal{F}^{-1}F_\mu$ and $F_\mu(x) = e^{2\pi i \mu(x)}$. By comparison with the standard Schrödinger setting, it is natural to refer to F_μ as the *generalized Fresnel function* associated with μ.

In light of the previous discussion, it would be desirable for the fundamental solution $E(t,\cdot)$ to belong to $\mathcal{F}L^1(\mathbb{R}^d)$ for any given $t > 0$ (that is, $F_\mu \in L^1(\mathbb{R}^d)$), but this is of course not the case. Nonetheless, our main result shows that E_μ is locally in $\mathcal{F}L^1(\mathbb{R}^d)$, uniformly with respect to translations in $\mathbb{R}^d$. More precisely:

Theorem 1.1. *Let $\mu\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ be a smooth, positively homogeneous function of degree $m \geq 2$, satisfying*

$$\det|\nabla^2\mu(x)| \neq 0 \quad \text{if} \quad |x| = 1.$$

Let $F_\mu(x) = e^{2\pi i\mu(x)}$ be [1][1] the generalized Fresnel function associated with μ and set $E_\mu = \mathcal{F}^{-1}F_\mu$. For every $g \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\sup_{y\in\mathbb{R}^d} \|g(\cdot - y)E_\mu\|_{\mathcal{F}L^1} < \infty. \tag{3}$$

As a consequence (cf. Corollary 3.6), there exists a constant $C > 0$ (depending on μ and g) such that

$$\|E(t,\cdot)\|_{W^{1,\infty}} := \sup_{y\in\mathbb{R}^d} \|g(\cdot - y)E(t,\cdot)\|_{\mathcal{F}L^1} \leq C\max\{t^{-2d/m}, t^{-d/m}\}, \qquad t > 0. \tag{4}$$

In the case of the Schrödinger equation (where $\mu(x) = |x|^2$), it is enough to choose g as a Gaussian function to realize by explicit computations that the bound (4) is actually optimal both as $t \to 0^+$ and $t \to +\infty$—see, e.g. [9],.

The condition appearing in (3) actually shows that E_μ belongs to the so-called *Wiener amalgam space* $W^{1,\infty}(\mathbb{R}^d)$—or, equivalently, that the generalized Fresnel function F_μ belongs to the *modulation space* $M^{1,\infty}(\mathbb{R}^d)$. This terminology comes from harmonic analysis in phase space [10–15], where wave packet decompositions of functions and operators are performed

[1]The value of F_μ at 0 is irrelevant, but to be definite we can set $F_\mu(0) = 1$.

and the regularity of distributions is measured in terms of mixed Lebesgue norms of the corresponding coefficients. To be more precise, recall that *Gabor wave packets* are obtained by means of phase space translations of a generating atom $g \in \mathcal{S}(\mathbb{R}^d)$, namely

$$\pi(x,\xi)g(y) := e^{2\pi i\xi\cdot y}g(y-x), \qquad (x,\xi) \in \mathbb{R}^{2d}.$$

The Gabor (or Bargmann, or short-time Fourier) transform of $\mathcal{S}'(\mathbb{R}^d)$ is then the phase space representation defined by

$$V_g f(x,\xi) := \langle f, \pi(x,\xi)g\rangle = \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot y} f(y)\overline{g(y-x)}\, dy.$$

It is then clear that (3) means that $V_g E_\mu(x,\xi) \in L^\infty_x(L^1_\xi)$, which is in fact the characterizing property of the Wiener amalgam space $W^{p,q}(\mathbb{R}^d)$ with $p = 1$ and $q = \infty$. For general $1 \leq p,q \leq \infty$, these are Banach spaces of distributions endowed with norms $\|f\|_{W^{p,q}} = \|V_g f(x,\xi)\|_{L^q_x(L^p_\xi)}$, which are independent of the choice of g in practice (i.e., up to equivalence constants). Interchanging the phase-space integration order captures the modulation space regularity: $M^{p,q}(\mathbb{R}^d)$ is the space of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\|f\|_{M^{p,q}} = \|V_g f(x,\xi)\|_{L^q_\xi(L^p_x)}$ is finite for some (hence any) non-trivial $g \in \mathcal{S}(\mathbb{R}^d)$—for instance, $F_\mu \in M^{1,\infty}(\mathbb{R}^d)$, since $W^{p,q}(\mathbb{R}^d) = \mathcal{F}M^{p,q}(\mathbb{R}^d)$. A review of the basic results of Gabor analysis is provided in Section 2.2.

The analysis of PDE with low-regular coefficients *via* suitable wave packets has a long tradition—some fundamental contributions are to be found in [16–18]. More generally, advanced techniques from harmonic analysis have proven to be highly effective to the study of higher-order dispersive PDE [19–22]. Although the literature on nonlinear dispersive PDE is extensive and marked by impressive, sophisticated wellposedness results in individual cases, our goal here is rather to obtain general results for nonspecific higher-order dispersive PDE with fairly irregular potentials.

We also stress that the analysis of the amalgam/modulation space regularity of oscillating functions such as F_μ is a largely investigated problem in harmonic analysis, in view of the relevance to the continuity of Schrödinger-type semigroups beyond Lebesgue spaces—see for instance [23–28], and also Corollary 3.6 below. In particular, it is well known that the unimodular function $e^{2\pi i|x|^m}$ belongs to $W^{1,\infty}(\mathbb{R}^d)$ if $m \leq 2$, and this condition is in fact optimal. Theorem 1.1 fills the gap on the phase space regularity in the case where $m \geq 2$, showing that $e^{2\pi i|x|^m} \in M^{1,\infty}(\mathbb{R}^d)$ in that case. The Fresnel function $e^{2\pi i|x|^2}$ thus marks the edge between the two regimes.

After this brief detour, let us come back to the original problem with fresh eyes—that is, studying local and global wellposedness of the higher-order dispersive problem in (1) in the context of modulation and amalgam spaces. We highlight that the analysis of nonlinear dispersive PDE has indeed benefited from ideas and techniques of Gabor analysis in the last decades—a comprehensive bibliography is inconceivable, we just mention [9,29–35] and the monographs [14,36–39], bearing witness to the broad scope of this approach, as well as [40–47] for additional related contributions.

As already anticipated, we plan to set up a standard Duhamel-type iterative scheme starting from Theorem 1.1. To this aim, we note from (4) that the map $(0,1] \ni t \mapsto \|E(t,\cdot)\|_{W^{1,\infty}}$ is integrable if $m > 2d$. A closer inspection of the details involved in the iteration argument actually shows that more general potentials in the class of the amalgam spaces can be taken into account, also in the presence of analytic nonlinearities. Our main result can thus be stated in full generality as follows—recall that $W^{\infty,1}(\mathbb{R}^d)$ denotes the Wiener amalgam space introduced above, and should not be mistaken for the Sobolev space.

Theorem 1.2. *Let μ be as in Theorem 1.1 with $m > 2d$. Given $V \in L^\infty([0,1]; W^{\infty,1}(\mathbb{R}^d))$ and an entire real-analytic [2][2] function $G\colon \mathbb{C} \to \mathbb{C}$, with $G(0) = 0$, consider the Cauchy problem*

$$\begin{cases} \frac{1}{2\pi i}\partial_t u = \mu(D)u + V(t,x)G(u) \\ u(0,x) = f(x) \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{5}$$

Consider $R > 0$. For every $f \in B_R := \{f \in \mathcal{F}L^1(\mathbb{R}^d) : \|f\|_{\mathcal{F}L^1} \le R\}$ there exist $T = T(R) \in (0,1]$ and a unique solution $u \in C^0([0,T]; W^{1,\infty}(\mathbb{R}^d))$, and the correspondence $f \mapsto u$ from B_R to $C^0([0,T]; W^{1,\infty}(\mathbb{R}^d))$ is Lipschitz continuous [3][3].

Moreover, if $G(u) = u$ and $V \in L^\infty([0,+\infty); W^{\infty,1}(\mathbb{R}^d))$, for every initial datum $f \in \mathcal{F}L^1(\mathbb{R}^d)$ there exists a unique global solution $u \in C^0([0,+\infty); W^{1,\infty}(\mathbb{R}^d))$.

Primary examples of potentials to which Theorem 1.2 applies are complex finite measures, which belong to $W^{\infty,1}(\mathbb{R}^d)$ according to [48, Proposition 3.4]. In fact, certain higher-order distributions are encompassed as well—such as (distributional) derivatives of order $\le (d-1)/2$ of the surface measure of any compact smooth hypersurface of $\mathbb{R}^d$, with non-zero Gaussian curvature at every point. See Example 5.2 below for further details in this connection.

We emphasize that the solution u of the problem (5) has the same spatial local regularity of the initial datum. To the best of our knowledge, this is the first time that point interactions can be approached *via* a phase space analysis angle beyond the usual self-adjoint extension approach. The role of Theorem 1.1 in the Duhamel machinery is thus clear: after scaling, the phase-space regularity of the generalized Fresnel factor yields the Wiener amalgam bounds for the fundamental solution $E_\mu(t,\cdot)$, which in turn determine both the functional setting for solution regularity and the class of rough potentials V that can be treated by multiplication estimates.

It is natural to wonder whether the assumption $m > 2d$ can be relaxed, at least in the linear case. One may expect that such improvement can possibly be obtained at the price of a regularity gain at the level of potentials, which in turn could be balanced by allowing less regular initial data. In fact, appropriate adjustments to the proof of Theorem 1.2 allowed us to obtain the following result.

Theorem 1.3. *Let $V \in L^\infty([0,+\infty); W^{p,1}(\mathbb{R}^d))$ for some $1 \le p \le \infty$, and let μ be as in Theorem 1.1 with $m \ge 2$ and $m > 2d/p'$, where p' denotes the Hölder conjugate index associated with p. Consider the Cauchy problem*

$$\begin{cases} \frac{1}{2\pi i}\partial_t u = \mu(D)u + V(t,x)u \\ u(0,x) = f(x), \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

[2] Recall that $G\colon \mathbb{C} \to \mathbb{C}$ is said to be entire real-analytic if $G(z)$ has an absolutely convergent Taylor series for every $z \in \mathbb{C}$ and $G(0) = 0$. This condition is satisfied in particular by polynomials in $z, \bar{z}$, as well as power-type nonlinearities such as $G(z) = \lambda|z|^{2k}z$ (with $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$) or $G(z) = \lambda z^k$ (with $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}_+$).

[3] The constant $T = T(R)$ in fact depends also on the norm of V in $L^\infty([0,1]; W^{\infty,1}(\mathbb{R}^d))$, on the analytic nonlinearity G and the constant in the relevant dispersive estimate.

For every initial datum $f \in \mathcal{F}L^q(\mathbb{R}^d)$ *with* $1 \le q \le p'$ *there exists a unique solution* $u \in C^0([0,+\infty); W^{q,\infty}(\mathbb{R}^d))$.

Theorems 1.2 and 1.3 should therefore be read as Cauchy theories in Wiener amalgam spaces. In the nonlinear case the solution is constructed in $C^0([0,T]; W^{1,\infty}(\mathbb{R}^d))$, whereas in the linear case one obtains global solutions in $C^0([0,+\infty); W^{q,\infty}(\mathbb{R}^d))$. We emphasize once more that the role of the higher-order Fresnel estimates is not limited to generalizing boundedness results for unperturbed Schrödinger-type propagators on modulation spaces: these estimates are crucial for running a Duhamel scheme with measure-valued and distributional perturbations in some suitable regimes.

We conclude by highlighting that some concrete choices of potentials that fit the assumptions of Theorem 1.3 are discussed in Example 5.2 below. Remarkably, the latter include cropped Coulomb-type singularities and measures associated with certain low-dimensional smooth manifolds (or even fractal sets), the prime instance being the surface measure of the sphere $S^{d-1} \subset \mathbb{R}^d$.

2. Preliminaries

2.1. Notation

The inner product on $L^2(\mathbb{R}^d)$ is defined by

$$\langle f,g\rangle = \int_{\mathbb{R}^d} f(y)\overline{g(y)}\, \mathrm{d}y, \qquad f,g \in L^2(\mathbb{R}^d).$$

This agreement extends to the duality pairing between a temperate distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ and a function $g \in \mathcal{S}(\mathbb{R}^d)$ in the Schwartz class by setting $\langle f,g\rangle = f(\overline{g})$.

The definition of the Fourier transform of $f \in L^2(\mathbb{R}^d)$ used in this note is

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot x} f(x)\, \mathrm{d}x,$$

with obvious extensions to temperate distributions. For $1 \le p \le \infty$ we denote by $\mathcal{F}L^p(\mathbb{R}^d)$ the space of temperate distributions in $\mathbb{R}^d$ whose (inverse) Fourier transform belongs to L^p, with the norm $\|f\|_{\mathcal{F}L^p} = \|\widehat{f}\|_{L^p}$.

Given $A, B \in \mathbb{R}$, we write $A \lesssim_\lambda B$ as a shorthand for the inequality $A \le C(\lambda)B$, where $C(\lambda) > 0$ is a constant that depends on the parameter λ. We also write $A \asymp_\lambda B$ to mean that both $A \lesssim_\lambda B$ and $B \lesssim_\lambda A$ hold.

The symbol $\langle x\rangle$ stands for the inhomogeneous norm of $x \in \mathbb{R}^d$, that is $\langle x\rangle := (1+|x|^2)^{1/2}$.

2.2. Basics of time-frequency analysis

We recall here some elementary facts of time-frequency (alias phase-space) analysis. Proofs and further details can be found in the monographs [14,36–38].

Given $x,\xi \in \mathbb{R}^d$, the translation and modulation operators T_x and M_ξ, acting on $f\colon \mathbb{R}^d \to \mathbb{C}$, are defined by

$$T_x f(y) := f(y-x), \qquad M_\xi f(y) := e^{2\pi i\xi\cdot y} f(y).$$

The composition $\pi(x,\xi) := M_\xi T_x$ is said to be the phase space shift of f along $(x,\xi) \in \mathbb{R}^d \times \widehat{\mathbb{R}^d} \simeq \mathbb{R}^{2d}$.

The short-time Fourier transform (STFT), or Gabor transform, of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ is defined by

$$V_g f(x,\xi) := \langle f, \pi(x,\xi) g\rangle = \mathcal{F}(f \cdot \overline{T_x g})(\xi), \qquad (x,\xi) \in \mathbb{R}^{2d}.$$

The Gabor transform is a continuous map $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \ni f \times g \to V_g f \in \mathcal{S}(\mathbb{R}^{2d})$, hence satisfying $|V_g f(x,\xi)| \lesssim_N \langle (x,\xi)\rangle^{-N}$ for every $N \in \mathbb{N}$.

Given $1 \le p, q \le \infty$, the modulation space $M^{p,q}(\mathbb{R}^d)$ is the space of all the distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that, for a given $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, the mixed Lebesgue norm

$$\|f\|_{M^{p,q}_{(g)}} := \|V_g f(x,\xi)\|_{L^q_\xi(L^p_x)} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|^p \,\mathrm{d}x \right)^{q/p} \mathrm{d}\xi \right)^{1/q} \tag{6}$$

is finite—with obvious modifications if $p, q = \infty$. When $q = p$ we write $M^p(\mathbb{R}^d)$ instead of $M^{p,p}(\mathbb{R}^d)$.

It is well known that $(M^{p,q}(\mathbb{R}^d), \|\cdot\|_{M^{p,q}_{(g)}})$ is a Banach space for every p, q, and the norms $\|\cdot\|_{M^{p,q}_{(g)}}$ and $\|\cdot\|_{M^{p,q}_{(\gamma)}}$ are equivalent for every choice of a different window $\gamma \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$—we thus omit the dependence on the window unless relevant, and write $\|\cdot\|_{M^{p,q}}$ from now on.

Swapping the integration order in (6) yields

$$\|f\|_{W^{p,q}_{(g)}} := \|V_g f(x,\xi)\|_{L^q_x(L^p_\xi)} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|^p \,\mathrm{d}\xi \right)^{q/p} \mathrm{d}x \right)^{1/q}. \tag{7}$$

The space of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\|f\|_{W^{p,q}_{(g)}} < \infty$ is again a Banach space, called the Wiener amalgam space $W^{p,q}(\mathbb{R}^d)$, and different choices of windows in (7) result in equivalent norms on $W^{p,q}(\mathbb{R}^d)$. We write $W^p(\mathbb{R}^d)$ instead of $W^{p,p}(\mathbb{R}^d)$.

We list below some properties of modulation and Wiener amalgam spaces that are needed in the note.

Proposition 2.1. *Consider $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$. The following properties hold.*

(1) We have the identification $W^{p,q}(\mathbb{R}^d) = \mathcal{F}M^{p,q}(\mathbb{R}^d)$, that is

$$f \in M^{p,q}(\mathbb{R}^d) \quad \Leftrightarrow \quad \widehat{f} \in W^{p,q}(\mathbb{R}^d).$$

Moreover, $M^p(\mathbb{R}^d) = W^p(\mathbb{R}^d)$.

(2) The pointwise multiplication is continuous $M^{p_1,q_1}(\mathbb{R}^d) \times M^{p_2,q_2}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$ if and only if

$$\frac{1}{p_1} + \frac{1}{p_2} \ge \frac{1}{p}, \qquad \frac{1}{q_1} + \frac{1}{q_2} \ge 1 + \frac{1}{q}.$$

In particular, $M^{\infty,1}(\mathbb{R}^d)$ is a Banach algebra for pointwise multiplication and every modulation space is a Banach module over it. In light of the previous item, the same constraints on the indices characterize continuity of convolution as a mapping $W^{p_1,q_1}(\mathbb{R}^d) \times W^{p_2,q_2}(\mathbb{R}^d) \to W^{p,q}(\mathbb{R}^d)$.

(3) The pointwise multiplication is continuous $W^{p_1,q_1}(\mathbb{R}^d) \times W^{p_2,q_2}(\mathbb{R}^d) \to W^{p,q}(\mathbb{R}^d)$ if and only if

$$\frac{1}{p_1} + \frac{1}{p_2} \ge 1 + \frac{1}{p}, \qquad \frac{1}{q_1} + \frac{1}{q_2} \ge \frac{1}{q}.$$

In particular, $W^{1,\infty}(\mathbb{R}^d)$ *is a Banach algebra for pointwise multiplication and every Wiener amalgam space is a Banach module over it.*
(4) *Given* $\lambda \neq 0$ *consider the dilation* $f_\lambda(x) := f(\lambda x)$*. There exists a constant* $C > 0$ *such that*

$$\|f_\lambda\|_{M^{p,q}} \leq C|\lambda|^{-d(1/p-1/q+1)}(1+\lambda^2)^{d/2}\|f\|_{M^{p,q}}.$$

(5) *Let* T *be a linear operator and* $A_1, A_2 > 0$ *constants such that*

$$\|Tf\|_{M^{p_1,q_1}} \leq A_1\|f\|_{M^{p_1,q_1}}, \qquad \forall f \in M^{p_1,q_1}(\mathbb{R}^d),$$

$$\|Tf\|_{M^{p_2,q_2}} \leq A_2\|f\|_{M^{p_2,q_2}}, \qquad \forall f \in M^{p_2,q_2}(\mathbb{R}^d).$$

Then, for all $0 < \theta < 1$ *and* $1 \leq p, q \leq \infty$ *such that*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2},$$

there exists a constant $C > 0$ *independent of* T *such that*

$$\|Tf\|_{M^{p,q}} \leq CA_1^{1-\theta}A_2^{\theta}\|f\|_{M^{p,q}}, \qquad \forall f \in M^{p,q}(\mathbb{R}^d).$$

(6) *Given* $X \subseteq \mathcal{S}'(\mathbb{R}^d)$*, set* $X_{\mathrm{comp}} := \{u \in X : \mathrm{supp}(u) \text{ is a compact subset of } \mathbb{R}^d\}$*. Then* $(M^{p,q}(\mathbb{R}^d))_{\mathrm{comp}} = (\mathcal{F}L^q(\mathbb{R}^d))_{\mathrm{comp}}$ *and* $(W^{p,q}(\mathbb{R}^d))_{\mathrm{comp}} = (\mathcal{F}L^p(\mathbb{R}^d))_{\mathrm{comp}}$.

3. Proof of Theorem 1.1

Let us start by introducing the central objects of this note, namely generalized Fresnel functions of order m; cf [49].

Definition 3.1. Given $m \in \mathbb{R}$, the class $A^m(\mathbb{R}^d)$ consists of smooth functions $\mu \in C^\infty(\mathbb{R}^d;\mathbb{R})$ such that, for all $\alpha \in \mathbb{N}^d$,

$$\sup_{x\in\mathbb{R}^d} \langle x\rangle^{|\alpha|-m}|\partial^\alpha \mu(x)| < \infty.$$

Given $\mu \in A^m(\mathbb{R}^d)$, the corresponding Fresnel function is defined by $F_\mu(x) := e^{2\pi i\mu(x)}$.

The following result elucidates some phase space features of the Fresnel function F_μ in terms of the corresponding Gabor transform.

Proposition 3.2. *Let* F_μ *be the Fresnel function associated with* $\mu \in A^m(\mathbb{R}^d)$*,* $m \geq 2$*. Let* $g \in C_c^\infty(\mathbb{R}^d)$ *be such that* $\mathrm{supp}(g) \subseteq B = \{y \in \mathbb{R}^d : |y| \leq 1\}$*.*

(i) We have

$$|V_g F_\mu(x,\xi)| \leq \|g\|_{L^1}, \qquad (x,\xi) \in \mathbb{R}^{2d}. \tag{8}$$

(ii) There exists $A > 0$ *such that, for all* $N \geq 0$*,*

$$\text{if} \quad |\xi - \nabla\mu(x)| \geq A|x|^{m-2} \quad \text{then} \quad |V_g F_\mu(x,\xi)| \leq C\langle\xi - \nabla\mu(x)\rangle^{-N}, \tag{9}$$

for some $C > 0$ *that depends on* N*.*

Proof. The pointwise bound in (8) is straightforward.

Concerning (9), let us separately discuss the cases where $|x| \leq 1$ and $|x| > 1$. It is easy to realize that if $|x| \leq 1$ then the function

$$h_x(y) := e^{2\pi i\mu(y)}\overline{g(y-x)}$$

belongs to $\mathcal{S}(\mathbb{R}^d)$, the corresponding seminorms being bounded with respect to x. Therefore, if $|x| \leq 1$ then

$$|V_g F_\mu(x,\xi)| = |\widehat{h_x}(\xi)| \lesssim_N \langle\xi\rangle^{-N},$$

for all $N \geq 0$, and this proves the claim since $\langle\xi - \nabla\mu(x)\rangle \asymp \langle\xi\rangle$ if $|x| \leq 1$. Assume now $|x| > 1$, and note that

$$V_g F_\mu(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\varphi_{x,\xi}(y)}\overline{g(y)}\, \mathrm{d}y,$$

where we set $\varphi(y) = \varphi_{x,\xi}(y) := (y+x)\cdot\xi - \mu(y+x)$. After noticing that $\nabla\varphi(y) = \nabla_y\varphi(y) = \xi - \nabla\mu(y+x)$, it is natural to consider the set

$$\Omega := \{(x,\xi) \in \mathbb{R}^{2d} : \quad |x| \geq 1, \quad |\xi - \nabla\mu(x)| \geq A|x|^{m-2}\},$$

where the constant $A > 0$ is chosen below. If $|y| \leq 1$ and $|x| \geq 1$ then

$$|\nabla\mu(y+x) - \nabla\mu(x)| \leq \sup_{|z|\leq 1} |\nabla^2\mu(z+x)| \leq C|x|^{m-2},$$

for some $C > 0$. Choosing $A > 2C$ yields, for $(x,\xi) \in \Omega$ and $|y| \leq 1$,

$$|\nabla\varphi(y)| \geq |\xi - \nabla\mu(x)| - |\nabla\mu(y+x) - \nabla\mu(x)| \geq \frac{1}{2}|\xi - \nabla\mu(x)| \geq \frac{A}{2}|x|^{m-2}. \tag{10}$$

Moreover, since $\mu \in A^m(\mathbb{R}^d)$ we have

$$|\partial^\alpha\varphi(y)| \lesssim |x|^{m-|\alpha|}, \qquad |\alpha| \geq 2, \quad |x| \geq 1, \quad |y| \leq 1. \tag{11}$$

Consider now the differential operator

$$L := \sum_{j=1}^d B_j^{(x,\xi)}(y)\, \partial_{y_j},$$

where

$$B_j^{(x,\xi)}(y) := \frac{\partial_{y_j}\varphi(y)}{-2\pi i|\nabla\varphi(y)|^2}, \qquad (x,\xi) \in \Omega, \quad |y| \leq 1.$$

A recursive argument shows that $\partial^\alpha B_j^{(x,\xi)}$, $\alpha \in \mathbb{N}^d$, is a (finite, complex) linear combination of terms such as

$$\frac{\partial^{\gamma_1}\varphi(y)\cdots\partial^{\gamma_k}\varphi(y)}{|\nabla\varphi(y)|^{k+1}} = \frac{\partial^{\gamma_1}\varphi(y)}{|\nabla\varphi(y)|}\frac{\partial^{\gamma_k}\varphi(y)}{|\nabla\varphi(y)|}\frac{1}{|\nabla\varphi(y)|},$$

for suitable $k = k(\alpha, j) \in \mathbb{N}$, $\gamma_1,\ldots,\gamma_k \in \mathbb{N}^d$ with $|\gamma_1|,\ldots,|\gamma_k| \geq 1$. Therefore, in light of (10) and (11) we infer that

$$\frac{|\partial^\gamma\varphi(y)|}{|\nabla\varphi(y)|} \leq \begin{cases} 1 & (|\gamma| = 1), \\ C_\gamma & (|\gamma| \geq 2), \end{cases} \qquad (x,\xi) \in \Omega, \quad |y| \leq 1,$$

for suitable constants $C_\gamma > 0$. Once again by virtue of (10), we have

$$\frac{1}{|\nabla\varphi(y)|} \le \frac{2}{|\xi - \nabla\mu(x)|}, \qquad (x,\xi) \in \Omega, \quad |y| \le 1,$$

hence we conclude that

$$|\partial^\alpha B_j^{(x,\xi)}(y)| \lesssim_\alpha \frac{1}{|\xi - \nabla\mu(x)|}, \qquad (x,\xi) \in \Omega, \quad |y| \le 1, \quad \alpha \in \mathbb{N}^d, \quad j = 1, \dots, d. \quad (12)$$

Note that

$$L^N e^{-2\pi i\varphi(y)} = e^{-2\pi i\varphi(y)}, \quad N \in \mathbb{N}.$$

Integration by parts $N \ge 1$ times then gives

$$|V_g F_\mu(x,\xi)| \le \int_{|y|\le 1} |(L^\top)^N \bar{g}(y)| \, \mathrm{d}y. \quad (13)$$

Arguing by induction on N yields that $(L^\top)^N$ is a linear partial differential operator whose coefficients are (finite, complex) linear combinations of terms of the type

$$\partial^{\gamma_1} B_{j_1}^{(x,\xi)}(y) \cdots \partial^{\gamma_N} B_{j_N}^{(x,\xi)}(y),$$

for suitable $j_1, \dots, j_N \in \{1, \dots, d\}$ and $\gamma_1, \dots, \gamma_N \in \mathbb{N}^d$. We have, as a result of (12),

$$|\partial^{\gamma_1} B_{j_1}^{(x,\xi)}(y) \cdots \partial^{\gamma_N} B_{j_N}^{(x,\xi)}(y)| \lesssim |\xi - \nabla\mu(x)|^{-N}, \qquad (x,\xi) \in \Omega, \quad |y| \le 1,$$

hence from (13) we obtain

$$|V_g F_\mu(x,\xi)| \lesssim |\xi - \nabla\mu(x)|^{-N},$$

and the claim for $N \ge 1$ follows after noticing that if $(x,\xi) \in \Omega$ then $|\xi - \nabla\mu(x)| \ge A|x|^{m-2} \gtrsim 1$, hence $|\xi - \nabla\mu(x)| \asymp \langle \xi - \nabla\mu(x)\rangle$. The case $N = 0$ is trivially encompassed by (8). □

In light of the previous findings, our next goal is to provide a characterization of F_μ in terms of membership to suitable modulation spaces. To this aim, we anticipate a technical result for later use.

Proposition 3.3. *Let $\mu\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ be a positively homogeneous C^2 function of degree $m \ge 2$, satisfying*

$$\det|\nabla^2\mu(x)| \ne 0 \quad \text{if} \quad |x| = 1.$$

For all $x_0 \in \mathbb{R}^d$ with $|x_0| = 1$ there exist an open conic neighborhood $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ of x_0 and a constant $C = C(x_0,\Gamma) > 0$ such that

$$|\nabla\mu(x) - \nabla\mu(y)| \ge C|x-y|(|x|^{m-2} + |y|^{m-2}), \quad \forall x, y \in \Gamma. \quad (14)$$

Proof. As a consequence of the inverse mapping theorem we have that for suitable $c > 0$, $r > 1$ and $\varepsilon > 0$, if $r^{-1} < |x|, |y| < r$ and $x, y \in \Gamma_\varepsilon := \left\{u \in \mathbb{R}^d \setminus \{0\} : \left|\frac{u}{|u|} - x_0\right| < \varepsilon\right\}$, then

$$|\nabla\mu(x) - \nabla\mu(y)| \ge c|x-y|.$$

This implies the inequality in (14) for $x, y \in \Gamma_\varepsilon$, $r^{-1} < |x|, |y| < r$. We claim that, by homogeneity, this implies that the same estimate in (14) holds if $x, y \in \Gamma_\varepsilon$ are such that

$r^{-2} < |x|/|y| < r^2$. Indeed, given that both sides of (14) are positively homogeneous of degree $m-1$ with respect to the pair $x, y \neq 0$, such inequality holds for a pair x, y if and only if it holds for the pair $\alpha x, \alpha y$ for some $\alpha > 0$. Since it has already been proved for $x, y \in \Gamma_\varepsilon$ with $r^{-1} < |x|, |y| < r$, it suffices to prove that, given $x, y \in \Gamma_\varepsilon$ with $r^{-2} < |x|/|y| < r^2$, there exists $\alpha > 0$ such that $r^{-1} < \alpha|x|, \alpha|y| < r$, that is, $\alpha \in (r^{-1}/|x|, r/|x|)$ and $\alpha \in (r^{-1}/|y|, r/|y|)$. These two intervals have non-empty intersection—if $|x| \le |y|$ we have $r^{-1}/|y| \le r^{-1}/|x| < r/|y| \le r/|x|$, and analogous conclusions hold if $|x| \ge |y|$.

It remains to prove the inequality in (14) for $x, y \in \Gamma_\varepsilon$ satisfying $|x|/|y| \ge r^2$ or $|y|/|x| \ge r^2$—the arguments being identical, we focus here on the first assumption. Therefore, combining this with triangle inequality we infer

$$|x-y|(|x|^{m-2}+|y|^{m-2}) \le \left(1+\frac{1}{r^2}\right)\left(1+\frac{1}{r^{2(m-2)}}\right)|x|^{m-1}.$$

On the other hand,

$$\begin{aligned}
|\nabla\mu(x)-\nabla\mu(y)| &= \left||x|^{m-1}\nabla\mu\left(\frac{x}{|x|}\right)-|y|^{m-1}\nabla\mu\left(\frac{y}{|y|}\right)\right| \\
&\ge \left||x|^{m-1}-|y|^{m-1}\right|\left|\nabla\mu\left(\frac{x}{|x|}\right)\right| - |y|^{m-1}\left|\nabla\mu\left(\frac{x}{|x|}\right)-\nabla\mu\left(\frac{y}{|y|}\right)\right| \\
&\ge |x|^{m-1}\left(\left(1-\frac{1}{r^{2(m-1)}}\right)\left|\nabla\mu\left(\frac{x}{|x|}\right)\right| - \frac{1}{r^{2(m-1)}}\sup_{\substack{u,v\in\Gamma_\varepsilon\\ |u|=|v|=1}}|\nabla\mu(u)-\nabla\mu(v)|\right).
\end{aligned}$$

Note that:

- The term $\sup|\nabla\mu(u)-\nabla\mu(v)|$ can be made arbitrarily small by tuning the choice of ε small enough.
- $|\nabla\mu(u)|$ is uniformly bounded away from zero if $|u|=1$. In fact, we have in general

$$|\nabla\mu(x)| \gtrsim |x|^{m-1}, \qquad |x| \neq 0,$$

as a consequence of Euler's homogeneous function theorem

$$\nabla^2\mu(x)x = (m-1)\nabla\mu(x), \qquad x \in \mathbb{R}^d \setminus \{0\},$$

the assumptions on the invertibility of the Hessian of μ on the unit sphere and the fact that $m \ge 2$.

Hence $|\nabla\mu(x)-\nabla\mu(y)| \gtrsim |x|^{m-1}$ for $x, y \in \Gamma_\varepsilon$ satisfying $|x|/|y| \ge r^2$, which concludes the proof. □

Remark 3.4. The claim of Proposition 3.3 fails to be true if $m < 2$, as showed by straightforward computations in the case where $\mu(x) = |x|^m$ with $m < 2$ and $d = 1$. Indeed, if $m < 2$, the estimate

$$|mx^{m-1}-m| \ge C|x-1|(x^{m-2}+1), \qquad x > 0$$

clearly cannot hold for some $C > 0$, as one sees by letting $x \to +\infty$. Also, the localization in cones is necessary: if $\mu(x) = x^3/3$, $x \in \mathbb{R}$, we have $|\mu'(x)-\mu'(y)| = |x^2-y^2|$, which vanishes (in particular) for $x = -y$, and therefore is not bounded below by $C|x-y|(|x|+|y|)$ for every $x, y \neq 0$.

We are now ready to locate the modulation space membership of higher-order Fresnel functions and prove Theorem 1.1.

Proof of Theorem 1.1. We show that the function $F_\mu(x) = e^{2\pi i\mu(x)}$ belongs to $M^{1,\infty}(\mathbb{R}^d)$, which is equivalent to the claim in view of Proposition 2.1.

Let $\chi : \mathbb{R}^d \to \mathbb{R}$ be a smooth bump function near the origin, namely satisfying $0 \le \chi(x) \le 1$ for all $x \in \mathbb{R}^d$, $\chi(x) = 0$ if $|x| \ge 2$ and $\chi(x) = 1$ if $|x| < 1$. Therefore, we have

$$F_\mu(x) = F_\mu^{(1)}(x) + F_\mu^{(2)}(x), \quad F_\mu^{(1)}(x) := \chi(x)e^{2\pi i\mu(x)}, \quad F_\mu^{(2)}(x) := (1-\chi(x))e^{2\pi i\mu(x)}.$$

Let us prove that $F_\mu^{(1)} \in M^{1,\infty}(\mathbb{R}^d)$ first. To this aim, it is enough to compute $V_gF_\mu^{(1)}$ with a smooth window function g with compact support, for instance in the unit ball, so that $V_gF_\mu^{(1)}(x,\xi) = 0$ if $|x| \ge 3$. On the other hand, we obtain the pointwise bound $|V_gF_\mu^{(1)}(x,\xi)| \le \|g\|_{L^1}$, implying the claim.

We focus now on $F_\mu^{(2)}$. Let $\tilde{\mu} : \mathbb{R}^d \to \mathbb{R}$ be any smooth function such that $\tilde{\mu}(x) = \mu(x)$ if $|x| \ge 1$. The companion Fresnel function is $\widetilde{F}_\mu(x) = e^{2\pi i\tilde{\mu}(x)}$, and since $1-\chi \in M^{\infty,1}(\mathbb{R}^d)$ is a bounded pointwise multiplier on any modulation space (cf. Proposition 2.1), it suffices to show that $\widetilde{F}_\mu \in M^{1,\infty}(\mathbb{R}^d)$, namely that

$$\sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{R}^d} |V_g\widetilde{F}_\mu(x,\xi)| \,\mathrm{d}x < \infty.$$

It is clear that $\tilde{\mu} \in A^m(\mathbb{R}^d)$, hence by Proposition 3.2 we have for every suitable window g and $N > 0$:

$$|V_g\widetilde{F}_\mu(x,\xi)| \lesssim_N \langle \xi - \nabla\tilde{\mu}(x)\rangle^{-N} \quad \text{if} \quad |\xi - \nabla\tilde{\mu}(x)| \ge A|x|^{m-2}, \tag{15}$$

for a suitable $A > 0$. Given $\xi \in \mathbb{R}^d$, consider the sets

$$\Omega_\xi^{(1)} := \{x \in \mathbb{R}^d : |\xi - \nabla\tilde{\mu}(x)| > A|x|^{m-2}\},$$

$$\Omega_\xi^{(2)} := \{x \in \mathbb{R}^d : |\xi - \nabla\tilde{\mu}(x)| \le A|x|^{m-2}\}.$$

We first prove that

$$\sup_{\xi\in\mathbb{R}^d} \int_{\Omega_\xi^{(2)}} |V_g\widetilde{F}_\mu(x,\xi)| \,\mathrm{d}x < \infty.$$

In fact, this claim follows at once by combining (8) for $\widetilde{F}_\mu$ with uniformity of the measures of $\Omega_\xi^{(2)}$ over $\xi \in \mathbb{R}^d$, that is

$$\sup_{\xi\in\mathbb{R}^d} |\Omega_\xi^{(2)}| < \infty.$$

The proof of the latter fact goes as follows. Let $\{\Gamma_j\}_{j=1,\dots,n}$ be a finite covering of $\mathbb{R}^d \setminus \{0\}$ consisting of open conic neighborhoods obtained as in Proposition 3.3. It is then enough to show that

$$\sup\{\operatorname{diam}(\Omega_\xi^{(2)} \cap \Gamma_j) : \xi \in \mathbb{R}^d, j = 1,\dots,n\} < \infty,$$

which is in turn equivalent to

$$\sup\left\{\operatorname{diam}(\{x \in \Omega_\xi^{(2)} \cap \Gamma_j, |x| \ge 1\}) : \xi \in \mathbb{R}^d, j = 1,\dots,n\right\} < \infty.$$

Note that if $x, y \in \Omega_\xi^{(2)} \cap \Gamma_j$ satisfy $|x|, |y| \geq 1$ then

$$|\xi - \nabla\tilde{\mu}(x)| = |\xi - \nabla\mu(x)| \leq A|x|^{m-2}, \qquad |\xi - \nabla\tilde{\mu}(y)| = |\xi - \nabla\mu(y)| \leq A|y|^{m-2}.$$

As a result, we have

$$|\nabla\mu(x) - \nabla\mu(y)| \leq A(|x|^{m-2} + |y|^{m-2}),$$

but by virtue of Proposition 3.3 we also have, for a suitable $B > 0$,

$$|\nabla\mu(x) - \nabla\mu(y)| \geq B|x-y|(|x|^{m-2} + |y|^{m-2}),$$

hence resulting in the bound $|x - y| \leq A/B$, which gives the claim.

It now remains only to prove that

$$\sup_{\xi\in\mathbb{R}^d} \int_{\Omega_\xi^{(1)}} |V_g\widetilde{F}_\mu(x,\xi)|\, \mathrm{d}x < \infty.$$

It follows from (15) and the definition of $\Omega_\xi^{(1)}$ that

$$|V_g\widetilde{F}_\mu(x,\xi)| \lesssim_N \langle x\rangle^{-(m-2)N}, \qquad x \in \Omega_\xi^{(1)},$$

which suffices to prove the claim in the case where $m > 2$ provided that N is chosen large enough. On the other hand, if $m = 2$ then (8) yields uniform boundedness of $V_g\widetilde{F}_\mu$ on $\Omega_\xi^{(1)}$, which implies that

$$\sup_{\xi\in\mathbb{R}^d} \int_{E_\xi^{(\leq 1)}} |V_g\widetilde{F}_\mu(x,\xi)|\, \mathrm{d}x < \infty, \qquad E_\xi^{(\leq 1)} := \{x \in \Omega_\xi^{(1)} : |x| \leq 1\}.$$

Therefore, it remains to obtain the analogous bound in the case where $E_\xi^{(\leq 1)}$ is replaced by $E_\xi^{(>1)} := \{x \in \Omega_\xi^{(1)} : |x| > 1\}$. Proposition 3.3 indicates that one should make use of a cone-based geometric framework. Namely, let $\{\Gamma_j\}_{j=1,\dots,n}$ be a finite covering of $\mathbb{R}^d \setminus \{0\}$ consisting of open conic neighborhoods obtained as in Proposition 3.3. By virtue of additivity, it is thus enough to show that

$$\sup_{\xi\in\mathbb{R}^d} \int_{E_\xi^{(>1)}\cap\Gamma_j} |V_g\widetilde{F}_\mu(x,\xi)|\, \mathrm{d}x < \infty, \qquad \forall j = 1,\dots,n.$$

Since $\tilde{\mu}(x) = \mu(x)$ if $|x| > 1$, we have $E_\xi^{(>1)}\cap\Gamma_j = \{x \in \Gamma_j : |x| > 1 \text{ and } |\xi - \nabla\mu(x)| \geq A\}$, which motivates the change of variables $v = \nabla\mu(x)$. The gradient map $\nabla\mu \colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d \setminus \{0\}$ is smooth, positively homogeneous of degree 1 and by Proposition 3.3 we have that Γ_j is bijectively mapped to $\Lambda_j := \nabla\mu(\Gamma_j)$ (indeed, the injectivity follows from (14)). Therefore, since $|\det \nabla^2\mu(x)| \geq \delta > 0$ for $|x| = 1$ by assumption, after resorting to (15) we obtain

$$\begin{aligned}\int_{E_\xi^{(>1)}\cap\Gamma_j} |V_g\widetilde{F}_\mu(x,\xi)|\, \mathrm{d}x &\lesssim_N \int_{E_\xi^{(>1)}\cap\Gamma_j} \langle\xi - \nabla\mu(x)\rangle^{-N}\, \mathrm{d}x \\ &\lesssim \int_{\Gamma_j} \langle\xi - \nabla\mu(x)\rangle^{-N}\, \mathrm{d}x \\ &\lesssim \int_{\Lambda_j} \langle\xi - v\rangle^{-N}\, \mathrm{d}v\end{aligned}$$

$$\leq \int_{\mathbb{R}^d} \langle v \rangle^{-N} \, \mathrm{d}v < \infty,$$

as long as $N > d$. ☐

Remark 3.5. Let us make some comments on Theorem 1.1.

- Given a symmetric matrix $Q \in \mathbb{R}^{d\times d}$, consider $\mu(x) = \frac{1}{2}Qx \cdot x$. After diagonalization, it is easy to realize that the assumption of invertibility of $\nabla^2\mu(x) = Q$ is actually necessary in order for $F_\mu = e^{2\pi i\mu}$ to belong to $M^{1,\infty}(\mathbb{R}^d)$. It is indeed well known that a linear and invertible change of variable yields a bounded operator in $M^{1,\infty}(\mathbb{R}^d)$ (see for instance [37, Proposition 2.3]). As a consequence, we can verify the claim assuming $\mu(x) = \frac{1}{2}\sum_{j=1}^k \varepsilon_j x_j^2$, with $\varepsilon_j \in \{-1, 1\}$, $0 \leq k \leq d$ (with $Q(x) \equiv 0$ if $k = 0$). If $k < d$ then the modulus short-time Fourier transform $|V_g F_\mu(x_1, \ldots, x_d, \xi_1, \ldots, \xi_d)|$, with $g = g_1 \otimes \ldots \otimes g_d$, is in fact independent of x_d, and therefore $\int_{\mathbb{R}} |V_g F_\mu(x_1, \ldots, x_d, \xi_1, \ldots, \xi_d)| \, \mathrm{d}x_d = \infty$ for every $x_1, \ldots, x_{d-1}, \xi_1, \ldots, \xi_d$. As a consequence, $F_\mu \notin M^{1,\infty}(\mathbb{R}^d)$.
- It is well known [23] that $e^{2\pi i|x|^m} \in W^{1,\infty}(\mathbb{R}^d)$ if $m \leq 2$, and this condition is in fact optimal. Indeed, if $e^{2\pi i|x|^m} \in W^{1,\infty}(\mathbb{R}^d)$ then by Proposition 2.1 the pointwise multiplication by $e^{2\pi i|x|^m}$ is a bounded operator on $W^{p,q}(\mathbb{R}^d)$ for every $1 \leq p, q \leq \infty$ —that is, the Fourier multiplier $e^{2\pi i|D|^m}$ is bounded on $M^{p,q}(\mathbb{R}^d)$ for every $1 \leq p, q \leq \infty$, which in turn happens only if $m \leq 2$ in light of [26, Theorem 1.2]. This result can now be complemented with Theorem 1.1, showing that $e^{2\pi i|x|^m} \in M^{1,\infty}(\mathbb{R}^d)$ if $m \geq 2$. In particular, at the threshold $m = 2$ we have that the Fresnel function $e^{2\pi i|x|^2}$ belongs to $W^{1,\infty}(\mathbb{R}^d) \cap M^{1,\infty}(\mathbb{R}^d)$ —this is of course known and can be easily verified by an explicit computation with a Gaussian window.
- Heuristic considerations lead us to believe that the assumption $m \geq 2$ is sharp. Indeed, the principle of duality of phases [50, Sec. VIII.5] shows that, if $d = 1$, the Fourier transform of $e^{2\pi i|x|^m}$ with $m > 1$ roughly behaves like $e^{ic|\xi|^{m'}}$ for some $c \in \mathbb{R} \setminus \{0\}$, where m' satisfies $\frac{1}{m} + \frac{1}{m'} = 1$ —see also [51] for an alternative derivation of this result. As argued above, the latter function notably fails to belong to $W^{1,\infty}(\mathbb{R}) = \mathcal{F}M^{1,\infty}(\mathbb{R})$ when $m' > 2$.

3.1. Some dispersive estimates for $\mu(D)$

Let us highlight an interesting consequence of Theorem 1.1, namely dispersive estimates for higher-order Schrödinger-type equations.

Corollary 3.6. *Let μ be as in Theorem 1.1. Consider the Cauchy problem*

$$\begin{cases} \frac{1}{2\pi i}\partial_t u = \mu(D)u \\ u(0,x) = f(x), \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

The corresponding propagator $U_\mu(t) := e^{2\pi it\mu(D)}$ is bounded $W^{\infty,1}(\mathbb{R}^d) \to W^{1,\infty}(\mathbb{R}^d)$, with

$$\|U_\mu(t)f\|_{W^{1,\infty}} \leq C \max\{t^{-d/m}, t^{-2d/m}\}\|f\|_{W^{\infty,1}}, \tag{16}$$

for a constant $C > 0$ that does not depend on t and f.

Proof. Set $F_\mu(x) = e^{2\pi i\mu(x)}$. First, notice that $U_\mu(t)$ coincides with the Fourier multiplier $F_\mu(t^{1/m}D)$ with symbol $F_\mu(t^{1/m}x) = e^{2\pi i\mu(t^{1/m}x)}$. Consider now the case where $t = 1$. Given

$f \in W^{\infty,1}(\mathbb{R}^d)$, in light of Proposition 2.1 and Theorem 1.1 we have

$$\|U_\mu(1)f\|_{W^{1,\infty}} = \|F_\mu \widehat{f}\|_{M^{1,\infty}} \lesssim \|F_\mu\|_{M^{1,\infty}} \|f\|_{W^{\infty,1}}.$$

The general case reduces to the former by resorting to dilation properties of the symbol F_μ on modulation spaces (see Proposition 2.1):

$$\|F_\mu(t^{1/m}\cdot)\|_{M^{1,\infty}} \lesssim t^{-2d/m}(1+t^{2/m})^{d/2}\|F_\mu\|_{M^{1,\infty}} \lesssim \begin{cases} t^{-d/m}\|F_\mu\|_{M^{1,\infty}} & (t \geq 1) \\ t^{-2d/m}\|F_\mu\|_{M^{1,\infty}} & (0 < t < 1), \end{cases}$$

and the claim follows. □

Remark 3.7. We emphasize that the initial datum f belongs to $W^{\infty,1}(\mathbb{R}^d)$, encompassing the case of complex finite measures—cf. [48, Proposition 3.4]. Moreover, in the Schrödinger setting ($m = 2$) we retrieve the asymptotics provided by well-known dispersive estimates on amalgam spaces, cf [9].. In fact, arguing as in [9], one can obtain Strichartz estimates in this general context.

4. Proof of Theorem 1.2

Let us now consider the problem of local wellposedness in the presence of potential perturbations and nonlinearities. Our strategy is centered on a standard abstract iteration argument, which is stated below for the convenience of the reader.

Proposition 4.1 ([8, Proposition 1.38]). *Let X and Y be two Banach spaces, and let $A\colon X \to Y$ be a bounded linear operator such that*

$$\|Au\|_Y \leq C_0\|u\|_X, \tag{17}$$

for all $u \in X$ and some $C_0 > 0$. Let $G\colon Y \to X$ be a nonlinear operator such that $G(0) = 0$ and

$$\|G(u) - G(v)\|_X \leq \frac{1}{2C_0}\|u - v\|_Y, \tag{18}$$

for all u, v in the ball $B_R(Y) = \{u \in Y : \|u\|_Y \leq R\}$ for some $R > 0$.

Then, for every $u_0 \in B_{R/2}(Y)$ there exists a unique solution $u \in B_R(Y)$ to the equation

$$u = u_0 + AG(u),$$

and the correspondence $u_0 \mapsto u$ is Lipschitz with constant at most 2, hence $\|u\|_Y \leq 2\|u_0\|_Y$.

We are now ready to prove our main wellposedness result for higher-order dispersive equations with potentials in $W^{\infty,1}(\mathbb{R}^d)$ and analytic nonlinearities.

Proof of Theorem 1.2. After setting $U_\mu(t) := e^{2\pi i t\mu(D)}$, let us start by rewriting the problem in integral form *via* Duhamel's formula:

$$u(t,x) = U_\mu(t)f(x) + 2\pi i \int_0^t U_\mu(t-s)\big[V(s,\cdot)G(u(s,\cdot))\big](x)\, ds.$$

Our goal is now to ensure that the assumptions of Proposition 4.1 are satisfied with $X = Y = C^0([0,T]; W^{1,\infty}(\mathbb{R}^d))$, with $0 < T \leq 1$ to be determined later, $u_0(t) = U_\mu(t)f$ and A being the Duhamel operator $Av(t) := 2\pi i \int_0^t U_\mu(t-s)V(s,\cdot)v(s,\cdot)\, ds$.

Let us first note that, in view of the embedding $\mathcal{F}L^1(\mathbb{R}^d) \hookrightarrow W^{1,\infty}(\mathbb{R}^d)$, we have

$$\|u_0(t)\|_{W^{1,\infty}} = \|U_\mu(t)f\|_{W^{1,\infty}} \lesssim \|U_\mu(t)f\|_{\mathcal{F}L^1} = \|f\|_{\mathcal{F}L^1}.$$

Moreover, the map $[0,+\infty) \ni t \mapsto u_0(t) \in W^{1,\infty}(\mathbb{R}^d)$ is continuous. Indeed, for $t, t_0 \in [0,+\infty)$, by dominated convergence we infer

$$\|U_\mu(t)f - U_\mu(t_0)f\|_{W^{1,\infty}} \lesssim \|(F_\mu(t^{1/m}\cdot) - F_\mu(t_0^{1/m}\cdot))\widehat{f}\|_{L^1} \to 0 \qquad \text{as} \qquad t \to t_0.$$

As a result, $u_0 \in X$ and $\|u_0\|_X \leq C\|f\|_{\mathcal{F}L^1}$ for a constant $C > 0$ that does not depend on T. Fix the data radius $R > 0$ from the statement and set $\rho := 2CR$, so that $\|u_0\|_X \leq \rho/2$ whenever $\|f\|_{\mathcal{F}L^1} \leq R$.

Concerning (18), the Banach algebra property of X under pointwise multiplication (Proposition 2.1) implies (cf. for instance [52]) that there exists $C_\rho > 0$ such that

$$\|G(u) - G(v)\|_X \leq C_\rho \|u - v\|_X \quad \text{if} \quad \max\{\|u\|_X, \|v\|_X\} \leq \rho.$$

When it comes to prove (17), by virtue of Proposition 2.1 and (16) in the case where $t \in [0,T]$, with $0 < T \leq 1$, we have

$$\begin{aligned}
\|Au(t,\cdot)\|_{W^{1,\infty}} &\lesssim \int_0^t \|U_\mu(t-s)V(s,\cdot)u(s,\cdot)\|_{W^{1,\infty}} \,\mathrm{d}s \\
&\lesssim \int_0^t |t-s|^{-2d/m} \|V(s,\cdot)u(s,\cdot)\|_{W^{\infty,1}} \,\mathrm{d}s \\
&\lesssim \left(\int_0^t |t-s|^{-2d/m} \,\mathrm{d}s\right) \|V\|_{L^\infty([0,1];W^{\infty,1}(\mathbb{R}^d))} \|u\|_X \\
&\leq C' \frac{m}{m-2d} T^{\frac{m-2d}{m}} \|V\|_{L^\infty([0,1];W^{\infty,1}(\mathbb{R}^d))} \|u\|_X,
\end{aligned}$$

for a constant $C' > 0$ depending on μ, where we used the assumption $m > 2d$. It is now enough to choose

$$T = \min\left\{\left(\frac{m-2d}{2mC'C_\rho\|V\|_{L^\infty([0,1];W^{\infty,1}(\mathbb{R}^d))}}\right)^{\frac{m}{m-2d}}, 1\right\}$$

to obtain

$$\|Au(t,\cdot)\|_{W^{1,\infty}} \leq \frac{1}{2C_\rho}\|u\|_X, \qquad 0 \leq t \leq T,$$

so that (17) is proved.

To complete the proof, let us also record that A maps $X = C^0([0,T]; W^{1,\infty}(\mathbb{R}^d))$ into itself. Indeed, for $u \in X$ let us set $w(s) = V(s,\cdot)u(s,\cdot)$; by Proposition 2.1 we infer $w \in L^\infty([0,T]; W^{\infty,1}(\mathbb{R}^d))$, since

$$\|w(s)\|_{W^{\infty,1}} \lesssim \|V\|_{L^\infty([0,1];W^{\infty,1})}\|u\|_X \quad \text{for a.e. } s \in [0,T].$$

Extend now w by zero outside $[0,T]$ and define $K(t) = \mathbf{1}_{(0,T]}(t)U_\mu(t)$. By (16), since $T \leq 1$ we obtain

$$\|K(t)\|_{\mathcal{L}(W^{\infty,1},W^{1,\infty})} \lesssim t^{-2d/m}\mathbf{1}_{(0,T]}(t),$$

and since $m > 2d$ we have $K \in L^1(\mathbb{R}; \mathcal{L}(W^{\infty,1}, W^{1,\infty}))$; here the required strong measurability of K follows from the multiplier representation of $U_\mu(t)$. The Duhamel operator for $0 \leq t \leq T$ can then be written as a vector-valued convolution with respect to the continuous bilinear multiplication $W^{\infty,1}(\mathbb{R}^d) \times \mathcal{L}(W^{\infty,1}(\mathbb{R}^d), W^{1,\infty}(\mathbb{R}^d)) \to W^{1,\infty}(\mathbb{R}^d)$ given by $f \cdot S = Sf$ [53, Appendix], namely:

$$Au(t) = 2\pi i \int_0^t U_\mu(t-s)w(s)\, \mathrm{d}s = 2\pi i(w * K)(t).$$

The desired claim then follows by the standard continuity of Banach-valued convolutions [53, Appendix, Theorem 1.9.9], namely

$$L^\infty(\mathbb{R}; W^{\infty,1}) * L^1(\mathbb{R}; \mathcal{L}(W^{\infty,1}, W^{1,\infty})) \hookrightarrow BUC(\mathbb{R}; W^{1,\infty}),$$

which in turn implies $Au \in C^0([0,T]; W^{1,\infty}(\mathbb{R}^d))$.

In the case where $G(u) = u$ and $V \in L^\infty([0,+\infty); W^{\infty,1}(\mathbb{R}^d))$, one can take $C_\rho = 1$, and the time $T \le 1$ does not depend on the initial datum. Assuming that u has been constructed on $[0, nT]$, for $t \in [nT, (n+1)T]$ we set

$$\Phi_n(t) = U_\mu(t)f + 2\pi i \int_0^{nT} U_\mu(t-s)V(s,\cdot)u(s,\cdot)\, \mathrm{d}s.$$

The same operator-valued convolution argument used above gives $\Phi_n \in C^0([nT,(n+1)T]; W^{1,\infty}(\mathbb{R}^d))$ and $\Phi_n(nT) = u(nT)$. We then solve

$$v(t) = \Phi_n(t) + 2\pi i \int_{nT}^t U_\mu(t-s)V(s,\cdot)v(s,\cdot)\, \mathrm{d}s, \qquad t \in [nT,(n+1)T],$$

and the corresponding Duhamel operator has the same contraction constant as before, since $t - s \le T \le 1$ and $\|V\|_{L^\infty([nT,(n+1)T];W^{\infty,1})} \le \|V\|_{L^\infty([0,+\infty);W^{\infty,1})}$. As a result, v extends u uniquely to $[0,(n+1)T]$, and iteration over $n \ge 0$ gives a unique global solution $u \in C^0([0,+\infty); W^{1,\infty}(\mathbb{R}^d))$. $\square$

5. Proof of Theorem 1.3

Proof of Theorem 1.3. Let $F_t^{(\mu)}$ be the symbol of the free propagator $U_\mu(t) = e^{2\pi i t\mu(D)}$, namely $F_t^{(\mu)}(\xi) = e^{2\pi i t\mu(\xi)}$. Note that Proposition 3.2 implies that $F_t^{(\mu)} \in M^\infty(\mathbb{R}^d)$, uniformly with respect to t—that is, there exists $C > 0$ such that $\|F_t^{(\mu)}\|_{M^\infty} \le C$ for every $t \in \mathbb{R}$. Combining this bound with (16) *via* complex interpolation (see Proposition 2.1), for all $1 \le p \le \infty$ we obtain

$$\|F_t^{(\mu)}\|_{M^{p,\infty}} \lesssim \max\{t^{-\frac{d}{mp}}, t^{-\frac{2d}{mp}}\}.$$

We point out that this estimate is used below with p' in place of p.

We can run again the machinery of Proposition 4.1 as in the proof of Theorem 1.2, this time with $X = Y = C^0([0,T]; W^{q,\infty}(\mathbb{R}^d))$, $u_0(t) = U_\mu(t)f$ and $T > 0$ to be determined later. In particular, we have now

$$\|u_0(t)\|_{W^{q,\infty}} = \|U_\mu(t)f\|_{W^{q,\infty}} \lesssim \|U_\mu(t)f\|_{\mathcal{F}L^q} = \|f\|_{\mathcal{F}L^q},$$

and the correspondence $[0,+\infty) \ni t \mapsto u_0(t) \in W^{q,\infty}(\mathbb{R}^d)$ is continuous, so that $u_0 \in X$ with $\|u_0\|_X \le C\|f\|_{\mathcal{F}L^q}$ for a constant $C > 0$ that does not depend on T.

Using the convolution property $W^{r,1}(\mathbb{R}^d) * W^{p',\infty}(\mathbb{R}^d) \hookrightarrow W^{q,\infty}(\mathbb{R}^d)$ with $1/p + 1/q = 1 + 1/r$ (see Proposition 2.1), for $0 < t < 1$ we infer

$$\|U_\mu(t)f\|_{W^{q,\infty}} = \|\left(\mathcal{F}^{-1}F_t^{(\mu)}\right) * f\|_{W^{q,\infty}}$$

$$\begin{aligned}&\lesssim \|F_t^{(\mu)}\|_{M^{p',\infty}}\|f\|_{W^{r,1}}\\&\lesssim t^{-\frac{2d}{mp'}}\|f\|_{W^{r,1}}.\end{aligned}$$

Moreover, the relationships $1/p+1/q = 1+1/r$ and $1 \le q \le p'$ imply that $1 \le r \le \infty$, and by Proposition 2.1 we infer that there exists $C > 0$ such that, for every $t \ge 0$ and $v \in W^{r,1}(\mathbb{R}^d)$

$$\|V(t,\cdot)v\|_{W^{r,1}} \le C\|V(t,\cdot)\|_{W^{p,1}}\|v\|_{W^{q,\infty}}.$$

Combining these bounds, for $t \in [0, T]$ we obtain

$$\begin{aligned}\|Au(t,\cdot)\|_{W^{q,\infty}} &\lesssim \int_0^t \|U_\mu(t-s)V(s,\cdot)u(s,\cdot)\|_{W^{q,\infty}}\, \mathrm{d}s\\&\lesssim \int_0^t |t-s|^{-\frac{2d}{mp'}}\|V(s,\cdot)u(s,\cdot)\|_{W^{r,1}}\, \mathrm{d}s\\&\lesssim \Big(\int_0^t |t-s|^{-\frac{2d}{mp'}}\, \mathrm{d}s\Big)\|V\|_{L^\infty([0,+\infty);W^{p,1}(\mathbb{R}^d))}\|u\|_X.\end{aligned}$$

The claim then follows as in the proof of Theorem 1.2 after choosing T conveniently small and iterating this argument. Indeed, it is clear from the above proof that the local time-step is independent of the size of the solution (as usual, for linear equations). □

Remark 5.1. The condition $q \le p'$ is quite natural, since it does ensure that the product Vu in the Cauchy problem is well-defined in the sense of distributions, since $V \in (\mathcal{F}L^p)_{\mathrm{loc}}(\mathbb{R}^d)$ and $u \in (\mathcal{F}L^{p'})_{\mathrm{loc}}(\mathbb{R}^d)$ imply $Vu \in (\mathcal{F}L^1)_{\mathrm{loc}}(\mathbb{R}^d)$.

Example 5.2. *It is worth mentioning some concrete examples of distributions that could serve as (time-independent) potentials in Theorem 1.3.*

- *Consider compactly supported Coulomb-type singularities as in $V(x) = |x|^{-\alpha}\psi(x)$, where $0 < \alpha < d$ and $\psi \in C_c^\infty(\mathbb{R}^d)$ satisfies $\psi \equiv 1$ in a neighborhood of the origin. In view of Proposition 2.1, V belongs to $W^{p,1}(\mathbb{R}^d)$ if $V \in \mathcal{F}L^p(\mathbb{R}^d)$, hence if $1/p' > \alpha/d$. In particular, the previous result holds provided that*

$$\frac{\alpha}{d} < \min\left\{\frac{1}{q}, \frac{m}{2d}\right\}, \qquad m \ge 2,$$

  —indeed, in that case there exists $1 \le p \le \infty$ such that $1/p' > \alpha/d$, $m > 2d/p'$ and $q < p'$ (note that $\alpha/d < 1$).
- *Let V be the surface measure of the sphere $S^{d-1} \subset \mathbb{R}^d$. It is well known [50] that for $\xi \in \mathbb{R}^d$ with $|\xi|$ sufficiently large the following asymptotic formula holds:*

$$\widehat{V}(\xi) = 2|\xi|^{-\frac{d-1}{2}}\cos\Big(2\pi\Big(|\xi| - \frac{d-1}{8}\Big)\Big) + O(|\xi|^{-\frac{d+1}{2}}).$$

  Hence $V \in \mathcal{F}L^p(\mathbb{R}^d)$ if and only if $1/p' > \frac{d+1}{2d}$ (for the "only if" part it suffices to estimate from below the relevant integral on the union of the annuli $\{-\frac{1}{8}+k \le |\xi| - \frac{d-1}{8} \le \frac{1}{8}+k\}$, with $k \in \mathbb{Z}$). Once again in view of Proposition 2.1, we conclude that $V \in W^{p,1}(\mathbb{R}^d)$ if and only if $1/p' > \frac{d+1}{2d}$, hence the previous result holds provided that

$$\frac{d+1}{2d} < \min\left\{\frac{1}{q}, \frac{m}{2d}\right\}$$

(which implies $m > 2$)—indeed, in that case there exists $1 \le p \le \infty$ such that $1/p' > \frac{d+1}{2d}$, $m > 2d/p'$ and $q < p'$. Similar conclusions can be inferred for surface measures of more general smooth compact hypersurfaces with non-zero Gaussian curvature at every point; indeed, if V is such a measure we have

$$|\widehat{V}(\xi)| \lesssim (1 + |\xi|)^{-(d-1)/2}, \qquad \xi \in \mathbb{R}^d,$$

see, e.g. [50],. Hence $V \in \mathcal{F}L^p(\mathbb{R}^d)$ if $1/p' > \frac{d+1}{2d}$. Moreover, up to suitable adjustments, we can consider smooth low-dimensional manifolds of finite type or even some fractal sets in $\mathbb{R}^d$ —see [54, Theorem 9A2] in this connection.

- The case where $p = \infty$, $q = 1$ in Theorem 1.3, corresponds to well-behaved potentials and rough initial data—indeed, the Feichtinger space $W^{1,1}(\mathbb{R}^d) = M^1(\mathbb{R}^d)$ consists of functions with local $\mathcal{F}L^1$ regularity and L^1 decay at infinity, while $\mathcal{F}L^\infty(\mathbb{R}^d)$ is known as the space of pseudomeasures, and of course contains the space of finite complex measures. The solution $u \in C^0([0, +\infty); W^{\infty,\infty}(\mathbb{R}^d))$ is therefore, for every $t \ge 0$, locally a pseudomeasure.
- Let V be a (distributional) derivative of order $\le (d-1)/2$ of the surface measure of a smooth compact hypersurface of $\mathbb{R}^d$, with non-zero Gaussian curvature at every point. Then, in light of the discussion above, V is a compactly supported distribution in $\mathcal{F}L^\infty(\mathbb{R}^d)$, and therefore belongs to $W^{\infty,1}(\mathbb{R}^d)$ by Proposition 2.1. As a consequence, Theorem 1.2 applies.

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No dataset has been analyzed or generated in connection with this note.

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