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Dynamical restriction for Schrödinger equations

Fabio Nicola

Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

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ABSTRACT

We prove a dynamical restriction principle, asserting that every restriction estimate satisfied by the Fourier transform in $\mathbb{R}^d$ is also valid for the propagator of certain Schrödinger equations. We consider smooth Hamiltonians with an at most quadratic growth, and also a class of nonsmooth Hamiltonians, encompassing potentials that are Fourier transforms of complex (finite) Borel measures. Roughly speaking, if the initial datum belongs to $L^p(\mathbb{R}^d)$, for p in a suitable range of exponents, the solution $u(t,\cdot)$ (for each fixed t, with the exception of certain particular values) can be meaningfully restricted to compact curved submanifolds of $\mathbb{R}^d$. The underlying property responsible for this phenomenon is the boundedness of the propagator $L^p \to (\mathcal{F}L^p)_{\mathrm{loc}}$, with $1 \le p \le 2$, which is derived from almost diagonalization and dispersive estimates in function spaces defined in terms of wave packet decompositions in phase space.

1. Introduction and discussion of the main results

The regularizing effect of the Schrödinger propagator has been object of extensive research and underlies notable *space-time* estimates, like local smoothing and Strichartz

E-mail address: fabio.nicola@polito.it.

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estimates (see, e.g., [17,40]), along with *fixed-time* estimates in weighted Sobolev spaces (see, e.g., [12]). The existing literature is really vast, and we do not attempt to recount the principal contributions here.

This note explores an additional instance of this regularizing effect, which, to our knowledge, has not been explicitly studied in the literature before.

Let $\mathcal{F}$ denote the Fourier transform in $\mathbb{R}^d$ and let ν be a positive Radon measure in $\mathbb{R}^d$ — for example, a surface measure of a compact hypersurface of $\mathbb{R}^d$. Let $L^p(\mathbb{R}^d, \nu)$ denote the associated Lebesgue spaces and use $L^p = L^p(\mathbb{R}^d)$ for the spaces corresponding to the Lebesgue measure.

The idea at the core of this note is highlighted by the following basic observation. Consider the free Schrödinger equation in $\mathbb{R} \times \mathbb{R}^d$,

$$i\partial_t u = -\frac{1}{2}\Delta u,$$

and let

$$U_0(t)f(x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} f(y)\, dy$$

be the corresponding propagator, that is, $u(t,x) = U_0(t)f(x)$ is the solution satisfying $u(0,x) = f(x)$. By expanding the exponent of the integral kernel, considering that the pointwise multiplication by $e^{ia|x|^2}$, with $a \in \mathbb{R}$, is bounded in $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d, \nu)$, we see that the following facts are equivalent for every $1 \le p, q \le \infty$, $t \neq 0$:

(a) $\mathcal{F} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d, \nu)$ continuously;
(b) $U_0(t) : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d, \nu)$ continuously.

The investigation of conditions on p, q, ν that ensure (a) is valid represents one of the main research areas in modern harmonic analysis — the so-called *restriction problem*. This issue has numerous connections with other problems and the literature on it is extensive. We refer the reader to the works [39,46] and the books [11,38] for an in-depth exploration of this captivating subject; see also [28] for a new, noncommutative perspective. In essence, the core philosophy can be summarized by stating that, even though the Fourier transform $\mathcal{F}f$ of a function $f \in L^p(\mathbb{R}^d)$, with $p > 1$, is typically not continuous, if p belongs to a certain range $[1, p_0)$, $\mathcal{F}f$ can still be meaningfully restricted to a *curved* k-dimensional submanifold of $\mathbb{R}^d$ — that is, $\mathcal{F}$ extends to a bounded operator from $L^p(\mathbb{R}^d)$ to some Lebesgue space $L^q(\mathbb{R}^d, \nu)$, where ν is the k-dimensional Hausdorff measure restricted to that submanifold. For example, the Stein-Tomas restriction theorem for the $(d-1)$-dimensional sphere $S^{d-1} \subset \mathbb{R}^d$ states that

$$\int_{S^{d-1}} |\mathcal{F}f|^2 d\nu \le C\|f\|_{L^p}^2, \quad f \in \mathcal{S}(\mathbb{R}^d),\ 1 \le p \le \frac{2(d+1)}{d+3}, \tag{1.1}$$

where ν is the surface measure (see e.g. [39,46]).

Hence, by the above observation, we see that *the same restriction phenomenon occurs for the solution $u(t,\cdot) = U_0(t)f$ of the free Schrödinger equation, with the initial datum $f \in L^p(\mathbb{R}^d)$, for the same range of exponents.* This exemplifies the widely recognized principle that the Schrödinger propagator frequently behaves like the Fourier transform (see, e.g., [45, page 519]).

One may wonder if a similar phenomenon persists when a potential is present. In this note, we affirmatively address this query for two classes of Hamiltonians, that is, smooth Hamiltonians with at most quadratic growth and a class of nonsmooth Hamiltonians, which includes potentials that are Fourier transforms of complex (finite) Borel measures. Hence, we provide a *dynamical* version of the restriction problem, in line with recent interest in exploring dynamic versions of classical results in harmonic analysis; refer, for example, to the contributions [4,10,16].

The following proposition shows that the underlying property responsible for this phenomenon is an $L^p(\mathbb{R}^d) \to (\mathcal{F}L^p)_{\mathrm{loc}}(\mathbb{R}^d)$ estimate, for $1 \le p \le 2$.

Proposition 1.1. *Let $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ be a linear continuous operator such that, for every function $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $1 \le r \le 2$, there exists a constant $C > 0$ such that*

$$\|\mathcal{F}(\varphi A f)\|_{L^r} \le C\|f\|_{L^r} \qquad f \in \mathcal{S}(\mathbb{R}^d). \tag{1.2}$$

Let $1 \le p \le 2$, $1 \le q \le \infty$, and let ν be a positive Radon measure in $\mathbb{R}^d$ with compact support, such that the Fourier transform extends to a bounded operator $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d,\nu)$. Then, for the same p, q, ν, there is a constant $C > 0$ such that

$$\|Af\|_{L^q(\mathbb{R}^d,\nu)} \le C\|f\|_{L^p(\mathbb{R}^d)} \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

The proof is immediate: write $\varphi A = \mathcal{F}^{-1}\mathcal{F}\varphi A$, with $\varphi = 1$ on the support of ν. This observation motivates the following definition.

Definition 1.2. We say that a continuous linear operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ has the restriction property if, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $r \in [1,2]$, the inequality (1.2) holds for some $C > 0$.

Hence, we will focus on the problem of proving the restriction property for the propagator of the aforementioned Schröedinger equations.

1.1. Smooth Hamiltonians

Consider a real-valued function $a(t,x,\xi)$, $t \in \mathbb{R}$, $x, \xi \in \mathbb{R}^d$, satisfying the following conditions.

(i) The map $\mathbb{R} \ni t \mapsto a(t,x,\xi)$ is continuous for every fixed $x, \xi \in \mathbb{R}^d$;

(ii) For every $t \in \mathbb{R}$, $a(t, x, \xi)$ is smooth with respect to x, ξ and, for every $T > 0$, $\alpha, \beta \in \mathbb{N}^d$ with $|\alpha| + |\beta| \geq 2$ there exist $C_{\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(t, x, \xi)| \leq C_{\alpha,\beta} \qquad |\alpha| + |\beta| \geq 2, \quad x, \xi \in \mathbb{R}^d, \ |t| \leq T.$$

Let $\chi_{t,s}$, $s, t \in \mathbb{R}$, be its associated Hamiltonian flow, i.e. $z(t) := \chi_{t,s}(z_0)$, $z_0 \in \mathbb{R}^{2d}$, is the solution of the equation

$$\dot{z}(t) = J\nabla_z a(t, z(t)),$$

with

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

satisfying the initial condition $z(s) = z_0$.

Consider the Cauchy problem in $\mathbb{R} \times \mathbb{R}^d$:

$$\begin{cases} i\partial_t u = a^w(t)u \\ u(s) = f \end{cases} \tag{1.3}$$

where $a^w(t)$ stands for the Weyl quantization of $a(t, \cdot)$. This problem is forward and backward globally wellposed in $L^2(\mathbb{R}^d)$ (see [41, Proposition 7.1]) and we will denote by $U(t, s)$ the corresponding propagator, that is, $u(t) = U(t, s)f$ is the solution to (1.3). We write $(x^{t,s}, \xi^{t,s}) = \chi_{t,s}(y, \eta)$. Let

$$C(t, s) := \inf_{y,\eta \in \mathbb{R}^{2d}} \Big| \det \frac{\partial x^{t,s}}{\partial \eta}(y, \eta) \Big|.$$

Theorem 1.3. *For every $T > 0$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $p \in [1, 2]$, there is a constant $C_0 > 0$ such that, for every $t, s \in [-T, T]$ such that $C(t, s) > 0$,*

$$\|\mathcal{F}(\varphi U(t, s)f)\|_{L^p} \leq C_0 C(t, s)^{-(\frac{2}{p}-1)} \|f\|_{L^p} \qquad f \in \mathcal{S}(\mathbb{R}^d). \tag{1.4}$$

In particular, for such t, s', $U(t, s)$ has the restriction property.

As an example, by combining this result with Proposition 1.1 and the aforementioned Stein-Tomas restriction estimate (1.1) we see that, for all $t \in \mathbb{R}$ such that $C(t, 0) > 0$, the solution $u(t, \cdot) = U(t, 0)f$ satisfies the estimate

$$\int_{S^{d-1}} |u(t, \cdot)|^2 d\nu \leq C_0 \|f\|_{L^p}^2, \quad f \in \mathcal{S}(\mathbb{R}^d), \ 1 \leq p \leq \frac{2(d+1)}{d+3},$$

with a constant C_0 depending on t.

Remark 1.4.

1) Estimate (1.4) does not extend to the "exceptional times" where $C(t,s)=0$. For example, for $t=s$, $U(s,s)$ is the identity operator, which obviously does not have any regularizing effect. Indeed, the estimate is expected to degenerate when $C(t,s)=0$. The blow-up rate $C(t,s)^{-(\frac{2}{p}-1)}$ is optimal in general, as one can see in the case of the free Schödinger equation, with initial datum at time $s=0$ given by $f_\lambda(x):=e^{-|\lambda x|^2}$, $\lambda>0$. Denoting by $u_\lambda(t,x)$ the corresponding solution, an explicit computation gives, with $t=\lambda^{-1}$,

$$\|\mathcal{F}(\varphi u_\lambda(\lambda^{-1},\cdot))\|_{L^p}\sim c_p\lambda^{-d(1-1/p)} \qquad \lambda\to+\infty,\ 1\leq p<\infty\ (c_p>0).$$

Since in this case $C(t,0)=|t|^{-d}$, this proves the claimed optimality.
2) For simplicity, we supposed $\varphi\in\mathcal{S}(\mathbb{R}^d)$, although it is possible to consider a more general multiplier φ that is not necessarily a Schwartz function. In Remark 3.2 we observe that, roughly speaking, $\mathcal{F}L^1$ regularity and a suitable decay of φ at infinity are in fact sufficient (determining the minimal assumptions on φ under which the estimate is expected to hold seems delicate and we do not pursue this question in this note). Observe that the estimate does not hold without any multiplier, as one already sees for the free Schrödinger equation, whose propagator $U(t)$ fails to be continuous $L^p(\mathbb{R}^d)\to\mathcal{F}L^p(\mathbb{R}^d)$ if $p\neq2$ and $t\neq0$.
3) The above result is, notably, of a global nature — it holds "beyond caustics" (the points where $C(t,s)=0$).
4) In contrast to the Strichartz and local smoothing estimates (see, e.g., [40]), the previous result does *not* involve time averaging.

1.2. Nonsmooth Hamiltonians

We now consider the same problem for certain nonsmooth Hamiltonians. While we defer a more thorough exploration of this topic for later, here we focus on the simple yet nontrivial instance of a time-independent Hamiltonian of the form

$$a(x,\xi)=a_2(x,\xi)+a_1(x,\xi)+a_0(x,\xi),$$

where

(iii) $a_2(x,\xi)$ is a real-valued quadratic form in $\mathbb{R}^d\times\mathbb{R}^d$ (that is, a second degree real-valued homogeneous polynomial);
(iv) $a_1\in C^\infty(\mathbb{R}^{2d})$ is a real-valued function satisfying

$$|\partial_x^\alpha\partial_\xi^\beta a_1(x,\xi)|\leq C_{\alpha,\beta} \qquad |\alpha|+|\beta|\geq1,\quad x,\xi\in\mathbb{R}^d;$$

(v)
$$a_0\in M^{\infty,1}(\mathbb{R}^{2d}).$$

The space $M^{\infty,1}(\mathbb{R}^{2d})$ is the so-called Sjöstrand class (after the work [37] on the Wiener property of pseudodifferential operators; see also [19,26]). Its definition will be recalled in Section 2. To get a sense of this space, the reader is notified that the functions in $M^{\infty,1}(\mathbb{R}^d)$ are bounded and have local $\mathcal{F}L^1$ regularity (uniformly with respect to translations). Our interest in this space, as a class of (possibly pseudodifferential) potentials, lies in the fact that if $V : \mathbb{R}^d \to \mathbb{C}$ is the Fourier transform of a complex (finite) Borel measure, then $a_0(x,\xi) := V(x)$ belongs to $M^{\infty,1}(\mathbb{R}^{2d})$ ([31]). This class of potentials is a popular choice in several contexts, for example, in the literature about the rigorous construction of the Feynman path integral (see, e.g., [1,22] and also [32] for a phase-space approach).

Let $Sp(d,\mathbb{R})$ be the group of real $2d \times 2d$ symplectic matrices. Consider the Hamiltonian flow associated with the quadratic Hamiltonian $a_2(x,\xi)$, that is,

$$\mathbb{R} \ni t \mapsto S_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix} \in Sp(d,\mathbb{R}),$$

and let $U(t) = e^{-ta^w}$ be the Schrödinger propagator associated with the Hamiltonian $a = a_2 + a_1 + a_0$, with a_0, a_1, a_2 as above.

Theorem 1.5. *Consider a Hamiltonian $a(x,\xi) = a_2(x,\xi) + a_1(x,\xi) + a_0(x,\xi)$, with a_0, a_1, a_2 as above. For every $T > 0$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $p \in [1,2]$, there exists a constant $C_0 > 0$ such that, for every $t \in [-T,T]$ such that $\det(B_t) \neq 0$,*

$$\|\mathcal{F}(\varphi U(t) f)\|_{L^p} \leq C_0 (\det(B_t))^{-(\frac{2}{p}-1)} \|f\|_{L^p} \qquad f \in \mathcal{S}(\mathbb{R}^d). \tag{1.5}$$

In particular, for such t's, $U(t)$ enjoys the restriction property.

1.3. Strategy of the proof

Let us briefly describe the strategy of the proof of Theorems 1.3 and 1.5. First, we prove a transference principle (Proposition 3.1), asserting that the restriction property holds for any operator A that satisfies a refined Hausdorff-Young — alias dispersive — inequality, namely $A : W^{p',p}(\mathbb{R}^d) \to W^{p,p'}(\mathbb{R}^d)$, $1 \leq p \leq 2$, where the spaces $W^{p,q}(\mathbb{R}^d)$ are the so-called *Wiener amalgam spaces*. They are defined in terms of wave packet decompositions in phase space and were first introduced by Feichtinger in [15] (their definition and properties will be recalled in Section 2 below). Hence, we are led to study the continuity $W^{p',p}(\mathbb{R}^d) \to W^{p,p'}(\mathbb{R}^d)$, $1 \leq p \leq 2$, of the above propagators (cf. (6.1)). This will be accomplished by exploiting certain almost diagonalization estimates with respect to wave packets, proved in [6,19,41]. In summary, we have the following implications, each of which appears to be of independent interest.

Almost diagonalization with respect to Gabor wave packets

$$\Downarrow$$

Dispersive estimates in Wiener amalgam spaces

$$\Downarrow$$

Dynamical restriction estimates

We observe that dispersive estimates in Wiener amalgam spaces, in the simpler case of a quadratic Hamiltonian, have already been obtained (using the explicit formula for the solution) in [7,8,23,24,43,48].

In Section 7 we will prove a microlocal version of the above results, for smooth Hamiltonians of the form $a_2(x,\xi)+a_1(x,\xi)$, with a_2 and a_1 as in Section 1.2, in terms of the so-called *global* (alias Gabor) wave-front set [20,33,34,36] (see also [2] for an analytic version). In fact, the almost diagonalization estimates mentioned earlier can be seen as a type of propagation of singularities, offering a phase-space-based explanation for the above regularization effect.

We restricted our focus to Hamiltonians of the type described in Sections 1.1 and 1.2 to illustrate the general approach while avoiding heavy technicalities, but the method of this note can similarly be used to address other Hamiltonians, as those considered in [47], including smooth magnetic potentials with at most linear growth, as well as other dispersive equations.

2. Notation and preliminaries

2.1. Notation

We denote by $C_c^\infty(\mathbb{R}^d)$ the space of smooth functions with compact support in $\mathbb{R}^d$ and by $\mathcal{S}(\mathbb{R}^d)$ the space of Schwartz functions, along with the space $\mathcal{S}'(\mathbb{R}^d)$ of temperate distributions. As in the Introduction, $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^d$, normalized as

$$\mathcal{F}f(\xi)=\int_{\mathbb{R}^d} e^{-ix\cdot\xi}f(x)\,dx,\qquad \xi\in\mathbb{R}^d.$$

We write $\mathcal{F}L^p(\mathbb{R}^d)$ for the space of temperate distributions whose (inverse) Fourier transform belongs to $L^p(\mathbb{R}^d)$, equipped with the obvious norm. Also, the notation $(\mathcal{F}L^p)_{\mathrm{loc}}(\mathbb{R}^d)$ stands for the space of all $f\in\mathcal{S}'(\mathbb{R}^d)$ such that $\varphi f\in\mathcal{F}L^p(\mathbb{R}^d)$, for every $\varphi\in C_c^\infty(\mathbb{R}^d)$, equipped with the obvious family of seminorms.

2.2. Modulation and Wiener amalgam spaces

For $x,\xi\in\mathbb{R}^d$, $g\in L^2(\mathbb{R}^d)$, define the phase-space shift $\pi(x,\xi)$ by

$$\pi(x,\xi)g(y)=e^{iy\cdot\xi}g(y-x)\qquad y\in\mathbb{R}^d. \tag{2.1}$$

If $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, the functions $\pi(x,\xi)g$ are sometimes called Gabor wave packets, or Gabor atoms, or even coherent states [18,27].

Given $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, we define the short-time Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ by

$$V_g f(z) = \langle f, \pi(z)g\rangle \qquad z \in \mathbb{R}^{2d}.$$

One can verify that, if $\|g\|_{L^2} = (2\pi)^{-d/2}$, $V_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is a (not onto) isometry, i.e., $V_g^* V_g = I$.

Then, the modulation spaces $M^{p,q}(\mathbb{R}^d)$, $1 \le p, q \le \infty$ (cf. [3,15,18]), are defined as the set of temperate distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}} := \Big(\int\limits_{\mathbb{R}^d} \|V_g f(\cdot,\xi)\|_{L^p}^q \, d\xi\Big)^{1/q} < \infty$$

(with obvious changes if $q = \infty$). Different g's give rise to equivalent norms.

Similarly, we consider their Fourier images $W^{p,q}(\mathbb{R}^d) = \mathcal{F}M^{p,q}(\mathbb{R}^d)$, $1 \le p, q \le \infty$, also called Wiener amalgam spaces and often denoted by $W(\mathcal{F}L^p, L^q)$ in the literature, which can be equivalently defined by requiring that

$$\|f\|_{W^{p,q}} := \Big(\int\limits_{\mathbb{R}^d} \|V_g f(x,\cdot)\|_{L^p}^q \, dx\Big)^{1/q} < \infty$$

(with obvious changes if $q = \infty$).

We observe that for $p = q$, $M^{p,p}(\mathbb{R}^d) = W^{p,p}(\mathbb{R}^d)$ and for $p = q = 2$, $M^{2,2}(\mathbb{R}^d) = W^{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Moreover, the reader should keep in mind that both $M^{q,p}(\mathbb{R}^d)$ and $W^{p,q}(\mathbb{R}^d)$ are constituted of (generalized) functions with local $\mathcal{F}L^p$ regularity and L^q decay at infinity. This makes the following properties intuitively clear (see the original papers [15,42] or the books [3,9,32] for the proofs).

Embeddings. If $1 \le p_1, p_2, q_1, q_2 \le \infty$, $W^{p_1,q_1}(\mathbb{R}^d) \subset W^{p_2,q_2}(\mathbb{R}^d)$ if and only if $p_1 \le p_2$ and $q_1 \le q_2$. Moreover, if $1 \le p \le 2$, $L^p(\mathbb{R}^d) \subset W^{p',p}(\mathbb{R}^d)$ and $W^{p,p}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$. All these inclusions are, in fact, continuous embeddings.

Convolution. If $1 \le p_1, p_2, q_1, q_2, p, q \le \infty$, and

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \qquad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} + 1,$$

we have

$$\|f * g\|_{W^{p,q}} \le C\|f\|_{W^{p_1,q_1}} \|g\|_{W^{p_2,q_2}}.$$

Pointwise multiplication. If $1 \le p_1, p_2, q_1, q_2, p, q \le \infty$, and

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1, \qquad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q},$$

we have

$$\|fg\|_{W^{p,q}} \leq C\|f\|_{W^{p_1,q_1}}\|g\|_{W^{p_2,q_2}}.$$

2.3. Pseudodifferential operators

We define the Weyl quantization of a temperate distribution $a(x,\xi)$ in $\mathbb{R}^{2d}$ formally as

$$a^w f(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\cdot\xi} a\Big(\frac{x+y}{2},\xi\Big) f(y)\, dyd\xi \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

For our purposes, the relevant classes of symbols will be the Hörmander class $S^0_{0,0}(\mathbb{R}^{2d})$, constituted of smooth functions $a \in C^\infty(\mathbb{R}^{2d})$ such that $\partial^\alpha a \in L^\infty(\mathbb{R}^d)$ for every multi-index $\alpha \in \mathbb{N}^{2d}$, and the class $M^{\infty,1}(\mathbb{R}^{2d})$ — the so-called Sjöstrand class, which represents the natural symbol class for a noncommutative version of the celebrated Wiener theorem [37].

According to our previous discussion, the functions in $M^{\infty,1}(\mathbb{R}^{2d})$ are bounded and have a local $\mathcal{F}L^1$ regularity. Moreover, we have $S^0_{0,0}(\mathbb{R}^{2d}) \subset M^{\infty,1}(\mathbb{R}^{2d})$.

Almost diagonalization. The issue of almost diagonalization of pseudodifferential and Fourier integral operators has a long tradition that dates back at least to pioneering work [13,19,29,37,41]. We do not attempt to list the huge number of subsequent contributions, but we address the interested reader to the books [9,30,32], the survey [44] and the references therein.

For future reference, it is important to recall that the above symbol classes can be characterized in terms of almost diagonalization estimates for the corresponding Weyl operators.

(a) If $a \in \mathcal{S}'(\mathbb{R}^{2d})$ then $a \in M^{\infty,1}(\mathbb{R}^{2d})$ if and only if a^w satisfies the estimate

$$|\langle a^w \pi(z)g, \pi(w)g\rangle| \leq H(w-z) \qquad z,w \in \mathbb{R}^{2d} \tag{2.2}$$

for some $H \in L^1(\mathbb{R}^{2d})$ and some (and therefore, every) $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ [19].

(b) If $a \in \mathcal{S}'(\mathbb{R}^{2d})$ then $a \in S^0_{0,0}(\mathbb{R}^{2d})$ if and only if

$$|\langle a^w \pi(z)g, \pi(w)g\rangle| \leq C_N(1+|w-z|)^{-N} \qquad z,w \in \mathbb{R}^{2d} \tag{2.3}$$

for every $N \in \mathbb{N}$ and some (and therefore, every) $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ [41] (that is, (2.2) holds with $H(z) = C_N(1+|z|)^{-N}$ for every $N \in \mathbb{N}$).

These characterizations imply that such symbol classes are closed with respect to the Weyl product (see below) and also imply at once the corresponding continuity properties.

Continuity. If $a \in M^{\infty,1}(\mathbb{R}^{2d})$ then $a^w : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$ continuously, for $1 \le p, q \le \infty$ [19] (in particular $a^w : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$). This is also an immediate consequence of the above characterization (a) and the formula (4.3) below, using Young's inequality for mixed norm spaces

$$L^1(\mathbb{R}^{2d}) * L^q(\mathbb{R}^d; L^p(\mathbb{R}^d)) \subset L^q(\mathbb{R}^d; L^p(\mathbb{R}^d)).$$

The same argument shows that if $a \in M^{\infty,1}(\mathbb{R}^{2d})$ then $a^w : W^{p,q}(\mathbb{R}^d) \to W^{p,q}(\mathbb{R}^d)$ continuously, for $1 \le p, q \le \infty$.

We will also make use of the following Banach algebra property [19,37].

(c) The set of Weyl operators a^w, with Weyl symbol $a \in M^{\infty,1}(\mathbb{R}^{2d})$, endowed with the norm $\|a^w\| := \|a\|_{M^{\infty,1}}$, is a Banach algebra of bounded operators in $L^2(\mathbb{R}^d)$. An equivalent norm is given by

$$|||a^w||| := \int_{\mathbb{R}^{2d}} \sup_{w \in \mathbb{R}^{2d}} |\langle a^w \pi(z+w)g, \pi(w)g\rangle| \, dz, \tag{2.4}$$

for any $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ (cf. item (a) above). That is, for $a \in \mathcal{S}'(\mathbb{R}^{2d})$,

$$C^{-1}\|a\|_{M^{\infty,1}} \le |||a^w||| \le C\|a\|_{M^{\infty,1}}$$

for some constant $C > 0$.

Incidentally, we observe that lower bounds of pseudodifferential operators for these symbol classes have also been studied; for example, the classical Fefferman-Phong inequality is known to hold for a nonnegative symbol with fourth derivatives in $M^{\infty,1}(\mathbb{R}^{2d})$ [26].

3. A transference principle

In this section we show that any operator satisfying a refined form of the Hausdorff-Young inequality enjoys the restriction property (see Definition 1.2).

The classical Hausdorff-Young inequality asserts that the Fourier transform extends to a bounded operator $L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$, for $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{p'} = 1$. In fact, Feichtinger proved (in the more general context of locally compact Abelian groups) that, for $1 \le p \le q \le \infty$, $\mathcal{F} : W^{q,p}(\mathbb{R}^d) \to W^{p,q}(\mathbb{R}^d)$ continuously [14]. Since $L^p(\mathbb{R}^d) \hookrightarrow W^{p',p}(\mathbb{R}^d)$ and $W^{p,p'}(\mathbb{R}^d) \hookrightarrow L^{p'}(\mathbb{R}^d)$ for $1 \le p \le 2$, this result represents an extension and an improvement of the Hausdorff-Young inequality.

We can state our first result.

Proposition 3.1 *(Transference principle). Let $1 \le p \le 2$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. There is a constant $C > 0$ such that, for every bounded operator $A : W^{p',p}(\mathbb{R}^d) \to W^{p,p'}(\mathbb{R}^d)$, we have*

$$\|\mathcal{F}(\varphi A f)\|_{L^p} \le C\|A\|_{W^{p',p}\to W^{p,p'}}\|f\|_{L^p} \qquad f \in \mathcal{S}(\mathbb{R}^d). \tag{3.1}$$

As a consequence, A has the restriction property (see Definition 1.2).

Proof of Proposition 3.1. Since $1 \le p \le 2$, we have $W^{p,p}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ and, moreover, the Fourier transform is bounded in $W^{p,p}(\mathbb{R}^d)$. Hence

$$\|\mathcal{F}(\varphi A f)\|_{L^p} \le C\|\varphi A f\|_{W^{p,p}}.$$

Observe that $\varphi \in \mathcal{S}(\mathbb{R}^d) \subset W^{1,q}(\mathbb{R}^d)$ for every $q \in [1, +\infty]$. Choosing q so that $\frac{1}{q} = \frac{1}{p} - \frac{1}{p'}$, by the pointwise multiplication property of Wiener amalgam spaces (see Section 2) we have

$$\begin{aligned}\|\varphi A f\|_{W^{p,p}} &\le C'\|\varphi\|_{W^{1,q}}\|Af\|_{W^{p,p'}}\\ &\le C''\|A\|_{W^{p',p}\to W^{p,p'}}\|f\|_{W^{p',p}}.\end{aligned}$$

Since $L^p(\mathbb{R}^d) \hookrightarrow W^{p',p}(\mathbb{R}^d)$ for $1 \le p \le 2$, this concludes the proof of (3.1). $\square$

Remark 3.2. From the above proof it follows that (3.1) holds true for $\varphi \in W^{1,q}(\mathbb{R}^d)$, with $\frac{1}{q} = \frac{1}{p} - \frac{1}{p'}$.

4. Almost diagonalization

The proof of Theorems 1.3 and 1.5 is based on some almost diagonalization estimates from [6,19,41]. We focus on almost diagonalization with respect to the Gabor wave plackets $\pi(z)g$, for $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, $z \in \mathbb{R}^{2d}$, following [5,30,41].

Let $\chi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be a smooth "tame" symplectomorphism, that is:

(a) χ is smooth;
(b) We have

$$|\partial^\alpha \chi(z)| \le C_\alpha \qquad z \in \mathbb{R}^{2d},\ |\alpha| \ge 1; \tag{4.1}$$

(c) χ preserves the symplectic structure of $\mathbb{R}^{2d}$, i.e. with $(x, \xi) = \chi(y, \eta)$, $dx \wedge d\xi = dy \wedge d\eta$.

The Hamiltonian flow $\chi_{t,s}$, for every fixed $t, s \in \mathbb{R}$, considered in Section 1.1, is a typical example of a tame symplectomorphism.

We consider a continuous linear operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ that satisfies the estimate

$$|\langle A\pi(z)g, \pi(w)g\rangle| \leq H(w - \chi(z)) \qquad z, w \in \mathbb{R}^{2d}, \tag{4.2}$$

for some $H \in L^1(\mathbb{R}^{2d})$, $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$.

Theorem 4.1. *Suppose that $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ satisfies (4.2) for some $H \in L^1(\mathbb{R}^d)$, $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, and χ as above. Let $(x, \xi) = \chi(y, \eta)$ and suppose*

$$B := \inf_{y,\eta \in \mathbb{R}^{2d}} \Big| \det \frac{\partial x}{\partial \eta}(y, \eta)\Big| > 0.$$

For $1 \leq p \leq q \leq \infty$, there exists a constant $C > 0$ depending only on g (and the choice of the window in the definition of the spaces $W^{p,q}$) such that

$$\|Af\|_{W^{p,q}} \leq CB^{-(\frac{1}{p}-\frac{1}{q})}\|H\|_{L^1}\|f\|_{W^{q,p}} \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

Proof. We can suppose that the function $g \in \mathcal{S}(\mathbb{R}^{2d}) \setminus \{0\}$ in (4.2) is normalized so that $\|g\|_{L^2} = (2\pi)^{-d/2}$. Moreover, we will use g as a window in the definition of the spaces $W^{p,q}(\mathbb{R}^d)$ (see Section 2).

We use the inversion formula $f = V_g^* V_g f$:

$$f(y) = \int_{\mathbb{R}^{2d}} V_g f(z)\pi(z)g(y)\, dz \qquad y \in \mathbb{R}^d.$$

Applying the operator A we have

$$V_g(Af)(w) = \int_{\mathbb{R}^{2d}} \langle A\pi(z)g, \pi(w)g\rangle V_g f(z)\, dz. \tag{4.3}$$

Using (4.2) we see that

$$|V_g(Af)(w)| \leq \int_{\mathbb{R}^{2d}} H(w - \chi(z))|V_g f(z)|\, dz.$$

Since $\chi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is a symplectomorphism, its derivative has a determinant equal to 1, so that

$$|V_g(Af)(w)| \leq \int_{\mathbb{R}^{2d}} H(w - z)|V_g f(\chi^{-1}(z))|\, dz.$$

By Young's inequality for mixed-norm Lebesgue spaces we deduce that

$$\|V_g(Af)\|_{L^q_x(L^p_\xi)} \leq \|H\|_{L^1}\|(V_g f)\circ\chi^{-1}\|_{L^q_x(L^p_\xi)}.$$

Therefore, it is sufficient to prove that

$$\|F\circ\chi^{-1}\|_{L^q_x(L^p_\xi)} \leq B^{-(\frac{1}{p}-\frac{1}{q})}\|F\|_{L^p_y(L^q_\eta)}$$

for (continuous) functions $F(y,\eta)$ on $\mathbb{R}^{2d}$. This is clear if $p=q$. By interpolation it is enough to prove the case $1\leq p<\infty$, $q=\infty$, namely

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|F(y(x,\xi),\eta(x,\xi))|^p\,d\xi \leq B^{-1}\int_{\mathbb{R}^d}\sup_{\eta\in\mathbb{R}^d}|F(y,\eta)|^p\,dy. \tag{4.4}$$

Now, we have

$$\int_{\mathbb{R}^d}|F(y(x,\xi),\eta(x,\xi))|^p\,d\xi \leq \int_{\mathbb{R}^d}\sup_{\eta\in\mathbb{R}^d}|F(y(x,\xi),\eta)|^p\,d\xi.$$

Moreover, by Hadamard's global inverse function theorem (see, e.g., [25, Theorem 6.2.4]), we can perform the change of variable $y=y(x,\xi)$ in the right-hand side, regarding x as a parameter. Indeed, since the inverse of a symplectic matrix $M\in Sp(d,\mathbb{R})$ is given by $J^{-1}M^tJ$, we see that

$$\frac{\partial y}{\partial \xi}(x,\xi) = -\Big(\frac{\partial x}{\partial \eta}\Big)^t(y,\eta).$$

Hence,

$$\Big|\det\frac{\partial y}{\partial \xi}(x,\xi)\Big| \geq B>0,$$

and we deduce

$$\int_{\mathbb{R}^d}\sup_{\eta\in\mathbb{R}^d}|F(y(x,\xi),\eta)|^p\,d\xi \leq B^{-1}\int_{\mathbb{R}^d}\sup_{\eta\in\mathbb{R}^d}|F(y,\eta)|^p\,dy,$$

which concludes the proof of (4.4). $\square$

5. Almost diagonalization of Schrödinger propagators

This section is devoted to almost diagonalization results for the Schrödinger equations considered in Sections 1.1 and 1.2.

We recall the following result from [41, Corollary 7.4] (there a Gaussian window was considered, but the same result holds for every $g\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}$) by the argument in the proof of [5, Lemma 3.3] — with sums over lattices replaced by integrals over $\mathbb{R}^{2d}$).

Theorem 5.1. *Let $a(x,\xi)$ be a Hamiltonian that satisfies assumptions (i) and (ii) of Section 1.1, let $\chi_{t,s}$ be the corresponding Hamiltonian flow and let $U(t,s)$ be the corresponding Schrödinger propagator. Let $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. Then for every $T > 0$ and $N \in \mathbb{N}$ there exists $C_N > 0$ such that*

$$|\langle U(t,s)\pi(z)g, \pi(w)g\rangle| \leq C_N(1 + |w - \chi_{t,s}(z)|)^{-N} \qquad z,w \in \mathbb{R}^{2d},\ t,s \in [-T,T].$$

We now focus on the class of nonsmooth Hamiltonians considered in Section 1.2. We restrict to time-independent Hamiltonians mainly for the sake of simplicity (as the proof is somewhat technical), and we expect that a time-dependent version of the result should also hold.

First, we need the following result, which follows from the proof of [6, Theorem 3.4].

Proposition 5.2. *Let $g \in \mathcal{S}(\mathbb{R}^d)$, with $\|g\|_{L^2} = (2\pi)^{-d/2}$. Let $A_j : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, $j = 1,2$, be continuous linear operators satisfying the estimates*

$$|\langle A_j\pi(z)g, \pi(w)g\rangle| \leq H_j(w - \mathcal{A}_j z), \qquad z,w \in \mathbb{R}^{2d}$$

for some $H_j \in L^1(\mathbb{R}^{2d})$, $\mathcal{A}_j \in Sp(d,\mathbb{R})$, $j = 1,2$.

Then

$$|\langle A_1A_2\pi(z)g, \pi(w)g\rangle| \leq H(w - \mathcal{A}_1\mathcal{A}_2 z), \qquad z,w \in \mathbb{R}^{2d},$$

where $H \in L^1(\mathbb{R}^{2d})$ is given[1] *by $H = ((H_1 \circ \mathcal{A}_1) * H_2) \circ \mathcal{A}_1^{-1}$. In particular, $\|H\|_{L^1} \leq \|H_1\|_{L^1}\|H_2\|_{L^1}$.*

Theorem 5.3. *Consider a Hamiltonian $a(x,\xi) = a_2(x,\xi) + a_1(x,\xi) + a_0(x,\xi)$, with a_0, a_1, a_2 satisfying assumptions (iii), (iv), and (v) of Section 1.2. Let $\mathbb{R} \ni t \mapsto S_t$ be the Hamiltonian flow of a_2 and let $U(t) = e^{-ita^w}$ be the corresponding Schrödinger propagator, as in Section 1.2. Let $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. Then*

$$|\langle U(t)\pi(z)g, \pi(w)g\rangle| \leq H_t(w - S_t z) \qquad z,w \in \mathbb{R}^{2d}, \quad t \in \mathbb{R} \tag{5.1}$$

for some time-dependent function $H_t \in L^1(\mathbb{R}^{2d})$, with $\|H_t\|_{L^1}$ locally bounded as a function of $t \in \mathbb{R}$.

Proof. This result was already proved in [6, Theorem 4.1] in the case $a_1 = 0$, hence $a = a_2 + a_0$, regarding the nonsmooth term $a_0 \in M^{\infty,1}(\mathbb{R}^{2d})$ as a perturbation. Later, that proof was slightly simplified in [32, Theorem 6.4.1]. There we worked in the so-called interaction picture and therefore we studied the evolution of the operator

[1] Here the matrices $\mathcal{A}_j$ are identified with the corresponding linear maps, $\circ$ stands for the composition of functions, and $*$ for the convolution.

$$e^{ita_2^w} a_0^w e^{-ita_2^w},$$

a key point being that this is still a pseudodifferential operator with symbol in $M^{\infty,1}(\mathbb{R}^{2d})$, as a consequence of the symplectic covariance of the Weyl quantization. This is no longer a priori guaranteed if a_2 is replaced by $a_2 + a_1$ in the above formula (in fact, this is still the case, as a consequence of the result we are proving, but we cannot use this fact yet). Consequently, we will use an approach similar to that in [6,32], but several modifications will be necessary at certain crucial points.

Let

$$U_1(t) = e^{-it(a_2+a_1)^w}$$

be the propagator associated with the Hamiltonian $a_2 + a_1$, and let $\sigma_t \in \mathcal{S}'(\mathbb{R}^{2d})$, $t \in \mathbb{R}$, be defined by

$$\sigma_t^w := U_1(t) a_0^w U_1(-t).$$

Since $a_0 \in M^{\infty,1}(\mathbb{R}^{2d})$, σ_0^w is a bounded operator in $L^2(\mathbb{R}^d)$ and σ_t^w is a strongly continuous family of bounded operators in $L^2(\mathbb{R}^d)$.

We will prove, in order, the following facts.

(a) For every $T > 0$ and $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$|\langle U_1(t)\pi(z)g, \pi(w)g\rangle| \le C_N(1 + |w - S_t z|)^{-N} \qquad z, w \in \mathbb{R}^{2d},\ t \in [-T, T]. \tag{5.2}$$

(b) We have

$$|\langle \sigma_t^w \pi(z)g, \pi(w)g\rangle| \le H_t(w - z) \qquad z, w \in \mathbb{R}^{2d},\ t \in \mathbb{R},$$

for some function $H_t \in L^1(\mathbb{R}^{2d})$, with $\|H_t\|_{L^1}$ locally bounded, as a function of t. Equivalently, for every $t \in \mathbb{R}$, $\sigma_t \in M^{\infty,1}(\mathbb{R}^{2d})$, and for every $T > 0$ there exists $C > 0$ such that

$$\|\sigma_t\|_{M^{\infty,1}} \le C, \qquad t \in [-T, T]. \tag{5.3}$$

(c) For every $t \in \mathbb{R}$, $k \ge 1$, the iterated integral

$$b_{t,k}^w := \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \sigma_{t_1}^w \sigma_{t_2}^w \dots \sigma_{t_k}^w \, dt_k\, dt_{k-1} \dots dt_1,$$

which converges in the strong topology of bounded operators in $L^2(\mathbb{R}^d)$, defines an operator with Weyl symbol $b_{t,k} \in M^{\infty,1}(\mathbb{R}^{2d})$, and for every $T > 0$ there exists $C > 0$ such that

$$\|b_{t,k}\|_{M^{\infty,1}} \le C^k/k!, \qquad t \in [-T,T]. \tag{5.4}$$

Let us prove (a). It follows from the almost diagonalization result in Theorem 5.1, applied to the Hamiltonian $a_2(x,\xi)+a_1(x,\xi)$, that for every $T>0$ and $N \in \mathbb{N}$ there exists $C_N>0$ such that

$$|\langle U_1(t)\pi(z)g, \pi(w)g\rangle| \le C_N(1+|w-\chi_t(z)|)^{-N} \qquad z,w \in \mathbb{R}^{2d},\ t\in[-T,T], \tag{5.5}$$

where χ_t is the Hamiltonian flow of the Hamiltonian $a_2(x,\xi)+a_1(x,\xi)$. Hence, it is sufficient to prove that (5.5) continues to hold with $\chi_t(z)$ replaced by $S_t z$, which is the flow associated only with $a_2(x,\xi)$. To see this, observe that

$$\frac{d}{dt}\chi_t(z) = J\nabla a_2(\chi_t(z)) + J\nabla a_1(\chi_t(z))$$

and similarly

$$\frac{d}{dt}S_t z = J\nabla a_2(S_t z).$$

Hence, since $\chi_0(z)=S_0 z = z$,

$$\chi_t(z) - S_t z = \int_0^t \Big(J\nabla a_2(\chi_\tau(z)) - J\nabla a_2(S_\tau z) + J\nabla a_1(\chi_\tau(z))\Big)\, d\tau.$$

Now, $J\nabla a_2$ is a linear map, whereas $J\nabla a_1$ has entries in $L^\infty(\mathbb{R}^d)$ by the assumption on a_1 in (iv), Section 1.2. Hence, by Gronwall's inequality we deduce that

$$|\chi_t(z) - S_t z| \le C|t|e^{C|t|}$$

for some constant $C>0$ *independent of* z. By the triangle inequality we therefore see that we can replace $\chi_t(z)$ by $S_t z$ in (5.5), which concludes the proof of (a).

Let us prove (b). Since $U_1(t)$ satisfies the estimate (5.2), while a_0^w satisfies (2.2), it follows from Proposition 5.2 that the composition σ_t^w enjoys the estimate in (b). The claimed equivalence in terms of the Weyl symbol σ_t, namely (5.3), is a consequence of item (c) in Section 2.3.

Now, prove (c). To verify $b_{t,k} \in M^{\infty,1}(\mathbb{R}^d)$, and to estimate its norm, we can use the equivalent norm in (2.4). We observe that, since the operators σ_t^w and the phase space shifts $\pi(z)$ are strongly continuous in $L^2(\mathbb{R}^d)$, the supremum

$$\sup_{w\in\mathbb{R}^{2d}} |\langle \sigma_{t_1}^w \sigma_{t_2}^w \dots \sigma_{t_k}^w \pi(z+w)g, \pi(w)g\rangle|$$

is a lower semicontinuous function of $t_1, \dots t_k, z$, hence Borel measurable. As a consequence, we can apply Fubini's theorem to exchange the integrals with respect to the t's and with respect to z, and we deduce that

$$|||b_{t,k}||| \leq \Big| \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} |||\sigma_{t_1}^w \sigma_{t_2}^w \dots \sigma_{t_k}^w||| \, dt_k \, dt_{k-1} \dots dt_1 \Big|$$

The desired estimate (5.4) then follows from the Banach algebra property and (5.3) (see item (c) in Section 2.3).

Now, regarding a_0^w as a perturbation of $(a_0 + a_1)^w$, bounded in $L^2(\mathbb{R}^d)$, by a standard argument from operator theory (cf., e.g., [32, Theorem 6.4.1]) we can write $U(t) = e^{-ita^w}$ as

$$U(t) = U_1(t) c_t^w, \tag{5.6}$$

where c_t^w is given by the so-called Dyson series

$$\begin{aligned} c_t^w &= I + \sum_{k=1}^{\infty} (-i)^k \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \sigma_{t_1}^w \sigma_{t_2}^w \dots \sigma_{t_k}^w \, dt_k \, dt_{k-1} \dots dt_1 \\ &= I + \sum_{k=1}^{\infty} (-i)^k b_{t,k}^w, \end{aligned}$$

which converges in the space $\mathcal{B}(L^2(\mathbb{R}^d))$ of bounded operators in $L^2(\mathbb{R}^d)$, endowed with the strong operator topology. However, by (5.4), this series also converges absolutely in the Banach algebra of pseudodifferential operators with Weyl symbol in $M^{\infty,1}(\mathbb{R}^{2d})$, which is continuously embedded in $\mathcal{B}(L^2(\mathbb{R}^d))$. Hence, we have $c_t \in M^{\infty,1}(\mathbb{R}^{2d})$, and $\|c_t\|_{M^{\infty,1}(\mathbb{R}^{2d})} \leq C(T)$ for $|t| \leq T$. This implies (see again item (c) from Section 2.3), that

$$|\langle c_t^w \pi(z) g, \pi(w) g \rangle| \leq H_t(w - z) \qquad z, w \in \mathbb{R}^{2d}, \ t \in \mathbb{R},$$

for some function $H_t \in L^1(\mathbb{R}^{2d})$, with $\|H_t\|_{L^1}$ locally bounded, as a function of t. Combining this latter estimate with (5.2), we deduce again from Proposition 5.2 that the operator $U(t)$, as given in (5.6), satisfies the estimate in (5.1) and this concludes the proof of Theorem 5.3. □

6. Proof of the main results (Theorems 1.3 and 1.5)

We are now able to prove our main results.

Proof of Theorem 1.3. As a consequence of the almost diagonalization result in Theorem 5.1 and Theorem 4.1 we deduce the following *dispersive estimate* for the propagator $U(t,s)$: for every $1 \le p \le q \le \infty$ there exists a constant $C_0 > 0$ such that

$$\|U(t,s)f\|_{W^{p,q}} \le C_0 C(t,s)^{-(\frac{1}{p}-\frac{1}{q})} \|f\|_{W^{q,p}} \tag{6.1}$$

for all $s,t \in [-T,T]$ such that $C(t,s) > 0$. The estimate (1.4) then follows from Proposition 3.1. $\square$

Proof of Theorem 1.5. The proof follows the same pattern as that of Theorem 1.3. That is, now we apply Theorem 5.3, Theorem 4.1 and Proposition 3.1. $\square$

7. Microlocal restriction estimates

Let $A : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ be a continuous linear operator, that satisfies the estimate

$$|\langle A\pi(z)g, \pi(w)g\rangle| \le C_N(1 + |w - \mathcal{A}z|)^{-N} \qquad z, w \in \mathbb{R}^{2d} \tag{7.1}$$

for some $\mathcal{A} \in Sp(d,\mathbb{R})$, $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and every $N \in \mathbb{N}$. For such an operator, we will prove a microlocal improvement of (3.1).

We begin by recalling from [20,34] the definition of the *global* (alias Gabor) wave front set $WF_G(f) \subset \mathbb{R}^{2d} \setminus \{0\}$ of a temperate distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, as well as its role in the problem of propagation of singularities.

Global wave front set. We say that $z_0 \in \mathbb{R}^{2d} \setminus \{0\}$ does not belong to $WF_G(f) \subset \mathbb{R}^{2d} \setminus \{0\}$ if there exists an open conic neighborhood $\Gamma \subset \mathbb{R}^{2d} \setminus \{0\}$ of z_0 such that $V_g f$ has rapid decay in Γ for some $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, namely

$$\sup_{z\in\Gamma} (1 + |z|)^N |V_g f(z)| < \infty.$$

Equivalently, $z_0 \notin WF_G(f)$ if there exists a smooth function $\psi(z)$ in $\mathbb{R}^{2d}$, positively homogeneous of degree 0 for $|z| \ge 1$ (in the sense that coincides for $|z| \ge 1$ with a function positively homogeneous of degree 0) and such that $\psi(\lambda z_0) \ne 0$ for $\lambda > 0$ large enough, such that $\psi^w f \in \mathcal{S}(\mathbb{R}^d)$.

It turns out that $WF_G(f)$ is a closed conic subset of $\mathbb{R}^{2d} \setminus \{0\}$, and $WF_G(f) = \emptyset$ if and only if $f \in \mathcal{S}(\mathbb{R}^d)$. Hence $WF_G(f)$ encodes both the lack of regularity and decay of f. Despite the similarity with the usual wave front set [21], there are some striking differences; for example, the global wave front set is invariant under phase-space shifts:

$$WF_G(\pi(z)f) = WF_G(f) \qquad z \in \mathbb{R}^{2d}.$$

Microlocality. Given $a \in \mathcal{S}'(\mathbb{R}^{2d})$, we define its cone support $\mathrm{conesupp}(a) \subset \mathbb{R}^{2d} \setminus \{0\}$ as the complementary in $\mathbb{R}^{2d} \setminus \{0\}$ of the set of the z_0's such that there exists an open conic neighborhood $\Gamma \subset \mathbb{R}^{2d} \setminus \{0\}$ of z_0 such that, for every $N \in \mathbb{N}$ and every multi-index α,

$$\sup_{z \in \Gamma} (1 + |z|)^N |\partial^\alpha a(z)| < \infty.$$

Therefore, $\operatorname{conesupp}(a)$ is a closed conic subset of $\mathbb{R}^{2d} \setminus \{0\}$. For example, if $\varphi(x)$ is a Schwartz function, regarded as a function in $\mathbb{R}^{2d}$ (independent of the covariable ξ) we have

$$\operatorname{conesupp}(\varphi) = \{0\} \times (\mathbb{R}^d \setminus \{0\}). \tag{7.2}$$

The reader is notified that the set $\operatorname{conesupp}(a)$ is called *microsupport* in [35] and alternative terminology is also used in the literature (see the footnote in [35, page 95]).

As expected, if $a \in S^0_{0,0}(\mathbb{R}^{2d})$ and $f \in \mathcal{S}'(\mathbb{R}^d)$ we have

$$WF_G(a^w f) \subset WF_G(f) \cap \operatorname{conesupp}(a). \tag{7.3}$$

This was proved in [20] for a in Shubin's symbol classes and in [34] for $a \in S^0_{0,0}(\mathbb{R}^{2d})$, except for a slightly different definition of $\operatorname{conesupp}(a)$. This set is defined there (following [20]) as the complementary in $\mathbb{R}^{2d} \setminus \{0\}$ of the set of z_0' such that there exists an open conic neighborhood $\Gamma \subset \mathbb{R}^{2d} \setminus \{0\}$ of z_0 such that

$$\overline{\Gamma \cap \operatorname{supp}(a)} \quad \text{is compact.}$$

With our definition of $\operatorname{conesupp}(a)$, (7.3) is slightly stronger (and perhaps more natural — for example, we recapture the basic fact that $a \in \mathcal{S}(\mathbb{R}^{2d})$ and $f \in \mathcal{S}'(\mathbb{R}^d)$ imply $a^w f \in \mathcal{S}(\mathbb{R}^d)$). However, (7.3) follows immediately from the corresponding result in [34], because if $a \in S^0_{0,0}(\mathbb{R}^{2d})$ has a rapid decay in an open conic neighborhood $\Gamma \subset \mathbb{R}^{2d} \setminus \{0\}$ of z_0, for every conic open neighborhood $\Gamma' \subset\subset \Gamma$ there exists $\tilde{a}(z)$ that vanishes for z in Γ' and $|z| \geq 1$, such that $a - \tilde{a} \in \mathcal{S}(\mathbb{R}^{2d})$, so that $(a - \tilde{a})^w f \in \mathcal{S}(\mathbb{R}^d)$ if $f \in \mathcal{S}'(\mathbb{R}^d)$.

Propagation of singularities. If $A : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is a linear continuous operator satisfying the almost diagonalization estimate (7.1) for every N, for $f \in \mathcal{S}'(\mathbb{R}^d)$ we have

$$WF_G(Af) \subset \mathcal{A}(WF_G(f)), \tag{7.4}$$

where the matrix $\mathcal{A}$ is identified with the corresponding linear map. Indeed, this is an immediate consequence of the formula (4.3): using (7.1) we obtain

$$|V_g(Af)(\mathcal{A}w)| \leq C_N \int_{\mathbb{R}^{2d}} \langle 1 + |w - z|)^{-N} |V_g f(z)| \, dz,$$

from which it is easy to show that if $V_g f(z)$ has a rapid decay in an open conic neighborhood Γ, and if Γ' is an open conic subset of $\mathbb{R}^{2d} \setminus \{0\}$, with $\Gamma' \subset\subset \Gamma$, then $V_g(Af)(\mathcal{A}w)$ has a rapid decay for $w \in \Gamma'$ (see [34]).

We can then state the promised microlocal improvement of (3.1).

Theorem 7.1. *Let $A : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ be a continuous linear operator satisfying the estimates (7.1) for every $N \in \mathbb{N}$ and some $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, where $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(d, \mathbb{R})$ is a symplectic matrix with $\det B \neq 0$. Let $\psi(z)$, with $z \in \mathbb{R}^{2d}$, be a smooth function, positively homogeneous of degree 0 for $|z| \geq 1$, with $\psi(z) = 1$ for z in an open conic neighborhood of $\mathcal{A}^{-1}(\{0\} \times (\mathbb{R}^d \setminus \{0\}))$ and $|z| \geq 1$.*

If $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $1 \leq p \leq 2$, there exists $C > 0$ and, for every $N \in \mathbb{N}$, there exists $C_N > 0$ such that

$$\|\mathcal{F}(\varphi A f)\|_{L^p} \leq C\|\psi^w f\|_{L^p} + C_N \|\langle x\rangle^{-N} f\|_{H^{-N}} \qquad f \in \mathcal{S}(\mathbb{R}^d),$$

where $\langle x\rangle := (1 + |x|^2)^{1/2}$ and $\|\cdot\|_{H^{-N}}$ denotes the (L^2-based) Sobolev norm of order $-N$ in $\mathbb{R}^d$.

Proof. We write $f = \psi^w f + (1-\psi)^w f$. The term $\mathcal{F}\varphi A \psi^w f$ can be estimated using the assumption (7.1), Theorem 4.1 with $q = p'$, and (3.1). Hence, it is sufficient to prove that for every $N > 0$,

$$\|\mathcal{F}(\varphi A(1-\psi)^w f)\|_{L^p} \leq C_N \|\langle x\rangle^{-N} f\|_{H^{-N}} \tag{7.5}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. In fact, we will prove in a moment that

$$f \in \mathcal{S}'(\mathbb{R}^d) \ \Rightarrow\ \varphi A(1-\psi)^w f \in \mathcal{S}(\mathbb{R}^d). \tag{7.6}$$

This shows that $\mathcal{F}\varphi A(1-\psi)^w$, as a continuous linear operator $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, in fact maps $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$. Consider, for every $N > 0$, the Banach space X_N of temperate distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\|\langle x\rangle^{-N} f\|_{H^{-N}} < \infty$, with the obvious norm. By the closed graph theorem, for every $N > 0$, $\mathcal{F}\varphi A(1-\psi)^w : X_N \to L^p(\mathbb{R}^d)$ continuously, which gives (7.5).

It remains to prove (7.6). Let $V = \{0\} \times (\mathbb{R}^d \setminus \{0\})$. By assumption we have

$$\mathrm{conesupp}(1-\psi) \cap \mathcal{A}^{-1}(V) = \emptyset.$$

On the other hand, using (7.2), (7.3) and (7.4) (where φ is regarded as a symbol in $S^0_{0,0}(\mathbb{R}^{2d})$ independent of the covariable ξ), we have

$$WF_G(\varphi A(1-\psi)^w f) \subset V \cap \mathcal{A}(\mathrm{conesupp}(1-\psi)) = \emptyset.$$

Hence, $\varphi A(1-\psi)^w f \in \mathcal{S}(\mathbb{R}^d)$, which concludes the proof of (7.6). $\square$

Corollary 7.2. *Let ν be a positive Radon measure in $\mathbb{R}^d$ with compact support, and let $1 \leq p \leq 2$, $1 \leq q \leq \infty$ such that $\mathcal{F} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d, \nu)$ continuously. Under the same* *notation and assumptions as in Theorem 7.1, there exists $C>0$ and, for every $N\in\mathbb{N}$, there exists $C_N>0$ such that*

$$\|Af\|_{L^q(\mathbb{R}^d,\nu)} \leq C\|\psi^w f\|_{L^p(\mathbb{R}^d)} + C_N\|\langle x\rangle^{-N} f\|_{H^{-N}(\mathbb{R}^d)}, \qquad f\in\mathcal{S}(\mathbb{R}^d). \tag{7.7}$$

Proof. The desired result follows from Theorem 7.1 by writing $\varphi A = \mathcal{F}^{-1}\mathcal{F}\varphi A$ with $\varphi = 1$ on the support of ν. $\square$

This result applies to the Schrödinger propagator $A = U_1(t)$ corresponding to a smooth Hamiltonian of the type $a_2(x,\xi) + a_1(x,\xi)$, with a_2 and a_1 satisfying assumptions (iii) and (iv) of Section 1.2. Indeed, by (5.2) we see that (7.1) holds with $\mathcal{A} = S_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix} \in Sp(d,\mathbb{R})$ — that is, the Hamiltonian flow associated with a_2 — and the estimate (7.7) therefore holds for $A = U_1(t)$, with constants depending on t, for all t such that $\det(B_t)\neq 0$.

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Data availability

No data was used for the research described in the article.

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