THE ISOPERIMETRIC INEQUALITY FOR PARTIAL SUMS OF TOEPLITZ EIGENVALUES IN THE FOCK SPACE

FABIO NICOLA, FEDERICO RICCARDI*, AND PAOLO TILLI

Abstract. We prove that, among all subsets $\Omega \subset \mathbb{C}$ having circular symmetry and prescribed measure, the ball is the only maximizer of the sum of the first κ eigenvalues ($\kappa \geq 1$) of the corresponding Toeplitz operator T_Ω on the Fock space $\mathcal{F}$. As a byproduct, we prove that, again among circularly symmetric sets of prescribed measure, balls maximize any Schatten p-norm of T_Ω for $p > 1$ (and minimize the corresponding quasinorm for $p < 1$), and that the second eigenvalue is maximized by a particular annulus. Moreover, we extend some of these results to general radial symbols in $L^p(\mathbb{C})$, with $p > 1$, characterizing those that maximize the sum of the first κ eigenvalues. We also show a symmetry breaking phenomenon for the second eigenvalue, when the assumption of circular symmetry is dropped.

1. Introduction

Recent years have seen a growing interest in optimization problems arising in the context of time-frequency analysis and complex analysis, for example concerning the maximal concentration of the short-time Fourier transform with Gaussian window [9, 23, 24] or the wavelet transform with respect to the Cauchy wavelet [29] and related localization operators [8, 21, 25, 30, 31], optimal estimates for functionals and contractive estimates on various reproducing kernel Hilbert spaces [2, 6, 7, 14, 15, 17, 20, 22]. A case of special interest is given by the Fock space $\mathcal{F}$ of entire functions $F : \mathbb{C} \to \mathbb{C}$ with finite norm

$$\|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dA(z),$$

where $dA(z)$ stands for the Lebesgue measure in the plane. In [24] it was proved that, for every normalized $F \in \mathcal{F}$ and every set $\Omega \subset \mathbb{C}$ of finite two-dimensional Lebesgue measure (denoted by $|\Omega|$, and simply called area or measure of Ω), the concentration inequality

$$\int_{\Omega} |F(z)|^2 e^{-\pi|z|^2}\, dA(z) \leq 1 - e^{-|\Omega|} \tag{1.1}$$

holds true, and equality is achieved if and only if Ω is (up to a negligible set) a ball centered at some $w \in \mathbb{C}$ and F is a normalized multiple of the reproducing kernel $e^{\pi \overline{w} z}$. This result can be seen as the optimal norm estimate $\|T_\Omega\|_{\mathcal{F}\to\mathcal{F}} \leq 1 - e^{-|\Omega|}$ for the Toeplitz

2020 *Mathematics Subject Classification.* 47B35, 47A75, 49R05, 30H20, 47A30.

Key words and phrases. Toeplitz operator, Fock space, extremal problems for eigenvalues, short-time Fourier transform, localization operator, Schatten class.

*Corresponding author.

operator $T_\Omega = P\chi_\Omega\colon \mathcal{F} \to \mathcal{F}$, where χ_Ω is the characteristic function of the set Ω and P is the projection operator from $L^2(\mathbb{C}, e^{-\pi|z|^2} dA(z))$ onto $\mathcal{F}$. Indeed T_Ω, also known as *time-frequency localization operator* [4], is self-adjoint, compact and positive on $\mathcal{F}$ (see e.g. [35]), and hence, labeling its eigenvalues in a decreasing way as

$$\lambda_1(\Omega) \geq \lambda_2(\Omega) \geq \cdots > 0, \tag{1.2}$$

and interpreting the left-hand side of (1.1) as a Rayleigh quotient, one obtains

$$\lambda_1(\Omega) \leq 1 - e^{-|\Omega|} \qquad \text{(with equality} \iff \Omega \text{ is a ball)}, \tag{1.3}$$

that is a "Faber–Krahn inequality" for the operator T_Ω: among all sets Ω of prescribed measure, the first eigenvalue $\lambda_1(\Omega)$ of T_Ω is maximized when Ω is a ball.

Remark 1.1. The classical Faber–Krahn inequality [13] states that, among sets of given measure, the ball *minimizes* the first eigenvalue $\lambda_1^D(\Omega)$ of the Dirichlet Laplacian, while in (1.3) the ball *maximizes* $\lambda_1(\Omega)$. As explained in Remark 1.3 of [24], however, the two Faber–Krahn inequalities go in the same direction: indeed, the spectrum (1.2) of T_Ω (where $\lambda_n(\Omega) \to 0$ and $\lambda_1(\Omega)$ is the *largest* eigenvalue) should not be compared with the spectrum of the Dirichlet Laplacian $-\triangle$ (where $\lambda_n^D(\Omega) \to \infty$ and $\lambda_1^D(\Omega)$ is the *smallest* eigenvalue) but, rather, with the spectrum of its *inverse* $(-\triangle)^{-1}$ as a compact operator on $H_0^1(\Omega)$, with *decreasing* eigenvalues $1/\lambda_n^D(\Omega) \to 0$. This parallelism between the eigenvalues $\lambda_n(\Omega)$ of T_Ω and the inverse eigenvalues $1/\lambda_n^D(\Omega) \to 0$ of the Laplacian will be pursued further in the following.

The study of the spectrum of localization operators was initiated in [4], and several results are now available concerning the eigenvalue distributions and other global spectral properties (see, e.g., [1, 16] and the references therein). However, contrarily to the case of the Laplacian (where, since the pioneering work of Pólya and Szegő [27], spectral optimization problems have become a well-established research field, see [3, 10, 13] for some modern accounts), much less is known about extremal properties of the eigenvalues $\lambda_n(\Omega)$, and the Faber–Krahn inequality in [24] was the first step in this direction (prior to [24], this had been proved in [8] but only for sets with circular symmetry).

The goal of this paper is to carry over the program initiated in [24] with the Faber–Krahn inequality (1.3), by investigating (in the spirit of [27]) further basic questions, such as: among all sets $\Omega \subset \mathbb{C}$ of *prescribed area*, which sets maximize the second eigenvalue $\lambda_2(\Omega)$? Which sets maximize the sum

$$\sum_{k=1}^{\mathrm{K}} \lambda_k(\Omega) \tag{1.4}$$

of the first K eigenvalues? Or, again, which sets maximize the Schatten norms

$$\left(\sum_{k=1}^{\infty} \lambda_k(\Omega)^p \right)^{\frac{1}{p}} \quad ? \tag{1.5}$$

One reason of interest is that the machinery developed in [24] for Faber–Krahn (though adaptable to contexts other than the Fock space, see, e.g., [6, 7, 15, 17, 29]) is *not suitable* to attack the aforementioned problems, since the differential inequality obtained in [24] is no longer available: therefore, new ideas and tools must be introduced. Moreover, contrary to (1.3), for the maximization of $\lambda_2(\Omega)$ circular symmetry *breaks down*, in that the optimal set Ω (if one exists) is no longer rotationally invariant around any point: this symmetry breaking suggests that it might be extremely difficult to characterize the sets Ω of prescribed area that maximize $\lambda_2(\Omega)$ (or the sum of the first κ eigenvalues, or the Schatten norms etc.) without additional constraints, and these questions remain (in their full generality) three challenging open problems. We are able, however, to solve these problems within the class of sets (of given area) that are *circularly symmetric*, i.e. measurable sets $\Omega \subset \mathbb{C}$ that, possibly after a translation, have the form

$$\Omega = \{z \in \mathbb{C} \quad \text{such that} \ \ |z| \in E\}, \quad \text{for some measurable set } E \subset \mathbb{R}^+. \tag{1.6}$$

Any such set Ω is obtained by rotating the set $E \subset \mathbb{R}^+$ (thought of as a subset of the positive real halfline) around the origin of the complex plane $\mathbb{C}$, and the area constraint $|\Omega| = s$ corresponds to the integral condition

$$2\pi \int_E r\,dr = s.$$

When Ω is *radial*, that is, circularly symmetric *around the origin* as in (1.6), it is well known [4, 32] that the eigenfunctions of T_Ω are the normalized monomials

$$e_k(z) = \frac{\pi^{k/2}\, z^k}{\sqrt{k!}}, \quad k = 0, 1, 2, \dots \tag{1.7}$$

(which also form an orthonormal basis of the Fock space $\mathcal{F}$), and the corresponding eigenvalues are given by the Rayleigh quotients

$$\langle T_\Omega\, e_k, e_k\rangle_{\mathcal{F}} = \int_\Omega |e_k(z)|^2 e^{-\pi|z|^2}\, dA(z), \quad k = 0, 1, 2, \dots \tag{1.8}$$

Of course, there is a one-to-one correspondence between these numbers and the eigenvalues labeled decreasingly as in (1.2), but which k corresponds to which n strongly depends on the set Ω: for instance, if Ω is a thin annulus of radius $R \gg 1$, then $\lambda_1(\Omega)$ is likely to correspond to some $k \sim \pi R^2$, since $e^{-\pi|z|^2}|e_j(z)|^2$ has its peaks along a circle of radius $\sqrt{j/\pi}$. If Ω is a ball, however, one has (cf. [4, 32])

$$\lambda_n(\Omega) = \int_\Omega |e_{n-1}(z)|^2 e^{-\pi|z|^2}\, dA(z) = \int_0^{\pi r^2} \frac{s^{n-1}}{(n-1)!} e^{-s}\, ds, \qquad n \ge 1, \tag{1.9}$$

where r is the radius of the ball, and in particular (cf. (1.3))

$$\lambda_1(B) = 1 - e^{-|B|}, \quad \text{for every ball } B \text{ of area } |B|. \tag{1.10}$$

The origin as center of symmetry is not privileged: the case where Ω has circular symmetry around an *arbitrary* point $w \in \mathbb{C}$ can be easily reduced to the case where $w = 0$, thanks to the following general fact.

Remark 1.2 (Translation invariance). If a measurable set $\Omega \subset \mathbb{C}$ is translated by a vector $w \in \mathbb{C}$, then the spectrum of T_Ω is unchanged, i.e. $\lambda_n(\Omega) = \lambda_n(\Omega + w)$ for every $n \geq 1$. This follows from the well-known fact (see e.g. [35]) that the operator $\mathcal{U}_w : \mathcal{F} \to \mathcal{F}$ defined by

$$\mathcal{U}_w F(z) := e^{-\frac{\pi|w|^2}{2} + \pi z \overline{w}} F(z - w), \quad F \in \mathcal{F},\ z \in \mathbb{C}, \tag{1.11}$$

is *unitary* (in fact, $|\mathcal{U}_w F(z)|^2 e^{-\pi|z|^2} = |F(z-w)|^2 e^{-\pi|z-w|^2}$), so that the quadratic forms induced by T_Ω and $T_{\Omega+w}$ are related by $\langle T_\Omega F, F\rangle_{\mathcal{F}} = \langle T_{\Omega+w} \mathcal{U}_w F, \mathcal{U}_w F\rangle_{\mathcal{F}}$. Thus, if F_n is an eigenfunction of T_Ω relative to $\lambda_n(\Omega)$, then $\mathcal{U}_w F_n$ is the corresponding eigenfunction for $T_{\Omega+w}$; in particular, if a set $\widehat{\Omega}$ has circular symmetry around an *arbitrary* point $w \in \mathbb{C}$ (i.e. if $\widehat{\Omega} = \Omega + w$ for some Ω as in (1.6)), then the eigenvalues $\lambda_n(\widehat{\Omega})$ are given by (1.8), while the eigenfunctions of $T_{\widehat{\Omega}}$ are the functions $\mathcal{U}_w e_k(z)$, cf. (1.7).

After these preliminaries, our main results can be stated as follows. Concerning the maximization of $\lambda_2(\Omega)$, among sets with rotational symmetry the optimal set is always a certain *annulus*, whose shape and size depend on the prescribed area:

Theorem 1.3 (Krahn–Szegő inequality for circularly symmetric sets)**.** *Let $\Omega \subset \mathbb{C}$ be a circularly symmetric set of finite area $s := |\Omega| > 0$. Then*

$$\lambda_2(\Omega) \leq \lambda_2(A_s) = e^{s/(1-e^s)} \left(1 - e^{-s}\right), \tag{1.12}$$

where A_s is the annulus $\{r < |z| < R\}$ of area s, with radii defined by the equations

$$\pi r^2 = \frac{s}{e^s - 1}, \quad \pi R^2 = \frac{s e^s}{e^s - 1}. \tag{1.13}$$

Moreover, equality in (1.12) occurs if and only if Ω is equivalent, up to a translation, to the annulus A_s.

Here and in the following we say that two measurable subsets $\Omega, \Omega' \subset \mathbb{C}$ are equivalent if their symmetric difference $\Omega \Delta \Omega' := (\Omega \setminus \Omega') \cup (\Omega' \setminus \Omega)$ has measure zero.

The name "Krahn–Szegő" is by formal analogy, along the lines of Remark 1.1, with the minimization problem for the second eigenvalue of the Laplacian ([3, 13]). As we will show, the optimal annulus A_s defined by (1.13) arises, in quite a natural way, as a superlevel set

$$A_s = \left\{ z \in \mathbb{C} \text{ such that } \ |e_1(z)|^2 e^{-\pi|z|^2} > t \right\} \tag{1.14}$$

where $e_1(z)$ is as in (1.8), and t chosen as to match the prescribed area $|A_s| = s$. For these annuli one has $\lambda_2(A_s) = \lambda_1(A_s)$, and the corresponding eigenfunction is any linear combination of $e_0(z)$ and $e_1(z)$. As we mentioned, however, these annuli are no longer optimal (at least for small area s) among planar sets without circular symmetry:

Proposition 1.4 (Symmetry breaking for $\lambda_2(\Omega)$)**.** *For every small enough $s > 0$, there exist sets $\Omega \subset \mathbb{C}$ of area s (such as the union of two equal disks sufficiently apart from each other) that violate (1.12), i.e. such that*

$$\lambda_2(\Omega) > e^{s/(1-e^s)} \left(1 - e^{-s}\right). \tag{1.15}$$

It is an open problem to establish whether (1.12) can also be violated by sets Ω of large measure s (if not, then A_s would maximize $\lambda_2(A_s)$ without any symmetry constraint). Moreover, without symmetry constraints, it is not even known (no matter how $s>0$ is chosen) whether a set Ω exists, that maximizes $\lambda_2(\Omega)$ among sets of area s (the union of two disjoint disks, which by the way optimizes the Krahn–Szegő inequality for the Laplacian, satisfies (1.15) for small s, but does *not* maximize $\lambda_2(\Omega)$ among sets of area s, see the proof of Proposition 1.4).

Summing up, among circular sets of a area s, $\lambda_1(\Omega)$ and $\lambda_2(\Omega)$ are maximized, respectively, by the ball (which is also optimal among arbitrary sets of area s) and by the annulus A_s. The sum $\lambda_1(\Omega)+\lambda_2(\Omega)$, however, is maximized again by the ball and, more generally, the same is true for the sum in (1.4):

Theorem 1.5 (Sum of the first K eigenvalues for circularly symmetric sets)**.** *Let $\Omega\subset\mathbb{C}$ be a circularly symmetric set of finite area $|\Omega|>0$. Then, for every integer $\mathrm{K}\geq 1$,*

$$\sum_{k=1}^{\mathrm{K}}\lambda_k(\Omega)\leq\sum_{k=1}^{\mathrm{K}}\lambda_k(\Omega^*)=\mathrm{K}-\sum_{k=0}^{\mathrm{K}-1}(\mathrm{K}-k)\frac{|\Omega|^k}{k!}e^{-|\Omega|}, \tag{1.16}$$

where Ω^* is the ball such that $|\Omega^*|=|\Omega|$. Moreover, equality holds if and only if Ω is equivalent (up to a translation) to Ω^*.

As is well known, the sum $\sum_{k=1}^{\mathrm{K}}\lambda_k(\Omega)$ is the trace of the operator T_Ω *restricted* to the space generated by its first K eigenfunctions; more precisely, if P_V is the orthogonal projector from $\mathcal{F}$ to any K-dimensional subspace $V\subset\mathcal{F}$ spanned by K orthogonal functions $F_1,\ldots,F_{\mathrm{K}}$, one has

$$\operatorname{tr}(P_VT_\Omega P_V)=\sum_{k=1}^{\mathrm{K}}\langle T_\Omega F_k,F_k\rangle=\sum_{k=1}^{\mathrm{K}}\int_\Omega|F_k(z)|^2e^{-\pi|z|^2}\,dA(z)\leq\sum_{k=1}^{\mathrm{K}}\lambda_k(\Omega), \tag{1.17}$$

with equality when V is spanned by the *first* K eigenfunctions of T_Ω. In fact, we obtain (1.16) as a consequence of the following more precise statement, which also takes into account the center of symmetry of Ω and (implicitly) the first K eigenfunctions of T_Ω:

Theorem 1.6. *Let $\Omega\subset\mathbb{C}$ be a set with finite measure $|\Omega|>0$, having circular symmetry around some point $w\in\mathbb{C}$. If $F_1,\ldots,F_{\mathrm{K}}$ are any K orthonormal functions in the Fock space $\mathcal{F}$, then*

$$\sum_{k=1}^{\mathrm{K}}\int_\Omega|F_k(z)|^2e^{-\pi|z|^2}\,dA(z)\leq\mathrm{K}-\sum_{k=0}^{\mathrm{K}-1}(\mathrm{K}-k)\frac{|\Omega|^k}{k!}e^{-|\Omega|}. \tag{1.18}$$

Moreover, equality is achieved if and only if Ω is equivalent to a ball centered at $w\in\mathbb{C}$ and, for some unitary matrix $U\in\mathbb{C}^{\mathrm{K}\times\mathrm{K}}$, there holds

$$\begin{pmatrix}F_1(z)\\ \vdots\\ F_{\mathrm{K}}(z)\end{pmatrix}=U\begin{pmatrix}\mathcal{U}_we_0(z)\\ \vdots\\ \mathcal{U}_we_{\mathrm{K}-1}(z)\end{pmatrix}\qquad\forall z\in\mathbb{C}, \tag{1.19}$$

where $\mathcal{U}_w : \mathcal{F} \to \mathcal{F}$ *is the unitary operator defined in* (1.11) *and the* e_k*'s are the monomials defined in* (1.7)*.*

The validity of (1.16) and (1.18) without the assumption that Ω has circular symmetry remains an *open problem*; in fact, we conjecture that here (contrary to Proposition 1.4) no symmetry breaking occurs, i.e. that the ball maximizes the sum $\sum_{k=1}^{\mathrm{K}} \lambda_k(\Omega)$ among *all sets* of given area (this would extend the Faber–Krahn inequality of [24], from $\lambda_1(\Omega)$ to the sum $\sum_{k=1}^{\mathrm{K}} \lambda_k(\Omega)$). With this respect, let us mention that also the corresponding problem for the eigenvalues of the Laplacian, i.e. the maximization of $\sum_{k=1}^{\mathrm{K}} \lambda_k^D(\Omega)^{-1}$ under an area constraint (cf. Remark 1.1) is still open when $\mathrm{K} > 1$ (see [13]; if, instead of the area constraint, one considers a conformal radius constraint on Ω, then the ball *minimizes* the same sum: this is the Pólya–Schiffer inequality, see [13, 26]). When $\mathrm{K} = \infty$, Luttinger [19] proved that, under an area constraint, the ball maximizes $\sum_{k=1}^{\infty} \lambda_k^D(\Omega)^{-1}$, but the same problem for localization operators is trivial, since the sum $\sum_{k=1}^{\infty} \lambda_k(\Omega)$, i.e. the trace of T_Ω, is equal to the area $|\Omega|$ for *any* measurable set Ω of finite area (see [35]). The problem becomes nontrivial (even with the circular symmetry constraint) if one considers infinite sums as in (1.5) with an exponent $p \neq 1$ or, more generally, the trace $\operatorname{tr} \Phi(T_\Omega)$ where where Φ is an arbitrary convex function:

Theorem 1.7 (Trace inequality for circularly symmetric sets)**.** *Let* $\Omega \subset \mathbb{C}$ *be a circularly symmetric set of finite area* $|\Omega| > 0$*, and let* $\Phi \colon (0,1) \to \mathbb{R}$ *be a convex function. Then* $\operatorname{tr} \Phi(T_\Omega) \le \operatorname{tr} \Phi(T_{\Omega^*})$*, that is,*

$$\sum_{k=1}^{\infty} \Phi(\lambda_k(\Omega)) \le \sum_{k=1}^{\infty} \Phi(\lambda_k(\Omega^*)), \tag{1.20}$$

where Ω^* *is the ball such that* $|\Omega^*| = |\Omega|$*. Moreover, if* $\operatorname{tr} \Phi(T_{\Omega^*})$ *is finite and if* Φ *is not affine in the interval* $(0, \lambda_1(\Omega^*))$*, then equality in* (1.20) *is achieved if and only if* Ω *is equivalent, up to a translation, to* Ω^**.*

In particular, letting $\Phi(t) = t^p$ when $p > 1$, or $\Phi(t) = -t^p$ when $p \in (0,1)$, one obtains the following result:

Corollary 1.8. *Among circularly symmetric sets* Ω *of given area, the ball maximizes all the Schatten norms in* (1.5) *when* $p > 1$*, and minimizes all the Schatten quasi-norms in* (1.5) *when* $p \in (0,1)$*.*

Again, the validity of these results without the assumption of circular symmetry is an open problem, except when $p = 2$ (Hilbert–Schmidt norm): indeed, in [21] it was proved that balls are maximizers, without the assumption that Ω is circularly symmetric.

We point out for future reference that, as already observed, for every set Ω of finite measure, there holds

$$\sum_{k=1}^{\infty} \lambda_k(\Omega) = |\Omega| < \infty \tag{1.21}$$

(in particular T_Ω is of trace class, so that all the above Schatten norms are finite, see [35]).

Finally we observe that, probably, a quantitative form of the above results could be established by refining these methods. Further, similar inquiries can be explored in higher dimensions or alternative frameworks, such as functions within weighted Bergman spaces or polynomial spaces. We intend to address these questions in future research.

2. Proof of Theorems 1.5 and 1.6

As explained in the Introduction, Theorem 1.5 follows from Theorem 1.6, whose proof requires some preliminary considerations and a lemma which might be of some interest in other contexts as well.

First, by Remark 1.2, it suffices to prove Theorem 1.6 when $w = 0$, i.e. when Ω is circularly symmetric around the origin: the general case then follows by replacing the κ orthogonal functions $\{F_j\}$ with $\{\mathcal{U}_w F_j\}$ which, as explained after (1.11), corresponds to a shift $\Omega \mapsto \Omega - w$, i.e. to the change of variable $z \to z - w$ in all the integrals in (1.18).

Now, given Ω and $F_1, \dots, F_\kappa$ as in Theorem 1.5 with $w = 0$, our goal is to prove that

$$\sum_{k=1}^{\kappa} \int_\Omega |F_k(z)|^2 e^{-\pi|z|^2}\, dA(z) \le \sum_{k=0}^{\kappa-1} \int_{\Omega^*} |e_k(z)|^2 e^{-\pi|z|^2}\, dA(z), \tag{2.1}$$

where Ω^* is the ball (centered at the origin) such that $|\Omega^*| = |\Omega|$, and the e_k's are the first κ monomials in (1.7). Since Ω^* is a ball, due to (1.9) the second sum in (2.1) coincides with $\sum_{k=1}^{\kappa} \lambda_k(\Omega^*)$ which in turn, by a direct computations of the integrals in (1.9), coincides with the right-hand side of (1.18): in other words, (2.1) is equivalent to (1.18). To complete the proof of Theorem 1.6, we will also prove that equality in (2.1) occurs only when $\Omega = \Omega^*$ and

$$\begin{pmatrix} F_1(z) \\ \vdots \\ F_\kappa(z) \end{pmatrix} = U \begin{pmatrix} e_0(z) \\ \vdots \\ e_{\kappa-1}(z) \end{pmatrix} \tag{2.2}$$

with U unitary matrix, which corresponds to (1.19) when $w = 0$.

Since Ω is circularly symmetric around the origin, every eigenvalue $\lambda_n(\Omega)$ can be found in (1.8) corresponding to some value of k (the correspondence $n \mapsto k_n$, as already mentioned, depends on Ω), and so, if we use (1.17) to bound the left-hand side of (2.1), we obtain that

$$\sum_{k=1}^{\kappa} \int_\Omega |F_k(z)|^2 e^{-\pi|z|^2}\, dA(z) \le \sum_{n=1}^{\kappa} \int_\Omega |e_{k_n}(z)|^2 e^{-\pi|z|^2}\, dA(z),$$

for some unknown integers $k_n \ge 0$, which we may sort in such a way that

$$0 \le k_1 < k_2 < \cdots < k_\kappa. \tag{2.3}$$

Thus, to prove (2.1), it suffices to prove that

$$\sum_{n=1}^{\kappa} \int_\Omega |e_{k_n}(z)|^2 e^{-\pi|z|^2}\, dA(z) \le \sum_{k=0}^{\kappa-1} \int_{\Omega^*} |e_k(z)|^2 e^{-\pi|z|^2}\, dA(z), \tag{2.4}$$

for every K-tuple of integers k_n satisfying (2.3). To simplify the notation in view of (2.4), given a vector of K integers $\alpha = (k_1, \ldots, k_{\mathrm{K}})$ with the k_i's as in (2.3) we let

$$u_\alpha(z) := e^{-\pi|z|^2} \sum_{n=1}^{\mathrm{K}} |e_{k_n}(z)|^2, \quad u_*(z) := e^{-\pi|z|^2} \sum_{k=0}^{\mathrm{K}-1} |e_k(z)|^2, \tag{2.5}$$

where $u_*(z)$ is just the particular case of u_α when $\alpha = (0, 1, \ldots, \mathrm{K}-1)$, as in the right-hand side of (2.4).

Remark 2.1. Since $u_\alpha(z)$ is a continuous radial function, any of its *superlevel sets*

$$E_t = \{z \in \mathbb{C} \mid u_\alpha(z) > t\}, \quad t > 0, \tag{2.6}$$

is an open set with circular symmetry, i.e. the union of a ball and countably many annuli (the ball and some annuli may be empty); in fact, since along every ray $u_\alpha(z)$ has finitely many critical points, there are finitely many annuli, so that every superlevel set has the form

$$\{|z| < R_0\} \cup \bigcup_{i=1}^{m} \{r_i < |z| < R_i\}, \quad 0 \le R_0 \le r_1 \le R_1 \le \cdots \le r_m \le R_m \tag{2.7}$$

for some $m \ge 1$ (the cases where $R_0 = 0$ or $r_i = R_i$ allow for empty sets). In view of (2.4), superlevel sets are relevant in that, for any set Ω of finite measure $|\Omega| > 0$, there holds

$$\int_\Omega u_\alpha(z)\, dA(z) \le \int_E u_\alpha(z)\, dA(z) \quad (E \text{ superlevelset of } u_\alpha \text{ such that } |E| = |\Omega|), \tag{2.8}$$

i.e. any integral increases on passing from a set Ω to a superlevel set with the same measure (this well known property –valid in general and not peculiar of u_α– is a particular case of the so called "bathtub principle", see [18]). Let us just observe that, since u_α has no flat zones, its distribution function $t \mapsto |E_t|$ (which maps the interval $(0, \max u_\alpha)$ surjectively onto $(0, +\infty)$) is continuous: therefore, for any set Ω of finite measure, one can always find a $t > 0$ such that $|E_t| = |\Omega|$, i.e. a superlevel set E of u_α always exists, that can be used in (2.8) matching the measure of Ω (it is easy to see that such E is unique, and that the inequality in (2.8) is strict unless $|E \triangle \Omega| = 0$).

Using the bathtub principle (2.8) (and the notation in (2.5) for u_*) we see that (2.4) follows from

$$\int_E u_\alpha(z)\, dA(z) \le \int_{\Omega^*} u_*(z)\, dA(z) \quad (E \text{ superlevel set of } u_\alpha \text{ such that } |E| = |\Omega^*|), \tag{2.9}$$

which we are going to prove in the following (on passing from (2.4) to (2.9) we loose track of Ω except for its measure, implicitly retained through the ball Ω^* of equal area; also note that, since u_* is radially decreasing, the ball Ω^* is a superlevel set of u_*, as E is of u_α).

The meaning of (2.9) is now clear: given any number $s > 0$, the integral of a function u_α of the kind (2.5), over its superlevel set E of measure $|E| = s$, is *maximized* when $\alpha = (0, 1, \ldots, \mathrm{K} - 1)$, i.e. when $u_\alpha = u_*$. We will prove this in a *constructive way*, modifying the initial vector $\alpha = (k_1, \ldots, k_{\mathrm{K}})$ through a sequence of moves: after each move (where suitable entries of α are decreased by 1), the integral of the new u_α (over its superlevel set of measure s) will increase, until α is eventually turned into the optimal vector $(0, 1, \ldots, \mathrm{K}-1)$.

Proof of (2.9). Consider an arbitrary vector $\alpha = (k_1, \ldots, k_{\mathrm{K}})$ with the k_n's as in (2.3), and assume that $\alpha \neq (0, 1, \ldots, \mathrm{K}-1)$. Due to (2.3), denoting by J the number of those subscripts $n \in \{1, \ldots, \mathrm{K}\}$ for which $k_n = n - 1$, we may either have $\mathrm{J} \geq 1$, in which case $k_n = n-1$ if $n \leq \mathrm{J}$ and $k_n > n-1$ if $n > \mathrm{J}$ (note that $\mathrm{J} < \mathrm{K}$ since $\alpha \neq (0, \ldots, \mathrm{K}-1)$), or we may have $\mathrm{J} = 0$, in which case $k_n > n-1$ for every n. Thus, the entries of the new vector

$$\widehat{\alpha} := \begin{cases} (k_1 - 1, \ldots, k_{\mathrm{K}} - 1) & \text{if } \mathrm{J} = 0 \\ (0, 1, \ldots, \mathrm{J}-1, \;\; k_{\mathrm{J}+1} - 1, \ldots, k_{\mathrm{K}} - 1) & \text{if } \mathrm{J} \geq 1 \end{cases} \tag{2.10}$$

still form an increasing sequence, as did the entries of α due to (2.3).

Lemma 2.2. *If E_α is any superlevel set of u_α, and $E_{\widehat{\alpha}}$ is the superlevel set of $u_{\widehat{\alpha}}$ such that $|E_{\widehat{\alpha}}| = |E_\alpha|$, we have*

$$\int_{E_\alpha} u_\alpha(z)\, dA(z) \leq \int_{E_{\widehat{\alpha}}} u_{\widehat{\alpha}}(z)\, dA(z). \tag{2.11}$$

Moreover, the inequality is strict if $\mathrm{J} > 0$, or if $\mathrm{J} = 0$ and $\widehat{\alpha} = (0, 1, \ldots, \mathrm{K}-1)$.

Proof. Recalling (2.5) and the previous definition of J, we can split

$$u_\alpha(z) = f(\pi|z|^2) + g(\pi|z|^2), \quad f(\sigma) := e^{-\sigma} \sum_{n=0}^{\mathrm{J}-1} \frac{\sigma^n}{n!}, \quad g(\sigma) := e^{-\sigma} \sum_{n=\mathrm{J}+1}^{\mathrm{K}} \frac{\sigma^{k_n}}{k_n!} \tag{2.12}$$

(we agree that $f \equiv 0$ if $\mathrm{J} = 0$) and, similarly,

$$u_{\widehat{\alpha}}(z) = f(\pi|z|^2) + \widehat{g}(\pi|z|^2), \qquad \widehat{g}(\sigma) := e^{-\sigma} \sum_{n=\mathrm{J}+1}^{\mathrm{K}} \frac{\sigma^{k_n - 1}}{(k_n - 1)!}. \tag{2.13}$$

By Remark 2.1, the structure of the set E_α is as in (2.7): therefore, letting $r_0 = 0$ to regard the ball in (2.7) as an annulus, using polar coordinates and then letting $\sigma = \pi r^2$ we have

$$\int_{E_\alpha} u_\alpha(z)\, dA(z) = \sum_{i=0}^{m} 2\pi \int_{r_i}^{R_i} r \left(f(\pi r^2) + g(\pi r^2) \right)\, dr = \sum_{i=0}^{m} \int_{a_i}^{b_i} \left(f(\sigma) + g(\sigma) \right)\, d\sigma, \tag{2.14}$$

where we have set $a_i = \pi r_i^2$ and $b_i = \pi R_i^2$. Now we claim that

$$g(a_i) - g(b_i) = f(b_i) - f(a_i) \leq 0 \quad \forall i \in \{1, \ldots, m\}. \tag{2.15}$$

This is obvious if $a_i = b_i$ (i.e. $r_i = R_i$ in (2.7), which corresponds to an empty annulus), while if $a_i < b_i$ (i.e. $r_i < R_i$ and the annulus is nonempty) the equality in (2.15) follows from the splitting (2.12) and fact that E_α is a superlevel set of u_α, hence u_α is *constant* along ∂E_α (in particular, u_α is constant along the boundary of the i-th annulus); finally, the inequality in (2.15) is *strict* if $\mathrm{J} > 0$ because $f(\sigma)$ is strictly decreasing in this case, whereas it is an equality if $\mathrm{J} = 0$ and $f(\sigma) \equiv 0$. Combining (2.15) with the fact that $a_0 = 0$

and hence $g(b_0) - g(a_0) = g(b_0) \geq 0$ (again, equality holds when $\text{J} = 0$, i.e. when $u_\alpha(0) = 0$ and the ball in (2.15) is empty so that $R_0 = b_0 = 0$), we finally have

$$\sum_{i=0}^{m} \big(g(a_i) - g(b_i)\big) \leq 0, \quad \text{with strict inequality if } \text{J} > 0. \tag{2.16}$$

Hence, observing that $g(\sigma) = \widehat{g}(\sigma) - g'(\sigma)$, integrating we have

$$\sum_{i=0}^{m} \int_{a_i}^{b_i} g(\sigma)\, d\sigma = \sum_{i=0}^{m} \int_{a_i}^{b_i} \widehat{g}(\sigma)\, d\sigma + \sum_{i=0}^{m} \big(g(a_i) - g(b_i)\big) \leq \sum_{i=0}^{m} \int_{a_i}^{b_i} \widehat{g}(\sigma)\, d\sigma, \tag{2.17}$$

and the inequality is strict if $\text{J} > 0$. Plugging this into (2.14) we get

$$\int_{E_\alpha} u_\alpha(z)\, dA(z) \leq \sum_{i=0}^{m} \int_{a_i}^{b_i} \left(f(\sigma) + \widehat{g}(\sigma)\right)\, d\sigma = \int_{E_\alpha} u_{\widehat{\alpha}}(z)\, dA(z), \tag{2.18}$$

where the last equality follows from (2.13), getting back to polar coordinates with the change of variable $\sigma = \pi r^2$ (exactly as we got (2.14) from (2.12), but in reverse order). From the bathtub principle (2.8) applied to $u_{\widehat{\alpha}}$, the last integral increases if we pass from E_α to the superlevel set $E_{\widehat{\alpha}}$ of $u_{\widehat{\alpha}}$ having the same measure, i.e.

$$\int_{E_\alpha} u_{\widehat{\alpha}}(z)\, dA(z) \leq \int_{E_{\widehat{\alpha}}} u_{\widehat{\alpha}}(z)\, dA(z), \tag{2.19}$$

which combined with (2.18) proves (2.11).

Finally, assume that (2.11) is an equality. Then equality holds also in (2.18) (hence $\text{J} = 0$ otherwise (2.17) would be strict) and in (2.19), which implies (as observed at the end of Remark 2.1 relative to (2.8)) that $|E_\alpha \triangle E_{\widehat{\alpha}}| = 0$, i.e., that E_α is already a superlevel set of $u_{\widehat{\alpha}}$. To conclude the proof, we now further assume that $\widehat{\alpha} = (0, 1, \ldots, \text{K} - 1)$ (hence $\alpha = (1, 2, \ldots, \text{K})$ by (2.10) since $\text{J} = 0$), and show that this leads to a contradiction. Indeed, with this assumption $u_{\widehat{\alpha}}(z) = u_*(z)$ (cf. (2.5)) and hence, as $u_*(z)$ is radially decreasing, $E_{\widehat{\alpha}}$ is a ball. Hence, from (2.5) (with $k_n = n$) and (1.9) (with $\Omega = E_\alpha$) we have

$$\int_{E_\alpha} u_\alpha(z)\, dA(z) = \sum_{n=1}^{\text{K}} \int_{E_\alpha} e^{-\pi|z|^2} |e_n(z)|^2\, dA(z) = \lambda_2(E_\alpha) + \cdots + \lambda_{\text{K}+1}(E_\alpha),$$

and, similarly,

$$\int_{E_\alpha} u_{\widehat{\alpha}}(z)\, dA(z) = \sum_{n=0}^{\text{K}-1} \int_{E_\alpha} e^{-\pi|z|^2} |e_n(z)|^2\, dA(z) = \lambda_1(E_\alpha) + \cdots + \lambda_{\text{K}}(E_\alpha).$$

Since (2.18) is now an equality, equating the last two sums we find $\lambda_{\text{K}+1}(E_\alpha) = \lambda_1(E_\alpha)$, which is impossible since, in the case of a ball, the eigenvalues in (1.9) are known to be *simple* (see [4, 32]). $\square$

With this lemma at hand, we can now prove (2.9) (and hence also (1.18), as explained after (2.1)). Starting with an arbitrary $\alpha \neq (0, 1, \ldots, \text{K} - 1)$ (otherwise (2.9) is trivial), we can apply (2.11) iteratively, replacing α with $\widehat{\alpha}$ after each iteration until eventually

$\widehat{\alpha} = (0, 1, \ldots, \mathrm{K}-1)$, thus proving (2.9) Now assume that equality occurs in (1.18), hence also in (2.9). If $\alpha \neq (0, 1, \ldots, \mathrm{K}-1)$, then each iteration of Lemma 2.2 must yield equality in (2.11) (otherwise in the end one would have strict inequality in (2.9)): then, as claimed by the lemma, at each iteration $\mathrm{J} = 0$ and $\widehat{\alpha} \neq (0, 1, \ldots, \mathrm{K}-1)$. But this is a contradiction, since the last iteration occurs precisely when $\widehat{\alpha} = (0, 1, \ldots, \mathrm{K}-1)$. Thus, equality in (2.9) occurs only if $\alpha = (0, 1, \ldots, \mathrm{K}-1)$, i.e. when $u_\alpha = u_*$ and $E = \Omega_*$ is a ball. Tracing back to (2.1) through (2.4), equality in (2.1) is possible only if the initial Ω is a ball and (as discussed after (1.17)) $F_1, \ldots, F_{\mathrm{K}}$ span the same subspace as $e_0, \ldots, e_{\mathrm{K}-1}$ (the first K eigenfunctions of the ball Ω). But since both sets of functions are orthonormal in $\mathcal{F}$, this condition is equivalent to (2.2). Hence Theorem 1.6 is completely proved. □

Remark 2.3. It follows from the proof of Lemma 2.2 that (2.18) reduces to an equality when $\mathrm{J} = 0$ (this condition is equivalent to $u_\alpha(0) = 0$, i.e. $k_1 > 0$ in (2.5), by (2.3)). Since the domain of integration E_α is an arbitrary superlevel set of u_α (playing no apparent role for $u_{\widehat{\alpha}}$), equality in (2.18) whenever $u_\alpha(0) = 0$ is somewhat surprising, even in the simplest case where $\mathrm{K} = 1$ in (2.5) (i.e. $\alpha = (k_1)$ with $k_1 > 0$) and in (2.10) (i.e. $\widehat{\alpha} = (k_1 - 1)$). In this case, writing k in place of k_1 and recalling (1.7), (2.18) amounts to

$$\int_{E_k} e^{-\pi|z|^2} |e_k(z)|^2 \, dA(z) = \int_{E_k} e^{-\pi|z|^2} |e_{k-1}(z)|^2 \, dA(z) \quad k = 1, 2, \ldots \tag{2.20}$$

where E_k is an arbitrary superlevel set of the form

$$E_k = \left\{ z \in \mathbb{C} \mid e^{-\pi|z|^2} |e_k(z)|^2 > t \right\} \quad \text{for some } t > 0. \tag{2.21}$$

When nonempty, the set E_k is clearly an open annulus whose inner and outer radii are the two positive solutions of the equation

$$e^{-\pi r^2} \frac{\pi^k r^{2k}}{k!} = t$$

(by (1.7), two distinct solutions exist if and only if $t < e^{-k} k^k / k!$, otherwise $E_k = \emptyset$). Equation (2.20) (which one can easily check integrating by parts) will play a central role in the next section.

3. Consequences of the main result

Theorems 1.5 and 1.6 have a number of interesting corollaries that we are going to state and prove in this section.

The first corollaries are simple applications of *majorization theory* (see [12]). Given two vectors $x = (x_k)_{k=1}^n$ and $y = (y_k)_{k=1}^n$ in $\mathbb{R}^n$, such that $x_k \geq x_{k+1}$ and $y_k \geq y_{k+1}$ $(1 \leq k < n)$, x is said to be "weakly majorized" by y (in symbols $x \prec_w y$) if

$$\sum_{k=1}^{\mathrm{K}} x_k \leq \sum_{k=1}^{\mathrm{K}} y_k, \quad \forall \mathrm{K} \in \{1, \ldots, n\} \tag{3.1}$$

("strong majorization" or simply "majorization", in symbols $x \prec y$, occurs if, in addition, there is equality in (3.1) for $\mathrm{K} = n$). In these terms, the validity of the inequality in (1.16) for arbitrary $\kappa \geq 1$ is equivalent to the weak majorizations

$$(\lambda_1(\Omega), \ldots, \lambda_n(\Omega)) \prec_w (\lambda_1(\Omega^*), \ldots, \lambda_n(\Omega^*)) \quad \forall n \geq 1. \tag{3.2}$$

By a well known theorem of Ky Fan [5], if $f : \mathbb{R}^n \to \mathbb{R}$ is a *symmetric gauge function* (i.e. f is a norm on $\mathbb{R}^n$, $f(x_1, \ldots, x_n) = f(|x_1|, \ldots, |x_n|)$ and $f(x) = f(Px)$ for every $x \in \mathbb{R}^n$ and every permutation matrix P), then $f(x) \leq f(y)$ whenever x, y are vectors with positive entries such that $x \prec_w y$. Thus, we obtain from Theorem 1.5 and (3.2):

Corollary 3.1. *Let $\Omega \subset \mathbb{C}$ be a circularly symmetric set of finite area $|\Omega| > 0$, and let Ω^* be the ball such that $|\Omega^*| = |\Omega|$. Then, for every $n \geq 1$ and every symmetric gauge function f on $\mathbb{R}^n$, there holds*

$$f\left(\lambda_1(\Omega), \ldots, \lambda_n(\Omega)\right) \leq f\left(\lambda_1(\Omega^*), \ldots, \lambda_n(\Omega^*)\right). \tag{3.3}$$

A possible application is the case of a weighted sum of the first n eigenvalues:

Corollary 3.2. *Let $\Omega \subset \mathbb{C}$ be a circularly symmetric set of finite area $|\Omega| > 0$, and let Ω^* be the ball such that $|\Omega^*| = |\Omega|$. If $n \geq 2$ and $t_1 \geq t_2 \geq \cdots \geq t_n > 0$, then*

$$\sum_{k=1}^{n} t_k \lambda_k(\Omega) \leq \sum_{k=1}^{n} t_k \lambda_k(\Omega^*), \tag{3.4}$$

and equality occurs if and only if Ω is equivalent (up to a translation) to Ω^*.

The inequality in (3.4) follows easily from Corollary 3.1, yet the following short proof also enables us to characterize the cases of equality.

Proof. Letting $\tau_n = t_n$ and $\tau_\kappa = t_\kappa - t_{\kappa+1}$ for $1 \leq \kappa < n$, there holds

$$\sum_{k=1}^{n} t_k \lambda_k(\Omega) = \sum_{\kappa=1}^{n} \tau_\kappa \sum_{k=1}^{\kappa} \lambda_k(\Omega) \leq \sum_{\kappa=1}^{n} \tau_\kappa \sum_{k=1}^{\kappa} \lambda_k(\Omega^*) = \sum_{k=1}^{n} t_k \lambda_k(\Omega^*) \tag{3.5}$$

where we have used (1.16) with $\kappa \in \{1, \ldots, n\}$ to estimate the inner sums (note $\tau_\kappa \geq 0$). Since however $\tau_n > 0$, if equality holds in (3.5) then in particular equality must hold in (1.16) when $\kappa = n$, so Ω is equivalent to Ω^* by Theorem 1.5. $\square$

Remark 3.3. The monotonicity assumption on the weights t_k cannot be dropped. For instance, if for some $k_0 < \kappa$ we have $t_1 = \cdots = t_{k_0} < t_{k_0+1}$ then (3.4) is false for certain sets Ω. Indeed, in this case one can check that the radial function

$$u(z) = \sum_{k=1}^{\kappa} t_k |e_{k-1}(z)|^2 e^{-\pi |z|^2}$$

(whose integral on the ball Ω^*, by (1.9), gives the right-hand side of (3.4)) is radially *increasing* in a neighborhood of $z = 0$. Hence, if $r > 0$ is small enough and Ω is the annulus $2r < |z| < \sqrt{5}\, r$ (so that Ω^* is the ball of radius r), then $\sup_{\Omega^*} u < \inf_\Omega u$ and hence

$$\sum_{k=1}^{n} t_k \lambda_k(\Omega^*) = \int_{\Omega^*} u(z)\, dA(z) < \int_\Omega u(z)\, dA(z) \leq \sum_{k=1}^{n} t_k \lambda_k(\Omega)$$

(the last inequality holds because the eigenfunctions of T_Ω are the monomials (1.7)).

The following result is another immediate consequence of the weak majorizations (3.2) (using [33, Theorem 15.16]).

Corollary 3.4. *Let $\Omega \subset \mathbb{C}$ be a circularly symmetric set of finite area $|\Omega| > 0$, and let Ω^* be the ball such that $|\Omega^*| = |\Omega|$. Then, for every $n \geq 1$ and every convex increasing function $\Phi : (0,1) \to \mathbb{R}$, there holds*

$$\sum_{k=1}^{n} \Phi(\lambda_k(\Omega)) \leq \sum_{k=1}^{n} \Phi(\lambda_k(\Omega^*)). \tag{3.6}$$

Letting $n \to \infty$ in (3.6) leads to inequalities of the kind (1.20), but under the extra assumption that Φ is increasing (the choice $\Phi(t) = t^p$ with $p \geq 1$). In fact, the proof of Theorem 1.7 is more delicate and requires the following extension of Karamata's inequality:

Proposition 3.5. *Let $\{x_k\}, \{y_k\} \subset (0,1)$ be two decreasing sequences ($k \geq 1$) such that*

$$\sum_{k=1}^{\mathrm{K}} x_k \leq \sum_{k=1}^{\mathrm{K}} y_k \quad \forall \mathrm{K} \geq 1, \quad \textit{and} \quad \sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty} y_k < \infty. \tag{3.7}$$

Then, for every convex function $\Phi\colon (0,1) \to \mathbb{R}$ it holds

$$\sum_{k=1}^{\infty} \Phi(x_k) \leq \sum_{k=1}^{\infty} \Phi(y_k). \tag{3.8}$$

Now assume, in addition, that Φ is not an affine function on the interval $(0, y_1]$. If the series $\sum \Phi(y_k)$ converges and equality occurs in (3.8), then the equality $\sum_{k=1}^{\mathrm{K}} x_k = \sum_{k=1}^{\mathrm{K}} y_k$ occurs for at least one value of $\mathrm{K} \geq 1$.

The classical Karamata's inequality (see [12, Theorem 108]) states that $\sum_{k=1}^{n} \Phi(x_k) \leq \sum_{k=1}^{n} \Phi(y_k)$, when $x = (x_k)$ and $y = (y_k)$ are vectors in $\mathbb{R}^n$ with decreasing entries such that $x \prec y$ (as defined after (3.1)), and (3.8) is a natural extension to the case of summable sequences (the conditions in (3.7) are the natural analogue of the strong majorization $x \prec y$ for finite vectors). Now, while the inequality (3.8) is essentially already known (see [28]), the statement about the equality case is likely to be new (note that Φ is not assumed to be strictly convex, but merely non-affine). Since it is not easily obtainable directly from (3.8), a full proof of Proposition 3.5 is given in the Appendix.

Proof of Theorem 1.7. Since Ω and Ω^* have equal measure, due to (1.21) we have

$$\sum_{k=1}^{\infty} \lambda_k(\Omega) = \sum_{k=1}^{\infty} \lambda_k(\Omega^*) < \infty \tag{3.9}$$

and hence, since the inequality in (1.16) holds for every $\mathrm{K} \geq 1$, we can apply Proposition 3.5 to the two decreasing sequences $x_k = \lambda_k(\Omega)$ and $y_k = \lambda_k(\Omega^*)$. Then (1.20) follows from (3.8) and, in case of equality (with finite sums and Φ not affine), by Proposition 3.5 equality occurs in (1.16) for at least one values of $\mathrm{K} \geq 1$, and the last claim of Theorem 1.5 implies that Ω is a ball. $\square$

Remark 3.6. Theorems 1.5 and 1.6 can be rephrased in terms of the short-time Fourier transform with Gaussian window $\varphi(x) = 2^{1/4}e^{-\pi x^2}$, defined as

$$\mathcal{V}f(x,\omega) = \int_{\mathbb{R}} f(t)\varphi(t-x)e^{-2\pi i\omega t}\,dt, \quad (x,\omega)\in\mathbb{R}^2,\ f\in L^2(\mathbb{R}).$$

Indeed, it is well known (cf. [11, Section 3.4]) that the Bargmann transform

$$\mathcal{B}f(z) \coloneqq 2^{1/4}\int_{\mathbb{R}} f(t)e^{\pi itz-\pi t^2-\pi|z|^2/2}\,dt, \quad z = x+i\omega\in\mathbb{C}$$

is a unitary operator from $L^2(\mathbb{R})$ on $\mathcal{F}$ and is related to the short-time Fourier transform via the relation

$$\mathcal{V}f(x,-\omega) = e^{\pi ix\omega}e^{-\pi|z|^2/2}\mathcal{B}f(z), \quad z = x+i\omega\in\mathbb{C}.$$

Moreover, the Hermite functions

$$h_k(t) = \frac{2^{1/4}}{\sqrt{k!}}\left(-\frac{1}{2\sqrt{\pi}}\right)^k e^{\pi t^2}\frac{\mathrm{d}^k}{\mathrm{d}t^k}(e^{-2\pi t^2}), \quad t\in\mathbb{R},\ k\in\mathbb{N}$$

are mapped by $\mathcal{B}$ into the normalized monomials in $\mathcal{F}$.

We can then consider the so-called time-frequency localization operator, or anti-Wick operators (see [4]), defined as

$$L_\Omega \coloneqq \mathcal{V}^*\chi_\Omega\mathcal{V}\colon L^2(\mathbb{R})\to L^2(\mathbb{R}),$$

where $\Omega\subset\mathbb{R}^2$ is a set of finite area. With the natural identification $\mathbb{R}^2\simeq\mathbb{C}$ it is easy to see that, if Ω is radial (i.e., circularly symmetric around the origin), then $L_\Omega = \mathcal{B}^*T_\Omega\mathcal{B}$ and therefore L_Ω has the same eigenvalues of T_Ω. Indeed, if $F = \mathcal{B}f$, $G = \mathcal{B}g$, with $f,g\in L^2(\mathbb{R})$, we have

$$\begin{aligned}\langle T_\Omega F, G\rangle_{\mathcal{F}} &= \int_\Omega F(z)\overline{G(z)}e^{-\pi|z|^2}\,dA(z) = \int_\Omega \mathcal{B}f(z)\overline{\mathcal{B}g(z)}e^{-\pi|z|^2}\,dA(z)\\ &= \int_\Omega \mathcal{V}f(x,-\omega)\overline{\mathcal{V}g(x,-\omega)}\,dx\,d\omega = \langle L_\Omega f, g\rangle_{L^2}.\end{aligned}$$

With this in mind, Theorems 1.5 and 1.6 can be easily rephrased in terms of the short-time Fourier transform and the operator L_Ω.

4. Maximizing the second eigenvalue

In this section we address the problem of maximizing the second eigenvalue $\lambda_2(\Omega)$ of T_Ω, among circularly symmetric sets Ω of given area. In particular, we will prove Theorem 1.3 and Proposition 1.4.

Proof of Theorem 1.3. Let Ω be as in the statement of the theorem and let $s = |\Omega|$ denote its area. By circular symmetry (which by Remark 1.2 we may assume to be *around the origin*), the eigenfunctions of T_Ω are the monomials in (1.7), and the eigenvalues $\lambda_n(\Omega)$ are given (in some unknown order but with the correct multiplicities) by the integrals in (1.8). In particular, letting for simplicity

$$u_k(z) := e^{-\pi|z|^2}|e_k(z)|^2 = e^{-\pi|z|^2}\frac{\pi^k|z|^{2k}}{k!}, \quad k = 0, 1, 2, \dots$$

we have that

$$\lambda_2(\Omega) \quad \text{is the second largest value of} \quad \int_\Omega u_k(z)\, dA(z), \ \text{ as } k \geq 0. \tag{4.1}$$

Denoting by E_k^s the superlevel set of u_k of measure s, that is, a set as in (2.21) where t is now chosen in such a way that $|E_k^s| = s$ (cf. the last sentence in Remark 2.1), we have from (2.20)

$$\int_{E_k^s} u_k(z)\, dA(z) = \int_{E_k^s} u_{k-1}(z)\, dA(z) < \int_{E_{k-1}^s} u_{k-1}(z)\, dA(z) \quad \forall k \geq 1, \tag{4.2}$$

where the inequality follows from the bathtub principle (2.8) and is strict because it holds $|E_k^s \triangle E_{k-1}^s| > 0$ (note that E_0^s is the ball of area s, while the other E_k^s are pairwise distinct annuli, as described after (2.21)).

If, starting from some $k \geq 2$, we iterate (4.2) $k-1$ times, we obtain the inequality

$$\int_{E_k^s} u_k(z)\, dA(z) < \Lambda := \int_{E_1^s} u_1(z)\, dA(z) = \int_{E_1^s} u_0(z)\, dA(z) \quad \forall k \geq 2 \tag{4.3}$$

(the former equality serves to define Λ, whereas the latter is a further application of (2.20) with $k = 1$), and combining the bathtub principle (2.8) with (4.3) we find

$$\int_\Omega u_k(z)\, dA(z) \leq \int_{E_k^s} u_k(z)\, dA(z) < \Lambda \quad \forall k \geq 2. \tag{4.4}$$

Moreover, now using only the bathtub principle, we have

$$\int_\Omega u_1(z)\, dA(z) \leq \int_{E_1^s} u_1(z)\, dA(z) = \Lambda \quad \text{(with strict inequality if } |\Omega \triangle E_1^s| > 0). \tag{4.5}$$

Now, a combination of (4.1) with (4.4) and (4.5) proves the following: $\lambda_2(\Omega) \leq \Lambda$, and the inequality is strict unless $|\Omega \triangle E_1^s| = 0$ (i.e., unless $\Omega = E_1^s$). When $\Omega = E_1^s$, this statement reveals that $\lambda_2(E_1^s) = \Lambda$ and, since the only integral not estimated in (4.4) or (4.5) is when $k = 0$, by the equality of the two integrals in (4.3) we see that also $\lambda_1(E_1^s) = \Lambda$. Summing up: the annulus E_1^s maximizes $\lambda_2(\Omega)$, among all circularly symmetric sets Ω of measure s. Since clearly $E_1^s = A_s$ as defined in (1.14), to complete the proof of Theorem 1.3 it remains to prove that the inner and outer radii of A_s (as defined in (1.14)) are those given by (1.13), and to check the explicit value of $\lambda_2(A_s)$ in (1.12).

The set A_s in (1.14) is clearly an annulus $\{r < |z| < R\}$ whose area $\pi(R^2 - r^2)$ must equal s, and we must also have $u_1(r) = u_1(R) = t$. Thus, since $u_1(\rho) = e^{-\pi\rho^2}\pi\rho^2$, setting $a = \pi r^2$ and $b = \pi R^2$ the annulus $A_s = E_1^s$ is determined by the system of two equations $b - a = s$ and $ae^{-a} = be^{-b}$. Since the only (positive) solution is given by $a := s/(e^s - 1)$

and $b := se^s/(e^s-1)$, (1.13) is proved. Finally, since $u_0(z) = e^{-\pi|z|^2}$ and $A_s = E_1^s$, we have from (4.3) that the explicit value of $\lambda_2(A_s) = \Lambda$ is given by

$$\int_{A_s} e^{-\pi|z|^2}\,dA(z) = 2\pi\int_r^R \rho e^{-\pi\rho^2}\,d\rho = \int_a^b e^{-\sigma}\,d\sigma = e^{-a}\left(1-e^{a-b}\right) = e^{-a}(1-e^{-s})$$

and, since $a = s/(e^s-1)$, the equality in (1.12) is proved. □

Remark 4.1. There is no analogue of Theorem 1.3 for the third eigenvalue, that is, it is not true that the annulus E_2^s (superlevel set of u_2 of measure s) maximizes $\lambda_3(\Omega)$, among circularly symmetric sets of area s. Indeed, since (generalizing (4.1)) $\lambda_n(E_2^s)$ is the n-*th largest* value among the integrals $\int_{E_2^s} u_k(z)\,dA(z)$ as $k \ge 0$, it is easy to check that

$$\lambda_1(E_2^s) = \int_{E_2^s} u_2(z)\,dA(z) = \int_{E_2^s} u_1(z)\,dA(z) = \lambda_2(E_2^s) > \lambda_3(E_2^s) = \int_{E_2^s} u_0(z)\,dA(z)$$

(the other integrals with $k \ge 3$ provide smaller values). Now, given $\varepsilon \in (0,s)$, consider the competitor $\Omega_\varepsilon := A_\varepsilon \cup B_\varepsilon$, where $A_\varepsilon \subset E_2^s$ is any smaller annulus of area $s-\varepsilon$, and B_ε is the ball of area ε (we choose ε so small that $A_\varepsilon \cap B_\varepsilon = 0$, in such a way that $|\Omega_\varepsilon| = s$). Since $e_0(z) = e^{-\pi|z|^2}$ is radially decreasing, replacing E_2^s with Ω_ε we will have

$$\int_{\Omega_\varepsilon} u_0(z)\,dA(z) > \int_{E_2^s} u_0(z)\,dA(z) = \lambda_3(E_2^s)$$

(the increase being of order $O(\varepsilon)$) whereas the two integrals of u_1 and u_2 will decrease a bit, but (for small enough ε) each of them will remain larger than $\lambda_3(E_2^s)$. In other words, we will have

$$\int_{\Omega_\varepsilon} u_k(z)\,dA(z) > \lambda_3(E_2^s), \quad k \in \{0,1,2\},$$

and since each of these three integrals is an eigenvalue of T_{Ω_ε}, we will have $\lambda_3(\Omega_\varepsilon) > \lambda_3(E_2^s)$.

Finally, a refinement of this argument shows that, for $k \ge 3$, the annulus E_{k-1}^s (superlevel set of u_{k-1} of measure s) is not a maximizer of $\lambda_k(\Omega)$.

We now recall some well known facts that will be used in the proof of Proposition 1.4 (see [35] for more details). As quickly mentioned after (1.1), the functions $K_w(z) = e^{\pi\overline{w}z}$ are the reproducing kernels of the Fock space $\mathcal{F}$, that is,

$$F(w) = \langle F, K_w\rangle_{\mathcal{F}} = \int_{\mathbb{C}} F(z)e^{\pi w\overline{z}}e^{-\pi|z|^2}\,dA(z), \quad \forall F \in \mathcal{F}, \quad \forall w \in \mathbb{C} \tag{4.6}$$

from which, choosing $F = K_w$ and then $F = K_{-w}$, one can compute the quantities

$$\|K_w\|_{\mathcal{F}}^2 = e^{\pi|w|^2}, \qquad \langle K_{-w}, K_w\rangle_{\mathcal{F}} = e^{-\pi|w|^2}. \tag{4.7}$$

Finally, since equality is achieved in (1.1) (as mentioned thereafter) when $F = K_w/||K_w||_{\mathcal{F}}$ and Ω is any ball centered at w, we have using (4.7)

$$\int_B e^{-\pi|z|^2}|K_w(z)|^2\,dA(z) = e^{\pi|w|^2}\left(1-e^{-|B|}\right) \quad \text{for every } B \text{ ball centered at } w. \tag{4.8}$$

Proof of Proposition 1.4. Given a radius $r > 0$, consider the domain $\Omega = B^+ \cup B^-$, where $B^\pm$ denote the two balls of radius r centered at $\pm w$, and $w \in \mathbb{R}$ satisfies $w > r$ so that $B^+ \cap B^- = \emptyset$. The idea of the proof is that, when $w \gg r$, the localization operator T_Ω (acting on $\mathcal{F}$) essentially behaves as a direct sum $T_{B^+} \oplus T_{B^-}$ acting on $\mathcal{F} \oplus \mathcal{F}$ and hence, when $w \to \infty$, its spectrum tends to split as $\sigma(T_\Omega) \sim \sigma(T_{B^+}) \cup \sigma(T_{B^-})$ so that, in the limit, $\lambda_1(\Omega) = \lambda_2(\Omega) = \lambda_1(B^\pm)$. In other words, since letting $s = |\Omega|$ (i.e. $s = 2\pi r^2$) one has $|B^\pm| = s/2$ and hence $\lambda_1(B^\pm) = 1 - e^{-s/2}$ by (1.9), one expects that

$$\lambda_2(\Omega) \sim 1 - e^{-s/2} \quad \text{as } w \to \infty. \tag{4.9}$$

This asymptotics would complete the proof: since by a Taylor expansion there holds

$$1 - e^{-s/2} > e^{s/(1-e^s)} \left(1 - e^{-s}\right) \quad \text{for small enough } s > 0, \tag{4.10}$$

we see that (1.15) would follow from (4.9), provided s is small enough and w very large.

In fact, instead of (4.9), we will prove the one-sided bound

$$\liminf_{w\to\infty} \lambda_2(\Omega) \geq 1 - e^{-s/2} \quad \text{for every fixed radius } r, \tag{4.11}$$

which combined with (4.10) proves (1.15): it will suffice to first fix the radius r (hence also the measure $s = 2\pi r^2$) small enough, so that the inequality in (4.10) holds true, and then choose w large enough, to guarantee that the resulting Ω satisfies (1.15).

So let us prove (4.11), with $\Omega = B^+ \cup B^-$ and $B^\pm$ balls of radius r centered at $\pm w$, as described above. From the max-min characterization of the eigenvalues, we have

$$\lambda_2(\Omega) = \max_{S_2} \min_{F \in S_2 \setminus \{0\}} \frac{\int_\Omega e^{-\pi|z|^2} |F(z)|^2 \, dA(z)}{\|F\|^2_{\mathcal{F}}}, \tag{4.12}$$

where the max is over all subspaces $S_2 \subset \mathcal{F}$ of dimension two. Thus, considering in particular the subspace S_2 spanned by the two reproducing kernels $k^\pm(z) := e^{\pm\pi w z}$, we have from (4.12)

$$\lambda_2(\Omega) \geq \min_{\substack{a,b\in\mathbb{C} \\ (a,b)\neq(0,0)}} \frac{\int_\Omega e^{-\pi|z|^2} \left|ak^+(z) + bk^-(z)\right|^2 \, dA(z)}{\|ak^+ + bk^-\|^2_{\mathcal{F}}}. \tag{4.13}$$

Using (4.7) (note $k^+ = K_w$ and $k^- = K_{-w}$), one can easily compute

$$\|ak^+ + bk^-\|^2_{\mathcal{F}} = \left(|a|^2 + |b|^2\right) e^{\pi w^2} + 2e^{-\pi w^2} \operatorname{Re} a\bar{b} \leq \left(|a|^2 + |b|^2\right) \left(e^{\pi w^2} + e^{-\pi w^2}\right). \tag{4.14}$$

On the other hand, writing $k^\pm$ and $d\nu$ as abbreviations for $k^\pm(z)$ and $e^{-\pi|z|^2} dA(z)$ in the integral, since by (4.8) $\int_{B^\pm} |k^\pm|^2 \, d\nu = e^{\pi w^2}(1 - e^{-s/2})$, we have

$$\begin{aligned}\int_\Omega \left|ak^+ + bk^-\right|^2 d\nu &> |a|^2 \int_{B^+} \left|k^+\right|^2 d\nu + |b|^2 \int_{B^-} \left|k^-\right|^2 d\nu - 2 \operatorname{Re} a\bar{b} \int_\Omega k^+ \, \overline{k^-} \, d\nu \\ &\geq \left(|a|^2 + |b|^2\right) \left(e^{\pi w^2} \left(1 - e^{-s/2}\right) - \int_\Omega \left|k^+ \, \overline{k^-}\right| \, d\nu\right)\end{aligned}$$

and, since when $z \in \Omega$ we have $|k^+(z)\overline{k^-(z)}| = |e^{\pi w(z-\overline{z})}| = |e^{2\pi wi \operatorname{Im} z}| = 1$, there holds

$$\int_\Omega \left|k^+\overline{k^-}\right| \, d\nu = \int_\Omega d\nu = 2\int_{B^+} e^{-\pi(x^2+y^2)}\, dxdy$$
$$\leq 2|B^+|\, e^{-\pi(w-r)^2} < 2\pi r^2 e^{-\pi w^2 + 2\pi wr}.$$

Plugging this into the previous estimate, we finally obtain

$$\int_\Omega \left|ak^+(z) + bk^-(z)\right|^2 \, d\nu > \left(|a|^2+|b|^2\right)\left(e^{\pi w^2}\left(1-e^{-s/2}\right) - 2\pi r^2 e^{-\pi w^2+2\pi wr}\right). \tag{4.15}$$

It is now easy to see that (4.15) and (4.14), combined with (4.13), yield (4.11). □

5. Extension to a general symbol

The class of Toeplitz operators, considered in the previous sections, can be easily extended to include symbols that are not necessarily characteristic functions. Namely, consider a measurable function $\sigma\colon \mathbb{C} \to \mathbb{C}$, say $\sigma \in L^p(\mathbb{C}) = L^p(\mathbb{C}, dA)$, for some $p \in [1, +\infty]$, where, as usual, dA denotes the Lebesgue measure in $\mathbb{C} \simeq \mathbb{R}^2$. Let P be the projection operator from $L^2(\mathbb{C}, e^{-\pi|z|^2} dA(z))$ onto $\mathcal{F}$. We then define the Toeplitz operator

$$T_\sigma := P\sigma\colon \mathcal{F} \to \mathcal{F}.$$

When $\sigma = \chi_\Omega$, for some measurable subset $\Omega \subset \mathbb{C}$, we recapture the operator T_Ω considered so far. The function σ is usually referred to as *symbol*, or *weight* of the operator. It is well known that if $\sigma \in L^p(\mathbb{C})$ for some $p \in [1, +\infty)$ then T_σ is bounded and compact in $\mathcal{F}$ ([4, 34]). Moreover if σ is, in addition, real-valued and nonnegative, then T_σ is also self-adjoint and nonnegative, and therefore its spectrum consists of a sequence of eigenvalues $\lambda_1(\sigma) \geq \lambda_2(\sigma) \geq \cdots \geq 0$, converging to zero. In this section, we are going to study the problem of maximizing the sum $\sum_{k=1}^{K} \lambda_k(\sigma)$ when σ is radial.

Observe that, in terms of the associated quadratic form, if $F \in \mathcal{F}$, we have

$$\langle T_\sigma F, F\rangle_{\mathcal{F}} = \int_{\mathbb{C}} \sigma(z)|F(z)|^2 e^{-\pi|z|^2}\, dA(z).$$

For a nonnegative symbol $\sigma \in L^p(\mathbb{C})$ $(p < \infty)$, by the so called "layer cake" representation

$$\sigma(z) = \int_0^\infty \chi_{\{\sigma > t\}}(z)\, dt, \quad z \in \mathbb{C},$$

we obtain, for an arbitrary $F \in \mathcal{F}$, that

$$\begin{aligned}\int_{\mathbb{C}} \sigma(z)|F(z)|^2 e^{-\pi|z|^2}\, dA(z) &= \int_{\mathbb{C}} \left(\int_0^\infty \chi_{\{\sigma>t\}}(z)\, dt\right)|F(z)|^2 e^{-\pi|z|^2}\, dA(z)\\ &= \int_0^\infty \left(\int_{\{\sigma>t\}} |F(z)|^2 e^{-\pi|z|^2}\, dA(z)\right) dt.\end{aligned} \tag{5.1}$$

Resuming the short notation T_Ω for T_{χ_Ω} used in the Introduction, (5.1) becomes

$$\langle T_\sigma F, F\rangle = \int_0^\infty \langle T_{\{\sigma>t\}} F, F\rangle\, dt, \tag{5.2}$$

which reveals T_σ as a *mixture* of the localization operators $T_{\{\sigma>t\}}$ on the superlevel sets of σ. When σ is radial, this allows us to prove that the sum of the first K eigenvalues increases, if we replace σ with its decreasing rearrangement σ^* defined by

$$\sigma^*(z) := \int_0^\infty \chi_{\{\sigma>t\}^*}(z)\, dt, \quad z \in \mathbb{C}, \tag{5.3}$$

where $\{\sigma > t\}^*$ is the open ball of center 0 and the same measure as $\{\sigma > t\}$ (it is well known that σ^* is radially decreasing and equimeasurable with σ, in particular $\|\sigma^*\|_p = \|\sigma\|_p$, see e.g. [12]).

Defining for $\mathrm{K} \geq 1$ the functions

$$G_{\mathrm{K}}(s) = \mathrm{K} - \sum_{k=0}^{\mathrm{K}-1} (\mathrm{K} - k) \frac{s^k}{k!} e^{-s} \tag{5.4}$$

(these are functions appearing in the right-hand side of (1.18)), we have the following result.

Theorem 5.1. *Assume $\sigma \in L^p(\mathbb{C})$ ($1 \leq p < \infty$) is a radial nonnegative symbol and let σ^* be its decreasing rearrangement as in (5.3). Then, for every $\mathrm{K} \geq 1$*

$$\sum_{k=1}^{\mathrm{K}} \lambda_k(\sigma) \leq \sum_{k=1}^{\mathrm{K}} \lambda_k(\sigma^*) = \int_0^\infty G_{\mathrm{K}}(\mu(t))\, dt, \tag{5.5}$$

where $\mu(t) = |\{\sigma > t\}|$ is the distribution function of σ. Moreover, equality in (5.5) is achieved if and only if $\sigma(z) = \sigma^*(z)$ for a.e. $z \in \mathbb{C}$. Finally, every eigenvalue $\lambda_n(\sigma^*)$ ($n \geq 1$) is simple, and the corresponding eigenfunction is the monomial $e_{n-1}(z)$ in (1.7).

Here, as usual, σ being *radial* means that $\sigma(z) = \sigma(|z|)$. By (5.1), however, Remark 1.2 carries over from sets $\Omega \subset \mathbb{C}$ to arbitrary symbols $\sigma \in L^p$, hence the theorem remains valid if σ is radial around some center $z_0 \in \mathbb{C}$ (with equality when $\sigma(z) = \sigma^*(z - z_0)$).

Proof. Since σ is radial, it is known (see [4]) that the eigenfunctions of T_σ are still the monomials (1.7) and its eigenvalues are therefore given (in some order) by the numbers

$$\langle T_\sigma\, e_k, e_k\rangle_{\mathcal{F}} = \int_{\mathbb{C}} \sigma(z)\, |e_k(z)|^2 e^{-\pi|z|^2}\, dA(z), \quad k = 0, 1, 2, \ldots \tag{5.6}$$

which, in particular, is true for T_{σ^*}. In this case, however, using (5.1) (written for σ^* first with $F = e_k$, $k \geq 1$, and then with $F = e_{k-1}$) we can compare the Rayleigh quotients

$$\begin{aligned}\langle T_{\sigma^*}\, e_k, e_k\rangle_{\mathcal{F}} &= \int_0^\infty \langle T_{\{\sigma^*>t\}} e_k, e_k\rangle\, dt = \int_0^\infty \lambda_{k+1}(\{\sigma^* > t\})\, dt \\ &> \int_0^\infty \lambda_k(\{\sigma^* > t\})\, dt = \int_0^\infty \langle T_{\{\sigma^*>t\}} e_{k-1}, e_{k-1}\rangle\, dt = \langle T_{\sigma^*}\, e_{k-1}, e_{k-1}\rangle_{\mathcal{F}},\end{aligned}$$

having used (1.9) with $\Omega = \{\sigma^* > t\}$ (which is a ball) and the fact that, for balls, every eigenvalue is simple (see [4, 32]). This monotonicity reveals that, similarly to (1.9),

$$\lambda_k(T_{\sigma^*}) = \langle T_{\sigma^*}\, e_{k-1}, e_{k-1}\rangle_{\mathcal{F}} = \int_0^\infty \lambda_k(\{\sigma^* > t\})\, dt, \tag{5.7}$$

thus proving the last claim of the theorem. In addition, since $\{\sigma^* > t\}$ is a ball and by equimeasurability $|\{\sigma^* > t\}| = \mu(t)$, the last equality in (5.5) follows from (5.7), (1.16) (last equality, with $\Omega = \{\sigma > t\}$) and (5.4).

Then, getting back to σ and the relative eigenvalues (5.6), let $k \to i_k$ be a bijection such that e_{i_k} is an eigenfunction of T_σ relative to the k-th eigenvalue $\lambda_k(\sigma)$. Letting $F = e_{i_k}$ in (5.2) and summing, we have

$$\begin{aligned}\sum_{k=1}^{\mathrm{K}} \lambda_k(\sigma) &= \sum_{k=1}^{\mathrm{K}} \langle T_\sigma e_{i_k}, e_{i_k}\rangle = \int_0^\infty \sum_{k=1}^{\mathrm{K}} \langle T_{\{\sigma>t\}} e_{i_k}, e_{i_k}\rangle \, dt \le \int_0^\infty \sum_{k=1}^{\mathrm{K}} \lambda_k(\{\sigma > t\})\, dt \\ &\le \int_0^\infty \sum_{k=1}^{\mathrm{K}} \lambda_k(\{\sigma > t\}^*)\, dt = \int_0^\infty \sum_{k=1}^{\mathrm{K}} \lambda_k(\{\sigma^* > t\})\, dt = \sum_{k=1}^{\mathrm{K}} \lambda_k(T_{\sigma^*})\end{aligned} \tag{5.8}$$

where the two inequalities follow, respectively, from (1.17) and from (1.16), and the last inequality follows from (5.7). This proves the inequality in (5.5), equality being possible only if equality holds, for a.e. t, also in (1.16) when $\Omega = \{\sigma > t\}$. As claimed after (1.16), this implies that $\{\sigma > t\}$ is a ball for a.e. t (hence for *every* t, as $\{\sigma > t\} = \bigcup_{\tau > t}\{\sigma > t\}$), that is, $\sigma = \sigma^*$ almost everywhere. Conversely, when $\sigma = \sigma^*$, the sets $\{\sigma > t\}$ are balls (hence there is equality in (1.17) and in (1.16) with $\Omega = \{\sigma > t\}$) and by (5.7) the bijection $k \to i_k$ is simply $i_k = k - 1$. All in all, when $\sigma = \sigma^*$ there is equality in (5.8) and hence also in (5.5). □

The integral in (5.5) provides an optimal upper bound, in terms of the distribution function $\mu(t) = |\{\sigma > t\}|$, for the sum of the first K eigenvalues of T_σ. It is then natural to investigate those symbols $\sigma \in L^p(\mathbb{C})$ that maximize this sum, under an L^p-norm constraint on σ. When $\mathrm{K} = 1$ (maximization of $\lambda_1(\sigma)$) this problem was studied in [25, 30], allowing for several simultaneous norm constraints. Here, when $\mathrm{K} > 1$, combining Theorem 5.1 with the techniques introduced in [25], we prove the following result (where, for simplicity, we restrict ourselves to the case of one L^p constraint with $p > 1$).

Theorem 5.2. *Let $p > 1$, $B > 0$ and $\mathrm{K} \ge 1$. Then, for all radial nonnegative symbols $\sigma \in L^p(\mathbb{C})$ such that $\|\sigma\|_p \le B$, it holds*

$$\sum_{k=1}^{\mathrm{K}} \lambda_k(\sigma) \le \sum_{k=1}^{\mathrm{K}} \lambda_k(\sigma_p), \tag{5.9}$$

where the optimal symbol $\sigma_p \in L^p(\mathbb{C})$ is given by

$$\sigma_p(z) = \frac{B}{c_p} \left(\sum_{k=0}^{\mathrm{K}-1} \frac{(\pi|z|^2)^k}{k!} \right)^{\frac{1}{p-1}} e^{-\frac{\pi|z|^2}{p-1}}, \quad z \in \mathbb{C}, \tag{5.10}$$

where $c_p > 0$ is defined by $c_p^p = \int_0^\infty (G_{\mathrm{K}}'(s))^{\frac{p}{p-1}}\, ds$, and G_{K} is the function in (5.4). Moreover, equality in (5.9) is achieved if and only if $\sigma(z) = \sigma_p(z)$ for a.e. $z \in \mathbb{C}$.

Proof. If σ is nonnegative and $\|\sigma\|_{L^p} \le B$, then its distribution function $\mu(t) = |\{\sigma > t\}|$ clearly belongs to the class of functions

$$\mathcal{C} := \left\{ \nu\colon (0,+\infty) \to [0,+\infty) \quad \text{such that} \quad \nu \text{ is decreasing and} \quad p\int_0^\infty t^{p-1}\nu(t)\,dt \le B^p \right\}.$$

Therefore, we have from (5.5)

$$\sum_{k=1}^{\mathrm{K}} \lambda_k(\sigma) \le \int_0^\infty G_{\mathrm{K}}(\mu(t))\,dt \le \sup_{\nu\in\mathcal{C}} \int_0^\infty G_{\mathrm{K}}(\nu(t))\,dt. \tag{5.11}$$

The idea of the proof is to show that the supremum in (5.11) is attained by a unique $\nu \in \mathcal{C}$ which is the distribution function of the admissible symbol σ_p defined in (5.10). When $\mathrm{K} = 1$ this was accomplished in [25], under the additional constraint that $\|\sigma\|_{L^\infty} \le A \in (0, \infty]$ (letting $A = \infty$ makes this constraint inactive, as in our case). Following step by step the proof of [25, Theorem 3.4], however, reveals that the very same argument (with the same B and in the particular case where $A = \infty$, but with the function $G = G_1$ replaced by G_{K}) can be used here (the only relevant properties of $G = G_1$ used in [25] are smoothness, strict monotonicity and concavity, which G_{K} satisfies). In this way, one proves that the supremum in (5.11) is attained by a unique $\mu_p \in \mathcal{C}$, that is implicitly defined by the condition that

$$\frac{B}{c_p} G'_{\mathrm{K}}(\mu_p(t))^{\frac{1}{p-1}} = t \quad \text{if} \quad 0 < t < \frac{B}{c_p}, \tag{5.12}$$

whereas $\mu_p(t) = 0$ for $t \ge B/c_p$. It is easy to see that μ_p is the distribution function of σ_p. Indeed, as σ_p is (strictly) radially decreasing, taking any $z \in \mathbb{C}$ and letting $t = \sigma_p(z)$, the superlevel set $\{\sigma_p > t\}$ is the open ball of radius $|z|$ whose measure is $\pi|z|^2$; on the other hand, computing $G'_{\mathrm{K}}(s)$ from (5.4), we can rewrite (5.10) as

$$t = \sigma_p(z) = \frac{B}{c_p} G'_{\mathrm{K}}(\pi|z|^2)^{\frac{1}{p-1}}$$

which, combined with (5.12), yields $G'_{\mathrm{K}}(\mu_p(t)) = G'_{\mathrm{K}}(\pi|z|^2)$, whence $\mu_p(t) = \pi|z|^2$ i.e. $\mu_p(t) = |\{\sigma_p > t\}|$ for every t in the range of σ_p (that is, for every $t \in (0, B/c_p]$ since $\max \sigma_p = B/c_p$). Finally, if $t > B/c_p$ we have $\{\sigma_p > t\} = \emptyset$ and in this case $\mu_p(t) = 0$ by definition: thus, summing up, we have $\mu_p(t) = |\{\sigma_p > t\}|$ for all $t > 0$, as claimed.

Now note that, when $\sigma = \sigma_p$, the two inequalities in (5.11) both become equalities: the former, because in this case $\sigma = \sigma^*$ (hence there is equality in (5.5) by Theorem 5.1), and the latter because in this case $\mu = \mu_p$ and μ_p achieves the supremum in (5.11). Thus, the supremum in (5.11) equals the right-hand side of (5.9), which then follows from (5.11). Finally, equality in (5.9) forces a double equality in (5.11): then necessarily $\mu = \mu_p$ (since μ_p is the only maximizer) and $\sigma = \sigma^*$ a.e. (by Theorem 5.1, (5.5) being an equality). Thus, both σ^* and σ_p have μ_p as distribution function: since both functions are radial and radially decreasing, $\sigma^* = \sigma_p$ a.e. and hence also $\sigma = \sigma_p$ a.e., as claimed. ∎

Appendix A. An extension of Karamata's inequality

Proof of Proposition 3.5. We first note that Φ, being convex, has a constant sign on $(0,\varepsilon)$ for some $\varepsilon>0$, hence each of the two series in (3.8) has a well-defined sum, not necessarily finite. More precisely, if $\Phi(0^+)$ (limit from the right) is nonzero, then the two series diverge, both to $+\infty$ or both to $-\infty$, and (3.8) is trivial in this case. Hence we may assume $\Phi(0^+)=0$ so that, extending Φ to $[0,1)$ by letting $\Phi(0):=0$, the right derivative $\Phi'(0^+)$ is well defined, with values in $[-\infty,+\infty)$. Moreover, defining $S_0:=0$ and, for $\mathrm{K}, k\geq 1$,

$$S_{\mathrm{K}} := \sum_{k=1}^{\mathrm{K}} (y_k - x_k)\,, \qquad c_k := \begin{cases} \dfrac{\Phi(y_k)-\Phi(x_k)}{y_k-x_k} & \text{if } x_k\neq y_k \\ \Phi'(x_k) \quad \text{(right derivative)} & \text{if } x_k = y_k, \end{cases}$$

summation by parts yields

$$\sum_{k=1}^{\mathrm{K}} \left(\Phi(y_k)-\Phi(x_k)\right) = \sum_{k=1}^{\mathrm{K}} c_k\,(S_k - S_{k-1}) = c_{\mathrm{K}} S_{\mathrm{K}} + \sum_{k=1}^{\mathrm{K}-1} (c_k - c_{k+1}) S_k, \quad \forall \mathrm{K}\geq 1. \tag{A.1}$$

Now two cases are possible:

(i) $\Phi'(0^+)$ is finite. In this case, Φ is Lipschitzian near 0^+ and hence the two series in (3.8) are both convergent, due to (3.7) and the fact that $\Phi(0)=0$. Moreover, by the convexity of Φ we have $c_k \geq c_{k+1} \geq \Phi'(0^+)$ $\forall k\geq 1$ (in particular the sequence c_k is bounded), whereas $S_{\mathrm{K}}\geq 0$ and $S_{\mathrm{K}}\to 0$ by (3.7). Hence, letting $\mathrm{K}\to\infty$ in (A.1) leads to

$$\sum_{k=1}^{\infty} \left(\Phi(y_k)-\Phi(x_k)\right) = \sum_{k=1}^{\infty} (c_k - c_{k+1}) S_k. \tag{A.2}$$

Since as already observed $(c_k - c_{k+1})S_k \geq 0$, (3.8) is proved, and equality occurs in (3.8) if and only if $(c_k-c_{k+1})S_k = 0$ for every $k\geq 1$. In addition, since the interval $(0,y_1]$ is the union of all the closed intervals $[x_k\wedge y_k, x_k\vee y_k]$ $(k\geq 1)$, the condition that all the c_k's are equal is equivalent to the condition that Φ, restricted to $(0,y_1]$, is affine. If this is not the case (i.e. $c_h - c_{h+1} > 0$ for at least one $h\geq 1$), then equality in (3.8) implies that $S_h = 0$, which completes the proof of Proposition 3.5.

(ii) $\Phi'(0^+) = -\infty$. In this case, $\Phi(x)<0$ if $x>0$ is small enough, hence the series in (3.8) might each be convergent or divergent to $-\infty$. Note that each of the functions

$$\Phi_n(t) := \max\{\Phi(t), -nt\}, \quad t\in[0,1), \quad n\geq 1$$

is convex on $[0,1)$ and such that $\Phi_n'(0^+) = -n$, hence it falls within case *(i)* for which Proposition 3.5 has already been proved. Thus, (3.8) with Φ_n in place of Φ gives

$$\sum_{k=1}^{\infty} \Phi_n(x_k) \leq \sum_{k=1}^{\infty} \Phi_n(y_k) \quad \forall n\geq 1. \tag{A.3}$$

Then, since $\Phi_n \searrow \Phi$ and (as observed in case *(i)*) the series in (A.3) are convergent, letting $n \to \infty$ in (A.3) one obtains (3.8) from monotone convergence (the limit series in (3.8), however, need not be both convergent).

To complete the proof, assume now that Φ is non-affine on $(0, y_1]$ and that equality occurs in (3.8), the two series being convergent. Let $a \in (0, y_1)$ be any point where Φ is differentiable, chosen small enough so that Φ, restricted to the interval $[a, y_1]$, is not affine, and consider the decomposition $\Phi(t) = \Phi_1(t) + \Phi_2(t)$, where

$$\Phi_2(t) = \begin{cases} \Phi(t) & \text{if } t \in [0, a] \\ \Phi(a) + \Phi'(a)(t-a) & \text{if } t \in [a, 1). \end{cases}$$

Clearly, both Φ_1 and Φ_2 are convex functions on $[0, 1)$, hence the first part of Proposition 3.5 (which has already been proved for any convex Φ) yields

$$\sum_{k=1}^{\infty} \Phi_1(x_k) \le \sum_{k=1}^{\infty} \Phi_1(y_k), \quad \sum_{k=1}^{\infty} \Phi_2(x_k) \le \sum_{k=1}^{\infty} \Phi_2(y_k). \tag{A.4}$$

Adding these inequality yields (3.8) again, but since we are assuming that (3.8) holds as an equality, there must be equality also in (A.4), in particular for Φ_1. Now observe that, since Φ_2 is affine on the interval $[a, 1]$ but Φ is not, on that interval $\Phi_1 = \Phi - \Phi_2$ is not affine either (in particular, Φ_1 is not affine on $(0, y_1]$). But since $\Phi_1'(0^+) = 0$, Φ_1 falls within case *(i)*, in which Proposition (3.5) has already been proved: then, since there is equality in (A.4), the last claim of Proposition (3.5) applies to Φ_1, hence $S_k = 0$ for at least one value of k. $\square$

Remark A.1. If $S_k = 0$ for some $k \ge 1$ and $x_{k+1} \vee y_{k+1} \le x_k \wedge y_k$, then equality occurs in (3.8) provided Φ is affine on the interval $(0, x_{k+1} \vee y_{k+1})$. More elaborate examples of equality cases can be constructed, with $S_k = 0$ for several non-consecutive values of k and Φ piecewise affine. In fact, with some effort, building on the previous proof it is possible to characterize all cases where equality occurs, but we will not pursue this here.

Acknowledgments

F.N. is a fellow of the Accademia delle Scienze di Torino and a member of the Società Italiana di Scienze e Tecnologie Quantistiche (SISTEQ).

Statements and Declarations

Competing interests. The authors have no financial or non-financial interests to disclose.

Data availability statement. This manuscript has no associated data.

References

[1] L. D. Abreu, K. Gröchenig, and J. L. Romero. On accumulated spectrograms. *Trans. Amer. Math. Soc.*, 368(5):3629–3649, 2016.

[2] M. Ángeles García-Ferrero and J. Ortega-Cerdá. Stability of the concentration inequality on polynomials. *Comm. Math. Phys.*, 406(5):Paper No. 112, 35, 2025.

[3] M. S. Ashbaugh and R. D. Benguria. Isoperimetric inequalities for eigenvalues of the Laplacian. In *Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday*, volume 76, Part 1 of *Proc. Sympos. Pure Math.*, pages 105–139. Amer. Math. Soc., Providence, RI, 2007.

[4] I. Daubechies. Time-frequency localization operators: a geometric phase space approach. *IEEE Trans. Inform. Theory*, 34(4):605–612, 1988.

[5] K. Fan. Maximum properties and inequalities for the eigenvalues of completely continuous operators. *Proc. Nat. Acad. Sci. U.S.A.*, 37:760–766, 1951.

[6] R. L. Frank. Sharp inequalities for coherent states and their optimizers. *Adv. Nonlinear Stud.*, 23(1):Paper No. 20220050, 28, 2023.

[7] R. L. Frank, F. Nicola, and P. Tilli. The generalized Wehrl entropy bound in quantitative form. *arXiv:2307.14089*, 2023. To appear in *J. Eur. Math. Soc.*

[8] A. Galbis. Norm estimates for selfadjoint Toeplitz operators on the Fock space. *Complex Anal. Oper. Theory*, 16(1):Paper No. 15, 13, 2022.

[9] J. Gómez, A. Guerra, J. P. G. Ramos, and P. Tilli. Stability of the Faber-Krahn inequality for the short-time Fourier transform. *Invent. Math.*, 236(2):779–836, 2024.

[10] D. S. Grebenkov and B.-T. Nguyen. Geometrical structure of Laplacian eigenfunctions. *SIAM Rev.*, 55(4):601–667, 2013.

[11] K. Gröchenig. *Foundations of time-frequency analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2001.

[12] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge, at the University Press, 1952. 2d ed.

[13] A. Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[14] D. Kalaj. Contraction property of certain classes of log-$\mathcal{M}$-subharmonic functions in the unit ball. *J. Funct. Anal.*, 286(1):Paper No. 110203, 29, 2024.

[15] A. Kulikov. Functionals with extrema at reproducing kernels. *Geom. Funct. Anal.*, 32(4):938–949, 2022.

[16] A. Kulikov. Exponential lower bound for the eigenvalues of the time-frequency localization operator before the plunge region. *Applied and Computational Harmonic Analysis*, 71:101639, 2024.

[17] A. Kulikov, F. Nicola, J. Ortega-Cerdà, and P. Tilli. A monotonicity theorem for subharmonic functions on manifolds. *Adv. Math.*, 479:Paper No. 110423, 18, 2025.

[18] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[19] J. M. Luttinger. Generalized isoperimetric inequalities. *J. Mathematical Phys.*, 14:586–593, 1973.

[20] F. Nicola. The wave function of stabilizer states and the Wehrl conjecture. *J. Math. Pures Appl. (9)*, 205:Paper No. 103778, 28, 2026.

[21] F. Nicola and F. Riccardi. The quantitative isoperimetric inequality for the Hilbert–Schmidt norm of localization operators. *J. Fourier Anal. Appl.*, 31(1):Paper No. 12,

2025.

[22] F. Nicola, F. Riccardi, and P. Tilli. The Wehrl-type entropy conjecture for symmetric $SU(N)$ coherent states: cases of equality and stability. *arXiv:2412.10940*, 2024.

[23] F. Nicola, J. L. Romero, and S. I. Trapasso. On the existence of optimizers for time-frequency concentration problems. *Calc. Var. Partial Differential Equations*, 62(1):Paper No. 21, 21, 2023.

[24] F. Nicola and P. Tilli. The Faber-Krahn inequality for the short-time Fourier transform. *Invent. Math.*, 230(1):1–30, 2022.

[25] F. Nicola and P. Tilli. The norm of time-frequency and wavelet localization operators. *Trans. Amer. Math. Soc.*, 376(10):7353–7375, 2023.

[26] G. Pólya and M. Schiffer. Convexity of functionals by transplantation. *J. Analyse Math.*, 3:245–346, 1954. With an appendix by Heinz Helfenstein.

[27] G. Pólya and G. Szegö. *Isoperimetric Inequalities in Mathematical Physics*, volume No. 27 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1951.

[28] G. S. Raju Kosuru and S. Saha. Characterizations of majorization on summable sequences. *Filomat*, 34(7):2193–2202, 2020.

[29] J. P. G. Ramos and P. Tilli. A Faber-Krahn inequality for wavelet transforms. *Bull. Lond. Math. Soc.*, 55(4):2018–2034, 2023.

[30] F. Riccardi. A new optimal estimate for the norm of time-frequency localization operators. *J. Funct. Anal.*, 287(6):Paper No. 110523, 23, 2024.

[31] F. Riccardi. An optimal estimate for the norm of wavelet localization operators. In *2025 International Conference on Sampling Theory and Applications (SampTA)*, pages 1–4, 2025.

[32] K. Seip. Reproducing formulas and double orthogonality in Bargmann and Bergman spaces. *SIAM J. Math. Anal.*, 22(3):856–876, 1991.

[33] B. Simon. *Convexity*, volume 187 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2011. An analytic viewpoint.

[34] M. W. Wong. *Wavelet transforms and localization operators*, volume 136 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2002.

[35] K. Zhu. *Analysis on Fock spaces*, volume 263 of *Graduate Texts in Mathematics*. Springer, New York, 2012.

(Fabio Nicola) Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Email address: fabio.nicola@polito.it

(Federico Riccardi) Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Email address: federico.riccardi@polito.it

(Paolo Tilli) Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Email address: paolo.tilli@polito.it