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ABSTRACT

The Clifford algebra of the endomorphisms of the exterior algebra of a countably dimensional vector space induces natural *bosonic* shadows, i.e. families of linear maps between the cohomologies of complex Grassmannians. The main result of this paper is to provide a determinantal formula expressing generating functions of such endomorphisms unifying several classical special cases. For example the action over a point recovers the Jacobi-Trudy formula in the theory of symmetric functions or the Giambelli's one in classical Schubert calculus, whereas the action of degree preserving endomorphisms takes into account a finite type version of the Date-Jimbo-Kashiwara-Miwa bosonic vertex operator representation of the Lie algebra $gl(\infty)$. The fermionic actions on (finite type) bosonic spaces are described in terms of the classical theory of symmetric functions. The main guiding principle is the fact that the exterior algebra is a (non irreducible) representation of the ring of symmetric functions, which is the way we use to spell the "finite type" Boson-Fermion correspondence.

Contents

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1. Introduction

1.1. Let $r, n \in (\mathbb{N} \cup \{\infty\})^2$ such that $0 \le r \le n$. The main purpose of this paper is to describe linear maps between singular cohomologies $B_{r,n}$ of complex Grassmannians $G(r, n)$ induced by homogeneous endomorphisms (basically the *free fermions* that the title alludes to) of the exterior algebra $\bigwedge \mathbb{Q}^n$. By *free fermions* one usually means the elements of the Clifford algebra $\mathcal{C}_\infty$ spanned by 1 and $(\psi_i, \psi_i^*)_{i\in\mathbb{Z}}$, subjects to the relations $\psi_i\psi_j + \psi_i\psi_j = 0$, $\psi_i^*\psi_j^* + \psi_i^*\psi_j^* = 0$ and $\psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}$, see e.g. [19] or [33, Section 1]. The *elementary endomorphisms* of the exterior algebra $\bigwedge \mathbb{Q}^n$ ($n \in \mathbb{N} \cup \{\infty\}$), when multiplied, are subject to the same commutation relations of the Clifford algebra $\mathcal{C}_\infty$ alluded to above. By wedging and contracting, they define linear maps $\bigwedge^r \mathbb{Q}^n \to \bigwedge^{r+s} \mathbb{Q}^n$ and, because of a certain *finite type boson-fermion correspondence*, linear maps $B_r \to B_{r+s}$, whence, up to a slight abuse of terminology, the title of the paper. The main motivation to pursue this goal is that it place itself at the crossroad of a number of subjects, such as the theory of symmetric functions, Schubert Calculus for Grassmannians, as well as the representation theory of Lie algebras of endomorphisms of infinite dimensional vector space à la Date, Jimbo, Kashiwara and Miwa(DJKM) [6].

The simplest cases are those corresponding to the pair $(1, n)$, for any n, and (r, ∞), for any r, which will be partly discussed separately via ad hoc arguments in section 3. The case (∞, ∞) is related to the DJKM bosonic vertex operator representation of the Lie algebra $gl_\infty(\mathbb{Q})$ of matrices $A : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$, with finitely many non-zero diagonals.

The general instance, especially the trickiest one for finite r and n, is taken into account by our main result, Theorem 6.7, built out of a new determinantal expression which on one hand can be seen as a generalization of the classical Giambelli's formula of Schubert calculus and, on the other, keeps into account, as a particular case, the structure of the $\mathbb{Q}$–polynomial ring in r indeterminates as a representation of the Lie algebra $gl_\infty(\mathbb{Q})$. To achieve the latter one may exploit suitable operators, carrying a classical flavor, whose limit for $r \to \infty$ is precisely those used by Date, Jimbo, Kashiwara and Miwa (DJKM), in [6], to obtain the bosonic vertex operator representation of the Lie algebra $gl_\infty(\mathbb{C})$ (see also [19]).

To this respect, it should be mentioned that Laksov and Thorup, in their reworking of Schubert calculus in terms of action on an exterior power, in the paper [26] affirm that "in the work of E. Date, M. Jimbo, M. Kashiwara, and T. Miwa [6] Schur functions appear in connection with exterior power in another context". The present paper, on the other hand, is a further indication, besides e.g. [5,4] that the context is indeed the same and that the celebrated *bosonic vertex operator representation*of Date, Jimbo, Kashiwara and Miwa could be seen as an instance of the general Schubert Calculus studied in [26] living on the Universal Grassmann Manifold (inductive limit of finite dimensional grassmannians) by Sato [32]

The underlying conceptual framework is easy. If $r \le n < \infty$, the $gl_n(\mathbb{Q})$-module structure of $B_{r,n}$ is induced by the standard representation of the $n \times n$ square matrices on the exterior power $\bigwedge^r \mathbb{Q}^n$, due to the linear isomorphism $B_{r,n} \to \bigwedge^r \mathbb{Q}^n$. The latter can be understood, as we will, as a sort of "finite dimensional" version of the *boson-fermion* correspondence, for which there are now many references in the literature (see e.g. the relatively recent papers [22,31]), and which reflects the fact that both spaces possess bases of the same cardinality. For finite r and n (in contrast to $r = n = \infty$ as in the DJKM picture), the determination of an explicit generating function of elementary matrices is tricky. One way to cope with the issues is proposed in [16]. On the other hand one can notice that $B_{r,n}$ is more generally a module over the algebra of the degree 0 homogeneous endomorphisms of $\bigwedge \mathbb{Q}^n$, a Lie subalgebra of $\mathrm{End}_{\mathbb{Q}}(\bigwedge \mathbb{Q}^n)$. This was studied in [2] for $n = \infty$, generalizing formulas deduced in [16]. The structure of $B = B_{\infty,\infty}$ as a representation of the endomorphisms of the exterior algebra $\bigwedge \mathbb{Q}^\infty$ is considered in [4]: the output are formulas involving a vertex operators description of the generating function which reduces to the DJKM ones when the exterior algebra endomorphisms are induced by $gl_\infty(\mathbb{Q})$ via the *trace representation* (see Sec. 4.1). Exploiting the remarkable fact that the exterior algebra of a countably infinite vector space is in turn a representation of the ring of symmetric functions, the aim of this paper is to provide a unified framework to all the previously mentioned situations. The finite-dimensional boson-fermion correspondence yields the general determinantal formula which Theorem 6.7 accounts for.

1.2. From a representation-theoretic perspective, the boson-fermion correspondence – first introduced in quantum field theory (see, e.g., [11])– is merely an isomorphism between two modules over the infinite-dimensional Heisenberg algebra (see e.g. [33, Section 1] for a concise enlightening overview). The former, the bosonic Fock space, is $B[q, q^{-1}]$, where B is the free polynomial algebra generated by infinitely many indeterminates $(x_1, x_2, \ldots)$ (the bosonic Fock space in charge 0), whereas the latter, the *fermionic* one, is usually understood as the linear span of the so-called charged partitions, which in turns can be thought of as monomials of a semi-infinite exterior power of an infinite dimensional vector space V (see [34,23,28] and also [17, Section 4.1]). In charge zero, the boson-fermion correspondence maps a basis element of B, parametrized by a partition, to an infinite exterior monomial attached to the same partition (see e.g. [25, Theorem 6.1]). The irreducible B-module structure of the fermionic space allows to deal with a wealth of related situations in mathematical physics and in (algebraic) geometry. Geometric realization of the fermionic Fock spaces and Clifford and Heisenberg operators, after Nakajima and Grojnowskij, are faced e.g. in [33], also in connection with the geometry of moduli spaces of framed torsion-free sheaves as in [28]. From the mathematical physics point of view, one must mention soliton equations, the celebrated Kadomtsev-Petshiasvili (KP) hierarchy, the bosonic vertex operators representation of infinite dimensional Lie algebras [6,19] and Jing's article [22], not to speak about the representation theory of Kac–Moody algebras. In particular, the KP-hierarchy rules equations of the group orbit of the highest weight vector of B into the projective space of the corresponding highest weight module; in addition, the fact that the polynomial solutions of the celebrated KP-hierarchy are parametrized by the points of the Sato's Universal Grassmann Manifold (UGM), as constructed e.g. in [32], can be recovered via the bosonic representation of the Clifford algebra $\mathcal{C}_\infty$ of free fermions; see [7,9,8,12,24] and [35,37,36] for further information on highest weight modules and other kind of representations of affine Kac-Moody Lie (super)algebras. Most of the above situations have been considered in their finite (type or dimensional) counterparts in previous work of one of us (e.g. [16], [2], [17], [4], [18]) and this paper set itself in the same stream.

1.3. To more precisely anticipate the shape of our results, let $n \in \mathbb{N} \cup \{\infty\}$, so that $V_n = \mathbb{Q}[X]/(X^n)$ is the $\mathbb{Q}$-algebra of all polynomials of degree $< n$ in one indeterminate X. It will be considered a $B_{1,n} := \dfrac{\mathbb{Q}[c_1]}{(c_1^n)}$-module via the map $c_1(p(X) + (X^n)) = Xp(X) + (X^n)$. Denote by V_n^* its (restricted) dual, spanned by $(\partial^j)_{0 \le j < n}$, where ∂^j denotes the unique linear form on V_n such that $\partial^j(X^i) = \delta^{ij}$. If $n = \infty$ we write $V := V_\infty = \mathbb{Q}[X]$. For all $(I, J) = (i_1, \ldots, i_h, j_1, \ldots, j_k) \in \mathbb{N}^h \times \mathbb{N}^k$, the formal words (free fermions)

$$X^I \partial^J := X^{i_1} \cdots X^{i_h} \partial^{j_1} \cdots \partial^{j_k} \tag{1.1}$$

induce homogeneous (elementary) endomorphisms of $\bigwedge V_n$ of degree $h - k$ (which is zero if $h < k$) i.e., for all $0 \le r < n$, a family of maps $\Phi_{(I,J),n} : \bigwedge^r V_n \to \bigwedge^{r+h-k} V_n$ defined by

$$\begin{aligned} \Phi_{(I,J),n}(u) :&= X^{i_1} \cdots X^{i_k} \partial^{j_1} \cdots \partial^{j_k}(u) \\ &= X^{i_1} \wedge \cdots \wedge X^{i_h} \wedge \partial^{j_1} \lrcorner (\ldots \lrcorner (\partial^{j_h} \lrcorner u)) \ldots), \quad \forall u \in \bigwedge^r V_n \end{aligned} \tag{1.2}$$

where if $\alpha \in V^*$, by $\alpha\lrcorner : \bigwedge V_n \to \bigwedge V_n$ one understands the unique odd derivation of the exterior algebra such that $\alpha \lrcorner u = \alpha(u)$. It is a derivation of degree -1 meaning that it maps $\bigwedge^k V_n$ to $\bigwedge^{k-1} V_n$ (Cf. Section 4.7).

The exterior algebra $\bigwedge V_n$ is an irreducible representation of the Clifford algebra spanned by the identity $1 := 1_{\bigwedge V_n}$ and all the words of type (1.1), subject to the relations $X^i\partial^j + \partial^j X^i = \delta^{ij}$, $X^iX^j = -X^jX^i$ and $\partial^i\partial^j = -\partial^j\partial^i$ holding in $\mathrm{End}_{\mathbb{Q}}(\bigwedge V_n)$. Let $(S_{\boldsymbol{\lambda}})_{\boldsymbol{\lambda}\in\mathcal{P}_{r,n}}$ be the Schur basis of $B_{r,n}$ (basically the Schur determinant of the Chern Classes of the universal quotient bundle over $G(r,n)$), where $\mathcal{P}_{r,n}$ denotes the set of all partitions $\boldsymbol{\lambda} := (n-r \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0)$, in turns isomorphic to $\bigwedge^r V_n$. Wedging with h vectors and contracting with k linear forms against elements of $\bigwedge^r V_n$ induce obvious vector space homomorphisms

$$\Phi_{(I,J),n}; B_{r,n} \longrightarrow B_{r+h-k,n} \tag{1.3}$$

which is our aim to encode all together into a generating function.

To this purpose, let

$$\mathbf{z}_h := (z_1,\ldots,z_h), \qquad \mathbf{w}_k := (w_1,\ldots,w_k) \quad \text{and} \quad \mathbf{t}_r := (t_1,\ldots,t_r)$$

be indeterminates over $\mathbb{Q}$. For each $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$ denote by $s_{\boldsymbol{\lambda}}(\mathbf{t}_r)$ the Schur polynomial (Section 2.5) attached to the partition $\boldsymbol{\lambda}$ and to the indeterminates $\mathbf{t}_r$, and

$$p_i(\mathbf{t}_r) := t_1^i + \cdots + t_r^i,$$

be the (symmetric polynomial) sum of the i^{th} powers of the indeterminates t_j. Our **main Theorem** 6.7 then gives an explicit precise description of a $B_{r+h-k,n}$-valued formal power series

$$\Phi_{(r,h,k)}(\mathbf{z}_h, \mathbf{w}_k^{-1}, \mathbf{t}_r) = (-1)^h \prod_{i=1}^{k} w_i^{-r+1} \cdot \mathbb{D}_{(r,h,k),n} \exp\left(\sum_{i\geq 1} x_i p_i(\mathbf{z}_h, \mathbf{t}_r) + \frac{p_i(\mathbf{t}_r)}{i\prod_{j=1}^{k} w_j^i}\right) \tag{1.4}$$

which we call *structural generating series*, enjoying the following property

> The coefficient of $\boxed{\mathbf{z}_h^I \mathbf{w}_k^{-J} s_{\boldsymbol{\lambda}}(\mathbf{t}_r)}$ in the expansion of $\Phi_{(r,h,k)}(\mathbf{z}_h, \mathbf{w}_k, \mathbf{t}_r)$ is the image in $B_{r+h-k,n}$ of $S_{\boldsymbol{\lambda}} \in B_{r,n}$ through the natural map $\Phi_{(I,J),n}$ (1.3).

The key block of (1.4) is the determinant $\mathbb{D}_{(r,h,k),n}$, whose explicit expression is displayed in (6.3). It has a number of relevant particular cases. If $h = k$ and $n = \infty$, formula (1.4) simplifies the main statement of [2], where one studied $B_r := B_{r,\infty}$ as a module over the Lie algebra of the endomorphisms of $\bigwedge^k V$. If in addition $h = k = 1$, one obtains the *finite type* counterpart of the Date-Jimbo-Kashiwara-Miwa (DJKM) representation of $gl_\infty(\mathbb{Q})$, as studied in [16]. Moreover, if $r \to \infty$, the formula provides a determinantal expression of the vertex operators occurring in the DJKM representation (cf. also Section 3, Example b)). The computations heavily rely on the use of the exponential of the trace representation of the Lie algebra $gl_n(\mathbb{Q})$, as in [2,4], which in [13] was phrased in terms of *Hasse-Schmidt derivation* on an exterior algebra. The general formula we obtain involves a determinant which may be considered as a deformed version of the usual Giambelli's one occurring in classical Schubert calculus. Our main formula also recovers, as a limit when $r, n \to \infty$, in the same spirit of previous work (e.g. [2,16,3]), the DJKM vertex operator representation of the Lie algebra $gl_\infty(\mathbb{Q})$, to emphasize that the latter is nothing but an extremal case of a general situation (sloganized in [16] as *the cohomology of the Grassmannian is a gl_n-module*) whose other extremal one is the usual multiplication of matrices by vectors.

The referee kindly pointed out that Theorem 6.7 above also admits a natural interpretation from the viewpoint of symmetric functions. We fully agree. Via the finite boson–fermion correspondence, the structural generating series acts on the Schur basis of $B_{r,n}$, so that the determinant $\mathbb{D}_{(r,h,k)}$ may be regarded as a deformation of the classical Jacobi–Trudi/Giambelli determinant. This perspective is closely related to the philosophy developed by Jing and Rozhkovskaya [21], in which the Clifford algebra action is constructed starting from generalized Jacobi–Trudi identities. The present paper adopts a different starting point, namely the action of the Clifford algebra on the exterior algebra, from which the corresponding determinantal identities on the bosonic side are recovered via the finite boson–fermion correspondence. The two viewpoints therefore appear to be complementary, and we believe that it would be worthwhile to make the relationship between the two approaches more explicit in future work.

1.4. About the content. The paper is organized as follows. Section 2 collects a few combinatorial preliminaries which are, on the other hand, well available in the literature, possibly with a slight change of notation. We also discuss a couple of special cases, to show the relationship of the present subject to the theory of symmetric functions as in [29, p. 96] and/or the vertex operators occurring in the description of the KP hierarchy (generating series of Bernstein operators like in [29, I.5, Example 29, formula (6)]). For the sequel, it is important to notice that we will be working with two rings of symmetric polynomials in two different set of variables, B_r and Λ_r. The variables of B_r are not explicitly invoked, while we use those of the ring Λ_r. We hope this will not cause too much confusion.

Section 4 provides further details about the $B_{r,n}$-module structure of the exterior power $\bigwedge^r V_n$. The key homomorphisms $\sigma_+(z), \overline{\sigma}_+(z) : \bigwedge V_n \to \bigwedge V_n[\![z]\!]$ will be introduced and rigorously defined in terms of exponential of linear differential operators on the exterior algebra, and they are related with the classical Chern polynomials of the tautological and quotient bundle in the universal sequence over the Grassmannian. In this section we also show (Proposition 4.2) how product of r *Schubert derivations* supplies a generating function for the basis elements of the r-th exterior power $\bigwedge^k \mathbb{Q}^n$, which is suited to study the $gl_n(\mathbb{Q})$-action of $B_{r,n}$, which is the finite dimensional shadow of the DJKM representation of $gl_\infty(\mathbb{Q})$.

In Section 5, the necessary combinatorics needed to work with the representation of the Clifford algebra of "free fermions" on an exterior algebra is developed. This yoga was already partially used in [4,2] for the bosonic representation of the Clifford algebra generalizing formulas due Date, Jimbo, Kashiwara and Miwa [6].

Section 6 is finally devoted to the statement and proof of our main formula. As anticipated, it involves a determinant that is as huge as the integer $r + h$, but whose expression has a very elegant and regular shape. Using Mathematica©, Example 7.5 fully works out the expression of the generating structural series of the action of the 4×4 square matrices against the cohomology of the Grassmannian $G(2, 4)$: it is the first instance of the DJKM representation in a finite dimensional context which is not just the multiplication of a matrix times a vector.

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2. Preliminaries and notation

2.1. If A is any algebra over some commutative ring R and $\mathbf{t}_r := (t_1, \dots, t_r)$ are indeterminates over A, the notation $A[\![\mathbf{t}_r]\!]$ stands for the algebra of formal power series in those indeterminates with the usual definition of product.

2.2. Let $V := \mathbb{Q}[X]$ be the vector space of polynomials with rational coefficients. Its natural basis is $\mathbf{X} := (X^i)_{i\geq 0}$. For each $n \in \mathbb{N} \cup \{\infty\}$, let

$$V_n := \frac{\mathbb{Q}[X]}{(X^{n-1})} \quad \text{if } n < \infty \quad \text{and} \quad V_\infty := V.$$

The linear maps

$$\partial^i := \frac{1}{i!}\left.\frac{d^i}{dX^i}\right|_{X=0} : V_n \mapsto \mathbb{Q}$$

define the linear forms $\boldsymbol{\partial} := (\partial^i)_{i\geq 0}$ of the (restricted) dual V_n^* of V_n, dual to $\mathbf{X} = (X^i)_{0\leq i<n}$, i.e. $\partial^j(X^i) = \delta_i^j$. We denote by $\mathbf{X}(z)$ and $\boldsymbol{\partial}(w^{-1})$ the generating functions of the bases $\mathbf{X}$ and $\boldsymbol{\partial}$ of V and V^* respectively, i.e.

$$\mathbf{X}(z) = \sum_{0\leq i<n} X^i z^i \qquad \text{and} \qquad \boldsymbol{\partial}(w^{-1}) = \sum_{0\leq i<n} \partial^i w^{-i} = \exp\left(w^{-1}\frac{d}{dX}\right). \tag{2.1}$$

2.3. Let $(r, n) \in (\mathbb{N} \cup \{\infty\})^2$ such that $0 \leq r \leq n$. To any such pair we attach three isomorphic vector spaces, as follows. The first one is $\mathbb{P}_{r,n} : \mathbb{Q}^{\mathcal{P}_{r,n}}$, the $\mathbb{Q}$-vector space freely generated by all the *partitions* of length at most r whose Young diagram is contained in a $r \times (n - r)$ rectangle; the second one is $B_{r,n}$, the universal decomposition algebra of the polynomial T^n as the product of two monic polynomials of degree r and $n - r$ (see e.g. [26,27]), and $\bigwedge^r V_n$, the r-th exterior power of an n-dimensional vector space. In all of the above, n may be infinite. The three spaces are related by the commutative diagram

$$\begin{tikzcd} & \mathbb{P}_{r,n} \arrow[dl, "\beta"'] \arrow[dr, "\phi"] & \\ B_{r,n} \arrow[rr, "\phi\circ\beta^{-1}"'] & & \bigwedge^r V_n \end{tikzcd} \tag{2.2}$$

where $\phi \circ \beta^{-1}$ can be understood as the *boson-fermion* correspondence, as will be explained in the sequel.

2.4. The space $\mathbb{P}_{r,n}$. A *partition* is a non-increasing sequence $\boldsymbol{\lambda} : \lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ of non negative integers with finite length $\ell(\boldsymbol{\lambda}) := \sharp\{i \,|\, \lambda_i \neq 0\}$. We denote by $\mathcal{P}$ the set of all partitions, by $\mathcal{P}_r := \{\boldsymbol{\lambda} \in \mathcal{P} \,|\, \ell(\boldsymbol{\lambda}) \leq r\}$ the set of all partitions of length at most r and by $\mathcal{P}_{r,n} := \{\boldsymbol{\lambda} \in \mathcal{P}_r \,|\, \lambda_1 \geq n-r\}$ the set of all partitions whose Young diagram is contained in a $r \times (n - r)$ rectangle. Clearly $\mathcal{P}_r = \mathcal{P}_{r,\infty}$ and $\mathcal{P} = \mathcal{P}_\infty$. Let $\mathbb{P}_{r,n} := \mathbb{Q}^{\mathcal{P}_{r,n}}$ be the $\mathbb{Q}$-vector space generated by all $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$ Each element $u \in \mathbb{P}_{r,n}$ can be written in *bosonic* ($\beta(u)$) or *fermionic* ($\phi(u)$) notation, in the sense specified below.

2.5. Exterior powers and algebra. The r-th exterior power of V_n is the vector space $\bigwedge^r V_n$ generated by all the expressions of the form $X^{i_1} \wedge \cdots \wedge X^{i_r}$, modulo the relations

$$X^{i_{\tau(1)}} \wedge \cdots \wedge X^{i_{\tau(n)}} = \mathrm{sgn}(\tau) X^{i_1} \wedge \cdots \wedge X^{i_r}, \quad \tau \in S_n$$

where $\mathrm{sgn}(\tau)$ denotes the sign of the permutation τ. Therefore, $\bigwedge^r V_n = \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}_{r,n}} \mathbb{Q}\cdot \mathbf{X}^r(\boldsymbol{\lambda})$, where

$$\mathbf{X}^r(\boldsymbol{\lambda}) = \mathbf{X}^r(\lambda_1,\ldots,\lambda_r) := X^{r-1+\lambda_1}\wedge\cdots\wedge X^{\lambda_r}, \tag{2.3}$$

whence the $\mathbb{Q}$-linear isomorphism $\phi : \mathbb{P}_{r,n} \to \bigwedge^r V_n$ mapping $\boldsymbol{\lambda}$ to $\mathbf{X}^r(\boldsymbol{\lambda})$.

The *exterior algebra* of V_n is $(\bigwedge V_n, \wedge)$, where $\bigwedge V_n = \bigoplus_{r\geq 0}\bigwedge^r V_n$ and

$$\wedge : \bigwedge^r V_n \times \bigwedge^s V_n \to \bigwedge^{r+s} V_n$$

is the (bilinear) the juxtaposition product.

Consider now the wedge product

$$\mathbf{X}(t_1)\wedge\cdots\wedge\mathbf{X}(t_r) \in (\bigwedge^r V_n)[\![t_1,\ldots,t_r]\!].$$

It is clearly a (possibly infinite) linear combination of $(\mathbf{X}^r(\boldsymbol{\lambda}))_{\boldsymbol{\lambda}\in\mathcal{P}_{r,n}}$ which is divisible by the Vandermonde determinant $\Delta_0(\mathbf{t}_r) := \prod_{i<j}(t_i - t_j)$. The equality

$$\mathbf{X}(t_1)\wedge\cdots\wedge\mathbf{X}(t_r) = \Delta_0(\mathbf{t}_r)\sum_{\boldsymbol{\lambda}\in\mathcal{P}_{r,n}} \mathbf{X}^r(\boldsymbol{\lambda}) s_{\boldsymbol{\lambda}}(\mathbf{t}_r)$$

defines the *Schur symmetric polynomials* $s_{\boldsymbol{\lambda}}(\mathbf{t}_r)$ in $\mathbf{t}_r = (t_1,\ldots,t_r)$. Therefore

$$\sum_{\boldsymbol{\lambda}\in\mathcal{P}_{r,n}} \mathbf{X}^r(\boldsymbol{\lambda}) s_{\boldsymbol{\lambda}}(\mathbf{t}_r) \in \big(\bigwedge^r V_n\big)[\![t_1,\ldots,t_r]\!]$$

is a generating function of the basis elements of $\bigwedge^r V_n$.[1] It is well known that $\big(s_{\boldsymbol{\lambda}}(\mathbf{t}_r)\big)_{\boldsymbol{\lambda}\in\mathcal{P}_r}$ forms a basis of the ring $\Lambda_r := \mathbb{Q}[\mathbf{t}_r]^{S_r}$ of the symmetric polynomials in the indeterminates $(t_1,\ldots,t_r)$.

2.6. The "bosonic space" $B_{r,n}$. Let $\mathbf{c} := (c_1, c_2, \ldots)$ be a sequence of indeterminates over $\mathbb{Q}$. To each $r \in \mathbb{N}\cup\{\infty\}$ we attach a $\mathbb{Q}$-algebra B_r. By definition $B_0 = \mathbb{Q}$ and, for $0 < r \leq \infty$, one set $B_r := \mathbb{Q}[c_1,\ldots,c_r]$. In addition one sets $B := B_\infty = \mathbb{Q}[\mathbf{c}]$ for the polynomial $\mathbb{Q}$-algebra in the infinitely many indeterminates (c_i). One considers the generic (Chern) monic polynomial[2]

$$\mathbf{c}_r(z) = 1 - c_1 z^r + \cdots + (-1)^r c_r z^r \in B_r[z]$$

and the formal (Segre) power series[3]

$$\mathbf{S}_r(z) = \sum_{i\in\mathbb{Z}} S_i z^i := \frac{1}{\mathbf{c}_r(z)} \in B_r[\![z]\!].$$

In other words:

$$\mathbf{c}_r(z)(1 + S_1 z + \cdots + S_{n-r} z^r) = 1$$

[1] Here $(\bigwedge V_n)[\![t_1,\ldots,t_r]\!]$ are the $\bigwedge V_n$-valued formal power series in the indeterminates $(t_1,\ldots,t_r)$.

[2] The indeterminate coefficients are denoted c_i, to evoke the Chern classes of the tautological bundle over the Grassmannian $G(r,n)$.

[3] The indeterminate coefficients are denoted S_i, to reminish the Segre classes of the tautological bundle over the Grassmannian $G(r,n)$.

the equality holding in $B_{r,n}[z]$ for finite n, or

$$\mathbf{c}_r(z)(1+\sum_{j\geq 1} S_j z^j) = 1$$

the equality holding in $B_r[\![z]\!]$ if $n=\infty$.

For $n \geq r \geq 0$, let $I_{r,n} := (S_{n-r+1}, \ldots, S_n)$ be the ideal of B_r generated by $S_{n-r+1}, \ldots, S_n$. Let

$$\pi_{r,n} : B_r \longrightarrow B_{r,n} = \frac{B_r}{I_{r,n}},$$

be the canonical projection. It is well known that $B_{r,n}$ is the universal decomposition algebra of the monomial T^n as the product of two monic polynomials, one of degree r (see e.g. [5]). This is in turn the presentation of the singular cohomology ring of the Grassmannian, once the indeterminates (c_i) are interpreted as the Chern classes of the tautological rank r bundle and the S_i's as the Chern classes of the universal quotient bundle over the Grassmannian $G(r,n)$ (or the Seegre classes of the tautological bundle). We denote $\mathbf{S}_{r,n} = (\pi_{r,n}(S_j))_{j\in\mathbb{Z}} = (1 = S_0, S_1, \ldots, S_{n-r})$ and

$$\mathbf{S}_{r,n}(z) = 1 + S_1 z + \cdots + S_{n-r} z^r \in B_{r,n}[z]$$

Let $\mathbf{x}_{r,n} = (x_1, x_2, \ldots)$ be the sequence of elements of $B_{r,n}$ defined via the equality

$$\exp\left(\sum_{i\geq 1} x_i z^i\right) = \mathbf{S}_{r,n}(z) = 1 + S_1 z + \cdots + S_{n-r} z^{n-r}. \tag{2.4}$$

In other words

$$\mathbf{x}_{r,n}(z) := \sum_{i\geq 1} x_i z^i = \log(1 + S_1 z + \cdots + S_{n-r} z^{n-r}) = \sum_{j\geq 1} \frac{1}{j}\Big(S_1 z + \cdots + S_{n-r} z^{n-r}\Big)^{j-1}.$$

Let $p_i(\mathbf{t}_r) := \sum_{j=1}^{r} t_j^i$. To each partition $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$ we associate $S_{\boldsymbol{\lambda}} \in \mathcal{P}_{r,n}$ defined as follows

$$\sum_{\boldsymbol{\lambda}\in\mathcal{P}_r} S_{\boldsymbol{\lambda}} \cdot s_{\boldsymbol{\lambda}}(\mathbf{t}_r) = \exp(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r)). \tag{2.5}$$

It is well known that $(S_{\boldsymbol{\lambda}} \,|\, \boldsymbol{\lambda} \in \mathcal{P}_{r,n})$ is a $\mathbb{Q}$-basis of $B_{r,n}$:

$$B_{r,n} := \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}_{r,n}} \mathbb{Q}\cdot S_{\boldsymbol{\lambda}}$$

so that $\exp\left(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r)\right)$ is a generating function of the basis $S_{\boldsymbol{\lambda}}$. This gives us the second alluded natural isomorphisms $\beta : \mathbb{P}_{r,n} \to B_{r,n}$, mapping $\boldsymbol{\lambda}$ to $S_{\boldsymbol{\lambda}}$.

Moreover, the equality

$$\exp\left(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r)\right) = \exp\left(\sum_{\lambda_1\geq 1} x_i t_1^{\lambda_1}\right)\cdots\exp\left(\sum_{\lambda_r\geq 1} x_i \mathbf{t}_r^{\lambda_r}\right), \tag{2.6}$$

along with (2.5), implies the celebrated Jacobi–Trudy formula

$$S_{\boldsymbol{\lambda}} = \det(S_{\lambda_j - j + i})$$

obtained by equating the coefficients of $s_{\boldsymbol{\lambda}}(\mathbf{t}_r)$ on both members of (2.6). Since the S_i's generate $B_{r,n}$ as a ring, the structure of $B_{r,n}$ as a $\mathbb{Q}$-algebra is determined once one knows all the products

$$S_i \cdot S_{\boldsymbol{\lambda}},$$

as linear combination of the basis $(S_{\boldsymbol{\lambda}})$. All such products are ruled by the obvious equality

$$\exp\left(\sum_{i\geq 1} x_i z^i\right)\exp\left(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r)\right) = \exp\left(\sum_{i\geq 1} x_i p_i(z,\mathbf{t}_r)\right) = \sum_{\boldsymbol{\lambda}\in\mathcal{P}_{r,n}} S_{\boldsymbol{\lambda}}\cdot s_{\boldsymbol{\lambda}}(z,\mathbf{t}_r). \tag{2.7}$$

In other words $S_i \cdot S_{\boldsymbol{\lambda}}$ is the coefficient of $z^i s_{\boldsymbol{\lambda}}(\mathbf{t}_r)$ in the most right hand side of (2.7) or, more practically, the coefficient of $z^i \mathbf{t}_r^{\boldsymbol{\lambda}}$ of

$$\Delta_0(\mathbf{t}_r)\cdot \exp\left(\sum_{i\geq 1} x_i p_i(z,\mathbf{t}_r)\right),$$

where $\mathbf{t}_r^{\boldsymbol{\lambda}} := t_1^{r-1+\lambda_1} t_2^{r-2+\lambda_2}\cdots t_r^{\lambda_r}$. The map $\beta : \mathbb{P}_{r,n} \to B_{r,n}$ maps any $\boldsymbol{\lambda}\in\mathbb{P}_{r,n}$ to its bosonic expression $S_{\boldsymbol{\lambda}}$.

2.7. If both $r = n = \infty$, one is then concerned with the polynomial ring $B := \mathbb{Q}[x_1, x_2, \ldots,]$ in infinitely many indeterminates. It corresponds to the cohomology ring of the Universal Grassmann Manifold (UGM), an *ind-variety* which Sato constructs in [32] as a direct limit of finite dimensional Grassmannians. As the UGM is a direct limit of finite dimensional Grassmannians, its cohomology is the inverse limit of the cohomologies of the finite dimensional ones, and coincides with B_∞.

3. Generating functions of linear maps: a few special cases

Recall that the purpose of this paper is to provide, inspired by the DJKM vertex operator representation of the Lie algebra $gl(\infty)$, explicit descriptions of maps $B_{r,n} \to B_{r+s.n}$ induced by elementary endomorphisms of the exterior algebra $\bigwedge V_n$, where $V_n = \mathbb{Q}[X]/(X^n)$.

It seems convenient to anticipate a few special cases which, though elementary, seems interesting in their own.

a) If $r = 1$, $B_{1,n} = \dfrac{\mathbb{Q}[c_1]}{(c_1^n)}$. It is obviously isomorphic to $V_n = \mathbb{Q}[X]/(X^n)$. We think of V_n as a $B_{1,n}$-module such that $c_1 X^i = X^{i+1} + (X^n)$.
Any matrix $A \in \mathrm{End}_{\mathbb{Q}}(V_n)$ can be seen as linear combination of $E_{ij} := X^i \otimes \partial^j$ (notation as in 1.3). Let $\mathcal{E}_{1,n} = \sum_{i,j=0}^n X^i z^i \otimes \partial^j w^{-j}$ which can be seen as the $\mathbb{Q}[z, w^{-1}]$ valued matrix

$$\begin{pmatrix} 1 & w^{-1} & \cdots & w^{-n+1} \\ z & zw^{-1} & \cdots & zw^{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ z^{n-1} & z^{n-1}w^{-1} & \cdots & z^{n-1}w^{-n+1} \end{pmatrix}$$

If X^k is any basis element of V_n one has

$$\mathcal{E}_{1,n}(z,w)X^k = \sum_{i=1}^k X^i z^i \delta_k^j = \sum_{i=1}^k c_1^i X^0 z^i w^{-k} = \frac{w^{-k}}{1-c_1 z} X^0$$

The expression can be further improved if, instead of acting just on X^k, we act on all the X^k at once $(1 \le k \le n)$, using a generating series, which provides the elegant formula below:

$$\mathcal{E}_{1,n}(z,w,t)X^0 := \mathcal{E}_{1,n}(z,w)\sum X^k t^k = \frac{1}{1-c_1z}\left(\frac{1-\dfrac{t^n}{w^n}}{1-\dfrac{t}{w}}\right)X^0 = \frac{1-\dfrac{t^n}{w^n}}{(1-c_1z)\left(1-\dfrac{t}{w}\right)}X^0 \tag{3.1}$$

Since working over $\mathbb{Q}$, we have the exponential form:

$$\begin{aligned}\mathcal{E}_{1,n}(z,w,t) &= \left(1-\frac{t^n}{w^n}\right)\exp\left(\sum_{i\ge1}\frac{1}{i}\left(c_1^n z^n + \frac{t^i}{w^i}\right)\right)\\ &= \exp\left(\sum_{i\ge1}\frac{1}{i}\left(c_1^n z^n + \frac{t^i}{w^i} - \frac{t^{ni}}{w^{ni}}\right)\right).\end{aligned} \tag{3.2}$$

We call (3.2) the *generating structural series* governing the multiplication of a matrix by a vector of V_n, identified with a polynomial of degree $\le n-1$. In other words

$$E_{ij}X^k = \Big([z^i w^{-j} t^k]\mathcal{E}_{1,n}(z,w,t)\Big)X^0.$$

Expression (3.2) is a particular case of the general formula (6.4) for $r = h = k = 1$ and $n < \infty$. If $n = \infty$ one has

$$\mathcal{E}_1(z,w,t) := \mathcal{E}_{1,\infty}(z,w,t) = \exp\left(\sum_{i\ge1}\frac{1}{i}\left(c_1^n z^n + \frac{t^i}{w^i}\right)\right),$$

another case taken into account by the same formula (6.4).

b) The second example we propose points to the relationship of our subject to vertex operators and the classical theory of symmetric functions at once. For sake of simplicity, let $n = \infty$, so working in the ring $B_r = \mathbb{Q}[c_1, \dots, c_r]$. The task is to describe the action of wedging a vector against a polynomial of B_r. To do so, one recalls that the basis element $S_{\boldsymbol{\lambda}}$ of B_r $(\boldsymbol{\lambda} \in \mathcal{P}_r)$ does correspond to the basis element $X^{r-1+\lambda_1} \wedge \cdots \wedge X^{\lambda_r} \in \bigwedge^r V$. We are so interested in computing $X^i \wedge S_{\boldsymbol{\lambda}}$, defined by

$$(X^i \wedge S_{\boldsymbol{\lambda}})\mathbf{X}^{r+1}(0) := X^i \wedge (S_{\boldsymbol{\lambda}}\mathbf{X}^r(0)) := X^i \wedge \mathbf{X}^r(\boldsymbol{\lambda}) = X^i \wedge X^{r-1+\lambda_1} \wedge \cdots \wedge X^{\lambda_r}. \tag{3.3}$$

Let us now stipulate the existence, established in Section 4.1, of an algebra homomorphism

$$\sigma_+(z) : \bigwedge V \to \bigwedge V[\![z]\!]$$

such that

$$\sigma_+(z)X^0 = \sum_{i\ge0} X^i z^i = \frac{X^0}{1-Xz}.$$

Denote by $\overline{\sigma}_+(z)$ the inverse of $\sigma_+(z)$ in the power series algebra $\mathrm{End}_{\mathbb{Q}}(\bigwedge V)[\![z]\!]$: it is a $\wedge$-homomorphism as well, i.e. $\overline{\sigma}_+(z)(u \wedge v) = \overline{\sigma}_+(z)u \wedge \overline{\sigma}_+(z)v$, and is the unique such that $\overline{\sigma}_+(z)u = u - (Xu)z$, for all $u \in V$. Therefore

$$\sigma_+(z)X^0 \wedge \left(X^{r-1+\lambda_1} \wedge \cdots \wedge X^{\lambda_r}\right) = \sigma_+(z)\Big(X^0 \wedge \overline{\sigma}_+(z)(X^{r-1+\lambda_1} \wedge \cdots \wedge X^{\lambda_r})\Big)$$
$$= \sigma_+(z)(X^0 \wedge \overline{\sigma}_+(z)X^{r-1+\lambda_1} \wedge \cdots \wedge \overline{\sigma}_+(z)X^{\lambda_r} \tag{3.4}$$

Formula (3.4) can be recast in two different ways. The former consists in considering the unique $\wedge$-algebra homomorphism $\overline{\sigma}_-(z) : \bigwedge V \to \bigwedge V[\![z^{-1}]\!]$ (like in [16,17]) such that

$$\overline{\sigma}_-(z)_{|V} = 1 - X^{-1}z^{-1},$$

where by X^{-1} one means the (locally) nilpotent endomorphism of V_n defined by $X^{-1}X^j = X^{j-1}$ if $j \geq 1$ and 0 otherwise. Then (3.4) can be written as

$$\sigma_+(z)X^0 \wedge X^{r-1+\lambda_1} \wedge \cdots \wedge X^{\lambda_r} = z^r\sigma_+(z)\overline{\sigma}_-(z)(X^{r+\lambda_1} \wedge \cdots X^{1+\lambda_r} \wedge X^0)$$
$$= z^r\Gamma_r(z)S_{(\lambda_1,\cdots,\lambda_r)}\mathbf{X}^{r+1}(0)$$

where one sets

$$(\Gamma_r(z)S_{(\lambda_1,\cdots,\lambda_r)})X^{r+1} \wedge \cdots \wedge X^1 \wedge X^0 := \sigma_+(z)\overline{\sigma}_-(z)\mathbf{X}^{r+1}(\lambda_1, \cdots, \lambda_r).$$

It is not that difficult to see, as shown in [14,15], that

$$\Gamma(z) := \lim_{r\to\infty} \Gamma_r(z) = \exp\left(\sum_{i\geq 1} x_i z^i\right) \exp\left(\sum_{i\geq 1} \frac{1}{iz^i}\frac{\partial}{dx_i}\right)$$

which is precisely the expression of one of the Bernstein (vertex) operators occurring in the description of the Boson-Fermion correspondence ([25,20] or [29, p. 96]). See also Rosas [31] for further combinatorial aspects of these vertex operators.

This paper takes however an alternative approach, which brings the subject back in the realm of the classical theory of symmetric functions, where vertex operators are clues of the presence of Vandermonde determinants. Let $\sigma_+(\mathbf{t}_r) = \sigma_+(t_1)\cdots\sigma_+(t_r)$ (a kind of *multivariate* Hasse-Schmidt derivation on the exterior algebra, see also [1]). One of our outputs is the remarkable equality (Proposition 4.2)

$$\sum_{\boldsymbol{\lambda}\in\mathcal{P}_2} \mathbf{X}^r(\boldsymbol{\lambda})s_{\boldsymbol{\lambda}}(\mathbf{t}_r) = \sigma_+(\mathbf{t}_r)\left(X^{r-1} \wedge \cdots \wedge X^0\right)$$

This time one wedges $\sigma_+(z)X^0$ against not just one basis element of $\bigwedge^r V$, but rather the generating series of the basis of $\bigwedge^r V$.

In this case we have that since

$$\sigma_+(z)X^0 \wedge \sigma_+(\mathbf{t}_r)\left[X^{r-1} \wedge \cdots \wedge X^0\right]$$

vanishes whenever $z = t_i$ for some $1 \leq i \leq r$, it can be written as

$$\prod_{i=1}^r (z-t_i)\sigma_+(z, t_1, \cdots, t_r)\left[X^r \wedge X^{r-1} \wedge \cdots \wedge X^1 \wedge X^0\right].$$

From a "bosonic" point of view, i.e. with B_r coefficients, it can be seen as

$$z^r\prod_{i=1}^r\left(1-\frac{t_i}{z}\right)\exp\left(\sum_{i\geq 1} x_ip_i(z,\mathbf{t}_r)\right) = z^r\exp\left(\sum_{i\geq 1}\frac{p_i(\mathbf{t}_r)}{z^i}\right)\exp\left(\sum_{i\geq 1} x_ip_i(z,\mathbf{t}_r)\right), \tag{3.5}$$

which is one more particular case of the main formula (6.4), putting $h = 1$ and $k = 0$. In passing, we also remark that

$$z^r \exp(\sum_{i\geq 1} x_i z^i)\Gamma_r(z)\exp\left(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r)\right) = z^r \exp(\sum_{i\geq 1} x_i z^i)\exp\left(\sum_{i\geq 1}\left(x_i + \frac{1}{z^i}\right)p_i(\mathbf{t}_r)\right)$$

whence

$$\Gamma_r(z)\exp\left(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r)\right) = \exp\left(\sum_{i\geq 1}\left(x_i + \frac{1}{z^i}\right)p_i(\mathbf{t}_r)\right). \tag{3.6}$$

For $r = \infty$, formula (3.6) is nothing but the sum over all the partitions of the right hand side of the formula stated in [30, Theorem 3.4]. It is a consequence of the commutation rule (similar to those proven within a similar framework see [2, Section 4] or [17, Section 8]).

$$\overline{\sigma}_-(z)\sigma_+(\mathbf{t}_r) = \left(\sum_{i\geq 1}\frac{p_i(\mathbf{t}_r)}{z^i}\right)\sigma_+(\mathbf{t}_r)\overline{\sigma}_-(z),$$

occurring also in a more general categorical context like in [10, formula (12), (13), (14)]. In sum, wedging a generating function of the basis of $\bigwedge^h V$ against a generating function of the basis of B_r, in the sense of (3.3), is the same as multiplying two exponentials.

4. The $B_{r,n}$–module structure of the exterior algebra

To generalize the examples of Section 3, we recall the natural module structure of the exterior algebra of a countably infinite dimensional $\mathbb{Q}$–vector space over the ring $B_{r,n}$.

4.1. Let $gl_n(\mathbb{Q}) := (\mathbb{Q}^{n\times n}, [\cdot,\cdot])$ be the Lie algebra of all the $\mathbb{Q}$-valued $n \times n$ square matrices with respect to the usual commutator. We understand it as

$$gl_n(\mathbb{Q}) := \bigoplus_{0\leq i,j<n} \mathbb{Q}\cdot E_{ij} \tag{4.1}$$

where, as customarily, we denote by E_{ij} the elementary $n \times n$ matrix whose entries are all zero but 1 in position (i, j). Therefore, due to (4.1), if $n = \infty$, by $gl_\infty(\mathbb{Q})$ we understand the Lie algebra of all the matrices of infinite size whose entries are all zero but finitely many.

The exterior algebra $\bigwedge V_n = \bigoplus_{r\geq 0}\bigwedge^r V_n$ is a $gl_n(\mathbb{Q})$-module via the *trace representation*. The trace $\mathrm{tr}(A)$ of $A \in gl_n(\mathbb{Q})$ is the unique derivation of the exterior algebra such that it restrict to the multiplication by A to the first degree of $\bigwedge V_n$, i.e.:

$$\begin{cases}\mathrm{tr}(A)(u\wedge v) &= \mathrm{tr}(A)u\wedge v + u\wedge \mathrm{tr}(A)v\\ \mathrm{tr}(A)u &= Au, \quad \forall u \in V_n = \bigwedge^1 V_n.\end{cases}$$

For all $j \geq 0$, let $X^j : V_n \to V_n$ denote the endomorphism given by multiplication by X^j, which for finite r can be identified with the elementary $n \times n$ nilpotent matrix of rank $n - j$, and let us denote by $\delta(X^j)$ its trace. For any indeterminate z, let:

$$\sigma_+(z) = \sum_{i\geq 0}\sigma_i z^i := \exp\left(\sum_{j\geq 1}\frac{\delta(X^j)}{j}z^j\right) : \bigwedge V_n \to \left(\bigwedge V_n\right)[\![z]\!].$$

Notice that $\mathbf{X}(z) = \sigma_+(z)X^0$. Because it is the exponential of a derivation, it is clearly a $\bigwedge$-algebra homomorphism, namely $\sigma_+(z)(u \wedge v) = \sigma_+(z)u \wedge \sigma_+(z)v$. We also use the notation

$$\overline{\sigma}_+(z) = \sum_{i\geq 0}(-1)^i\overline{\sigma}_i z^i = \exp\left(-\sum_{j\geq 1}\frac{\delta(X^j)}{j}z^j\right) : \bigwedge V_n \to \left(\bigwedge V_n\right)[\![z]\!]$$

which is clearly the inverse of $\sigma_+(z)$ in the algebra $\mathrm{End}_{\mathbb{Q}}(\bigwedge V_n)[\![z]\!]$, i.e.

$$\sigma_+(z)\overline{\sigma}_+(z) = \overline{\sigma}_+(z)\sigma_+(z) = 1,$$

implying the following Cayley-Hamilton relations (see e.g. [18]):

$$\sigma_+(z)(\overline{\sigma}_+(z)u \wedge v) = u \wedge \sigma_+(z)v, \tag{4.2}$$

$$\overline{\sigma}_+(z)(\sigma_+(z)u \wedge v) = u \wedge \overline{\sigma}_+(z)v. \tag{4.3}$$

We also denote

$$\sigma_+(\mathbf{t}_r) = \prod_{i=1}^{r}\sigma_+(t_i) = \sigma_+(t_1)\cdots\sigma_+(t_r),$$

the product in $\mathrm{End}_{\mathbb{Q}}(\bigwedge V_n)[\![\mathbf{t}_r]\!]$ regarded as an algebra over $\mathrm{End}_{\mathbb{Q}}(\bigwedge V_n)$. We have:

4.2. Proposition. *For all $r \geq 1$, the following equality holds:*

$$\sigma_+(t_1)X^0 \wedge \cdots \wedge \sigma_+(t_r)X^0 = \Delta_0(\mathbf{t}_r)\sigma_+(\mathbf{t}_r)\mathbf{X}^r(0). \tag{4.4}$$

Proof. For $r = 1$, we set $\Delta_0(t_1) = 1$, so that formula (4.4) holds in this case. For $r = 2$ we have, using the Cayley Hamilton relations:

$$\begin{aligned}\sigma_+(t_1)X^0 \wedge \sigma_+(t_2)X^0 &= \sigma_+(t_1,t_2)\big(\overline{\sigma}_+(t_2)X^0 \wedge \overline{\sigma}_+(t_1)X^0\big)\\ &= \sigma_+(t_1,t_2)\big[(X^0 - t_1X^1) \wedge (X^0 - t_2X^1)\big]\\ &= (t_1 - t_2)\sigma_+(t_1,t_2)X^1 \wedge X^0\end{aligned} \tag{4.5}$$

and thus the property holds for $r = 2$. Then we argue by induction. Let us suppose that

$$\sigma_+(t_1)X^0 \wedge \cdots \wedge \sigma_+(t_{r-1})X^0 = \Delta_0(\mathbf{t}_{r-1})\sigma_+(\mathbf{t}_r)\mathbf{X}^{r-1}(0).$$

Then, on one hand $\sigma_+(t_1)X^0 \wedge \cdots \wedge \sigma_+(t_{r-1})X^0 \wedge \sigma_+(t_r)X^0$ is clearly a multiple of $\prod_{i<j}(t_i - t_j)$. On the other hand, using induction:

$$\sigma_+(t_1)X^0 \wedge \cdots \wedge \sigma_+(t_{r-1})X^0 \wedge \sigma_+(t_r)X^0 = \Delta_0(\mathbf{t}_{r-1})\sigma_+(\mathbf{t}_{r-1})\mathbf{X}^{r-1} \wedge \sigma_+(t_r)X^0$$

Invoking the Cayley-Hamilton relations (4.2) we get, from the last side:

$$\begin{aligned}&\Delta_0(\mathbf{t}_{r-1})\sigma_+(\mathbf{t}_{r-1})\mathbf{X}^{r-1} \wedge \sigma_+(t_r)X^0\\ &= \Delta_0(\mathbf{t}_{r-1})\sigma_+(\mathbf{t}_r)\overline{\sigma}_+(t_r)\mathbf{X}^{r-1}(0) \wedge \overline{\sigma}_+(\mathbf{t}_r \setminus t_r)X^0\\ &= \sum_{i=0}^{r-1}(-1)^i t^i\mathbf{X}^{r-1}(1^i) \wedge (X^0 - e_1(\mathbf{t}_{r-1}X^1 + \cdots + (-1)^{r-1}e_{r-1}(\mathbf{t}_{r-1})X^{r-1}.\end{aligned} \tag{4.6}$$

The above expression is clearly a multiple of $\mathbf{X}^r(0)$, which is divisible by $\prod_{i=1}^r (t_i - t_r)$. To determine its coefficient one looks at that of $t_1 \cdots t_{r-1} = e_{r-1}(\mathbf{t}_{r-1})$ in (4.6). In all cases one gets a summand of the form:

$$(-1)^{r-1} t_1 \cdots t_{r-1} \mathbf{X}^{r-1}(0) \wedge X^{r-1}. \tag{4.7}$$

Now:

$$\mathbf{X}^{r-1}(0) \wedge X^{r-1} = (-1)^{r-1} X^{r-1} \wedge \mathbf{X}^{r-1}(0) = (-1)^{r-1} \mathbf{X}^r(0).$$

Therefore

$$(-1)^{r-1} e_{r-1}(\mathbf{t}_{r-1}) = (-1)^{r-1} t_1 \cdots t_{r-1} \mathbf{X}^{r-1}(0) \wedge X^{r-1} = t_1 \cdots t_{r-1} \mathbf{X}^r(0),$$

proving that (4.6) is equal to $\Delta_0(\mathbf{t}_r)\mathbf{X}^r(0)$, as claimed. □

4.3. Proposition. *The equality*

$$x_i \cdot \mathbf{X}^r(\boldsymbol{\lambda}) := \delta(X^i)\mathbf{X}^r(\boldsymbol{\lambda}), \qquad \boldsymbol{\lambda} \in \mathcal{P}_{r,n}$$

makes $\bigwedge^r V_n$ *into a free* $B_{r,n}$*-module of rank* 1 *generated by* $\mathbf{X}^r(0) = X^{r-1} \wedge X^{r-2} \wedge \cdots \wedge X^0$*, such that* $\mathbf{X}^r(\boldsymbol{\lambda}) = S_{\boldsymbol{\lambda}} \cdot \mathbf{X}^r(0)$.

Proof. Since S_i is a polynomial in $\mathbf{x}_{r,n} = (x_1, x_2, \ldots)$, it is enough to show that

$$S_i(S_j u) = (S_i S_j)u, \qquad (\forall u \in \bigwedge^r V_n).$$

We will check it through generating functions:

$$\exp\left(\sum_{i\geq 1} x_i z^i\right) \exp\left(\sum_{i\geq 1} x_i w^i\right) \mathbf{X}^r(\boldsymbol{\lambda}) = \exp\left(\sum_{i\geq 1} x_i z^i\right) \exp\left(\sum_{j\geq 1} \frac{\delta(X^j)}{j} w^j\right) \mathbf{X}^r(\boldsymbol{\lambda})$$
$$= \exp\left(\sum_{i\geq 1} \frac{\delta(X^i)}{i} z^i\right) \exp\left(\sum_{j\geq 1} \frac{\delta(X^j)}{j} w^j\right) \mathbf{X}^r(\boldsymbol{\lambda}) = \exp\left(\sum_{i\geq 1} \frac{\delta(X^i)}{i} (z^i + w^i)\right) \mathbf{X}^r(0)$$
$$= \exp\left(\sum_{i\geq 1} x_i (z^i + w^i)\right) \mathbf{X}^r(0) = \left(\exp(\sum_{i\geq 1} x_i z^i) \exp(\sum_{i\geq 1} x_i w^i)\right) \mathbf{X}^r(0).$$

The assignment $x_i \mapsto \frac{\delta(X^i)}{i}\mathbf{X}^r(0)$ is clearly injective. To prove that it is surjective, it is enough to show that $\mathbf{X}^r(\boldsymbol{\lambda}) = S_{\boldsymbol{\lambda}} \cdot \mathbf{X}^r(0)$ and we do that all at once by using generating functions:

$$\sum_{\boldsymbol{\lambda} \in \mathcal{P}_{r,n}} \mathbf{X}^r(\boldsymbol{\lambda}) s_{\boldsymbol{\lambda}}(\mathbf{t}_r) = \sigma_+(\mathbf{t}_r)\mathbf{X}^r(0) = \exp\left(\sum_{i\geq 1} \frac{\delta(X^i)}{i} p_i(\mathbf{t}_r)\right) \mathbf{X}^r(0)$$
$$= \exp\left(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r)\right) \mathbf{X}^r(0) = \sum_{\boldsymbol{\lambda} \in \mathcal{P}_{r,n}} S_{\boldsymbol{\lambda}} \mathbf{X}^r(0) s_{\boldsymbol{\lambda}}(\mathbf{t}_r).$$

More explicitly, we have the equality:

$$\mathbf{X}^r(\boldsymbol{\lambda}) = X^{r-1+\lambda_1} \wedge \cdots \wedge X^{\lambda_r} = \begin{vmatrix} S_{\lambda_1} & S_{\lambda_2-1} & \cdots & S_{\lambda_r-r+1} \\ S_{\lambda_1+1} & S_{\lambda_2} & \cdots & S_{\lambda_r-r+2} \\ \vdots & \vdots & \ddots & \vdots \\ S_{\lambda_1+r-1} & S_{\lambda_2+r-1} & \cdots & S_{\lambda_r} \end{vmatrix} \cdot X^{r-1} \wedge X^{r-2} \wedge \cdots \wedge X^0, \qquad (4.8)$$

which can also be seen as the action of the word $X^{r-1+\lambda_1} \wedge \cdots \wedge X^{\lambda_r}$ on $B_0 = \mathbb{Q}$! □

4.4. Corollary. *The isomorphism $B_{r,n} \mapsto \bigwedge^r V_n$ makes $B_{r,n}$ into a representation of the Lie algebra $gl_n(\mathbb{Q})$.*

Proof. Indeed, the action $\circledast$ defined by

$$(A \circledast P)\mathbf{X}^r(0) = \mathrm{tr}(A)(P \cdot \mathbf{X}^r(0)), \qquad (4.9)$$

satisfies $[A, B] \circledast P = A \circledast (B \circledast P) - B \circledast (A \circledast P)$ because $\mathrm{tr}([A, B]) = [\mathrm{tr}(A), \mathrm{tr}(B)]$, as well known. □

4.5. We recall the formalism of two-rows-determinant introduced in [2] to deal with contractions. Let $(a_0, \ldots, a_{r-1})$ be any set of indeterminates (or rational numbers) and $(v_0, v_1, \ldots, v_{r-1}) \in V_n^r$. Then we let

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_{r-1} \\ v_0 & v_1 & \ldots & v_{r-1} \end{vmatrix} = \sum_{j=0}^{r-1} (-1)^j a_j \mathbf{v}_{I\setminus\{j\}} \qquad (4.10)$$

where $I = \{0, \ldots, r-1\}$ and $\mathbf{v}_{I\setminus\{j\}}$ is the wedge product of $v_0, v_1, \ldots, v_{r-1}$ with v_j removed. For example

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ v_0 & v_1 & v_2 \end{vmatrix} = a_0\, v_1 \wedge v_2 - a_1\, v_0 \wedge v_2 + a_2\, v_0 \wedge v_1.$$

4.6. Remark. Notice that the output of expression (4.10) changes sign whenever two columns are exchanged. Hence, the linearity with respect to the first entry of the second row implies the linearity with respect to all the other arguments.

4.7. The *contraction* of a vector of $\bigwedge^r V_n$ against a linear form α can be defined using diagram (4.10) by declaring:

$$\alpha \lrcorner v := \begin{vmatrix} \alpha(v) \\ v \end{vmatrix} = \alpha(v)$$

and, for $r \geq 2$

$$\alpha \lrcorner (v_1 \wedge v_2 \wedge \cdots \wedge v_r) = \begin{vmatrix} \alpha(v_1) & \cdots & \alpha(v_r) \\ v_1 & \cdots & v_r \end{vmatrix}. \qquad (4.11)$$

4.8. Proposition. *Let $r \leq n-1$. For each $(a_1, \ldots, a_r)$ and a partition $\lambda_1 \geq \cdots \geq \lambda_r$, the following formula holds*

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_r \\ X^{r-1+\lambda_1} & X^{r-2+\lambda_2} & \ldots & X^{\lambda_r} \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & \cdots & a_r \\ S_{\lambda_1+1} & S_{\lambda_2} & \cdots & S_{\lambda_r+r-2} \\ \vdots & \vdots & \ddots & \vdots \\ S_{\lambda_1+r-1} & S_{\lambda_2+r-2} & \cdots & S_{\lambda_r} \end{vmatrix} X^{r-2} \wedge \cdots \wedge X^0 \qquad (4.12)$$

Proof. It is a consequence of (4.8). Given a partition $\boldsymbol{\lambda} = (\lambda_1 \geq \cdots \geq \lambda_r)$, let $\boldsymbol{\lambda}^{(j)}$ be the partition of length at most $r-1$ obtained by removing λ_j from $\boldsymbol{\lambda}$. Then

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_r \\ X^{r-1+\lambda_1} & X^{r-2+\lambda_2} & \ldots & X^{\lambda_r} \end{vmatrix} = \sum_{i=1}^{r}(-1)^{j-1}a_j\mathbf{X}^{r-1}(\boldsymbol{\lambda}^{(j)})$$

$$= (\sum_{i=1}^{r}(-1)^{j-1}a_jS_{\boldsymbol{\lambda}^{(j)}})\mathbf{X}^{r-1}(0)$$

which is precisely the expansion of the determinant occurring on the right hand side of (4.12). □

4.9. Corollary. *Let* $\mathbf{w}_k := (w_1, \ldots, w_k)$ *be a* k*-tuple of indeterminates and keep the same notation as in* (2.1)*. Consider the map*

$$\partial(\mathbf{w}_k^{-1})\lrcorner : \bigwedge^r V_n \to \bigwedge^{r-k} V_n[\![w_1^{-1}, \ldots, w_k^{-1}]\!]$$

defined by

$$\partial(\mathbf{w}_k^{-1})\lrcorner u = \partial(\mathbf{w}_{k-1}^{-1})\lrcorner(\partial(w_1^{-1})\lrcorner u).$$

Then

$$\boldsymbol{\partial}(\mathbf{w}_k^{-1})\lrcorner\mathbf{X}^r(\boldsymbol{\lambda}) := \boldsymbol{\partial}(w_k^{-1})\lrcorner\left(\boldsymbol{\partial}(w_{k-1}^{-1})\lrcorner\cdots\lrcorner\partial(w_1^{-1})\lrcorner\mathbf{X}^r(\boldsymbol{\lambda})\right)$$

$$= \begin{vmatrix} w_1^{-r+1-\lambda_1} & w_1^{-r+2-\lambda_2} & \cdots & w_1^{-\lambda_r} \\ \vdots & \vdots & \ddots & \vdots \\ w_k^{-r+1-\lambda_1} & w_k^{-r+2-\lambda_2} & \cdots & w_k^{-\lambda_r} \\ X^{r-1+\lambda_1} & X^{r-2+\lambda_2} & \ldots & X^{\lambda_r} \end{vmatrix}$$

$$= w_1^{-r+1}w_2^{-r+2}\cdots w_k^{-r+k}\begin{vmatrix} w_1^{-\lambda_1} & w_1^{1-\lambda_2} & w_1^{2-\lambda_3} & \cdots & w_1^{r-1+\lambda_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_k^{1-k-\lambda_1} & w_k^{2-k-\lambda_2} & w_k^{-k+3-\lambda_3} & \cdots & w_k^{r-k+\lambda_r} \\ S_{\lambda_1+k} & S_{\lambda_2+k-1} & S_{\lambda_3+k-2} & \cdots & S_{\lambda_r+k+r-3} \\ \vdots & \vdots & \ddots & \vdots & \\ S_{\lambda_1+r-1} & S_{\lambda_2+r-2} & S_{\lambda_2-r+1} & \cdots & S_{\lambda_r} \end{vmatrix} X^{r-1-k}\wedge\cdots\wedge X^0.$$

Proof. One first notice that $\boldsymbol{\partial}(w_i^{-1})X^j = w_i^{-j}$ and then merges it in the rephrasing (4.11) of the contraction, eventually using Proposition 4.8. □

5. The action of the Clifford algebra

Let $V_n^* := \bigoplus_{j=0}^{n-1}\mathbb{Q}\cdot\partial^j$ be the restricted dual of V_n (which, for $n < \infty$ is the usual dual). Let us consider the canonical Clifford Algebra $\mathcal{C}(V)$ supported on $V \oplus V^*$, defined by the canonical non-degenerate bilinear form

$$\langle u \oplus \alpha, v \oplus \beta\rangle = \beta(u) + \alpha(v).$$

Write $\boldsymbol{\partial}^r(\boldsymbol{\mu}) = \partial^{r-1+\mu_1} \wedge \cdots \partial^{\mu_r} \in \bigwedge^r V_n^*$. An element (a word) $\mathbf{X}^h(\boldsymbol{\lambda})\boldsymbol{\partial}^k(\boldsymbol{\mu})$ of the Clifford algebra $\mathcal{C}(V)$, in *normal form*, acts on the exterior algebra $\bigwedge V$ by wedging and contracting:

$$\mathbf{X}^h(\boldsymbol{\lambda})\boldsymbol{\partial}^k(\boldsymbol{\mu})(u) := \mathbf{X}^h(\boldsymbol{\lambda}) \wedge (\boldsymbol{\partial}^k(\boldsymbol{\mu}) \lrcorner u) \qquad (\forall u \in \bigwedge V). \tag{5.1}$$

If an element is not in normal form, one uses the well known and easy to check relations:

$$\left[X^i, \partial^j\right]_+ := X^i\partial^j + \partial^j X^i = \delta_{ij}\mathbf{1}_{\bigwedge V}.$$

The X^i and ∂^j plays the same role as the generators of the free-fermions Clifford algebra, usually denoted by ψ_i, ψ_i^* (see e.g. [19] or [33, Section 1]). Notice that if $n < \infty$, the Clifford algebra is naturally isomorphic to the algebra of the endomorphisms $gl(\bigwedge V_n)$ of $\bigwedge V_n$, isomorphic to $\bigwedge V_n \otimes \bigwedge V_n^*$. If $n = \infty$ the action is different, because the image of an element of the form $u \otimes 1$ is in general an infinite linear combination of the basis elements of $\bigwedge V_n$, contrarily to the image of any element of $u \otimes \alpha$, with $\alpha \neq 1$.

5.1. Let $\mathbf{z}_h = (z_1, \ldots, z_h)$ and $\mathbf{w}_k^{-1} = (w_1^{-1}, \ldots, w_k^{-1})$ be, respectively, an h-tuple and a k-tuple of indeterminates. Let

$$\mathcal{E}_{(r,h,k);n}(\mathbf{z}_h, \mathbf{w}_k^{-1}) := \sum_{(\boldsymbol{\lambda},\boldsymbol{\mu}) \in \mathcal{P}_{h,n} \times \mathcal{P}_{k,n}} \mathbf{X}^h(\boldsymbol{\lambda}) \otimes \boldsymbol{\partial}^k(\boldsymbol{\mu}) s_{\boldsymbol{\lambda}}(\mathbf{z}_h) s_{\boldsymbol{\mu}}(\mathbf{w}_k^{-1}).$$

Thus, for each $r, h, k \geq 0$, the Clifford algebra $\mathcal{C}(V)$ induces vector space homomorphisms

$$B_r \longrightarrow B_{r+h-k}[\![\mathbf{z}_h, \mathbf{w}_k^{-1}]\!]$$

defined by

$$(\mathcal{E}_{h,k,n}(\mathbf{z}_h, \mathbf{w}_k^{-1}))\mathbf{X}^r(0) = \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}} \mathbf{X}^h(\boldsymbol{\lambda}) \wedge \left(\boldsymbol{\partial}^k(\boldsymbol{\mu}) \lrcorner \mathbf{X}^r(0)\right) s_{\boldsymbol{\lambda}}(\mathbf{z}_k) s_{\boldsymbol{\mu}}(\mathbf{w}_k^{-1})$$

which is clearly zero if $r + h - k < 0$. This enables us to define a graded linear homomorphims

$$\star : B_{r,n} \to B_{r+h-k,n} \tag{5.2}$$

through the equality:

$$\mathbf{X}^h(\boldsymbol{\lambda})\boldsymbol{\partial}^k(\boldsymbol{\mu}) \star P := \left[s_{\boldsymbol{\lambda}}(\mathbf{z}_h) s_{\boldsymbol{\mu}}(\mathbf{w}_k^{-1})\right] \left(\mathcal{E}_{(r,h,k),n}(\mathbf{z}_h, \mathbf{w}_k^{-1})P\right) \in \bigwedge^{r+h-k} V_n[\![\mathbf{z}^k, \mathbf{w}_k^{-1}]\!].$$

Recall that $E_{i,j} \in gl_n(\mathbb{Q})$ denotes the elementary matrix with all entries zero but 1 in position (i, j), acting on $u \in V_n$ as $E_{i,j}(u) = (X^i \otimes \partial^j)(u) = X^i\partial^j(u)$

5.2. Proposition. *For all* $u \in \bigwedge V_n$, $\mathrm{tr}(E_{ij})u = X^i \wedge (\partial^j \lrcorner u)$

Proof. If $u \in V_n$, then $\mathrm{tr}(E_{ij})(u) = E_{ij}(u) = X^i \wedge \partial^j(u)$ and the property holds. Let us assume that it holds for each $v \in \bigwedge^j V_n$ for all $0 \leq j \leq r-1$. Then if $u \in V_n$ and $v \in \bigwedge V_n$ is arbitrarily chosen:

$$\begin{aligned}\mathrm{tr}(E_{ij})(u \wedge v) &= \mathrm{tr}(E_{ij})u \wedge v + u \wedge \mathrm{tr}(E_{ij})v = X^i\partial^j(u) \wedge v + u \wedge X^i \wedge (\partial^j \lrcorner v) \\ &= X^i\partial^j(u) \wedge v - X^i \wedge \partial^j \lrcorner v = X^i \wedge (\partial^j \lrcorner (u \wedge v)).\end{aligned}$$

Because of each element of $\bigwedge V_n$ is a finite linear combination of monomials of the form $u \wedge v$, with $u \in V_n$, the property is proven. $\square$

If $h = k = 1$, put $\mathbf{z}_1 = z$ and $\mathbf{w}_1^{-1} = w^{-1}$. Then $\mathcal{E}_{r,n}(z, w) := \mathcal{E}_{(r,1,1);n}(z, w)$ is the generating function of the elementary endomorphisms of V_n:

$$\mathcal{E}_{r,n}(z,w) := \sum_{0\leq i,j<n} E_{i,j} z^i w^{-j} \in gl_n(\mathbb{Q})[[z, w^{-1}]].$$

$$\mathcal{E}_{r,n}(z,w) = \sum_{0\leq i,j<n} X^i z^i w^{-j} \otimes \partial^j w^{-j} = \mathbf{X}(z) \otimes \boldsymbol{\partial}(w^{-1}) = \sigma_+(z)X^0 \otimes \boldsymbol{\partial}(w^{-1})$$

where $\boldsymbol{\partial}(w^{-1}) = \sum_{i\geq 0} \partial^i w^{-i}$ is a generating function of the dual basis of $\mathbf{X}$. It follows that the action of $\mathcal{E}_{r,n}(z, w)$ is given by

$$\Big(\mathcal{E}_n(z,w) \star S_{\boldsymbol{\lambda}}\Big)\mathbf{X}^r(0) = \operatorname{tr}(\mathcal{E}_n(z,w))\mathbf{X}^r(\boldsymbol{\lambda})$$

a particular case of (5.1).

5.3. Proposition. *For all* $u \in \bigwedge V_n$

$$\operatorname{tr}(\mathcal{E}_{r,n}(z,w))u = \sigma_+(z)X^0 \wedge \boldsymbol{\partial}(w^{-1}) \lrcorner u$$

Proof. It is a straightforward consequence of Proposition 5.2. $\square$

6. The main theorem

6.1. Our main goal is to explicitly compute the action of $\mathcal{E}_{(r,h,k);n}(\mathbf{z}_h, \mathbf{w}_k^{-1})$ against the generating function $\exp(\mathbf{x}_{r,n}(\mathbf{t}_r))$ of the $\mathbb{Q}$-basis of $B_{r,n}$. Recall that (Proposition 4.2)

$$\exp(\mathbf{x}_{r,n}(\mathbf{t}_r))\mathbf{X}^r(0) = \sigma_+(\mathbf{t}_r)\mathbf{X}^r(0).$$

To this purpose, one needs one more piece of notation.

6.2. Let $\mathfrak{S}_r$ be the symmetric group on r-elements and $\mathbf{t}_r = (t_1, \ldots, t_r)$ be an r-tuple of indeterminates and let

$$\Lambda_r := \mathbb{Q}[\mathbf{t}_r]^{\mathfrak{S}_r}$$

be the ring of symmetric polynomials in $\mathbf{t}_r$. For an arbitrary indeterminate ζ over $\mathbb{Q}[\mathbf{t}_r]$, let

$$E_r(\mathbf{t}_r, \zeta) := 1 - e_1(\mathbf{t}_r)\zeta + \cdots + (-1)^r e_r(\mathbf{t}_r)\zeta^r = \prod_{i=1}^{r}(1 - t_i\zeta) \in \mathbb{Q}[\mathbf{t}_r],$$

which is the generating function of the *elementary symmetric polynomials*. The equality

$$H_r(\mathbf{t}_r, \zeta) = \sum_{j\in\mathbb{Z}} h_j(\mathbf{t}_r)\zeta^j = \frac{1}{E_r(\mathbf{t}_r, \zeta)}$$

defines the complete symmetric polynomials $h_j(\mathbf{t}_r)$ in $\mathbf{t}_r$. Let

$$p_i(\mathbf{t}_r) := t_1^i + \cdots + t_r^i$$

be the i-th symmetric Newton power sum. They are related to $e_i(\mathbf{t}_r)$ and $h_j(\mathbf{t}_r)$ via the equality

$$\exp\left(\sum_{i\geq 1}\frac{p_i(\mathbf{t}_r)}{i}\zeta^i\right)=\frac{1}{E_r(\mathbf{t}_r,\zeta)}=\sum_{j\in\mathbb{Z}}h_j(\mathbf{t}_r)\zeta^j,$$

holding in the algebra $\Lambda_r[\![\xi]\!]$.

By the fundamental theorem of symmetric functions, $\Lambda_r{=}\mathbb{Q}[e_1(\mathbf{t}_r),\dots,e_r(\mathbf{t}_r)]$. The ring Λ_r is also generated as a $\mathbb{Q}$-algebra by $(h_1(\mathbf{t}_r),\dots,h_r(\mathbf{t}_r))$, so that $h_j(\mathbf{t}_r)$ is a weighted homogeneous polynomial of degree j in $h_1(\mathbf{t}_r),\dots,h_r(\mathbf{t}_r)$. The same holds for $(p_1(\mathbf{t}_r),\dots,p_r(\mathbf{t}_r))$, i.e. they are algebraically independent and each $p_i(\mathbf{t}_r)$ is a homogeneous polynomial expression of weighted degree i in $(p_j(\mathbf{t}_r))_{1\leq j\leq r}$.

6.3. Definition. For all $(j,m)\in\mathbb{N}\times\mathbb{N}$, define symmetric polynomials $U_{j,m}(\mathbf{t}_r)$ via the equality:

$$\sum_{j\geq 0}U_{j,m}(\mathbf{t}_r)\zeta^j=E_r(\mathbf{t}_r,\zeta)\sum_{i\geq m}h_i(\mathbf{t}_r)\zeta^i.$$

To get a feeling of how they explicitly look like is easy. For instance

$$U_{0,m}(h_m(\mathbf{t}_r))=h_m(\mathbf{t}_r),\qquad U_{1,m}(h_m(\mathbf{t}_r))=h_{m+1}(\mathbf{t}_r)-e_1(\mathbf{t}_r)h_m(\mathbf{t}_r),$$
$$U_{2,m}(\mathbf{t}_r)=h_{m+2}(\mathbf{t}_r)-e_1(\mathbf{t}_r)h_{m+1}(\mathbf{t}_r)+e_2(\mathbf{t}_r)h_m(\mathbf{t}_r).$$

In general: $U_{j,m}(\mathbf{t}_r)=h_{m+j}-e_1(\mathbf{t}_r)h_{m+j-1}(\mathbf{t}_r)+\cdots+(_1)^re_r(\mathbf{t}(r)h_m(\mathbf{t}_r)$

6.4. Proposition.

1. *For all* $m\geq 0$, $U_{r+j,m}(\mathbf{t}_r)=0$ *for all* $j\geq 0$*;*
2. *The sum of the first* m *terms of* $H_r(\mathbf{t}_r,\zeta)$ *is*

$$\sum_{j=0}^{m-1}h_j(\mathbf{t}_r)\zeta^j=\frac{1}{E_r(\mathbf{t}_r,\zeta)}(1-y_m\zeta^m)$$

where

$$y_m=U_{0,m}(\mathbf{t}_r)+U_{1,m}(\mathbf{t}_r)\zeta+\cdots+U_{r-1,m}(\mathbf{t}_r)\zeta^{r-1}\tag{6.1}$$

Proof. To prove the first part, notice that for all $j\geq 0$

$$U_{r+j,m}(\mathbf{t}_r)=h_{m+r+i}(\mathbf{t}_r)-e_1(\mathbf{t}_r)h_{m+r+i-1}(\mathbf{t}_r)+\cdots+(-1)^rh_{m+i}(\mathbf{t}_r)$$

which vanishes due to the relation $E_r(\mathbf{t}_r,\zeta)H_r(\mathbf{t}_r,\zeta)=1$ (all the coefficients of the positive powers of the product vanish). The second part is a consequence of the first one. We have

$$\begin{aligned}1&=E_r(\mathbf{t}_r,\zeta)\sum_{j=0}^{m-1}h_j(\mathbf{t}_r)\zeta^j+E_r(\mathbf{t}_r,\zeta)\sum_{j\geq m}h_j(\mathbf{t}_r)\zeta^j\\&=E_r(\mathbf{t}_r,\zeta)\sum_{j=0}^{m-1}h_j(\mathbf{t}_r)\zeta^j+\zeta^m\Big(U_{0,m}(\mathbf{t}_r)+U_{1,m}(\mathbf{t}_r)\zeta+\cdots+U_{r-1,m}(\mathbf{t}_r)\zeta^{r-1}\Big)\end{aligned}$$

due to the vanishing of $U_{j,m}(\mathbf{t}_r)$ for all $j\geq r$. Therefore

$$\sum_{j=0}^{m-1} h_j(\mathbf{t}_r)\zeta^j = \frac{1}{E_r(\mathbf{t}_r,\zeta)}(1-y_m\zeta^m) = \exp\left(\sum_{j\geq 1}\frac{p_j(\mathbf{t}_r)}{j}\zeta^j\right)(1-y_m\zeta^m)$$

where y_m is as in (6.1). □

6.5. One still needs to define linear combinations $\mathbb{S}_j(\mathbf{t}_r) \in B_{r,n} \otimes \Lambda_r$ of the polynomials $S_j \in B_{r,n}$ with coefficients in Λ_r. They are defined through the equality

$$\sum_{j\geq -r} \mathbb{S}_j(\mathbf{t}_r)\zeta^j := E_r(\mathbf{t}_r,\zeta^{-1})\sum_{j\geq 0} S_j\zeta^j. \tag{6.2}$$

Equating the coefficients of the same power of ζ on both members of (6.2), one obtains the first values of $\mathbb{S}_i(\mathbf{t}_r)$, and the feeling of what they look like:

$$\mathbb{S}_{-r}(\mathbf{t}_r) = 1; \qquad \mathbb{S}_{-r+1}(\mathbf{t}_r) = -e_1(\mathbf{t}_r)S_0 + e_2(\mathbf{t}_r)S_1; \quad \mathbb{S}_0(\mathbf{t}_r) = S_0 - e_1(\mathbf{t}_r)S_1 + \cdots + (-1)^r e_r(\mathbf{t}_r)S_r$$

or

$$\mathbb{S}_n(z) = S_n - zS_{n+1}, \qquad \mathbb{S}_n(z,w) = S_n - (z+w)S_{n+1} + zw\cdot S_{n+2}.$$

Again, keeping into account that $S_j = 0$ for $j < 0$:

$$\mathbb{S}_{-2}(\mathbf{t}_r) = e_2(\mathbf{t}_r) - e_3(\mathbf{t}_r)S_1 + \cdots + (-1)^{r+1}e_r(\mathbf{t}_r)S_{r-2}.$$

6.6. Let us finally consider the $(r+h)\times(r+h)$ square matrix given by:

$$\mathbb{D}_{(r,h,k),n} =$$
$$\begin{vmatrix}
0 & \cdots & 0 & 1-y_{n-r+1}w_1^{-n+r-1} & w_1\left(1-y_{n-r+2}w_1^{-n+r-2}\right) & \cdots & w_1^{r-1}\left(1-y_nw_1^{-n}\right)\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & 1-y_{n-r+1}w_k^{-n+r-1} & w_k\left(1-y_{n-r+2}w_k^{-n+r-2}\right) & \cdots & w_k^{r-1}\left(1-y_nw_k^{-n}\right)\\
\mathbb{S}_{-r+k-1}(\mathbf{t}_r) & \cdots & \mathbb{S}_{-r+k-h}(\mathbf{t}_r) & \mathbb{S}_{-k+1}(\mathbf{z}_h) & \mathbb{S}_{-k}(\mathbf{z}_h) & \cdots & \mathbb{S}_{-r+k}(\mathbf{z}_h)\\
\mathbb{S}_{-r+k-2}(\mathbf{t}_r) & \cdots & \mathbb{S}_{-r+k-h+1}(\mathbf{t}_r) & \mathbb{S}_{-k+2}(\mathbf{z}_h) & \mathbb{S}_{-k+1}(\mathbf{z}_h) & \cdots & \mathbb{S}_{-r+k+1}(\mathbf{z}_h)\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\mathbb{S}_{h-1}(\mathbf{t}_r) & \cdots & \mathbb{S}_0(\mathbf{t}_r) & \mathbb{S}_{r-1}(\mathbf{z}_h) & \mathbb{S}_{r-2}(\mathbf{z}_h) & \cdots & \mathbb{S}_0(\mathbf{z}_h)
\end{vmatrix} \tag{6.3}$$

Recalling that (Cf. (5.2) for the definition of $\star$)

$$\left(\mathcal{E}_{(r,h,k),n}(\mathbf{z}_h,\mathbf{w}_k)\star\exp(\sum_{i\geq 1}x_ip_i(\mathbf{t}_r))\right)\mathbf{X}^r(0) := \mathcal{E}_{(r,h,k),n}(\mathbf{z}_h,\mathbf{w}_k)\sigma_+(\mathbf{t}_r)\mathbf{X}^r(0)$$

we are now in position to state our main theorem.

6.7. Theorem.

$$\mathcal{E}_{(r,h,k),n}(\mathbf{z}_h,\mathbf{w}_k)\star\exp(\sum_{i\geq 1}x_ip_i(\mathbf{t}_r))$$

$$= (-1)^h \prod_{i=1}^{k} w_i^{-r+1} \cdot \mathbb{D}_{(r,h,k),n} \exp\left(\sum_{i\geq 1} x_i p_i(\mathbf{z}_h) + \frac{p_i(\mathbf{t}_r)}{i\prod_{j=1}^{k} w_j^i}\right) \exp(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r)) \tag{6.4}$$

Proof. The first step of the proof consists in using the equality

$$\mathcal{E}_{(r,h,k),n}(\mathbf{z}_h, \mathbf{w}_k) \star \exp(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r))\mathbf{X}^r(0) = \sigma_+(\mathbf{z}_h)\mathbf{X}^r(0) \wedge \partial(\mathbf{w}_k^{-1}) \lrcorner \sigma_+(\mathbf{t}_r)\mathbf{X}^r(0)$$

$$= \sigma_+(\mathbf{z}_h)\mathbf{X}^r(0) \wedge \begin{vmatrix} \partial(w_1^{-1})(\sigma_+(\mathbf{t}_r)X^{r-1}) & \cdots & \partial(w_1^{-1})(\sigma_+(\mathbf{t}_r)X^0) \\ \vdots & \ddots & \vdots \\ \partial(w_k^{-1})(\sigma_+(\mathbf{t}_r)X^{r-1}) & \cdots & \partial(w_k^{-1})(\sigma_+(\mathbf{t}_r)X^0) \\ \sigma_+(\mathbf{t}_r)X^{r-1} & \cdots & \sigma_+(\mathbf{t}_r)X^0 \end{vmatrix}$$

from which

$$\mathcal{E}_{(r,h,k),n}(\mathbf{z}_h, \mathbf{w}_k) \star \exp(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r))\mathbf{X}^r(0)$$

$$= (-1)^h \begin{vmatrix} 0 & \cdots & 0 & \partial(w_1^{-1})(\sigma_+(\mathbf{t}_r)X^{r-1}) & \cdots & \partial(w_1^{-1})(\sigma_+(\mathbf{t}_r)X^0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \partial(w_k^{-1})(\sigma_+(\mathbf{t}_r)X^{r-1}) & \cdots & \partial(w_k^{-1})(\sigma_+(\mathbf{t}_r)X^0) \\ \sigma_+(\mathbf{z}_h)X^{h-1} & \cdots & \sigma_+(\mathbf{z}_h)X^0 & \sigma_+(\mathbf{t}_r)X^{r-1} & \cdots & \sigma_+(\mathbf{t}_r)X^0 \end{vmatrix}$$

$$= (-1)^h \sigma_+(\mathbf{z}_k)\sigma_+(\mathbf{t}_r) \begin{vmatrix} 0 & \cdots & 0 & \partial(w_1^{-1})(\sigma_+(\mathbf{t}_r)X^{r-1}) & \cdots & \partial(w_1^{-1})(\sigma_+(\mathbf{t}_r)X^0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \partial(w_k^{-1})(\sigma_+(\mathbf{t}_r)X^{r-1}) & \cdots & \partial(w_k^{-1})(\sigma_+(\mathbf{t}_r)X^0) \\ \overline{\sigma}_+(\mathbf{t}_r)X^{h-1} & \cdots & \overline{\sigma}_+(\mathbf{t}_r)X^0 & \overline{\sigma}_+(\mathbf{z}_h)X^{r-1} & \cdots & \overline{\sigma}_+(\mathbf{z}_h)X^0 \end{vmatrix}. \tag{6.5}$$

Now, for all $0 \leq j \leq r-1$ and $1 \leq i \leq k$

$$\begin{aligned} \boldsymbol{\partial}(w_i^{-1})(\sigma_+(\mathbf{t}_r)X^{r-1-j}) &= \boldsymbol{\partial}(w_i^{-1})(\sum_{0\leq i<n-r+j+1} h_i(\mathbf{t}_r)X^{r-1-j+i}) \\ &= \sum_{0\leq i<n-r+j+1} h_i(\mathbf{t}_r)w_i^{-r+1+j-i} \\ &= w_i^{-r+1+j} \sum_{0\leq i<n-r+j+1} h_i(\mathbf{t}_r)w_i^{-i} \\ &= w_i^{-r+1+j} \frac{1}{E_r(\mathbf{t}_r, w_i^{-1})} \left(1 - y_{n-r+j+1} w_i^{-n+r-j-1}\right) \end{aligned}$$

Keeping into account that

$$\frac{1}{E_r(\mathbf{t}_r, w_i^{-1})} = \exp(-\log(E_r(\mathbf{t}_r, w_i^{-1}))) = \exp\left(\sum_{j\geq 1} \frac{p_j(\mathbf{t}_r)}{j w_i^j}\right)$$

and that

$$\sigma_+(\mathbf{t}_r) \bigwedge^r V_n = \exp\left(\sum x_i p_i(\mathbf{t}_r)\right) \bigwedge^r V_n.$$

Formula (6.5) can be written as

$$(-1)^h \prod_{i=1}^{k} w_i^{-r+1} \exp\left(\sum_{j\geq 1} x_j p_j(\mathbf{z}_h) + \frac{p_j(\mathbf{t}_r)}{j\prod_{i=1}^{r} w_i^j}\right) \exp\left(\sum_{i\geq 1} x_j p_j(\mathbf{t}_r)\right) \widetilde{D}_{(r,h,k),n}$$

where using Proposition 6.4.

$$\widetilde{D}_{(r,h,k),n} =$$

$$\begin{vmatrix} 0 & \cdots & 0 & 1-\dfrac{y_{n-r+1}}{w_1^{n-r+1}} & w_1\left(1-\dfrac{y_{n-r+2}}{w_1^{n-r+2}}\right) & \cdots & w_1^{r-1}\left(1-\dfrac{y_n}{w_1^n}\right) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1-\dfrac{y_{n-r+1}}{w_k^{n-r+1}} & w_k\left(1-\dfrac{y_{n-r+2}}{w_k^{n-r+2}}\right) & \cdots & w_k^{r-1}\left(1-\dfrac{y_n}{w_k^n}\right) \\ \overline{\sigma}_+(\mathbf{t}_r)X^{h-1} & \cdots & \overline{\sigma}_+(\mathbf{t}_r)X^0 & \overline{\sigma}_+(\mathbf{z}_h)X^{r-1} & \overline{\sigma}_+(\mathbf{z}_h)X^{r-2} & \cdots & \overline{\sigma}_+(\mathbf{z}_h)X^0 \end{vmatrix} \tag{6.6}$$

We are only left to express $\widetilde{D}_{(r,h,k),n}$ in the shape proclaimed in the statement. To this purpose we first notice that

$$\overline{\sigma}_+(\mathbf{t}_r)X^{h-j} = X^0 - e_1(\mathbf{t}_r)X^1 + \cdots + (-1)^r e_r(\mathbf{t}_r)X^{h-j+r}$$

and, for all $0 \leq j \leq r-1$

$$\overline{\sigma}_+(\mathbf{z}_h)X^{r-j} = X^{r-j} - e_1(\mathbf{z}_h)X^{r-j+1} + \cdots + (-1)^h e_h(\mathbf{z}_h)X^{r-j+h}.$$

Then we invoke Remark 4.6 and Corollaries 4.9 to put (6.6) in the shape (6.3). $\square$

7. Special cases

7.1. Let $k=0$. Then for all h, r, n, one has

$$\sigma_+(\mathbf{z}_h)\mathbf{X}^h(0) \wedge \sigma_+(\mathbf{t}_r)\mathbf{X}^r(0) = \prod_{i,j}(z_i - t_j)\sigma_+(\mathbf{z}_h, \mathbf{t}_r)\mathbf{X}^{h+k}(0).$$

The proof consists in observing that in the localization $\mathbb{Q}[\mathbf{z}_h, \mathbf{t}_r]$ at the Vandermonde $\Delta_0(\mathbf{z}_h, \mathbf{t}_r)$, one has

$$\begin{aligned} \sigma_+(\mathbf{z}_h)\mathbf{X}^h(0) \wedge \sigma_+(\mathbf{t}_r)\mathbf{X}^r(0) &= \frac{1}{\Delta_0(\mathbf{z}_h)\Delta_0(\mathbf{t}_r)}\sigma_+(z_1)X^0 \wedge \cdots \wedge \sigma_+(z_h)X^0 \wedge \sigma_+(t_1)X^0 \wedge \cdots \wedge \sigma_+(t_r) \\ &= \frac{\Delta_0(\mathbf{z}_h, \mathbf{t}_r)}{\Delta_0(\mathbf{z}_h)\Delta_0(\mathbf{t}_r)}\sigma_+(\mathbf{z}_h, \mathbf{t}_r)\mathbf{X}^{r+h}(0) \\ &= \prod_{i,j}(z_i - t_j)\sigma_+(\mathbf{z}_h, \mathbf{t}_r)\mathbf{X}^{r+h}(0) \end{aligned}$$

as desired. This is one of the possible spelling of the Cauchy formula.

7.2. Let now $h=0$ and $k=r$. Then for any finite n

$$\boldsymbol{\partial}(\mathbf{w}_r^{-1}) \lrcorner \overline{\sigma}_+(\mathbf{t}_r)\mathbf{X}^r(0) =$$

$$\prod_{i=1}^{r} w_i^{-r+1} \exp\left(\frac{p_i(\mathbf{t}_r)}{i\prod_{j=1}^{r} w_j^i}\right) \begin{vmatrix} 1-\dfrac{y_{n-r+1}}{w_1^{n-r+1}} & w_1\left(1-\dfrac{y_{n-r+2}}{w_1^{n-r+2}}\right) & \cdots & w_1^{r-1}\left(1-\dfrac{y_n}{w_1^n}\right) \\ \vdots & \vdots & \ddots & \vdots \\ 1-\dfrac{y_{n-r+1}}{w_r^{n-r+1}} & w_r\left(1-\dfrac{y_{n-r+2}}{w_r^{n-r+2}}\right) & \cdots & w_r^{r-1}\left(1-\dfrac{y_n}{w_r^n}\right) \end{vmatrix} \exp(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r)). \tag{7.1}$$

If $n=\infty$, formula (7.1) simplifies into:

$$\prod_{i=1}^{r} w_i^{-r+1} \exp\left(\frac{p_i(\mathbf{t}_r)}{i\prod_{j=1}^{r} w_j^i}\right) \Delta_0(\mathbf{w}_r) \exp(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r))$$

where $\Delta_0(\mathbf{w}_r)$ is the Vandermonde determinant $\prod_{i>j}(w_i - w_j)$.

7.3. Putting $h = k = 1$ in expression (6.4), one gets the formula describing $B_{r,n}$ as $gl_n(\mathbb{Q})$-module, i.e., putting $\mathcal{E}_{r,n}(z,w) := \mathcal{E}_{((r,1,1);n}(z)$:

$$\begin{aligned} &\mathcal{E}_{r,n}(z,w) \star \exp(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r)) \\ &= -w^{-r+1} \cdot \mathbb{D}_{r,n} \exp\left(\sum_{i\geq 1} x_i z^i + \frac{p_i(\mathbf{t}_r)}{iw^i}\right) \exp(\sum_{i\geq 1} x_i p_i(\mathbf{t}_r)) \end{aligned} \tag{7.2}$$

where

$$\mathbb{D}_{r,n} = \begin{vmatrix} 0 & 1-y_{n-r+1}w^{-n+r-1} & w\left(1-y_{n-r+2}w^{-n+r-2}\right) & \cdots & w^{r-1}\left(1-y_n w^{-n}\right) \\ \mathbb{S}_{-r+1}(\mathbf{t}_r) & \mathbb{S}_0(z) & \mathbb{S}_{-1}(z) & \cdots & \mathbb{S}_{-r+1}(z) \\ \mathbb{S}_{-r+2}(\mathbf{t}_r) & \mathbb{S}_1(z) & \mathbb{S}_0(z) & \cdots & \mathbb{S}_{-r+2}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{S}_0(\mathbf{t}_r) & \mathbb{S}_{r-1}(z) & \mathbb{S}_{r-2}(z) & \cdots & \mathbb{S}_0(z) \end{vmatrix} \tag{7.3}$$

7.4. In turn, putting $r = 1$ in formula (7.3), then $B_{1,n} = \dfrac{\mathbb{Q}[c_1]}{(c_1^n)}$ and $S_i = c_1^i$. Thus:

$$\mathbb{D}_{1,n} = \begin{vmatrix} 0 & 1-y_n w^{-n} \\ S_0(t) & S_0(z) \end{vmatrix} = \begin{vmatrix} 0 & 1-y_n w^{-n} \\ 1-tS_1 & 1-zS_1 \end{vmatrix} = -(1-y_n w^{-n})(1-c_1 t)$$

In this case, cf. (6.1), $y_n = t^n$. Then formula (7.2) reads as:

$$\begin{aligned} \mathcal{E}_{1,n}(z,w) \star (1 + h_1 t + \cdots + h_{n-1}t^{n-1}) &= \frac{1}{1-c_1 z}\frac{1}{1-\dfrac{t}{w}}\frac{1}{1-c_1 t}(1-t^n w^{-n})(1-c_1 t) \\ &= \frac{1}{1-c_1 z}\frac{1}{1-\dfrac{t}{w}}(1-y_n w^{-n}) = \frac{1-y_n w^{-n}}{(1-c_1 z)\left(1-\dfrac{t}{w}\right)} \end{aligned}$$

which coincides with (3.1) that we already computed by hand in the introduction and which describes the usual practice of multiplying a matrix by a vector. For example

$$E_{23} \star c_1^4 = \left[z^2 w^{-3} t^4\right](1 + c_1 z + c_1^2 z^2 + \cdots)\left(1 + \frac{t}{w} + \frac{t^2}{w^2} + \cdots + \frac{t^{n-1}}{w^{n-1}}\right) = 0$$

and

$$E_{23} \star c_1^3 = \left[z^2 w^{-3} t^3\right]\left\{(1 + c_1 z + c_1^2 z^2 + \cdots)\left(1 + \frac{t}{w} + \frac{t^2}{w^2} + \cdots + \frac{t^{n-1}}{w^{n-1}}\right)\right\} = c_1^2.$$

7.5. The case of $B_{2,4}$. The ring

$$B_{2,4} = \frac{\mathbb{Q}[c_1, c_2]}{(S_3, S_4)} = \frac{\mathbb{Q}[c_1, c_2]}{(c_1^2 - 2c_1c_2, c_1^4 - c_1^2 c_2)}$$

may be interpreted as the cohomology ring of the complex Grassmannian $\mathbf{G}(1, \mathbb{P}^3)$ of lines in $\mathbb{P}^3$. In this case c_1, c_2 would be the Chern classes of the universal quotient bundle. Put $r = 2$ and $n = 4$ in formula (7.3). One has

$$\mathbb{D}_{2,4} = \begin{vmatrix} 0 & 1 - y_3 w^3 & w(1 - y_4 w^{-4} \\ \mathbb{S}_{-1}(t_1, t_2) & \mathbb{S}_0(z) & \mathbb{S}_{-1}(z) \\ \mathbb{S}_0(t_1, t_2) & \mathbb{S}_1(z) & \mathbb{S}_0(z) \end{vmatrix}$$

whose expansion gives

$$= \begin{vmatrix} 0 & 1 - h_3(\mathbf{t}_2)w^{-4} + e_2(\mathbf{t}_2)h_2(\mathbf{t}_2)w^{-5} & w(1 - h_4(\mathbf{t}_2)w^{-5} + e_2(\mathbf{t}_2)h_3(\mathbf{t}_2)w^{-6}) \\ -e_1(\mathbf{t}_2) + e_2(\mathbf{t}_2)S_1 & 1 - S_1 z & -z \\ 1 - e_1(\mathbf{t}_2)S_1 + e_2(\mathbf{t}_2)S_2 & S_1 - zS_2 & 1 - S_1 z) \end{vmatrix}.$$

Therefore, using the fact that $n = 4$, we have

$$\mathcal{E}_{2,4}(z, w) \star \exp\left(\sum_{i \geq 1} x_i (t_1^i + t_2^i)\right) = \mathcal{E}_{2,4}(z, w) \star \sum_{\boldsymbol{\lambda} \in \mathcal{P}_{2,4}} S_{\boldsymbol{\lambda}} \cdot s_{\boldsymbol{\lambda}}(t_1, t_2)$$

$$= \frac{w^{-1}(1 + S_1 t_1 + S_2 t_1^2)(1 + S_1 t_2 + S_2 t_2^2)(1 + S_1 z + S_2 z^2)}{1 - \dfrac{t_1 + t_2}{w}} \cdot \mathbb{D}_{2,4}.$$

Further simplifications are obtained by observing that

$$\frac{\mathbb{D}_{2,4}}{1 - \dfrac{t_1 + t_2}{w}} = \begin{vmatrix} 0 & 1 + \dfrac{h_1(\mathbf{t}_2)}{w} + \dfrac{h_2(\mathbf{t}_2)}{w^2} & w\left(1 + \dfrac{h_1(\mathbf{t}_2)}{w} + \dfrac{h_2(\mathbf{t}_2)}{w^2} + \dfrac{h_3(\mathbf{t}_3)}{w^3}\right) \\ -e_1(\mathbf{t}_2) + e_2(\mathbf{t}_2)S_1 & 1 - S_1 z & -z \\ 1 - e_1(\mathbf{t}_2)S_1 + e_2(\mathbf{t}_2)S_2 & S_1 - zS_2 & 1 - S_1 z \end{vmatrix}.$$

Using Mathematica, we have explicitly computed formula (6.4), using (6.3), for $r = 2$ and $n = 4$ and the output is:

$$\mathcal{E}_{2,4}(z, w) \star \exp \sum_{i \geq 1} x_i p_i(\mathbf{t}_2) =$$

$$1+S_1s_1(\mathbf{t}_2)+S_2\cdot s_2(\mathbf{t}_2)+z[(S_1^2-S_2)\cdot s_1(\mathbf{t}_2)+(S_1^3-S_1S_2)\cdot s_2(\mathbf{t}_2)]+z^2[-(S_1^2-S_2)\cdot s_1(\mathbf{t}_2)+S_2^2\cdot s_2(\mathbf{t}_2)]$$
$$+w^{-1}[S_1\cdot s_1(\mathbf{t}_2)+S_2\cdot s_{(1,1)}(\mathbf{t}_2)]+w^{-2}\left[-s_{(1,1)}(\mathbf{t}_2)+S_2\cdot s_{(2,2)}(\mathbf{t}_2)\right]+w^3[-s_{(2,1}(\mathbf{t}_2)-S_1\cdot s_{(2,2)}(\mathbf{t}_2)]$$
$$+\frac{z}{w}\left[1+(S_1^2-S_2)\cdot s_{(1,1)}(\mathbf{t}_2)+S_1S_2\cdot s_{(2,1)}(\mathbf{t}_2)\right]+\frac{z}{w^2}[s_{(1,1)}(\mathbf{t}_2)+S_1S_2\cdot s_{(2,2)}(\mathbf{t}_2)]$$
$$+\frac{z}{w^3}(s_2(\mathbf{t}_2)-(S_1^2-S_2)\cdot s_{(2,2)}(\mathbf{t}_2))+\frac{z^2}{w}\left[S_1+S_2^2s_{(2,1)}(\mathbf{t}_2)\right]+\frac{z^2}{w^2}\left[S_1\cdot s_1(\mathbf{t}_2)+(S_1^2-S_2)\cdot s_{(1,1)}(\mathbf{t}_2)\right]$$
$$+\frac{z^2}{w^3}\left[S_1s_2(\mathbf{t}_2)(S_1^2-S_2)\cdot s_{(2,1)}(\mathbf{t}_2)\right]+\frac{z^3}{w}\left[S_2-S_2^2\cdot s_{(1,1)}(\mathbf{t}_2)\right]+\frac{z^3}{w^2}\left[S_2\cdot s_1(\mathbf{t}_2)+S_1S_2\cdot s_{(1,1)}(\mathbf{t}_2)\right]$$
$$+\frac{z^3}{w^3}\left[S_2\cdot s_2(\mathbf{t}_2)+S_1S_2\cdot s_{(2,1)}(\mathbf{t}_2)+S_2^2\cdot s_{(2,2)}(\mathbf{t}_2)\right].$$

More explicitly

$$=1+c_1(t_1+t_2)+(c_1^2-c_2)\cdot(t_1^2+t_1t_2+t_2^2)+z\left[c2\cdot(t_1+t_2)+(c_1^3-c_1c_2)\cdot(t_1^2+t_1t_2+t_2^2)\right]$$
$$+z^2[-c_2\cdot(t_1+t_2)+(c_1^4-2c_1^2c_2+c_2^4)\cdot(t_1^2+t_1t_2+t_2^2)]+\frac{1}{w}[c_1\cdot(t_1+t_2)+(c_1^2-c_2)t_1t_2]$$
$$+\frac{1}{w^2}\left[-t_1t_2+(c_1^2-c_2)\cdot t_1^2t_2^2\right]+\frac{1}{w^3}[-(t_1^2t_2+t_1t_2^2)-c_1\cdot t_1^2t_2^2]$$
$$+\frac{z}{w}\left[1+c_2\cdot t_1t_2+(c_1^3-c_1c_2)\cdot(t_1^2t_2+t_1t_2^2)\right]+\frac{z}{w^2}[t_1t_2+c_1(c_1^2-c_2)\cdot t_1^2t_2^2]$$
$$+\frac{z}{w^3}(s_2(\mathbf{t}_2)-c_2\cdot t_1^2t_2^2)+\frac{z^2}{w}\left[c_1+(c_1^4+2c_1^2c_2+c_2^2)(t_1^2t_2+t_1t_2^2)\right]+\frac{z^2}{w^2}\left[c_1\cdot(t_1+t_2)+c_2\cdot(t_1t_2)\right]$$
$$+\frac{z^2}{w^3}\left[c_1(t_1^2+t_1t_2+t_2^2)+c_2(t_1^2t_2+t_1t_2^2)\right]+\frac{z^3}{w}\left[(c_1^2-c_2)-(c_1^4-2c_1^2c_2+c_2^2)\cdot s_{(1,1)}(\mathbf{t}_2)\right]$$
$$+\frac{z^3}{w^2}\left[(c_1^2-c_2)\cdot(t_1+t_2)+S_1S_2\cdot t_1t_2\right]$$
$$+\frac{z^3}{w^3}\left[(c_1^2-c_2)\cdot(t_1^2+t_1t_2+t_2^2)+(c_1^3-c_1c_2)\cdot(t_1^2t_2+t_1t_2^2)+(c_1^2-c_2)^2)t_1^2t_2^2\right]$$

and, clearly, $E_{i,j}\star S_{(\lambda_1,\lambda_2)}=\left[z^iw^{-j}s_{(\lambda_1,\lambda_2)}\right]\left(\mathcal{E}_{2,4}(z,w)\star\exp\sum_{i\geq1}x_ip_i(\mathbf{t}_2)\right)$ or, which is the same, and somehow more effective:

$$E_{i,j}\star S_{(\lambda_1,\lambda_2)}=\left[z^iw^{-j}t_1^{1+\lambda_1}t_2^{\lambda_2}\right]\left((t_1-t_2)\cdot\mathcal{E}_{2,4}(z,w)\star\exp\sum_{i\geq1}x_ip_i(\mathbf{t}_2)\right).$$

7.6. Let us consider now formulas (7.2) and (7.3) for $n=\infty$, considering the $gl_\infty(\mathbb{Q})$-structure of B_r. In this case, one has

$$\mathcal{E}_r(z,w)\star\exp(\sum_{i\geq1}x_ip_i(\mathbf{t}_r))$$
$$=-w^{-r+1}\cdot\mathbb{D}_r\exp\left(\sum_{i\geq1}x_iz^i+\frac{p_i(\mathbf{t}_r)}{iw^i}\right)\exp(\sum_{i\geq1}x_ip_i(\mathbf{t}_r)),\tag{7.4}$$

where $\mathbb{D}_r:=\mathbb{D}_{r,\infty}$ simplifies into

$$\mathbb{D}_r = \begin{vmatrix} 0 & 1 & w & \cdots & w^{r-1} \\ \mathbb{S}_{-r+1}(\mathbf{t}_r) & \mathbb{S}_0(z) & \mathbb{S}_{-1}(z) & \cdots & \mathbb{S}_{-r+1}(z) \\ \mathbb{S}_{-r+2}(\mathbf{t}_r) & \mathbb{S}_1(z) & \mathbb{S}_0(z) & \cdots & \mathbb{S}_{-r+2}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{S}_0(\mathbf{t}_r) & \mathbb{S}_{r-1}(z) & \mathbb{S}_{r-2}(z) & \cdots & \mathbb{S}_0(z) \end{vmatrix} \tag{7.5}$$

and the generators (x_i) of the ring B_r are given by formula (2.4) for $n = \infty$. Let us notice that determinant (7.5) is nothing but the "bosonic" counterpart of

$$w^{-r+1}\sigma_+(z)X^0 \wedge \partial(w^{-1})\lrcorner\sigma_+(\mathbf{t}_r)\mathbf{X}^r(0) = w^{-r+1}\sigma_+(\mathbf{t}_r)\sigma_+(z)\left[\overline{\sigma}_+(\mathbf{t}_r)X^0 \wedge \overline{\sigma}_+(z)\overline{\sigma}_+(w)\mathbf{X}^{r-1}(0)\right].$$

The latter expression admits a determinantal description as well. For instance, for $r = 2$, we have

$$\overline{\sigma}_+(\mathbf{t}_2)X^0 \wedge \overline{\sigma}_+(z,w)X^0 = \begin{vmatrix} e_2(t_1,t_2) & e_2(z,w) & 1 \\ -e_1(t_1,t_2) & -e_1(z,w) & -c_1 \\ 1 & 1 & c_2 \end{vmatrix} X^1 \wedge X^0.$$

Thus

$$\begin{aligned} &\mathcal{E}(z,w)\exp\left(\sum_{i\geq 1} x_i(t_1^i + t_2^i)\right) \\ &= \frac{1}{w}\exp\left[\sum_{i\geq 1}\left(x_i\left(t_1^i + t_2^i + z^i\right) + \frac{t_1^i + t_2^i}{iw^i}\right)\right]\begin{vmatrix} e_2(t_1,t_2) & e_2(z,w) & 1 \\ -e_1(t_1,t_2) & -e_1(z,w) & -c_1 \\ 1 & 1 & c_2 \end{vmatrix}. \end{aligned}$$

CRediT authorship contribution statement

Letterio Gatto: Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Malihe Yousofzadeh:** Writing – review & editing, Writing – original draft, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The authors declare that the data supporting the findings of this study are available within the paper. All the article listed in the final reference list are publicly available either on journals or in the `ArXiv` data repository.

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