Geometrically Non-Linear Thermo-Elastic Analysis of Variable Angle Tow Composite Plates Thorugh Layerwise Models

Materials Research Proceedings 69 (2026) 1187-1191 https://doi.org/10.21741/9781644904251-204

Geometrically Non-Linear Thermo-Elastic Analysis of Variable Angle Tow Composite Plates Thorugh Layerwise Models

Francesca Bracaglia[1,a*], Alfonso Pagani[1,b], Enrico Zappino[1,c] and Erasmo Carrera[1,d]

[1]Mul2 Lab, DIMEAS, Politecnico di Torino, Corso Duca degli Abruzzi 24, Torino , Italy

[a]francesca.bracaglia@polito.it, [b]alfonso.pagani@polito.it, [c]enrico.zappino@polito.it, [d]erasmo.carrera@polito.it

Keywords: Thermo-Elastic Analysis, Geometrically Nonlinear Analysis, Variable Stiffness Composite, Thermal Stresses

Abstract. Composite materials are widely used in the engineering field due to their valuable specific properties and the adaptability of fiber orientation and lay-up to specific load conditions. The tailorability of composite materials is increased with Variable Angle Tows (VAT) composite materials, in which fiber deposition can follow a curved path. These properties lead to a higher anisotropy, and non-linear analyses are required to ensure a good description of the behaviour. The study focuses on the post-buckling configuration of the VAT model and examines the impact of curved fiber paths. The analyses are conducted to compare high-order theory for describing the nonlinear behaviour of the VAT composite structure. Additionally, the post-buckling shape of VAT plates is thoroughly explored, leading to the selection of the most accurate model for characterizing plate behaviour.

Introduction

Composite materials present valuable properties, such as high specific strength and stiffness, thermal stability, and the adaptability of lamination and lay-up to manage the mechanical and thermal properties of the structure [1]. Thanks to advanced manufacturing techniques, the capabilities of the constant-stiffness composite laminate are enhanced by the introduction of Variable Angle Tow (VAT) composite materials, which have been developed in recent years [2]. The VATs are characterized by the curvilinear path of the reinforced fibers. Consequently, the properties of VATs vary continuously within each lamina, depending on the curvature imposed on the fibers, opening a vast range of possibilities for the adaptability of the structures to the specific load conditions [3]. Due to their high tailoring capability, VATs are mainly studied in the field of optimization to adapt fiber orientation to different loads or geometric configurations.

Additionally, the remarkable stability under thermal and high-temperature conditions enables the use of this class of material in applications where temperature is a primary load factor. Among the key concerns in VAT plate analysis, thermal stability is a focal point, as demonstrated in [4]. Furthermore, this class of materials, which has been extensively studied in the field of linear or linearized analysis under mechanical and thermal loads, lacks studies in the nonlinear thermal field. The reported tailoring capabilities are suitable for adapting the fiber deposition to the load distribution. However, the post-buckling and high-deflection configurations can be significantly affected by the curved deposition. Under variation in the applied temperature, the presence of curved fibers can lead to unexpected phenomena, and nonlinear investigations are necessary to ensure a correct level of confidence in the results.

This paper utilizes the Carrera Unified Formulation (CUF) to develop a high-order plate theory within the finite element method framework, offering novel insights into the intricate three-dimensional phenomena exhibited by VAT structures [5]. The governing equations are obtained from the Principle of Virtual Displacement (PVD), and the nonlinear problem is solved through

Materials Research Proceedings 69 (2026) 1187-1191 https://doi.org/10.21741/9781644904251-204

the Newton-Raphson procedure in conjunction with the arc-length constraint. The geometrical nonlinearity is described using the full Green-Lagrange strain tensor, and its accuracy is compared with that of the simplified von Kármán nonlinearities model. The analyses adopt a layer-wise approach, and a comparison to the equivalent single-layer model is also presented. It is demonstrated that the layer-wise model in combination with high-order theories provides an accurate description of the through-the-thickness primary unknowns, facilitating a detailed analysis of nonlinear thermal stresses within the component.

Non-linear thermal formulation

The thermo-mechanical field is expressed in a decoupled approach where the unknowns of the problem are the displacement components, which are expressed through the CUF and FEM as the combination of nodal displacements $\boldsymbol{q}_{i\tau}$, shape functions $\boldsymbol{N}_i(x,y)$, and expansion functions $\boldsymbol{F}_\tau(z)$, as reported in Eq. (1). More details about CUF can be found in [6].

$$\boldsymbol{u}(x,y,z) = \boldsymbol{F}_\tau(z)\,\boldsymbol{u}_\tau(x,y) = \boldsymbol{F}_\tau(z)\boldsymbol{N}_i(x,y)\,\boldsymbol{q}_{i\tau} \quad \tau = 1,2,\dots,M \quad i = 1,2,\dots,N_n \qquad (1)$$

Where the double indices mean sum, M and N_n are the number of expansion terms and the number of nodes respectively.

The VAT orientation is considered to vary linearly, following the notation by Gürdal et al. [7]; the lamination is denoted as $[\Phi < T_0, T_1 >]$, where Φ is the reference system rotation angle from the x-axis, T_0 and T_1 are the fiber orientations in the center and at the edge of the plate, respectively.

It is possible to describe the strain and the stresses as the sum of the mechanical linear and non-linear terms and the thermal component, as reported in Eq.s (2), (3), where the nonlinearities are incorporated in the formulation using the full Green-Lagrange strain tensor, see [8].

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_m + \boldsymbol{\varepsilon}_\theta = (\boldsymbol{b}_l + \boldsymbol{b}_{nl})\boldsymbol{u} - \boldsymbol{\alpha}\,\Delta T \qquad (2)$$
$$\boldsymbol{\sigma} = \boldsymbol{C}\,\boldsymbol{\varepsilon} = \boldsymbol{C}[(\boldsymbol{b}_l + \boldsymbol{b}_{nl})\,\boldsymbol{u} - \boldsymbol{\alpha}\,\Delta T] = \boldsymbol{\sigma}_l + \boldsymbol{\sigma}_{nl} - \boldsymbol{\beta}\Delta T \qquad (3)$$

Where the vector $\boldsymbol{\beta}$ is the thermo-mechanical coupling coefficient. Due to the presence of VAT the fiber orientation vary from a point to another and consequently matrix $\boldsymbol{C}$ becomes a function of the position, $\boldsymbol{C}(x, \mathrm{y})$ as well as vector $\boldsymbol{\beta}\,(x,y)$.

The governing equations are obtained though the PVD as reported in Eq.s (4), (5).

$$\delta L_{int} = \int_V \delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma} = \int_V \delta\boldsymbol{\varepsilon}_m^T\boldsymbol{C}\boldsymbol{\varepsilon}_m dV - \int_V \delta\boldsymbol{\varepsilon}_m^T\,\boldsymbol{\beta}\Delta T\,dV = \delta\boldsymbol{q}^T\,\boldsymbol{K}_s\boldsymbol{q} - \delta\boldsymbol{q}^T\,\boldsymbol{F}_\theta = 0 \qquad (4)$$
$$\delta\boldsymbol{q} : \boldsymbol{K}_S\,\boldsymbol{q} - \boldsymbol{F}_\theta = 0 \qquad (5)$$

Where $\boldsymbol{K}_S$ is the secant stiffness matrix which depends only on the mechanical components and $\boldsymbol{F}_\theta$ is the nonlinear vector of thermal loads. Eq. (5) describes a nonlinear set of equations, and it is necessary to solve it in an iterative procedure. To use the Newton-Raphson method, the residual nodal force vector is defined as in Eq. (6). Furthermore, it is necessary to linearize the equation through a Taylor expansion about the known solution as reported in Eq. (7).

$$\boldsymbol{\varphi}_{res} = \boldsymbol{K}_S\,\boldsymbol{q} - \boldsymbol{F}_\theta \qquad (6)$$
$$\boldsymbol{\varphi}_{res}(\boldsymbol{q} + \delta\boldsymbol{q}, \Delta T + \delta\Delta T) = \boldsymbol{\varphi}_{res}(\boldsymbol{q}, \Delta T) + \left.\frac{\partial\boldsymbol{\varphi}_{res}}{\partial\boldsymbol{q}}\right|_q \delta\boldsymbol{q} + \left.\frac{\partial\boldsymbol{\varphi}_{res}}{\partial\Delta T}\right|_{\Delta T} \delta\lambda(\Delta T)_{ref} = 0 \qquad (7)$$

From Eq. (7) it is possible to define the tangent stiffness matrix as $\boldsymbol{K}_T = \frac{\partial \boldsymbol{\varphi}_{res}}{\partial \boldsymbol{q}}\Big|_q$ and the tangent load vector as $\widetilde{\boldsymbol{F}}_\theta = -\frac{\partial \boldsymbol{\varphi}_{res}}{\partial \Delta T}\Big|_{\Delta T} (\Delta T)_{ref}$ which is composed of a linear and a nonlinear part. Substituting these definitions into Eq. (7) it is possible to write Eq.s (8) and (9).

$$\boldsymbol{\varphi}_{res}(\boldsymbol{q} + \delta\boldsymbol{q}, \Delta T + \delta\Delta T) = \boldsymbol{\varphi}_{res}(\boldsymbol{q}, \Delta T) + \boldsymbol{K}_T\delta\boldsymbol{q} - \left(\widetilde{\boldsymbol{F}}_{l_\theta} + \widetilde{\boldsymbol{F}}_{nl_\theta}\right)\delta\lambda = 0 \tag{8}$$

$$\boldsymbol{K}_T\delta\boldsymbol{q} = \widetilde{\boldsymbol{F}}_\theta\delta\lambda - \boldsymbol{\varphi}_{res}(\boldsymbol{q}, \Delta T) \tag{9}$$

Here, the loading scaling parameter is introduced as a new variable; as a result, a constraint equation is needed to close the problem. The obtained system can be written as in Eq. (10).

$$\begin{cases} \boldsymbol{K}_T\delta\boldsymbol{q} = \widetilde{\boldsymbol{F}}_\theta\delta\lambda - \boldsymbol{\varphi}_{res} \\ c(\delta\boldsymbol{q}, \delta\lambda) = 0 \end{cases} \tag{10}$$

The path following method is adopted, where it is possible to act on both $\delta\boldsymbol{q}$ and $\delta\lambda$, see [9].

Numerical results

The study is conducted on a rectangular plate with lamination $[90 \pm < 60, 90 >]_{2s}$. The plate edges are $a = 300$ mm and $b = 150$ mm with a thickness of 1.0176 mm. The material properties are E_1=181GPa, E_2= E_3=10.27GPa, G_{12}= G_{13}=7.17GPa ν_{12}= 0.28, $\alpha_1 = -0.11$ x $10^{-6}/°C$, $\alpha_2 = \alpha_3 = 25.2$ x $10^{-6}/°C$. The plate is subjected to uniform thermal load. The edges are fixed against normal displacements, and they are allowed to translate in the tangential direction. A convergence analysis is conducted on the plate, for which a mesh of 5 x 10 Q9 elements is selected. A validation is presented in Fig. 1 comparing the displacement in the center of the plate. Present results are obtained with LE2 and TE2 models and are compared to reference [10].

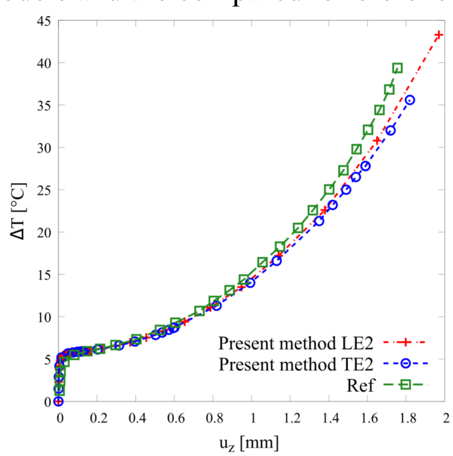

Figure 1: Displacement obtained through present method LE2 and TE2 compared to the reference [10].

From Fig. 1 it is possible to say that the displacements are in close agreement with the reference, which model is based on von Kármán-type nonlinearities and the Rayleigh-Ritz energy method. The discrepancy between the two results may be attributable to the differing formulations of the problem. In the present work the analysis is conducted to an higher level of temperature where the equilibrium curve change in shape due to the change of the post-buckling shape. From Fig. 2 it is possible to note the primary buckling and the secondary buckling of the plate.

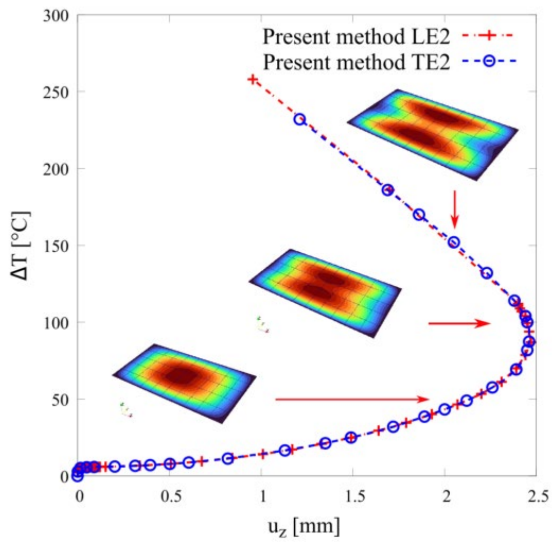

Figure 2: Displacement equilibrium curve obtained though the present method TE2 and LE2

Figure 3 shows the stress distribution for σ_x and σ_z along the thickness of the plate in Point A (75, 30) mm, considering the center of the plate as the origin of the reference system The results obtained demonstrate that, in the case of out-of-plane stresses, the interlaminar continuity is not guaranteed by the Taylor mode. Moreover, it is evident that a discrepancy in the results is present even when in-plane stresses are taken into consideration.

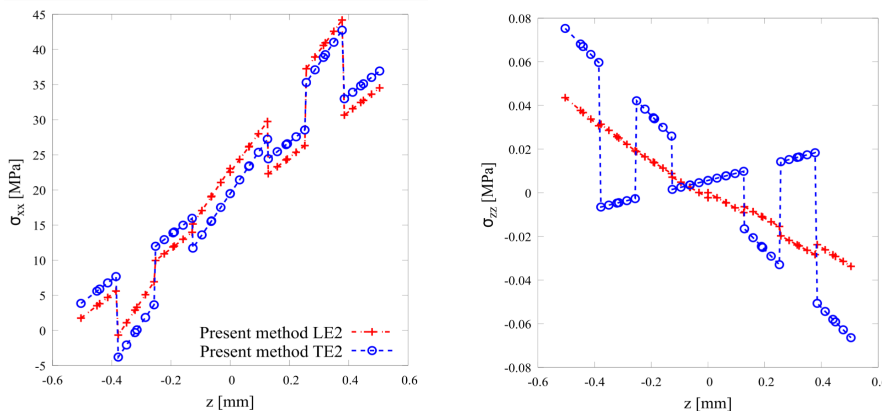

Figure 3: Stress distribution along the thickness Point A (75,30) mm. Comparison between LE2 and TE2 models.

Conclusions

The present work aims to show a nonlinear formulation for the post-buckling configuration of a rectangular plate made of a variable angle tow composite material. The governing equations are obtained from the principle of virtual displacement, and a comparison is conducted between the Lagrange and Taylor models for displacements and stresses. The findings indicate that the displacement description is accurate, even when the Taylor model is taken into consideration. However, to achieve optimal stress results, it is necessary to employ a Lagrange model in conjunction with the Layer wise description.

References

[1] Peters ST. Handbook of composites. 2nd ed. Boston, MA: Springer US; 1998. https://doi.org/10.1007/978-1-4615-6389-1

[2] Hyer MW, Charette RF. Use of curvilinear fiber format in composite structure design. AIAA J 1991;29(6):10115. https://doi.org/10.2514/3.10697

[3] A. Pagani, A. Racionero Sánchez-Majano & M. Petrolo. Influence of Structural Theories on Optimal Fiber Distributions in Tow-Steered Composites Considering Local Strain and Stress. Aerotecnica Missili & Spazio (2025). https://doi.org/10.1007/s42496-025-00275-3

[4] G. Manickam, M. Haboussi, M. D'Ottavio, V. Kulkarni, A. Chettiar, V. Gunasekaran. Nonlinear thermo-elastic stability of variable stiffness curvilinear fibres based layered composite beams by shear deformable trigonometric beam model coupled with modified constitutive equations. International journal of Non-Linear Mechanics, 148 (2023) 104303. https://doi.org/10.1016/j.ijnonlinmec.2022.104303

[5] F. Bracaglia, R. Masia, A. Pagani, E. Zappino, E. Carrera. Thermal buckling of variable stiffness composite laminates using high order plate finite elements. Composite Structures 345 (2024) 118393. https://doi.org/10.1016/j.compstruct.2024.118393

[6] E. Carrera, M. Cinefra, M. Petrolo, and E. Zappino. Finite Element Analysis of Structures through Unified Formulation. John Wiley & Sons, Chichester, West Sussex, UK, 2014. https://doi.org/10.1002/9781118536643

[7] Z. Gu ̈rdal, B.F. Tatting, C.K. Wu. Variable stiffness composite panels: Effects of stiffness variation on the in-plane and buckling response. Composites Part A: Applied Science and Manufacturing, 39(5):911-922, May 2008. https://doi.org/10.1016/j.compositesa.2007.11.015

[8] P. Chiaia, A. Pagani, M. Cinefra and E. Carrera. Analysis of transversely isotropic compressible and nearly-incompressible soft material structures by high order unified finite elements. Mechanics of Advanced Materials and Structures. 2023: 31(27), 9451-9467. https://doi.org/10.1080/15376494.2023.2273962

[9] M. A. Crisfield. An arc-length method including line searches and accelerations. International Journal for Numerical Methods in Engineering, 19(9):1269-1289, September 1983. https://doi.org/10.1002/nme.1620190902

[10] X . Chen, G. Nie and X. Yang. Thermal postbuckling analysis of variable angle tow composite cylindrical panels. Journal of Thermal Stresses,(2021) 44:7, 850-882. https://doi.org/10.1080/01495739.2021.1923429