A Convex Formulation of the Multi-Commodity Dynamic Traffic Assignment Problem

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Davide Sipione, *Graduate Student Member, IEEE*, Giacomo Como, *Member, IEEE*, and Gustav Nilsson, *Senior Member, IEEE*

***Abstract*—We consider a multi-commodity Dynamic Traffic Assignment (DTA) problem formulated as a network flow control problem on the Cell Transmission Model (CTM). The objective is to design optimal control policies using variable speed limits, ramp metering, and dynamic routing to regulate traffic evolution over time on a given limited-capacity transportation network. Even simple instances of DTA problems on the CTM are known to give rise to non-convex optimal control formulations. Nevertheless, a single-commodity DTA formulation has recently been proposed that admits a tight convex relaxation, enabling tractable optimal control synthesis. The single-commodity formulation, however, is structurally restrictive, as it effectively allows only a single destination. To address this limitation, we develop a multi-commodity CTM model in which each commodity is associated with potentially distinct sets of off-ramps. By extending the convexification approach developed for the single-commodity case, we establish a tight convex relaxation of the multi-commodity DTA problem on the CTM model. This relaxation relies on concave, commodity-specific demand functions and concave aggregate supply functions for every cell, which ensure convexity of the resulting optimal control problem. Our proposed formulation requires commodity-dependent implementation of variable speed limits and dynamic routing policies.**

***Index Terms*—Dynamic traffic assignment problem, convex optimal control, transportation networks.**

I. Introduction

THE Dynamic Traffic Assignment (DTA) problem, first introduced in [1], has become a cornerstone of transportation research [2]. While originally proposed for planning purposes, it has increasingly been leveraged in the context of optimal real-time feedback control [3].

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Davide Sipione is with the Department of Mathematical Sciences, Politecnico di Torino, 10129 Turin, Italy (e-mail: davide.sipione@polito.it).

Giacomo Como is with the Department of Mathematical Sciences, Politecnico di Torino, 10129 Turin, Italy, and also with the Department of Automatic Control, Lund University, 22100 Lund, Sweden (e-mail: giacomo.como@polito.it).

Gustav Nilsson is with Univ. Grenoble Alpes, Inria, CNRS, Grenoble INP, GIPSA-Lab, 38000 Grenoble, France (e-mail: gustav.nilsson@inria.fr).

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A macroscopic, first-order traffic model suitable for addressing the DTA problem is the celebrated Cell Transmission Model (CTM) [4]. Straightforward formulations of the DTA problem with CTM dynamics are known to yield non-convex formulations, mainly due to congestion effects [5]. However, in [6] a linear program is obtained by relaxing the supply and demand constraints of the CTM. These results are extended in [7] where a tight and convex relaxation of the DTA problem is formulated, for arbitrary network topology, arbitrary concave demand and supply functions, and convex cost functions.

However, none of these studies account for the heterogeneity of traffic flows. As autonomous vehicles become increasingly more common and with the ever-growing traffic demand, a shift in how freeway networks are managed and optimized has become necessary. Several studies have attempted to solve a multi-class/multi-commodity DTA problem. In [8] the population is split into controllable and selfish agents and the resulting nonlinear optimal control problem is then solved using the discrete adjoint method and a multi-start approach. Similarly, in [9] and [10] the population is split into the sets of compliant and non-compliant users, where the former follow the imposed routing blindly, and the latter employ a logit choice function at diverge junctions. However, these studies lack theoretical optimality guarantees.

In this letter, we employ a macroscopic model for heterogeneous traffic flows, based on an extension of the CTM. We focus on dynamical multi-commodity (MC) flow networks, modeled as dynamical systems driven by mass conservation laws on directed multigraphs. The model is then used to improve traffic performance through optimal control techniques. Specifically, we formulate a MC-DTA problem where the control action —variable speed limits, ramp metering, and dynamic routing— can be applied separately to every commodity. However, straightforward formulations of the MC-DTA problem lead to non-convex instances. To tackle this issue, we extend single-commodity results [7], to provide a tight and convex relaxation of the MC-DTA problem, assuming both concavity of the demand and supply functions and convexity of the cost function. Different from the single-commodity case [7], where the same routing has to be deployed to all traffic, it is possible to model each origin-destination pair as one commodity, and impose constraints to ensure that all the demands will reach their desired destination. Moreover, we draw insights from Pontryagin's Maximum Principle for the optimality of the proposed solution.

The rest of the letter is organized as follows. The dynamical MC flow network model is introduced in Section II. In Section III, we formalize the MC-DTA problem and provide a tight convex relaxation. In Section IV, we present some illustrative examples and compare the efficiency of the MC-DTA solution with respect to the solution of an optimal control problem where the routing is exogenously fixed and only variable speed limits and ramp metering can be controlled [11].

Notation: The sets of real, non-negative real, and positive real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$, respectively. For non-empty finite sets $\mathcal{A}$ and $\mathcal{B}$, $\mathbb{R}^{\mathcal{A}}$ is the set of $|\mathcal{A}|$-dimensional real vectors and $\mathbb{R}^{\mathcal{A}\times\mathcal{B}}$ is the set of $|\mathcal{A}|\times|\mathcal{B}|$ real matrices, whose entries are indexed by elements of $\mathcal{A}$ and $\mathcal{A}\times\mathcal{B}$, respectively.

II. A Controlled Dynamical Multi-Commodity CTM-Based Flow Network Model

In this section, we introduce a macroscopic model for dynamic multi-commodity traffic flows on networks.

A. Multi-Commodity Flow Network Model

We model a transportation network as a finite directed multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ denote the set of nodes and edges, respectively. From a physical point of view, edges —referred to as *cells*— represent sections of roads, while nodes correspond to junctions that connect subsequent cells. For every cell i in $\mathcal{E}$, let σ_i, τ_i in $\mathcal{V}\cup\{w\}$ denote its tail and head nodes, respectively. Here w is a virtual node representing the external world. We assume that there are no self-loops, i.e., $\sigma_i \neq \tau_i$ for every cell i in $\mathcal{E}$. The sets of on-ramps and off-ramps are denoted by $\mathcal{O} = \{i \in \mathcal{E} : \sigma_i = w\}$ and $\mathcal{S} = \{i \in \mathcal{E} : \tau_i = w\}$, respectively. The set of adjacent pairs of cells is denoted by $\mathcal{A} = \{(i, j) \in \mathcal{E}\times\mathcal{E} : \tau_i = \sigma_j \neq w\}$.

The network is shared by a nonempty finite set of commodities $\mathcal{K}$, which may be restricted to different subgraphs of $\mathcal{G}$. To every commodity k in $\mathcal{K}$ we associate a set $\mathcal{E}_k \subseteq \mathcal{E}$ of utilizable cells. We assume that, for every commodity k in $\mathcal{K}$, the sets of utilizable on-ramps $\mathcal{O}_k = \mathcal{O}\cap\mathcal{E}_k$ and off-ramps $\mathcal{S}_k = \mathcal{S}\cap\mathcal{E}_k$ are nonempty and that every utilizable cell i in $\mathcal{E}_k$ belongs to a path from some on-ramp o in $\mathcal{O}_k$ to some off-ramp r in $\mathcal{S}_k$. To each commodity is associated its own set of adjacent pairs of cells $\mathcal{A}_k$, i.e., $\mathcal{A}_k = \{(i, j) \in \mathcal{E}_k\times\mathcal{E}_k : \tau_i = \sigma_j \neq w\}$.

The state space of the system is

$$\mathcal{X} := \left\{x \in \mathbb{R}_+^{\mathcal{E}\times\mathcal{K}} : \ x_{ik} = 0\ ,\ \forall i \in \mathcal{E}\setminus\mathcal{E}_k\right\} .$$

Every entry $x_i^{(k)}$ of a state x in $\mathcal{X}$ represents the traffic volume of commodity k on cell i. Borrowing from the fundamental diagram of traffic flows, two families of functions are introduced. The maximum possible outflow $d_i^{(k)}(x_i^{(k)})$ from a cell i depends on the traffic volume $x_i^{(k)}$ through a demand function $d_i^{(k)} : \mathbb{R}_+ \to \mathbb{R}_+$. The maximum possible inflow $s_i\left(\sum_{k\in\mathcal{K}} w_i^{(k)} x_i^{(k)}\right)$ into a cell i depends on a weighted sum of all traffic volumes in cell i through a shared supply function $s_i : \mathbb{R}_+ \to \mathbb{R}_+$. Note that $w_i^{(k)} > 0$ is a positive weight for the traffic volume in cell i that is allowed to be dependent on the commodity, in case different commodities will occupy different amounts of space in the cell.

Assumption 1:

(i) The supply function of every cell i in $\mathcal{E}\setminus\mathcal{O}$ is continuously differentiable, concave, and non-increasing on $[0, x_i^{\text{jam}}]$, where $x_i^{\text{jam}} = \inf\{\rho > 0 : s_i(\rho) = 0\}$.

(ii) For every cell i in $\mathcal{E}_k$ and commodity k in $\mathcal{K}$, the demand function

$$d_i^{(k)}(x_i^{(k)}) = \min\left\{d_{i,1}^{(k)}(x_i^{(k)}), \dots, d_{i,n_i^{(k)}}^{(k)}(x_i^{(k)})\right\}$$

is the minimum of a positive finite number $n_i^{(k)}$ of continuously differentiable, concave, and non-decreasing functions $d_{i,h}^{(k)} : \mathbb{R}_+ \to \mathbb{R}_+$. Moreover, $d_i^{(k)}(0) = 0$.

The following example shows demand and supply functions that satisfy Assumption 1.

Example 1: Linear saturated demand and affine saturated supply functions

$$d_i^{(k)}(\xi) = \min\left\{\frac{v_i^{(k)}\xi}{L_i}, C_i^{(k)}\right\}\ ,\ s_i(\xi) = \max\left\{0, w_i(x_i^{\text{jam}} - \xi)\right\},$$

where $v_i^{(k)} \geq 0$ is the speed at which flow travels through the cell, $L_i > 0$ is the length of the cell, $C_i^{(k)} > 0$ is the commodity-specific maximum outflow capacity of the cell, $w_i > 0$ is the congestion wave speed.

B. Controlled Dynamical Multi-Commodity Flow Network

The flow of commodity k entering the network is modeled as an exogenous piecewise-continuous time-varying inflow $\lambda_i^{(k)}(t) \geq 0$ at each on-ramp i in $\mathcal{O}_k$. For every other cell i in $\mathcal{E}_k\setminus\mathcal{O}_k$ the exogenous inflow $\lambda_i^{(k)}(t)$ is identically set to 0. We denote by $\lambda^{(k)}(t) = (\lambda_i^{(k)}(t))_{i\in\mathcal{E}}$ the vector of exogenous inflows of commodity k across all cells and refer to $\lambda = (\lambda_i^{(k)})_{i\in\mathcal{E},k\in\mathcal{K}}$ as the *exogenous inflow array*.

Control is actuated both through variable speed limits combined with ramp metering and dynamic routing. The former determine the maximum outflow of each commodity k from each cell i as a fraction of the demand $d_i^{(k)}(x_i^{(k)})$, while the latter controls how outflow splits at diverge junctions. On the one hand, variable speed limits and ramp metering are modeled as a family α of piecewise-continuous functions $\alpha_i^{(k)} : [0, T] \to [0, 1]$ for i in $\mathcal{E}$ and k in $\mathcal{K}$, so that the maximum outflow of commodity k from cell i is

$$\bar{d}_i^{(k)}(\alpha_i^{(k)}, x_i^{(k)}) = \min\left\{d_{i,1}^{(k)}(x_i^{(k)}), \dots, \alpha_i^{(k)} d_{i,h_i^{(k)}}^{(k)}(x_i^{(k)}),\right.$$
$$\left.\dots, d_{i,n_i^{(k)}}^{(k)}(x_i^{(k)})\right\},$$

where the control acts only on the $h_i^{(k)}$-th demand component, for some $h_i^{(k)}$ in $\{1, \dots, n_i^{(k)}\}$. On the other hand, the turning ratios $M_{ij}^{(k)} \geq 0$ —representing the fraction of outflow of commodity k routed from cell i to cell j— must satisfy:

$$\sum_{j\in\mathcal{E}} M_{ij}^{(k)} = \begin{cases} 1 & i \in \mathcal{E}_k\setminus\mathcal{S}_k \\ 0 & i \in \mathcal{S}_k \end{cases}\ ; \tag{1}$$

$$M_{ij}^{(k)} = 0, \quad \forall (i, j) \in \mathcal{E}\times\mathcal{E}\setminus\mathcal{A}_k\ . \tag{2}$$

Equation (1) ensures that only off-ramps are allowed to send flow to the external world, while Equation (2) implies that flow can only travel through adjacent cells. For every commodity

k in $\mathcal{K}$, we can then define the set of admissible routing matrices $\mathcal{R}_k = \{M^{(k)} \in \mathbb{R}_+^{\mathcal{E}\times\mathcal{E}} : (1),(2)\}$. Dynamic routing is then specified as a family R of piecewise-continuous functions $R^{(k)} : [0,T] \to \mathcal{R}_k$, for k in $\mathcal{K}$. We shall refer to the pair (α, R) as a *feasible control action*.

To describe the MC flow dynamics, mass conservation dictates that the variation of the traffic volume $x_i^{(k)}$ of commodity k in $\mathcal{K}$ in a cell i in $\mathcal{E}$ is equal to the difference between the total inflow to and the total outflow from that cell, i.e.,

$$\dot{x}_i^{(k)} = \lambda_i^{(k)} + \sum_{j\in\mathcal{E}_k} f_{ji}^{(k)}(x,\alpha,R) - z_i^{(k)}(x,\alpha,R)\,, \tag{3}$$

where $f_{ji}^{(k)}(x,\alpha,R)$ denotes the flow sent from cell j to cell i and $z_i^{(k)}(x,\alpha,R)$ denotes the total outflow from cell i as

$$z_i^{(k)}(x,\alpha,R) = \begin{cases} \sum_{j\in\mathcal{E}_k} f_{ij}^{(k)}(x,\alpha,R) & i \in \mathcal{E}_k \setminus \mathcal{S}_k \\ \bar{d}_i^{(k)}(\alpha_i^{(k)}, x_i^{(k)}) & i \in \mathcal{S}_k\,, \end{cases} \tag{4}$$

both expressed as functions of the current state x. Let $\dot{x}^{(k)} = (\dot{x}_i^{(k)})_{i\in\mathcal{E}}$ and $\dot{x} = (\dot{x}_i^{(k)})_{i\in\mathcal{E},k\in\mathcal{K}}$ denote the vector of commodity k traffic variation and the matrix of traffic variation, respectively. We are now left with modeling the dependence of the cell-to-cell flows $f_{ij}^{(k)}$ on the state x. For this, we employ a First-In-First-Out (FIFO) allocation rule. Formally, we set

$$f_{ij}^{(k)}(x,\alpha,R) = \gamma_i^{(k)}(x,\alpha,R) R_{ij}^{(k)} \bar{d}_i^{(k)}(\alpha_i^{(k)}, x_i^{(k)})\,, \tag{5}$$

where $\gamma_i^{(k)}(x,\alpha,R) = \max\{\tau \in [0,1] : (6)\}$,

$$\tau \cdot \max_{\substack{j\in\mathcal{E}:\\ R_{ij}^{(k)}>0}} \sum_{\bar{k}\in\mathcal{K}} \sum_{h\in\mathcal{E}} R_{hj}^{(\bar{k})} \bar{d}_h^{(\bar{k})}(\alpha_h^{(\bar{k})}, x_h^{(\bar{k})}) \le s_j \left(\sum_{\bar{k}\in\mathcal{K}} w_j^{(\bar{k})} x_j^{(\bar{k})} \right). \tag{6}$$

Equation (6) implies that in a diverge junction, if for a commodity k in $\mathcal{K}$ a downstream cell i in $\mathcal{E}_k$ is congested then the flow directed to every downstream cell in $\mathcal{E}_k$ is limited accordingly. Moreover, in ordinary junctions, the flow is split proportionally among commodities that share that cell when congestion downstream occurs.

Definition 1: Given a nonempty finite directed multigraph $\mathcal{G} = (\mathcal{V},\mathcal{E})$, supply functions s_i and demand functions $d_i^{(k)}$ satisfying Assumption 1, an exogenous inflow array λ, and a feasible control action (α, R), a controlled Dynamical Multi-Commodity Flow Network (DMCFN) is the control system defined by Equations (3)–(6).

Remark 1: Our model assumes that commodity-specific demand functions $d_i^{(k)}$ are only dependent on the state of the same commodity. Physically, this corresponds to scenarios in which, when the flow is not limited by congestion through the supply functions, there is enough space for commodities to overtake each other, i.e., each link has multiple lanes. This modeling approach has previously been used in, e.g., [12].

We establish well-posedness of the initial value problem.

Lemma 1: For every initial condition $x(0)$ in $\mathcal{X}$ and feasible control action (α, R), there exists a unique solution $x(t)$ of the controlled DMCFN (3)–(6).

Proof: For every commodity k in $\mathcal{K}$ and cell i in $\mathcal{E}$,

$$g_i^{(k)}(t,x) = \lambda_i^{(k)}(t) + \sum_j f_{ji}^{(k)}(x,\alpha(t),R(t)) - z_i^{(k)}(x,\alpha(t),R(t))$$

is piecewise-continuous in $t \in \mathbb{R}_+$ and globally Lipschitz in $x \in \mathcal{X}$, due to Assumption 1. From the Picard-Lindelöf theorem it follows that, for every initial state $x(0)$ in $\mathcal{X}$, there exists a unique solution $x : \mathbb{R}_+ \to \mathcal{X}$ of (3)–(6). $\square$

C. Optimal Multi-Commodity Dynamic Traffic Assignment

We are interested in optimizing the time-integral of a running cost function $\varphi(x,z)$, which depends on the traffic volume x and cell outflow z, over a finite time horizon $[0,T]$.

Assumption 2: The running cost function $\varphi(x,z)$ is convex, differentiable, non-decreasing in x, non-increasing in z, and such that $\varphi(0,0) = 0$.

For every feasible control action (α, R), let

$$J(\alpha,R) = \int_0^T \varphi(x(t),z(t))dt\,,$$

be the total cost associated with the corresponding controlled DMCFN defined by Equations (3)–(6). Then, we aim to solve the following optimal control problem.

Problem 1 (Continuous time MC-DTA):

$$J^* = \min_{(\alpha,R)} J(\alpha,R)\,, \tag{7}$$

where the minimization runs over all feasible control actions.

Note that Problem 1 is non-convex in general, mainly due to congestion constraints at diverge junctions [5].

Example 2: The Total Travel Time (TTT) running cost function

$$\varphi(x,z) = \sum_{i\in\mathcal{E}} \sum_{k\in\mathcal{K}} x_i^{(k)}\,, \tag{8}$$

satisfies Assumption 2. In this case, the total cost

$$\int_0^T \varphi(x,z)dt = \sum_{i\in\mathcal{E}} \sum_{k\in\mathcal{K}} \int_0^T x_i^{(k)}(t)dt\,,$$

represents the total time spent by all vehicles traveling through the network during the time span $[0,T]$.

III. Tight Convexification of the MC-DTA Problem

In this section, we first introduce a convex relaxation of Problem 1 and then show that it is tight. The key insight behind this relaxation is to shift the optimization variables from the control action variables $\alpha_i^{(k)}(t)$ and $R_{ij}^{(k)}(t)$ to the traffic volume variables $\bar{x}^{(k)}(t)$ in $\mathbb{R}_+^{\mathcal{E}_k}$ and the traffic flow variables $\bar{f}^{(k)}(t)$ in $\mathbb{R}_+^{\mathcal{A}_k}$ and $\mu_i^{(k)}$ in $\mathbb{R}_+^{\mathcal{E}_k}$, such that $\mu_i^{(k)} = 0$ for every i in $\mathcal{E}_k \setminus \mathcal{S}_k$. Regarding the traffic flow variables, $\bar{f}$ is the flow between cells within the network, while μ captures the flow that leaves the network.

The introduced optimization variables must satisfy the following demand and supply constraints, which relax the controlled DMCFN equations (3)–(6) by replacing equalities with the following inequalities for every cell i in $\mathcal{E}$:

(i) demand constraints

$$\mu_i^{(k)}(t) + \sum_{j\in\mathcal{E}_k} \bar{f}_{ij}^{(k)}(t) \le d_{i,h}^{(k)}(\bar{x}_i^{(k)}(t))\,, \tag{9}$$

for every commodity k in $\mathcal{K}$ and $h = 1, \ldots, n_i^{(k)}$;

(ii) supply constraints for every cell i in $\mathcal{E} \setminus \mathcal{O}$

$$\sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{E}_k} \bar{f}_{ji}^{(k)}(t) \leq s_i \left(\sum_{k\in\mathcal{K}} w_i^{(k)} \bar{x}_i^{(k)}(t) \right) . \quad (10)$$

Let $\bar{u}(t)$ be the concatenation of vectors $(\bar{f}(t), \mu(t))$ and write the demand and supply constraints (9)–(10) and the constraint $\mu_i^{(k)} = 0$ for every i in $\mathcal{E} \setminus \mathcal{S}_k$ in compact vector form as $D\bar{u}(t) \leq h(\bar{x}(t))$, where $D \in \mathbb{R}_+^{r\times m}$ with $r = |\mathcal{E} \setminus \mathcal{O}| + \sum_{k\in\mathcal{K}} \left(|\mathcal{E}_k \setminus \mathcal{S}_k| + \sum_{i\in\mathcal{E}_k} n_i^{(k)} \right)$ and $m = \sum_{k\in\mathcal{K}} (|\mathcal{A}_k| + |\mathcal{E}_k|)$. Then, for every feasible pair $(\bar{x}, \bar{u})$, let

$$\bar{J}(\bar{x}, \bar{u}) = \int_0^T \bar{\varphi}(\bar{x}(t), \bar{u}(t))dt ,$$

be the associated total cost, where $\bar{\varphi}(\bar{x}, \bar{u}) := \varphi(\bar{x}, \zeta(\bar{u}))$ and $(\zeta(\bar{u}))_i^{(k)} = \sum_{j\in\mathcal{E}_k} \bar{f}_{ij}^{(k)} + \mu_i^{(k)}$. Let also

$$q_i^{(k)}(t, \bar{x}, \bar{u}) = \lambda_i^{(k)}(t) + \sum_{j\in\mathcal{E}_k} \bar{f}_{ji}^{(k)} - \sum_{j\in\mathcal{E}_k} \bar{f}_{ij}^{(k)} - \mu_i^{(k)} ,$$

and $q(t, \bar{x}, \bar{u}) = (q_i^{(k)}(t, \bar{x}, \bar{u}))_{i\in\mathcal{E}, k\in\mathcal{K}}$. Then, the relaxation of the MC-DTA problem is defined as follows in terms of the linear control system

$$\dot{\bar{x}} = q(t, \bar{x}, \bar{u}) , \quad (11)$$

with mixed constraints

$$D\bar{u}(t) - h(\bar{x}(t)) \leq 0 , \qquad \forall 0 \leq t \leq T . \quad (12)$$

Problem 2 (Relaxation of the continuous time MC-DTA):

$$\bar{J}^* = \min_{(\bar{x},\bar{u})} \ \bar{J}(\bar{x}, \bar{u})$$
$$\text{subject to} \ \ (11), (12), \ \bar{x}(0) = \bar{x}^0 .$$

First, we prove the existence of an optimal control.

Proposition 1: Given a running cost function satisfying Assumption 2, for every initial condition x^0 in $\mathcal{X}$, Problem 2 admits an optimal solution.

Proof: Let $A \subseteq [0, T] \times \mathcal{X}$ denote the set of admissible time-state pairs for the control system (11) with constraints (12). For every non-on-ramp cell i in $\mathcal{E}_k \setminus \mathcal{O}$, admissible states satisfy $0 \leq \sum_{k\in\mathcal{K}} w_i^{(k)} \bar{x}_i^{(k)} \leq x_i^{\text{jam}}$. On the other hand, for every on-ramp j in $\mathcal{O}_k$, we have $0 \leq \bar{x}_j^{(k)}(t) \leq \bar{x}_j^{(k),0} + \int_0^T \lambda_j^{(k)}(s)ds$. Therefore, all state components are uniformly bounded on $[0, T]$ hence A is bounded. Since q is linear, we have that A is closed. Hence, A is compact. Moreover, for every t in $[0, T]$, the set $A(t) = \{\bar{x} \in \mathcal{X} : (t, \bar{x}) \in A\}$ is convex. It is possible to verify that the set of admissible flows $\mathcal{U}(x) = \left\{\bar{u} \in \mathbb{R}_+^m \mid D\bar{u} - h(\bar{x}) \leq 0\right\}$ is compact and convex.

The set of admissible time-state-flow triples $M = \{(t, \bar{x}, \bar{u}) \in [0, T] \times \mathcal{X} \times \mathbb{R}_+^m : (t, \bar{x}) \in A , \bar{u} \in \mathcal{U}(x)\}$ is also compact. To see this, let $U = \prod_{\ell=1}^m [0, \bar{u}_\ell^{\max}]$, where for every $\ell = 1, \ldots, m$ we let $\bar{u}_\ell^{\max} = \max_{(t,\bar{x})\in A} \max_{\bar{u}\in U(x)} u_\ell$. Since h is continuous and every $\bar{u}$ is bounded by some demand or supply constraint it holds that $\bar{u}_\ell^{\max} < \infty$ for every ℓ, so that U is a compact set. Hence, $M \subseteq A \times U$ and since $D\bar{u} - h(\bar{x})$ is continuous in (x, u) also M is compact. Then, the set

$$\tilde{Q}(t, \bar{x}) = \left\{ \begin{pmatrix} q(t, \bar{x}, \bar{u}) \\ \phi^0 \end{pmatrix} \middle| \ \bar{u} \in \mathcal{U}(\bar{x}) , \bar{\phi}^0 \geq \bar{\varphi}(\bar{x}, \bar{u}) \right\}$$

is convex for every $\bar{x}$ in $\mathcal{X}$, since $\bar{\varphi}(\bar{x}, \bar{u})$ is convex in $(\bar{x}, \bar{u})$ and $q(t, \bar{x}, \bar{u})$ is affine in $\bar{u}$. Hence, the assumptions of [13, Theorem 9.3.i] hold and Problem 2 admits an optimal solution. □

Notice that Problem 2 is a convex relaxation of Problem 1.

Proposition 2: Problem 2 is convex, and every feasible solution of Problem 1 is also feasible for Problem 2.

Proof: First, notice that for every choice of control parameters $\alpha_i^{(k)}(t)$ in $[0, 1]$ and turning ratios $R_{ij}^{(k)}(t)$ the demand (9) and supply constraints (10) are inevitably satisfied. Moreover, every feasible solution of Problem 1 is also feasible for Problem 2. Then, for convexity, notice that if both $\left(\bar{x}^0(t), \bar{u}^0(t)\right)$ and $\left(\bar{x}^1(t), \bar{u}^1(t)\right)$ satisfy the constraints, then for every β in $[0, 1]$, also $\left(\bar{x}^\beta(t), \bar{u}^\beta(t)\right)$ does, where $\bar{x}^\beta = (1 - \beta)\bar{x}^0 + \beta\bar{x}^1$ and $\bar{u}^\beta = (1 - \beta)\bar{u}^0 + \beta\bar{u}^1$. This implies that

$$\int_0^T \bar{\varphi}(\bar{x}^\beta(t), \bar{u}^\beta(t))dt \leq (1 - \beta) \int_0^T \bar{\varphi}(\bar{x}^0(t), \bar{u}^0(t))dt + \beta \int_0^T \bar{\varphi}(\bar{x}^1(t), \bar{u}^1(t))dt \quad (13)$$

Since $\bar{\varphi}(\bar{x}, \bar{u})$ is convex by assumption, then inequality (13) is always satisfied. Moreover, the concavity of the demand and supply functions implies the convexity of the constraints (9) and (10) and the other constraints are linear equalities. □

It remains to establish the tightness of the above convex relaxation. In particular, we prove that for every solution of Problem 2, there exists a corresponding selection of control action (α, R) such that equations (3)–(6) are satisfied. To this end, consider a solution $(\bar{x}, \bar{u})$ and let

$$\alpha_i^{(k)}(t) = \frac{\mu_i^{(k)}(t) + \sum_{j\in\mathcal{E}_k} \bar{f}_{ij}^{(k)}(t)}{d_{i,h_i^{(k)}}^{(k)}(\bar{x}_i^{(k)}(t))}, \qquad \forall i \in \mathcal{E}_k , \quad (14)$$

$$R_{ij}^{(k)}(t) = \begin{cases} \dfrac{\bar{f}_{ij}^{(k)}(t)}{\sum_{l\in\mathcal{E}} \bar{f}_{il}^{(k)}(t) + \mu_i^{(k)}(t)} & \forall (i, j) \in \mathcal{A}_k \\ 0 & \forall (i, j) \in \mathcal{E} \times \mathcal{E} \setminus \mathcal{A}_k \end{cases} , \quad (15)$$

with the convention that $\alpha_i^{(k)} = 1$ if $\mu_i^{(k)} + \sum_{j\in\mathcal{E}} \bar{f}_{ij}^{(k)} = d_{i,h_i^{(k)}}^{(k)}(\bar{x}_i^{(k)}) = 0$. If $\sum_{j\in\mathcal{E}} \bar{f}_{ij}^{(k)} + \mu_i^{(k)} = 0$, then the turning ratios are uniform over the set of outgoing cells, i.e. $R_{ij}^{(k)} = |\{l \in \mathcal{E}_k : (i, l) \in \mathcal{A}_k\}|^{-1}$. The intuition behind (14) and (15) is to set the speed limit to the fraction of maximum demand actually leaving the cell, and the routing to the ratio of flow over total outflow of each commodity.

Proposition 3: For a feasible solution $(\bar{x}(t), \bar{u}(t))$ of Problem 2, let the control action (α, R) be defined as in (14) and (15), respectively. Then,

(i) (α, R) is a feasible solution of Problem 1;

(ii) $J(\alpha, R) \leq \bar{J}(\bar{x}, \bar{u})$.

Proof: Let $(\bar{x}(t), \bar{u}(t))$ be a feasible solution. It follows from the demand constraints (9) that the outflow is less than or equal to the demand, i.e., $\mu_i^{(k)} + \sum_{j\in\mathcal{E}} \bar{f}_{ij}^{(k)} \leq d_i^{(k)}(\bar{x}_i^{(k)})$. Hence, when selecting the control parameters as in (14), the outflow of every cell corresponds exactly to the demand. Formally, $\mu_i^{(k)} + \sum_{j\in\mathcal{E}_k} \bar{f}_{ij}^{(k)} = \bar{d}_i^{(k)}(\alpha_i^{(k)}, \bar{x}_i^{(k)})$ for all $i \in \mathcal{E}_k$. Moreover, it follows that $\bar{f}_{ij}^{(k)} = R_{ij}^{(k)} \sum_{l \in \mathcal{E}_k} \bar{f}_{il}^{(k)} = R_{ij}^{(k)} \bar{d}_i^{(k)}(\alpha_i^{(k)}, \bar{x}_i^{(k)})$ for all (i, j) in $\mathcal{A}_k$, which implies that for every j in $\mathcal{E}_k \setminus \mathcal{S}_k$,

$$\sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{E}_k} R_{ij}^{(k)} \bar{d}_i^{(k)}(\alpha_i^{(k)}, \bar{x}_i^{(k)}) \leq s_j \left(\sum_{k \in \mathcal{K}} w_j^{(k)} \bar{x}_j^{(k)} \right) .$$

The last inequality follows from the supply constraint (10), which implies that there is no congestion, i.e., $\gamma_i^{(k)} = 1$, for every k in $\mathcal{K}$ and i in $\mathcal{E}_k$. After that it is easy to verify that the choice of routing matrices $R^{(k)}(t)$ is in $\mathcal{R}_k$ for every k and t. Hence, $x = \bar{x}$, $f = \bar{f}$ for every cell, and $z = \mu$ for every off-ramp cell. Moreover, the equations (3), (4), (5), (6), are satisfied so that (α, R) is a solution of the original Problem 1. To prove (ii), notice that setting α and R as in (14), (15) induces the same values of traffic volumes and flows to the original problem, which implies $J(\alpha, R) \leq \bar{J}(\bar{x}, \bar{u})$. □

Theorem 1: Suppose the cost function satisfies Assumption 2 and for every initial condition x^0 in $\mathcal{X}$ set α^* and R^* as in (14) and (15), respectively. Then the following statements are equivalent.

(i) (α^*, R^*) is an optimal solution of Problem 1;
(ii) $(\bar{x}^*, \bar{u}^*)$ is an optimal solution of Problem 2;
(iii) there exists $\chi : [0, T] \to \mathbb{R}^{\mathcal{E} \times \mathcal{K}}$, $\eta : [0, T] \to \mathbb{R}_+^r$ such that $(\bar{x}^*, \bar{u}^*)$ satisfy the PMP conditions

$$H(\bar{x}, \bar{u}, \chi) = -\bar{\varphi}(\bar{x}, \bar{u}) + \langle \chi, \dot{\bar{x}} \rangle , \quad (16)$$

$$\eta^\top (D\bar{u}^* - h(\bar{x}^*)) = 0 , \quad (17)$$

$$\dot{\chi}_i^{(k)} = \frac{\partial \bar{\varphi}(\bar{x}^*, \bar{u}^*)}{\partial \bar{x}_i^{(k)}} - \frac{\partial h(\bar{x}^*)}{\partial \bar{x}_i^{(k)}}^\top \eta , \quad (18)$$

$$\frac{\partial}{\partial \bar{u}} H(\bar{x}^*, \bar{u}^*, \chi) = D^\top \eta , \quad (19)$$

$$\max_{\bar{u} \in \mathbb{R}_+^m : D\bar{u} \leq h(\bar{x})} H(\bar{x}^*, \bar{u}, \chi) = H(\bar{x}^*, \bar{u}^*, \chi) , \quad (20)$$

$$\chi(T) = 0 , \qquad \chi(t) \neq 0 , \quad \forall 0 \leq t < T . \quad (21)$$

Proof: We first focus on proving the equivalence of (i) and (ii). Let $(\bar{x}^*, \bar{u}^*)$ be the optimal control solution of Problem 2, whose existence is proven in Proposition 1. From Proposition 3, we know that we can map every feasible solution $(\bar{x}, \bar{u})$ of Problem 2 to a feasible solution (α, R) of Problem 1, such that $J(\alpha, R) \leq \bar{J}(\bar{x}, \bar{u})$. By taking the infimum on both sides we get $\inf J(\alpha, R) \leq \bar{J}^*$ and from Proposition 2 we know that every solution of the original problem is feasible for the relaxed one, such that $\bar{J}^* \leq \inf J(\alpha, R)$. Combining the two inequalities yields $J^* = \inf J(\alpha, R) = \bar{J}^*$. Thus, Problem 1 admits an optimal solution (α^*, R^*).

We are now left to prove that (ii) is equivalent to (iii). This reduces to proving that the PMP conditions are both necessary and sufficient for optimality. For necessity, since the functions s_i and $d_{i,h}^{(k)}$ are differentiable by Assumption 1, the continuity assumptions of the Pontryagin Maximum Principle with mixed constraints [14] are satisfied. To verify the regularity condition in [14, Definition 2], let $\mathcal{B}_a^n$ be the ball of dimension n and radius a with center at the origin. We need to prove that there exists $\delta > 0$ such that $\mathcal{B}_\delta^r \subseteq D\mathcal{B}_1^m + Du^* - h(x^*) + \mathbb{R}_+^r$. To verify this, it is sufficient to prove that $\mathrm{Im}(D) \cap \mathbb{R}_{++}^r \neq \emptyset$. Indeed, if this is the case, then there exists v in $(\mathbb{R}_{++}^r \cap D\mathcal{B}_1^m)$, so that it is sufficient to take $\delta = \min_i v_i > 0$. When $D \in \mathbb{R}_+^{r \times m}$, as is our case, it is sufficient to verify that $D\mathbf{1} > 0$, which is true in our setting since D is a nonnegative matrix with no null rows (as every constraint involves at least one control variable). Lastly, sufficiency follows by exploiting convexity of the problem and noting that the PMP conditions match those in [15, Corollary 1.(c)]. □

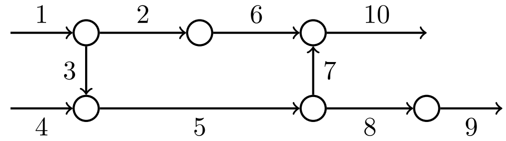

Fig. 1. Synthetic network with 2 on-ramps and 2 off-ramps.

IV. Examples

A. An Analytical Example

We exploit the PMP conditions (16)–(21) in a simple setting to understand how the optimal controller behaves. To this end, consider the diverge junction composed of cells 1, 2, and 3 in Figure 1, and assume that in this example cells 2 and 3 are off-ramps.

The network is shared by two commodities, namely a and b, but commodity b is restricted to cells 1 and 2, while commodity a can travel freely through the network. For simplicity, we consider the following demand and supply functions $d_i^{(k)}(x_i^{(k)}) = x_i^{(k)}$ and $s_i(\rho) = \gamma - \rho$ with weights $w_i^{(a)} = w_i^{(b)} = 1$. Consider the TTT running cost function (8). Let χ denote the adjoint state so that the Hamiltonian (16) is

$$H(x, f, \chi) = c(x, \chi, \lambda) + \kappa_{12}^{(a)} f_{12}^{(a)} + \kappa_{12}^{(b)} f_{12}^{(b)} + \kappa_{13}^{(a)} f_{13}^{(a)} , \quad (22)$$

where $\kappa_{ij}^{(k)} = \chi_j^{(k)} - \chi_i^{(k)}$ and $c(x, \chi, \lambda) = -x_1^{(a)} - x_1^{(b)} - x_2^{(a)} - x_2^{(b)} - x_3^{(a)} + \chi_1^{(a)} \lambda_1^{(a)} + \chi_1^{(b)} \lambda_1^{(b)} - \chi_2^{(a)} x_2^{(a)} - \chi_2^{(b)} x_2^{(b)} - \chi_3^{(a)} x_3^{(a)}$. With $\eta_i \geq 0$ for $i = 1, \ldots, 4$, (17) reduces to

$$\eta_1 \big(f_{12}^{(a)} + f_{13}^{(a)} - x_1^{(a)} \big) = 0 \, , \; \eta_2 \big(f_{12}^{(b)} - x_1^{(b)} \big) = 0 \, ,$$
$$\eta_3 \big(f_{12}^{(a)} + f_{12}^{(b)} - \gamma + x_2^{(a)} + x_2^{(b)} \big) = 0 \, , \quad \eta_4 \big(f_{13}^{(a)} - \gamma + x_3^{(a)} \big) = 0 \, ,$$

and the adjoint dynamics (18) read

$$\dot{\chi}_1^{(a)} = 1 - \eta_1 \, , \quad \dot{\chi}_1^{(b)} = 1 - \eta_2 \, , \quad \dot{\chi}_2^{(a)} = 1 + \chi_2^{(a)} + \eta_3 \, ,$$
$$\dot{\chi}_2^{(b)} = 1 + \chi_2^{(b)} + \eta_3 \, , \quad \dot{\chi}_3^{(a)} = 1 + \chi_3^{(a)} + \eta_4 \, .$$

Whenever $\chi_2^{(a)} \geq 0$, we have $\dot{\chi}_2^{(a)} \geq 1$. Since $\chi_2^{(a)}(T) = 0$ by (21), this implies that $\chi_2^{(a)}(t) < 0$ for every t in $[0, T)$. Then, non-negativity of the η_i's and the stationarity condition (19) imply that

$$0 \leq \eta_1(t) + \eta_3(t) = \kappa_{12}^{(a)}(t) = \chi_2^{(a)}(t) - \chi_1^{(a)}(t) < -\chi_1^{(a)}(t) , \quad (23)$$

so that $\chi_1^{(a)}(t) < 0$, for every $0 \leq t < T$. Similarly, we find that, for $0 \leq t < T$, $\chi_2^{(b)}(t) < 0$, so that

$$0 \leq \eta_2(t) + \eta_3(t) = \kappa_{12}^{(b)}(t) = \chi_2^{(b)}(t) - \chi_1^{(b)}(t) < -\chi_1^{(b)}(t) , \quad (24)$$

and $\chi_1^{(b)}(t) < 0$. Moreover,

$$0 \leq \eta_1(t) + \eta_4(t) = \kappa_{13}^{(a)} . \quad (25)$$

It then follows from (22) and (25) that the maximum in (20) is achieved necessarily with $f_{13}^{(a)} = \min\{x_1^{(a)}, \gamma - x_3^{(a)}\}$. On the

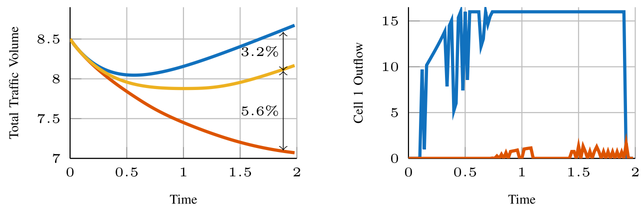

Fig. 2. Left: Total traffic volume over time, in blue uncontrolled dynamics, in yellow control as in [11], and in red the optimal control. Right: Cell 1 Outflow over time: in blue commodity a and in red commodity b.

other hand, to determine the value of $f_{12}^{(a)}$ and $f_{12}^{(b)}$ that achieve the maximum in (20), we exploit (23) and (24) by considering separately the following cases:

(i) if $f_{13}^{(a)} = x_1^{(a)}$, then $f_{12}^{(a)} = 0$, because the demand of commodity a has already been fully satisfied, and $f_{12}^{(b)} = \min\{x_1^{(b)}, \gamma - x_2^{(a)} - x_2^{(b)}\}$.

(ii) if $f_{13}^{(a)} = \gamma - x_3^{(a)}$ and $2\gamma - x_2^{(a)} - x_2^{(b)} - x_3^{(a)} \geq x_1^{(a)} + x_1^{(b)}$, i.e., the sum of the supplies is bigger than the sum of the demands, then $f_{12}^{(a)} = x_1^{(a)} - f_{13}^{(a)}$ and $f_{12}^{(b)} = x_1^{(b)}$.

(iii) if $f_{13}^{(a)} = \gamma - x_3^{(a)}$ and $2\gamma - x_2^{(a)} - x_2^{(b)} - x_3^{(a)} < x_1^{(a)} + x_1^{(b)}$, i.e., the sum of the supplies is smaller than the sum of the demands, then $\eta_1 = \eta_2 = 0$ by (17). Hence, (19) implies that $\kappa_{12}^{(a)} - \kappa_{12}^{(b)} = \eta_3 - \eta_3 = 0$, which in turn implies that any control such that $f_{12}^{(a)} + f_{12}^{(b)} = \gamma - x_2^{(a)} - x_2^{(b)}$ is optimal. In particular, in this case an optimal strategy is to block one of the commodities and assign the whole available supply in cell 2 to the other commodity.

B. A Numerical Example

We consider the synthetic network in Fig. 1, shared by commodities a and b, with $\mathcal{O}_a = \{1\}$, $\mathcal{O}_b = \{1, 4\}$, $\mathcal{S}_a = \{9, 10\}$, $\mathcal{S}_b = \{10\}$, and cell 3 unavailable to commodity b. We set $d_i^{(a)}(x_i^{(a)}) = 4x_i^{(a)}$, $d_i^{(b)}(x_i^{(b)}) = 2x_i^{(b)}$, $s_i(\rho) = 4 - \rho$, $w_i^{(a)} = 4$, and $w_i^{(b)} = 2$. Finally, we use constant inflows $\lambda_1^{(a)} = \lambda_1^{(b)} = \lambda_4^{(b)} = 1$, initial conditions $x_i^{(k)} = 0.5$, the TTT cost (8), and horizon $T = 2$.

We compare the performance of the optimal solution of the MC-DTA problem both with an uncontrolled MC network flow dynamics and with the solution of the related optimal control problem discussed in [11] where only variable speed limits and ramp metering $\alpha(t)$ can be designed while the routing $R(t)$ is exogenously fixed. The optimal solutions are obtained numerically by solving discrete-time versions[1] of the optimal control problems, which result in finite-dimensional convex problems, and using the software CVX [16]. The total traffic volume over time is reported on the left side of Figure 2. The control with fixed routing improves performance by 3.2% with respect to the uncontrolled scenario, while adding dynamic routing provides an additional 5.6% improvement. It is worth observing that the simulation indicates that it is optimal to block the outflow of one of the two commodities from cell 1, consistently with case (iii) discussed in the analytical example of Section IV.A.

[1]Specifically, the time interval is split in 100 subintervals of equal length 0.02.

V. Conclusion

We have proposed a dynamical multi-commodity flow network model and formulated a dynamical traffic assignment problem, which admits a tight convex relaxation, thus paving the way for the use of available convex optimization solvers. Future research should aim to develop a distributed algorithm to solve the problem and to account for implementation issues, such as discretized control actions.

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