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On the Power of One-tailed Tests when the Test Statistic is Monotone in the Hazard Rate Stochastic Order

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Abstract:

- A direct consequence of the Karlin-Rubin Theorem is that, under appropriate assumptions on the family of distributions corresponding to the statistic used, one-tailed tests for a population parameter based on one-sided rejection regions are at least as powerful as those based on interval rejection regions. Fundamentally, it requires the family of distributions associated with the statistic to be increasing with the parameter in the likelihood ratio stochastic order. In this note it is shown that the same conclusion can be obtained by replacing the likelihood ratio order with the weaker hazard rate order, thus extending its validity to more cases.

Keywords:

- *one-tailed tests; Karlin-Rubin theorem; Neyman-Pearson lemma; hazard rate stochastic order; likelihood ratio stochastic order; usual stochastic order.*

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- 62F03, 60E15.

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1. INTRODUCTION

Let $\vec{X}$ be a random sample from a population whose distribution is in the parametric family $\{P_\theta, \theta \in \Theta\}$ and let $T(\vec{X})$ be a sufficient statistic for the parameter θ, with probability density function (pdf) or probability mass function (pmf) belonging to the family $\{g(\cdot \mid \theta) : \theta \in \Theta\}$. As a direct consequence of the well-known Karlin-Rubin Theorem (Karlin and Rubin, 1956), if the family $\{g(\cdot \mid \theta) : \theta \in \Theta\}$ satisfies the monotone likelihood ratio property, then one-tailed tests of the form $H_0 : \theta \leq \theta_0$ and $H_1 : \theta > \theta_0$, when based on the rejection region $\{T(\vec{X}) > t_0\}$, are at least as powerful as when based on any rejection region $\{T(\vec{X}) \in (t_1, t_2])\}$ while keeping the significance level α unchanged.

The main purpose of this note is to show that such a statement remains valid under weaker assumptions on the distribution of $T(\vec{X})$, namely when its laws satisfy a monotonicity property based on the hazard rate order rather than on the stronger likelihood ratio order, thus expanding the class of statistics to which the statement applies. To this end, some useful notions are introduced below. Note that here the terms *increasing* and *decreasing* are used in a non-strict sense.

Recall that, given a random variable X, its survival function is defined as $\overline{F}_X(t) = P(X > t)$, its distribution function is defined as $F_X(t) = 1 - \overline{F}_X(t)$, its pdf is defined as the function f_X such that $\int_a^b f_X(t)dt = P(X \in [a, b])$, if the random variable is continuous, and its pmf is defined as $p_X(x) = P(X = x)$, if the random variable is discrete. Moreover, the support of X, $S(X)$, is defined as the set of points in which $\overline{F}$ strictly decreases.

1.1. Location stochastic orders

In order to compare the location of two random variables, several criteria have been proposed in the literature. One of the most common approaches is to use stochastic orders, (see Belzunce et al., 2015; Mosler and Scarsini, 2012; Shaked and Shanthikumar, 2007). In the following, the main location stochastic orders that are necessary for the development of the paper are recalled, with the convention that $\frac{a}{0}$ equals $+\infty$ for any $a \neq 0$.

Definition 1.1. (Shaked and Shanthikumar, 2007) Let X and Y be two random variables. Then,

- If $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for any $t \in \mathbb{R}$, X is said to be smaller than or equal to Y in the *usual stochastic order*, and it is denoted as $X \leq_{st} Y$.
- If $\frac{\overline{F}_Y(t)}{\overline{F}_X(t)}$ is increasing in t on $S(X) \cup S(Y)$, X is said to be smaller than or equal to Y in the *hazard rate order*, and it is denoted as $X \leq_{hr} Y$.
- If X and Y are continuous [discrete] and $\frac{f_Y(t)}{f_X(t)}$ $\left[\frac{p_Y(t)}{p_X(t)}\right]$ is increasing in t on $S(X) \cup S(Y)$, X is said to be smaller than or equal to Y in the *likelihood ratio order*, and it is denoted as $X \leq_{lr} Y$.

Each of the latter stochastic orders requires stronger or weaker conditions for X and Y to be comparable. In particular, the following result describes the implications between the different notions.

Theorem 1.1. *(Shaked and Shanthikumar, 2007) Let X and Y be two random variables. Then,*

$$X \leq_{lr} Y \implies X \leq_{hr} Y \implies X \leq_{st} Y$$

Sometimes, more than a pair of random variables should be compared in some stochastic sense. In this context, given a family of random variables $\{X_\theta, \theta \in \Theta\}$, having laws $\{P_\theta, \theta \in \Theta\}$ with $\Theta \subseteq \mathbb{R}$, we say that the family is increasing in θ with respect to the likelihood ratio order if

$$X_{\theta_0} \leq_{lr} X_{\theta_1},$$

for any $\theta_0, \theta_1 \in \Theta$ such that $\theta_0 \leq \theta_1$. This property corresponds to the Monotone Likelihood Ratio property (shortly, MLR) mentioned in the Karlin-Rubin Theorem, as stated in the following definition.

Definition 1.2. (Casella and Berger, 2024) A family of pdfs or pmfs $\{g(\cdot \mid \theta) : \theta \in \Theta\}$ for a univariate random variable T with a real-valued parameter θ has a monotone likelihood ratio (MLR) if, for every $\theta_2 > \theta_1$, $g(t \mid \theta_2)/g(t \mid \theta_1)$ is a monotone function of t on $\{t : g(t \mid \theta_1) > 0 \text{ or } g(t \mid \theta_2) > 0\}$. Note that $c/0$ is defined as ∞ if $0 < c$.

The concepts of being increasing with respect to the hazard rate order and the usual stochastic order can be defined analogously. Clearly, as a consequence of Theorem 1.1, these properties are weaker than the MLR property.

1.2. Hypothesis testing

Let $\{P_\theta, \theta \in \Theta\}$ be a family of probability laws over $\mathbb{R}$. Consider a family of random variables $\{X_\theta, \theta \in \Theta\}$ such that X_θ has law P_θ for any $\theta \in \Theta$. Moreover, consider a finite sample $\vec{X}$ of independent observations drawn from a population having distribution in the family $\{P_\theta, \theta \in \Theta\}$, where θ is unknown. A natural problem is to use the value of $\vec{X}$ to estimate θ or to perform hypothesis tests on it.

As in parameter estimation, hypothesis tests are widely used in almost every scientific field to draw conclusions from measurements. Hence, we focus just on hypothesis tests associated with the value of an unknown one-dimensional parameter $\theta \in \Theta$. In this case, the null and alternative hypotheses are of the form

$$H_0 : \theta \in \Theta_0, \qquad H_1 : \theta \in \Theta_1$$

with $\Theta_0, \Theta_1 \subset \Theta$ and $\Theta_0 \cap \Theta_1 = \emptyset$, assuming that θ belongs to either Θ_0 or Θ_1. Then, the usual procedure is, when having n-dimensional samples, to construct a rejection region $R \subset \mathbb{R}^n$ for the sample based on a statistic $T(\vec{X})$. If the sample falls in the rejection region, then the null hypothesis is rejected and, therefore, the alternative hypothesis is accepted. With the rejection region fixed, we say that the associated test has significance α if

$$\sup_{\theta \in \Theta_0} P_\theta(\vec{X} \in R) = \alpha.$$

Moreover, the power function of the test is defined as the function

$$\beta(\theta) = P_\theta(\vec{X} \in R)$$

for any $\theta \in \Theta_1$.

In general, the value of α is fixed and one aims to choose the rejection region such that the associated power function is as large as possible. We refer the reader to Casella and Berger (2024); Rohatgi and Saleh (2015) for more details in this regard.

2. A COROLLARY OF THE KARLIN-RUBIN THEOREM

As stated above, for a fixed value of α, one aims to construct the most powerful test. In this context, a test is said to be the Uniform Most Powerful (UMP) if, for any $\theta_1 \in \Theta_1$, its power function in θ_1 is always greater than for any other test with the same significance α (Casella and Berger, 2024). The Karlin-Rubin Theorem states some sufficient conditions for a test to be UMP. The following theorem is exactly Theorem 8.3.17 in Casella and Berger (2024).

Theorem 2.1. *(Casella and Berger, 2024) Consider testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Suppose that T is a sufficient statistic for θ and the family of pdfs or pmfs $\{g(\cdot \mid \theta) : \theta \in \Theta\}$ of T satisfies the MLR property. Then, for any t_0, the test that rejects H_0 if and only if $T > t_0$ is a UMP level α test, where $\alpha = P_{\theta_0}(T > t_0)$.*

Note that the Karlin-Rubin Theorem implies, in particular, that when the MLR property is satisfied, then for a given significance level α, any test with associated rejection region $R' = \{\vec{x} \in \mathbb{R}^n \mid T(\vec{x}) \in (t_1, t_2]\}$ is no more powerful than the test with rejection region $R = \{\vec{x} \mid T(\vec{x}) > t_0\}$. Taking this into account, the following result is a direct consequence.

Corollary 2.1. *Let Θ be a real interval and let $\{X_\theta, \theta \in \Theta\}$ be a family of random variables. Let $T : \mathbb{R}^n \to \mathbb{R}$ be a sufficient statistic for θ such that its family of pdfs or pmfs is increasing in θ with respect to the likelihood ratio order. Consider the hypothesis*

$$H_0 : \theta \leq \theta_0 \quad H_1 : \theta > \theta_0.$$

Let the rejection regions R and R' be defined as

$$R = \{\vec{x} \mid T(\vec{x}) > t_0\} \quad \text{and} \quad R' = \{\vec{x} \mid T(\vec{x}) \in (t_1, t_2]\},$$

with $\sup_{\theta \leq \theta_0} P_\theta(\vec{X} \in R) = \sup_{\theta \leq \theta_0} P_\theta(\vec{X} \in R') = \alpha$. Then it holds that

$$\beta(\theta_1) = P_{\theta_1}(\vec{X} \in R) \geq P_{\theta_1}(\vec{X} \in R') = \beta'(\theta_1)$$

for any $\theta_1 > \theta_0$.

As shown in the next section, the final assertion of Corollary 2.1 is actually satisfied under a weaker assumption on the family of pdfs or pmfs of the statistic T.

3. MAIN RESULTS

Corollary 2.1 requires the family $\{g(\cdot \mid \theta) : \theta \in \Theta\}$ to be increasing in θ with respect to the likelihood ratio order. A natural question is whether the result still holds when weaker stochastic orders, such as the hazard rate order or the usual stochastic order for the laws of $T(\vec{X})$, are considered. Let us start with the hazard rate order and with the case of two simple hypotheses.

Lemma 3.1. *Let Θ be a real interval and let $\{X_\theta, \theta \in \Theta\}$ be a family of random variables. Let $T : \mathbb{R}^n \to \mathbb{R}$ be a statistic such that its laws are increasing in θ with respect to the hazard rate order. Consider the hypothesis*

$$H_0 : \theta = \theta_0 \quad H_1 : \theta = \theta_1$$

with $\theta_0 \leq \theta_1$. Then, for any rejection regions R and R' defined as

$$R = \{\vec{x} \mid T(\vec{x}) > t_0\} \quad \text{and} \quad R' = \{\vec{x} \mid T(\vec{x}) \in (t_1, t_2]\},$$

with $P_{\theta_0}(\vec{X} \in R) = P_{\theta_0}(\vec{X} \in R') = \alpha \in (0, 1)$, it holds that

$$\beta(\theta_1) = P_{\theta_1}(\vec{X} \in R) \geq P_{\theta_1}(\vec{X} \in R') = \beta'(\theta_1).$$

Proof: Denote as $\overline{F}_0$ the survival function of $T(\vec{X})$ whether the sample comes from P_{θ_0} and as $\overline{F}_1$ the survival function of $T(\vec{X})$ whether the sample comes from P_{θ_1}. Consider the following three cases.

1) $t_1 \leq t_0 \leq t_2$. One has that

$$\begin{aligned}
\alpha =& P_{\theta_0}(T(\vec{X}) > t_0) = \overline{F}_0(t_0), \\
\alpha =& P_{\theta_0}(T(\vec{X}) \in (t_1, t_2]) = \overline{F}_0(t_1) - \overline{F}_0(t_2), \\
\beta(\theta_1) =& P_{\theta_1}(T(\vec{X}) > t_0) = \overline{F}_1(t_0), \\
\beta'(\theta_1) =& P_{\theta_1}(T(\vec{X}) \in (t_1, t_2]) = \overline{F}_1(t_1) - \overline{F}_1(t_2).
\end{aligned}$$

Since the laws of $T(\vec{X})$ are increasing in θ with respect to the hazard rate order, one gets

$$\overline{F}_1(u)\overline{F}_0(v) \leq \overline{F}_0(u)\overline{F}_1(v),$$

for any $u, v \in \mathbb{R}$ such that $u \leq v$. In particular,

$$\overline{F}_1(t_1)\overline{F}_0(t_0) \leq \overline{F}_0(t_1)\overline{F}_1(t_0), \quad \overline{F}_1(t_0)\overline{F}_0(t_2) \leq \overline{F}_0(t_0)\overline{F}_1(t_2).$$

It follows

$$\overline{F}_1(t_1)\overline{F}_0(t_0) - \overline{F}_0(t_0)\overline{F}_1(t_2) \leq \overline{F}_0(t_1)\overline{F}_1(t_0) - \overline{F}_1(t_0)\overline{F}_0(t_2),$$

i.e.,

$$\overline{F}_0(t_0)\left(\overline{F}_1(t_1)-\overline{F}_1(t_2)\right) \le \overline{F}_1(t_0)\left(\overline{F}_0(t_1)-\overline{F}_0(t_2)\right).$$

Therefore, $\alpha\beta'(\theta_1) \le \beta(\theta_1)\alpha$ and $\beta'(\theta_1) \le \beta(\theta_1)$.

2) $t_1 \le t_2 \le t_0$. Consider the rejection regions,

$$R_2 = \{\vec{x} \mid T(\vec{x}) > t_2\} \quad \text{and} \quad R_2' = \{\vec{x} \mid T(\vec{x}) \in (t_1, t_0]\}.$$

Note that both rejection regions have the same significance level, being

$$\begin{aligned} P_{\theta_0}(T(\vec{X}) > t_2) =& \overline{F}_0(t_2) - \overline{F}_0(t_0) + \overline{F}_0(t_0) = \alpha + \overline{F}_0(t_2) - \overline{F}_0(t_0), \\ P_{\theta_0}(T(\vec{X}) \in (t_1, t_0]) =& \overline{F}_0(t_1) - \overline{F}_0(t_2) + \overline{F}_0(t_2) - \overline{F}_0(t_0) = \alpha + \overline{F}_0(t_2) - \overline{F}_0(t_0). \end{aligned}$$

Therefore, case 1) can be applied, obtaining that the inequality

$$P_{\theta_1}(T(\vec{X}) > t_2) \ge P_{\theta_1}(T(\vec{X}) \in (t_1, t_0])$$

holds if and only if

$$\overline{F}_1(t_2) - \overline{F}_1(t_0) + \overline{F}_1(t_0) \ge \overline{F}_1(t_1) - \overline{F}_1(t_2) + \overline{F}_1(t_2) - \overline{F}_1(t_0),$$

i.e., if and only if

$$\overline{F}_1(t_2) - \overline{F}_1(t_0) + \beta(\theta_1) \ge \beta'(\theta_1) + \overline{F}_1(t_2) - \overline{F}_1(t_0),$$

hence $\beta(\theta_1) \ge \beta'(\theta_1)$.

3) $t_0 \le t_1 \le t_2$. In this case, observe that $R' \subset R$, so that $\beta(\theta_1) \ge \beta'(\theta_1)$.

□

It should be noted that the latter result can be viewed as a version of the Neyman-Pearson Lemma (see Casella and Berger, 2024) associated with Corollary 2.1, but based on the ratio of survival functions rather than on the likelihood ratio.

The previous result can be extended to hypotheses of the form $H_0 : \theta \le \theta_0$ and $H_1 : \theta > \theta_0$ when the distribution of the statistic is continuous at $\theta = \theta_0$.

Theorem 3.1. *Let Θ be a real interval and let $\{X_\theta, \theta \in \Theta\}$ be a family of random variables. Let $T : \mathbb{R}^n \to \mathbb{R}$ be a statistic such that its laws are increasing in θ with respect to the hazard rate order. Consider the hypothesis*

$$H_0 : \theta \le \theta_0, \quad H_1 : \theta > \theta_0,$$

and assume that $T(\vec{X})$ has continuous distribution in P_{θ_0}. Let the rejection regions R and R' be defined as

$$R = \{\vec{x} \mid T(\vec{x}) > t_0\} \quad \text{and} \quad R' = \{\vec{x} \mid T(\vec{x}) \in (t_1, t_2]\},$$

and such that $\sup_{\theta \le \theta_0} P_\theta(\vec{X} \in R) = \sup_{\theta \le \theta_0} P_\theta(\vec{X} \in R') = \alpha \in (0, 1)$. Then it holds that

$$\beta(\theta_1) = P_{\theta_1}(\vec{X} \in R) \ge P_{\theta_1}(\vec{X} \in R') = \beta'(\theta_1)$$

for any $\theta_1 > \theta_0$.

Proof: First, notice that the laws of $T(\vec{X})$ are increasing in θ with respect to the hazard rate order, so applying Theorem 1.1, it is also increasing in θ with respect to the usual stochastic order. Therefore,

$$\alpha = \sup_{\theta \leq \theta_0} P_\theta(T(\vec{X}) > t_0) = P_{\theta_0}(T(\vec{X}) > t_0).$$

Now fix any θ_1 such that $\theta_1 > \theta_0$ and consider the hypotheses $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$. Notice that

$$P_{\theta_0}(\vec{X} \in R) = \alpha,$$

and

$$P_{\theta_0}(\vec{X} \in R') \leq \alpha.$$

Consider a new rejection region $R'' = \{\vec{x} \mid T(\vec{x}) \in (s_1, s_2]\}$ such that $(t_1, t_2] \subseteq (s_1, s_2]$ and fulfilling $P_{\theta_0}(\vec{X} \in R'') = \alpha$. Such a rejection region R'' exists by the continuity of the distribution of $T(\vec{X})$ when $\theta = \theta_0$. In particular, since $P_{\theta_0}(\vec{X} \in R') \leq \alpha < 1$, one may enlarge the interval continuously until its P_{θ_0}-probability reaches α. Notice also that $R' \subseteq R''$.

Then, it is possible to apply Lemma 3.1 considering the rejection regions R and R'', obtaining

$$\beta(\theta_1) = P_{\theta_1}(\vec{X} \in R) \geq P_{\theta_1}(\vec{X} \in R'') = \beta''(\theta_1).$$

Finally, notice that since $R' \subseteq R''$ one has

$$\beta(\theta_1) \geq \beta''(\theta_1) = P_{\theta_1}(\vec{X} \in R'') \geq P_{\theta_1}(\vec{X} \in R') = \beta'(\theta_1),$$

thus the result holds. □

Apart from dropping sufficiency of the statistic T and adding continuity assumptions of its law in θ_0, it is immediate to observe that the main difference between Theorem 3.1 and Corollary 2.1 consists in replacing the likelihood ratio order by the hazard rate order. Therefore, it applies to more general families of distributions for T, since the required assumptions are weaker. We remark that the assumption of a continuous distribution of $T(\vec{X})$ when $\theta = \theta_0$ could be avoided if one uses randomized tests. In the next example, a family of statistics that is increasing in the hazard rate order but not in the likelihood ratio order is provided, so that Theorem 3.1 can be applied.

Example 3.1. Let $\theta > 0$ and consider X_θ having density function

$$f_\theta(t) = \begin{cases} \frac{1}{2\theta} & \text{if } t \in [0, \frac{\theta}{2}], \\ \frac{3}{2\theta} & \text{if } t \in [\frac{\theta}{2}, \theta]. \end{cases}$$

Therefore, its survival function is

$$\overline{F}_\theta(t) = \begin{cases} 1 - \frac{t}{2\theta} & \text{if } t \in [0, \frac{\theta}{2}], \\ \frac{3}{4} - \frac{3}{2\theta}(t - \frac{\theta}{2}) & \text{if } t \in [\frac{\theta}{2}, \theta]. \end{cases}$$

Consider $\theta_0 \leq \theta_1$. Then, if $\frac{\theta_1}{2} \leq \theta_0$ we have

$$\frac{\overline{F}_{\theta_0}(t)}{\overline{F}_{\theta_1}(t)} = \begin{cases} \frac{1-\frac{t}{2\theta_0}}{1-\frac{t}{2\theta_1}} & \text{if } t \in [0, \frac{\theta_0}{2}], \\ \frac{\frac{3}{4}-\frac{3}{2\theta_0}(t-\frac{\theta_0}{2})}{1-\frac{t}{2\theta_1}} & \text{if } t \in [\frac{\theta_0}{2}, \frac{\theta_1}{2}], \\ \frac{\frac{3}{4}-\frac{3}{2\theta_0}(t-\frac{\theta_0}{2})}{\frac{3}{4}-\frac{3}{2\theta_1}(t-\frac{\theta_1}{2})} & \text{if } t \in [\frac{\theta_1}{2}, \theta_0], \\ 0 & \text{if } t \in [\theta_0, \theta_1). \end{cases}$$

We now verify that this ratio is decreasing. Since it is continuous, it is sufficient to check the derivative in each of the different intervals. Starting from the first one, we have that

$$\frac{d}{dt}\frac{1-\frac{t}{2\theta_0}}{1-\frac{t}{2\theta_1}} = \frac{-\frac{1}{2\theta_0}(1-\frac{t}{2\theta_1}) + \frac{1}{2\theta_1}(1-\frac{t}{2\theta_0})}{(1-\frac{t}{2\theta_1})^2} = \frac{\frac{1}{2}(\frac{1}{\theta_1}-\frac{1}{\theta_0})}{(1-\frac{t}{2\theta_1})^2} \leq 0.$$

Similarly for the other two, we have

$$\frac{d}{dt}\frac{\frac{3}{4}-\frac{3}{2\theta_0}(t-\frac{\theta_0}{2})}{1-\frac{t}{2\theta_1}} = \frac{\frac{3}{2}(\frac{1}{2\theta_1}-\frac{1}{\theta_0})}{(1-\frac{t}{2\theta_1})^2} \leq 0, \qquad \frac{d}{dt}\frac{\frac{3}{4}-\frac{3}{2\theta_0}(t-\frac{\theta_0}{2})}{\frac{3}{4}-\frac{3}{2\theta_1}(t-\frac{\theta_1}{2})} = \frac{\frac{9}{4}(\frac{1}{\theta_1}-\frac{1}{\theta_0})}{(\frac{3}{4}-\frac{3}{2\theta_1}(t-\frac{\theta_1}{2}))^2} \leq 0.$$

If $\frac{\theta_1}{2} > \theta_0$, then

$$\frac{\overline{F}_{\theta_0}(t)}{\overline{F}_{\theta_1}(t)} = \begin{cases} \frac{1-\frac{t}{2\theta_0}}{1-\frac{t}{2\theta_1}} & \text{if } t \in [0, \frac{\theta_0}{2}], \\ \frac{\frac{3}{4}-\frac{3}{2\theta_0}(t-\frac{\theta_0}{2})}{1-\frac{t}{2\theta_1}} & \text{if } t \in [\frac{\theta_0}{2}, \theta_0], \\ 0 & \text{if } t \in [\theta_0, \theta_1), \end{cases}$$

which is also decreasing by computing the same derivatives.

It follows that $\{X_\theta, \theta \in \mathbb{R}^+\}$ is increasing in θ with respect to the hazard rate order. The expectation of X_θ is $\frac{5}{8}\theta$, thus $\frac{8}{5}\bar{X}_\theta$ is a reasonable statistic (and estimator) for θ. Now, notice that each X_θ has increasing failure rate (IFR), i.e. $\frac{f_\theta(t)}{\overline{F}_\theta(t)}$ is increasing (since $f_\theta(t)$ is increasing and $\overline{F}_\theta(t)$ is decreasing). Then, it is possible to use Theorem 1.B.4. in Shaked and Shanthikumar (2007) to obtain that the family of laws for $\{\frac{8}{5}\bar{X}_\theta, \theta \in \mathbb{R}^+\}$ is increasing in θ with respect to the hazard rate order.

This is no longer true when considering the likelihood ratio order. For it, consider the simple case of a sample of size $n = 2$, letting X'_θ denote a random variable that is independent of X_θ and has its same distribution. The distribution function of $Z_\theta = X_\theta + X'_\theta$ can be written as

$$F_{Z_\theta}(t) = \begin{cases} \frac{t^2}{8\theta^2} & \text{if } t \in \left[0, \frac{\theta}{2}\right], \\ \frac{1}{16} + \frac{3}{4\theta^2}\left(t-\frac{\theta}{2}\right)^2 - \frac{1}{8\theta^2}(\theta-t)^2 & \text{if } t \in \left[\frac{\theta}{2}, \theta\right], \\ \frac{7}{16} - \frac{3}{4\theta^2}\left(\frac{3\theta}{2}-t\right)^2 + \frac{9}{8\theta^2}(t-\theta)^2 & \text{if } t \in \left[\theta, \frac{3\theta}{2}\right], \\ 1 - \frac{9}{8\theta^2}(2\theta-t)^2 & \text{if } t \in \left[\frac{3\theta}{2}, 2\theta\right]. \end{cases}$$

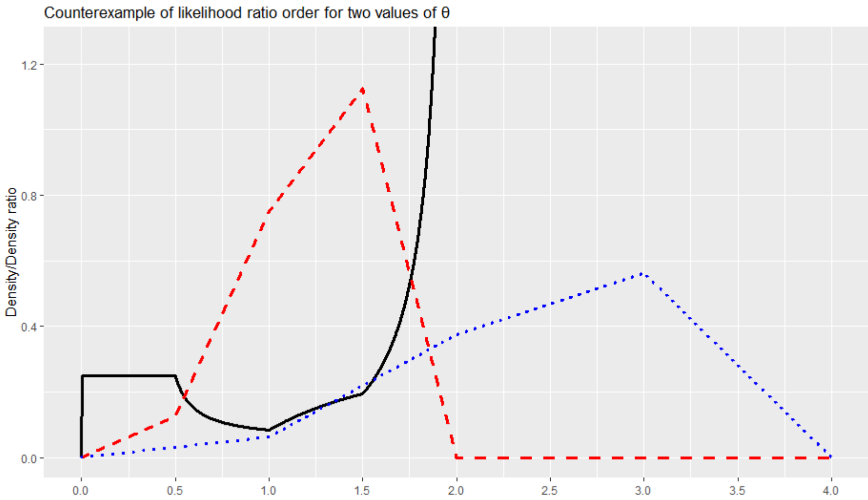

Figure 1: Densities of the sum of two independent variables for the cases $\theta = 2$ (blue dotted line) and $\theta = 1$ (red dashed line) in Example 3.1, as well as their ratio (black solid line), which is not increasing. Big values of the ratio are not shown in the figure.

Therefore, its density is

$$f_{Z_\theta}(t) = \begin{cases} \frac{t}{4\theta^2} & \text{if } t \in \left[0, \frac{\theta}{2}\right], \\ \frac{3}{2\theta^2}\left(t - \frac{\theta}{2}\right) + \frac{1}{4\theta^2}(\theta - t) & \text{if } t \in \left[\frac{\theta}{2}, \theta\right], \\ \frac{3}{2\theta^2}\left(\frac{3\theta}{2} - t\right) + \frac{9}{4\theta^2}(t - \theta) & \text{if } t \in \left[\theta, \frac{3\theta}{2}\right], \\ \frac{9}{4\theta^2}(2\theta - t) & \text{if } t \in \left[\frac{3\theta}{2}, 2\theta\right]. \end{cases}$$

This family of densities, and therefore also the family of densities of $\{\frac{8}{5}\bar{X}_\theta, \theta \in \mathbb{R}^+\} = \{\frac{4}{5}Z_\theta, \theta \in \mathbb{R}^+\}$ is not increasing with respect to the likelihood ratio order (i.e., it is not MLR). In fact, letting for example $\theta_0 = 1$ and $\theta_1 = 2$ one gets

$$\frac{f_{Z_{\theta_1}}(0.5)}{f_{Z_{\theta_0}}(0.5)} = \frac{1}{4}, \qquad \frac{f_{Z_{\theta_1}}(1)}{f_{Z_{\theta_0}}(1)} = \frac{1}{12}, \qquad \frac{f_{Z_{\theta_1}}(1.5)}{f_{Z_{\theta_0}}(1.5)} = \frac{7}{36},$$

thus the ratio is not monotone in t, see Figure 1.

Therefore, Corollary 2.1 cannot be applied in this case. However, as it has been shown before, $\{\frac{8}{5}\bar{X}_\theta, \theta \in \mathbb{R}^+\}$ is increasing in θ with respect to the hazard rate order, so Theorem 3.1 can be applied.

Finally, one may ask whether a similar result can be proved for the usual stochastic order, which is weaker than the hazard rate order. However, in this case the result no longer holds, as the following example illustrates.

Example 3.2. Let the statistic T be based on a sample of size $n = 1$, and let $T(X) = X$, where X can have distribution in the set $\{P_\theta, \theta \in \{0, 1\}\}$, for which the corresponding

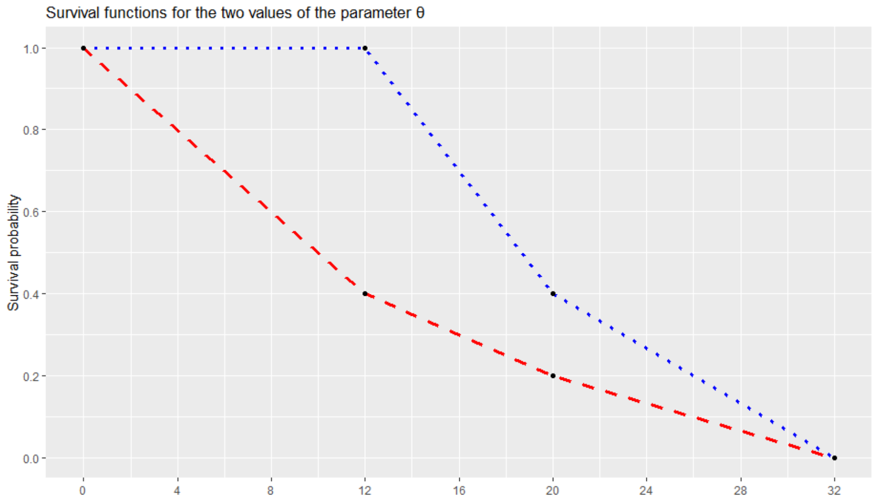

Figure 2: Survival functions for P_0 (red dashed line) and P_1 (blue dotted line) in Example 3.2

survival functions are

$$\overline{F}_0(x) = \begin{cases} 1, & \text{if } x \leq 0; \\ (20-x)/20, & \text{if } x \in (0,12]; \\ (28-x)/40, & \text{if } x \in (12,20]; \\ (32-x)/60, & \text{if } x \in (20,32]; \\ 0, & \text{if } x > 32, \end{cases} \quad \text{and} \quad \overline{F}_1(x) = \begin{cases} 1, & \text{if } x \leq 12; \\ (76-3x)/40, & \text{if } x \in (12,20]; \\ (32-x)/30, & \text{if } x \in (20,32]; \\ 0, & \text{if } x > 32, \end{cases}$$

whose graphs are shown in Figure 2. Since $\overline{F}_0(t) \leq \overline{F}_1(t)$ for any t, then $P_0(T(X) > t) \leq P_1(T(X) > t)$, i.e., the monotonicity in usual stochastic order is satisfied by the family of laws of T.

It is straightforward to verify that the rejection regions $R = \{x \in \mathbb{R} \mid T(x) > 20\}$ and $R' = \{x \in \mathbb{R} \mid T(x) \in (12, 20]\}$ have the same significance level α when $\theta = 0$.

$$\begin{aligned} \alpha =& P_0(T(X) > 20) = \overline{F}_0(20) = 0.2, \\ \alpha =& P_0(T(X) \in (12,20]) = \overline{F}_0(12) - \overline{F}_0(20) = 0.2. \end{aligned}$$

However, we can observe that

$$\begin{aligned} \beta(1) =& P_1(T(X) > 20) = \overline{F}_1(20) = 0.4, \\ \beta'(1) =& P_1(T(X) \in (12,20]) = \overline{F}_1(12) - \overline{F}_1(20) = 0.6, \end{aligned}$$

thus $\beta(1) < \beta'(1)$, and the property stated in Lemma 3.1 is no longer satisfied.

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