This dissertation studies the role of topology and geometry in the analysis of complex systems, with a focus on higher-order dynamical models and neural networks. The thesis is divided into two main parts, which address distinct problems but share a common methodological foundation rooted in applied topology and discrete geometry.

The first part investigates dynamical processes on higher-order networks, modeled using simplicial complexes and hypergraphs. After introducing the necessary background in algebraic topology and topological signal processing, we study Kuramoto-like synchronization models defined on simplicial complexes, in which oscillators are associated with simplices of arbitrary dimension and interact via the complex's adjacency structure. The main contributions of this part consist in an analytical study of the equilibrium configurations of these models, their dependence on the strength of the interaction terms, and the stability properties of the resulting phase-locked states. In the process, particular attention is devoted to disentangling the meaning of the different interaction mechanisms appearing in the equations, clarifying their interpretation and their respective impact on collective synchronization. This analysis highlights how the topology of the underlying simplicial complex constrains the set of admissible equilibria and shapes their stability properties.

Building on this operator-based viewpoint, we introduce a renormalization-group framework for higher-order networks based on a novel class of operators, the cross-order Laplacians, which describe diffusion processes between simplices of different dimensions. Using spectral and entropic observables derived from these operators, we define higher-order notions of scale-invariance and propose an explicit higher-order Laplacian renormalization scheme. The method is validated on synthetic models with known hierarchical structure and applied to real-world datasets, revealing order-specific scale-invariant signatures and nontrivial higher-order organization.

The second part focuses on the topology and geometry of the parameter space of ReLU neural networks trained by gradient flow. Exploiting the rescaling symmetries induced by homogeneous activations, we characterize the training-invariant parameter space as an algebraic variety constrained by conserved quantities. For shallow networks, we show that this invariant set can be disconnected, leading to a topological obstruction to training that depends on initialization and can reduce the network's effective expressivity. We derive exact conditions for disconnectedness and analytical estimates for the probability of encountering such obstructions under standard initialization schemes.

We then extend the analysis to general feedforward architectures modeled as directed acyclic graphs. The invariant set is described as an algebraic variety whose topology depends on architectural features such as bottlenecks and flow imbalance. We give a complete characterization of its connectedness properties, identify the singularities of this space corresponding to effective subnetworks, and show how regularization schemes, particularly nuclear norm regularization, promote convergence toward such configurations, providing a geometry-motivated form of structured pruning.