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DOCTORAL DISSERTATION

**Complexity Hierarchies
in
Generalized Descriptive Set Theory**

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Abstract

Within the framework of generalized descriptive set theory, we conduct a systematic analysis of several hierarchies of classes of sets and functions. At the level of sets, we study the Borel hierarchy, the difference hierarchy, and the Wadge hierarchy; at the level of functions, we examine the Baire class hierarchy as well as the class of Borel-measurable functions.

We aim for maximal generality in two directions. First, we adopt a cardinal-independent approach, working with uncountable cardinals κ under the sole assumption that $2^{<\kappa} = \kappa$, thereby encompassing both regular and singular cardinals. Second, whenever feasible, we move beyond the generalized Cantor and Baire spaces, carrying out the analysis in broader classes of topological spaces.

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Chapter 1

Introduction

Classical descriptive set theory

Descriptive set theory provides canonical methods, through hierarchies and reducibility notions, to assess the topological complexity of subsets and functions on topological spaces. This field systematically studies natural classes of definable sets and functions, providing structural information for families of such objects. The opening statement in Kechris’ book [Kec95], the primary reference for descriptive set theory, perfectly captures its essence:

Descriptive set theory is the study of “definable sets” in Polish (i.e., separable completely metrizable) spaces. In this theory, sets are classified in hierarchies, according to the complexity of their definitions, and the structure of the sets in each level of these hierarchies is systematically analyzed.

Classical descriptive set theory thus selects a particular class of spaces—the Polish spaces—which turn out to be ubiquitous in mathematics. Fundamental examples include the Cantor space ${}^\omega 2$ of all infinite binary sequences, and the Baire space ${}^\omega \omega$ of all infinite sequences of natural numbers, both endowed with the product topology. The standard basis for ${}^\omega \omega$ consists of the basic open sets $N_s = \{x \in {}^\omega \omega \mid s \subseteq x\}$, where s ranges over finite sequences of natural numbers, and similarly for the Cantor space.

The Cantor and Baire spaces play a fundamental role within classical descriptive set theory. For several results, it is sufficient to focus on these canonical spaces, which are well-studied and have rich combinatorial structure. However, much of descriptive set theory’s success in applications throughout mathematics stems from its applicability to the broader class of Polish spaces, which arise naturally in analysis, topology, algebra, and many other areas.

As mentioned in Kechris’ definition, definable objects in descriptive set theory—both sets and functions—are classified in hierarchies of complexity. There are many such hierarchies, depending on the definition or parameters used to compare the objects under consideration; thus, each hierarchy captures a different aspect of complexity.

Hierarchies of sets. The most fundamental hierarchy is the Borel hierarchy, which stratifies the Borel σ -algebra generated by the open sets of a Polish space X . The hierarchy begins with the open sets $\Sigma_1^0(X)$ and closed sets $\Pi_1^0(X)$, and proceeds through transfinite levels by taking countable unions and intersections. At each countable ordinal $\xi < \omega_1$, the classes $\Sigma_\xi^0(X)$ and $\Pi_\xi^0(X)$ are defined inductively: $\Sigma_\xi^0(X)$ consists of countable unions of sets in $\bigcup_{\eta < \xi} \Pi_\eta^0(X)$, while $\Pi_\xi^0(X)$ consists of complements of sets in $\Sigma_\xi^0(X)$; the ambiguous class $\Delta_\xi^0(X) = \Sigma_\xi^0(X) \cap \Pi_\xi^0(X)$ contains the sets belonging to both classes. The Borel sets are precisely those sets that appear at some level of this hierarchy.

Beyond Borel sets, one encounters the projective sets, defined as those obtained from Borel sets by repeatedly applying projection (equivalently, taking continuous images) and complementation. The projective sets of a Polish space X are organized into an infinite hierarchy of length ω , known as the projective hierarchy. At the first level, the analytic sets $\Sigma_1^1(X)$ are exactly the continuous images of Borel sets, while the co-analytic sets $\Pi_1^1(X)$ are their complements. The bi-analytic sets are those belonging to $\Delta_1^1(X) = \Sigma_1^1(X) \cap \Pi_1^1(X)$. Recursively, for each $1 < n < \omega$, the sets in $\Sigma_{n+1}^1(X)$ are continuous images of sets in $\Pi_n^1(X)$, and the sets in $\Pi_{n+1}^1(X)$ are precisely the complements of sets in $\Sigma_{n+1}^1(X)$. The connection between the Borel and projective hierarchy is given by the famous Souslin's theorem, that states that all bi-analytic sets are Borel.

A refinement of the Borel hierarchy is provided by the difference hierarchy. Roughly speaking, for each level $1 \leq \xi < \omega_1$ of the Borel hierarchy, the difference hierarchy over $\Pi_\xi^0(X)$ (respectively, over $\Sigma_\xi^0(X)$) classifies sets according to the minimum number of differences needed to express them as boolean combinations of sets in $\Pi_\xi^0(X)$ (respectively, in $\Sigma_\xi^0(X)$). The classes are denoted by $\mathcal{D}_\alpha(\Pi_\xi^0(X))$, where α measures the number of nested difference operations used to build the sets in the class. This yields a strictly finer stratification within each Borel class. The connection between the difference hierarchy and the Borel hierarchy is given by the following fundamental theorem, due to Hausdorff and Kuratowski [Kec95, Theorem 22.27]:

Theorem 1.0.1. *Let X be a Polish space. For any $1 \leq \xi < \omega_1$,*

$$\Delta_{\xi+1}^0(X) = \bigcup_{1 \leq \alpha < \omega_1} \mathcal{D}_\alpha(\Pi_\xi^0(X)).$$

A highly refined hierarchy of definable sets is the Wadge hierarchy, which classifies sets using continuous reducibility. A set A is Wadge reducible to a set B (written $A \leq_W B$) if there exists a continuous function f such that $A = f^{-1}(B)$. The relation \leq_W is a pre-order, and if we quotient by the corresponding equivalence relation, the resulting equivalence classes are partially ordered. This partial order is called the Wadge hierarchy. The Wadge hierarchy is typically studied on the Baire space (or the Cantor space), mainly because on these spaces the continuous functions are exactly those that arise from strategies in a certain infinite game on the natural numbers. Assuming the axiom of determinacy AD (or working with Borel sets in ZFC), this partial order on ${}^\omega\omega$ is well-founded and has antichains of size at most 2. Therefore, the Wadge hierarchy provides a semi-linear classification of subsets of ${}^\omega\omega$ by their complexity with respect to continuous reducibility.

The Wadge hierarchy provides a refinement of both the difference hierarchy and the Borel hierarchy. More precisely, each class appearing in these two hierarchies forms an initial segment of the Wadge hierarchy. Moreover, the first ω_1 levels of the Wadge hierarchy are occupied precisely by the differences hierarchy over $\mathbf{\Pi}_1^0(\omega\omega)$, so that the sets of countable Wadge rank are the $\mathbf{\Delta}_2^0(\omega\omega)$ sets. Then, at level ω_1 , there are the the classes $\{\mathbf{\Sigma}_2^0(\omega\omega) \setminus \mathbf{\Pi}_2^0(\omega\omega), \mathbf{\Pi}_2^0(\omega\omega) \setminus \mathbf{\Sigma}_2^0(\omega\omega)\}$. From this point on the Wadge hierarchy is strictly finer than the one induced by the Hausdorff difference analysis.

Hierarchies of functions. Parallel to the classification of sets, classical descriptive set theory also studies hierarchies of functions between Polish spaces. The Baire class hierarchy stratifies functions according to how they can be obtained from continuous functions through transfinite iteration of pointwise limits. Baire class 1 functions are pointwise limits of sequences of continuous functions; more generally, for each countable ordinal ξ , the Baire class ξ functions are those obtainable as pointwise limits of sequences of functions from lower Baire classes.

Closely related are the Borel functions (also called Borel measurable functions), defined as the functions such that the preimage of every open set is Borel. A fundamental theorem of classical descriptive set theory establishes that a function between Polish spaces is Borel measurable if and only if it is Baire class ξ for some $\xi < \omega_1$. Moreover, the precise relationship between the two notions can be made explicit: given two Polish spaces X and Y , a function $f: X \rightarrow Y$ is Baire class ξ if and only if the preimage via f of every open subset of Y belongs to $\mathbf{\Sigma}_{\xi+1}^0(X)$. This equivalence highlights a link between hierarchies of functions and hierarchies of sets, showing that a function's complexity can be assessed either through its construction via pointwise limits, or through the Borel complexity of the preimages of open sets.

Generalized descriptive set theory: a brief history

Given the remarkable achievements of classical descriptive set theory, it is natural to ask whether its framework can be extended to more general contexts.

Generalized descriptive set theory can be understood, roughly speaking, as the higher analogue of classical descriptive set theory obtained by replacing all occurrences of the first infinite cardinal ω with an uncountable cardinal κ (or its cofinality). This leads to the study of spaces of κ -sequences with binary values, such as the generalized Cantor space ${}^\kappa 2$, or with values in κ , such as the generalized Baire space ${}^\kappa \kappa$, both endowed with a natural topology which here we call bounded topology (see Section 2.2).

Initially, the theory focused on uncountable cardinals κ satisfying $\kappa^{<\kappa} = \kappa$ (equivalently, κ is regular and $2^{<\kappa} = \kappa$). This restriction ensures that certain techniques from the classical setting generalize naturally to the new context. Moreover, it entails that these generalized spaces exhibit a key property analogous to the classical framework: they both have weight κ , mirroring how $\omega\omega$ and $\omega 2$ are second countable in the classical setting. This approach appeared several times in the literature, beginning with the seminal work of Vaught [Vau75] in 1975 and later, in the 1990s, in foundational works by Väänänen [Vä91, MV93]. Due to striking applications and deep connections with other

well-established areas of mathematical logic, such as Shelah’s stability in model theory [FHK14, HM17, HKM17, MMR21, Mor22, Mor23a], generalized descriptive set theory has gained considerable attention in modern set theory, and the search for solid foundations that would enable further applications became an important priority.

A systematic development of the theory began approximately 15 years ago, initiating a productive period in which the literature flourished and rigorous foundations for the subject were established. During this period, researchers also began relaxing the initial restrictions, with the objective of expanding the theory in the most general and consistent manner possible. This generalization proceeded along three main directions: (A) encompassing singular cardinals, (B) developing Polish-like spaces in the generalized setting, and (C) introducing alternative topologies beyond the bounded topology.

- (A) **Encompass singular cardinals.** Dimonte and Motto Ros [DMR25] introduced an alternative approach to generalized descriptive set theory focusing on uncountable cardinals with countable cofinality, thereby opening the possibility of encompassing singular cardinals of any cofinality in the study of the subject. Examples of such works include [Dim23, Ago22, ACRP25, MRP25, DPT24, DIL23, BDM25]. Prior to this development, singular cardinals appeared only in [DV11] and [AMR22].

In this framework, the replacement of ω with κ is not as straightforward as in the regular case: when κ is singular, the correct and coherent choice is sometimes $\text{cof}(\kappa)$ instead of κ . For example, the correct generalizations of the Cantor and Baire spaces are ${}^\kappa 2$ and ${}^{\text{cof}(\kappa)} \kappa$, both equipped with the bounded topology.

- (B) **Involve Polish-like spaces.** As already mentioned, much of the efficacy of descriptive set theory in applications across mathematics arises from the natural occurrence of Polish spaces in a wide variety of settings. Therefore, concentrating only on ${}^{\text{cof}(\kappa)} \kappa$ and ${}^\kappa 2$ constitutes a potential limitation, and finding generalized versions of Polish spaces has been an important objective. This was first done in [CS16, Gal19], and later in [Ago22] and subsequent works such as [AMRS23]. Nowadays the literature features a thorough study of the classes of Polish-like spaces that allow a meaningful development of the theory [MR13, AMRS23, AT18, Gal16, DW96, Ago22, DMR25].

There are many natural candidates for Polish-like spaces, that are described in detail in [Ago22]. In broad terms, they can be divided into two classes. The first consists of generalizations of the notion of metric by replacing \mathbb{R} with a totally ordered Abelian group \mathbb{G} of degree κ , yielding a notion of generalized metric (called \mathbb{G} -metric) that is suitable in this context (see Section 2.4). The second comes from a well-known game-theoretic characterization of Polish spaces as strong Choquet spaces [Kec95, Section 8]. Indeed, the strong Choquet game has natural generalizations to higher cardinals by simply allowing the two players to play for κ -many rounds. Depending on the different possible choices for winning conditions (which only appear in the uncountable case), one obtains distinct notions: either strong κ -Choquet spaces [CS16] or strong fair κ -Choquet spaces [Ago22].

- (C) **Introduce other topologies.** In most works in this area, the spaces $\text{cof}^{(\kappa)}\kappa$ and ${}^{\kappa}2$ are equipped with the bounded topology, which is particularly natural due to its applications in model theory and infinitary logics [Vau75, MV93, SV00, She01, SV02, She04, DV11, FHK14, MMR21, Mor23b]. However, meaningful variants of this classical framework have emerged in the literature. For example, works in general topology [Kur66, Ü82, Kra04, Ili12, CK13] typically employ either the product or the box topology, and recent developments have shown that the product topology can be indispensable in certain contexts within generalized descriptive set theory, as demonstrated in [AMR22, Sections 12 and 13] and [DMR25, Sections 5 and 6]. Later, [HKS22] introduced the ideal topology, which refines both the bounded and the product topologies, and provides a broader framework in which the bounded topology appears as a particular instance.

Set up and main results

This thesis aligns with the branch of generalized descriptive set theory that seeks to establish a rigorous foundation for the field, and aims at providing a treatment that is as comprehensive as possible.

Throughout this thesis, we work in the theory $\text{ZF} + \text{AC}_{\kappa}({}^{\kappa}2)$, unless stated otherwise. Furthermore, except in Chapter 2 and in Sections 4.1, 4.2, and 4.3.7, we additionally assume the full Axiom of Choice AC.

We adopt a cardinal-independent approach, in line with direction (A) described above. Our assumption on κ is the following:

$$\kappa \text{ is an uncountable cardinal satisfying } 2^{<\kappa} = \kappa.$$

The only exception is Chapter 7, where we pursue direction (C): in that context, we assume that κ is an uncountable *regular* cardinal and *drop the assumption* $2^{<\kappa} = \kappa$.

Whenever possible, we also follow Direction (B), working with arbitrary \mathbb{G} -metrizable spaces of weight at most κ . However, certain sections concentrate specifically on the generalized Cantor and Baire spaces, while others have a larger scope and deal with regular Hausdorff topological spaces. Since different settings are involved, at the beginning of each chapter, we state explicitly the assumptions in force.

Here is a synopsis of the chapters of the thesis and their content. This serves as an introductory overview; for further details we refer the reader to the introduction of each chapter.

Chapter 2: Preliminaries

Here we present all the fundamental concepts, definitions, and preliminary results that will be employed in the next chapters of this thesis, and subsequently used without further comment.

Chapter 3: Generalized Borel sets

In generalized descriptive set theory, κ^+ -Borel sets provide the natural higher-level counterpart of classical Borel sets. Although κ^+ -Borel subsets of the

generalized Cantor and Baire spaces have been extensively studied for regular cardinals κ satisfying $\kappa^{<\kappa} = \kappa$, two significant aspects of the κ^+ -Borel hierarchy have remained largely unexplored: the first concerns its behavior on *subspaces* of the generalized Baire space and on more general Polish-like spaces; the second concerns the entire framework at singular cardinals, especially those of uncountable cofinality.

In this chapter, we develop the basic theory of generalized Borel sets and their hierarchy, aiming for maximal generality in two directions. First, we assume only that κ satisfies $2^{<\kappa} = \kappa$, thereby encompassing singular cardinals. Second, we move beyond subspaces of ${}^{\text{cof}(\kappa)}\kappa$ to consider the more general context of regular Hausdorff spaces of weight at most κ . Our analysis shows that even Polish-like properties are not essential for analyzing the κ^+ -Borel hierarchy, and that many of the key results hold under remarkably mild topological assumptions.

We begin by introducing the γ -Borel hierarchy, a natural stratification of γ -Borel sets, and give a systematic presentation in full generality for any ordinal γ . This includes the notion of order $\text{ord}_\gamma(X)$, which measures the length of the γ -Borel hierarchy on a given space X . We establish key properties of the κ^+ -Borel hierarchy: optimal closure properties of its pointclasses, properness (meaning that new sets appear at every level of the hierarchy), conditions ensuring that the hierarchy does not collapse, the existence of universal and complete sets, and structural properties such as generalized reduction and separation properties. When κ is singular, we introduce and analyze an alternative hierarchy called the κ -Borel hierarchy, which naturally arises from the fact that κ^+ -algebras and κ -algebras coincide in this case. We precisely characterize the connection between these two hierarchies, showing that even when the κ^+ -Borel and κ -Borel sets coincide, the κ -Borel hierarchy is strictly finer and provides a more informative classification.

In the second part of this chapter, we construct via forcing various models demonstrating unexpected behaviors of the κ^+ -Borel hierarchy on closed subspaces of ${}^\kappa\kappa$ (for regular κ). We resolve three central problems affirmatively: we show that, consistently, there exist κ -thin closed sets (i.e., closed sets containing no κ -perfect subset) whose hierarchy has maximal length κ^+ ; there exist closed sets of cardinality greater than κ whose hierarchy has minimal length 2; and, most strikingly, there exist closed sets whose hierarchy has length strictly between 2 and κ^+ . This last result stands in stark contrast with classical descriptive set theory, where the Perfect Set Property for Borel sets eliminates such intermediate cases.

Finally, in the third part of the chapter, we introduce and study the difference hierarchy over a boldface pointclass $\mathbf{\Gamma}$. After presenting some fundamental properties and results, we establish conditions under which the hierarchy does not collapse, and we prove that the generalized Hausdorff–Kuratowski theorem fails.

Chapter 4: The Wadge hierarchy on the generalized Cantor space

Within classical descriptive set theory, the structure of the Wadge hierarchy on the Cantor spaces ${}^\omega 2$ is by now thoroughly analyzed (see, e.g., [And07] and [AC13]). Working in ZFC, the Wadge hierarchy restricted to Borel subsets of ${}^\omega 2$ is well-founded and satisfies the Semi-Linear Ordering principle SLO_W ,

i.e., for any two (Borel) sets A and B , either $A \leq_W B$ or $\neg B \leq_W A$. Both of these fundamental facts follow from Borel determinacy and the characterization of continuous reducibility in terms of winning strategies in a suitable infinite (Gale–Stewart) game on ω . When one moves to generalized descriptive set theory, and studies the Wadge hierarchy on the generalized Cantor space ${}^\kappa 2$ for uncountable κ , the situation is more involved. The main difficulty is the lack of a higher analogue of Borel determinacy, so many of the game-theoretic tools central to the classical theory no longer apply directly (see Section 2.9).

In this chapter we conduct a systematic study of the Wadge hierarchy on ${}^\kappa 2$, treating both regular and singular cardinals. We first analyze the degrees at the base of the hierarchy, proving that SLO_W holds for open and closed sets. Next, we show that the initial κ^+ -many levels of the Wadge hierarchy are exactly described by the difference hierarchy over closed sets, in analogy with the classical case. More precisely, we establish that \leq_W restricted to sets in the difference hierarchy over $\kappa^+ \text{-}\mathbf{\Pi}_1^0$ is well-founded, that every antichain throughout these first levels has size at most 2, and that selfdual degrees and non-selfdual pairs alternate. Furthermore, we show that at every limit level of cofinality smaller than κ there is exactly one selfdual degree, while at limit levels of cofinality κ the situation is as follows: if κ is not weakly compact, there is again a unique selfdual degree, but if κ is weakly compact, then instead we have a non-selfdual pair.

The central theorem of the chapter shows that once we move beyond the difference hierarchy over $\kappa^+ \text{-}\mathbf{\Pi}_1^0$, the Wadge hierarchy behaves pathologically: both SLO_W and well-foundedness already fail for the class $\kappa^+ \text{-}\mathbf{\Delta}_2^0({}^\kappa 2)$. This occurs for both regular and singular cardinals κ , though the arguments in the two cases are essentially different.

Finally, we prove that SLO_L already fails for clopen subsets of ${}^\kappa 2$, and that the Lipschitz hierarchy on clopen sets is not well-founded.

Chapter 5: Generalized Baire class functions

This chapter covers the study of κ^+ -Borel measurable functions and their stratification, examining fundamental questions about Borel functions within our generalized framework. In classical descriptive set theory, two keystone results due to Lebesgue, Hausdorff, and Banach establish that: (1) Borel functions coincide with the closure under pointwise limits of continuous functions (under suitable hypotheses on the spaces), and (2) the Baire class hierarchy stratifies Borel functions according to their complexity, with Baire class ξ functions being precisely the $\Sigma_{\xi+1}^0$ -measurable functions. Moving to generalized descriptive set theory, we discover that the analysis of pointwise limits becomes significantly more subtle. When κ is uncountable and regular, the closure under κ -limits of continuous functions captures only the $\kappa^+ \text{-}\Sigma_n^0$ -measurable functions for finite n , and thus it is far from exhausting all κ^+ -Borel measurable functions. As a remedy we consider limits over directed sets. We prove that the class of κ^+ -Borel measurable functions between κ -metrizable spaces of weight at most κ coincides with the closure under pointwise D -limits of continuous functions, where D varies among all directed sets of size at most κ . Moreover, we show that it suffices to use sequential limits together with limits over the partial order Fin_κ of finite subsets of κ ordered by inclusion, or even just a single kind of non-sequential limit, that we denote by $\widehat{\text{Fin}}_\kappa$. Obtaining a higher analogue of

the Baire class hierarchy stratification theorem is even more delicate, with serious obstacles arising at limit levels of cofinality smaller than κ . Nevertheless, when κ is a regular cardinal, we isolate a suitable notion of generalized κ -Baire class ξ functions and prove that, for κ -metrizable spaces of weight at most κ , a function is κ -Baire class ξ if and only if it is κ^+ - $\Sigma_{\xi+1}^0$ -measurable. We show that our definition is close to optimal through various counterexamples. The case of singular cardinals requires further adjustments, but analogous results can still be obtained. We also establish other structural results including a characterization of κ -Baire class 1 functions in terms of limits of surprisingly simple Lipschitz functions called κ -full functions, in analogy with [MR09b], and a characterization of κ^+ - Σ_{ξ}^0 -measurable functions in terms of uniform limits of simpler functions.

Chapter 6: Characterizing classes of functions via games

In this chapter, we study several generalized reduction games that provide game-theoretic characterizations of specific classes of partial functions (continuous, Lipschitz, Baire class 1, and piecewise continuous) of the form ${}^\mu\nu \rightarrow {}^\mu\nu$, where ν, μ are cardinals with $\nu \geq 2$ and $\mu \geq \omega$. Each of these games can be regarded as a variant of the generalized Gale–Stewart game. There are two ways of defining these games and their associated strategies, which we prove to be equivalent in each case under consideration.

The first approach consists in slightly modifying the rules of the generalized Gale–Stewart game by equipping Player **II** with a more powerful extended alphabet — for example, allowing Player **II** to pass, erase some previous moves, or restart the entire play from the beginning. This approach is inspired by [MR11] and [Sem07]. A second convenient approach, inspired by [Car14], consists in modifying the game rules to allow Player **II** to play a sequence (rather than a single ordinal) at each round. This alternative formulation yields the same classes of functions as the first approach. Of course, even though the statements are often natural generalizations of their classical counterparts, new definitions and proofs are required in most cases.

Chapter 7: On Borel sets in ideal topologies

This chapter studies the ideal topology $\tau_{\mathcal{I}}$ on ${}^\kappa 2$ and ${}^\kappa \kappa$, first defined in [HKS22], where κ is a regular uncountable cardinal and \mathcal{I} is an ideal extending the bounded ideal. This topology is a refinement of the bounded topology and offers a broader framework in which the bounded topology appears as a particular instance. After establishing basic properties of the ideal topology and characterizing continuous functions on these spaces via monotone and domain-increasing functions, we provide a comprehensive description of the \mathcal{I} -Borel hierarchy, that is, the κ^+ -Borel hierarchy on spaces equipped with $\tau_{\mathcal{I}}$. We establish natural features including the property of being increasing, the optimal closure properties of the pointclasses $\mathcal{I}\text{-}\Sigma_{\alpha}^0$, $\mathcal{I}\text{-}\Pi_{\alpha}^0$, and $\mathcal{I}\text{-}\Delta_{\alpha}^0$, and the length of the hierarchy. Most importantly, we answer affirmatively the main open question from [HKS22]: we prove that the \mathcal{I} -Borel hierarchy does not collapse for any ideal \mathcal{I} extending the bounded ideal, establishing that \mathcal{I} -Borel sets form a strict hierarchy of length κ^+ . Then, we introduce and analyze the class of \mathcal{I} -analytic sets. We prove that when the ideal \mathcal{I} contains an unbounded set, the class of \mathcal{I} -analytic sets coincides with the entire powerset of the space,

a dramatic collapse that occurs for all reasonable definitions of \mathcal{I} -analyticity. This reveals a fundamental distinction between the bounded topology and general ideal topology settings, as analytic sets behave quite differently from their classical counterparts in this context.

Publications and collaborations

Most of the results in Chapter 3 also appear in the article “Generalized Borel sets,” jointly written with Claudio Agostini, Nick Chapman, and Luca Motto Ros [ACRP25]. Chapter 4 is part of an ongoing project with Luca Motto Ros and Philipp Schlicht. Chapter 5 is based on the article “Generalized Baire class functions,” written in collaboration with Luca Motto Ros [MRP25]. Finally, the results in Chapter 7 are taken from the article “On Borel sets in ideal topologies,” co-authored by Miguel Moreno [MP25].

Chapter 2

Preliminaries

We present here, for the reader's benefit, a collection of definitions, notations, and elementary facts that will be required in subsequent sections. We assume some familiarity with set theory and general topology, and we adopt standard notation in those fields. For all undefined notions, we refer the reader to [Kec95, Jec03]. We also refer to [AMR22] for the generalized descriptive set theory notation adopted in this thesis.

For a cardinal κ and a set X , the axiom $\text{AC}_\kappa(X)$ of κ -choice over X is defined as follows:

For every sequence $(A_\alpha)_{\alpha < \kappa}$ of nonempty subsets of X there is a function $f: \kappa \rightarrow X$ such that $f(\alpha) \in A_\alpha$ for all $\alpha < \kappa$.

Any such function f is called a choice function for the given sequence of sets $(A_\alpha)_{\alpha < \kappa}$. Clearly, whenever there exists a surjection from a set X onto a set Y , and $\mu \leq \kappa$, $\text{AC}_\kappa(X)$ entails $\text{AC}_\mu(Y)$. The (full) κ -choice axiom AC_κ states that $\text{AC}_\kappa(X)$ holds for every set X .

In this preliminary chapter (and throughout the entire thesis, except for Chapter 7), we work under the assumption that

$$\kappa \text{ is an uncountable cardinal satisfying } 2^{<\kappa} = \kappa.$$

Concerning the axiomatic framework, unless otherwise specified we will assume

$$\text{ZF} + \text{AC}_\kappa({}^\kappa 2).$$

Note that this mirrors the usual setting in classical descriptive set theory, whose basics are developed in $\text{ZF} + \text{AC}_\omega(\mathbb{R})$, and allows for the use of our results in models in which the full Axiom of Choice AC does not necessarily hold. Moreover, observe that the hypothesis $2^{<\kappa} = \kappa$ itself provides a very weak instance of the Axiom of Choice, because it guarantees that for every $\lambda < \kappa$, the set ${}^\lambda 2$ admits a well-ordering of order type at most κ .

We let Ord be the class of all ordinals, and Card be the subclass of cardinals. If \preceq is a well-order on a set X , we let $\text{ot}(X, \preceq)$ be its order type, i.e. the unique $\alpha \in \text{Ord}$ that is order isomorphic to (X, \preceq) . For $\alpha, \beta \geq \omega$, we denote by α^+ the least cardinal strictly larger than α , and let $\text{cof}_\beta^\alpha = \{\gamma < \alpha \mid \text{cof}(\gamma) = \beta\}$. For any $S \subseteq \alpha$ and $\gamma < \alpha$, we also write $S_{[\gamma]} = \{\delta < \alpha \mid \gamma + \delta \in S\}$. Given any set X , we denote by id_X the identify function on X . If $A \subseteq X^2$, we denote by A_x

the vertical section determined by $x \in X$, i.e. we set $A_x = \{y \in X \mid (x, y) \in A\}$. Moreover, for every cardinal ν we set $[X]^\nu = \{Y \subseteq X \mid |Y| = \nu\}$. The projection (on the first coordinate) of a subset $A \subseteq X \times Y$ of a cartesian product is $p(A) = \{x \in X \mid \exists y \in Y (x, y) \in A\}$.

2.1 Topology

Recall that a topology on a set X is a subset $\tau \subseteq \mathcal{P}(X)$ containing both the empty set \emptyset and X , and closed under arbitrary unions and finite intersections. Sets belonging to τ are termed open, while those of the form $X \setminus U$ for $U \in \tau$ are dubbed closed. A set is called clopen if it is simultaneously closed and open and $\text{cl}_X(V) = \bigcap \{X \setminus U \mid U \in \tau, U \cap V = \emptyset\}$ denotes the closure of V in X ; when the ambient space X is understood from the context, we simply write $\text{cl}(V)$. A point x is isolated in X if there is an open set $U \subseteq X$ such that $U = \{x\}$, and a topological space is perfect if it has no isolated points. If any two distinct points $x, y \in X$ can be separated by open sets, i.e., there exist $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$, then the topology is **Hausdorff**. If for every $x \in X$ and every open neighborhood $U \in \tau$ of x , there exists $V \in \tau$ satisfying $x \in V \subseteq \text{cl}(V) \subseteq U$, then the topology is **regular**. A basis for a topology is a family $\mathcal{B} \subseteq \tau$ such that for every $x \in U$ and each open neighborhood $U \in \tau$ of x , there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Let X be a topological space admitting a well-orderable basis. The **weight** of X is the smallest cardinality of a basis for its topology, and is denoted by $w(X)$, or $w(X, \tau)$ when we want to specify the topology τ on X . A subset $Y \subseteq X$ is dense (in X) if $\text{cl}(Y) = X$. The **density character** of (X, τ) is the least cardinality of a dense subset of X . Given $\nu \in \text{Card}$, we say that the space X is **ν -additive** if the intersection of any collection of fewer than ν -many open subsets of X is still an open set. In particular, every topological space is ω -additive, and it is easy to show that if a regular space X is ν -additive for some $\nu > \omega$, then X is also zero-dimensional, that is, it admits a basis consisting of clopen sets. It is easy to check that if a Hausdorff space X is ν -additive for $\nu > w(X)$, then X is discrete; if X is not discrete, instead, the set of infinite cardinals ν such that X is ν -additive has a maximum, which we call the **additivity of X** . Finally, given a regular cardinal μ , a topological space X is called **μ -Lindelöf** if every open cover of X admits a subcover whose cardinality is $< \mu$. In this terminology, X is compact precisely when it is ω -Lindelöf. *All topological spaces in this work are assumed to be regular and Hausdorff, unless otherwise specified.*

A function $f: X \rightarrow Y$ between the topological spaces X and Y is continuous if the inverse image of each open set is open. It is open (resp. closed) if the image of each open (resp. closed) set is open (resp. closed). It is a homeomorphism if it is a bijection and is both continuous and open. Finally, f is an embedding if it is a homeomorphism onto its image $f(X)$, where the latter is equipped with the subspace topology inherited from Y . The product of any two functions $f: X \rightarrow Y$ and $g: V \rightarrow W$ is the map $f \times g: X \times V \rightarrow Y \times W$ sending $(x, v) \in X \times V$ to $(f(x), g(v))$. If all of X, Y, V , and W are topological spaces, then $f \times g$ is continuous if and only if so are f and g .

As mentioned in the introduction, generalized descriptive set theory is primarily concerned with the study of the generalized Cantor and Baire spaces, that are constructed as follows. Given a set A and $\gamma \in \text{Ord}$, the set

${}^\gamma A = \{x \mid x: \gamma \rightarrow A\}$ is the Cartesian product of γ -many copies of A , that is, ${}^\gamma A = \prod_{\alpha < \gamma} A_\alpha$ where $A_\alpha = A$ for every $\alpha < \gamma$. We also set ${}^{<\gamma} A = \bigcup_{\beta < \gamma} {}^\beta A$. Let $\mu \in \text{Card}$. Any set of the form ${}^\mu A$ is naturally equipped with the **bounded topology** τ_b , i.e. the smallest topology generated by the family $\{\mathbf{N}_s \mid s \in {}^{<\mu} A\}$, where $\mathbf{N}_s = \{x \in {}^\mu A \mid s \subseteq x\}$. When $X \subseteq {}^\mu A$, we write $\mathbf{N}_s^X = \mathbf{N}_s \cap X$. It is straightforward to see that each \mathbf{N}_s is clopen, making the topology τ_b zero-dimensional. Suppose that μ is infinite, that A is a well-orderable set with at least two elements, and that $|A|^{<\mu}$ is itself well-orderable. Under these assumptions, the space $({}^\mu A, \tau_b)$ is regular Hausdorff, and both its weight and its density character are equal to $|A|^{<\mu}$, while its additivity is $\text{cof}(\mu)$. Typically, this will be used in the cases $A = 2$ with $\mu = \kappa$, or $A = \kappa$ with $\mu = \text{cof}(\kappa)$. In these situations, the assumption $2^{<\kappa} = \kappa$ already ensures that $|A|^{<\mu}$ is well-defined and in fact equals κ . The **generalized Baire space** and the **generalized Cantor space** are, respectively, $({}^{\text{cof}(\kappa)} \kappa, \tau_b)$ and $({}^\kappa 2, \tau_b)$. The reference to τ_b will be dropped in most cases. The space ${}^\kappa 2$ is homeomorphic to a superclosed subset of ${}^{\text{cof}(\kappa)} \kappa$, and it is indeed homeomorphic to the whole ${}^{\text{cof}(\kappa)} \kappa$ if and only if κ is not weakly compact. (See [AMR22, Proposition 6.6(a)] or [HN74] for the regular case and [DMR25, Theorem 3.2.2] for the case $\text{cof}(\kappa) = \omega$; the latter can easily be generalized to cover the case of singular cardinals of arbitrary cofinality.)

2.2 Trees and sequences

For ordinals α and β , we write ${}^\alpha \beta$ for the collection of all functions from α to β , we set ${}^{<\alpha} \beta = \bigcup_{\gamma < \alpha} {}^\gamma \beta$, and for each $s \in {}^\alpha \beta$ we let $\text{lh}(s) = \alpha$ be its length. For $\gamma \leq \text{lh}(s)$, the notation $s \upharpoonright \gamma$ denotes the restriction of s to the domain γ , while $s \hat{\ } t$ stands for the concatenation of sequences s and t ; following standard notation, when $t = \langle \gamma \rangle$ is a sequence of length 1 we write $s \hat{\ } \gamma$ and $\gamma \hat{\ } s$ rather than $s \hat{\ } \langle \gamma \rangle$ and $\langle \gamma \rangle \hat{\ } s$. We also let $s_{[\gamma]}$ be the (unique) sequence such that $s \upharpoonright \gamma \hat{\ } s_{[\gamma]} = s$. We write $s \subseteq t$ if s is an initial segment of t ; two sequences s and t are called comparable if $s \subseteq t$ or $t \subseteq s$, and incomparable, written $s \perp t$, otherwise. We denote by $s \wedge t$ the \subseteq -maximal sequence u such that $u \subseteq t$ and $u \subseteq s$. For any ordinal γ , we denote by $i^{(\gamma)}$ the constant sequence of length γ and value i everywhere.

For $A \subseteq {}^\alpha \beta$ (or $A \subseteq {}^{<\alpha} \beta$) and $t \in {}^{<\alpha} \beta$, we define $t \hat{\ } A = \{t \hat{\ } x \mid x \in A\}$ and $A_{[t]} = \{x \mid t \hat{\ } x \in A\}$. Clearly, $A_{[t]} = \{s_{[\text{lh}(t)]} \mid t \subseteq s \in A\}$.

We say that a sequence s is increasing if $s(i) \leq s(j)$ for every $i < j < \text{lh}(s)$, and that s is strictly increasing if $s(i) < s(j)$ for every $i < j < \text{lh}(s)$. Moreover, s is continuous if $s(i) = \bigcup_{j < i} s(j)$ for every limit ordinal $i < \text{lh}(s)$. Similarly, given a limit ordinal γ and a sequence $(s_i)_{i < \gamma}$, we say that $(s_i)_{i < \gamma}$ is increasing if $s_i \subseteq s_j$ for every $i < j < \gamma$, and that $(s_i)_{i < \gamma}$ is strictly increasing if $s_i \subsetneq s_j$ for every $i < j < \gamma$; finally $(s_i)_{i < \gamma}$ is continuous if for every limit ordinal $i < \gamma$, $s_i = \bigcup_{j < i} s_j$.

A set $\mathcal{T} \subseteq {}^{<\alpha} \beta$ is called **tree** if it is closed under initial segments; its **body** is the set $[\mathcal{T}] = \{x \in {}^\alpha \beta \mid \forall \gamma < \alpha (x \upharpoonright \gamma \in \mathcal{T})\}$. Elements of a tree are called **nodes**. We call a tree \mathcal{T} **pruned** if for all nodes $s \in \mathcal{T}$ there is $x \in [\mathcal{T}]$ such that $s \subseteq x$, and we denote by $\partial \mathcal{T} = \{t \in {}^{<\alpha} \beta \mid t \notin \mathcal{T} \wedge t \upharpoonright \delta \in \mathcal{T} \text{ for all } \delta < \text{lh}(t)\}$ the **boundary of the tree** \mathcal{T} . For a cardinal λ , we say that \mathcal{T} is ($<$) λ -**splitting** if for every $t \in \mathcal{T}$ there is a node $s \in \mathcal{T}$ which extends t and is

$(<)\lambda$ -splitting in \mathcal{T} , i.e. it has $(<)\lambda$ -many immediate successors. We say that \mathcal{T} is **splitting** if it is 2-splitting.

Notice that, for cardinals $\nu \geq 2$ and $\mu \geq \omega$, the bounded topology τ_b is the unique topology on ${}^\mu\nu$ whose closed sets coincide with the sets of the form $[\mathcal{T}]$, for $\mathcal{T} \subseteq {}^\mu\nu$ a pruned tree. Every subset $A \subseteq {}^\mu\nu$ determines a unique pruned tree $\mathcal{T}_A = \{x \upharpoonright \alpha \mid x \in A \wedge \alpha < \mu\}$, called the **tree of A** , satisfying $[\mathcal{T}_A] = \text{cl}(A)$; in particular, if A is closed then $[\mathcal{T}_A] = A$. We can strengthen the notion of closed set A by imposing additional conditions to its associated tree \mathcal{T}_A . Specifically, a tree $\mathcal{T} \subseteq {}^{<\mu}\nu$ is called **superclosed** when it is both pruned and $<\mu$ -closed, meaning that for every limit ordinal $\alpha < \mu$ and every $s \in {}^\alpha\nu$, if $s \upharpoonright \beta \in \mathcal{T}$ for every $\beta < \alpha$, then $s \in \mathcal{T}$. A set $C \subseteq {}^\mu\nu$ is **superclosed** when it has the form $C = [\mathcal{T}]$ for some superclosed tree $\mathcal{T} \subseteq {}^{<\mu}\nu$. We call \mathcal{T} **μ -perfect** if it is superclosed and splitting. A set $C \subseteq {}^\mu\nu$ is **strongly μ -perfect** when it has the form $C = [\mathcal{T}]$ for some μ -perfect tree $\mathcal{T} \subseteq {}^{<\mu}\nu$.

2.3 Baire category

Let μ be an infinite cardinal. We say that a subset A of a topological space (X, τ) is **μ -meager** if it is the union of μ -many nowhere dense sets, where a set is nowhere dense if it is disjoint from some open dense subset of X (equivalently, its closure has empty interior). We call a set $A \subseteq X$ **μ -comeager** if its complement is μ -meager. Any subset of a μ -meager set remains μ -meager. When $\mu \leq \mu'$ are infinite cardinal any set $A \subseteq X$ which is μ -(co)meager is necessarily μ' -(co)meager.

Definition 2.3.1. Let μ be an infinite cardinal. A topological space X is said to be **μ -Baire** if the intersection of μ -many open dense subsets of X is dense in X .

Analogously to the classical Baire space notion (corresponding to the case $\mu = \omega$), it is easy to verify that the space X is μ -Baire exactly when every μ -comeager subset of X is dense, or equivalently when no non-empty open subset of X is μ -meager. Additionally, when $\mu \leq \mu'$, every μ' -Baire space is necessarily μ -Baire, and every open subspace of a μ -Baire space is itself μ -Baire.

Remark 2.3.2. If a topological space X contains a dense subset $D \subseteq X$ that is μ -Baire, then X is μ -Baire. To see this, let $(U_\alpha)_{\alpha < \mu}$ with each U_α open dense in X for every $\alpha < \mu$. Since $U_\alpha \cap D$ is open and dense in D for every $\alpha < \mu$ and D is μ -Baire, $\bigcap_{\alpha < \mu} (U_\alpha \cap D)$ is dense in D , thus it is dense in X . As $\bigcap_{\alpha < \mu} (U_\alpha \cap D) \subseteq \bigcap_{\alpha < \mu} U_\alpha$, we are done.

From now on we will work with $\mu = \kappa$ regular.

Proposition 2.3.3. *Assume κ is regular. Given any superclosed tree $\mathcal{T} \subseteq {}^{<\kappa}2$, $[\mathcal{T}]$ is a κ -Baire space. In particular, $[\mathcal{T}]$ is not κ -meager in itself.*

Proof. Let $(U_\alpha)_{\alpha < \kappa}$ be a sequence of dense open subsets of $[\mathcal{T}]$. To show that $\bigcap_{\alpha < \kappa} U_\alpha$ is dense in $[\mathcal{T}]$, given any $s \in {}^{<\kappa}2$, we need to find $x \in [\mathcal{T}]$ such that $s \subseteq x$ and $x \in U_\alpha$ for every $\alpha < \kappa$. So, fix $s \in \mathcal{T}$. We construct a sequence $(s_\alpha)_{\alpha < \kappa}$ with each $s_\alpha \in {}^{<\kappa}2$ by induction on α as follows.

To begin with, set $s_0 = s$ and consider U_0 open dense in $[\mathcal{T}]$. Then, $U_0 \cap N_{s_0}^{[\mathcal{T}]} \neq \emptyset$ is open, therefore there exists an extension $s_0 \subseteq s_1 \in \mathcal{T}$ such that

$N_{s_1}^{[\mathcal{T}]} \subseteq U_0 \cap N_{s_0}^{[\mathcal{T}]}$. Assume s_α has already been defined. Since U_α is open dense, $U_\alpha \cap N_{s_\alpha}^{[\mathcal{T}]} \neq \emptyset$ is open therefore we find $s_\alpha \subseteq s_{\alpha+1} \in \mathcal{T}$ such that $N_{s_{\alpha+1}}^{[\mathcal{T}]} \subseteq U_\alpha \cap N_{s_\alpha}^{[\mathcal{T}]}$. When α is limit, take $s_\alpha = \bigcup_{\beta < \alpha} s_\beta$. Since \mathcal{T} is $< \kappa$ -closed, $s_\alpha \in \mathcal{T}$, and since $N_{s_\alpha}^{[\mathcal{T}]}$ is non-empty open, we can continue the process. Let $x = \bigcup_{\alpha < \kappa} s_\alpha$. Then, $x \in [\mathcal{T}]$ because for every $\beta < \kappa$ there exists $\alpha < \kappa$ such that $x \upharpoonright \beta \subseteq s_\alpha \in \mathcal{T}$, and $x \in U_\alpha$ for every $\alpha < \kappa$ by construction. \square

As a consequence, we get the following result which already appeared in [AMR22].

Corollary 2.3.4. *Assume κ is regular. The spaces ${}^\kappa 2$ and ${}^\kappa \kappa$ are κ -Baire, therefore they are not κ -meager in themselves.*

2.4 Generalized metrics

Metric spaces are central in classical descriptive set theory. When moving to the generalized context, we can continue to use classical metrics if $\text{cof}(\kappa) = \omega$ ([DMR25]), while if $\text{cof}(\kappa) > \omega$ then \mathbb{R} -valued metrics need to be replaced with \mathbb{G} -metrics, that is, metrics taking value in a totally ordered (abelian) group \mathbb{G} with degree $\text{cof}(\kappa)$ ([AMRS23, Ago22]). Since it turns out that in the latter case the choice of \mathbb{G} is irrelevant, we usually speak of $\text{cof}(\kappa)$ -metrizable spaces rather than \mathbb{G} -metrizable spaces. For the same reason, we can safely assume that \mathbb{G} is always a field; for the sake of definiteness, we indeed stipulate that $\mathbb{G} = \mathbb{R}$ if $\text{cof}(\kappa) = \omega$, and $\mathbb{G} = \text{cof}(\kappa)\text{-}\mathbb{Q}$, where $\text{cof}(\kappa)\text{-}\mathbb{Q}$ is Asperò-Tsaprounis' ordered field of $\text{cof}(\kappa)$ -rationals ([AT18]), if $\text{cof}(\kappa) > \omega$. We also fix once and for all a strictly decreasing sequence $(r_\alpha)_{\alpha < \text{cof}(\kappa)}$ coinital in \mathbb{G}^+ : this is done by letting $\hat{r} = 2$ if $\text{cof}(\kappa) = \omega$ and $\hat{r} = \omega$ if $\text{cof}(\kappa) > \omega$, and then setting $r_\alpha = \hat{r}^{-\alpha}$.

It is well-known that if $\text{cof}(\kappa) > \omega$, all $\text{cof}(\kappa)$ -metrizable spaces are $\text{cof}(\kappa)$ -additive, and they indeed admit a compatible \mathbb{G} -ultrametric. (This follows e.g. from Theorem 2.4.1.) Moreover, most of the metric related notions can be naturally adapted to generalized metrics: this includes Cauchy-completeness, which in this case refers to Cauchy sequences of length $\text{cof}(\kappa)$ rather than countable sequences. A stronger notion of completeness is obtained by requiring that the intersection of any decreasing sequence of closed balls is non-empty. A \mathbb{G} -metric satisfying this property is called **spherically complete**. One can show that spherically complete \mathbb{G} -metrics are always Cauchy-complete, but the converse might fail — indeed there are even spaces which admit a compatible Cauchy-complete \mathbb{G} -metric, but no spherically complete \mathbb{G} -metric is compatible with their topology.

Both the generalized Cantor and Baire spaces admit natural spherically complete \mathbb{G} -ultrametrics which are compatible with their (bounded) topology. In the case of ${}^{\text{cof}(\kappa)}\kappa$, for distinct $x, y \in {}^{\text{cof}(\kappa)}\kappa$ we set $d(x, y) = r_\alpha$, where $\alpha < \text{cof}(\kappa)$ is smallest such that $x \upharpoonright \alpha \neq y \upharpoonright \alpha$. The case of ${}^\kappa 2$ is similar: we fix a strictly increasing sequence $(\kappa_\alpha)_{\alpha < \text{cof}(\kappa)}$ of ordinals cofinal in κ , and then for distinct $x, y \in {}^\kappa 2$ we set $d(x, y) = r_\alpha$ for the smallest $\alpha < \text{cof}(\kappa)$ such that $x \upharpoonright \kappa_\alpha \neq y \upharpoonright \kappa_\alpha$. When considering ${}^{\text{cof}(\kappa)}\kappa$ and ${}^\kappa 2$ as \mathbb{G} -metric spaces, we always tacitly refer to these specific \mathbb{G} -ultrametrics.

A very convenient result in the context of $\text{cof}(\kappa)$ -metrizable spaces is the following.

Theorem 2.4.1 ([Sik50, AMRS23, Ago22]). (AC) *Suppose that $\text{cof}(\kappa) > \omega$. For any space X of weight at most κ , the following are equivalent:*

- (1) X is $\text{cof}(\kappa)$ -metrizable;
- (2) X is $\text{cof}(\kappa)$ -ultrametrizable;
- (3) X is homeomorphic to a subset of ${}^{\text{cof}(\kappa)}\kappa$.

Moreover, X is a spherically complete $\text{cof}(\kappa)$ -(ultra)metrizable space if and only if it is homeomorphic to a superclosed subset of ${}^{\text{cof}(\kappa)}\kappa$.

Given the importance that the spaces from Theorem 2.4.1 play in this thesis, we denote by \mathcal{M}_κ the collection of all $\text{cof}(\kappa)$ -metrizable spaces of weight at most κ .

With a little care, one can strengthen the implication (2) \Rightarrow (3) of Theorem 2.4.1 and get the following result, whose proof is a higher analogue of [MR09b, Theorem 4.1] and works also when $\text{cof}(\kappa) = \omega$.

Proposition 2.4.2. *Let (X, d) be a \mathbb{G} -ultrametric space of weight at most κ . Then, there is a topological embedding $h: X \rightarrow {}^{\text{cof}(\kappa)}\kappa$ such that $d(h(x), h(y)) \leq d(x, y)$, for all $x, y \in X$.*

Moreover, we can further ensure that if d is spherically complete, then the range of h is a superclosed subset of ${}^{\text{cof}(\kappa)}\kappa$.

Proof. We define the inverse $f = h^{-1}$ of h by building a scheme $\{B_s \mid s \in <{}^{\text{cof}(\kappa)}\kappa\}$ on X such that for every $\alpha < \text{cof}(\kappa)$ and $s \in <{}^{\text{cof}(\kappa)}\kappa$:

- (i) $B_s \subseteq B_t$ whenever $t \subseteq s$;
- (ii) $\{B_t \mid \text{lh}(t) = \alpha\}$ is a covering of X such that $B_t \cap B_{t'} = \emptyset$ for all distinct $t, t' \in {}^\alpha\kappa$;
- (iii) if $\text{lh}(s)$ is a successor ordinal and $B_s \neq \emptyset$, then B_s is the open ball $B_s = B_d(x, r_{\text{lh}(s)})$, for some/any $x \in B_s$; if $\text{lh}(s)$ is a limit ordinal, then $B_s = \bigcap_{\beta < \text{lh}(s)} B_{s \upharpoonright \beta + 1}$.

Condition (iii) ensures that $\text{diam}(B_s) \leq r_{\text{lh}(s)}$ whenever $\text{lh}(s)$ is a successor ordinal. Together with condition (i), this implies that the map f canonically induced by the scheme, which is defined by letting $f(x)$ be the unique element in $\bigcap_{\alpha < \text{cof}(\kappa)} B_{x \upharpoonright \alpha}$ if the latter is non-empty (otherwise $f(x)$ is undefined), is well-defined and continuous. Condition (ii) entails that f is a bijection on its domain. Moreover, f is an open map because each B_s is clopen by condition (iii) and $\text{cof}(\kappa)$ -additivity of X , together with the fact that (ii) grants that $f(N_s \cap \text{dom}(f)) = B_s$. Therefore $h = f^{-1}: X \rightarrow {}^{\text{cof}(\kappa)}\kappa$ is a topological embedding, and we want to check that $d(x, y) \leq d(f(x), f(y))$ for every $x, y \in \text{dom}(f) \subseteq {}^{\text{cof}(\kappa)}\kappa$. This is clear if $x = y$, so suppose that $x \neq y$. Let $\alpha < \text{cof}(\kappa)$ be smallest such that $x \upharpoonright \alpha \neq y \upharpoonright \alpha$, so that $d(x, y) = r_\alpha$. Necessarily, α is a successor ordinal. Since $f(x)$ witnesses $B_{x \upharpoonright \alpha} \neq \emptyset$, the latter is an open ball of radius r_α by condition (iii). Suppose towards a contradiction that $d(f(x), f(y)) < r_\alpha$. Then, $f(y) \in B_{x \upharpoonright \alpha}$. On the other hand, $f(y) \in B_{y \upharpoonright \alpha}$ by definition of f , hence $B_{x \upharpoonright \alpha} \cap B_{y \upharpoonright \alpha} \neq \emptyset$. This contradicts (ii) and the choice of α . Therefore $d(x, y) = r_\alpha \leq d(f(x), f(y))$. The fact that the image of h is superclosed when d is spherically complete easily follows from (ii) and (iii).

It remains to construct the scheme $\{B_s \mid s \in {}^{<\text{cof}(\kappa)}\kappa\}$, and this is done by recursion on $\text{lh}(s)$. Set $B_\emptyset = X$, and $B_s = \bigcap_{\beta < \text{lh}(s)} B_{s \uparrow \beta+1}$ if $\text{lh}(s)$ is limit. (Notice that in the latter case condition (ii) is automatically satisfied since by inductive hypothesis it already holds at all levels $\alpha < \text{lh}(s)$.) Consider now a successor ordinal $\alpha = \beta + 1$, and assume that by inductive hypothesis B_s has been defined for all $s \in {}^\beta \kappa$. Fix any $s \in {}^\beta \kappa$. If $B_s = \emptyset$, then we set $B_{s \frown i} = \emptyset$ for all $i < \kappa$. If instead $B_s \neq \emptyset$, then $B_d(x, r_\alpha) \subseteq B_s$ for every $x \in B_s$; this is because by condition (iii) we have that $B_s = B_d(x, r_\beta)$ if β is a successor ordinal, or $B_s = \bigcap_{\gamma < \beta} B_d(x, r_{\gamma+1})$ if β is limit. Fix an enumeration $(x_i)_{i < \kappa}$, possibly with repetitions, of a dense subset D of B_s . We define $B_{s \frown i}$ by recursion on $i < \kappa$, ensuring along the construction that $B_{s \frown i}$ is an open ball with radius r_α . First we set $B_{s \frown 0} = B_d(x_0, r_\alpha)$. For $i > 0$, we let $A = B_s \setminus \bigcup_{j < i} B_{s \frown j}$ and distinguish two cases. If $A = \emptyset$, then we let $B_{s \frown i} = \emptyset$. If instead $A \neq \emptyset$, then $A = \bigcup_{x \in A} B_d(x, r_\alpha)$ is a non-empty open set, and we can set $B_{s \frown i} = B_d(x_k, r_\alpha)$ for the smallest $k < \kappa$ such that $x_k \in A$. Conditions (i) and (iii) are satisfied by construction. Condition (ii), instead, follows from the fact that it holds at level β by inductive hypothesis, together with the fact that $\{B_{s \frown i} \mid i < \kappa\}$ is a covering of B_s (by density of D) and by construction $B_{s \frown i} \cap B_{s \frown j} = \emptyset$ if $i \neq j$. \square

Remark 2.4.3. Note that the argument used in the proof of Proposition 2.4.2 does not require AC, even though we consider an arbitrary (\mathbb{G} -ultrametric) space X of weight at most κ . This is because every μ -metrizable space of weight at most κ is a surjective image of ${}^\mu \kappa$, and if $\mu \leq \kappa$, then ${}^\mu \kappa$ itself is a surjective image of ${}^\kappa 2$. Consequently, in the context of that proposition, ${}^\kappa 2$ surjects onto X , and hence $\text{AC}_\kappa({}^\kappa 2)$ yields $\text{AC}_\kappa(X)$. This is precisely what we need to obtain a dense subset of cardinality at most κ from a base of X of size at most κ .

It might be worth recording that Proposition 2.4.2 entails an analogue of Theorem 2.4.1 for the case $\text{cof}(\kappa) = \omega$.

Theorem 2.4.4. *Suppose that $\text{cof}(\kappa) = \omega$. For any space X of weight at most κ , the following are equivalent:*

- (1) X is metrizable and $\dim(X) = 0$, i.e., X has Lebesgue covering dimension 0;
- (2) X is ultrametrizable;
- (3) X is homeomorphic to a subset of ${}^\omega \kappa$.

Moreover, X is a completely metrizable space with $\dim(X) = 0$ if and only if it is completely ultrametrizable, if and only if it is homeomorphic to a closed subset of ${}^\omega \kappa$.

Proof. The equivalence between (1) and (3) is standard (see e.g. [DMR25, Proposition 3.3.2]). The implication (3) \Rightarrow (2) is obvious, while the reverse implication (2) \Rightarrow (3) follows from Proposition 2.4.2. \square

We conclude this section by introducing the definition of Lipschitz function in the setting of \mathbb{G} -metric spaces. This notion will be useful later on, for instance in Section 2.7. Note that such definition makes sense since we assumed that \mathbb{G} is a field.

Definition 2.4.5. A map f between two \mathbb{G} -metric spaces (X, d_X) and (Y, d_Y) is **Lipschitz (with constant $R \in \mathbb{G}^+$)** if for all $x, y \in X$

$$d_Y(f(x), f(y)) \leq R \cdot d_X(x, y).$$

2.5 Limits

Let X be a topological space admitting a well-orderable basis, and let (D, \leq) be a directed set, still denoted by D . A point $x \in X$ is a **D -limit** of a D -net $(x_d)_{d \in D}$ of points from X if for every open neighborhood U of x there is $d \in D$ such that $x_{d'} \in U$ for all $d' \in D$ with $d' \geq d$. When this happens, we write $x = \lim_{d \in D} x_d$. An important case often considered in (generalized) metrizable spaces is when $D = \alpha$ is a limit ordinal, i.e. **sequential limits** (or **α -limits**).

It is well-known that D -limits capture the notion of topological closure: a set $C \subseteq X$ is (topologically) closed in an arbitrary topological space X if and only if $\lim_{d \in D} x_d \in C$ for all directed sets D and all nets $(x_d)_{d \in D}$ of points from C . If X has weight κ , then without loss of generality one can restrict the attention to directed sets D of size at most κ , while if X is $\text{cof}(\kappa)$ -metrizable, then it is enough to consider $\text{cof}(\kappa)$ -limits. Occasionally we will also consider restricted forms of D -limits, i.e. we will restrict to D -limits satisfying some special extra properties — see Section 5.3.

Limits can be composed in the obvious way: if D, D' are two directed sets and $(x_{d,d'})_{d \in D, d' \in D'}$ is a family of points of X , we let $\lim_{d \in D} \lim_{d' \in D'} x_{d,d'}$ be the point (if it exists) $x = \lim_{d \in D} x_d$, where in turn $x_d = \lim_{d' \in D'} x_{d,d'}$ for all $d \in D$. The previous double limit operator will often be denoted by $\lim_D \circ \lim_{D'}$.

The notion of D -limit can be lifted to functions by taking pointwise limits: if f and $(f_d)_{d \in D}$ are functions between topological spaces X and Y , then we write $f = \lim_{d \in D} f_d$ if $f(x) = \lim_{d \in D} f_d(x)$ for all $x \in X$. All the previous concepts and notations can be adapted to pointwise limits of functions in the obvious way.

If \mathcal{F} is a collection of function from X to Y , we let

$$D\text{-lim } \mathcal{F} = \left\{ \lim_{d \in D} f_d \mid f_d \in \mathcal{F} \right\}$$

be the collection of all pointwise D -limits of functions from \mathcal{F} . This can be extended to arbitrary families of limits by setting, for every collection \mathcal{D} of directed sets, $\mathcal{D}\text{-lim } \mathcal{F} = \bigcup_{D \in \mathcal{D}} D\text{-lim } \mathcal{F}$.

Remark 2.5.1. Let κ be an infinite cardinal, and consider any cofinal set $D \subseteq \kappa$. If $f = \lim_{\alpha \in \kappa} f_\alpha$, then $f = \lim_{\alpha \in D} f_\alpha$ too. Conversely, suppose that $f = \lim_{\alpha \in D} f_\alpha$. For each $\alpha \in \kappa$, let $f_\alpha = f_{\alpha'}$ where $\alpha' \in D$ is smallest such that $\alpha \leq \alpha'$. Then, the sequence $(f_\alpha)_{\alpha \in \kappa}$ extends $(f_\alpha)_{\alpha \in D}$, and $f = \lim_{\alpha \in \kappa} f_\alpha$. It easily follows that $\kappa\text{-lim } \mathcal{F} = \text{cof}(\kappa)\text{-lim } \mathcal{F}$ for every family of functions \mathcal{F} .

2.6 Boldface pointclasses and universal sets

A **pointclass** Γ is a class-function assigning to every non-empty topological space X a non-empty family $\Gamma(X) \subseteq \mathcal{P}(X)$. The dual of Γ is the pointclass $\check{\Gamma}$ defined by $\check{\Gamma}(X) = \{X \setminus A \mid A \in \Gamma(X)\}$, while the ambiguous pointclass Δ_Γ

associated to Γ is obtained by setting $\Delta_\Gamma(X) = \Gamma(X) \cap \check{\Gamma}(X)$. A pointclass Γ is said to be **boldface** if it is closed under continuous preimages, that is: $f^{-1}(B) \in \Gamma(X)$ whenever $B \in \Gamma(Y)$ and $f: X \rightarrow Y$ is continuous. Obviously, if Γ is boldface, then so are $\check{\Gamma}$ and Δ_Γ . A pointclass Γ is **hereditary** if $\Gamma(Y) = \{A \cap Y \mid A \in \Gamma(X)\}$ for every $Y \subseteq X$. Notice that if Γ is a boldface pointclass, then it is hereditary if and only if for every $Y \subseteq X$ and $B \in \Gamma(Y)$ there is $A \in \Gamma(X)$ such that $B = A \cap Y$. Finally, a boldface pointclass Γ is called **non-selfdual on X** if $\Gamma(X) \neq \check{\Gamma}(X)$, and it is called **selfdual on X** otherwise. When Γ is non-selfdual on X , we say that a set $A \subseteq X$ is a **proper $\Gamma(X)$ -set** (or **$\Gamma(X)$ -proper**) if $A \in \Gamma(X) \setminus \check{\Gamma}(X)$. A function $f: X \rightarrow Y$ between two topological spaces is **Γ -measurable** if $f^{-1}(U) \in \Gamma(X)$ for every open $U \subseteq Y$.

Let Γ be a boldface pointclass, X be a topological space, and $A \in \Gamma(X)$. A **Γ -covering** of A is a family $\{A_i \mid i \in I\}$, for some index set I , such that $\bigcup_{i \in I} A_i = A$ and $A_i \in \Gamma(X)$ for all $i \in I$. A Γ -covering $\{A_i \mid i \in I\}$ is said to be **disjoint** if $A_i \cap A_j = \emptyset$ for all distinct $i, j \in I$. (But notice that some A_i might be empty.) Finally, a **Γ -partition** of A is a disjoint Γ -covering of A all of whose elements are non-empty. When the class Γ is irrelevant, we simply drop it from the above terminology.

Let X, Y be topological spaces, and let Γ be a boldface pointclass. A set $A \subseteq X$ is **κ -hard for Γ** if for all $B \in \Gamma(\kappa^2)$ there is a continuous function $f: \kappa^2 \rightarrow X$ such that $B = f^{-1}(A)$; A is **κ -Lipschitz-hard for Γ** if there is such an f which is Lipschitz. If moreover $A \in \Gamma(X)$, then we say that A is **κ -complete for Γ** (or **κ -Lipschitz-complete for Γ** , respectively).

A set $\mathcal{U} \subseteq Y \times X$ is **Y -universal for $\Gamma(X)$** if $\mathcal{U} \in \Gamma(Y \times X)$ and $\Gamma(X) = \{\mathcal{U}_y \mid y \in Y\}$, where $\mathcal{U}_y = \{x \in X \mid (y, x) \in \mathcal{U}\}$ is the vertical section of \mathcal{U} at y . It is clear that if \mathcal{U} is Y -universal for $\Gamma(X)$, then its complement $\mathcal{U}^c = (Y \times X) \setminus \mathcal{U}$ is Y -universal for $\check{\Gamma}(X)$. Note that when Γ is boldface, in order to show that $\mathcal{U} \in \Gamma(Y \times X)$ is Y -universal for $\Gamma(X)$, it is enough to check that $\mathcal{U} \in \Gamma(Y \times X)$ and that for every $A \in \Gamma(X)$ there is $y \in Y$ such that $A = \mathcal{U}_y$; indeed, the remaining condition “ $\mathcal{U}_y \in \Gamma(X)$ for all $y \in Y$ ” already follows from the fact that each \mathcal{U}_y is the preimage of \mathcal{U} via the continuous map $x \mapsto (y, x)$.

We report three technical lemmas concerning the existence of universal and complete sets. They are basically folklore, but since we could not trace them back in the literature in the form we need, we give full proofs for the reader’s convenience.

Lemma 2.6.1. *Let Γ be a hereditary boldface pointclass, and let X, Y , and Z be topological spaces such that there is a topological embedding $f: Z \rightarrow Y$. If there is a Z -universal set \mathcal{U}' for $\Gamma(X)$, then there is a Y -universal set \mathcal{U} for $\Gamma(X)$.*

Proof. The map $g = f \times \text{id}_X$ is a topological embedding of $Z \times X$ into $Y \times X$, therefore $g[\mathcal{U}'] \in \Gamma(f[Z] \times X)$ because Γ is boldface. Since it is also hereditary, there is $\mathcal{U} \in \Gamma(Y \times X)$ such that $\mathcal{U} \cap (f[Z] \times X) = g[\mathcal{U}']$. Such a \mathcal{U} is as desired. \square

Lemma 2.6.2. *Let Γ be a hereditary boldface pointclass, and let X, Y be topological spaces. Suppose that there is an embedding $f: \kappa^2 \rightarrow X$. If $\mathcal{U} \subseteq Y \times X$ is Y -universal for $\Gamma(X)$, then \mathcal{U} is κ -complete for Γ . Furthermore, if $Y \times X$ is homeomorphic to X , then there is a subset of X which is κ -complete for Γ .*

Proof. Fix any $A \in \mathbf{\Gamma}(\kappa^2)$. Then, $f[A] \in \mathbf{\Gamma}(f[\kappa^2])$, hence there is $B \in \mathbf{\Gamma}(X)$ such that $B \cap f[\kappa^2] = f[A]$. Let $\bar{y} \in Y$ be such that $B = \mathcal{U}_{\bar{y}}$: then the continuous map $g: \kappa^2 \rightarrow Y \times X$ defined by $g(x) = (\bar{y}, f(x))$ is such that $g^{-1}(\mathcal{U}) = A$. This shows that \mathcal{U} is κ -complete for $\mathbf{\Gamma}$. If moreover, $h: Y \times X \rightarrow X$ is a homeomorphism, then $h \circ g: \kappa^2 \rightarrow X$ is continuous and such that $(h \circ g)^{-1}(h(\mathcal{U})) = A$. It easily follows that $h(\mathcal{U}) \subseteq X$ is κ -complete for $\mathbf{\Gamma}$ as well. \square

Lemma 2.6.3. *Let X be a topological space, and let $\mathbf{\Gamma}$ be a boldface pointclass that is selfdual on X . Then, there is no X -universal set for $\mathbf{\Gamma}(X)$.*

Proof. Towards a contradiction, suppose that there exists a X -universal set \mathcal{U} for $\mathbf{\Gamma}(X)$. Let $f: X \rightarrow X \times X$ be such that $f(x) = (x, x)$, and let $D = \{x \in X \mid f(x) \notin \mathcal{U}\}$. Note that $D = X \setminus f^{-1}(\mathcal{U})$. Since $\mathbf{\Gamma}$ is boldface and $\mathbf{\Gamma}(X) = \mathbf{\Gamma}(X)$, then $D \in \mathbf{\Gamma}(X)$. Hence there is $y_0 \in X$ such that $\mathcal{U}_{y_0} = D$ by universality of \mathcal{U} . But then

$$(y_0, y_0) \in \mathcal{U} \Leftrightarrow y_0 \in \mathcal{U}_{y_0} \Leftrightarrow y_0 \in D \Leftrightarrow f(y_0) \notin \mathcal{U} \Leftrightarrow (y_0, y_0) \notin \mathcal{U},$$

a contradiction. \square

2.7 Approximating functions

Consider the cardinals $\nu \geq 2$ and $\mu \geq \omega$. Similarly to the classical setting (see [Kec95, Proposition 2.6]), every continuous function from ${}^\mu\nu$ into itself can be represented with a monotone and length-increasing function $\varphi: <{}^\mu\nu \rightarrow <{}^\mu\nu$; such a φ is called approximating function for f .

Definition 2.7.1. Let $\mathcal{T} \subseteq <{}^\mu\nu$ be a tree and let $\varphi: \mathcal{T} \rightarrow <{}^\mu\nu$ be a function. We say that φ is

- **monotone** if $s \subseteq t \Rightarrow \varphi(s) \subseteq \varphi(t)$ for all $s, t \in \mathcal{T}$;
- **α -Lipschitz**, for a given $\alpha < \mu$, if it is monotone and for all $s \in \mathcal{T}$

$$\text{lh}(s) \leq \text{lh}(\varphi(s)) + \alpha; \tag{2.7.1}$$

- **continuous** if it is monotone and for all $x \in [\mathcal{T}]$

$$\text{lh}\left(\bigcup_{\gamma < \mu} \varphi(x \upharpoonright \gamma)\right) = \mu. \tag{2.7.2}$$

Moreover, we say that φ is **strictly monotone** if $s \subsetneq t \Rightarrow \varphi(s) \subsetneq \varphi(t)$ for all $s, t \in \mathcal{T}$, and that φ is **continuous*** (**α -Lipschitz***) if it is continuous (α -Lipschitz) and for all increasing sequences $(s_i)_{i < \lambda}$ in \mathcal{T} with $\lambda < \mu$ limit ordinal, $\varphi(\bigcup_{i < \lambda} s_i) = \bigcup_{i < \lambda} \varphi(s_i)$.

Clearly, if φ is α -Lipschitz and $\alpha \leq \beta < \mu$, then φ is β -Lipschitz, and every α -Lipschitz function is necessarily continuous.

Notice also that one can always extend a partial α -Lipschitz (or continuous) function $\varphi: \mathcal{T} \rightarrow <{}^\mu\nu$ to a total α -Lipschitz (or continuous) function $\bar{\varphi}: <{}^\mu\nu \rightarrow <{}^\mu\nu$. Indeed, it is enough to set $\bar{\varphi}(s) = \varphi(s)$ for $s \in \mathcal{T}$, and $\bar{\varphi}(s) = \left(\bigcup_{\gamma < \beta} \varphi(s \upharpoonright \gamma)\right) \cap 0^{(\text{lh}(s))}$ for $s \notin \mathcal{T}$, where $\beta = \sup\{\gamma + 1 < \text{lh}(s) \mid s \upharpoonright \gamma \in \mathcal{T}\}$.

If φ is monotone, let

$$D_\varphi = \{x \in [\mathcal{T}] \mid \forall \beta < \mu \exists \gamma < \mu \text{lh}(\varphi(x \upharpoonright \gamma)) \geq \beta\}$$

and

$$f_\varphi: D_\varphi \rightarrow {}^\mu\nu, \quad x \mapsto \bigcup_{\gamma < \mu} \varphi(x \upharpoonright \gamma)$$

be the function induced by φ . Notice that a monotone φ is continuous precisely when $D_\varphi = [\mathcal{T}]$. It is easy to see that $f_\varphi: D_\varphi \rightarrow {}^\mu\nu$ is continuous, and that D_φ is the intersection of μ -many open sets. Conversely:

Proposition 2.7.2. *Let $f: A \rightarrow {}^\mu\nu$ be a continuous function with $A \subseteq {}^\mu\nu$. Then, there is a monotone $\varphi: \mathcal{T}_A \rightarrow <{}^\mu\nu$ such that $f_\varphi \supseteq f$ and $A \subseteq D_\varphi \subseteq [\mathcal{T}_A]$.*

Proof. For every $s \in \mathcal{T}_A$, we define $X_s = \{t \in <{}^\mu\nu \mid f(\mathbf{N}_s \cap A) \subseteq \mathbf{N}_t\}$. Note that X_s is a chain; indeed if t and t' are incomparable, then $\mathbf{N}_t \cap \mathbf{N}_{t'} = \emptyset$, and if $t \subseteq t' \in X_s$, then $t \in X_s$. For $s \in \mathcal{T}_A$, we let $\varphi(s)$ be the longest $t \in X_s$ such that $\text{lh}(t) \leq \text{lh}(s)$. Clearly, φ is monotone, and $A \subseteq D_\varphi \subseteq \text{cl}(A)$, since f is continuous on A and φ is defined on \mathcal{T}_A . It remains to prove that $f_\varphi(x) = f(x)$, for every $x \in A$. By continuity of f , for every $\alpha < \mu$ there exists $\beta \geq \alpha$ such that $f(\mathbf{N}_{x \upharpoonright \beta} \cap A) \subseteq \mathbf{N}_{f(x) \upharpoonright \alpha}$. By definition of $\varphi(x \upharpoonright \beta)$, it must be the case that $f(x) \upharpoonright \alpha \subseteq \varphi(x \upharpoonright \beta)$: the result follows. \square

We recall from Section 2.4 that if μ is regular, then the space ${}^\mu\nu$ admits a natural \mathbb{G} -ultrametric that is compatible with its (bounded) topology. Specifically, for distinct $x, y \in {}^\mu\nu$, we define $d(x, y) = r_\alpha$, where $\alpha < \mu$ is the least ordinal such that $x \upharpoonright \alpha \neq y \upharpoonright \alpha$.

Proposition 2.7.3. *Assume that μ is regular. Let $f: A \rightarrow {}^\mu\nu$ be a function with $A \subseteq {}^\mu\nu$. Then, f is Lipschitz with constant r_α^{-1} if and only if there is an α -Lipschitz function $\varphi: \mathcal{T}_A \rightarrow <{}^\mu\nu$ such that $f_\varphi = f$.*

Proof. First, let φ be α -Lipschitz such that $f_\varphi = f$. Given $x, y \in A$, our goal is to show that $d(f(x), f(y)) \leq r_\alpha^{-1} \cdot d(x, y)$. The result is trivial if $d(x, y) \geq r_\alpha$, because $d(f(x), f(y)) \leq 1 \leq r_\alpha^{-1} \cdot d(x, y)$, so we assume that $d(x, y) < r_\alpha$. Let $s \in \mathcal{T}_A$ be longest such that $s \subseteq x$ and $s \subseteq y$: then $d(x, y) = r_{\text{lh}(s)+1}$, and $\text{lh}(s) \leq \text{lh}(\varphi(s)) + \alpha$ by choice of φ . Since $f(x), f(y) \in \mathbf{N}_{\varphi(s)}$, $d(f(x), f(y)) \leq r_{\text{lh}(\varphi(s))+1}$. Moreover, by choice of the field \mathbb{G} and of the sequence $(r_\alpha)_{\alpha < \mu}$, the condition $\text{lh}(s) \leq \text{lh}(\varphi(s)) + \alpha$ implies $r_{\text{lh}(s)+1} \geq r_{\text{lh}(\varphi(s))+1} \cdot r_\alpha$, so $d(x, y) \geq r_\alpha \cdot d(f(x), f(y))$, and we conclude that $d(f(x), f(y)) \leq r_\alpha^{-1} \cdot d(x, y)$ as desired.

Vice versa, let f be Lipschitz with constant r_α^{-1} . For every $s \in \mathcal{T}_A$, we define $X_s = \{t \in <{}^\mu\nu \mid f(\mathbf{N}_s \cap A) \subseteq \mathbf{N}_t\}$. Note that each X_s is a chain. For every $s \in \mathcal{T}_A$, we claim that the longest $t \in X_s$ is such that $\text{lh}(s) \leq \text{lh}(t) + \alpha$. Then, it is sufficient to set $\varphi(s) = t$ to conclude. Indeed, let $s \in \mathcal{T}_A$ and suppose towards a contradiction that $\text{lh}(s) > \text{lh}(t) + \alpha$. Then, there must be $x, y \in A \cap \mathbf{N}_s$ such that $d(x, y) \leq r_{\text{lh}(s)+1}$ and $d(f(x), f(y)) = r_{\text{lh}(t)+1}$. But this implies

$$d(x, y) \leq r_{\text{lh}(s)+1} < r_\alpha \cdot r_{\text{lh}(t)+1} = r_\alpha \cdot d(f(x), f(y)),$$

thus $d(f(x), f(y)) > r_\alpha^{-1} \cdot d(x, y)$, a contradiction. \square

Remark 2.7.4. Let μ be regular. If $f: A \rightarrow {}^\mu\nu$ is a continuous (α -Lipschitz) function with $A \subseteq {}^\mu\nu$, then we can assume that its approximation $\varphi: \mathcal{T}_A \rightarrow <{}^\mu\nu$ is continuous* (α -Lipschitz*). Indeed, for any $\lambda < \mu$ and for any increasing sequence $(s_i)_{i < \lambda}$ in \mathcal{T}_A , the function φ satisfies $\bigcup_{i < \lambda} \varphi(s_i) \subseteq \varphi(\bigcup_{i < \lambda} s_i)$ by monotonicity. It is therefore sufficient to consider the function $\varphi': \mathcal{T}_A \rightarrow <{}^\mu\nu$ defined for any $s \in \mathcal{T}_A$ as follows. Set $\varphi'(\emptyset) = \varphi(\emptyset)$ and set $\varphi'(s) = \varphi(s)$ if $\text{lh}(s)$ is a successor ordinal, and $\varphi'(s) = \bigcup_{i < \text{lh}(s)} \varphi(s \upharpoonright i)$ if $\text{lh}(s)$ is a limit ordinal. In particular, to see that if φ is α -Lipschitz then φ' remains α -Lipschitz, consider $s \in <{}^\mu\nu$ with $\text{lh}(s)$ limit:

$$\text{lh}(s) = \sup_{i < \text{lh}(s)} \text{lh}(s \upharpoonright i) \leq \sup_{i < \text{lh}(s)} \text{lh}(\varphi(s \upharpoonright i)) + \alpha = \text{lh}(\varphi'(s)) + \alpha.$$

In light of Proposition 2.7.3, we give the following definition.

Definition 2.7.5. Let $\alpha < \mu$. Given a set $A \subseteq {}^\mu\nu$, a function $f: A \rightarrow {}^\mu\nu$ is **α -Lipschitz** if $f = f_\varphi$ for some α -Lipschitz $\varphi: \mathcal{T}_A \rightarrow <{}^\mu\nu$. Equivalently: for all $\beta < \mu$ there exists $\gamma_\beta < \mu$ such that $\beta \leq \gamma_\beta + \alpha$ and $x \upharpoonright \beta = y \upharpoonright \beta$ implies $f(x) \upharpoonright \gamma_\beta = f(y) \upharpoonright \gamma_\beta$ for all $x, y \in A$.

Throughout this thesis, we will refer to 0-Lipschitz functions (in the sense of Definitions 2.7.1 and 2.7.5) simply as “Lipschitz” functions. Note the by Proposition 2.7.3, if μ is regular and $f: {}^\mu\nu \rightarrow {}^\mu\nu$ is a 0-Lipschitz function, then f is a Lipschitz function with constant 1 in the sense of Definition 2.4.5, where ${}^\mu\nu$ is equipped with its natural \mathbb{G} -metric introduced in Section 2.4.

2.8 Reducibilities

As in classical descriptive set theory, the study of generalized Wadge reducibility naturally leads one to consider additional types of reducibilities (for instance, Lipschitz reducibility or κ^+ -Borel reducibility). In this section, we present a very general framework for comparing subsets, adapted from [AM03], which encompasses several different notions of reducibility.

Consider a topological space X and a collection $\mathcal{F} \subseteq {}^X X$ of functions closed under composition and containing the identity function and all constant functions, i.e.,

$$\begin{aligned} f, g \in \mathcal{F} &\Rightarrow f \circ g \in \mathcal{F}, \\ \text{id}_X &\in \mathcal{F}, \\ \mathbf{c}_x &\in \mathcal{F} \text{ for every } x \in X, \end{aligned} \tag{2.8.1}$$

where \mathbf{c}_x denotes the constant function taking value $x \in X$. The collection \mathcal{F} is called **reducibility (on X)**.

Given subsets $A, B \subseteq X$, we say that A is **\mathcal{F} -reducible** to B , and write

$$A \leq_{\mathcal{F}}^X B,$$

if there exists some $f \in \mathcal{F}$ such that $x \in A \Leftrightarrow f(x) \in B$ for every $x \in X$. The function f is then called a **reduction** of A to B . Observe that f simultaneously serves as a reduction of $\neg A$ to $\neg B$, where \neg denotes complementation in X . When such f exists, the problem of deciding whether $x \in A$ reduces to establishing if $f(x) \in B$: this captures the idea that, provided f is sufficiently well-behaved, the complexity of A is no greater than that of B .

By our assumptions on \mathcal{F} , the relation $\leq_{\mathcal{F}}^X$ is reflexive and transitive, that is, a preorder on $\mathcal{P}(X)$. When the space X is clear from the context, we drop the reference to it, simply writing $\leq_{\mathcal{F}}$. We also define

$$\begin{aligned} A \equiv_{\mathcal{F}} B &\Leftrightarrow A \leq_{\mathcal{F}} B \wedge B \leq_{\mathcal{F}} A, \\ A <_{\mathcal{F}} B &\Leftrightarrow A \leq_{\mathcal{F}} B \wedge B \not\leq_{\mathcal{F}} A. \end{aligned}$$

The relation $\equiv_{\mathcal{F}}$ is an equivalence relation on $\mathcal{P}(X)$, whose equivalence classes are called \mathcal{F} -degrees, and $[A]_{\mathcal{F}} = \{B \subseteq X \mid B \equiv_{\mathcal{F}} A\}$ is the \mathcal{F} -degree of A . The preorder $\leq_{\mathcal{F}}$ induces a partial order on the collection of the \mathcal{F} -degrees, called the \mathcal{F} -hierarchy on X .

We call a set A \mathcal{F} -selfdual when $A \leq_{\mathcal{F}} \neg A$ (equivalently, $A \equiv_{\mathcal{F}} \neg A$), otherwise it is \mathcal{F} -non-selfdual. Since selfduality is invariant under $\equiv_{\mathcal{F}}$, this notion extends to \mathcal{F} -degrees. If $A \not\leq_{\mathcal{F}} B$ and $B \not\leq_{\mathcal{F}} A$ we say that A and B (equivalently, $[A]_{\mathcal{F}}$ and $[B]_{\mathcal{F}}$) are \mathcal{F} -incomparable. The dual of $[A]_{\mathcal{F}}$ is $[\neg A]_{\mathcal{F}}$, and a pair of incomparable degrees of the form $\{[A]_{\mathcal{F}}, [\neg A]_{\mathcal{F}}\}$ is called a \mathcal{F} -non-selfdual pair. Observe that $[X]_{\mathcal{F}} = \{X\}$ together with $[\emptyset]_{\mathcal{F}} = \{\emptyset\}$ form a \mathcal{F} -non-selfdual pair and, since \mathcal{F} contains all constant functions, these are least within the \mathcal{F} -degrees, that is $\emptyset <_{\mathcal{F}} A$ and $X <_{\mathcal{F}} A$ for all $A \neq \emptyset, X$.

We call $[A]_{\mathcal{F}}$ a **successor degree** when there is a $B <_{\mathcal{F}} A$ such that for no $C \subseteq X$ we have $B <_{\mathcal{F}} C <_{\mathcal{F}} A$. (In this case the degree $[B]_{\mathcal{F}}$ is an immediate predecessor of $[A]_{\mathcal{F}}$.) When an \mathcal{F} -degree is not a successor and it is neither $[X]_{\mathcal{F}}$ nor $[\emptyset]_{\mathcal{F}}$, then we say it is a limit degree.

When $\mathcal{F} \subseteq \mathcal{G}$ are families of functions satisfying (2.8.1), then the pre-order $\leq_{\mathcal{G}}$ is coarser than $\leq_{\mathcal{F}}$, yielding:

$$\begin{aligned} A \leq_{\mathcal{F}} B &\Rightarrow A \leq_{\mathcal{G}} B, \\ A \text{ is } \mathcal{F}\text{-selfdual} &\Rightarrow A \text{ is } \mathcal{G}\text{-selfdual}, \\ [A]_{\mathcal{F}} &\subseteq [A]_{\mathcal{G}}. \end{aligned}$$

If \mathcal{F} is the collection of all continuous functions, then every occurrence of “ \mathcal{F} ” in subscripts is replaced by “ \mathbf{W} ”, honoring William W. Wadge [Wad83], who was the first to investigate the structural properties of continuous reducibility as such. If instead \mathcal{F} is the collection of all Lipschitz functions (i.e., the 0-Lipschitz functions introduced in Definition 2.7.5) we write “ \mathbf{L} ”. In the case where \mathcal{F} is the collection of κ^+ -**Bor**-measurable functions, we use “ κ^+ -**Bor**”.

2.9 Determinacy in the generalized context

Games are fundamental tools in classical descriptive set theory, and the prototype from which all such games derive is the Gale–Stewart game (see [Kec95, Section 20], or Chapter 6 for its generalized version). In other words, although the rules may differ from one game to another, each of them can in fact be reformulated as a Gale–Stewart game.

Gale–Stewart games are infinite, two-player, zero-sum, perfect-information games of length ω , denoted $G_X^\omega(A)$, where X is a non-empty set and $A \subseteq {}^\omega X$. Players **I** and **II** alternate choosing elements x_0, x_1, x_2, \dots from X for ω -many turns. At the end of a run, a sequence $(x_i)_{i < \omega} \in {}^\omega X$ has been played; Player **I** wins if $(x_i)_{i < \omega} \in A$, otherwise Player **II** wins. A strategy for a player is a

function determining their moves based on the history of previous moves; a strategy is winning if it guarantees victory regardless of the opponent’s play. A set $A \subseteq {}^\omega X$ is determined if one (and only one) of the players has a winning strategy in $G_X^\omega(A)$.

Recall that the Axiom of Determinacy AD_X for games on X is the statement “Every $A \subseteq {}^\omega X$ is determined”. When $X = \omega$, this is the usual Axiom of Determinacy AD , and when $X = {}^\omega\omega$ we get the Axiom of Real Determinacy $AD_{\mathbb{R}}$, which is much stronger than AD . The Axiom of Determinacy AD is incompatible with the full Axiom of Choice AC^1 because, among other things, it implies many regularity properties for sets of reals which are known to be inconsistent with AC . In fact, AD entails the existence of ω_1 -sequences of distinct reals, and hence implies that ${}^\omega 2$ cannot be well-ordered. Nonetheless, AD is compatible with weaker forms of AC . Kechris has shown [Kec84] that $AD \Rightarrow DC(\mathbb{R})^2$ under the additional assumption that $V = L(\mathbb{R})$, and by deep results of Woodin, assuming large cardinals, both AD and $AD_{\mathbb{R}}$ are consistent with $ZF + DC(\mathbb{R})$ (see, e.g., [Kan09]).

In his celebrated work [Mar75], Martin established within ZFC that all Borel sets are determined. In contrast, in generalized descriptive set theory there are no corresponding determinacy results, not even for (generalized) Borel sets. Indeed, assuming $2^{<\kappa} = \kappa$ and working either in the generalized Cantor space ${}^\kappa 2$ or in the generalized Baire space ${}^{\text{cof}(\kappa)}\kappa$ with $\text{cof}(\kappa) > \omega$, even clopen games may fail to be determined. If instead $\text{cof}(\kappa) = \omega$ but $\kappa > \omega$, then the games are determined whenever $A, B \subseteq {}^\omega\kappa$ are open or closed, yet there exist sets in $\kappa^+ \text{-}\Delta_2^0({}^\omega\kappa)$ for which the games are not determined. A thorough discussion of the failure of κ^+ -Borel determinacy in the case $\text{cof}(\kappa) = \omega$ is given in [DMR25, Section 4.5].

2.10 Generalized Lipschitz and Wadge games

In this section we generalize two important games introduced by Wadge on ω : the Lipschitz game and Wadge game (see [Wad83], [Kec95, section 21], [And03], or [And07]).

Consider the cardinals $\nu \geq 2$ and $\mu \geq \omega$. The games we are defining are two-person, perfect-information games on ν of length μ . Let $A, B \subseteq {}^\mu\nu$.

The **generalized Lipschitz game** $G_{\perp}^{\mu,\nu}(A, B)$ is played as follows. There are two players, **I** and **II**, which take turns in picking ordinals $x_i < \nu$ and $y_i < \nu$ (respectively) in the i -th round, with **I** starting first, for μ -many rounds. At limit levels, Player **I** starts again.

I	x_0	x_1	\cdots	x_α	\cdots
II	y_0	y_1	\cdots	y_α	\cdots

Once the sequences $x = (x_i)_{i < \mu}$ and $y = (y_i)_{i < \mu}$ are selected, Player **II** wins if $x \in A \Leftrightarrow y \in B$, and Player **I** wins otherwise.

A strategy for Player **II** is defined as a monotone function $\sigma : <{}^\mu\nu \setminus \{\emptyset\} \rightarrow <{}^\mu\nu$ such that $\text{lh}(\sigma(s)) = \text{lh}(s)$ and $y_i = \sigma((x_j)_{j \leq i})(i)$ for every $i < \mu$. Similarly, we define a strategy for Player **I** as a function $\tau : <{}^\mu\nu \rightarrow <{}^\mu\nu$ such that

¹In fact, AD is even incompatible with BPI —the principle asserting that for every boolean algebra B , any proper ideal on B can be extended to a prime ideal—which is known to be strictly weaker than AC .

² $DC(\mathbb{R})$ is the principle of Dependent Choice on the reals

$\text{lh}(\tau(s)) = \text{lh}(s) + 1$ and $x_i = \tau((y_j)_{j < i})(i)$ for every $1 < i < \mu$. We say that a strategy is winning if it guarantees victory for the corresponding player against any strategy of the opponent. Clearly, it cannot happen that both **I** and **II** have a winning strategy.

Note that strategy for Player **II** in $G_L^{\mu, \nu}(A, B)$ is a Lipschitz function as in Definition 2.7.1 hence it induces a Lipschitz function $f_\sigma: {}^\mu\nu \rightarrow {}^\mu\nu$ such that $A = f_\sigma^{-1}(B)$. Conversely, given any Lipschitz function $f: {}^\mu\nu \rightarrow {}^\mu\nu$, its approximation $\varphi_f: <{}^\mu\nu \rightarrow <{}^\mu\nu$ induces a strategy for Player **II** simply defining $\sigma(s) = \varphi_f(s) \upharpoonright \text{lh}(s)$ for any $s \in <{}^\mu\nu$. Finally, a strategy for **I** yields a Lipschitz function $f_\sigma: {}^\mu\nu \rightarrow {}^\mu\nu$ such that ${}^\mu\nu \setminus B = f^{-1}(A)$. Therefore:

II wins $G_L^{\mu, \nu}(A, B)$ if and only if $A \leq_L B$.
 If **I** wins $G_L^{\mu, \nu}(A, B)$, then ${}^\mu\nu \setminus B \leq_L A$.

The **generalized Wadge game** $G_W^{\mu, \nu}(A, B)$ is a variant of the generalized Lipschitz game where **II** is allowed to pass (i.e., not to play) at any round, provided that she produces a sequence of length μ at the end of the game, otherwise she loses. Formally, this is a game on $\nu \cup \{\mathbf{P}\}$, where **P** represents the move ‘‘pass’’ and does not belong to ν . All moves of **II** which are not **P** are called active moves. An example of a run of $G_W^{\mu, \nu}(A, B)$ is

I	x_0	x_1	\dots	x_α	$x_{\alpha+1}$	\dots
II	P	y_0	\dots	P	\dots	y_α
		\dots	P	\dots	\dots	P
		\dots	\dots	\dots	\dots	\dots

As in the case of the generalized Lipschitz game, once the sequences $x = \langle x_i \mid i < \mu \rangle$ and $y = \langle y_i \mid i < \mu \rangle$ are selected, Player **II** wins if $x \in A \Leftrightarrow y \in B$, and Player **I** wins otherwise.

A strategy for Player **II** is defined as a monotone function $\sigma: <{}^\mu\nu \setminus \{\emptyset\} \rightarrow <{}^\mu(\nu \cup \{\mathbf{P}\})$ such that $\text{lh}(\sigma(s)) = \text{lh}(s)$ and $y_i = \sigma((x_j)_{j \leq i})(i)$ for every $i < \mu$. Similarly, we define a strategy for Player **I** as a function $\tau: <{}^\mu(\nu \cup \{\mathbf{P}\}) \rightarrow <{}^\mu\nu$ such that $\text{lh}(\tau(s)) = \text{lh}(s) + 1$ and $x_i = \tau((y_j)_{j < i})(i)$ for every $1 < i < \mu$. A strategy for **II** is called legal if is such that, given any $x \in {}^\mu\nu$, $\sigma(x \upharpoonright \alpha + 1)(\alpha) \in \nu$ for μ -many α 's. Given a legal strategy for **II**, the interpretation function $\iota_W: <{}^\mu(\nu \cup \{\mathbf{P}\}) \rightarrow <{}^\mu\nu$ defined below will be used to recover only the active moves from Player **II**'s play. We recursively define ι_W on sequences in $<{}^\mu\nu$. We set $\iota_W(\emptyset) = \emptyset$,

$$\iota_W(s \frown \langle z \rangle) = \begin{cases} \iota_W(s) & \text{if } z = \mathbf{P}; \\ \iota_W(s) \frown \langle z \rangle & \text{otherwise;} \end{cases}$$

and $\iota_W(s) = \bigcup_{i < \text{lh}(s)} \iota_W(s \upharpoonright i)$ if $\text{lh}(s)$ is a limit ordinal. Note that ι_W is monotone.

Given a legal strategy σ for Player **II** in $G_W^{\mu, \nu}(A, B)$, then $\iota_W \circ \sigma$ is a continuous function as in Definition 2.7.1 hence, by Proposition 2.7.2, a winning strategy σ for **II** induces a continuous function $f_{\iota_W \circ \sigma}: {}^\mu\nu \rightarrow {}^\mu\nu$ such that $A = f_{\iota_W \circ \sigma}^{-1}(B)$. Conversely, given any continuous function $f: {}^\mu\nu \rightarrow {}^\mu\nu$, its approximation $\varphi_f: <{}^\mu\nu \rightarrow <{}^\mu\nu$ induces a strategy σ for Player **II** defined by recursion on $\text{lh}(s) < \mu$ as follows. For for any $s \in <{}^\mu\nu$ and $\alpha < \nu$, let

$$\sigma(s \frown \alpha) = \begin{cases} \sigma(s) \frown \varphi_f(s \frown \alpha)(\text{lh}(s) + 1) & \text{if } \text{lh}(s \frown \alpha) \leq \text{lh}(\varphi_f(s \frown \alpha)); \\ \sigma(s) \frown \mathbf{P} & \text{otherwise;} \end{cases}$$

if $\text{lh}(s)$ is a limit ordinal, let $\sigma(s) = \bigcup_{i < \text{lh}(s)} \sigma(s \upharpoonright i)$.

On the other hand, given a strategy τ for **I** in the generalized Wadge game, the function $\tau \circ \iota_W$ is such that $\text{lh}(\iota_W(s)) \leq \text{lh}(s) \leq \text{lh}(\iota_W(\tau(s)))$, hence it is Lipschitz (see Definition 2.7.1). Therefore, a winning strategy τ for **I** yields a Lipschitz function $f_{\tau \circ \iota_W} : {}^\mu\nu \rightarrow {}^\mu\nu$ such that ${}^\mu\nu \setminus B = f_{\tau \circ \iota_W}^{-1}(A)$. Summarizing:

$$\begin{aligned} \text{II wins } G_W^{\mu,\nu}(A, B) &\text{ if and only if } A \leq_W B. \\ \text{If I wins } G_W^{\mu,\nu}(A, B), &\text{ then } {}^\mu\nu \setminus B \leq_L A. \end{aligned}$$

When the space ${}^\mu\nu$ is clear from the context, we drop it from all the notation above; therefore, with the notation $G_W(A, B)$ (or $G_L(A, B)$) for $A, B \subseteq {}^\mu\nu$, we always refer to the game on ${}^\mu\nu$. We also fix each player's gender for purposes of convenience: Player **II** is regarded as feminine and thus referred to as “she”, while Player **I** is regarded as masculine and is titled “he”.

If we want to consider a more general notion of game on subsets $X, Y \subseteq {}^\mu\nu$ with $A \subseteq X$ and $B \subseteq Y$, then we will use the notation $G_W^{X,Y}(A, B)$ (or $G_L^{X,Y}(A, B)$). This latter version of the generalized Wadge (Lipschitz) game is just like $G_W^{\mu,\nu}(A, B)$ ($G_L^{\mu,\nu}(A, B)$), except that Player **I** and Player **II** must make moves so that the partial play is always in \mathcal{T}_X and \mathcal{T}_Y , respectively. When X and Y superclosed, the first player to break this rule loses outright. A legal position is just a node of \mathcal{T}_X or \mathcal{T}_Y , and a move that extends a legal position to another legal position is called a legal move. The definition of strategy must be modified accordingly so that they yield only legal moves.

2.11 The Semi-Linear Ordering principle (SLO)

If \mathcal{F} is a family of functions on a space X as in (2.8.1), the notion of **Semi-Linear Ordering Principle** for \mathcal{F} -maps is $\text{SLO}_{\mathcal{F}}(X)$:

$$\forall A, B \subseteq X (A \leq_{\mathcal{F}} B \vee \neg B \leq_{\mathcal{F}} A). \quad (2.11.1)$$

If $\mathcal{F} \subseteq \mathcal{G}$ then $\text{SLO}_{\mathcal{F}}(X) \Rightarrow \text{SLO}_{\mathcal{G}}(X)$, because $\leq_{\mathcal{G}}$ is coarser than $\leq_{\mathcal{F}}$ (see Section 2.8).

If Γ is a pointclass, we use the notation $\text{SLO}_{\mathcal{F}}(\Gamma(X))$ to express that:

$$\forall A, B \in \Gamma(X) (A \leq_{\mathcal{F}} B \vee \neg B \leq_{\mathcal{F}} A). \quad (2.11.2)$$

Observe that $\text{SLO}_{\mathcal{F}}(X)$ ensures that every antichain has size at most 2. Indeed, if $A \not\leq_{\mathcal{F}} B$ and $B \not\leq_{\mathcal{F}} A$, then by $\text{SLO}_{\mathcal{F}}(X)$ we obtain $\neg B \leq_{\mathcal{F}} A$ and $\neg A \leq_{\mathcal{F}} B$, hence $A \equiv_{\mathcal{F}} \neg B$. Equivalently, under $\text{SLO}_{\mathcal{F}}(X)$, for each $A \subseteq X$ we have that either $A \leq_{\mathcal{F}} \neg A$ (so that A , and its Wadge degree $[A]_{\mathcal{F}}$, are selfdual), or else A and $\neg A$ are incomparable with respect to $\leq_{\mathcal{F}}$ (so A is non-selfdual, and $\{[A]_{\mathcal{F}}, [\neg A]_{\mathcal{F}}\}$ is a maximal antichain, i.e., a non-selfdual pair). Consequently, the ordering induced on the \mathcal{F} -degrees is “almost” linear (and becomes linear if each degree is identified with its dual), which explains the terminology.

In the classical context, i.e., when $X = {}^\omega\omega$ or $X = {}^\omega 2$, it is well known that the preorders \leq_L and \leq_W can be characterized in terms of games, via the Lipschitz game and the Wadge game, respectively [Wad83]. These games are particular instances of Gale–Stewart games (see Section 2.9) and they are a central tool in classical Wadge reducibility theory. In particular, if the Wadge

(respectively Lipschitz) game is determined for every element of a non-empty class \mathfrak{S} of subsets of ${}^\omega\omega$, then $\text{SLO}_W(\mathfrak{S})$ (respectively $\text{SLO}_L(\mathfrak{S})$) holds. This fact, first established by Wadge [Wad83], is commonly referred to as the Wadge Lemma. Martin's celebrated result [Mar75] proving Borel determinacy in ZFC yields the Wadge Lemma for all Borel sets; under $\text{ZF} + \text{DC}(\mathbb{R}) + \text{AD}$, this result extends further to the full powerset.

In contrast, in generalized descriptive set theory, although the preorders \leq_L and \leq_W can still be characterized in terms of generalized games (see Section 2.10), there are no corresponding determinacy results available (see again Section 2.9). As a consequence, a higher-level analogue of the Wadge Lemma is not obtainable in this generalized setting, not even when restricted to generalized Borel sets. In particular, we currently lack any a priori knowledge regarding whether SLO_W or SLO_L holds, even for relatively simple sets. Nonetheless, the generalized Lipschitz and Wadge games defined in Section 2.10 continue to play a central role in the analysis of these hierarchies, and we will frequently employ them throughout this work.

We report three results concerning Wadge reductions, SLO_W , and κ -complete sets; the proofs are the natural generalizations of the ones used in the classical setting. The first result concerns how Wadge structures relate when considered on different topological spaces.

Lemma 2.11.1 ([And07, Proposition 28]). *Let X, Y be topological spaces. If $r : Y \rightarrow X$ is a retraction, then the map $A \mapsto r^{-1}(A)$ is an embedding of the Wadge structure of X into the Wadge structure of Y . In fact,*

$$\text{SLO}_W(Y) \Rightarrow \text{SLO}_W(X).$$

The next two results are essentially folklore, yet they will play a crucial role in our study of the generalized Wadge hierarchy.

Fact 2.11.2. *Let X be a topological space, Γ be a boldface pointclass and $A \subseteq X$.*

- (1) *If A is κ -hard for Γ , then $X \setminus A$ κ -hard for $\check{\Gamma}$.*
- (2) *If A is κ -hard for Γ and $A \leq_W B$, then B is κ -hard for Γ .*

Moreover, if Γ is non-selfdual:

- (3) *If A is κ -hard for Γ then $A \notin \check{\Gamma}(X)$.*
- (4) *If A is κ -complete for Γ then $A \in \Gamma(X) \setminus \check{\Gamma}(X)$.*

The above points hold similarly if we consider κ -Lipschitz-hard and κ -Lipschitz-complete sets instead.

The reverse implications in (3) and (4) in Lemma 2.11.2 may fail in general, but they hold if we assume $X = {}^\kappa 2$ and $\text{SLO}_W(\Gamma({}^\kappa 2))$. Indeed, assume $\text{SLO}_W(\Gamma({}^\kappa 2))$, let $A \notin \check{\Gamma}({}^\kappa 2)$ and $B \in \Gamma({}^\kappa 2)$. Then, either $B \leq_W A$ or $\neg A \leq_W B$, but the latter is impossible since $A \notin \check{\Gamma}({}^\kappa 2)$ implies $\neg A \notin \Gamma({}^\kappa 2)$. Furthermore, the following result holds.

Lemma 2.11.3. *Let Γ be a boldface pointclass that is not selfdual, and suppose that $\text{SLO}_W(\Delta_\Gamma(\kappa^2))$ holds. Then, $\text{SLO}_W(\Gamma(\kappa^2) \cup \check{\Gamma}(\kappa^2))$ is equivalent to the fact that every $\Gamma(\kappa^2)$ -proper set is κ -complete for Γ .*

An analogous statement holds if one substitutes SLO_W with SLO_L and replaces κ -complete by κ -Lipschitz-complete.

Proof. We have already observed that $\text{SLO}_W(\Gamma(\kappa^2))$ implies that every $\Gamma(\kappa^2)$ -proper set is κ -complete for Γ . Assume now that every $\Gamma(\kappa^2)$ -proper set is κ -complete for Γ . It suffices to prove that $\text{SLO}_W(\Gamma(\kappa^2))$ holds, since $\text{SLO}_W(\check{\Gamma}(\kappa^2))$ then follows immediately. Let $A, B \in \Gamma(\kappa^2)$. If $A, B \in \Delta_\Gamma(\kappa^2)$, the result follows by $\text{SLO}_W(\Delta_\Gamma(\kappa^2))$. If $B \in \Gamma(\kappa^2) \setminus \check{\Gamma}(\kappa^2)$, then B is κ -complete for Γ , hence $A \leq_W B$. If $B \in \Delta_\Gamma(\kappa^2)$ and $A \in \Gamma(\kappa^2) \setminus \check{\Gamma}(\kappa^2)$, then A is κ -complete for Γ , hence $\neg B \leq_W A$ because $\neg B \in \Delta_\Gamma(\kappa^2) \subseteq \Gamma(\kappa^2)$. \square

2.12 Wadge classes

An alternative (yet equivalent) framework for presenting and studying the Wadge hierarchy on a space X is provided by Wadge classes. A boldface pointclass Γ is a **Wadge class on X** when it is of the form

$$A \downarrow_X = \{B \subseteq X \mid B \leq_W A\}$$

for some $A \subseteq X$; any element $A \in \Gamma$ satisfying $\Gamma(X) = A \downarrow_X$ is called X -complete (with respect to Γ), and we say that A generates $\Gamma(X)$. Indeed, if $X = \kappa^2$ and $\Gamma(\kappa^2) = A \downarrow_{\kappa^2}$, then A is κ -complete for Γ . Observe that $\text{SLO}_W(\kappa^2)$ ensures that every non-selfdual boldface pointclass Γ is a Wadge class on κ^2 (Lemma 2.11.3). Conversely, selfdual boldface pointclasses need not be Wadge classes on κ^2 .

The Wadge hierarchy (as defined in Section 2.8) is clearly isomorphic to the structure formed by all Wadge classes ordered by inclusion, as witnessed by the map $[A]_W \mapsto A \downarrow_X$. In particular, $A \leq_W B$ if and only if $A \downarrow_X \subseteq B \downarrow_X$, therefore $\text{SLO}_W(X)$ implies that for any Wadge classes Γ and Δ ,

$$\Gamma \subseteq \Delta \quad \vee \quad \check{\Delta} \subseteq \Gamma.$$

Moreover, $A \subseteq X$ is a selfdual set precisely when $A \downarrow_X$ is selfdual as a boldface pointclass, and a non-selfdual pair $\{[A]_W, [\neg A]_W\}$ corresponds to a pair of distinct non-selfdual boldface pointclasses $(\Gamma(X), \check{\Gamma}(X))$ where $\Gamma(X) = A \downarrow_X$. Each Wadge class Γ induces a coarse Wadge class $\Gamma^* = \Gamma \cup \check{\Gamma}$.

2.13 More on universal sets

Following [And07], we consider higher analogues of some results concerning universal sets.

We remind the reader that, throughout this work, a function $f: \kappa^2 \rightarrow \kappa^2$ is referred to as ‘‘Lipschitz’’ when it is 0-Lipschitz in the sense of Definition 2.7.5.

Since a continuous (Lipschitz) function $\kappa^2 \rightarrow \kappa^2$ is completely determined by a continuous (respectively, Lipschitz) approximating function $<\kappa^2 \rightarrow <\kappa^2$ (see Section 2.7), and since these objects can be coded as subsets of κ , a map $x \mapsto f_x$ can be defined so that

$$\{f_x \mid x \in \kappa^2\}$$

is the collection of all continuous (respectively, Lipschitz) functions $f : {}^\kappa 2 \rightarrow {}^\kappa 2$. We call such a map a parametrization or coding of the continuous (respectively, Lipschitz) functions. In the next lemma, we fix once and for all a specific parametrization of Lipschitz functions.

Lemma 2.13.1. *There is a bijection*

$${}^\kappa 2 \rightarrow \{f : {}^\kappa 2 \rightarrow {}^\kappa 2 \mid f \text{ is Lipschitz}\} : x \mapsto \ell_x$$

such that the map

$${}^\kappa 2 \times {}^\kappa 2 \rightarrow {}^\kappa 2 : (x, y) \mapsto \ell_x(y)$$

is continuous.

Proof. To simplify the notation, given an ordinal β , let $\text{Succ}(\beta)$ be the collection of all successor ordinals up to β . Fix $\langle s_\alpha \mid \alpha < \kappa \rangle$ an enumeration without repetitions of $\{s \in <{}^\kappa 2 \mid \text{lh}(s) \in \text{Succ}(\kappa)\}$ such that $s_\alpha \subset s_\beta \Rightarrow \alpha < \beta$. For any $x \in {}^\kappa 2$, let $\varphi_x : <{}^\kappa 2 \rightarrow <{}^\kappa 2$ be defined recursively by $\varphi_x(\emptyset) = \emptyset$, $\varphi_x(s \hat{\ } \langle i \rangle) = \varphi_x(s) \hat{\ } \langle x(\alpha) \rangle$ for $\alpha < \kappa$ unique such that $s_\alpha = s \hat{\ } \langle i \rangle$, and $\varphi_x(s) = \bigcup_{i < \text{lh}(s)} \varphi_x(s \upharpoonright i)$ if $\text{lh}(s)$ is a limit ordinal. Each φ_x is well-defined and Lipschitz.

We first want to show that $\varphi_x \neq \varphi_{x'}$ for all distinct $x, x' \in {}^\kappa 2$. Let $\alpha < \kappa$ be the smallest such that $x(\alpha) \neq x'(\alpha)$, and let i be such that $i + 1 = \text{lh}(s_\alpha)$. Then, $\varphi_x(s_\alpha) \neq \varphi_{x'}(s_\alpha)$, since these two sequences differ on their last element:

$$(\varphi_x(s_\alpha))(i) = x(\alpha) \neq x'(\alpha) = (\varphi_{x'}(s_\alpha))(i).$$

We now show that $x \mapsto \varphi_x$ is surjective. Suppose $\varphi : <{}^\kappa 2 \rightarrow <{}^\kappa 2$ is Lipschitz. For every $\alpha < \kappa$, let i_α be the last element of the sequence $\varphi(s_\alpha)$. Then, $x \in {}^\kappa 2$ defined as $x(\alpha) = i_\alpha$ is such that $\varphi = \varphi_x$. In conclusion,

$${}^\kappa 2 \rightarrow \{\varphi : <{}^\kappa 2 \rightarrow <{}^\kappa 2 \mid \varphi \text{ is Lipschitz}\}, \quad x \mapsto \varphi_x$$

is a bijection. Letting $\ell_x : {}^\kappa 2 \rightarrow {}^\kappa 2$ be the Lipschitz functions induced by φ_x , it is not hard to see that the evaluation map $(x, y) \mapsto \ell_x(y)$ is continuous. \square

Remark 2.13.2. There is no parametrization $\{f_x \mid x \in {}^\kappa 2\}$ of continuous functions such that the evaluation map ${}^\kappa 2 \times {}^\kappa 2 \rightarrow {}^\kappa 2 : (x, y) \mapsto f_x(y)$ is continuous. Indeed, consider a continuous function $g : {}^\kappa 2 \rightarrow {}^\kappa 2$ such that $g(x) \neq x$ for every $x \in {}^\kappa 2$, e.g. $g(x)(i) = 1 - x(i)$ for every $i < \kappa$, and apply the diagonal argument to $g(f_x(x))$. Since g is continuous and the f_x 's are a parametrization of all continuous functions, there exists $y \in {}^\kappa 2$ such that $g(f_x(x)) = f_y(x)$ for every $x \in {}^\kappa 2$. However, choosing $y = x$ results in a contradiction.

Proposition 2.13.3. *Let Γ be a boldface pointclass such that $\Gamma({}^\kappa 2)$ is a non-selfdual Wadge class on ${}^\kappa 2$ generated by a set $A \subseteq {}^\kappa 2$. If $\Gamma({}^\kappa 2) = \{B \subseteq {}^\kappa 2 \mid B \leq_L A\}$, then there is a ${}^\kappa 2$ -universal set for $\Gamma({}^\kappa 2)$.*

Proof. Let $\{\ell_x \mid x \in {}^\kappa 2\}$ be a parametrization of all Lipschitz functions ${}^\kappa 2 \rightarrow {}^\kappa 2$ as in Lemma 2.13.1. We claim that $\mathcal{U} = \{(x, y) \in {}^\kappa 2 \times {}^\kappa 2 \mid \ell_x(y) \in A\}$ is ${}^\kappa 2$ -universal for $\Gamma({}^\kappa 2)$. Clearly, $\mathcal{U} \in \Gamma({}^\kappa 2 \times {}^\kappa 2)$, so it remains to show that $\{\mathcal{U}_x \mid x \in {}^\kappa 2\} = \Gamma({}^\kappa 2)$. This follows from $\{\mathcal{U}_x \mid x \in {}^\kappa 2\} = \{\ell_x^{-1}(A) \mid x \in {}^\kappa 2\} = \{B \subseteq {}^\kappa 2 \mid B \leq_L A\} = \Gamma({}^\kappa 2)$. \square

Proposition 2.13.3 also holds if we modify the assumptions as follows:

Proposition 2.13.4. *Let $\mathbf{\Gamma}$ be a boldface pointclass such that $\mathbf{\Gamma}^{(\kappa 2)}$ is a non-selfdual Wadge class on ${}^{\kappa}2$. Then, $\text{SLO}_L(\mathbf{\Gamma}^{(\kappa 2)})$ implies the existence of a ${}^{\kappa}2$ -universal set for $\mathbf{\Gamma}^{(\kappa 2)}$.*

Proof. Assume that $\mathbf{\Gamma}^{(\kappa 2)}$ is generated by a set A . Then $A \in \mathbf{\Gamma}^{(\kappa 2)} \setminus \check{\mathbf{\Gamma}}^{(\kappa 2)}$ and since $\text{SLO}_L(\mathbf{\Gamma}^{(\kappa 2)})$ holds, then A is κ -Lipschitz-complete for $\mathbf{\Gamma}$ by Lemma 2.11.3. This implies that $\mathbf{\Gamma}^{(\kappa 2)} = \{B \subseteq {}^{\kappa}2 \mid B \leq_L A\}$, so we conclude by Proposition 2.13.3. \square

Chapter 3

Generalized Borel Sets

The generalized Cantor and Baire spaces have been extensively studied for κ a (necessarily regular) cardinal satisfying $\kappa^{<\kappa} = \kappa$, yielding interesting results on their κ^+ -Borel subsets (see e.g. [MV93, FHK14], among many others). However, two significant aspects remain largely unexplored: first, the study of the κ^+ -Borel hierarchy for *subspaces* of the generalized Baire space, or for more general Polish-like spaces; and second, the entire framework of generalized descriptive set theory at singular cardinals, a new direction that has only recently begun to attract attention (see [AMR22] for some initial results, and [DMR25] for an extensive treatment of the countable cofinality case). In particular, the study of hierarchies of definable sets for subspaces of ${}^{\text{cof}(\kappa)}\kappa$ when κ is a singular cardinal of uncountable cofinality constitutes a completely novel area that needed systematic study. This chapter addresses all these issues.

The first part of this chapter, i.e. Sections 3.2–3.4, develops the foundational aspects of the theory of generalized Borel sets, while the second part of the chapter, corresponding to Sections 3.5–3.6, is concerned with somewhat unexpected behaviors of the κ^+ -Borel hierarchy on the arguably nicest kind of Polish-like spaces, namely, closed subspaces of ${}^{\text{cof}(\kappa)}\kappa$. Finally, the third part of the chapter, that is, Section 3.7, introduces and studies the difference hierarchy. We pursue maximal generality in two directions. First, we work with cardinals κ satisfying only $2^{<\kappa} = \kappa$, dropping the commonly assumed regularity condition, and thus including singular cardinals. Second, we extend beyond subspaces of ${}^{\text{cof}(\kappa)}\kappa$ to the broader setting of regular Hausdorff spaces of weight at most κ . This aligns with recent advances in generalized descriptive set theory, where many classical results have been successfully extended to various classes of Polish-like spaces, rather than restricting the attention solely to ${}^{\text{cof}(\kappa)}\kappa$ or ${}^\kappa 2$ (see [CS16, Gal19, Ago22, AMRS23]). Actually, our analysis indicates that even imposing such weaker Polish-like properties onto the space is not necessary for the study of its κ^+ -Borel hierarchy, and many results hold under remarkably mild conditions. In particular, one can safely work with arbitrary subspaces of ${}^\kappa 2$ and ${}^{\text{cof}(\kappa)}\kappa$ and forget about all the restrictions stated in the various results. Furthermore, one can even work with arbitrary T_0 spaces of weight at most κ , if only the global behavior of the hierarchy and its infinite levels are under question (see Proposition 3.2.23).

Let us now describe the content of the chapter in more detail.

In Section 3.2 we introduce the γ -Borel hierarchy (see Section 3.2.1), a natural stratification of the γ -Borel sets that is a foundational tool in descriptive

set theory and plays a central role in both the classical case ($\gamma = \omega_1$) and its generalization to uncountable cardinals ($\gamma = \kappa^+$). Despite its widespread use, a systematic presentation of the γ -Borel hierarchy in full generality has been lacking. We address this gap by providing a formal treatment of all relevant definitions, and establishing the key preliminary results necessary for the development of the theory. This includes natural features of the γ -Borel hierarchy, like the property of being increasing (Section 3.2.2), the length of the hierarchy on a given space (Section 3.2.3) and on its subspaces (Section 3.2.4), or the existence of universal sets (Section 3.2.5). In particular, we introduce the crucial concept of order $\text{ord}_\gamma(X)$ of the γ -Borel hierarchy on a space X (Definition 3.2.12), which measures its length.

In Section 3.3 (and, partially, in Section 3.4), we provide a comprehensive description of the κ^+ -Borel hierarchy on an arbitrary regular Hausdorff space X of weight at most κ , and we establish its fundamental properties. In particular:

- We determine the optimal closure properties of the pointclasses κ^+ - $\Sigma_\alpha^0(X)$, κ^+ - $\Pi_\alpha^0(X)$, and κ^+ - $\Delta_\alpha^0(X)$ appearing in the κ^+ -Borel hierarchy (Proposition 3.3.1, Proposition 3.3.2, and Corollary 3.4.9).
- We prove that the κ^+ -Borel hierarchy is proper, in the sense that new κ^+ -Borel subsets of X appear at each level of the hierarchy up to $\text{ord}_{\kappa^+}(X)$ (Corollary 3.3.5 and Corollary 3.3.6).
- We study the existence of universal and complete sets for the various levels of interest (Proposition 3.3.7).
- We provide higher analogues of structural properties, such as the reduction property, the separation property, and alike, for the various pointclasses appearing in the κ^+ -Borel hierarchy (Section 3.3.1).

Interestingly enough, while for a regular κ all these results can be proved without auxiliary tools, when κ is singular we need to pass through the study of an alternative hierarchy, discussed below. This explains why all the results for the regular case already appear in Section 3.3, while in a few cases their counterparts for the singular case have to be postponed to Section 3.4.

We also provide a sufficient condition for the κ^+ -Borel hierarchy to be non-collapsing, i.e. to ensure that $\text{ord}_{\kappa^+}(X)$ attains the maximal value κ^+ :

Theorem (Theorem 3.3.9). *Let X be a regular Hausdorff topological space of weight at most κ . If there is a κ^+ -Borel embedding of ${}^\kappa 2$ into X , then the κ^+ -Borel hierarchy on X does not collapse.*

When κ is regular, it is consistent that the converse of Theorem 3.3.9 holds for all κ^+ -Borel subspaces of ${}^\kappa \kappa$, and thus for all topological spaces of weight at most κ that are κ^+ -Borel isomorphic to a κ^+ -Borel subset of ${}^\kappa \kappa$;¹ this is due to the fact that there are models of set theory in which the κ -Perfect Set Property holds for all κ^+ -Borel subsets of ${}^\kappa \kappa$ (see [Sch17]). This will be complemented in Section 3.5, where we show that, consistently, there can be even closed subsets of ${}^\kappa \kappa$ which do not contain a κ^+ -Borel isomorphic copy of ${}^\kappa 2$, yet their κ^+ -Borel hierarchy does not collapse.

¹Such spaces have been referred to as standard Borel κ -spaces, see e.g. [MR13, Section 3].

As anticipated, in Section 3.4 we also introduce and analyze an alternative hierarchy for κ^+ -Borel sets that naturally arises when κ is a singular cardinal. Indeed, for such cardinals the notions of κ^+ -algebra and κ -algebra coincide, allowing the collection of κ^+ -Borel subsets of a space to be equivalently described as the smallest κ -algebra containing all its open sets. This leads to a new hierarchy, called κ -Borel hierarchy, which is the focus of this section, along with its relationship to the κ^+ -Borel hierarchy. One of our main results, Theorem 3.4.1, precisely characterizes the connection between these two hierarchies for spaces whose open sets can be written as unions of $\text{cof}(\kappa)$ -many closed sets — the latter is not a strong restriction, since such condition is naturally satisfied by most spaces studied in the literature on generalized descriptive set theory, including all subspaces of ${}^{\text{cof}(\kappa)}\kappa$ and, more generally, all $(\omega, \text{cof}(\kappa))$ -Nagata–Smirnov spaces (see Fact 3.1.1). It is remarkable that, even when the κ^+ -Borel and κ -Borel sets coincide, the two families stratify in two different hierarchies, with the κ -Borel hierarchy being strictly finer than the κ^+ -Borel hierarchy; thus, the κ -Borel hierarchy provides a more informative classification of κ^+ -Borel sets when κ is singular. Theorem 3.4.1 also allows us to extend various results from the κ^+ -Borel hierarchy to the κ -Borel hierarchy. For example, it yields the bounds

$$\text{ord}_{\kappa^+}(X) \leq \text{ord}_{\kappa}(X) \leq 2 \cdot \text{ord}_{\kappa^+}(X)$$

relating the lengths of the two hierarchies (see Corollary 3.4.2), hence the κ^+ -Borel hierarchy collapses if and only if the κ -Borel hierarchy does (Corollary 3.4.3). We also study additional fundamental properties of the κ -Borel hierarchy analogous to those listed above for the κ^+ -Borel hierarchy, including its optimal closure properties (Propositions 3.4.4 and 3.4.8), its properness (Propositions 3.4.5 and 3.4.6), the conditions for its collapse (Proposition 3.4.7), and the existence of universal and complete sets (Proposition 3.4.10). From these, we derive the counterparts for a singular cardinal κ of the missing results about the κ^+ -Borel hierarchy obtained in Section 3.3 for the regular case (Corollary 3.4.9).

With Section 3.5 and Section 3.6, we dive into the second part of the chapter, where we return to the case of regular cardinals κ , and we construct via forcing various models of set theory in which the κ^+ -Borel hierarchy exhibits specific behaviors. Our aim is to address three central problems, each of which we resolve affirmatively.

The first problem is motivated by the fact that if $X \subseteq {}^{\kappa}\kappa$ contains a κ -perfect subset, then $\text{ord}_{\kappa^+}(X) = \kappa^+$; this is well-known in the classical setting $\kappa = \omega$, and follows from our Theorem 3.3.9 if κ is uncountable.

Problem 1. *Can there be a κ^+ -Borel (or even closed) set $X \subseteq {}^{\kappa}\kappa$ which contains no κ -perfect subset, yet $\text{ord}_{\kappa^+}(X) = \kappa^+$?*

The reader might wonder which notion of κ -perfect set is involved in Problem 1. In fact, several reasonable notions of a κ -perfect set have been considered in generalized descriptive set theory, each with their own specific nuances, depending on the context. For the purposes of this chapter, we adopt the following definitions. A set $X \subseteq {}^{\kappa}\kappa$ is κ -perfect if it is closed and homeomorphic to the generalized Cantor space ${}^{\kappa}2$, and it is κ -thin if it has no κ -perfect subset (Definition 3.5.1). This choice does not rule out the other possibilities

that have been considered elsewhere: in fact, Corollary 3.5.10 yields that for κ^+ -Borel sets $X \subseteq {}^\kappa\kappa$, our definition gives rise to the same notion of κ -Perfect Set Property given by most of the other definitions of κ -perfect set proposed in literature: indeed, X has a κ -perfect subset if and only if there is κ^+ -Borel injection from ${}^\kappa 2$ into X . In particular, this aligns our formulation of Problem 1 with Theorem 3.3.9.

Problem 2 is instead motivated by the easy observation that if $|X| \leq \kappa$, then $\text{ord}_{\kappa^+}(X) \leq 2$ because every subset of X can be written as a union of length at most κ of singletons.

Problem 2. *Can there be a κ^+ -Borel (or even closed) set $X \subseteq {}^\kappa\kappa$ such that $|X| > \kappa$, yet $\text{ord}_{\kappa^+}(X) = 2$?*

In the classical setting $\kappa = \omega$, the Perfect Set Property makes questions about the order of the Borel hierarchy on Borel sets X moot: either X is countable, and hence $\text{ord}_{\omega_1}(X) \leq 2$, or it contains a copy of the whole Cantor space ${}^\omega 2$, and thus $\text{ord}_{\omega_1}(X) = \omega_1$. In the uncountable setting, however, the fact that the κ -Perfect Set Property may consistently fail for κ^+ -Borel (and even closed) sets makes the next problem worth investigating.

Problem 3. *Can there be a κ^+ -Borel (or even closed) set $X \subseteq {}^\kappa\kappa$ with $2 < \text{ord}_{\kappa^+}(X) < \kappa^+$?*

In Section 3.5, we investigate the interplay between the κ -Perfect Set Property and the order of the κ^+ -Borel hierarchy on definable subspaces of the generalized Baire space. We first deal with matters of definability, and lay the groundwork for connecting the κ -Perfect Set Property to the interpretation of a definable set in generic extensions. This is done via Theorem 3.5.8, due to Lücke [Lüc12]. We observe that we may pass to generic extensions by $< \kappa$ -closed forcings without adding new elements to κ -thin κ^+ -Borel sets; in other words, every code for a κ -thin κ^+ -Borel set yields the same subset of ${}^\kappa\kappa$ when interpreted in a $< \kappa$ -closed forcing extension. We also state Corollary 3.5.11, anticipating a positive answer to Problem 2. We then turn our attention to combining the work of Hamkins, Kunen and Miller to prove Lemma 3.5.14, in analogy with what has been shown about the Cohen real model in [Mil95, Theorem 14.3]. Together with Theorem 3.5.12, this shows (Corollary 3.5.15) that the ground model's generalized Baire space ${}^\kappa\kappa \cap V$ is a κ -thin closed set with order κ^+ in every forcing extension $V[G]$ by a small forcing, giving a positive answer to Problem 1 and showing that the sufficient conditions provided in Theorem 3.3.9 are not necessary in general, as already anticipated.

Constructing a space $X \subseteq {}^\kappa\kappa$ with $2 < \text{ord}_{\kappa^+}(X) < \kappa^+$ to solve Problem 3 is significantly more difficult. In Section 3.6, we partially generalize work of Miller [Mil79] surrounding the modification of the order of the Borel hierarchy on a given space X . The primary tool here is Miller's α -forcing, the use of which allows for fine control over specific levels of the hierarchy. Following the proof of [Mil79, Theorem 34], we use of an iteration of α -forcing to obtain:

Theorem (Theorem 3.6.19). *Let $X \subseteq {}^\kappa\kappa$ be such that $|X| > \kappa$, and let $1 < n < \omega$. Then, there exists a $< \kappa$ -closed, κ^+ -c.c. forcing extension $V[G]$ of V such that $V[G] \models \text{ord}_{\kappa^+}(X) = n$.*

Finally, combining this theorem with the results in Section 3.5, we derive the following:

Theorem (Corollary 3.6.20). *Let $X \subseteq {}^\kappa\kappa$ be a κ -thin κ^+ -Borel set with $|X| > \kappa$, and let $1 < n < \omega$. Then, there is a $<\kappa$ -closed, κ^+ -c.c. forcing extension $V[G]$ of V such that $V[G] \models X^{V[G]} = X \wedge \text{ord}_{\kappa^+}(X) = n$.*

In the above theorem, “ $X^V = X^{V[G]}$ ” means that given a κ^+ -Borel code for X (in V), its re-interpretation in $V[G]$ gives rise to exactly the same set X . In particular, this means that if X was a closed subset of ${}^\kappa\kappa$ in V , then it stays closed also when moving to the generic extension $V[G]$, and similarly for the other classes in the κ^+ -Borel hierarchy of ${}^\kappa\kappa$. Thus we obtain that, consistently, there are closed (hence nice Polish-like) subspaces of ${}^\kappa\kappa$ whose κ^+ -Borel hierarchy has length strictly between 2 and κ^+ , in stark contrast with the situation in classical descriptive set theory.

Finally, in Section 3.7, for a boldface pointclass $\mathbf{\Gamma}$, we give a comprehensive description of the difference hierarchy over $\mathbf{\Gamma}$ on an arbitrary regular Hausdorff space X of weight at most κ , and we derive its fundamental properties within the framework of generalized descriptive set theory. In particular, we analyze the existence of universal sets for the different levels of the difference hierarchy (Proposition 3.7.7), and we establish conditions ensuring that the hierarchy does not collapse, i.e., that it reaches the maximal length κ^+ (Corollary 3.7.9). Most notably, and in contrast with the classical setting, we show that Hausdorff–Kuratowski theorem (Theorem 1.0.1) in its generalized version fails (Theorem 3.7.14). More precisely, we prove that the difference hierarchy over closed sets does not cover the entire class $\kappa^+-\mathbf{\Delta}_2^0$, and in Examples 3.7.12 and 3.7.13 we exhibit sets that are $\kappa^+-\mathbf{\Delta}_2^0$ but cannot be written as differences of closed sets.

3.1 Setup and preliminaries

Throughout the chapter, we work in ZFC and we assume that

$$\kappa \text{ is an uncountable cardinal satisfying the condition } 2^{<\kappa} = \kappa.$$

As mentioned in the introduction, generalized descriptive set theory is primarily concerned with the study of the generalized Cantor and Baire spaces. However, recent advances in generalized descriptive set theory have pushed its scope beyond the two spaces mentioned above, and several classes of Polish-like spaces that can be considered as higher analogues of classical Polish spaces have been isolated — we refer the reader to [Ago22, AMRS23] for a thorough discussion on this. What matters for our purposes is that all such spaces are regular Hausdorff and have weight at most κ : this justifies our restriction to spaces with these features from Section 3.2 onward. When κ is singular, there is another condition that is shared by most Polish-like spaces considered so far and that is relevant to us, namely, the property of being a $(\omega, \text{cof}(\kappa))$ -Nagata–Smirnov space.² The class of $(\omega, \text{cof}(\kappa))$ -Nagata–Smirnov spaces is very wide and includes most spaces considered in previous literature in generalized descriptive set theory, like all subspaces of ${}^\kappa 2$ and ${}^{\text{cof}(\kappa)}\kappa$, all \mathbb{G} -metrizable spaces for a totally ordered group \mathbb{G} of degree $\text{cof}(\kappa)$, and all spaces of weight at most

²Let X be a topological space. A (regular Hausdorff) topological space is called a $(\omega, \text{cof}(\kappa))$ -Nagata–Smirnov if it has a basis for the topology which is the union of $\text{cof}(\kappa)$ -many locally finite families, where a family \mathcal{F} of subsets of X is said to be locally finite if every point $x \in X$ has a neighborhood U that intersects finitely many elements of \mathcal{F} .

$\text{cof}(\kappa)$. Thus, assuming that the space is $(\omega, \text{cof}(\kappa))$ -Nagata–Smirnov is not particularly restrictive either. In any case, the only thing that we will use about such spaces is that, as shown in [Ago22, Lemma 2.2.48]:

Fact 3.1.1. *Let X be a regular Hausdorff $(\omega, \text{cof}(\kappa))$ -Nagata–Smirnov space. Then, every open subset of X can be written as a union of $\text{cof}(\kappa)$ -many closed sets.*

The latter property is the one that will be explicitly stated in our results, when needed. In contrast, it is interesting to notice that any further “Polish-like requirement” that one can add to the spaces under considerations, like being (completely) $\text{cof}(\kappa)$ -metrizable or being spherically complete, does not have a direct impact on the theory of generalized Borel sets that we are going to develop.

For the results of Sections 3.5 and 3.6, we assume the reader to be familiar and comfortable with the theory of iterated forcing. Our notation for forcing is standard. The statement $q \leq p$ means “ q is stronger than p ”; in addition, we strive to follow Goldstern’s alphabet convention, i.e., stronger conditions should come later in the alphabet. If φ is a statement in the forcing language, we say that a condition p decides φ if either $p \Vdash \varphi$ or $p \Vdash \neg\varphi$ holds. Likewise, if $\dot{\tau}$ is name for a ground model object, we say that p decides $\dot{\tau}$ if there exists an $x \in V$ with $p \Vdash \dot{\tau} = \check{x}$, where \check{x} is the standard (check) name for a ground model object. We write $\mathbb{P} \Vdash \varphi$ if the statement φ is forced by every condition in \mathbb{P} . As usual, \parallel and \perp denote the compatibility and incompatibility relation, respectively.

Definition 3.1.2. We say a forcing notion \mathbb{P} is

- **$<\kappa$ -closed** if for every decreasing sequence $(p_i)_{i<\delta}$ with $\delta < \kappa$ we can find $q \in \mathbb{P}$ such that $q \leq p_i$ for all $i < \delta$;
- **κ^+ -c.c.** if every antichain in \mathbb{P} has size at most κ ;
- **κ -linked** if there are $(\mathbb{P}_i)_{i<\kappa}$ such that $\mathbb{P} = \bigcup_{i<\kappa} \mathbb{P}_i$ and for each $i < \kappa$, any $p, q \in \mathbb{P}_i$ are compatible;
- **well-met** if for any two compatible conditions $p, q \in \mathbb{P}$ there exists a greatest lower bound $r = p \wedge q$, that is, $r \leq p, q$ and for every r' with $r' \leq p, q$ we have $r' \leq r$.

Fact 3.1.3 (Folklore). *Let $(\mathbb{P}_\gamma, \dot{Q}_\gamma)_{\gamma < \gamma^*}$ be a $<\kappa$ -supported iteration such that for every $\gamma < \gamma^*$, \dot{Q}_γ is forced to be $<\kappa$ -closed, well-met, and κ -linked. Then, \mathbb{P}_{γ^*} satisfies the κ^+ -c.c. and is $<\kappa$ -closed.*

Proof. It is well-known that such iterations are $<\kappa$ -closed. For the κ^+ -c.c., see for example [Kun11, Lemma V.5.14]; the proof is formulated for the case $\kappa = \omega_1$, but transfers seamlessly to all uncountable $\kappa = \kappa^{<\kappa}$. \square

3.2 γ -Borel sets, and their hierarchy

Although we will mostly be concerned with γ -Borel sets for $\gamma = \kappa^+$ (Sections 3.3, 3.5, and 3.6) or, when κ is singular, for $\gamma = \kappa$ (Section 3.4), in this section we develop the theory in full generality for an arbitrary ordinal $\gamma > \omega$.

3.2.1 The γ -Borel hierarchy

Let $\gamma \in \text{Ord}$. A γ -**algebra** on a set X is a family of subsets of X which is closed under the operations of complementation and well-ordered unions of length less than γ . When X is a topological space, the γ -algebra generated by the topology of X , denoted by $\gamma\text{-Bor}(X)$, is the smallest γ -algebra on X containing all its open sets; its elements are called γ -**Borel** sets. Equivalently, $\gamma\text{-Bor}(X)$ is the smallest collection of subsets of X containing all open and closed sets and closed under intersections and unions of length less than γ . When needed, we might add a reference to the topology τ of X in the notation, and write e.g. $\gamma\text{-Bor}(X, \tau)$. Letting vary X over all non-empty topological spaces we get the pointclass $\gamma\text{-Bor}$, which is easily seen to be boldface and hereditary.

A function $f: X \rightarrow Y$ between two topological spaces X and Y is γ -**Borel** if it is γ -**Bor**-measurable; this is the same as requiring that $f^{-1}(A) \in \gamma\text{-Bor}(X)$ for every $A \in \gamma\text{-Bor}(Y)$. A γ -**Borel isomorphism** between X and Y is a bijection $f: X \rightarrow Y$ such that both f and f^{-1} are γ -Borel; X and Y are then γ -**Borel isomorphic** if there is a γ -Borel isomorphism between them. A γ -**Borel embedding** $f: X \rightarrow Y$ is an injective function which is a γ -Borel isomorphism as a function from X to $f[X]$.

Lemma 3.2.1. *Let $f: X \rightarrow Y$ be a function between two topological spaces X and Y , with Y Hausdorff. If f is γ -Borel, then its graph $\text{Gr}(f) = \{(x, y) \in X \times Y \mid f(x) = y\}$ is a γ -Borel subset of $X \times Y$. If f is continuous, then $\text{Gr}(f)$ is closed.*

Proof. The diagonal $\Delta = \{(y, y) \in Y \times Y\}$ of Y is closed in $Y \times Y$ because Y is Hausdorff. The function $f \times \text{id}_Y: X \times Y \rightarrow Y \times Y$ is γ -Borel (respectively, continuous) if and only if f is γ -Borel (respectively, continuous). Since $\text{Gr}(f) = (f \times \text{id}_Y)^{-1}(\Delta)$, the result follows. \square

Given a set X , a family $\mathcal{A} \subseteq \mathcal{P}(X)$, and $\mu \in \text{Card}$, we let

$$\begin{aligned} (\mathcal{A})_{\sigma_{<\mu}} &= \left\{ \bigcup_{\alpha < \beta} A_\alpha \mid \beta < \mu, A_\alpha \in \mathcal{A} \right\}, \quad \text{and} \\ (\mathcal{A})_{\delta_{<\mu}} &= \left\{ \bigcap_{\alpha < \beta} A_\alpha \mid \beta < \mu, A_\alpha \in \mathcal{A} \right\}. \end{aligned}$$

To simplify the notation, we also set $(\mathcal{A})_{\sigma_\mu} = (\mathcal{A})_{\sigma_{<\mu^+}}$ and $(\mathcal{A})_{\delta_\mu} = (\mathcal{A})_{\delta_{<\mu^+}}$.

In the classical case $\kappa = \omega$, the collection of Borel subsets of a topological space X is stratified in a hierarchy formed by the classes $\Sigma_\alpha^0(X)$, $\Pi_\alpha^0(X)$, and $\Delta_\alpha^0(X)$, where α ranges over non-zero ordinals. Following [AMR22, Section 2.4], a similar construction can be carried out for the collection of γ -Borel sets, for any $\gamma > \omega$: we call it the γ -**Borel hierarchy**.

Definition 3.2.2. For every topological space X , the following classes are defined by recursion on the ordinal $\alpha \geq 1$:

$$\begin{aligned} \gamma\text{-}\Sigma_1^0(X) &= \{U \subseteq X \mid U \text{ is open}\} & \gamma\text{-}\Pi_1^0(X) &= \{C \subseteq X \mid C \text{ is closed}\} \\ \gamma\text{-}\Sigma_\alpha^0(X) &= \left(\bigcup_{1 \leq \beta < \alpha} \gamma\text{-}\Pi_\beta^0(X) \right)_{\sigma_{<\gamma}} & \gamma\text{-}\Pi_\alpha^0(X) &= \{X \setminus A \mid A \in \gamma\text{-}\Sigma_\alpha^0(X)\}. \end{aligned}$$

We also set $\gamma\text{-}\Delta_\alpha^0(X) = \gamma\text{-}\Sigma_\alpha^0(X) \cap \gamma\text{-}\Pi_\alpha^0(X)$.

Notice that $\gamma\text{-}\mathbf{\Pi}_\alpha^0(X) = \left(\bigcup_{1 \leq \beta < \alpha} \gamma\text{-}\mathbf{\Sigma}_\beta^0(X) \right)_{\delta < \gamma}$. When it is important to specify the topology τ of X , we might add a reference to it in the notation and write e.g. $\gamma\text{-}\mathbf{\Sigma}_\alpha^0(X, \tau)$.

Letting X vary over all non-empty topological spaces, we get the boldface pointclass $\gamma\text{-}\mathbf{\Sigma}_\alpha^0$, its dual $\gamma\text{-}\mathbf{\Pi}_\alpha^0$, and the associated ambiguous pointclass $\gamma\text{-}\mathbf{\Delta}_\alpha^0$. The pointclasses $\gamma\text{-}\mathbf{\Sigma}_\alpha^0$ and $\gamma\text{-}\mathbf{\Pi}_\alpha^0$ are also hereditary, while in general $\gamma\text{-}\mathbf{\Delta}_\alpha^0$ fails to have such property.

Let γ^* be the smallest regular cardinal such that $\gamma^* \geq \gamma$, that is, $\gamma^* = \gamma$ if γ is already a regular cardinal, and $\gamma^* = |\gamma|^+$ otherwise. Then, it is easy to check that

$$\gamma\text{-}\mathbf{Bor}(X) = \bigcup_{1 \leq \alpha < \gamma^*} \gamma\text{-}\mathbf{\Sigma}_\alpha^0(X) = \bigcup_{1 \leq \alpha < \gamma^*} \gamma\text{-}\mathbf{\Pi}_\alpha^0(X) = \bigcup_{1 \leq \alpha < \gamma^*} \gamma\text{-}\mathbf{\Delta}_\alpha^0(X). \quad (3.2.1)$$

Remark 3.2.3. Let $\tau \subseteq \tau'$ be two topologies on a set X , and let $\alpha, \gamma, \gamma' \in \text{Ord}$ be such that $\alpha \geq 1$ and $\gamma \leq \gamma'$. Then, $\gamma\text{-}\mathbf{\Sigma}_\alpha^0(X, \tau) \subseteq \gamma'\text{-}\mathbf{\Sigma}_\alpha^0(X, \tau')$, and hence also $\gamma\text{-}\mathbf{\Pi}_\alpha^0(X, \tau) \subseteq \gamma'\text{-}\mathbf{\Pi}_\alpha^0(X, \tau')$, $\gamma\text{-}\mathbf{\Delta}_\alpha^0(X, \tau) \subseteq \gamma'\text{-}\mathbf{\Delta}_\alpha^0(X, \tau')$, and $\gamma\text{-}\mathbf{Bor}(X, \tau) \subseteq \gamma'\text{-}\mathbf{Bor}(X, \tau')$.

Moreover, when γ is not a cardinal, then $\gamma\text{-}\mathbf{Bor}(X, \tau) = |\gamma|^+\text{-}\mathbf{Bor}(X, \tau)$ and the γ -Borel hierarchy coincides, level by level, with the $|\gamma|^+$ -Borel hierarchy.

Thus it would not be restrictive to assume that γ is always cardinal, and the only relevant distinction is whether such cardinal is regular or singular. Moreover, we get the following easy fact.

Remark 3.2.4. Let X be any topological space, and $\alpha \geq 1$.

- If γ is a regular cardinal, then $\gamma\text{-}\mathbf{\Sigma}_\alpha^0(X)$ is closed under unions of length less than γ .
- If γ is a singular cardinal, then $\gamma\text{-}\mathbf{\Sigma}_\alpha^0(X)$ is closed under unions of length less than $\text{cof}(\gamma)$.
- If γ is not a cardinal, then $\gamma\text{-}\mathbf{\Sigma}_\alpha^0(X)$ is closed under unions of length less than $|\gamma|^+$, and in particular under $|\gamma|$ -sized unions.

Dually, we obtain closure properties of $\gamma\text{-}\mathbf{\Pi}_\alpha^0(X)$ under intersections.

Lemma 3.2.5. *Suppose that $f: X \rightarrow Y$ is a γ -Borel isomorphism between the topological spaces X and Y , and let $\beta \geq 1$ be such that both f and f^{-1} are $\gamma\text{-}\mathbf{\Sigma}_\beta^0$ -measurable. Then, if³ $\alpha \geq \beta \cdot \omega$ we have*

$$A \in \gamma\text{-}\mathbf{\Sigma}_\alpha^0(X) \Leftrightarrow f[A] \in \gamma\text{-}\mathbf{\Sigma}_\alpha^0(Y),$$

and similarly for $\gamma\text{-}\mathbf{\Sigma}_\alpha^0$ replaced by $\gamma\text{-}\mathbf{\Pi}_\alpha^0$ and $\gamma\text{-}\mathbf{\Delta}_\alpha^0$.

Moreover,

$$A \in \bigcup_{1 \leq \alpha < \beta \cdot \omega} \gamma\text{-}\mathbf{\Sigma}_\alpha^0(X) \Leftrightarrow f[A] \in \bigcup_{1 \leq \alpha < \beta \cdot \omega} \gamma\text{-}\mathbf{\Sigma}_\alpha^0(Y).$$

³An ordinal δ is additively closed if $\delta_0 + \delta_1 < \delta$ for all $\delta_0, \delta_1 < \delta$. Then, $\beta \cdot \omega$ is the smallest additively closed ordinal greater than β .

Proof. Fix any $\alpha \geq 1$, and let $\alpha' \in \text{Ord}$ be such that $\alpha = 1 + \alpha'$. Arguing by induction on α' , one can easily prove that $f[A] \in \gamma\text{-}\Sigma_{\beta+\alpha'}^0(Y)$ for each $A \in \gamma\text{-}\Sigma_\alpha^0(X)$ because f^{-1} is $\gamma\text{-}\Sigma_\beta^0$ -measurable. If instead $\alpha \geq \beta \cdot \omega \geq \omega$, then $\beta + \alpha' = \beta + \alpha = \alpha$, and thus we get $A \in \gamma\text{-}\Sigma_\alpha^0(X) \Rightarrow f[A] \in \gamma\text{-}\Sigma_\alpha^0(Y)$. If $\alpha < \beta \cdot \omega$, then $\beta + \alpha' < \beta \cdot \omega$ and we get

$$A \in \bigcup_{1 \leq \alpha < \beta \cdot \omega} \gamma\text{-}\Sigma_\alpha^0(X) \Rightarrow f[A] \in \bigcup_{1 \leq \alpha < \beta \cdot \omega} \gamma\text{-}\Sigma_\alpha^0(Y).$$

The reverse implications can be proved by repeating the same argument for f^{-1} . \square

3.2.2 When the hierarchy is increasing

It is desirable that the γ -Borel hierarchy is increasing (as it happens, for example, in the countable case for all Polish spaces), in the following sense.

Definition 3.2.6. We say that the γ -Borel hierarchy on a topological space on X is **increasing** (respectively, **increasing above level** $\delta \in \text{Ord}$) if $\gamma\text{-}\Sigma_\alpha^0(X) \subseteq \gamma\text{-}\Sigma_\beta^0(X)$ holds for every $\beta > \alpha \geq 1$ (respectively, for every $\beta > \alpha \geq \delta$).

In this respect, the only problematic case, as already noticed in [AMR22, Lemma 2.2], is when $\alpha = 1$ and $\beta = 2$.

Lemma 3.2.7. *Let X be any topological space. Then, the γ -Borel hierarchy on X is increasing above level 2, and $\gamma\text{-}\Sigma_1^0(X) \subseteq \gamma\text{-}\Pi_2^0(X) \subseteq \gamma\text{-}\Sigma_3^0(X)$. In particular, the γ -Borel hierarchy on X is increasing if and only if $\gamma\text{-}\Sigma_1^0(X) \subseteq \gamma\text{-}\Sigma_2^0(X)$.*

Remark 3.2.8. Note however that certain inclusions hold regardless of whether the γ -Borel hierarchy is increasing or not. In particular, for all ordinals $\alpha \leq \beta$ we have $\gamma\text{-}\Sigma_\alpha^0(X) \cup \gamma\text{-}\Pi_\alpha^0(X) \subseteq \gamma\text{-}\Sigma_\beta^0(X) \cup \gamma\text{-}\Pi_\beta^0(X)$ and $\gamma\text{-}\Delta_\alpha^0(X) \subseteq \gamma\text{-}\Delta_\beta^0(X)$.

Since the boldface pointclasses $\gamma\text{-}\Sigma_\alpha^0$ are hereditary, we get that if the γ -Borel hierarchy on X is increasing and $Y \subseteq X$, then the γ -Borel hierarchy on Y is increasing, too. The property of being increasing is responsible for the γ -Borel hierarchy behaving as expected with respect to inclusion. Indeed, by Lemma 3.2.7 we have that for every $\beta > \alpha \geq 2$

$$\gamma\text{-}\Sigma_\alpha^0(X), \gamma\text{-}\Pi_\alpha^0(X) \subseteq \gamma\text{-}\Delta_\beta^0(X) \subseteq \gamma\text{-}\Sigma_\beta^0(X), \gamma\text{-}\Pi_\beta^0(X) \quad (3.2.2)$$

and

$$\gamma\text{-}\Sigma_{\beta+1}^0(X) = (\gamma\text{-}\Pi_\beta^0(X))_{\sigma < \gamma}, \quad (3.2.3)$$

but when the γ -Borel hierarchy on X is increasing, then both (3.2.2) and (3.2.3) hold unconditionally at all levels of the γ -Borel hierarchy. All these observations will often be used implicitly in proofs.

Remark 3.2.9. It is easy to check that if the γ -Borel hierarchy on X is increasing, then for every $\alpha \geq 1$ we have that $\gamma\text{-}\Sigma_\alpha^0(X)$ is closed under finite intersections and $\gamma\text{-}\Pi_\alpha^0(X)$ is closed under finite unions. Together with Remark 3.2.4, this implies that $\gamma\text{-}\Delta_\alpha^0(X)$ is an ω -algebra.

In light of the discussion above, it is natural to ask for which topological spaces X we have $\gamma\text{-}\Sigma_1^0(X) \subseteq \gamma\text{-}\Sigma_2^0(X)$, so that the corresponding γ -Borel hierarchy is increasing. Notably, it turns out that when $\gamma = \kappa^+$ such condition is satisfied in virtually all topological spaces that are relevant to generalized descriptive set theory, since they all have weight at most κ .

Lemma 3.2.10. *Let X be a regular space with $w(X) < \gamma$. Then, $\gamma\text{-}\Sigma_1^0(X) \subseteq \gamma\text{-}\Sigma_2^0(X)$.*

Proof. Let \mathcal{B} be a basis for the topology of X such that $|\mathcal{B}| < \gamma$. For any open set $U \subseteq X$, consider the family of closed sets $\mathcal{C}_U = \{\text{cl}(B) \mid B \in \mathcal{B}, \text{cl}(B) \subseteq U\}$, which has size at most $|\mathcal{B}|$. Then, $\bigcup \mathcal{C}_U \subseteq U$ by definition, while $U \subseteq \bigcup \mathcal{C}_U$ by regularity of the topology of X . Thus $U \in \gamma\text{-}\Sigma_2^0(X)$, and we are done. \square

Since from Section 3.3 onward all spaces will be assumed to be regular Hausdorff and of weight at most κ , this means that the κ^+ -Borel hierarchy on each of them will always be increasing.⁴ As for the γ -Borel hierarchy when $\gamma = \kappa$ is a singular cardinal, the condition $\kappa\text{-}\Sigma_1^0(X) \subseteq \kappa\text{-}\Sigma_2^0(X)$ is no longer automatically true if X has weight κ , and that is one reason why (a slightly stronger form of) it will explicitly appear as an hypothesis in almost all statements of Section 3.4. However, notice that since we consider this hierarchy only when $\text{cof}(\kappa) < \kappa$, then by Fact 3.1.1 such conditions are verified in most spaces of interest, including all subspaces of ${}^\kappa 2$ or ${}^{\text{cof}(\kappa)} \kappa$. In the latter cases, one can additionally show that every open set can be written as a union of $\text{cof}(\kappa)$ -many clopen sets (see the proof of Lemma 3.3.16).

Fact 3.2.11. *If $X \subseteq {}^\kappa 2$, then any open subset of X can be written as a union of $\text{cof}(\kappa)$ -many clopen sets.*

3.2.3 When the hierarchy collapses

A basic parameter measuring the behavior of the γ -Borel hierarchy is its length. A standard result in classical descriptive set theory is that if X is an uncountable Polish space, then its (ω_1) -Borel hierarchy has maximal length, that is, $\Sigma_\alpha^0(X) \subsetneq \Sigma_\beta^0(X)$ holds for all $1 \leq \alpha < \beta < \omega_1$, and thus the length of the hierarchy is ω_1 . When moving to the general case of γ -Borel hierarchies, we know from (3.2.1) that an upper bound for its length is γ^* , the smallest regular cardinal above γ . However, for some specific topological space X it might happen that the γ -Borel hierarchy on X is shorter than that, in which case the following notion becomes relevant.

Definition 3.2.12. Let X be a topological space. The **order** of the γ -Borel hierarchy on X is

$$\text{ord}_\gamma(X) = \min \{ \alpha \in \text{Ord} \mid \gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\mathbf{Bor}(X) \}.$$

⁴For the readers interested in non-regular spaces or spaces of weight greater than κ , we mention that in any case most results of the chapter about the κ^+ -Borel hierarchy are still valid if we drop the topological requirements on the space and we explicitly assume $\kappa^+\text{-}\Sigma_1^0(X) \subseteq \kappa^+\text{-}\Sigma_2^0(X)$ in order to guarantee that such hierarchy is increasing.

As usual, a reference to the topology τ of X will be added to the notation when needed. Notice that in the definition of $\text{ord}_\gamma(X)$ we can equivalently replace $\gamma\text{-}\Sigma_\alpha^0(X)$ with $\gamma\text{-}\Pi_\alpha^0(X)$ or $\gamma\text{-}\Delta_\alpha^0(X)$, and that if $\text{ord}_\gamma(X) = \alpha$, then $\gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\Pi_\alpha^0(X)$. However, the latter equality is not sufficient to ensure $\text{ord}_\gamma(X) \leq \alpha$: if e.g. $\gamma = \kappa$ is a singular cardinal, it is possible that the κ -Borel hierarchy on a space X does not collapse, yet $\kappa\text{-}\Sigma_\alpha^0(X) = \kappa\text{-}\Pi_\alpha^0(X)$ for many $1 \leq \alpha < \kappa^+$ (see Theorem 3.4.1).

By the previous discussion, $\text{ord}_\gamma(X) \leq \gamma^*$ always holds. When $\text{ord}_\gamma(X) < \gamma^*$ we say that the γ -Borel hierarchy on X **collapses**. The fact that the γ -Borel hierarchy collapses automatically transfer to subspaces, and it is independent of the chosen topology if the space has small enough weight.

Proposition 3.2.13. *Let (X, τ) be a topological space with $\text{ord}_\gamma(X, \tau) < \gamma^*$.*

- (1) *If $Y \subseteq X$, then $\text{ord}_\gamma(Y, \tau \upharpoonright Y) \leq \text{ord}_\gamma(X, \tau) < \gamma^*$.*
- (2) *If $w(X, \tau) < \gamma^*$, then for every topology τ' on X such that $\gamma\text{-}\mathbf{Bor}(X, \tau') = \gamma\text{-}\mathbf{Bor}(X, \tau)$ we have $\text{ord}_\gamma(X, \tau') < \gamma^*$ too.*

Proof. Item (1) follows from the fact that the pointclasses $\gamma\text{-}\mathbf{Bor}$ and $\gamma\text{-}\Sigma_\alpha^0$ are hereditary. As for (2), let \mathcal{B} be a basis for τ with $|\mathcal{B}| < \gamma^*$, and let $\alpha < \gamma^*$ be such that $\text{ord}_\gamma(X, \tau) = 1 + \alpha$. Since γ^* is regular and $\tau \subseteq \gamma\text{-}\mathbf{Bor}(X, \tau) = \gamma\text{-}\mathbf{Bor}(X, \tau')$, there is $1 \leq \beta < \gamma^*$ such that $\mathcal{B} \subseteq \gamma\text{-}\Sigma_\beta^0(X, \tau')$. By recursion on α , one thus obtain $\gamma\text{-}\Sigma_{1+\alpha}^0(X, \tau) \subseteq \gamma\text{-}\Sigma_{\beta+\alpha}^0(X, \tau')$. Since $\gamma\text{-}\mathbf{Bor}(X, \tau') = \gamma\text{-}\mathbf{Bor}(X, \tau) = \gamma\text{-}\Sigma_{1+\alpha}^0(X, \tau)$, this proves that $\text{ord}_\gamma(X, \tau') \leq \beta + \alpha < \gamma^*$. \square

Corollary 3.2.14. *If X and Y are two γ -Borel isomorphic topological spaces of weight smaller than γ^* , then $\text{ord}_\gamma(X) < \gamma^* \Leftrightarrow \text{ord}_\gamma(Y) < \gamma^*$.*

Proof. Let $f: X \rightarrow Y$ be a γ -Borel isomorphism. Suppose that $\text{ord}_\gamma(X) < \gamma^*$. Let τ be the topology of X , and let τ' be the pull-back of the topology of Y along f . Then, $\gamma\text{-}\mathbf{Bor}(X, \tau) = \gamma\text{-}\mathbf{Bor}(X, \tau')$ because f is a γ -Borel isomorphism, hence $\text{ord}_\gamma(X, \tau') < \gamma^*$ by Proposition 3.2.13(2), and thus also $\text{ord}_\gamma(Y) < \gamma^*$ because $f: (X, \tau') \rightarrow Y$ is a homeomorphism by definition of τ' . The other implication can be proved in a similar way, using f^{-1} instead of f . \square

The following result provides several criteria for the collapse of the γ -Borel hierarchy on a space X .

Proposition 3.2.15. *Let X be a topological space. Then, for every $\alpha \geq 1$, the following are equivalent:*

- (1) $\text{ord}_\gamma(X) \leq \alpha$;
- (2) *for every $\beta \geq \alpha$, we have $\gamma\text{-}\Sigma_\beta^0(X) = \gamma\text{-}\Pi_\beta^0(X) = \gamma\text{-}\Delta_\beta^0(X) = \gamma\text{-}\mathbf{Bor}(X)$;*
- (3) *for some $\beta > \alpha$, we have $\gamma\text{-}\Delta_\alpha^0(X) = \gamma\text{-}\Sigma_\beta^0(X)$ (equivalently, $\gamma\text{-}\Delta_\alpha^0(X) = \gamma\text{-}\Pi_\beta^0(X)$);*
- (4) *for some $\beta > \alpha$, we have $\gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\Sigma_\beta^0(X)$ (equivalently, $\gamma\text{-}\Pi_\alpha^0(X) = \gamma\text{-}\Pi_\beta^0(X)$).*

- (5) the γ -Borel hierarchy on X is increasing above α and for some $\beta > \alpha$, one of $\gamma\text{-}\Sigma_\alpha^0(X)$, $\gamma\text{-}\Pi_\alpha^0(X)$, or $\gamma\text{-}\Delta_\alpha^0(X)$ coincides with one of $\gamma\text{-}\Sigma_\beta^0(X)$ or $\gamma\text{-}\Pi_\beta^0(X)$;
- (6) $\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X)$ and the class $\gamma\text{-}\Sigma_\alpha^0(X)$ is closed under intersections shorter than γ (equivalently, $\gamma\text{-}\Pi_\alpha^0(X)$ is closed under unions shorter than γ);
- (7) $\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X)$ and the class $\gamma\text{-}\Delta_\alpha^0(X)$ is closed under intersections shorter than γ (i.e. $\gamma\text{-}\Delta_\alpha^0(X)$ is a γ -algebra).

If γ is not a singular cardinal, the above conditions are also equivalent to:

- (8) the γ -Borel hierarchy on X is increasing above α and for some $\beta > \alpha$, $\gamma\text{-}\Delta_\alpha^0(X)$ coincides with $\gamma\text{-}\Delta_\beta^0(X)$;
- (9) for some $\beta > \alpha$, we have $\gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\Delta_\beta^0(X)$ (equivalently, $\gamma\text{-}\Pi_\alpha^0(X) = \gamma\text{-}\Delta_\beta^0(X)$);
- (10) $\gamma\text{-}\Sigma_\alpha^0$ is selfdual on X , i.e. $\gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\Pi_\alpha^0(X)$;

Proof. (1) \Rightarrow (2). Without loss of generality, we may assume $\text{ord}_\gamma(X) = \alpha$, as if (2) holds for every $\beta \geq \text{ord}_\gamma(X)$, then it holds for every $\beta \geq \alpha \geq \text{ord}_\gamma(X)$. As already observed, $\gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\mathbf{Bor}(X)$ is equivalent to $\gamma\text{-}\Delta_\alpha^0(X) = \gamma\text{-}\mathbf{Bor}(X)$. By Remark 3.2.8, for every $\beta \geq \alpha$ we get $\gamma\text{-}\Delta_\alpha^0(X) \subseteq \gamma\text{-}\Delta_\beta^0(X) \subseteq \gamma\text{-}\mathbf{Bor}(X) = \gamma\text{-}\Delta_\alpha^0(X)$, and hence also $\gamma\text{-}\Sigma_\beta^0(X) = \gamma\text{-}\Pi_\beta^0(X) = \gamma\text{-}\Delta_\beta^0(X) = \gamma\text{-}\mathbf{Bor}(X)$.

(2) \Rightarrow (3). Obvious.

(3) \Rightarrow (4). Let $\beta > \alpha$ be such that $\gamma\text{-}\Delta_\alpha^0(X) = \gamma\text{-}\Sigma_\beta^0(X)$, so that also $\gamma\text{-}\Delta_\alpha^0(X) = \gamma\text{-}\Pi_\alpha^0(X)$. By Remark 3.2.8,

$$\gamma\text{-}\Delta_\alpha^0(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X) \subseteq \gamma\text{-}\Sigma_\beta^0(X) \cup \gamma\text{-}\Pi_\beta^0(X) = \gamma\text{-}\Delta_\alpha^0(X),$$

hence $\gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\Delta_\alpha^0(X) = \gamma\text{-}\Sigma_\beta^0(X)$.

(4) \Rightarrow (5). By Lemma 3.2.7, we only need to consider the case $\alpha = 1$ and show that $\gamma\text{-}\Sigma_1^0(X) \subseteq \gamma\text{-}\Sigma_2^0(X)$ (the other condition is trivially satisfied because it is a weakening of (4)). Let $\beta > \alpha = 1$ be a witness for (4). Then, $\gamma\text{-}\Pi_1^0(X) \subseteq \gamma\text{-}\Sigma_\beta^0(X) = \gamma\text{-}\Sigma_1^0(X)$, and therefore $\gamma\text{-}\Sigma_1^0(X) = \gamma\text{-}\Delta_1^0(X) \subseteq \gamma\text{-}\Delta_2^0(X) \subseteq \gamma\text{-}\Sigma_2^0(X)$ by Remark 3.2.8.

(5) \Rightarrow (6). Assume that (5) holds, as witnessed by $\beta > \alpha$. Let first $\Gamma(X) = \gamma\text{-}\Sigma_\beta^0(X)$, and assume that one of $\gamma\text{-}\Delta_\alpha^0(X)$, $\gamma\text{-}\Pi_\alpha^0(X)$, or $\gamma\text{-}\Sigma_\alpha^0(X)$ equals $\Gamma(X)$.

We first show that, necessarily, $\gamma\text{-}\Delta_\alpha^0(X) = \gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\Pi_\alpha^0(X)$. If either $\gamma\text{-}\Delta_\alpha^0(X) = \Gamma(X)$ or $\gamma\text{-}\Pi_\alpha^0(X) = \Gamma(X)$, then in both cases $\Gamma \subseteq \gamma\text{-}\Pi_\alpha^0(X)$. Therefore $\gamma\text{-}\Sigma_\alpha^0(X) \subseteq \Gamma(X) \subseteq \gamma\text{-}\Pi_\alpha^0(X)$ (since we assumed that the γ -Borel hierarchy is increasing above α), and thus $\gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\Pi_\alpha^0(X) = \gamma\text{-}\Delta_\alpha^0(X)$. If instead $\gamma\text{-}\Sigma_\alpha^0(X) = \Gamma(X)$, then by $\beta > \alpha$ we get $\gamma\text{-}\Pi_\alpha^0(X) \subseteq \Gamma(X) = \gamma\text{-}\Sigma_\alpha^0(X)$, and thus $\gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\Pi_\alpha^0(X) = \gamma\text{-}\Delta_\alpha^0(X)$ again.

Thus in all three cases we must have $\gamma\text{-}\Delta_\alpha^0(X) = \gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\Pi_\alpha^0(X) = \Gamma = \check{\Gamma}$. In particular, $\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X) \cup \gamma\text{-}\Pi_\alpha^0(X) = \gamma\text{-}\Sigma_\alpha^0(X)$ by Remark 3.2.8. Moreover, if $(A_i)_{i < \delta}$ is a sequence of sets in $\gamma\text{-}\Sigma_\alpha^0(X)$ (for

some $\delta < \gamma$), then $\bigcap_{i < \delta} A_i \in \gamma\text{-}\Sigma_\alpha^0(X)$ because $\gamma\text{-}\Pi_{\alpha+1}^0(X) \subseteq \check{\Gamma}$ by $2 \leq \alpha+1 \leq \beta$ and Lemma 3.2.7, and we already showed that $\check{\Gamma} = \gamma\text{-}\Sigma_\alpha^0(X)$.

This concludes the proof in the case $\Gamma = \gamma\text{-}\Sigma_\beta^0(X)$. The case $\Gamma(X) = \gamma\text{-}\Pi_\beta^0(X)$ can be treated in a similar way, hence we are done.

(6) \Rightarrow (7). It suffices to show that (6) implies $\gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\Delta_\alpha^0(X)$. If $\alpha = 1$, the inclusion $\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\Sigma_1^0(X)$ easily yields the desired equality. Suppose now that $\alpha \geq 2$. By Lemma 3.2.7 and the assumption $\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X)$, we have $\bigcup_{1 \leq \beta < \alpha} \gamma\text{-}\Pi_\beta^0(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X)$, so that $\gamma\text{-}\Pi_{\alpha+1}^0(X) = (\gamma\text{-}\Sigma_\alpha^0(X))_{\delta < \gamma}$. Since the latter equals $\gamma\text{-}\Sigma_\alpha^0(X)$ by (6), we get $\gamma\text{-}\Pi_\alpha^0(X) \subseteq \gamma\text{-}\Pi_{\alpha+1}^0(X) = \gamma\text{-}\Sigma_\alpha^0(X)$ (by $\alpha \geq 2$ and Lemma 3.2.7), and we are done.

(7) \Rightarrow (1). Notice that $\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X)$ implies $\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X) \subseteq \gamma\text{-}\Pi_\alpha^0(X)$ because $\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X)$ is closed under complements, and thus $\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X) \subseteq \gamma\text{-}\Delta_\alpha^0(X)$. Then, $\gamma\text{-}\Delta_\alpha^0(X)$ is a γ -algebra containing all open sets, hence $\gamma\text{-}\mathbf{Bor}(X) \subseteq \gamma\text{-}\Delta_\alpha^0(X)$ and so $\text{ord}_\gamma(X) \leq \alpha$.

Assume now that γ is not a singular cardinal.

(1) \Rightarrow (8). By Remark 3.2.8, $\text{ord}_{\kappa^+}(X) \leq \alpha$ implies $\gamma\text{-}\mathbf{Bor}(X) \subseteq \gamma\text{-}\Delta_\alpha^0(X)$, hence also $\gamma\text{-}\Delta_\alpha^0(X) = \gamma\text{-}\Delta_\beta^0(X)$ for any $\beta \geq \alpha$. As for increasingness, by Lemma 3.2.7 it is enough to consider that case $\alpha = 1$ and show that $\gamma\text{-}\Sigma_1^0(X) \subseteq \gamma\text{-}\Sigma_2^0(X)$. In this case, (1) reads as $\gamma\text{-}\mathbf{Bor}(X) = \gamma\text{-}\Delta_1^0(X)$, hence $\gamma\text{-}\Sigma_1^0(X) = \gamma\text{-}\Delta_1^0(X) = \gamma\text{-}\Delta_2^0(X) \subseteq \gamma\text{-}\Sigma_2^0(X)$ by Remark 3.2.8 again.

(8) \Rightarrow (9). Follows from the fact that when the γ -Borel hierarchy on X is increasing above α , $\gamma\text{-}\Delta_\alpha^0(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X) \subseteq \gamma\text{-}\Delta_\beta^0(X)$.

(9) \Rightarrow (10). Obvious.

(10) \Rightarrow (1). By Remark 3.2.4, the class $\gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\Pi_\alpha^0(X)$ is a γ -algebra because γ is not a singular cardinal. By Remark 3.2.8, we have $\gamma\text{-}\Sigma_1^0(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X) \cup \gamma\text{-}\Pi_\alpha^0(X) = \gamma\text{-}\Sigma_\alpha^0(X)$, therefore $\gamma\text{-}\mathbf{Bor}(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X)$ and thus $\text{ord}_\gamma(X) \leq \alpha$. \square

Notice that the assumption “the γ -Borel hierarchy on X is increasing above α ” is automatically satisfied if either $\alpha \geq 2$ or $\gamma\text{-}\Sigma_1^0(X) \subseteq \gamma\text{-}\Sigma_2^0(X)$, and the assumption “ $\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X)$ ” is automatically satisfied if either $\alpha \geq 3$, or $\alpha = 2$ and $\gamma\text{-}\Sigma_1^0(X) \subseteq \gamma\text{-}\Sigma_2^0(X)$. In particular, we get:

Corollary 3.2.16. *Let X be a topological space whose γ -Borel hierarchy is increasing, and suppose that γ is not a singular cardinal. Then, the γ -Borel hierarchy on X collapses if and only if two of the pointclasses appearing in it coincide.*

Nevertheless, in the general case that the assumptions “the γ -Borel hierarchy on X is increasing above α ” in items (5) and (8) and “ $\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X)$ ” in items (6) and (7) are necessary. For example, set $\gamma = \omega$, $X = \mathbb{R}$, and $\alpha = 2$. Then, $\omega\text{-}\Sigma_1^0(\mathbb{R}) = \omega\text{-}\Pi_2^0(\mathbb{R})$, $\omega\text{-}\Pi_1^0(\mathbb{R}) = \omega\text{-}\Sigma_2^0(\mathbb{R})$, and $\omega\text{-}\Delta_1^0(\mathbb{R}) = \omega\text{-}\Delta_2^0(\mathbb{R}) = \{\emptyset, \mathbb{R}\}$; moreover, all the mentioned classes are closed under finite (i.e. shorter than ω) unions and intersections, yet $\text{ord}_\omega(\mathbb{R}) \geq 3$.

Also, the assumption that γ is not singular is necessary for the equivalence between (10) and (1), as in this case for certain values of $\alpha \geq 1$ it can happen that $\gamma\text{-}\Sigma_\alpha^0(X) = \gamma\text{-}\Pi_\alpha^0(X)$, even though the γ -Borel hierarchy does not collapse (see Theorem 3.4.1). In contrast, we will show that under certain mild

assumptions on X and γ , items (8) and (9) are equivalent to (1) also in the singular case (see Proposition 3.4.7).

Finally, we notice the following specific collapsing phenomenon, that will be relevant for some of the results in Sections 3.3 and 3.4.

Lemma 3.2.17. *Let X be a Hausdorff topological space. If the γ -Borel hierarchy on X is increasing, then the following are equivalent:*

- (1) $\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X) = \gamma\text{-}\Delta_2^0(X)$,
- (2) $\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X) = \gamma\text{-}\mathbf{Bor}(X) = \mathcal{P}(X)$,
- (3) X has at most one non-isolated point.⁵

Proof. (1) \Rightarrow (3). We prove the contrapositive. Suppose that there are two distinct non-isolated points $x, y \in X$. Since X is Hausdorff, there exist disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$. Consider the set $A = \{p\} \cup V \setminus \{q\}$. Then, A is neither open nor closed by the choice of x and y . However, we have that $\{x\}$ and $\{y\}$ are closed sets (because X is Hausdorff), the γ -Borel hierarchy on X is increasing by hypothesis, and $\gamma\text{-}\Delta_2^0(X)$ is an ω -algebra by Remark 3.2.9. Therefore $A \in \gamma\text{-}\Delta_2^0(X) \setminus (\gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X))$, and so (1) fails.

(3) \Rightarrow (2). If X is discrete (2) is trivially satisfied, so assume that X has exactly one non-isolated point x and consider an arbitrary $A \subseteq X$. If $x \notin A$, then A is open because it consists of isolated points. For the same reason, if $x \in A$ then $X \setminus A$ is open, and so A is closed. Therefore $\mathcal{P}(X) \subseteq \gamma\text{-}\Sigma_1^0(X) \cup \gamma\text{-}\Pi_1^0(X)$ and we are done.

(2) \Rightarrow (1). Obvious. \square

3.2.4 Decomposition theorems

In analogy with the Cantor-Bendixson theorem, we show that there is a canonical decomposition of each topological space, separating a part on which the γ -Borel hierarchy collapses from a part where the γ -Borel hierarchy does not collapse even locally. Recall that γ^* denotes the smallest regular cardinal above γ .

Proposition 3.2.18. *Let X be a topological space with $w(X) < \gamma^*$. Then, X can be partitioned into an open subspace C and a closed subspace P such that the γ -Borel hierarchy collapses on C , but it does not collapse on every non-empty open subset of P . Moreover, such a decomposition of X is unique.*

Proof. Given a basis \mathcal{B} of X with $|\mathcal{B}| = w(X)$, let $\mathcal{B}' = \{B \in \mathcal{B} \mid \text{ord}_\gamma(B) < \gamma^*\}$. Let $C = \bigcup \mathcal{B}'$ and $P = X \setminus C$. We claim that C and P are as required.

For each $B \in \mathcal{B}'$, let $1 \leq \alpha_B < \gamma^*$ be such that $\gamma\text{-}\Sigma_{\alpha_B}^0(B) = \gamma\text{-}\mathbf{Bor}(B)$, and let $\alpha = \sup\{\alpha_B + 1 \mid B \in \mathcal{B}'\}$. In this way, $\gamma\text{-}\Delta_{\alpha_B}^0(B) = \gamma\text{-}\mathbf{Bor}(B)$ and $2 \leq \alpha < \gamma^*$. Take now any $A \in \gamma\text{-}\mathbf{Bor}(C)$. For every $B \in \mathcal{B}'$, we have $A \cap B \in \gamma\text{-}\mathbf{Bor}(B) = \gamma\text{-}\Delta_{\alpha_B}^0(B) \subseteq \gamma\text{-}\Delta_\alpha^0(B) \subseteq \gamma\text{-}\Pi_\alpha^0(B)$ by Lemma 3.2.7. Since $\gamma\text{-}\Pi_\alpha^0$ is hereditary, there is $A'_B \in \gamma\text{-}\Pi_\alpha^0(C)$ such that $A'_B \cap B = A \cap B$, and since $B \in \gamma\text{-}\Sigma_1^0(C) \subseteq \gamma\text{-}\Pi_2^0(C) \subseteq \gamma\text{-}\Pi_\alpha^0(C)$ by Lemma 3.2.7, we get that

⁵Spaces satisfying one of these equivalent conditions are usually called Door spaces. See [Kel75, Chapter 2, Exercise C, page 76], which already states the equivalence (2) \Leftrightarrow (3).

$A'_B \cap B \in \gamma\text{-}\Pi_\alpha^0(C)$ too. It follows that $A = \bigcup_{B \in \mathcal{B}'} (A'_B \cap B)$ is a union of at most $w(X)$ -many $\gamma\text{-}\Pi_\alpha^0(C)$ sets. By our hypotheses, $w(X) \leq \gamma$. We distinguish two cases. If $w(X) < \gamma$, then $A \in \gamma\text{-}\Sigma_{\alpha+1}^0(C) \subseteq \gamma\text{-}\Sigma_{\alpha+3}^0(C)$. If $w(X) = \gamma$, then γ needs to be a singular cardinal. Write $A = \bigcup_{i < \gamma} A_i$ with $A_i \in \gamma\text{-}\Pi_\alpha^0(C)$, and fix an increasing sequence of ordinals $(\gamma_j)_{j < \text{cof}(\gamma)}$ cofinal in γ . Then, $A'_j = \bigcup_{i < \gamma_j} A_i \in \gamma\text{-}\Sigma_{\alpha+1}^0(C) \subseteq \gamma\text{-}\Pi_{\alpha+2}^0(C)$ for every $j < \text{cof}(\gamma)$, hence $A = \bigcup_{j < \text{cof}(\gamma)} A'_j \in \gamma\text{-}\Sigma_{\alpha+3}^0(C)$ because $\text{cof}(\gamma) < \gamma$. Since we showed that in all cases $\gamma\text{-}\mathbf{Bor}(C) \subseteq \gamma\text{-}\Sigma_{\alpha+3}^0(C)$, this means that $\text{ord}_\gamma(C) \leq \alpha + 3 < \gamma^*$.

Next suppose towards a contradiction that there is an open set $U \subseteq X$ such that $U \cap P \neq \emptyset$ and $\text{ord}_\gamma(U \cap P) < \gamma^*$. Without loss of generality we may assume that $U \in \mathcal{B}$. Let $\beta = \max\{3, \text{ord}_\gamma(C), \text{ord}_\gamma(U \cap P)\}$, so that $3 \leq \beta < \gamma^*$ and $\text{ord}_\gamma(U \setminus P) \leq \beta$ too by Proposition 3.2.13(1). Every $A \subseteq U$ can be written as $A = (A \cap P) \cup (A \setminus P)$. Since $U \cap P, U \setminus P \in \gamma\text{-}\Delta_3^0(U) \subseteq \gamma\text{-}\Delta_\beta^0(U)$, using that both $\gamma\text{-}\mathbf{Bor}$ and $\gamma\text{-}\Pi_\beta^0$ are hereditary pointclasses we easily get that each $A \in \gamma\text{-}\mathbf{Bor}(U)$ is the union of two $\gamma\text{-}\Delta_\beta^0(U)$ sets. This shows that $\text{ord}_\gamma(U) \leq \beta + 1 < \gamma^*$, and hence $U \in \mathcal{B}'$, contradicting $U \cap P \neq \emptyset$.

For the uniqueness part, suppose that C' and P' are disjoint subset of X such that $X = C' \cup P'$, C' is open, $\text{ord}_\gamma(C') < \gamma^*$, and $\text{ord}_\gamma(U \cap P') = \gamma^*$ for every open $U \subseteq X$ such that $U \cap P' \neq \emptyset$. Then, we must have $C' \subseteq C$, as C can equivalently be described as the union of all open sets $V \subseteq X$ such that $\text{ord}_\gamma(V) < \gamma^*$. If the inclusion were proper, then $C \cap P' \neq \emptyset$ because $P' = X \setminus C'$. But then $\text{ord}_\gamma(C \cap P') \leq \text{ord}_\gamma(C) < \gamma^*$ by Proposition 3.2.13(1), contradicting the choice of P' . This shows that $C' = C$, and hence also $P' = P$. \square

Proposition 3.2.18 admits a refined level-by-level version that, although a bit more technical, will be useful later on.

Proposition 3.2.19. *Suppose that the γ -Borel hierarchy on X is increasing. Let $\lambda < \gamma^*$ and $1 \leq \beta < \alpha < \gamma^*$ be such that both $\gamma\text{-}\Sigma_\beta^0(X)$ and $\gamma\text{-}\Sigma_\alpha^0(X)$ are closed under unions of size λ , and let $\mathcal{B} \subset \gamma\text{-}\Sigma_\beta^0(X)$ be such that $|\mathcal{B}| \leq \lambda$. Then, X can be partitioned into a $\gamma\text{-}\Sigma_\beta^0(X)$ set C and a $\gamma\text{-}\Pi_\beta^0(X)$ set $P = X \setminus C$ such that $\text{ord}_\gamma(C) \leq \alpha$, while $\text{ord}_\gamma(B \cap P) = \text{ord}_\gamma(B) > \alpha$ for every $B \in \mathcal{B}$ such that $B \cap P \neq \emptyset$. Moreover, if $\text{ord}_\gamma(X) > \alpha$, then $P \neq \emptyset$ and $\text{ord}_\gamma(X) = \text{ord}_\gamma(P) > \alpha$.*

Proof. The proof follows closely that of Proposition 3.2.18. Let $\mathcal{B}' = \{B \in \mathcal{B} \mid \text{ord}_\gamma(B) \leq \alpha\}$, and let $C = \bigcup \mathcal{B}'$ and $P = X \setminus C$. Then, $C \in \gamma\text{-}\Sigma_\beta^0(X)$ because we assumed that $\gamma\text{-}\Sigma_\beta^0(X)$ is closed under unions of λ , and thus $P \in \gamma\text{-}\Pi_\beta^0(X)$.

We know that $\gamma\text{-}\Sigma_\alpha^0(X)$ is closed under finite intersections (Remark 3.2.9) and unions of size λ (by hypothesis), hence the same is true of $\gamma\text{-}\Sigma_\alpha^0(C)$ because $\gamma\text{-}\Sigma_\alpha^0$ is hereditary. This easily gives that $\text{ord}_\gamma(C) \leq \alpha$: if $A \in \gamma\text{-}\mathbf{Bor}(C)$ then for every $B \in \mathcal{B}'$ we have $A \cap B \in \gamma\text{-}\mathbf{Bor}(B) = \gamma\text{-}\Sigma_\alpha^0(B)$, hence there is $A_B \in \gamma\text{-}\Sigma_\alpha^0(C)$ such that $A \cap B = A_B \cap B$, and since $B \in \gamma\text{-}\Sigma_\alpha^0(C)$ too (because $B \in \gamma\text{-}\Sigma_\beta^0(X) \subseteq \gamma\text{-}\Sigma_\alpha^0(X)$ and $B \subseteq C$), we obtain $A = \bigcup_{B \in \mathcal{B}'} (A_B \cap B) \in \gamma\text{-}\Sigma_\alpha^0(C)$.

Now consider $D \in \gamma\text{-}\mathbf{Bor}(X)$, and let $\alpha' = \text{ord}_\gamma(D \cap P)$. For every $A \in \gamma\text{-}\mathbf{Bor}(D)$ we get $A \cap C \in \gamma\text{-}\Sigma_\alpha^0(D \cap C)$ (since $\text{ord}_\gamma(D \cap C) \leq \text{ord}_\gamma(C) \leq \alpha$)

and $A \cap P \in \gamma\text{-}\Sigma_{\alpha'}^0(D \cap P)$. Since $P \in \gamma\text{-}\Pi_{\beta}^0(X) \subseteq \gamma\text{-}\Delta_{\alpha}^0(X)$, the usual computation yields $A = (A \cap C) \cup (A \cap P) \in \gamma\text{-}\Sigma_{\max(\alpha, \alpha')}^0(D)$. Hence

$$\text{ord}_{\gamma}(D \cap P) \leq \text{ord}_{\gamma}(D) \leq \max\{\alpha, \text{ord}_{\gamma}(D \cap P)\}. \quad (3.2.4)$$

Applying (3.2.4) to $D = B \in \mathcal{B}$, we get that if $B \cap P \neq \emptyset$, then $\text{ord}_{\gamma}(B \cap P) > \alpha$ (as otherwise $\text{ord}_{\gamma}(B) \leq \alpha$, and so $B \subseteq C$ by construction), and therefore also $\text{ord}_{\gamma}(B \cap P) = \text{ord}_{\gamma}(B) > \alpha$.

If instead we apply (3.2.4) to $D = X$, we get that if $\text{ord}_{\gamma}(X) > \alpha$ then also $\text{ord}_{\gamma}(X) = \text{ord}_{\gamma}(P) > \alpha$, which in particular implies $P \neq \emptyset$. \square

3.2.5 Universal sets

We now turn the attention to the existence of universal sets for the classes appearing in the γ -Borel hierarchy. The key lemma, which is essentially folklore (see e.g. the proof of [Kec95, Theorem 22.3]), is the following.

Lemma 3.2.20. *Let $\mu \leq \lambda$ be infinite cardinals, and let X be a topological space.*

- (1) *If $w(X) \leq \lambda$, then there is a λ^2 -universal set \mathcal{U} for $\gamma\text{-}\Sigma_1^0(X)$.*
- (2) *Suppose that for each $\beta < \mu$ we are given a boldface pointclass $\mathbf{\Gamma}_{\beta}$ such that $\emptyset \in \mathbf{\Gamma}_{\beta}(X)$ and $\mathbf{\Gamma}_{\beta}(X)$ admits a λ^2 -universal set \mathcal{U}_{β} . Then, there is also a λ^2 -universal set \mathcal{U} for*

$$\left(\bigcup_{\beta < \mu} \mathbf{\Gamma}_{\beta}(X) \right)_{\sigma_{\mu}}.$$

Proof. (1) Fix a basis $\{U_{\alpha} \mid \alpha < \lambda\}$ for X , and let

$$\mathcal{U} = \bigcup_{s \in < \lambda^2} \left(\mathbf{N}_s \times \bigcup \{U_{\alpha} \mid \alpha < \text{lh}(s), s(\alpha) = 1\} \right).$$

Clearly, $\mathcal{U} \in \gamma\text{-}\Sigma_1^0(\lambda^2 \times X)$. Given $U \subseteq X$ open, let $y \in \lambda^2$ be such that $y(\alpha) = 1 \Leftrightarrow U_{\alpha} \subseteq U$: then $\mathcal{U}_y = U$ because $U = \bigcup \{U_{\alpha} \mid y(\alpha) = 1\}$ by choice of y . This shows that \mathcal{U} is λ^2 -universal for $\gamma\text{-}\Sigma_1^0(X)$.

(2) Let $\langle \cdot, \cdot \rangle: \lambda \times \lambda \rightarrow \lambda$ be the usual Gödel pairing function, and let $(\cdot)_0, (\cdot)_1: \lambda \rightarrow \lambda$ be its left and right inverses, so that $\langle (\delta)_0, (\delta)_1 \rangle = \delta$ for every $\delta < \lambda$. Recall also that since μ is a cardinal, then $\langle \cdot, \cdot \rangle$ maps $\mu \times \mu$ onto μ .

For any $\delta < \mu$, let $f_{\delta}: \lambda^2 \rightarrow \lambda^2$ be defined by $f_{\delta}(x)(i) = x(\langle \delta, i \rangle)$ for every $i < \lambda$. Notice that each f_{δ} is continuous. Define $\mathcal{U} \subseteq \lambda^2 \times X$ by

$$(y, x) \in \mathcal{U} \Leftrightarrow \exists \delta < \mu [(f_{\delta}(y), x) \in \mathcal{U}_{(\delta)_0}].$$

For each $\delta < \mu$, the set $\{(y, x) \in \lambda^2 \times X \mid (f_{\delta}(y), x) \in \mathcal{U}_{(\delta)_0}\}$ belongs to $\mathbf{\Gamma}_{(\delta)_0}(\lambda^2 \times X)$ because it is the preimage of $\mathcal{U}_{(\delta)_0}$ under the continuous function $f_{\delta} \times \text{id}_X$, thus $\mathcal{U} \in \left(\bigcup_{\beta < \mu} \mathbf{\Gamma}_{\beta}(\lambda^2 \times X) \right)_{\sigma_{\mu}}$. Let now $A = \bigcup_{i < \mu} A_i$ for $A_i \in \bigcup_{\beta < \mu} \mathbf{\Gamma}_{\beta}(X)$. Let $\iota: \mu \rightarrow \mu$ be an injective map such that $A_i \in \mathbf{\Gamma}_{(\iota(i))_0}(X)$.

For each $\delta < \mu$, let $y_\delta \in {}^\lambda 2$ be such that $(\mathcal{U}_{(\delta)_0})_{y_\delta} = A_i$ if $\delta = \iota(i)$ for some $i < \mu$, and $(\mathcal{U}_{(\delta)_0})_{y_\delta} = \emptyset$ otherwise. For $\mu \leq \delta < \lambda$, pick an arbitrary $y_\delta \in {}^{<\lambda} 2$. Finally, let y be the only element of ${}^\lambda 2$ such that $f_\delta(y) = y_\delta$ for every $\delta < \lambda$. Since $\mathcal{U}_y = \bigcup_{\delta < \mu} (\mathcal{U}_{(\delta)_0})_{y_\delta}$, we get $\mathcal{U}_y = A$ and we are done. \square

Recall once again that γ^* is the least regular cardinal above γ , and that it is an upper bound for $\text{ord}_\gamma(X)$.

Corollary 3.2.21. *Let λ be an infinite cardinal, and let X be a topological space such that $w(X) \leq \lambda$. Assume that γ is not a limit cardinal, and that $\gamma \leq \lambda^+$. Then, for every $1 \leq \alpha < \gamma^*$ there are ${}^\lambda 2$ -universal sets for both $\gamma\text{-}\Sigma_\alpha^0(X)$ and $\gamma\text{-}\Pi_\alpha^0(X)$.*

Proof. By the assumption on γ , without loss of generality we may assume that $\gamma = \mu^+$ is a successor cardinal (see Remark 3.2.3), so that $\mu \leq \lambda$. In this situation, $\gamma\text{-}\Sigma_\alpha^0(X) = \left(\bigcup_{1 \leq \beta < \alpha} \gamma\text{-}\Pi_\beta^0(X) \right)_{\sigma_\mu}$ for every $\alpha > 1$.

We proceed by induction on $1 \leq \alpha < \gamma^* = \mu^+$. If $\alpha = 1$, Lemma 3.2.20(1) provides us with a ${}^\lambda 2$ -universal set \mathcal{U} for $\gamma\text{-}\Sigma_1^0(X)$, and hence \mathcal{U}^c is ${}^\lambda 2$ -universal for $\gamma\text{-}\Pi_1^0(X)$. Suppose now that $\alpha > 1$. Let $\rho: \mu \rightarrow (\alpha \setminus \{0\})$ be a surjection, and for every $\beta < \mu$ let $\Gamma_\beta = \gamma\text{-}\Pi_{\rho(\beta)}^0(X)$. Then, $\gamma\text{-}\Sigma_\alpha^0(X) = \left(\bigcup_{\beta < \mu} \Gamma_\beta(X) \right)_{\sigma_\mu}$. The inductive hypothesis ensures that we can apply Lemma 3.2.20(2) and obtain a ${}^\lambda 2$ -universal set \mathcal{U} for $\gamma\text{-}\Sigma_\alpha^0(X)$, so that \mathcal{U}^c is universal for $\gamma\text{-}\Pi_\alpha^0(X)$ and we are done. \square

3.2.6 More spaces

Lemma 3.2.5 and Lemma 3.2.7 can be used to show that the infinite levels of the γ -Borel hierarchy on a space X of weight smaller than γ^* do not depend too much on the chosen topology, in the sense that the original topology of X can be enhanced without altering the classes $\gamma\text{-}\Sigma_\alpha^0(X)$ (and hence also $\gamma\text{-}\Pi_\alpha^0(X)$ and $\gamma\text{-}\Delta_\alpha^0(X)$) for $\alpha \geq \omega$. For example, we can increase the additivity of the space under certain hypotheses on its weight, which are satisfied if e.g. $w(X) = \kappa$ and $\nu = \text{cof}(\kappa)$.

Proposition 3.2.22. *Let (X, τ) be a topological space of weight λ , and let ν be a cardinal such that $\lambda^{<\nu} < \gamma$. Let τ' be the smallest ν -additive topology refining τ . Then, for every $\alpha \geq \omega$, we have*

$$\gamma\text{-}\Sigma_\alpha^0(X, \tau) = \gamma\text{-}\Sigma_\alpha^0(X, \tau'),$$

and analogously for $\gamma\text{-}\Pi_\alpha^0$ and $\gamma\text{-}\Delta_\alpha^0$. Moreover,

$$\bigcup_{1 \leq n < \omega} \gamma\text{-}\Sigma_n^0(X, \tau) = \bigcup_{1 \leq n < \omega} \gamma\text{-}\Sigma_n^0(X, \tau').$$

Proof. By Lemma 3.2.5, it is enough to check that the identity $\text{id}_X: (X, \tau) \rightarrow (X, \tau')$ is a γ -Borel isomorphism such that both it and its inverse are $\gamma\text{-}\Sigma_3^0$ -measurable. Fix a basis \mathcal{B} for τ of size λ . Then, a basis for τ' is $\mathcal{B}' = \{\bigcap \mathcal{A} \mid \mathcal{A} \subseteq \mathcal{B}, |\mathcal{A}| < \nu\}$; in particular, $w(X, \tau') \leq \lambda^{<\nu}$. On the one hand, $\text{id}_X^{-1}: (X, \tau') \rightarrow (X, \tau)$ is continuous because $\tau' \supseteq \tau$, and thus it is also $\gamma\text{-}\Sigma_3^0$ -measurable. On the other hand, since $\nu \leq \lambda^{<\nu} < \gamma$ we have that $\mathcal{B}' \subseteq \gamma\text{-}\Pi_2^0(X, \tau)$, thus $\tau' \subseteq \gamma\text{-}\Sigma_3^0(X, \tau)$ and id_X is $\gamma\text{-}\Sigma_3^0$ -measurable too. \square

Another evidence of the fact that, when restricting to the infinite levels of the γ -Borel hierarchy, the only relevant parameter is the weight of the space is given by the following result.

Proposition 3.2.23. *Let (X, τ) be a T_0 topological space of weight at most λ , and suppose that $\max\{\mu, \text{cof}(\mu)^+\} \leq \gamma$ for $\mu = 2^{<\lambda}$. Then, τ can be refined to a topology τ' such that:*

- (1) (X, τ') embeds into ${}^\lambda 2$, hence τ' is regular Hausdorff, $(\omega, \text{cof}(\lambda))$ -Nagata-Smirnov, $\text{cof}(\lambda)$ -additive, and $w(X, \tau') \leq \mu$;
- (2) τ' has a basis of size at most μ that consists of $\lambda\text{-}\Delta_3^0(X, \tau)$ sets;
- (3) $\gamma\text{-}\Sigma_\alpha^0(X, \tau) = \gamma\text{-}\Sigma_\alpha^0(X, \tau')$ for all $\alpha \geq \omega$, $\gamma\text{-}\mathbf{Bor}(X, \tau) = \gamma\text{-}\mathbf{Bor}(X, \tau')$, and also $\bigcup_{1 \leq n < \omega} \gamma\text{-}\Sigma_n^0(X, \tau) = \bigcup_{1 \leq n < \omega} \gamma\text{-}\Sigma_n^0(X, \tau')$.

Proof. Let $\mathcal{B} = \{B_\alpha \mid \alpha < \lambda\}$ be an enumeration (possibly with repetitions) of a basis for τ . Let $f: X \rightarrow {}^\lambda 2$ be defined by $f(x)(\alpha) = 1$ if and only if $x \in B_\alpha$. Since X is T_0 , the map f is injective, and it is open on its image because $f[B_\alpha] = \{y \in f[X] \mid y(\alpha) = 1\}$ is open in $f[X]$. For every basic clopen set N_s of ${}^\kappa 2$ we have that $f^{-1}(N_s) = A \cap C$, where

$$A = \bigcap \{B_\alpha \mid \alpha < \text{lh}(s) \wedge s(\alpha) = 1\}, \quad \text{and}$$

$$C = \bigcap \{X \setminus B_\alpha \mid \alpha < \text{lh}(s) \wedge s(\alpha) = 0\}.$$

Then, $A \in \lambda\text{-}\Pi_2^0(X, \tau)$ and $C \in \lambda\text{-}\Pi_1^0(X, \tau)$, hence $A \cap C \in \lambda\text{-}\Delta_3^0(X, \tau)$ by Lemma 3.2.7. Let τ' be the topology generated by $\mathcal{B}' = \mathcal{B} \cup \{f^{-1}([s]) \mid s \in {}^{<\lambda} 2\}$. Then, by the above computation and Lemma 3.2.7, the basis of τ' obtained by closing \mathcal{B}' under finite intersections has size at most $2^{<\lambda} = \mu$ and consists of $\lambda\text{-}\Delta_3^0(X, \tau)$ sets. Moreover, $f: (X, \tau') \rightarrow {}^\lambda 2$ is an embedding because $f^{-1}([s]) \in \mathcal{B}'$ for every $s \in {}^{<\lambda} 2$ and $f[B]$ is open in $f[X]$ for every $B \in \mathcal{B}'$.

Finally, consider the map $\text{id}_X: (X, \tau) \rightarrow (X, \tau')$. Its inverse $\text{id}_X^{-1}: (X, \tau') \rightarrow (X, \tau)$ is continuous because $\tau' \supseteq \tau$. Since $\gamma \geq \mu \geq \lambda$ and $|\mathcal{B}'| \leq \mu$, every τ' -open set U is a union of μ -many $\mu\text{-}\Delta_3^0(X, \tau)$ sets by Remark 3.2.3. We distinguish two cases. If μ is regular, then $\gamma \geq \text{cof}(\mu)^+ = \mu^+ > \mu$, and so $U \in \mu^+\text{-}\Sigma_3^0(X, \tau) \subseteq \gamma\text{-}\Sigma_3^0(X, \tau)$ by Remark 3.2.3 again. If instead μ is singular, then $U \in \gamma\text{-}\Sigma_5^0(X, \tau)$. Indeed, if $U = \bigcup_{i < \mu} U_i$ with $U_i \in \mu\text{-}\Delta_3^0(X, \tau) \subseteq \gamma\text{-}\Delta_3^0(X, \tau)$ and $(\mu_j)_{j < \text{cof}(\mu)}$ is an increasing sequence of ordinals cofinal in μ , then $V_j = \bigcup_{i < \mu_j} U_i \in \gamma\text{-}\Sigma_3^0(X, \tau) \subseteq \gamma\text{-}\Pi_4^0(X, \tau)$ for every $j < \text{cof}(\mu)$ (because $\gamma \geq \mu > \mu_j$), and hence $U = \bigcup_{j < \text{cof}(\mu)} V_j \in \gamma\text{-}\Sigma_5^0(X, \tau)$ because $\gamma > \text{cof}(\mu)$. This shows that id_X is either $\gamma\text{-}\Sigma_3^0$ -measurable or $\gamma\text{-}\Sigma_5^0$ -measurable. In all cases, by Lemma 3.2.5 we get that $\gamma\text{-}\Sigma_\alpha^0(X, \tau) = \gamma\text{-}\Sigma_\alpha^0(X, \tau')$ for all $\alpha \geq \omega$, $\bigcup_{1 \leq n < \omega} \gamma\text{-}\Sigma_n^0(X, \tau) = \bigcup_{1 \leq n < \omega} \gamma\text{-}\Sigma_n^0(X, \tau')$, and hence also $\gamma\text{-}\mathbf{Bor}(X, \tau) = \gamma\text{-}\mathbf{Bor}(X, \tau')$. \square

To see the relevance of Proposition 3.2.23 in the context of this chapter, notice that setting $\lambda = \kappa$ we obtain that $\mu = \kappa$ too (because we assumed $2^{<\kappa} = \kappa$), and thus the above result can be applied with $\gamma = \kappa^+$, and also with $\gamma = \kappa$ if κ is singular. On the one hand, this implies that for results that do not depend

on the initial levels of the κ^+ -Borel (or of the κ -Borel) hierarchy, without loss of generality we can work only with subspaces of ${}^\kappa 2$. This applies e.g. to the question of whether such hierarchies collapse (Proposition 3.2.13), and thus it justifies our apparent restriction in scope while analyzing the connection between the κ -Perfect Set Property and the collapse of the κ^+ -Borel hierarchy in Section 3.5. On the other hand, Proposition 3.2.23 guarantees that all results contained herein touching on the infinite levels of the κ^+ -Borel hierarchy hold in full generality for all T_0 topological spaces of weight at most κ , without necessarily assuming them to be Hausdorff, nor regular.

As a corollary of Proposition 3.2.23, we obtain an elegant proof of the following well-known fact, which says that κ^+ -Borel spaces can be identified with the subspaces of the generalized Cantor space ${}^\kappa 2$.

Proposition 3.2.24. *The following are equivalent, for any κ^+ -algebra \mathcal{B} on a set X :*

- (1) (X, \mathcal{B}) is a κ^+ -Borel space, i.e. \mathcal{B} separates points and is generated by a subfamily \mathcal{A} of size at most κ ;
- (2) \mathcal{B} is generated by a T_0 topology of weight at most κ ;
- (3) \mathcal{B} is generated by a regular Hausdorff topology of weight at most κ ;
- (4) there is an injection $f: X \rightarrow {}^\kappa 2$ such that $A \in \mathcal{B} \Leftrightarrow f[A] \in \kappa^+\text{-Bor}(f[X])$, for every $A \subseteq X$.

Proof. The implications (4) \Rightarrow (3), (3) \Rightarrow (2), and (2) \Rightarrow (1) are obvious. To see (1) \Rightarrow (4), notice that since \mathcal{B} separates points, so does \mathcal{A} . Then, the topology τ generated by \mathcal{A} is T_0 . Thus an application of Proposition 3.2.23 with $\lambda = \mu = \kappa$ and $\gamma = \kappa^+$ yields the desired result. \square

3.3 The κ^+ -Borel hierarchy

As customary in generalized descriptive set theory, from this point onward (and unless stated otherwise),

all topological spaces are assumed to be regular Hausdorff and of weight at most κ .

In this section, we fix once and for all such a space X , and we study its κ^+ -Borel hierarchy. Notice however that, in view of Proposition 3.2.23, all results would remain valid for arbitrary T_0 spaces of weight at most κ if we restrict the attention to the infinite levels $\alpha \geq \omega$. Also, recall that the κ^+ -Borel hierarchy on X is increasing by Lemma 3.2.10 (applied with $\gamma = \kappa^+$). This will be tacitly used throughout the section.

We begin with the closure properties of the pointclasses appearing in the κ^+ -Borel hierarchy on X . Notice that $\text{ord}_{\kappa^+}(X) \leq \kappa^+$ because κ^+ , being a successor cardinal, is regular.

Proposition 3.3.1. *Given any $1 < \alpha < \kappa^+$, let $\hat{\alpha} = \text{cof}(\kappa)$ if α is a successor ordinal, and $\hat{\alpha} = \text{cof}(\alpha)$ if α is limit. Then,*

- (1) $\kappa^+\text{-}\Sigma_\alpha^0(X)$ is closed under unions of length κ and intersections of size smaller than $\hat{\alpha}$;

- (2) $\kappa^+ \text{-}\mathbf{\Pi}_\alpha^0(X)$ is closed under intersections of length κ and unions of size smaller than $\hat{\alpha}$;
- (3) $\kappa^+ \text{-}\mathbf{\Delta}_\alpha^0(X)$ is closed under complements and both unions and intersections of size smaller than $\hat{\alpha}$, that is, $\kappa^+ \text{-}\mathbf{\Delta}_\alpha^0(X)$ is a $\hat{\alpha}$ -algebra.

Furthermore, the same is true for $\alpha = 1$ if X is $\text{cof}(\kappa)$ -additive.⁶

Proof. Part (2) follows from (1) by taking complements, and part (3) follows from (1) and (2). Thus it is enough to prove (1).

The fact that $\gamma\text{-}\mathbf{\Sigma}_\alpha^0(X)$ is closed under unions of size κ already follows from Remark 3.2.4 applied with $\gamma = \kappa^+$, so let us consider closure under intersections shorter than $\hat{\alpha}$. We argue by induction on $1 < \alpha < \kappa^+$.

Consider first the case where $\alpha = \beta + 1$ is a successor ordinal (which covers in particular the base case $\alpha = 2$). Let $\lambda < \text{cof}(\kappa)$ and $(A_i)_{i < \lambda}$ be a sequence of sets in $\kappa^+ \text{-}\mathbf{\Sigma}_\alpha^0(X)$. By definition, for each $i < \lambda$ we have that $A_i = \bigcup_{j < \kappa} A_{i,j}$ with $A_{i,j} \in \kappa^+ \text{-}\mathbf{\Pi}_\beta^0(X)$ (here we implicitly use that the κ^+ -Borel hierarchy on X is increasing). Then, we can write

$$\bigcap_{i < \lambda} A_i = \bigcap_{i < \lambda} \left(\bigcup_{j < \kappa} A_{i,j} \right) = \bigcup_{s \in {}^\lambda \kappa} \left(\bigcap_{i < \lambda} A_{i,s(i)} \right).$$

Since $\kappa^+ \text{-}\mathbf{\Pi}_\beta^0(X)$ is closed under intersections of size κ (Remark 3.2.4), we have $\bigcap_{i < \lambda} A_{i,s(i)} \in \kappa^+ \text{-}\mathbf{\Pi}_\beta^0(X)$ for all $s \in {}^\lambda \kappa$. By choice of λ , we get $|{}^\lambda \kappa| = \kappa^\lambda \leq \kappa^{< \text{cof}(\kappa)} = \kappa$, where the last equality follows from $2^{< \kappa} = \kappa$. Thus $\bigcap_{i < \lambda} A_i = \bigcup_{s \in {}^\lambda \kappa} \left(\bigcap_{i < \lambda} A_{i,s(i)} \right) \in \kappa^+ \text{-}\mathbf{\Sigma}_\alpha^0(X)$, as desired.

Now let α be a limit ordinal, and let us prove that $\kappa^+ \text{-}\mathbf{\Sigma}_\alpha^0(X)$ is closed under intersections shorter than $\text{cof}(\alpha)$. Fix $\lambda < \text{cof}(\alpha)$ and a sequence $(A_i)_{i < \lambda}$ of sets in $\kappa^+ \text{-}\mathbf{\Sigma}_\alpha^0(X)$. For every $i < \lambda$ there is a sequence $(A'_{j,\ell})_{\ell < \kappa}$ of sets in $\bigcup_{1 \leq \beta < \alpha} \gamma \text{-}\mathbf{\Pi}_\beta^0(X)$ such that $A_i = \bigcup_{\ell < \kappa} A'_{i,\ell}$. Fix an increasing sequence $(\beta_j)_{j < \text{cof}(\alpha)}$ of ordinals cofinal in α , and for each $j < \text{cof}(\alpha)$ let $A_{i,j} = \bigcup \{A'_{i,\ell} \mid \ell < \kappa, A'_{i,\ell} \in \gamma \text{-}\mathbf{\Pi}_{\beta_j}^0(X)\}$. Then, $A_{i,j} \in \gamma \text{-}\mathbf{\Sigma}_{\beta_j+1}^0(X) \subseteq \gamma \text{-}\mathbf{\Pi}_{\beta_j+2}^0(X)$, and $A_i = \bigcup_{j < \text{cof}(\alpha)} A_{i,j}$. By the same computation we used before, we can write

$$\bigcap_{i < \lambda} A_i = \bigcap_{i < \lambda} \left(\bigcup_{j < \text{cof}(\alpha)} A_{i,j} \right) = \bigcup_{s \in {}^\lambda \text{cof}(\alpha)} \left(\bigcap_{i < \lambda} A_{i,s(i)} \right).$$

Fix any $s \in {}^\lambda \text{cof}(\alpha)$, and let $\bar{\beta} = \sup\{\beta_{s(i)} + 2 \mid i < \lambda\}$. Since $\lambda < \text{cof}(\alpha)$, we have $\bar{\beta} < \alpha$, and $\bigcap_{i < \lambda} A_{i,s(i)} \in \gamma \text{-}\mathbf{\Pi}_{\bar{\beta}}^0(X) \subseteq \bigcup_{1 \leq \beta < \alpha} \gamma \text{-}\mathbf{\Pi}_\beta^0(X)$ because $\gamma \text{-}\mathbf{\Pi}_{\bar{\beta}}^0(X)$ is closed under intersections of length κ . It is thus enough to show that $\text{cof}(\alpha)^\lambda = |{}^\lambda \text{cof}(\alpha)| \leq \kappa$. If κ is regular, then $\lambda < \text{cof}(\alpha) \leq \kappa$, hence⁷ $\text{cof}(\alpha)^\lambda \leq \kappa^{< \kappa} = \kappa$ and we are done. If κ is singular, then $\lambda < \text{cof}(\alpha) < \kappa$, thus $\text{cof}(\alpha)^\lambda \leq 2^{\text{cof}(\alpha)} < \kappa$ because in the singular case $2^{< \kappa} = \kappa$ is equivalent to κ being strong limit, hence we are done again.

Finally, if $\alpha = 1$ then $\kappa^+ \text{-}\mathbf{\Sigma}_1^0(X)$ is closed under unions of any size, and it is closed under intersections shorter than $\text{cof}(\kappa)$ if and only if X is $\text{cof}(\kappa)$ -additive. This concludes our proof. \square

⁶More generally, the closure properties of $\kappa^+ \text{-}\mathbf{\Sigma}_1^0(X)$ under intersections (or, equivalently, of $\kappa^+ \text{-}\mathbf{\Pi}_1^0(X)$ under unions) precisely correspond, by definition, to the additivity of the space.

⁷Recall that $\kappa^{< \kappa} = \kappa$ is equivalent to κ being regular and such that $2^{< \kappa} = \kappa$.

When κ is regular, it is not hard to see that many of the closure properties stated in Proposition 3.3.1 are optimal if $\text{ord}_{\kappa^+}(X) > \alpha$.

Proposition 3.3.2. *Suppose that κ is regular. Given any $1 \leq \alpha < \text{ord}_{\kappa^+}(X)$, let $\hat{\alpha} = \text{cof}(\kappa) = \kappa$ if α is a successor ordinal, and $\hat{\alpha} = \text{cof}(\alpha)$ if α is limit. Then,*

- (1) $\kappa^+-\Sigma_\alpha^0(X)$ is neither closed under complements, nor under intersections of size $\hat{\alpha}$;
- (2) $\kappa^+-\Pi_\alpha^0(X)$ is neither closed under complements, nor under unions of size $\hat{\alpha}$;
- (3) if $\alpha > 1$, then $\kappa^+-\Delta_\alpha^0(X)$ is not closed under unions or intersections of size $\hat{\alpha}$, and the same is true for $\alpha = 1$ if $X \subseteq \kappa^2$.

Notice that part (3) may fail if $\alpha = 1$: if e.g. the space X is connected, then $\kappa^+-\Delta_1^0(X) = \{\emptyset, X\}$ is closed under arbitrary unions and intersections.

Proof. Suppose first that $1 \leq \alpha < \kappa^+$ is such that $\hat{\alpha} = \kappa$. If $\kappa^+-\Sigma_\alpha^0(X)$ (equivalently: $\kappa^+-\Pi_\alpha^0(X)$) were closed under complements, or if $\alpha > 1$ (so that $\kappa^+-\Sigma_1^0(X) \cup \kappa^+-\Pi_1^0(X) \subseteq \kappa^+-\Sigma_\alpha^0(X)$ because the κ^+ -Borel hierarchy on X is increasing) and either $\kappa^+-\Sigma_\alpha^0(X)$ were closed under intersections of size κ (equivalently: $\kappa^+-\Pi_\alpha^0(X)$ were closed under unions of size κ), or $\kappa^+-\Delta_\alpha^0(X)$ were closed under either unions or intersections of size κ , then $\text{ord}_{\kappa^+}(X) \leq \alpha$ by Proposition 3.2.15, against our assumptions. Next, consider the case $\alpha = 1$. If $\kappa^+-\Sigma_\alpha^0(X)$ were closed under intersections of size κ (equivalently: $\kappa^+-\Pi_\alpha^0(X)$ were closed under unions of size κ), then each singleton $\{x\} \subseteq X$ would be open because $w(X) \leq \kappa$; thus X would be discrete, against $\alpha = 1 < \text{ord}_{\kappa^+}(X)$. Now further assume that $X \subseteq \kappa^2$. If $\kappa^+-\Delta_1^0(X)$ were closed under intersections (equivalently: unions) of size κ , then each singleton $\{x\} \subseteq X$ would be clopen because X has a clopen basis of size at most κ ; this again implies that X is discrete, contradicting $\alpha = 1 < \text{ord}_{\kappa^+}(X)$. This concludes the proof when α is such that $\hat{\alpha} = \kappa$.

Assume now that $1 \leq \alpha < \text{ord}_{\kappa^+}(X)$ is limit and $\hat{\alpha} = \text{cof}(\alpha) < \kappa$. The fact that $\kappa^+-\Sigma_\alpha^0(X)$ and $\kappa^+-\Pi_\alpha^0(X)$ are not closed under complements follows again from Proposition 3.2.15. As for the other closure properties, notice that $\kappa^+-\Pi_\alpha^0(X)$ is closed under unions of size $\hat{\alpha}$ if and only if $\kappa^+-\Sigma_\alpha^0(X)$ is closed under intersections of size $\hat{\alpha}$. Also, either condition implies that $\kappa^+-\Delta_\alpha^0(X)$ is closed under unions of size $\hat{\alpha}$, since $\kappa^+-\Sigma_\alpha^0(X)$ is always closed under unions of size κ by Proposition 3.3.1(1). Thus, it is enough to show that $\kappa^+-\Delta_\alpha^0(X)$ is not closed under unions of size $\hat{\alpha}$. Arguing as in the limit case of the proof of Proposition 3.3.1, we get that every $A \in \kappa^+-\Sigma_\alpha^0(X)$ can be written as $A = \bigcup_{i < \hat{\alpha}} A_i$ with $A_i \in \bigcup_{1 \leq \beta < \alpha} \kappa^+-\Pi_\beta^0(X) \subseteq \kappa^+-\Delta_\alpha^0(X)$. Thus we would have that if $\kappa^+-\Delta_\alpha^0(X)$ were closed under unions of size $\hat{\alpha}$, then $\kappa^+-\Sigma_\alpha^0(X) = \kappa^+-\Delta_\alpha^0(X)$, which again contradicts $\alpha < \text{ord}_{\kappa^+}(X)$ by Proposition 3.2.15. \square

Remark 3.3.3. One may wonder if the closure under κ -sized unions of $\kappa^+-\Sigma_\alpha^0(X)$ (equivalently: the closure under κ -sized intersections of $\kappa^+-\Pi_\alpha^0(X)$) is optimal as well when κ is regular and $1 < \alpha < \text{ord}_{\kappa^+}(X)$. This is the case if e.g. $\alpha = \beta + 1$ is a successor ordinal and κ is a regular cardinal

such that $2^\kappa = \kappa^+$. Indeed, every $A \in \kappa^+ \text{-}\Pi_{\alpha+1}^0(X)$ can be written as $A = \bigcap_{i < \kappa} \bigcup_{j < \kappa} A_{i,j}$ with $A_{i,j} \in \kappa^+ \text{-}\Pi_\beta^0(X)$, and hence $A = \bigcup_{s \in {}^\kappa \kappa} \bigcap_{i < \kappa} A_{i,s(i)}$ by the usual computation. Since $\kappa^+ \text{-}\Pi_\beta^0(X)$ is closed under κ -sized intersections (Remark 3.2.4), this means that every $A \in \kappa^+ \text{-}\Pi_{\alpha+1}^0(X)$ is a union of 2^κ -many $\kappa^+ \text{-}\Pi_\beta^0(X)$ sets, hence if $\kappa^+ \text{-}\Sigma_\alpha^0(X)$ were closed under unions of size $\kappa^+ = 2^\kappa$ we would have $\kappa^+ \text{-}\Pi_{\alpha+1}^0(X) = \kappa^+ \text{-}\Sigma_\alpha^0(X)$, contradicting $\text{ord}_{\kappa^+}(X) > \alpha$.

Proposition 3.3.2 remains true also when κ is singular, but the argument is more delicate and is postponed to Corollary 3.4.9, where it will be derived from an analogous result concerning the κ -Borel hierarchy.

Proposition 3.3.4. *Suppose that $1 \leq \alpha \leq \text{ord}_{\kappa^+}(X)$, and that $\alpha \neq \text{ord}_{\kappa^+}(X)$ if either $\text{ord}_{\kappa^+}(X) = \kappa^+$ or $\text{ord}_{\kappa^+}(X) = 2$. Then,*

$$\bigcup_{1 \leq \beta < \alpha} (\kappa^+ \text{-}\Sigma_\beta^0(X) \cup \kappa^+ \text{-}\Pi_\beta^0(X)) \subsetneq \kappa^+ \text{-}\Delta_\alpha^0(X). \quad (3.3.1)$$

Moreover, (3.3.1) holds for $\alpha = \text{ord}_{\kappa^+}(X) = 2$ as well if X does not have exactly one non-isolated point.

Proof. If $\alpha = 1$, then (3.3.1) reduces to $\kappa^+ \text{-}\Delta_1^0(X) \neq \emptyset$, which is true because e.g. $X \in \kappa^+ \text{-}\Delta_1^0(X)$. If instead $\alpha = 2 = \text{ord}_{\kappa^+}(X)$, then X is not discrete by $\text{ord}_{\kappa^+}(X) > 1$, so under our additional assumption on X for this specific case we conclude that X must have at least two non-isolated point, and (3.3.1) is satisfied by Lemma 3.2.17. Therefore from this point onward we can assume that $\text{ord}_{\kappa^+}(X) \geq 3$ and $\alpha \geq 2$. Recall also that $\kappa^+ \text{-}\Sigma_\alpha^0(X)$ is always closed under unions of size κ (Remark 3.2.4), and that the κ^+ -Borel hierarchy is increasing on X (Lemma 3.2.10).

Assume first that $\alpha = \alpha' + 1$ is a successor ordinal: then it is enough to show that there is $A \in \kappa^+ \text{-}\Delta_\alpha^0(X) \setminus (\kappa^+ \text{-}\Sigma_{\alpha'}^0(X) \cup \kappa^+ \text{-}\Pi_{\alpha'}^0(X))$. Let $P \subseteq X$ be the closed set obtained by applying Proposition 3.2.19 with $\gamma = \kappa^+ = \gamma^*$, $\lambda = \kappa$, $\beta = 1 < \alpha'$, and \mathcal{B} a basis for X of size at most κ . The space P is non-empty because $\text{ord}_{\kappa^+}(X) > \alpha'$, and it is not discrete because $\text{ord}_{\kappa^+}(P) = \text{ord}_{\kappa^+}(X) > 1$. Pick two distinct points $x_0, x_1 \in P$, and let $B_0, B_1 \in \mathcal{B}$ be disjoint open neighborhoods of x_0 and x_1 , respectively, so that $B_0 \cap P$ and $B_1 \cap P$ are both non-empty. (Here we use that P is not a singleton because it is not discrete.) Since $\text{ord}_{\kappa^+}(B_i \cap P) > \alpha'$ for both $i = 0$ and $i = 1$, by Proposition 3.2.15 there are $A_0 \in \kappa^+ \text{-}\Sigma_{\alpha'}^0(B_0 \cap P) \setminus \kappa^+ \text{-}\Pi_{\alpha'}^0(B_0 \cap P)$ and $A_1 \in \kappa^+ \text{-}\Pi_{\alpha'}^0(B_1 \cap P) \setminus \kappa^+ \text{-}\Sigma_{\alpha'}^0(B_1 \cap P)$. Then, $A = A_0 \cup A_1$ is as desired. Indeed, since $\kappa^+ \text{-}\Sigma_{\alpha'}^0$ and $\kappa^+ \text{-}\Pi_{\alpha'}^0$ are hereditary, there are $A'_0 \in \kappa^+ \text{-}\Sigma_{\alpha'}^0(X)$ and $A'_1 \in \kappa^+ \text{-}\Pi_{\alpha'}^0(X)$ such that $A_0 = A'_0 \cap (B_0 \cap P)$ and $A_1 = A'_1 \cap (B_1 \cap P)$. Since $\kappa^+ \text{-}\Delta_\alpha^0(X)$ is at least an ω -algebra by Remark 3.2.9, we get $A \in \kappa^+ \text{-}\Delta_\alpha^0(X)$. On the other hand, if $A \in \kappa^+ \text{-}\Sigma_{\alpha'}^0(X)$, then $A_1 = A \cap (B_1 \cap P) \in \kappa^+ \text{-}\Sigma_{\alpha'}^0(B_1 \cap P)$, contradicting the choice of A_1 ; similarly, if $A \in \kappa^+ \text{-}\Pi_{\alpha'}^0(X)$, then $A_0 = A \cap (B_0 \cap P) \in \kappa^+ \text{-}\Pi_{\alpha'}^0(B_0 \cap P)$, contradicting the choice of A_0 .

Suppose now that α is limit. In this case, we can assume $X \subseteq {}^\kappa 2$, by Proposition 3.2.23 (applied with $\gamma = \kappa^+$ and $\lambda = \kappa$, so that also $\mu = \kappa$). We let $\mathcal{B} = \{N_s \cap X \mid s \in {}^{<\kappa} 2\}$ be the canonical basis for X , which consists of clopen sets and has size at most κ . Let also $(\alpha_j)_{j < \text{cof}(\alpha)}$ be an increasing sequence of ordinals cofinal in α such that $\alpha_0 > 2$.

Claim 3.3.4.1. There is a family $(C_j)_{j < \text{cof}(\alpha)}$ of non-empty pairwise disjoint closed subsets of X such that $\text{ord}_{\kappa^+}(C_j) > \alpha_j$ for each $j < \text{cof}(\alpha)$.

Proof of the Claim. We distinguish two cases. Suppose first there is an ordinal $1 \leq \alpha' < \alpha$ such that $\text{ord}_{\kappa^+}(B) < \alpha$ implies $\text{ord}_{\kappa^+}(B) \leq \alpha'$ for every $B \in \mathcal{B}$. Let $P \subseteq X$ be the closed set obtained by applying Proposition 3.2.19 with $\gamma = \kappa^+ = \gamma^*$, $\lambda = \kappa$, $\beta = 1 < \alpha'$, and \mathcal{B} the canonical basis for X . Since $\text{ord}_{\kappa^+}(X) > \alpha'$, we have $P \neq \emptyset$ and $\text{ord}_{\kappa^+}(P) = \text{ord}_{\kappa^+}(X) \geq \alpha$. In particular, $\text{ord}_{\kappa^+}(P) \geq 3$, hence $|P| > \kappa$ and P is not discrete. Notice also that for every $s \in {}^{<\kappa}2$, if $\mathbf{N}_s \cap P \neq \emptyset$ then $\text{ord}_{\kappa^+}(\mathbf{N}_s \cap P) = \text{ord}_{\kappa^+}(\mathbf{N}_s \cap X) > \alpha'$, and hence $\text{ord}_{\kappa^+}(\mathbf{N}_s \cap P) \geq \alpha > \alpha_j$ (for every $j < \text{cof}(\alpha)$) by case assumption and choice of α' . Thus it is enough to set $C_j = \mathbf{N}_{s_j} \cap P$, for a suitable choice of $s_j \in {}^{<\kappa}2$. For this we distinguish two subcases.

If κ is regular, then we can pick a limit point $x \in P$ and a sequence $(x_j)_{j < \kappa}$ of points from $P \setminus \{x\}$ converging to x because P is not discrete. For every $j < \kappa$, let $s_j = x_j \upharpoonright \beta_j$ for $\beta_j < \kappa$ smallest such that $x \upharpoonright \beta_j \neq x_j \upharpoonright \beta_j$, and notice that x_j witnesses $\mathbf{N}_{s_j} \cap P \neq \emptyset$. By regularity of κ , without loss of generality we may assume that $\beta_i \neq \beta_j$ for distinct $i, j < \kappa$, so that $\mathbf{N}_{s_i} \cap \mathbf{N}_{s_j} = \emptyset$. Then, the sequence $(C_j)_{j < \text{cof}(\alpha)}$ with $C_j = \mathbf{N}_{s_j} \cap P$ is as desired. If instead κ is singular, then it is strong limit by $2^{<\kappa} = \kappa$, and moreover $\text{cof}(\alpha) < \kappa$ because $\alpha < \kappa^+$. Let $(\kappa_i)_{i < \text{cof}(\kappa)}$ be an increasing sequence of ordinals cofinal in κ , and for each $i < \text{cof}(\kappa)$ let $\mathcal{B}_i = \{\mathbf{N}_s \cap P \mid s \in {}^{\kappa_i}2 \wedge \mathbf{N}_s \cap P \neq \emptyset\}$. The elements of a given \mathcal{B}_i are clearly pairwise disjoint. If $|\mathcal{B}_i| < \text{cof}(\alpha)$ for every $i < \text{cof}(\kappa)$, then $|P| \leq \text{cof}(\alpha)^{\text{cof}(\kappa)} < \kappa$, a contradiction. Therefore it is enough to pick $i < \text{cof}(\kappa)$ such that $|\mathcal{B}_i| \geq \text{cof}(\alpha)$, and let $(C_j)_{j < \text{cof}(\alpha)}$ be an enumeration without repetitions of a large enough subfamily of \mathcal{B}_i .

The remaining case is when for every ordinal $\alpha' < \alpha$ there is $B \in \mathcal{B}$ such that $\alpha' < \text{ord}_{\kappa^+}(B) < \alpha$. For $j < \text{cof}(\alpha)$, recursively pick $B_j \in \mathcal{B}$ such that $\alpha_j < \text{ord}_{\kappa^+}(B_j) < \alpha$ and $\text{ord}_{\kappa^+}(B_j) > \sup_{i < j} \text{ord}_{\kappa^+}(B_i)$. Since $\text{ord}_{\kappa^+}(\bigcup_{i < j} B_i) = \sup_{i < j} \text{ord}_{\kappa^+}(B_i) < \text{ord}_{\kappa^+}(B_j)$, we have that $C_j = B_j \setminus \bigcup_{i < j} B_i \neq \emptyset$. Moreover, arguing as in the proof of Proposition 3.2.19 one can check that since both $\bigcup_{i < j} B_i$ and C_j belong to $\kappa^+ \text{-}\mathbf{\Delta}_2^0(X)$, then $\text{ord}_{\kappa^+}(B_j) \leq \max\{\text{ord}_{\kappa^+}(\bigcup_{i < j} B_i), \text{ord}_{\kappa^+}(C_j), 2\}$. In conclusion, $\text{ord}_{\kappa^+}(C_j) = \text{ord}_{\kappa^+}(B_j) > \alpha_j$ by $\text{ord}_{\kappa^+}(B_j) > \text{ord}_{\kappa^+}(\bigcup_{i < j} B_i)$ and $\text{ord}_{\kappa^+}(B_j) > \alpha_j \geq \alpha_0 > 2$. \square

For each $j < \text{cof}(\alpha)$, pick a set $A_j \in \kappa^+ \text{-}\mathbf{\Sigma}_{\alpha_j}^0(C_j) \setminus \kappa^+ \text{-}\mathbf{\Pi}_{\alpha_j}^0(C_j)$: we claim that $A = \bigcup_{j < \text{cof}(\alpha)} A_j$ witnesses (3.3.1). To see this, for each $j < \text{cof}(\alpha)$ use the fact that $\kappa^+ \text{-}\mathbf{\Sigma}_{\alpha_j}^0$ is hereditary to find $A'_j \in \kappa^+ \text{-}\mathbf{\Sigma}_{\alpha_j}^0(X)$ such that $A_j = A'_j \cap (\mathbf{N}_{s_j} \cap P)$. Then,

$$A = \bigcup_{j < \text{cof}(\alpha)} (A'_j \cap C_j),$$

$$X \setminus A = \left(X \setminus \bigcup_{j < \text{cof}(\alpha)} C_j \right) \cup \bigcup_{j < \text{cof}(\alpha)} (C_j \setminus A'_j),$$

hence both A and $X \setminus A$ belong to $\kappa^+ \text{-}\mathbf{\Sigma}_{\alpha}^0(X)$ by Proposition 3.3.1 and $2 < \alpha < \kappa^+$ (which also entails $\text{cof}(\alpha) \leq \kappa$), so that $A \in \kappa^+ \text{-}\mathbf{\Delta}_{\alpha}^0(X)$. Suppose towards a contradiction that $A \in \kappa^+ \text{-}\mathbf{\Sigma}_{\beta}^0(X) \cup \kappa^+ \text{-}\mathbf{\Pi}_{\beta}^0(X)$ for some $1 \leq \beta < \alpha$.

Let $j < \text{cof}(\alpha)$ be such that $\alpha_j > \beta$, so that $A \in \kappa^+ \text{-}\mathbf{\Pi}_{\alpha_j}^0(X)$. Then, $A_j = A \cap C_j \in \kappa^+ \text{-}\mathbf{\Pi}_{\alpha_j}^0(C_j)$, contradicting the choice of A_j . \square

A fundamental question to be addressed is whether the κ^+ -Borel hierarchy always needs to be **proper**, that is, if the inclusions in (3.2.2), when setting $\gamma = \kappa^+$, are strict for all relevant $\alpha, \beta < \text{ord}_{\kappa^+}(X)$. We are now going to show that this is always the case.

Corollary 3.3.5. *For every $1 \leq \alpha < \text{ord}_{\kappa^+}(X)$, we have that*

- (1) $\kappa^+ \text{-}\mathbf{\Delta}_{\alpha}^0(X) \subsetneq \kappa^+ \text{-}\mathbf{\Sigma}_{\alpha}^0(X)$ (equivalently, $\kappa^+ \text{-}\mathbf{\Delta}_{\alpha}^0(X) \subsetneq \kappa^+ \text{-}\mathbf{\Pi}_{\alpha}^0(X)$), and
- (2) $\bigcup_{1 \leq \beta < \alpha} (\kappa^+ \text{-}\mathbf{\Sigma}_{\beta}^0(X) \cup \kappa^+ \text{-}\mathbf{\Pi}_{\beta}^0(X)) \subsetneq \kappa^+ \text{-}\mathbf{\Delta}_{\alpha}^0(X)$.

Proof. Part (1) follows directly from Proposition 3.2.15, so we only need to prove part (2).

If $\text{ord}_{\kappa^+}(X) = 1$ there is nothing to prove. If $\text{ord}_{\kappa^+}(X) > \alpha = 1$, then the result follows from the fact that, for example, $X \in \kappa^+ \text{-}\mathbf{\Delta}_1^0(X)$. Finally, suppose that $1 < \alpha < \text{ord}_{\kappa^+}(X)$. Then, $\text{ord}_{\kappa^+}(X) \geq 3$, and hence X has at least two non-isolated points by Lemma 3.2.17. The result then follows from Proposition 3.3.4. \square

Another natural question related to the properness of the κ^+ -Borel hierarchy is the following. Suppose that $1 \leq \text{ord}_{\kappa^+}(X) < \kappa^+$. (Examples of spaces with this property will be provided in Section 3.6 for finite values of $\text{ord}_{\kappa^+}(X)$, and in a follow-up to this work for infinite values.) Is there any set $A \in \kappa^+ \text{-}\mathbf{\Delta}_{\text{ord}_{\kappa^+}(X)}^0(X) = \kappa^+ \text{-}\mathbf{Bor}(X)$ that does not belong to lower levels of the κ^+ -Borel hierarchy on X ? Lemma 3.2.17 and Proposition 3.3.4 allow us to answer to this question as well: the answer is affirmative if and only if the space does not have exactly one non-isolated point.

Corollary 3.3.6. *Suppose that $\text{ord}_{\kappa^+}(X) < \kappa^+$. Then, the following are equivalent:*

- (1) there is $A \in \kappa^+ \text{-}\mathbf{\Delta}_{\text{ord}_{\kappa^+}(X)}^0(X) \setminus \bigcup_{1 \leq \beta < \text{ord}_{\kappa^+}(X)} (\kappa^+ \text{-}\mathbf{\Sigma}_{\beta}^0(X) \cup \kappa^+ \text{-}\mathbf{\Pi}_{\beta}^0(X))$;
- (2) X does not have exactly one non-isolated point.

In particular, (1) holds whenever $\text{ord}_{\kappa^+}(X) \neq 2$.

When $\text{ord}_{\kappa^+}(X) = 2$, condition (1) from Corollary 3.3.6 may fail even for closed subsets X of ${}^{\kappa}2$. For example, let $X = \{x\} \cup \{x_i \mid i < \kappa\}$, where for every $i < \kappa$ we let $x_i(j) = x(j)$ if $j \neq i$ and $x_i(i) = 1 - x(i)$. Since X consists of a κ -sequence of isolated points converging to x , it is easy to see that it is as required.

We next move to the question of when the κ^+ -Borel hierarchy on X does not collapse. In classical descriptive set theory, the non-collapse of the Borel hierarchy of an uncountable Polish space X follows from the Perfect Set Property and the fact that $(\omega_1 \text{-})\mathbf{\Sigma}_{\alpha}^0(X)$ has a ${}^{\omega}2$ -universal set for each $1 \leq \alpha < \omega_1$ (see [Kec95, Section 22A]). In the generalized context, the non-collapse of the κ^+ -Borel hierarchy on ${}^{\kappa}2$ has been first obtained in [AMR22, Proposition 4.19] for an arbitrary infinite cardinal κ , i.e. without assuming $2^{<\kappa} = \kappa$. This required different methods, because if the condition $2^{<\kappa} = \kappa$ fails, then it might

happen that neither $\kappa^+-\Sigma_\alpha^0({}^\kappa 2)$ nor $\kappa^+-\Pi_\alpha^0({}^\kappa 2)$ have a ${}^\kappa 2$ -universal set, for any $1 \leq \alpha < \kappa^+$ ([AMR22, Corollary 4.16]). However, restoring our assumption $2^{<\kappa} = \kappa$ allows us to follow more closely the classical argument.

Proposition 3.3.7. *For each $1 \leq \alpha < \kappa^+$ there are ${}^\kappa 2$ -universal sets for both $\kappa^+-\Sigma_\alpha^0(X)$ and $\kappa^+-\Pi_\alpha^0(X)$. Therefore there are subsets of ${}^\kappa 2$ that are κ -complete for $\kappa^+-\Sigma_\alpha^0$ and $\kappa^+-\Pi_\alpha^0$, respectively. In contrast, there is no ${}^\kappa 2$ -universal set for $\kappa^+-\Delta_\alpha^0({}^\kappa 2)$ or $\kappa^+-\mathbf{Bor}({}^\kappa 2)$.*

Moreover, ${}^\kappa 2$ can systematically be replaced by X in all the above statements if ${}^\kappa 2$ embeds into X .

Proof. For the first part, apply Corollary 3.2.21 with $\lambda = \kappa$ and $\gamma = \kappa^+$ to get the desired ${}^\kappa 2$ -universal sets, and when $X = {}^\kappa 2$ apply Lemma 2.6.2 to obtain the existence of the complete sets, or Lemma 2.6.3 to obtain the non-existence of ${}^\kappa 2$ -universal sets for the selfdual classes (relatively to the space ${}^\kappa 2$ itself).

Assume now that there is an embedding $f: {}^\kappa 2 \rightarrow X$. Then, Lemma 2.6.1 and the first part ensure that there are X -universal sets for both $\kappa^+-\Sigma_\alpha^0(X)$ and $\kappa^+-\Pi_\alpha^0(X)$. As for complete sets, let $A' \subseteq {}^\kappa 2$ be κ -complete for $\kappa^+-\Sigma_\alpha^0$, and let $A \in \kappa^+-\Sigma_\alpha^0(X)$ be such that $A \cap f[{}^\kappa 2] = f[A']$: then A is κ -complete for $\kappa^+-\Sigma_\alpha^0$ as well. The case of $\kappa^+-\Pi_\alpha^0(X)$ is similar. The non-existence of X -universal sets for $\kappa^+-\Delta_\alpha^0(X)$ and $\kappa^+-\mathbf{Bor}(X)$ follows from Lemma 2.6.3 again. \square

Corollary 3.3.8. *The κ^+ -Borel hierarchy on ${}^\kappa 2$ does not collapse.*

By Proposition 3.2.13(1), this implies that the κ^+ -Borel hierarchy does not collapse on any space that contains a homeomorphic copy of ${}^\kappa 2$. Actually, the same conclusion holds in greater generality.

Theorem 3.3.9. *If there is a κ^+ -Borel embedding $f: {}^\kappa 2 \rightarrow X$, then the κ^+ -Borel hierarchy on X does not collapse.*

Proof. Assume towards a contradiction that $\text{ord}_{\kappa^+}(X) < \kappa^+$, and let $Y = f[X]$. Then, $\text{ord}_{\kappa^+}(Y) < \kappa^+$ by Proposition 3.2.13(1), hence $\text{ord}_{\kappa^+}({}^\kappa 2) < \kappa^+$ by Corollary 3.2.14, against Corollary 3.3.8. \square

Theorem 3.3.9 provides an important sufficient condition for the non-collapse of the κ^+ -Borel hierarchy on a space X , but in Corollary 3.5.15 we will show that, consistently, there may be very simple spaces (in fact, even closed subspaces of ${}^\kappa \kappa$, for κ regular) such that ${}^\kappa 2$ does not κ^+ -Borel embed into them, yet they have a non-collapsing κ^+ -Borel hierarchy.

As a consequence of Theorem 3.3.9, we get that if e.g. all κ^+ -Borel subsets of $\text{cof}^{(\kappa)}\kappa$ (or equivalently: of ${}^\kappa 2$) satisfy a very weak form of the κ -Perfect Set Property, then the κ^+ -Borel hierarchy does not collapse on every topological space of weight at most κ that is κ^+ -Borel isomorphic to a κ^+ -Borel subset of $\text{cof}^{(\kappa)}\kappa$ of size greater than κ . (The weak κ -Perfect Set Property we are alluding to is the following: Either $|A| \leq \kappa$, or there is a κ^+ -Borel embedding of ${}^\kappa 2$ into A . See [AMRS23] and [DMR25] for other variants and their relationships when κ is regular or $\text{cof}(\kappa) = \omega$, respectively.) This scenario is consistent if κ is regular ([Sch17]), and is necessarily true if κ has countable cofinality ([DMR25]) — the case of a singular cardinal κ of uncountable cofinality, instead, has not yet been studied. However, it is also known that if κ is regular, then it is consistent

to have κ^+ -Borel (or even closed) sets X that do not satisfy any form of κ -Perfect Set Property (see Section 3.5): for such sets, the non-collapse of the κ^+ -Borel hierarchy cannot be derived merely from the existence of ${}^{\kappa}2$ -universal sets, and indeed we will show that it is consistent to have $2 < \text{ord}_{\kappa^+}(X) < \kappa^+$ (see Section 3.6).

We conclude this section with a brief digression on κ^+ -Borel measurable functions, which we study in detail in Chapter 5. Recall that a function $f: X \rightarrow Y$ is $\mathbf{\Gamma}$ -measurable if $f^{-1}(U) \in \mathbf{\Gamma}(X)$ for every open $U \subseteq Y$. In particular, f is called κ^+ -Borel measurable if it is κ^+ -**Bor**-measurable or, equivalently, if $f^{-1}(B) \in \kappa^+$ -**Bor**(X) for every $B \in \kappa^+$ -**Bor**(Y). We denote by $\mathcal{B}(X, Y)$ the collection of all κ^+ -Borel measurable functions from X to Y , and for each ordinal $\alpha \geq 1$ we let $\mathcal{M}_\alpha(X, Y)$ be the collection of all κ^+ - Σ_α^0 -measurable functions from X to Y . Clearly, $\mathcal{B}(X, Y) = \mathcal{M}_{\text{ord}_{\kappa^+}(X)}(X, Y)$. In particular, we always have $\mathcal{B}(X, Y) = \mathcal{M}_{\kappa^+}(X, Y)$, and if $\text{w}(Y) \leq \kappa$ then $\mathcal{B}(X, Y) = \bigcup_{1 \leq \alpha < \kappa^+} \mathcal{M}_\alpha(X, Y)$. This means that the classes $\mathcal{M}_\alpha(X, Y)$ stratify the κ^+ -Borel measurable functions in a hierarchy of length at most κ^+ . In analogy with Definition 3.2.12, we introduce the following parameter, that measures the length of such hierarchy.

Definition 3.3.10. Let X and Y be topological spaces. The **order** of the hierarchy of κ^+ -Borel measurable functions from X to Y is

$$\text{ord}_{\kappa^+}^{\text{Fun}}(X, Y) = \min \{ \alpha \in \text{Ord} \mid \mathcal{M}_\alpha(X, Y) = \mathcal{B}(X, Y) \}.$$

By the previous discussion, $\text{ord}_{\kappa^+}^{\text{Fun}}(X, Y) \leq \kappa^+$; when $\text{ord}_{\kappa^+}^{\text{Fun}}(X, Y) < \kappa^+$ we say that the hierarchy of κ^+ -Borel measurable functions from X to Y **collapses**. The next proposition shows that if Y contains at least two points, then this happens precisely when the κ^+ -Borel hierarchy of sets on X collapses.

Proposition 3.3.11. Let Y be a Hausdorff topological space such that $|Y| \geq 2$. Then,

$$\text{ord}_{\kappa^+}(X) = \text{ord}_{\kappa^+}^{\text{Fun}}(X, Y),$$

Moreover, the hierarchy of κ^+ -Borel measurable functions is proper, meaning that $\mathcal{M}_\alpha \setminus \bigcup_{1 \leq \beta < \alpha} \mathcal{M}_\beta(X, Y) \neq \emptyset$ for every $1 \leq \alpha < \kappa^+$ with $\alpha \leq \text{ord}_{\kappa^+}^{\text{Fun}}(X, Y)$.

Proof. The inequality $\text{ord}_{\kappa^+}^{\text{Fun}}(X, Y) \leq \text{ord}_{\kappa^+}(X)$ is obvious, so let us prove the reverse inequality. Fix distinct points $y_0, y_1 \in Y$, and take any $A \in \kappa^+$ -**Bor**(X). Let $f_A: X \rightarrow Y$ be defined by $f_A(x) = y_0$ if $x \in A$, and $f_A(x) = y_1$ otherwise. Then, $f_A \in \mathcal{B}(X, Y) = \mathcal{M}_{\text{ord}_{\kappa^+}^{\text{Fun}}(X, Y)}(X, Y)$; since $A = f_A^{-1}(U)$ for any open $U \subseteq Y$ such that $y_0 \in U$ and $y_1 \notin U$, we then get $A \in \kappa^+$ - $\Sigma_{\text{ord}_{\kappa^+}^{\text{Fun}}(X, Y)}^0(X)$. The previous argument shows that κ^+ -**Bor**(X) $\subseteq \kappa^+$ - $\Sigma_{\text{ord}_{\kappa^+}^{\text{Fun}}(X, Y)}^0(X)$, and thus $\text{ord}_{\kappa^+}(X) \leq \text{ord}_{\kappa^+}^{\text{Fun}}(X, Y)$, as desired.

We now move to properness. To simplify the notation, let $\bar{\alpha} = \text{ord}_{\kappa^+}(X) = \text{ord}_{\kappa^+}^{\text{Fun}}(X, Y)$. Given $1 \leq \alpha < \bar{\alpha}$, let $A \in \kappa^+$ - $\Delta_\alpha^0(X) \setminus \bigcup_{1 \leq \beta < \alpha} \kappa^+$ - $\Sigma_\beta^0(X)$; the existence of such an A is granted by Proposition 3.3.4. Then, $f_A \in \mathcal{M}_\alpha(X, Y)$, but $f_A \notin \mathcal{M}_\beta(X, Y)$ for any $1 \leq \beta < \alpha$ because otherwise $A = f_A^{-1}(U)$ would belong to κ^+ - $\Sigma_\beta^0(X)$. The same argument works if $\alpha = \bar{\alpha} < \kappa^+$ too, except that when $\bar{\alpha} = 2$, instead of using Proposition 3.3.4 to find the desired A use the fact that κ^+ - $\Delta_2^0(X) \setminus \kappa^+$ - $\Sigma_1^0(X)$ is non-empty because X must contain a proper closed set (otherwise it would be discrete, against $\text{ord}_{\kappa^+}(X) = 2$). \square

An extensive study of the structure of the κ^+ -Borel measurable functions with respect to various kinds of limits, and the suitable notion of generalized Baire class α function can be found in Chapter 5. Here we briefly touch on this subject at the end of Section 3.6.

3.3.1 Structural properties

Following [DMR25], we consider higher analogues of the structural properties from [Kec95, Section 22.C].

Given sets X, Y , and $P \subseteq X \times Y$, a **uniformization** of P is a subset $P^* \subseteq P$ such that for all $x \in X$

$$\exists y P(x, y) \iff \exists! y P^*(x, y).$$

We recall that \mathcal{M}_κ denotes the collection of all $\text{cof}(\kappa)$ -metrizable spaces of weight at most κ (see Section 2.4).

Definition 3.3.12. Let Γ be a boldface pointclass.

- (1) Γ has the **separation property** if for every $X \in \mathcal{M}_\kappa$ and all disjoint sets $A, B \in \Gamma(X)$, there is $C \in \Delta_\Gamma(X)$ such that $A \subseteq C$ and $C \cap B = \emptyset$.
- (2) Γ has the **κ -separation property** if for every $X \in \mathcal{M}_\kappa$ and every sequence of sets $(A_i)_{i < \kappa}$ from $\Gamma(X)$ satisfying $\bigcap_{i < \kappa} A_i = \emptyset$, there is a sequence $(B_i)_{i < \kappa}$ of sets from $\Delta_\Gamma(X)$ such that $\bigcap_{i < \kappa} B_i = \emptyset$ and $A_i \subseteq B_i$ for every $i < \kappa$.
- (3) Γ has the **reduction property** if for every $X \in \mathcal{M}_\kappa$ and every $A, B \in \Gamma(X)$, there are disjoint sets $A^*, B^* \in \Gamma(X)$ such that $A^* \subseteq A$, $B^* \subseteq B$, and $A^* \cup B^* = A \cup B$.
- (4) Γ has the **κ -reduction property** if for every $X \in \mathcal{M}_\kappa$ and every sequence $(A_i)_{i < \kappa}$ of sets from $\Gamma(X)$, there is a sequence $(A_i^*)_{i < \kappa}$ of pairwise disjoint sets from $\Gamma(X)$ such that $\bigcup_{i < \kappa} A_i^* = \bigcup_{i < \kappa} A_i$ and $A_i^* \subseteq A_i$ for every $i < \kappa$.
- (5) Γ has the **ordinal κ -uniformization property** if for every $X \in \mathcal{M}_\kappa$ and every $R \in \Gamma(X \times \kappa)$, there is a uniformization of R in $\Gamma(X \times \lambda)$.

Clearly, if Γ has the κ -separation property then it has the separation property, and if Γ has the κ -reduction property then it has the reduction property. To state the relationships among the other structural properties we need one more definition.

Definition 3.3.13. A boldface pointclass Γ is **κ -reasonable** if for every $X \in \mathcal{M}_\kappa$, every set I with $|I| \leq \kappa$, and every family $(A_i)_{i \in I}$ of subsets of X , we have that $\forall i \in I (A_i \in \Gamma(X))$ if and only if $A \in \Gamma(X \times I)$, where

$$A = \{(x, i) \in X \times I \mid x \in A_i\}.$$

If a boldface pointclass $\Gamma \supseteq \kappa^+ \text{-}\Delta_1^0$ is such that $\Gamma(X)$ is closed under unions of size κ and intersections with clopen sets (for every $X \in \mathcal{M}_\kappa$), then Γ is κ -reasonable, which implies that also $\bar{\Gamma}$ and Δ_Γ are κ -reasonable. Therefore, all of $\kappa^+ \text{-}\Sigma_\alpha^0$, $\kappa^+ \text{-}\Pi_\alpha^0$, and $\kappa^+ \text{-}\Delta_\alpha^0$ are κ -reasonable.

The following is the higher analogue of [Kec95, Proposition 22.15], and can be proved using similar arguments.

Proposition 3.3.14. *Let Γ be a boldface pointclass.*

- (1) *If Γ has the reduction property, then $\check{\Gamma}$ has the separation property.*
- (2) *If Γ is closed under unions of length κ and has the κ -reduction property, then $\check{\Gamma}$ has the κ -separation property.*
- (3) *If Γ is κ -reasonable, then Γ has the κ -reduction property if and only if Γ has the ordinal κ -uniformization property.*
- (4) *If there is a κ^2 -universal set for $\Gamma(\kappa^2)$, then Γ cannot have both the reduction and the separation properties.*

The separation property admits a technical variation that will be used later.

Corollary 3.3.15. *(Folklore) Suppose that Γ has the separation property. Let $X \in \mathcal{M}_\kappa$ be such that $\Gamma(X)$ is closed under unions and intersections of size at most ν , for some cardinal ν . Let $C \in \Delta_\Gamma(X)$ and $(P_i)_{i < \nu}$ be a family of pairwise disjoint non-empty $\Gamma(X)$ -subsets of C . Then, there is a Δ_Γ -partition $\{C_i \mid i < \nu\}$ of C such that $P_i \subseteq C_i$ for every $i < \nu$.*

Proof. For each $i < \nu$, let $D_i = \bigcup_{j \neq i} P_j$. Then, P_i and D_i are disjoint Γ -sets. Applying the separation property we get $C'_i \in \Delta_\Gamma(X)$ such that $P_i \subseteq C'_i$ and $C'_i \cap P_j = \emptyset$ for every $j < \nu$ different from i . Since our hypotheses imply that $\Delta_\Gamma(X)$ is a ν^+ -algebra, it is enough to let $C_0 = (C \cap C'_0) \cup (C \setminus \bigcup_{j < \nu} C'_j)$ and $C_i = (C \cap C'_i) \setminus \bigcup_{j < i} C'_j$, for $0 < i < \nu$. \square

We now want to prove the analogue in generalized descriptive set theory of [Kec95, Theorem 22.16]. The straightforward generalization of the original argument would require $\kappa^+ \text{-}\Sigma_\alpha^0(X)$ to be closed under intersections of length less than κ , which is true if κ is regular and α is either a successor ordinal or a limit ordinal with cofinality κ , but fails in all other cases (see Proposition 3.3.2 and Corollary 3.4.9). As a remedy, following [DMR25, Proposition 4.2.1], which proves the same result for the case $\text{cof}(\kappa) = \omega$, we can exploit the following observation.

Lemma 3.3.16. *Assume that $\text{cof}(\kappa) > \omega$. Let $X \in \mathcal{M}_\kappa$ and $1 \leq \alpha < \kappa^+$.*

- (1) *If α is a successor ordinal and $A \in \kappa^+ \text{-}\Sigma_\alpha^0(X)$, then $A = \bigcup_{i < \text{cof}(\kappa)} A_i$ for some family $(A_i)_{i < \text{cof}(\kappa)}$ of sets in $\kappa^+ \text{-}\Delta_\alpha^0(X)$. Moreover, we can assume that $A_i \subseteq A_j$ for all $i \leq j < \text{cof}(\kappa)$ or, alternatively, that the sets A_i are pairwise disjoint.*
- (2) *If α is a limit ordinal, $(\alpha_i)_{i < \text{cof}(\alpha)}$ is a strictly increasing sequence cofinal in α , and $A \in \kappa^+ \text{-}\Sigma_\alpha^0(X)$, then $A = \bigcup_{i < \text{cof}(\alpha)} A_i$ for some family $(A_i)_{i < \text{cof}(\alpha)}$ such that $A_i \in \kappa^+ \text{-}\Delta_{\alpha_i+2}^0$ for all $i < \text{cof}(\alpha)$. Moreover, we can assume that $A_i \subseteq A_j$ for all $i \leq j < \text{cof}(\alpha)$ or, alternatively, that the sets A_i are pairwise disjoint.*
- (3) *If $\nu < \kappa$ and $A_j \in \kappa^+ \text{-}\Sigma_\alpha^0(X)$ for every $j < \nu$, then $\bigcap_{j < \nu} A_j \in \kappa^+ \text{-}\Delta_{\alpha+1}^0(X)$.*

The dual results obtained by replacing $\kappa^+ \text{-}\Sigma_\alpha^0$ with $\kappa^+ \text{-}\Pi_\alpha^0$ and swapping the role of unions and intersections hold as well.

Proof. The dual results can be obtained by taking complements and using De Morgan's rules, so we only consider the case of the classes $\kappa^+ \text{-}\Sigma_\alpha^0(X)$. By Theorem 2.4.1, we can assume that $X \subseteq \text{cof}(\kappa)_\kappa$.

We argue by induction on $1 \leq \alpha < \kappa^+$. For the basic case $\alpha = 1$, observe that every $A \in \kappa^+ \text{-}\Sigma_1^0(X)$ can be written as $\bigcup_{i < \text{cof}(\kappa)} A_i$, where

$$A_i = \bigcup \{ \mathbf{N}_s \cap X \mid \text{lh}(s) = i, \mathbf{N}_s \cap X \subseteq A \}.$$

Each A_i belongs to $\kappa^+ \text{-}\Delta_1^0(X)$ because

$$X \setminus A_i = \bigcup \{ \mathbf{N}_t \cap X \mid \text{lh}(t) = i, \mathbf{N}_t \cap X \not\subseteq A \}.$$

Clearly, $A_i \subseteq A_j$ whenever $i \leq j$. If instead we want the sets A_i to be pairwise disjoint, we replace each A_i with $A_i \setminus \bigcup_{j < i} A_j$: such sets are still clopen because $\kappa^+ \text{-}\Delta_1^0(X)$ is a $\text{cof}(\kappa)$ -algebra (because X , being a member of \mathcal{M}_κ , is $\text{cof}(\kappa)$ -additive). This proves (1). Part (2) needs not to be considered in the case $\alpha = 1$. As for (3), it suffices to prove that $\bigcap_{j < \nu} A_j \in \kappa^+ \text{-}\Sigma_2^0(X)$. Using (1), write every A_j as $A_j = \bigcup_{i < \text{cof}(\kappa)} A_{j,i}$ with $A_{j,i} \in \kappa^+ \text{-}\Delta_1^0(X)$. Then,

$$\bigcap_{j < \nu} A_j = \bigcap_{j < \nu} \bigcup_{i < \text{cof}(\kappa)} A_{j,i} = \bigcup_{s \in {}^\nu \text{cof}(\kappa)} \bigcap_{j < \nu} A_{j,s(j)}.$$

If κ is regular, then our assumption $2^{<\kappa} = \kappa$ entails $\kappa^{<\kappa} = \kappa$, and thus $|{}^\nu \text{cof}(\kappa)| \leq \kappa^{<\kappa} = \kappa$ because $\text{cof}(\kappa) = \kappa$. If κ is singular, then $2^{<\kappa}$ entails that κ is strong limit, and thus $|{}^\nu \text{cof}(\kappa)| \leq \lambda^\lambda = 2^\lambda < \kappa$ for $\lambda = \max\{\text{cof}(\kappa), \nu\} < \kappa$. Therefore in all cases $|{}^\nu \text{cof}(\kappa)| \leq \kappa$, and thus $\bigcap_{j < \nu} A_j \in \kappa^+ \text{-}\Sigma_2^0(X)$, as desired.

Let now $\alpha > 1$. Part (1) is relevant only when α is successor, so assume that $\alpha = \alpha' + 1$. Write $A = \bigcup_{j < \kappa} B_j$ with $B_j \in \kappa^+ \text{-}\Pi_{\alpha'}^0(X)$. Fix a strictly increasing sequence $(\kappa_i)_{i < \text{cof}(\kappa)}$ cofinal in κ . Then, each $A_i = \bigcup_{j < \kappa_i} B_j$ belongs to $\kappa^+ \text{-}\Delta_{\alpha'+1}^0(X) = \kappa^+ \text{-}\Delta_\alpha^0(X)$ because (3) holds at level α' by inductive hypothesis, and clearly $A = \bigcup_{i < \text{cof}(\kappa)} A_i$. Moreover, if $i \leq j < \text{cof}(\kappa)$ then $A_i \subseteq A_j$ by construction. If instead we want the sets A_i to be pairwise disjoint, then we again replace each A_i with $A_i \setminus \bigcup_{j < i} A_j$: since $\kappa^+ \text{-}\Delta_\alpha^0(X)$ is a $\text{cof}(\kappa)$ -algebra (because α is a successor ordinal), this works.

If α is limit, we need instead to prove part (2). By definition, $A = \bigcup_{j < \kappa} B_j$, where $B_j \in \bigcup_{\alpha' < \alpha} \kappa^+ \text{-}\Pi_{\alpha'}^0(X)$. For all $i < \text{cof}(\alpha)$, set $A_i = \bigcup \{ B_j \mid B_j \in \kappa^+ \text{-}\Pi_{\alpha_i}^0(X) \}$. Then, $A_i \in \kappa^+ \text{-}\Sigma_{\alpha_i+1}^0(X) \subseteq \kappa^+ \text{-}\Delta_{\alpha_i+2}^0(X)$. Moreover, $A_i \subseteq A_j$ if $i \leq j$, and $A = \bigcup_{i < \text{cof}(\alpha)} A_i$. If instead we want the sets A_i to be pairwise disjoint, we once again replace each A_i with $A_i \setminus \bigcup_{j < i} A_j$, which belongs to $\kappa^+ \text{-}\Delta_{\alpha_i+2}^0(X)$ because the sets A_i are in $\kappa^+ \text{-}\Sigma_{\alpha_i+1}^0(X)$.

Finally, we prove (3). Again, we only need to check that $\bigcap_{j < \nu} A_j \in \kappa^+ \text{-}\Sigma_{\alpha+1}^0$. For all $i < \nu$, apply⁸ (1) or (2), depending on whether α is a

⁸This can be done because in the previous two paragraphs we already proved that (1) and (2) hold at level α .

successor or a limit ordinal, to get $A_j = \bigcup_{i < \lambda} B_{j,i}$ with $B_{j,i} \in \kappa^+ \text{-}\Delta_\alpha^0(X)$ and $\lambda = \text{cof}(\kappa)$ if α is successor or $\lambda = \text{cof}(\alpha)$ if α is limit. Arguing as in the case $\alpha = 1$, we have $\bigcap_{j < \nu} A_j = \bigcup_{s \in \nu \lambda} \bigcap_{j < \nu} B_{j,s(j)}$, and we have to check that in all cases $|\nu \lambda| \leq \kappa$. If κ is regular, then $|\nu \lambda| \leq \kappa^{< \kappa} = \kappa$ because $\lambda \leq \kappa$. If instead κ is singular, then $\lambda < \kappa$: therefore $|\nu \lambda| \leq 2^{\max\{\nu, \lambda\}} < \kappa$ because κ is strong limit. \square

Theorem 3.3.17. *For any $1 < \alpha < \kappa^+$, the boldface pointclass $\kappa^+ \text{-}\Sigma_\alpha^0$ has the ordinal κ -uniformization property, and thus the κ -reduction property, but it does not have the κ -separation property. The class $\kappa^+ \text{-}\Pi_\alpha^0$ has the κ -separation property, but not the κ -reduction property.*

The same is true for $\alpha = 1$ if either $\text{cof}(\kappa) > \omega$, or $\text{cof}(\kappa) = \omega$ and we restrict the attention to spaces $X \in \mathcal{M}_\kappa$ with $\dim(X) = 0$ (equivalently: to ultrametrizable spaces of weight at most κ).

Proof. The case $\text{cof}(\kappa) = \omega$ has already been treated⁹ in [DMR25, Proposition 4.4.4]. We show that the same argument can be adapted to deal with the remaining cases, so from now on assume $\text{cof}(\kappa) > \omega$. By Proposition 3.3.14, it is enough to show that $\kappa^+ \text{-}\Sigma_\alpha^0$ has the ordinal κ -uniformization property. Fix any $X \in \mathcal{M}_\kappa$: since we assumed $\text{cof}(\kappa) > \omega$, by Theorem 2.4.1 we can suppose that $X \subset {}^{\text{cof}(\kappa)}\kappa$. We distinguish three cases.

First suppose that $\alpha = 1$. For any $i < \text{cof}(\kappa)$, we say that a set $A \subseteq X$ is i -clopen if there is $S \subseteq {}^i\kappa$ such that $A = \bigcup \{N_s \cap X \mid s \in S\}$. It is easy to check that i -clopen sets are closed under complements, and arbitrary unions and intersections. Moreover, if $j \leq i$ then every j -clopen set is also i -clopen. Fix any $R \in \kappa^+ \text{-}\Sigma_1^0(X \times \kappa)$. For every $i < \text{cof}(\kappa)$ and $\gamma < \kappa$, let

$$R_i^\gamma = \bigcup \{N_s \cap X \mid s \in {}^i\kappa \wedge N_s \times \{\gamma\} \subseteq R\},$$

and notice that $\bigcup_{i < \text{cof}(\kappa)} \bigcup_{\gamma < \kappa} R_i^\gamma$ coincides with the projection of R on the first coordinate. Each R_i^γ is i -clopen, hence so is $R_i = \bigcup_{\gamma < \kappa} R_i^\gamma$. The set

$$Q_i^\gamma = R_i^\gamma \setminus \left(\bigcup_{i' < i} R_{i'} \cup \bigcup_{\gamma' < \gamma} R_i^{\gamma'} \right)$$

is i -clopen too, hence

$$\begin{aligned} Q^* &= \{(x, \gamma, i) \in X \times \kappa \times \text{cof}(\kappa) \mid x \in Q_i^\gamma\} \\ &= \bigcup_{i < \text{cof}(\kappa)} \bigcup_{\gamma < \kappa} (Q_i^\gamma \times \{\gamma\} \times \{i\}) \in \kappa^+ \text{-}\Sigma_1^0(X \times \kappa \times \text{cof}(\kappa)). \end{aligned}$$

Finally, the set

$$R^* = \{(x, \gamma) \in X \times \kappa \mid \exists i < \text{cof}(\kappa) ((x, \gamma, i) \in Q^*)\}$$

is open and uniformizes R .

Next we consider the case where $\alpha = \alpha' + 1 > 1$ is a successor ordinal. Consider $R \in \kappa^+ \text{-}\Sigma_\alpha^0(X \times \kappa)$, and write it as $R = \bigcup_{i < \kappa} R_i$ with $R_i \in \kappa^+ \text{-}\Pi_{\alpha'}^0(X \times \kappa)$. Let

$$Q = \{(x, \gamma, i) \in X \times \kappa \times \kappa \mid (x, \gamma) \in R_i\},$$

⁹Formally, [DMR25, Proposition 4.4.4] states the result just for κ -Polish spaces, i.e. completely metrizable spaces of weight at most κ . However, the proof goes through also for the more general class \mathcal{M}_κ .

and notice that $Q \in \kappa^+ \text{-}\Pi_{\alpha'}^0(X \times \kappa \times \kappa)$ because $\kappa^+ \text{-}\Pi_{\alpha'}^0$ is κ -reasonable. Endow $\kappa \times \kappa$ with the Gödel well-ordering \preceq , and define

$$Q^* = \{(x, \gamma, i) \in X \times \kappa \times \kappa \mid (x, \gamma, i) \in Q \wedge \forall (\gamma', i') \prec (\gamma, i) ((x, \gamma', i') \notin Q)\}.$$

There are less than κ -many pairs $(\gamma', i') \prec (\gamma, i)$, and since the intersection of less than κ -many sets in $\kappa^+ \text{-}\Sigma_{\alpha'}^0(X \times \kappa \times \kappa)$ is in $\kappa^+ \text{-}\Delta_{\alpha'+1}^0(X \times \kappa \times \kappa) = \kappa^+ \text{-}\Delta_{\alpha}^0(X \times \kappa \times \kappa)$ by Lemma 3.3.16(3), we get that $Q^* \in \kappa^+ \text{-}\Delta_{\alpha}^0(X \times \kappa \times \kappa) \subseteq \kappa^+ \text{-}\Sigma_{\alpha}^0(X \times \kappa \times \kappa)$. Therefore, the set

$$R^* = \{(x, \gamma) \in X \times \kappa \mid \exists i < \text{cof}(\kappa) ((x, \gamma, i) \in Q^*)\}.$$

belongs to $\kappa^+ \text{-}\Sigma_{\alpha}^0(X \times \kappa)$ and uniformizes R .

Finally, assume that α is a limit ordinal, and let $R \in \kappa^+ \text{-}\Sigma_{\alpha}^0(X \times \kappa)$. By Lemma 3.3.16(2), there are an increasing sequence $(\alpha_i)_{i < \text{cof}(\alpha)}$ of ordinals smaller than α and sets $R_i \in \kappa^+ \text{-}\Delta_{\alpha_i}^0(X \times \kappa)$ such that $R = \bigcup_{i < \text{cof}(\alpha)} R_i$. For $i < \text{cof}(\alpha)$, let $Q^i \subseteq X \times \kappa \times \text{cof}(\alpha)$ be defined by

$$Q^i = \{(x, \gamma, i') \in X \times \kappa \times \text{cof}(\alpha) \mid i' \leq i \wedge (x, \gamma) \in R_{i'}\},$$

and set $Q = \bigcup_{i < \text{cof}(\alpha)} Q^i = \{(x, \gamma, i) \in X \times \kappa \times \text{cof}(\alpha) \mid (x, \gamma) \in R_i\}$. Since each pointclass $\kappa^+ \text{-}\Sigma_{\alpha_i}^0$ is κ -reasonable, then $Q^i \in \kappa^+ \text{-}\Sigma_{\alpha_i}^0(X)$, and thus $Q \in \kappa^+ \text{-}\Sigma_{\alpha}^0(X)$. Let

$$Q^* = \{(x, \gamma, i) \in X \times \kappa \times \text{cof}(\alpha) \mid (x, \gamma, i) \in Q \wedge \forall i' < i \forall \gamma' < \kappa ((x, \gamma', i') \notin Q^i) \wedge \forall \gamma' < \gamma ((x, \gamma', i) \notin Q^i)\}.$$

For any fixed pair $(\gamma, i) \in \kappa \times \text{cof}(\alpha)$, the set

$$\{x \in X \mid \forall i' < i \forall \gamma' < \kappa ((x, \gamma', i') \notin Q^i) \wedge \forall \gamma' < \gamma ((x, \gamma', i) \notin Q^i)\}$$

belongs to $\kappa^+ \text{-}\Pi_{\alpha_i}^0(X)$. Since $\kappa^+ \text{-}\Pi_{\alpha_i}^0(X) \subseteq \kappa^+ \text{-}\Sigma_{\alpha}^0(X)$ and $\kappa^+ \text{-}\Sigma_{\alpha}^0$ is κ -reasonable, $Q^* \in \kappa^+ \text{-}\Sigma_{\alpha}^0(X \times \kappa \times \text{cof}(\alpha))$. As before, it follows that the set R^* consisting of those $(x, \gamma) \in X \times \kappa$ such that $(x, \gamma, i) \in Q^*$ for some $i < \text{cof}(\alpha)$ is the desired uniformization of R in $\kappa^+ \text{-}\Sigma_{\alpha}^0(X \times \kappa)$. \square

By Corollary 3.3.15, we thus get:

Corollary 3.3.18. *Let $X \in \mathcal{M}_{\kappa}$ and $1 < \alpha < \kappa^+$. Let $C \in \kappa^+ \text{-}\Delta_{\alpha}^0(X)$, and let $\{P_0, \dots, P_n\} \subseteq \kappa^+ \text{-}\Pi_{\alpha}^0(X)$ be a finite family of pairwise disjoint non-empty subsets of C . Then, there is a $\kappa^+ \text{-}\Delta_{\alpha}^0$ -partition $\{C_0, \dots, C_n\}$ of C such that $P_i \subseteq C_i$ for every $i \leq n$.*

The same is true for $\alpha = 1$ if either $\text{cof}(\kappa) > \omega$, or $\text{cof}(\kappa) = \omega$ and we restrict the attention to the subclass of \mathcal{M}_{κ} consisting of all ultrametrizable spaces.

3.4 An alternative hierarchy when κ is singular

Recall that, starting from Section 3.3, we agreed that all topological spaces are assumed to be regular Hausdorff and of weight at most κ . Fix such a

space X . When κ is singular, we have that $\kappa^+\text{-Bor}(X) = \kappa\text{-Bor}(X)$. Indeed, $\kappa\text{-Bor}(X) \subseteq \kappa^+\text{-Bor}(X)$ by Remark 3.2.3. For the other inclusion, it is enough to check that $\kappa\text{-Bor}(X)$ is closed under unions of length κ . Let $A = \bigcup_{i < \kappa} A_i$, where $A_i \in \kappa\text{-Bor}(X)$ for every $i < \kappa$, and fix any sequence of ordinals $(\kappa_j)_{j < \text{cof}(\kappa)}$ cofinal in κ . Then, for all $j < \text{cof}(\kappa)$, we have $A'_j = \bigcup_{i < \kappa_j} A_i \in \kappa\text{-Bor}(X)$ because $\kappa_j < \kappa$, hence $A = \bigcup_{j < \text{cof}(\kappa)} A'_j \in \kappa\text{-Bor}(X)$ because $\text{cof}(\kappa) < \kappa$.

Thus, when κ is singular, the κ^+ -Borel sets can naturally be stratified into an alternative hierarchy: the κ -Borel hierarchy. The study of such hierarchy, together with its interplay with the κ^+ -Borel hierarchy studied in Section 3.3, is the subject of this section. To this aim, we will often assume $\kappa\text{-}\Sigma_1^0(X) \subseteq \text{cof}(\kappa)^+\text{-}\Sigma_2^0(X)$. On the one hand, this is due to the fact that in order to guarantee that the κ -Borel hierarchy is increasing, we need to have at least $\kappa\text{-}\Sigma_1^0(X) \subseteq \kappa\text{-}\Sigma_2^0(X)$, and this is no longer granted by the other tacit hypotheses on X (as it was the case for the κ^+ -Borel hierarchy). On the other hand, the fact that every open set can be written as a $\text{cof}(\kappa)$ -sized union of closed sets allows for a neater presentation of the combinatorial properties of the κ -Borel hierarchy. By Fact 3.1.1, such condition is anyway satisfied by virtually all spaces of interest in generalized descriptive set theory, including all subspaces of ${}^\kappa 2$ and ${}^{\text{cof}(\kappa)} \kappa$, so it does not really affect the intended scope of this thesis. However, we also remark that by Proposition 3.2.23, all our results are valid for arbitrary T_0 spaces of weight at most κ if we only consider infinite levels $\alpha \geq \omega$.

We begin with a result describing the relationship between the κ -Borel hierarchy and the κ^+ -Borel hierarchy. From this, we will derive various other properties of the κ -Borel hierarchy and, complementing some results obtained in Section 3.3, on the κ^+ -Borel hierarchy when κ is singular.

Recall that every ordinal α can be written uniquely as $\alpha = \gamma + n$, with $n < \omega$ and either $\gamma = 0$ or γ limit. Accordingly, we say that α is even (respectively, odd) if n is even (respectively, odd). If $\alpha = \gamma + n$ is even, to simplify the notation we set $\frac{\alpha}{2} = \gamma + \frac{n}{2}$. Notice that $\frac{2:\alpha}{2} = \alpha$ for every ordinal α , and that if $\alpha = 0$ or α is limit, then $\frac{\alpha}{2} = \frac{2:\alpha}{2} = \alpha$.

Theorem 3.4.1. *Assume that κ is singular, and $\kappa\text{-}\Sigma_1^0(X) \subseteq \text{cof}(\kappa)^+\text{-}\Sigma_2^0(X)$. Given $\alpha < \kappa^+$, let $\hat{\alpha} = \text{cof}(\alpha)$ if α is limit, and $\hat{\alpha} = \text{cof}(\kappa)$ otherwise.*

(1) *If α is even, then*

$$\kappa\text{-}\Sigma_{1+\alpha}^0(X) = \left(\bigcup_{\beta < \alpha} \kappa\text{-}\Pi_{1+\beta}^0(X) \right)_{\sigma_{\hat{\alpha}}} = \kappa^+\text{-}\Sigma_{1+\frac{\alpha}{2}}^0(X),$$

and dually for $\kappa\text{-}\Pi_{1+\alpha}^0(X)$.

(2) *If α is odd, then $\kappa\text{-}\Sigma_{1+\alpha}^0(X) = \kappa\text{-}\Pi_{1+\alpha}^0(X) = \kappa\text{-}\Delta_{1+\alpha}^0(X)$.*

Proof. By induction on $\alpha < \kappa^+$. The case $\alpha = 0$ is obvious, hence assume $\alpha > 0$.

We first consider (1), so assume that α is even and write it as $\alpha = \gamma + 2n$ for some $n < \omega$ and $\gamma = 0$ or γ limit. If $n = 0$, then $\alpha = \gamma > 0$ is limit, hence $\frac{\alpha}{2} = \alpha$. Recalling also Remark 3.2.3, the inclusions $\left(\bigcup_{\beta < \alpha} \kappa\text{-}\Pi_{1+\beta}^0(X) \right)_{\sigma_{\hat{\alpha}}} \subseteq \kappa\text{-}\Sigma_{1+\alpha}^0(X) \subseteq \kappa^+\text{-}\Sigma_{1+\alpha}^0(X)$ are obvious, because

$\hat{\alpha} = \text{cof}(\alpha) < \kappa$. The inductive hypothesis and the fact that, being α limit, $\beta < \alpha \Rightarrow 2 \cdot \beta < \alpha$, ensure that $\kappa^+ \text{-}\Sigma_{1+\beta}^0(X) = \kappa \text{-}\Sigma_{1+2\cdot\beta}^0(X)$, and hence $\bigcup_{\beta < \alpha} \kappa \text{-}\Sigma_{1+\beta}^0(X) = \bigcup_{\beta < \alpha} \kappa^+ \text{-}\Sigma_{1+\beta}^0(X)$. Arguing as in the limit case of the proof of Proposition 3.3.1, every $A \in \kappa^+ \text{-}\Sigma_{1+\alpha}^0(X)$ can be written as $A = \bigcup_{j < \text{cof}(\alpha)} A_j$ with $A_j \in \bigcup_{\beta < \alpha} \kappa^+ \text{-}\Sigma_{1+\beta}^0(X)$ for every $j < \text{cof}(\alpha)$. Since $\hat{\alpha} = \text{cof}(\alpha)$, we get $A \in \left(\bigcup_{\beta < \alpha} \kappa \text{-}\Pi_{1+\beta}^0(X) \right)_{\sigma_{\hat{\alpha}}}$. This shows that $\kappa^+ \text{-}\Sigma_{1+\alpha}^0(X) \subseteq \left(\bigcup_{\beta < \alpha} \kappa \text{-}\Pi_{1+\beta}^0(X) \right)_{\sigma_{\hat{\alpha}}}$, as desired.

Assume now that $n = m + 1$, so that $\alpha = \gamma + 2m + 2$ and $\frac{\alpha}{2} = \gamma + m + 1$. The inclusion $\left(\bigcup_{\beta < \alpha} \kappa \text{-}\Pi_{1+\beta}^0(X) \right)_{\sigma_{\hat{\alpha}}} \subseteq \kappa \text{-}\Sigma_{1+\alpha}^0(X)$ is again obvious because $\hat{\alpha} = \text{cof}(\kappa) < \kappa$. Since by inductive hypothesis $\kappa \text{-}\Pi_{1+\gamma+2m}^0(X) = \kappa^+ \text{-}\Pi_{1+\gamma+m}^0(X)$, we get $\kappa \text{-}\Sigma_{1+\gamma+2m+1}^0(X) \subseteq \kappa^+ \text{-}\Sigma_{1+\gamma+m+1}^0(X)$. As $\gamma + 2m + 1$ is odd, $\kappa \text{-}\Pi_{1+\gamma+2m+1}^0(X) = \kappa \text{-}\Sigma_{1+\gamma+2m+1}^0(X)$ by inductive hypothesis again. Therefore $\kappa \text{-}\Pi_{1+\gamma+2m+1}^0(X) \subseteq \kappa^+ \text{-}\Sigma_{1+\gamma+m+1}^0(X)$, and since the latter is closed under unions of size κ by Theorem 3.3.1(1), it follows that

$$\kappa \text{-}\Sigma_{1+\alpha}^0(X) = \kappa \text{-}\Sigma_{1+\gamma+2m+2}^0(X) \subseteq \kappa^+ \text{-}\Sigma_{1+\gamma+m+1}^0(X) = \kappa^+ \text{-}\Sigma_{1+\frac{\alpha}{2}}^0(X).$$

Let now $A \in \kappa^+ \text{-}\Sigma_{1+\frac{\alpha}{2}}^0(X) = \kappa^+ \text{-}\Sigma_{1+\gamma+m+1}^0(X)$, and let $A_i \in \kappa^+ \text{-}\Pi_{1+\gamma+m}^0(X)$ be such that $A = \bigcup_{i < \kappa} A_i$. Fix a sequence $(\kappa_j)_{j < \text{cof}(\kappa)}$ cofinal in κ , and let $B_j = \bigcup_{i < \kappa_j} A_i$ for every $j < \text{cof}(\kappa)$. By inductive hypothesis, $\kappa^+ \text{-}\Pi_{1+\gamma+m}^0(X) = \kappa \text{-}\Pi_{1+\gamma+2m}^0(X)$, hence $B_j \in \kappa \text{-}\Sigma_{1+\gamma+2m+1}^0(X) = \kappa \text{-}\Pi_{1+\gamma+2m+1}^0(X)$ because $\gamma + 2m + 1$ is odd. Since $A = \bigcup_{j < \text{cof}(\kappa)} B_j$ and $\hat{\alpha} = \text{cof}(\kappa)$, we conclude that $A \in \left(\bigcup_{\beta < \alpha} \kappa \text{-}\Pi_{1+\beta}^0(X) \right)_{\sigma_{\hat{\alpha}}}$. This shows that $\kappa^+ \text{-}\Sigma_{1+\frac{\alpha}{2}}^0(X) \subseteq \left(\bigcup_{\beta < \alpha} \kappa \text{-}\Pi_{1+\beta}^0(X) \right)_{\sigma_{\hat{\alpha}}}$ as desired.

We now move to (2), so assume that α is odd, and let δ be its predecessor, which is an even ordinal. It is enough to prove that $\kappa \text{-}\Pi_{1+\alpha}^0(X) \subseteq \kappa \text{-}\Sigma_{1+\alpha}^0(X)$. Let $A \in \kappa \text{-}\Pi_{1+\delta+1}^0(X)$, and let $\lambda < \kappa$ and $A_i \in \kappa \text{-}\Sigma_{1+\delta}^0(X)$ be such that $A = \bigcap_{i < \lambda} A_i$. We claim that for every $i < \lambda$ there are $\kappa \text{-}\Pi_{1+\delta}^0(X)$ sets $A_{i,j}$, for $j < \hat{\delta}$, such that $A_i = \bigcup_{j < \hat{\delta}} A_{i,j}$. If $\delta = 0$ this is true by the assumption $\kappa \text{-}\Sigma_1^0(X) \subseteq \text{cof}(\kappa)^+ \text{-}\Sigma_2^0(X)$. If $\delta > 0$, then this follows by the inductive hypothesis together with $\bigcup_{\beta < \delta} \kappa \text{-}\Pi_{1+\beta}^0(X) \subseteq \kappa \text{-}\Pi_{1+\delta}^0(X)$. Therefore

$$A = \bigcap_{i < \lambda} \left(\bigcup_{j < \hat{\delta}} A_{i,j} \right) = \bigcup_{s \in {}^\lambda(\hat{\delta})} \left(\bigcap_{i < \lambda} A_{i,s(i)} \right).$$

Since δ is even, $\kappa \text{-}\Pi_{1+\delta}^0(X) = \kappa^+ \text{-}\Pi_{1+\frac{\delta}{2}}^0(X)$ by inductive hypothesis, and since the latter is closed under intersections of size κ by Proposition 3.3.1(2), we have $\bigcap_{i < \lambda} A_{i,s(i)} \in \kappa \text{-}\Pi_{1+\delta}^0(X)$ for every $s \in {}^\lambda(\hat{\delta})$. Since κ is strong limit (because it is singular and $2^{<\kappa} = \kappa$) and $\hat{\delta}, \lambda < \kappa$, we have $|{}^\lambda(\hat{\delta})| < \kappa$, and so $A \in \kappa \text{-}\Sigma_{1+\delta+1}^0(X)$, as desired. \square

Corollary 3.4.2. *Suppose that κ is singular, and $\kappa\text{-}\Sigma_1^0(X) \subseteq \text{cof}(\kappa)^+\text{-}\Sigma_2^0(X)$. Then, for every $\alpha < \kappa^+$*

$$\text{ord}_{\kappa^+}(X) \leq 1 + \alpha \quad \Leftrightarrow \quad \text{ord}_{\kappa}(X) \leq 1 + 2 \cdot \alpha.$$

In particular, if one of $\text{ord}_{\kappa^+}(X)$ or $\text{ord}_{\kappa}(X)$ is a limit ordinal α , then $\text{ord}_{\kappa^+}(X) = \text{ord}_{\kappa}(X) = \alpha$.

Restricting the attention to infinite ordinals α , and making use of Proposition 3.2.23 and Theorem 3.3.9, we also obtain:

Corollary 3.4.3. *Suppose that κ is singular. Then, the κ -Borel hierarchy on X collapses if and only if the same happens for the κ^+ -Borel hierarchy on X .*

In particular, the κ -Borel hierarchy does not collapse on ${}^{\kappa}2$, nor on any other space X such that there is a κ^+ -Borel embedding of ${}^{\kappa}2$ into X .

We now move to the closure properties of the pointclasses appearing in the κ -Borel hierarchy on X .

Proposition 3.4.4. *Suppose that κ is singular, and $\kappa\text{-}\Sigma_1^0(X) \subseteq \text{cof}(\kappa)^+\text{-}\Sigma_2^0(X)$. Given $0 < \alpha < \kappa^+$, let $\hat{\alpha} = \text{cof}(\alpha)$ if α is limit, and $\hat{\alpha} = \text{cof}(\kappa)$ otherwise. Then,*

- (1) *If α is even, then $\kappa\text{-}\Sigma_{1+\alpha}^0(X)$ is closed under unions of size κ and intersections of size smaller than $\hat{\alpha}$. Therefore $\kappa\text{-}\Pi_{1+\alpha}^0(X)$ is closed under intersections of size κ and unions of size smaller than $\hat{\alpha}$, and $\kappa\text{-}\Delta_{1+\alpha}^0(X)$ is a $\hat{\alpha}$ -algebra.*
- (2) *If α is odd, then $\kappa\text{-}\Sigma_{1+\alpha}^0(X) = \kappa\text{-}\Pi_{1+\alpha}^0(X) = \kappa\text{-}\Delta_{1+\alpha}^0(X)$ is a $\text{cof}(\kappa)$ -algebra.*

Furthermore, the same is true for $\alpha = 0$ if X is $\text{cof}(\kappa)$ -additive.¹⁰

Proof. Part (1) follows from Theorem 3.4.1(1) and Proposition 3.3.1, once we notice that α is limit if and only if $1 + \frac{\alpha}{2}$ is limit. Part (2) follows instead from Theorem 3.4.1(2) and Remark 3.2.4. The additional part concerning the case $\alpha = 0$ is obvious. \square

We will show in Proposition 3.4.8 that the closure properties stated in Proposition 3.4.4 are optimal. In order to do this, we first need to consider properness of the κ -Borel hierarchy, and determine when the inclusions in (3.2.2) are strict for $\gamma = \kappa$. The following is the analogue of Corollary 3.3.5 for the κ -Borel hierarchy.

Proposition 3.4.5. *Suppose that κ is singular, and $\kappa\text{-}\Sigma_1^0(X) \subseteq \text{cof}(\kappa)^+\text{-}\Sigma_2^0(X)$. Let $\alpha < \kappa^+$ be such that $1 + \alpha < \text{ord}_{\kappa}(X)$.*

- (1) *If α is even, then $\kappa\text{-}\Delta_{1+\alpha}^0(X) \subsetneq \kappa\text{-}\Sigma_{1+\alpha}^0(X)$ (equivalently, $\kappa\text{-}\Delta_{1+\alpha}^0(X) \subsetneq \kappa\text{-}\Pi_{1+\alpha}^0(X)$).*
- (2) *$\bigcup_{\beta < \alpha} (\kappa\text{-}\Sigma_{1+\beta}^0(X) \cup \kappa\text{-}\Pi_{1+\beta}^0(X)) \subsetneq \kappa\text{-}\Delta_{1+\alpha}^0(X)$.*

¹⁰Recall that, more generally, the closure properties of $\kappa^+\text{-}\Sigma_1^0(X)$ under intersections (and of $\kappa^+\text{-}\Pi_1^0(X)$ under unions) are completely determined by the additivity of the space.

Proof. Part (1) follows from Theorem 3.4.1(1) and Corollary 3.3.5.

The proof of part (2) follows the one of Proposition 3.3.4. For $\alpha = 0$, it is enough to observe that $\kappa\text{-}\Delta_1^0(X) \neq \emptyset$, as witnessed by X . If $\alpha = 1$, so that $\text{ord}_\kappa(X) > 2$, the result follows from Lemma 3.2.17. Assume now $\alpha \geq 2$.

If $\alpha > 1$ is odd, let α' be its predecessor, so that α' is an even ordinal not smaller than 2. By Proposition 3.4.4(1), the class $\kappa\text{-}\Sigma_{1+\alpha'}^0(X)$ is closed under unions of size κ . Apply Proposition 3.2.19 to the ordinals $1 = \beta < 1 + \alpha' < \kappa^+$ for $\gamma = \lambda = \kappa$ and \mathcal{B} any basis for X satisfying $|\mathcal{B}| \leq \kappa$, and let $P \subseteq X$ be the closed set obtained this way. Then, $\text{ord}_\kappa(B \cap P) > 1 + \alpha'$ for every $B \in \mathcal{B}$ such that $B \cap P \neq \emptyset$, and since $\text{ord}_\kappa(X) > 1 + \alpha'$, we have $P \neq \emptyset$ and $\text{ord}_\kappa(P) > 1 + \alpha'$, which in particular implies that P contains at least two distinct points x_0 and x_1 . Let $B_0, B_1 \in \mathcal{B}$ be disjoint open sets such that $x_0 \in B_0$ and $x_1 \in B_1$. Since α' is even, by Proposition 3.2.15 and Theorem 3.4.1(1) there are $A_0 \in \kappa\text{-}\Sigma_{1+\alpha'}^0(B_0 \cap P) \setminus \kappa\text{-}\Pi_{1+\alpha'}^0(B_0 \cap P)$ and $A_1 \in \kappa\text{-}\Pi_{1+\alpha'}^0(B_1 \cap P) \setminus \kappa\text{-}\Sigma_{1+\alpha'}^0(B_1 \cap P)$. Let $A = A_0 \cup A_1$. Since $\kappa\text{-}\Sigma_{1+\alpha'}^0$ and $\kappa\text{-}\Pi_{1+\alpha'}^0$ are hereditary, and since $\kappa\text{-}\Delta_{1+\alpha}^0(X)$ is at least an ω -algebra by Remark 3.2.9, one easily gets $A \in \kappa\text{-}\Delta_{1+\alpha}^0(X)$. Moreover, $A \in \bigcup_{\beta < \alpha} (\kappa\text{-}\Sigma_{1+\beta}^0(X) \cup \kappa\text{-}\Pi_{1+\beta}^0(X)) = \kappa\text{-}\Sigma_{1+\alpha'}^0(X) \cup \kappa\text{-}\Pi_{1+\alpha'}^0(X)$ is forbidden by the choice of A_0 and A_1 , hence we are done.

Let us move to the even case. Consider first the case when α is a successor ordinal, so that $\alpha = \beta + 2$ for some even ordinal $\beta \geq 2$. Fix any increasing sequence of ordinals $(\kappa_i)_{i < \text{cof}(\kappa)}$ cofinal in κ .

Claim 3.4.5.1. There is $x \in X$ and a partition $(B_i)_{i < \text{cof}(\kappa)}$ of $X \setminus \{x\}$ such that $B_i \in \kappa\text{-}\Delta_3^0(X)$ and $\text{ord}_\kappa(B_i) > 1 + \alpha$ for every $i < \text{cof}(\kappa)$.

Proof of the Claim. Let $f: X \rightarrow {}^\kappa 2$ be defined as in the proof of Proposition 3.2.23, so that f is injective and $\kappa\text{-}\Sigma_3^0$ -measurable (because $w({}^\kappa 2) = 2^{<\kappa} = \kappa$ and $\kappa\text{-}\Sigma_3^0(X)$ is closed under unions of size κ by Proposition 3.4.4). Let

$$S = \{s \in {}^{<\kappa} 2 \mid \text{ord}_\kappa(f^{-1}(\mathbf{N}_s)) \leq 1 + \alpha\},$$

and let $O = \bigcup \{\mathbf{N}_s \mid s \in S\}$.

If $C = f[X] \setminus O$ were discrete, then it would have size at most κ , so that $|f^{-1}(C)| \leq \kappa$ too. This would mean that $f^{-1}(C) \in \kappa^+\text{-}\Sigma_2^0(X) = \kappa\text{-}\Sigma_3^0(X)$, and also $\text{ord}_\kappa(f^{-1}(C)) \leq 3$. In particular, $\{f^{-1}(\mathbf{N}_s) \mid s \in S\} \cup \{f^{-1}(C)\}$ would be a κ -sized covering of X consisting of $\kappa\text{-}\Sigma_3^0$ -subspaces with order at most $1 + \alpha$, and hence $\text{ord}_\kappa(X) \leq 1 + \alpha$ because $\kappa\text{-}\Sigma_3^0(X) \subseteq \kappa\text{-}\Sigma_{1+\alpha}^0(X)$ and $\kappa\text{-}\Sigma_{1+\alpha}^0(X)$ is closed under finite intersections and unions of size κ (Proposition 3.4.4). This contradicts the fact that $1 + \alpha < \text{ord}_\kappa(X)$, and so we conclude that C is not discrete.

Let $y \in C$ be a non-isolated point in C , and set $x = f^{-1}(y)$. Recursively construct a sequence of ordinals $(\beta_i)_{i < \text{cof}(\kappa)}$ cofinal in κ such that $U_i \cap C \neq \emptyset$ for $U_i = \mathbf{N}_{y \upharpoonright \sup_{j < i} \beta_j} \setminus \mathbf{N}_{y \upharpoonright \beta_i}$. Finally, let $B_i = f^{-1}(U_i)$. Then, $B_i \in \kappa\text{-}\Delta_3^0(X)$ because U_i is clopen, and $(B_i)_{i < \text{cof}(\kappa)}$ is a partition of $X \setminus \{x\}$ because $(U_i)_{i < \text{cof}(\kappa)}$ is a partition of ${}^\kappa 2 \setminus \{y\}$. Moreover, since the clopen set U_i intersects C , there is $s \in {}^\kappa 2 \setminus S$ such that $\mathbf{N}_s \subseteq U_i$. Therefore $f^{-1}(\mathbf{N}_s) \subseteq B_i$, and hence $\text{ord}_\kappa(B_i) > 1 + \alpha$ by $s \notin S$ and Proposition 3.2.13(1). \square

Claim 3.4.5.2. If $\text{ord}_\kappa(Y) > 1 + \alpha$, then for every cardinal $\lambda < \kappa$ there is $C \in \kappa\text{-}\Sigma_{1+\beta+1}^0(Y) \setminus (\kappa\text{-}\Pi_{1+\beta}^0(Y))_{\sigma_\lambda}$.

Proof of the Claim. Suppose not, so that $\kappa\text{-}\Sigma_{1+\beta+1}^0(Y) = (\kappa\text{-}\Pi_{1+\beta}^0(Y))_{\sigma_\lambda}$. Since $\beta+1$ is odd, by Theorem 3.4.1 we have $\kappa\text{-}\Sigma_{1+\beta+1}^0(Y) = \kappa\text{-}\Pi_{1+\beta+1}^0(Y)$ and $\kappa\text{-}\Sigma_{1+\alpha}^0(Y) = (\kappa\text{-}\Pi_{1+\beta+1}^0(Y))_{\sigma_{\text{cof}(\kappa)}}$. Therefore $\kappa\text{-}\Sigma_{1+\alpha}^0(Y) \subseteq (\kappa\text{-}\Pi_{1+\beta}^0(Y))_{\sigma_\mu}$ for $\mu = \lambda \cdot \text{cof}(\kappa) < \kappa$, hence $\kappa\text{-}\Sigma_{1+\alpha}^0(Y) = \kappa\text{-}\Sigma_{1+\beta+1}^0(Y) = \kappa\text{-}\Delta_{1+\beta+1}^0(Y)$ is a κ -algebra (in fact, a κ^+ -algebra) because $\kappa\text{-}\Sigma_{1+\alpha}^0(Y)$ is closed under unions of size κ by Proposition 3.4.4(1). It follows that $\kappa\text{-}\mathbf{Bor}(Y) \subseteq \kappa\text{-}\Sigma_{1+\beta+1}^0(Y)$, contradicting $\text{ord}_\kappa(Y) > 1 + \alpha$. \square

Let $(B_i)_{i < \text{cof}(\kappa)}$ be as in Claim 3.4.5.1. For each $i < \text{cof}(\kappa)$, apply Claim 3.4.5.2 with $Y = B_i$ and $\lambda = \kappa_i$ in order to find $A_i \in \kappa\text{-}\Sigma_{1+\beta+1}^0(B_i) \setminus (\kappa\text{-}\Pi_{1+\beta}^0(B_i))_{\sigma_{\kappa_i}}$. Let $A = \bigcup_{i < \text{cof}(\kappa)} A_i$: we claim that $A \in \kappa\text{-}\Delta_{1+\alpha}^0(X) \setminus \kappa\text{-}\Sigma_{1+\beta+1}^0(X)$, which is enough because $\beta+1$ is odd and hence $\kappa\text{-}\Sigma_{1+\beta+1}^0(X) = \kappa\text{-}\Pi_{1+\beta+1}^0(X)$ by Theorem 3.4.1(2). On the one hand, using the fact that $\kappa\text{-}\Sigma_{1+\beta+1}^0(X)$ is hereditary and the closure properties of $\kappa\text{-}\Sigma_{1+\alpha}^0(X)$ provided in Proposition 3.4.4(1), it is easy to see that both A and

$$X \setminus A = \{x\} \cup \bigcup_{i < \text{cof}(\kappa)} (B_i \setminus A_i)$$

belong to $\kappa\text{-}\Sigma_{1+\alpha}^0(X)$ because $1 + \alpha \geq 3$. On the other hand, if $A \in \kappa\text{-}\Sigma_{1+\beta+1}^0(X)$, then there would be $\lambda < \kappa$ such that $A \in (\kappa\text{-}\Pi_{1+\beta}^0(X))_{\sigma_\lambda}$. Let $i < \text{cof}(\kappa)$ be such that $\lambda \leq \kappa_i$. Then, $A_i = A \cap B_i \in (\kappa\text{-}\Pi_{1+\beta}^0(B_i))_{\sigma_{\kappa_i}}$, against the choice of A_i .

Finally, let α be limit. Since in this case $1 + \alpha = 1 + 2 \cdot \alpha$, we have $\text{ord}_{\kappa^+}(X) > 1 + \alpha = \alpha$ by $\text{ord}_\kappa(X) > 1 + \alpha$ and Corollary 3.4.2. Moreover, since $\frac{\alpha}{2} = \alpha$ we have $\kappa\text{-}\Delta_{1+\alpha}^0(X) = \kappa^+\text{-}\Delta_\alpha^0(X)$ and

$$\bigcup_{\beta < \alpha} (\kappa\text{-}\Sigma_{1+\beta}^0(X) \cup \kappa\text{-}\Pi_{1+\beta}^0(X)) = \bigcup_{1 \leq \beta < \alpha} (\kappa^+\text{-}\Sigma_\beta^0(X) \cup \kappa^+\text{-}\Pi_\beta^0(X))$$

by the fact that the κ -Borel hierarchy is increasing and Theorem 3.4.1(1). Therefore, the result follows from Corollary 3.3.5(2). \square

At the current stage, it is not yet clear whether it can happen that $2 < \text{ord}_\kappa(X) < \kappa^+$ if $\omega < \text{cof}(\kappa) < \kappa$. Nevertheless, we record the following result that complements Proposition 3.4.5(2) for the case $\alpha = \text{ord}_\kappa(X)$ and is the analogue of Corollary 3.3.6 for the κ -Borel hierarchy.

Proposition 3.4.6. *Suppose that κ is singular, and $\kappa\text{-}\Sigma_1^0(X) \subseteq \text{cof}(\kappa)^+\text{-}\Sigma_2^0(X)$. The following are equivalent:*

- (1) *there is $A \in \kappa\text{-}\Delta_{\text{ord}_\kappa(X)}^0(X) \setminus \bigcup_{1 \leq \beta < \text{ord}_\kappa(X)} (\kappa\text{-}\Sigma_\beta^0(X) \cup \kappa\text{-}\Pi_\beta^0(X))$,*
- (2) *X does not have exactly one non-isolated point.*

In particular, (1) holds whenever $\text{ord}_{\kappa^+}(X) \neq 2$.

Proof. If X has exactly one non-isolated point, then $\text{ord}_\kappa(X) = 2$ and $\kappa\text{-}\Delta_2^0(X) = \kappa\text{-}\Sigma_1^0(X) \cup \kappa\text{-}\Pi_1^0(X)$ by Lemma 3.2.17. This shows (1) \Rightarrow (2).

We now show that (2) \Rightarrow (1), using ideas and methods developed in the proofs of Propositions 3.3.4 and 3.4.5. If $\text{ord}_\kappa(X) = 1$, then (1) reduces to

$\kappa\text{-}\Delta_1^0(X) \neq \emptyset$, which is true because e.g. $X \in \kappa\text{-}\Delta_1^0(X)$. If $\text{ord}_\kappa(X) = 2$, then X is not discrete and thus X has at least two non-isolated points by (2): therefore (1) holds by Lemma 3.2.17.

If $\text{ord}_\kappa(X) = 1 + \alpha$ with α odd, then the result follows from the fact that the proof of Proposition 3.4.5 in the odd case works also under the assumption that α is such that $1 + \alpha = \text{ord}_\kappa(X)$.

If instead $\text{ord}_\kappa(X) = 1 + \alpha$ for α an even successor ordinal, then we use the fact that the predecessor α' of α is odd, and hence the class $\kappa\text{-}\Sigma_{1+\beta}^0$ is selfdual on X . Thus, if $\kappa\text{-}\Delta_{\text{ord}_\kappa(X)}^0(X) = \kappa\text{-}\Delta_{1+\alpha'+1}^0(X) \subseteq \kappa\text{-}\Sigma_{1+\alpha'}^0(X) \cup \kappa\text{-}\Pi_{1+\alpha'}^0(X)$, then we would have $\kappa\text{-}\mathbf{Bor}(X) \subseteq \kappa\text{-}\Sigma_{1+\alpha'}^0(X) \cup \kappa\text{-}\Pi_{1+\alpha'}^0(X) \subseteq \kappa\text{-}\Sigma_{1+\alpha'}^0(X)$, and hence $\text{ord}_\kappa(X) \leq 1 + \alpha' < 1 + \alpha$, a contradiction.

Finally, suppose that $\text{ord}_\kappa(X) = 1 + \alpha$ for α limit. Then, by Corollary 3.4.2, $\text{ord}_{\kappa^+}(X) = \text{ord}_\kappa(X) = 1 + \alpha = \alpha$. Moreover, $\kappa\text{-}\Delta_{\text{ord}_\kappa(X)}^0(X) = \kappa^+\text{-}\Delta_{\text{ord}_{\kappa^+}(X)}^0(X)$ and

$$\bigcup_{1 \leq \beta < \text{ord}_\kappa(X)} (\kappa\text{-}\Sigma_\beta^0(X) \cup \kappa\text{-}\Pi_\beta^0(X)) = \bigcup_{1 \leq \beta < \text{ord}_{\kappa^+}(X)} (\kappa^+\text{-}\Sigma_\beta^0(X) \cup \kappa^+\text{-}\Pi_\beta^0(X))$$

by the fact that the κ -Borel hierarchy is increasing and Theorem 3.4.1(1). Therefore the result follows from Corollary 3.3.6. \square

We are now ready to prove a counterpart of Proposition 3.2.15 for the special case of a singular cardinal $\gamma = \kappa$ (which requires the additional assumption $\kappa\text{-}\Sigma_1^0(X) \subseteq \text{cof}(\kappa)^+\text{-}\Sigma_2^0(X)$). The major difference is that condition (9) can be included in the list only when α is even: for example, by Theorem 3.4.1(2), for every odd α we have $\kappa\text{-}\Sigma_{1+\alpha}^0(\kappa^2) = \kappa\text{-}\Pi_{1+\alpha}^0(\kappa^2)$, yet the κ -Borel hierarchy on κ^2 does not collapse by Corollary 3.4.3.

Proposition 3.4.7. *Suppose that κ is singular, and $\kappa\text{-}\Sigma_1^0(X) \subseteq \text{cof}(\kappa)^+\text{-}\Sigma_2^0(X)$. Then, for every $\alpha < \kappa^+$, the following are equivalent:*

- (1) $\text{ord}_\kappa(X) \leq 1 + \alpha$;
- (2) for every $\beta \geq \alpha$, $\kappa\text{-}\Sigma_{1+\beta}^0(X) = \kappa\text{-}\Pi_{1+\beta}^0(X) = \kappa\text{-}\Delta_{1+\beta}^0(X) = \kappa\text{-}\mathbf{Bor}(X)$;
- (3) for some $\beta > \alpha$, one of $\kappa\text{-}\Sigma_{1+\alpha}^0(X)$, $\kappa\text{-}\Pi_{1+\alpha}^0(X)$, or $\kappa\text{-}\Delta_{1+\alpha}^0(X)$ coincides with one of $\kappa\text{-}\Sigma_{1+\beta}^0(X)$ or $\kappa\text{-}\Pi_{1+\beta}^0(X)$;
- (4) the class $\kappa\text{-}\Sigma_{1+\alpha}^0(X)$ is closed under intersections of size κ (equivalently, $\kappa\text{-}\Pi_{1+\alpha}^0(X)$ is closed under unions of size κ);
- (5) the class $\kappa\text{-}\Sigma_{1+\alpha}^0(X)$ is closed under intersections shorter than κ (equivalently, $\kappa\text{-}\Pi_{1+\alpha}^0(X)$ is closed under unions shorter than κ);
- (6) for some $\beta > \alpha$, one of $\kappa\text{-}\Sigma_{1+\alpha}^0(X)$, $\kappa\text{-}\Pi_{1+\alpha}^0(X)$, or $\kappa\text{-}\Delta_{1+\alpha}^0(X)$ coincides with $\kappa\text{-}\Delta_{1+\beta}^0(X)$.

Moreover, if $\alpha \geq 1$ the above conditions are also equivalent to

- (7) the class $\kappa\text{-}\Delta_{1+\alpha}^0(X)$ is closed under intersections of size κ , and hence it is a κ^+ -algebra;

- (8) the class $\kappa\text{-}\Delta_{1+\alpha}^0(X)$ is closed under intersections shorter than κ , and hence it is a κ -algebra.

If instead α is even, then we can add to the list of equivalent conditions the following one:

- (9) the boldface pointclass $\kappa\text{-}\Sigma_{1+\alpha}^0$ is selfdual on X , i.e. $\kappa\text{-}\Sigma_{1+\alpha}^0(X) = \kappa\text{-}\Pi_{1+\alpha}^0(X)$.

Notice that, if we additionally require $X \subseteq {}^{\kappa}2$, (7) and (8) become equivalent to the other conditions even when $\alpha = 0$.

Proof. The equivalence among conditions (1), (2), (3) and, under the assumption $\alpha \geq 1$, (7) and (8) follows from Proposition 3.2.15, as every κ -algebra is also a κ^+ -algebra (by singularity of κ), the κ -Borel hierarchy is increasing (because we assumed $\kappa\text{-}\Sigma_1^0(X) \subseteq \text{cof}(\kappa)^+\text{-}\Sigma_2^0(X)$), and hence also $\kappa\text{-}\Sigma_1^0(X) \cup \kappa\text{-}\Pi_1^0(X) \subseteq \kappa\text{-}\Sigma_{1+\alpha}^0(X)$ if $\alpha \geq 1$.

(2) \Rightarrow (4). Follows from the fact that $\kappa\text{-}\mathbf{Bor}(X) = \kappa^+\text{-}\mathbf{Bor}(X)$ because κ is singular.

(4) \Rightarrow (5). Obvious.

(5) \Rightarrow (6). Under (5), we have

$$\kappa\text{-}\Sigma_{1+\alpha}^0(X) \subseteq \kappa\text{-}\Delta_{1+\alpha+1}^0(X) \subseteq \kappa\text{-}\Pi_{1+\alpha+1}^0(X) \subseteq \kappa\text{-}\Sigma_{1+\alpha}^0(X).$$

(6) \Rightarrow (1). If $\alpha = 0$ the result is trivial, as in all cases we would get $\kappa\text{-}\Delta_1^0(X) = \kappa\text{-}\Delta_{1+\beta}^0(X)$, and hence X , being Hausdorff, would be discrete because $\kappa\text{-}\Pi_1^0(X) \subseteq \kappa\text{-}\Delta_{1+\beta}^0(X) = \kappa\text{-}\Delta_1^0(X)$. Thus we can suppose that $\alpha \geq 1$. We prove the contrapositive, so assume $\text{ord}_{\kappa}(X) > 1 + \alpha$ and fix an arbitrary $\beta > \alpha$. We distinguish two cases. If $1 + \beta < \text{ord}_{\kappa}(X)$, then $\kappa\text{-}\Delta_{1+\beta}^0(X) \setminus (\kappa\text{-}\Sigma_{1+\alpha}^0(X) \cup \kappa\text{-}\Pi_{1+\alpha}^0(X)) \neq \emptyset$ by Proposition 3.4.5. Assume now that $\text{ord}_{\kappa}(X) \leq 1 + \beta$. Since $2 \leq 1 + \alpha < \text{ord}_{\kappa}(X)$, then Proposition 3.4.6(1) applies and $\kappa\text{-}\Delta_{\text{ord}_{\kappa}(X)}^0(X) \setminus (\kappa\text{-}\Sigma_{1+\alpha}^0(X) \cup \kappa\text{-}\Pi_{1+\alpha}^0(X)) \neq \emptyset$. Since $\kappa\text{-}\Delta_{\text{ord}_{\kappa}(X)}^0(X) \subseteq \kappa\text{-}\Delta_{1+\beta}^0(X)$ by case assumption, we are done.

Finally, let α be even. The implication (2) \Rightarrow (9) is obvious, so to conclude the proof it is enough to show that (9) \Rightarrow (1). Since α is even, $\kappa\text{-}\Sigma_{1+\alpha}^0(X)$ is closed under unions of size κ by Proposition 3.4.4(1), and hence it is a κ^+ -algebra by (9). Since $\kappa\text{-}\Sigma_1^0(X) \subseteq \kappa\text{-}\Sigma_{1+\alpha}^0(X)$, we get in particular that $\kappa\text{-}\mathbf{Bor}(X) \subseteq \kappa\text{-}\Sigma_{1+\alpha}^0(X)$, and hence $\text{ord}_{\kappa}(X) \leq 1 + \alpha$. \square

We can now come back to the closure properties of the pointclasses in the κ -Borel hierarchy (Proposition 3.4.4), and prove their optimality.

Proposition 3.4.8. *Suppose that κ is singular, and $\kappa\text{-}\Sigma_1^0(X) \subseteq \text{cof}(\kappa)^+\text{-}\Sigma_2^0(X)$. Let α be such that $1 + \alpha < \text{ord}_{\kappa}(X)$, and let once again $\hat{\alpha} = \text{cof}(\alpha)$ if α is limit, and $\hat{\alpha} = \text{cof}(\kappa)$ otherwise.*

- (1) If α is even, then $\kappa\text{-}\Sigma_{1+\alpha}^0(X)$ is not closed under complements or intersections of size $\hat{\alpha}$, $\kappa\text{-}\Pi_{1+\alpha}^0(X)$ is not closed under complements or unions of size $\hat{\alpha}$. If furthermore $\alpha > 0$, then $\kappa\text{-}\Delta_{1+\alpha}^0(X)$ is not closed under unions or intersections of size $\hat{\alpha}$, and the same is true for $\alpha = 0$ if $X \subseteq {}^{\kappa}2$.

- (2) If α is odd, then $\kappa\text{-}\Sigma_{1+\alpha}^0(X) = \kappa\text{-}\Pi_{1+\alpha}^0(X) = \kappa\text{-}\Delta_{1+\alpha}^0(X)$ is not closed under unions or intersections of size $\text{cof}(\kappa)$.

As already observed, $\kappa\text{-}\Delta_1^0(X)$ might instead be closed under arbitrary unions and intersections, depending on the space X .

Proof. We begin with the case when α is even. Suppose first that $\alpha = 0$. If $\kappa\text{-}\Sigma_1^0(X)$ were closed under intersections of size $\text{cof}(\kappa)$ (equivalently: if $\kappa\text{-}\Pi_1^0(X)$ were closed under unions of size $\text{cof}(\kappa)$), then every singleton $\{x\}$ would also be open, as our assumptions entail that $\kappa\text{-}\Pi_1^0(X) \subseteq \text{cof}(\kappa)^+\text{-}\Pi_2^0(X)$ and singletons are closed. But this means that X is discrete, contradicting the assumption $\text{ord}_\kappa(X) > 1$. As for $\kappa\text{-}\Delta_1^0(X)$, if it were closed under unions (or equivalently, intersections) of size $\text{cof}(\kappa)$ and $X \subseteq {}^\kappa 2$, then by Fact 3.2.11 we would have $\kappa\text{-}\Sigma_1^0(X) \subseteq \kappa\text{-}\Delta_1^0(X)$, and hence X would again be discrete, against $\text{ord}_\kappa(X) > 1$.

Suppose now that $\alpha \geq 2$ is even. Arguing as in the proof of Proposition 3.3.2, the fact that $\kappa\text{-}\Pi_{1+\alpha}^0(X)$ is closed under intersections of size κ by Proposition 3.4.4(1) yields that it is enough to show that $\kappa\text{-}\Delta_{1+\alpha}^0(X)$ is not closed under intersections of size $\hat{\alpha}$. Suppose towards a contradiction that this is not true. Since $\bigcup_{\beta < \alpha} \kappa\text{-}\Sigma_{1+\beta}^0(X) \subseteq \kappa\text{-}\Delta_{1+\alpha}^0(X)$, applying Theorem 3.4.1(1) we get that

$$\kappa\text{-}\Pi_{1+\alpha}^0(X) = \left(\bigcup_{\beta < \alpha} \kappa\text{-}\Sigma_{1+\alpha}^0(X) \right)_{\delta_{\hat{\alpha}}} \subseteq \kappa\text{-}\Delta_{1+\alpha}^0(X).$$

This easily leads to $\kappa\text{-}\Sigma_{1+\alpha}^0(X) = \kappa\text{-}\Pi_{1+\alpha}^0(X)$, which contradicts Proposition 3.4.7 since we assumed $\text{ord}_\kappa(X) > 1 + \alpha$.

Suppose now that α is odd. It is enough to consider the case of unions. Since $\alpha + 1$ is an even successor ordinal, by Theorem 3.4.1(1) we get that if $\alpha\text{-}\Pi_1^0(\kappa)$ were closed under unions of size $\text{cof}(\kappa)$ then $\kappa\text{-}\Sigma_{1+\alpha+1}^0(X) = (\kappa\text{-}\Pi_{1+\alpha}^0(X))_{\sigma_{\text{cof}(\kappa)}} = \kappa\text{-}\Pi_{1+\alpha}^0(X)$, which, by $\text{ord}_\kappa(X) > 1 + \alpha$, contradicts again Proposition 3.4.7. \square

From Proposition 3.4.8 and Theorem 3.4.1(1) we can derive the optimality of the closure properties of the pointclasses in the κ^+ -Borel hierarchy when κ is singular, thus complementing Proposition 3.3.2.

Corollary 3.4.9. *Suppose that κ is singular, and $\kappa\text{-}\Sigma_1^0(X) \subseteq \text{cof}(\kappa)^+\text{-}\Sigma_2^0(X)$. For $1 \leq \alpha < \text{ord}_{\kappa^+}(X)$, let $\hat{\alpha} = \text{cof}(\kappa)$ if α is a successor ordinal, and $\hat{\alpha} = \text{cof}(\alpha)$ if α is limit. Then,*

- (1) $\kappa^+\text{-}\Sigma_\alpha^0(X)$ is not closed under complements and intersections of size $\hat{\alpha}$;
- (2) $\kappa^+\text{-}\Pi_\alpha^0(X)$ is not closed under complements and unions of size $\hat{\alpha}$;
- (3) if $\alpha > 0$, then $\kappa^+\text{-}\Delta_\alpha^0(X)$ is not closed under unions or intersections of size $\hat{\alpha}$, and the same is true for $\alpha = 0$ if $X \subseteq {}^\kappa 2$.

We conclude this section by briefly discussing the existence of universal and complete sets for the various pointclasses in the κ -Borel hierarchy. The following result is the counterpart of Proposition 3.3.7.

Proposition 3.4.10. *Suppose that κ is singular, and $\kappa\text{-}\Sigma_1^0(X) \subseteq \text{cof}(\kappa)^+\text{-}\Sigma_2^0(X)$. Let $\alpha < \kappa^+$.*

- (1) *If α is even, then there are κ^2 -universal sets for both $\kappa\text{-}\Sigma_{1+\alpha}^0(X)$ and $\kappa\text{-}\Pi_{1+\alpha}^0(X)$. Therefore there exists subsets of κ^2 that are κ -complete for $\kappa\text{-}\Sigma_{1+\alpha}^0$ and $\kappa\text{-}\Pi_{1+\alpha}^0$. In contrast, there is no κ^2 -universal set for $\kappa\text{-}\Delta_{1+\alpha}^0(\kappa^2)$ or $\kappa\text{-}\mathbf{Bor}(\kappa^2)$.*
- (2) *If α is odd, then there are neither κ^2 -universal sets for $\kappa\text{-}\Sigma_{1+\alpha}^0(\kappa^2)$ nor subsets of κ^2 that are κ -complete for $\kappa\text{-}\Sigma_{1+\alpha}^0$, and the same for $\kappa\text{-}\Pi_{1+\alpha}^0$ and $\kappa\text{-}\Delta_{1+\alpha}^0$.*

Moreover, κ^2 can be replaced by X in all the above statements as soon as κ^2 embeds into X .

Proof. Part (1) follows from Theorem 3.4.1(1) and Proposition 3.3.7, so let us move to part (2).

Assume that α is odd, so that $\kappa\text{-}\Sigma_{1+\alpha}^0(\kappa^2) = \kappa\text{-}\Pi_{1+\alpha}^0(\kappa^2) = \kappa\text{-}\Delta_{1+\alpha}^0(\kappa^2)$ by Theorem 3.4.1(2). Then, none of those classes can have a κ^2 -universal set by Lemma 2.6.3.

Next, let us consider complete sets. Let β be such that $\alpha = \beta + 1$. Then, $\kappa\text{-}\Sigma_{1+\alpha}^0(X) = (\kappa\text{-}\Pi_{1+\beta}^0(X))_{\sigma < \kappa} = \bigcup_{\lambda < \kappa} (\kappa\text{-}\Pi_{1+\beta}^0(X))_{\sigma_\lambda}$.

Claim 3.4.10.1. For every $\lambda < \kappa$, the class $(\kappa\text{-}\Pi_{1+\beta}^0(X))_{\sigma_\lambda}$ has a κ^2 -universal set.

Indeed, β is even and hence, by part (1), has a κ^2 -universal set. Applying Lemma 3.2.20(2), we easily get the desired result.

Towards a contradiction, assume now that $A \subseteq \kappa^2$ is κ -complete for $\kappa\text{-}\Sigma_{1+\alpha}^0$. Let $\lambda < \kappa$ be such that $A \in (\kappa\text{-}\Pi_{1+\beta}^0(\kappa^2))_{\sigma_\lambda}$. Since the latter is closed under continuous preimages, we get that $\kappa\text{-}\Sigma_{1+\alpha}^0(\kappa^2) = (\kappa\text{-}\Pi_{1+\beta}^0(\kappa^2))_{\sigma_\lambda}$. Since $\kappa\text{-}\Sigma_{1+\alpha}^0$ is a boldface pointclass that is selfdual on κ^2 , while $(\kappa\text{-}\Pi_{1+\beta}^0(\kappa^2))_{\sigma_\lambda}$ has a κ^2 -universal set by Claim 3.4.10.1, this contradicts Lemma 2.6.3.

Finally, the additional part can be inferred arguing as in the second part of the proof of Proposition 3.3.7. \square

The proof of Proposition 3.4.10 also gives the interesting information that if κ^2 embeds into X , then for every odd ordinal $\alpha = \beta + 1 < \kappa^+$ and every cardinal $\lambda < \kappa$,

$$(\kappa\text{-}\Pi_{1+\beta}^0(X))_{\sigma_\lambda} \subsetneq \kappa\text{-}\Sigma_{1+\alpha}^0(X).$$

In other words: the class $\kappa\text{-}\Sigma_{1+\alpha}^0(X)$ cannot be obtained by considering only unions of a *fixed* cardinality below κ . This is in stark contrast with what happens when α is even: by Theorem 3.4.1(1), in that case it is enough to consider unions of a fixed size $\hat{\alpha} < \kappa$.

3.5 κ -Thin κ^+ -Borel sets in $<\kappa$ -closed forcing extensions

The initial motivation for the results in this section is Theorem 3.3.9, which, in full alignment with classical results, provides a sufficient criterion for the non-collapse of the κ^+ -Borel hierarchy: the existence of a κ -perfect subset. While this sufficient condition is also necessary in the classical case, it fails in the generalized context due to the breakdown of the κ -Perfect Set Property in generalized descriptive set theory. This failure leaves open the following question: suppose X is a space of weight at most κ such that its κ^+ -Borel hierarchy does not collapse. Is there necessarily a κ^+ -Borel embedding of ${}^\kappa 2$ into X ?

We will show that when κ is a regular cardinal, this is not the case. From this point onward, the regularity of κ will always be tacitly assumed. Furthermore, we will restrict our attention to subspaces of ${}^\kappa \kappa$, for two reasons. First, providing examples among subspaces of ${}^\kappa \kappa$ is more interesting than simply constructing a topological space with no additional structure. Also, this allows to deal with matters of definability and provide results that hold in general for a wide class of subspaces of ${}^\kappa \kappa$. Nevertheless, notice that these results can be further extended to other T_0 topological spaces of weight at most κ anyway, thanks to Proposition 3.2.24.

In the classical setting, a set has the Perfect Set Property if either it is countable, or it contains a copy of the Cantor space ${}^\omega 2$. Due to the compactness of the latter and the fact that uncountable analytic sets contains a closed set homeomorphic to ${}^\omega 2$, there are several equivalent reformulation of the second alternative in Perfect Set Property which range from “there is a Borel injection of ${}^\omega 2$ into the set” to “the set contains a closed set homeomorphic to ${}^\omega 2$ ”. In general, when moving to the generalized context the equivalence among the analogues of the above variations ceases to exist: however, in Corollary 3.5.10 we are going to show that it survives if we restrict the attention to κ^+ -Borel sets.

Throughout this section, we will be working with the following version of κ -perfectness.

Definition 3.5.1. We say that a set $P \subseteq {}^\kappa \kappa$ is κ -**perfect** if it is closed and homeomorphic to the generalized Cantor space ${}^\kappa 2$, and call a set κ -**thin** if it has no κ -perfect subset.

Accordingly, we will say that a set $A \subseteq {}^\kappa \kappa$ has the κ -**Perfect Set Property** if either $|A| \leq \kappa$, or A contains a κ -perfect set. Moreover, we say that a boldface pointclass $\mathbf{\Gamma}$ has the κ -Perfect Set Property if there are no κ -thin sets in $\mathbf{\Gamma}({}^\kappa \kappa)$ of size greater than κ , that is, if every $A \in \mathbf{\Gamma}({}^\kappa \kappa)$ has the κ -Perfect Set Property. This is the strongest form of the property that can be formulated in purely topological terms.

3.5.1 Robust codes for κ^+ -Borel sets

Recall that a tree is a set of sequences closed under initial segments. Given a set A , $\mu \in \text{Card}$, and a tree $T \subseteq <^\mu A$, we define the body of T as

$$[T]_A^\mu = \{x \in {}^\mu A \mid \forall \alpha < \mu (x \upharpoonright \alpha \in T)\}.$$

When μ and/or A are clear from the context, we suppress them from the notation and write $[T]$ instead of $[T]_A^\mu$. In most cases, we will have either $\mu = \kappa$ or $\mu = \omega$. If T is a tree, then for every $t \in T$ we denote by $\text{Succ}_T(t)$ the set of immediate successors of t in T ; when clear from the context, we may suppress the subscript T .

A tree $S \subseteq {}^{<\omega}A$ is called well-founded if $[S] = \emptyset$. Every well-founded tree S naturally carries a rank function, defined as $\text{rk}_S(s) = 0$ if $s \in S$ is a leaf (i.e. $\text{Succ}_S(s) = \emptyset$), and $\text{rk}_S(s) = \sup\{\text{rk}_S(t) + 1 \mid t \in \text{Succ}_S(s)\}$ otherwise. Denote by S^α (respectively: $S^{<\alpha}$, or $S^{>\alpha}$) the set of nodes $s \in S$ such that $\text{rk}_S(s) = \alpha$ (respectively: $\text{rk}_S(s) < \alpha$, or $\text{rk}_S(s) > \alpha$). In particular, S^0 is the set of leaves of the tree S .

In classical descriptive set theory, Borel codes provide a way of interpreting a Borel set in different transitive models of set theory. The same is true in the generalized setting. Let $X \subseteq {}^\kappa\kappa$, and let $S \subseteq {}^{<\omega}A$ be a well-founded tree with $|A| \leq \kappa$. Given a function $f: S^0 \rightarrow {}^{<\kappa}\kappa$, we may assign to each $s \in S$ its **interpretation** $\mathcal{I}_{S,X}^f(s)$ as a κ^+ -Borel subset of X via

$$\mathcal{I}_{S,X}^f(s) = \begin{cases} \mathbf{N}_{f(s)} \cap X & \text{if } s \in T^0 \\ \bigcap_{t \in \text{Succ}_S(s)} X \setminus \mathcal{I}_{S,X}^f(t) & \text{otherwise.} \end{cases}$$

The space X will always be clear from the context, and it will systematically be omitted from the notation; when reasonable, also the reference to the tree S will be dropped, resulting in the lighter notations $\mathcal{I}_S^f(s)$ or $\mathcal{I}^f(s)$. Note that if $\text{rk}_S(s) = \beta > 0$, then $\mathcal{I}_S^f(s) \in \kappa^+ \text{-}\mathbf{\Pi}_\beta^0(X)$, while if $\text{rk}_S(s) = 0$, then $\mathcal{I}_S^f(s) \in \kappa^+ \text{-}\mathbf{\Delta}_1^0(X)$. Conversely, if $B \in \kappa^+ \text{-}\mathbf{\Pi}_\beta^0(X)$ for some $1 \leq \beta < \kappa^+$, then there is $\langle S, f \rangle$ such that $\text{rk}_S(\emptyset) = \beta$ and $B = \mathcal{I}_S^f(\emptyset)$.

We call the pair $\langle S, f \rangle$ a κ^+ -**Borel code** for the set $B = \mathcal{I}_S^f(\emptyset)$; in view of the above discussion, when $\text{rk}_S(\emptyset) = \beta$ we also say that $\langle S, f \rangle$ is a $\kappa^+ \text{-}\mathbf{\Pi}_\beta^0$ **code** for B . We would like to emphasize that although κ^+ -Borel sets can generally be coded in many different ways, and these different codings are equivalent for most practical purposes, it is convenient for this and the next section that we work with this specific coding.

If $\langle S, f \rangle$ is a κ^+ -Borel code for $B = \mathcal{I}_S^f(\emptyset) \in \kappa^+ \text{-}\mathbf{Bor}(X)$ and V' is a transitive model of ZFC containing $\langle S, f \rangle$, then B can be re-interpreted in V' giving rise to the κ^+ -Borel set $B^{V'} = (\mathcal{I}_S^f(\emptyset))^{V'}$. (The superscript V' will be dropped from the notation if clear from the context.) In particular, if $B \in \kappa^+ \text{-}\mathbf{\Pi}_\beta^0(X)$ and we choose a κ^+ -Borel code $\langle S, f \rangle$ for B such that $\text{rk}_S(\emptyset) = \beta$, then $B^{V'}$ is $\kappa^+ \text{-}\mathbf{\Pi}_\beta^0(X)$ in V' as well. Moreover, by $\kappa \text{-}\mathbf{\Sigma}_1^1$ -absoluteness (Lemma 3.5.3), the choice of the κ^+ -Borel code $\langle S, f \rangle$ for B is irrelevant whenever V' is a generic extension by a $<\kappa$ -closed forcing, so that in such a situation one can refer to $B^{V'}$ without having to specify which κ^+ -Borel code is used to re-interpret B in V' .

In this section we are mostly concerned with κ^+ -Borel sets that are κ -thin. A convenient way to control such property is to view κ^+ -Borel sets as Suslin_κ sets, i.e. as projections of the bodies $[T]$ of trees of the form $T \subseteq {}^{<\kappa}(\kappa \times W)$, as in this case a close connection can be made between a strengthening of the κ -Perfect Set Property and fresh elements in $<\kappa$ -closed forcing extensions. This connection is provided by Theorem 3.5.8, which is due to Philipp Lücke

[Lüc12] and proved in the restricted context of $\kappa\text{-}\Sigma_1^1$ (i.e. κ -analytic) sets. However, the proof given by Lücke easily extends to all Suslin_κ sets, yielding the slight generalization stated here. The proof ultimately relies on some classical arguments, for which we refer the reader to [Kun83, Lemma 3.4].

To introduce some notation, suppose that $T \subseteq {}^{<\kappa}(\kappa \times W)$ is a tree, where W is any set. Let $\pi[T]$ denote the projection of the body of T onto its first coordinate, that is:

$$\pi[T] = \{x \in {}^\kappa\kappa \mid \exists y \in {}^\kappa W ((x, y) \in [T])\}.$$

Likewise, let $\pi(s, w) = s$ for every pair $(s, w) \in T$; clearly, $\pi(T) \subseteq {}^{<\kappa}\kappa$ is still a tree.

Definition 3.5.2. Given sets W and $X \subseteq {}^\kappa\kappa$, we say that X is W -**Suslin** $_\kappa$ if there exists a tree $T \subseteq {}^{<\kappa}(\kappa \times W)$ such that $X = \pi[T]$.

We often do not care about the specifics of the second coordinate in the tree; in that case, we simply write Suslin_κ for sets that are W - Suslin_κ for some W .¹¹ The most prominent members of this class are the κ -analytic (or $\kappa\text{-}\Sigma_1^1$) sets, which are obtained setting $W = \kappa$.

Suslin_κ sets $X = \pi[T]$ can again be naturally re-interpreted in any transitive model V' of ZFC containing T as $(\pi[T])^{V'}$; however, we should be careful to remind ourselves that this interpretation is far from being robust, and without any absoluteness results it depends heavily on the chosen tree T . Indeed, in the generalized setting we lack the rich array of absoluteness results we are accustomed to when studying the real line. For example, there is no hope for even $\kappa\text{-}\Sigma_1^1$ -absoluteness to hold between any two transitive models of ZFC, since for a tree $T \subseteq {}^{<\kappa}\kappa$ the statement “ T has a branch of length κ ” will in general not be absolute. On the bright side, the next lemma shows that at least for $<\kappa$ -closed forcing extensions, $\kappa\text{-}\Sigma_1^1$ -absoluteness and more generally, Suslin_κ -absoluteness, holds. The lemma is well-known and likely folklore, a proof can for example also be found in [FKK16, Lemma 2.7].

Lemma 3.5.3. *Let \mathbb{P} be a $<\kappa$ -closed forcing notion, G a \mathbb{P} -generic filter, $W \in V$ and $T \subseteq {}^{<\kappa}W$ a tree. If $V[G] \models [T] \neq \emptyset$, then $V \models [T] \neq \emptyset$.*

Recall that a map φ between two trees T and T' is **order preserving** if $s \subseteq t \Rightarrow \varphi(s) \subseteq \varphi(t)$ for every $s, t \in T$. Similarly, φ is **strict order preserving** if $s \subsetneq t \Rightarrow \varphi(s) \subsetneq \varphi(t)$ for every $s, t \in T$, and it **preserves incompatibility** if $s \perp t \Rightarrow \varphi(s) \perp \varphi(t)$ for all $s, t \in T$. If φ is strict order preserving and preserves also incompatibility, then it is called **order embedding**; equivalently, φ is an order embedding if it is injective and $s \subsetneq t \Leftrightarrow \varphi(s) \subsetneq \varphi(t)$, for all $s, t \in T$. Similar terminology will be used also for *partial* maps between T and T' .

It $T, T' \subseteq {}^{<\mu}A$, to every strict order preserving function $\varphi: T \rightarrow T'$ we can associate a function $f_\varphi: [T] \rightarrow [T']$ by setting $f_\varphi(x) = \bigcup_{\alpha < \mu} \varphi(x \upharpoonright \alpha)$, for every $x \in [T]$. Notice that for every strict order preserving function $\varphi: T \rightarrow T'$ there is a *continuous*¹² strict order preserving function $\varphi': T \rightarrow T'$ such that $f_\varphi = f_{\varphi'}$; indeed, it is enough to define $\varphi'(s) = \bigcup_{t \subsetneq s} \varphi(t)$ for all $s \in T$ of limit

¹¹ Suslin_κ sets are not to be confused for the established notion of a κ -Suslin set, i.e. the projection of a tree on ${}^{<\omega}(\omega \times \kappa)$.

¹²By continuous, we mean that if $s \in T$ has limit length δ , then $\varphi'(s) = \bigcup_{\alpha < \delta} \varphi(s \upharpoonright \alpha)$.

length, and $\varphi'(s) = \varphi(s)$ for all other $s \in T$. Also, if φ is an order embedding, then f_φ is a topological embedding, i.e. a homeomorphism onto its image.

Following Lücke, we introduce the following strengthening of the κ -Perfect Set Property for Suslin_κ sets.

Definition 3.5.4. Let $T \subseteq {}^{<\kappa}(\kappa \times W)$ be a tree. We say $\varphi: {}^{<\kappa}2 \rightarrow T$ is an **\exists -perfect order embedding** if for each $s, t \in {}^{<\kappa}2$:

- φ is a strict order preserving map;
- $\pi \circ \varphi$ preserves incompatibility.

Lemma 3.5.5. Let $T \subseteq {}^{<\kappa}(\kappa \times W)$ be a tree, and let $\varphi: {}^{<\kappa}2 \rightarrow T$ be an order preserving function. The following are equivalent:

- (1) φ is \exists -perfect order embedding;
- (2) $\pi \circ \varphi$ is an order embedding of ${}^{<\kappa}2$ into $\pi(T)$;
- (3) both φ and $\pi \upharpoonright \varphi({}^{<\kappa}2)$ are order embeddings.

Proof. (1) \Rightarrow (2). Obvious, since π is strict order preserving.

(2) \Rightarrow (3). Since $\pi \circ \varphi$ is, in particular, injective, so are φ and $\pi \upharpoonright \varphi({}^{<\kappa}2)$. Thus, since φ is order preserving by assumption, it also preserves the strict order. Now, π is always order preserving, so in order for $\pi \circ \varphi$ to preserve incompatibility, φ needs to preserve incompatibility as well. Thus φ is an order embedding. It follows that $\pi \upharpoonright \varphi({}^{<\kappa}2)$ is an order embedding as well (the argument is similar).

(3) \Rightarrow (1). If φ and $\pi \upharpoonright \varphi({}^{<\kappa}2)$ are both order embeddings, so is $\pi \circ \varphi$. In particular, φ is strict order preserving, and $\pi \circ \varphi$ preserves incompatibility. \square

Lemma 3.5.6. Let $T, T' \subseteq {}^{<\kappa}A$ for some A . Let $\varphi: T \rightarrow T'$ be an order embedding and suppose that T is $<\kappa$ -splitting, i.e. that $|\text{Succ}_T(s)| < \kappa$ for all $s \in T$. Then, $f_\varphi([T])$ is closed in ${}^\kappa A$ (with respect to the bounded topology).

Proof. We need to prove that ${}^\kappa A \setminus f_\varphi([T])$ is open, so pick any $x \in {}^\kappa A \setminus f_\varphi([T])$. Let $S = \{s \in T \mid \varphi(s) \subseteq x\}$. Then, S is a chain because φ preserves incompatibility, therefore we can set $\bar{s} = \bigcup S$. Since φ is strict order preserving, we need to have $\text{lh}(\bar{s}) < \kappa$, as otherwise $f_\varphi(\bar{s}) = x$, against our choice for x . We distinguish two cases.

If $\bar{s} \notin T$, then for all $y \in [T]$ there is $\alpha < \text{lh}(\bar{s})$ such that $y \upharpoonright \alpha \perp \bar{s} \upharpoonright \alpha$, therefore $N_{x \upharpoonright \beta} \cap f_\varphi([T]) = \emptyset$ for $\beta = \sup\{\text{lh}(\varphi(s)) \mid s \in S\}$.

Suppose now that $\bar{s} \in T$. For each $t \in \text{Succ}_T(\bar{s})$ there is $\beta_t < \kappa$ such that $\varphi(t) \upharpoonright \beta_t \perp x \upharpoonright \beta_t$, by the definition of \bar{s} . Since $|\text{Succ}_T(\bar{s})| < \kappa$, by regularity of κ we have $\beta = \sup\{\beta_t \mid t \in \text{Succ}_T(\bar{s})\} < \kappa$. Then, $N_{x \upharpoonright \beta} \cap f_\varphi([T]) \neq \emptyset$. \square

Corollary 3.5.7. Let $T \subseteq {}^{<\kappa}(\kappa \times W)$ be a tree. If there is an \exists -perfect order embedding $\varphi: {}^{<\kappa}2 \rightarrow T$, then $\pi[T]$ contains a κ -perfect subset.

Proof. By Lemma 3.5.5, the map $\pi \circ \varphi: {}^{<\kappa}2 \rightarrow \pi(T) \subseteq {}^{<\kappa}\kappa$ is an order embedding. Since ${}^{<\kappa}2$ is obviously $<\kappa$ -splitting, we get that $P = f_{\pi \circ \varphi}({}^{<\kappa}2)$ is closed in ${}^\kappa\kappa$ by Lemma 3.5.6. Moreover, $f_{\pi \circ \varphi}$ witnesses that P is homeomorphic to ${}^{<\kappa}2$. Therefore it is enough to prove that the κ -perfect set P is contained in $\pi[T]$. But this easily follows from $f_\varphi({}^{<\kappa}2) \subseteq [T]$ and the fact that $f_{\pi \circ \varphi} = f_\pi \circ f_\varphi$. \square

As mentioned, Lücke observed that there is a tight connection between the possibility of adding new elements to a set of the form $\pi[T]$ via a $<\kappa$ -closed forcing notion, and the fact that there is an \exists -perfect order embedding into T (in which case $\pi(T)$ is not κ -thin by Corollary 3.5.7).

Theorem 3.5.8 ([Lüc12, Lemma 7.6]). *Let W be a set and $T \subseteq {}^{<\kappa}(\kappa \times W)$ a tree. Then, the following are equivalent:*

- (1) *Every $<\kappa$ -closed forcing which adds a new element of ${}^{\kappa}2$ adds a new element to $\pi[T]$.*
- (2) *There exists a $<\kappa$ -closed forcing which adds a new element to $\pi[T]$.*
- (3) *There is an \exists -perfect order embedding into T .*

When restricting the attention to κ^+ -Borel sets, there is a tension between two ways of coding them. On the one hand, every κ^+ -Borel set B is κ -analytic, and thus it is of the form $B = \pi[T]$ for some tree $T \subseteq {}^{<\kappa}(\kappa \times \kappa)$; this point of view allows us to exploit Theorem 3.5.8 in order to determine whether B contains a κ -perfect subset. However, this way of coding κ^+ -Borel sets is not very stable. Without a large enough fragment of absoluteness, we cannot even ensure that for two trees $T, T' \subseteq {}^{<\kappa}(\kappa \times \kappa)$ with $\pi[T] = \pi[T']$, the same equality continues to hold in $<\kappa$ -closed forcing extensions (because it is a κ - Π_2^1 statement). On the other hand, κ - Σ_1^1 -absoluteness (Lemma 3.5.3) guarantees that the statement “ $\langle S_1, f_1 \rangle$ and $\langle S_2, f_2 \rangle$ code the same κ^+ -Borel set” is absolute for $<\kappa$ -closed forcing extensions, so it is worth stating Theorem 3.5.8 in a form that fits the setting of κ^+ -Borel codes (Corollary 3.5.10). To this aim, one can exploit the following construction, which is borrowed from Lücke and Schlicht’s proof of the following result (see [LS15, Lemma 1.11]). For the reader’s convenience, we recount the proof.

Lemma 3.5.9. *For every (code for a) κ^+ -Borel set $B \subseteq {}^{\kappa}\kappa$ there exists a tree $T \subseteq {}^{<\kappa}(\kappa \times \kappa)$ such that $B = \pi[T]$, and moreover*

$$B^{V[G]} = (\pi[T])^{V[G]}$$

holds in all $<\kappa$ -closed forcing extensions $V[G]$.

Proof. Let $\langle S, f \rangle$ be the chosen code for B , and fix an enumeration $e: S \rightarrow \kappa$ such that $s \subsetneq t \implies e(s) < e(t)$. Let us call a pair $(y, z) \in {}^S 2 \times {}^S S$ **correct** if the following conditions hold:

- (1) $y(\emptyset) = 1$, and for each $s \in S^{>0}$ we have $y(s) = 0 \iff \exists t \in \text{Succ}_S(s) (y(t) = 1)$.
- (2) for each $s \in s \in S^{>0}$ with $y(s) = 0$, we have $z(s) \in \text{Succ}_S(s)$ and $y(z(s)) = 1$; otherwise, $z(s) = \emptyset$.¹³

¹³It bears mentioning that this diverges slightly from Lücke and Schlicht’s proof, which even implies that the projection $\pi \upharpoonright [T]$ can be chosen to be injective. However, this is not relevant for our arguments.

Consider now the set

$$C = \left\{ (x, y, z) \in {}^\kappa\kappa \times {}^S2 \times {}^SS \mid \begin{array}{l} (y, z) \text{ is correct} \wedge \\ \forall s \in S^0 (y(s) = 1 \Leftrightarrow x \in \mathbf{N}_{f(s)}) \end{array} \right\}. \quad (3.5.1)$$

If we identify S2 and SS with ${}^\kappa 2$ and ${}^\kappa\kappa$, respectively, via the bijection e , one can construe C as a closed subset of ${}^\kappa(\kappa \times (2 \times \kappa))$, and hence also as a closed subset of ${}^\kappa(\kappa \times \kappa)$. Indeed, condition (2) is clearly closed. As for (1), as stated it seems more complicated because of the implication $y(s) = 0 \Rightarrow \exists t \in \text{Succ}_S(s) (y(t) = 1)$: however, it is not hard to see that it is closed *relatively to the closed set determined by (2)* because z can be used to extract a witness of the existential quantification appearing in the problematic implication.

The first coordinate in a correct pair (y, z) models the interpretation function \mathcal{I}_S^f associated to the κ^+ -Borel code $\langle S, f \rangle$, hence $B = \pi[T]$ where $T \subseteq {}^{<\kappa}(\kappa \times \kappa)$ is a tree such that $C = [T]$. Moreover, the equality defining C in (3.5.1) is a $\kappa\text{-}\mathbf{\Pi}_1^1$ statement, therefore $C^{V[G]} = [T]^{V[G]}$ is still the set of triples such that (y, z) is correct and $\forall s \in S^0 (y(s) = 1 \Leftrightarrow x \in \mathbf{N}_{f(s)})$ (in the sense of $V[G]$). But this entails $B^{V[G]} = (\pi[T])^{V[G]}$, as desired. \square

The codes for κ^+ -Borel sets can also be used to canonically re-interpret κ^+ -Borel measurable functions in $<\kappa$ -closed forcing extentions. Indeed, suppose that $T \subseteq {}^{<\kappa}\kappa$ is a tree, and that $f: [T] \rightarrow {}^\kappa\kappa$ is κ^+ -Borel measurable. Suppose that \mathbb{P} is a $<\kappa$ -closed forcing notion, and that G is \mathbb{P} -generic over V . For each $t \in {}^{<\kappa}\kappa$ fix a κ^+ -Borel code $\langle S_t, f_t \rangle$ for $B_t = f^{-1}(\mathbf{N}_t)$, so that the latter can be re-interpreted in $V[G]$ as $B_t^{V[G]} = (\mathcal{I}_{S_t}^{f_t}(\emptyset))^{V[G]}$. (By $\kappa\text{-}\Sigma_1^1$ -absoluteness, the actual choice of such codes is irrelevant.) Working in $V[G]$, we can then canonically define a function $f^{V[G]}: ([T])^{V[G]} \rightarrow ({}^\kappa\kappa)^{V[G]}$ by setting for every $x \in ([T])^{V[G]}$

$$f^{V[G]}(x) = \bigcup \left\{ t \in {}^{<\kappa}\kappa \mid x \in (B_t)^{V[G]} \right\}.$$

Since the statement (the quantifier $\exists!$ means “there exists exactly one”)

$$\forall x \in [T] \forall i < \kappa \exists! t \in {}^i\kappa (x \in B_t)$$

is $\kappa\text{-}\mathbf{\Pi}_1^1$, hence absolute, $f^{V[G]}$ is a well-defined function. We also note that injectivity of f is $\kappa\text{-}\mathbf{\Pi}_1^1$.

Corollary 3.5.10. *For every (code for a) κ^+ -Borel sets $B \subseteq {}^\kappa\kappa$, the following are equivalent:*

- (1) *Every $<\kappa$ -closed forcing which adds a new element of ${}^\kappa 2$ adds a new element to B .*
- (2) *There exists a $<\kappa$ -closed forcing which adds a new element to B .*
- (3) *B contains a κ -perfect subset.*
- (4) *There is κ^+ -Borel injection of ${}^\kappa 2$ into B .*

Proof. The implications (1) \Rightarrow (2) and (3) \Rightarrow (4) are trivial. For the implication (2) \Rightarrow (3), apply Theorem 3.5.8 to the tree T constructed in Lemma 3.5.9 to find an \exists -perfect embedding into T . By Corollary 3.5.7, this implies (3).

It remains to show (4) \Rightarrow (1). Let $f: {}^\kappa 2 \rightarrow B$ be a κ^+ -Borel injection. Suppose that $V[G]$ is a forcing extension obtained via a $<\kappa$ -closed forcing, and that there is $\bar{x} \in ({}^\kappa 2)^{V[G]} \setminus ({}^\kappa 2)^V$. Since the formula $\forall x \in {}^\kappa 2 (f(x) \in B)$ is κ - Π_1^1 , by κ - Σ_1^1 -absoluteness we have $f^{V[G]}(\bar{x}) \in B^{V[G]}$. Likewise, $f^{V[G]}$ is injective because f is, therefore $f^{V[G]}(\bar{x}) \notin \text{ran}(f)$. Let $\bar{y} \in ({}^\kappa \kappa)^V$ be any element of $B \setminus \text{ran}(f)$. The statement “ $\forall x \in {}^\kappa 2 (f(x) \neq \bar{y})$ ” is κ - Π_1^1 , hence we have $f^{V[G]}(\bar{x}) \neq \bar{y}$. Summing up, $f^{V[G]}(\bar{x}) \in B^{V[G]} \setminus B^V$, as desired. \square

As a by-product, Corollary 3.5.10 shows that all common variants of the κ -Perfect Set Property coincide on κ^+ -Borel sets. Indeed, (3) corresponds to the strongest form of the property (namely, the one explicitly considered in this chapter), while (4) corresponds to the weakest one (see also [LMRS16, Lemma 2.9]).

From Corollary 3.5.10, the construction of a closed set $X \subseteq {}^\kappa \kappa$ with $|X| > \kappa$ and $\text{ord}_{\kappa^+}(X) = 2$ can be achieved by a straightforward generalization of standard arguments involving an iteration of Solovay’s almost disjoint forcing [JS70]. A generalization of almost disjoint forcing for uncountable κ has appeared in [Lüc12], essentially providing an answer to Problem 2 among other related results surrounding κ - Σ_1^1 - and κ - Δ_1^1 -definability.

Corollary 3.5.11. *Consistently, there exists a closed subset $X \subseteq {}^\kappa \kappa$ with $|X| > \kappa$ and $\text{ord}_{\kappa^+}(X) = 2$.*

We do not provide an explicit proof of Corollary 3.5.11 at this point, since the next section will develop a more general approach subsuming the use of almost disjoint forcing.

3.5.2 A κ -thin closed set of order κ^+

The rest of this section is devoted to the existence of κ -thin definable sets. Note that by the results of Schlicht [Sch17], it is consistent relative to an inaccessible above κ that every set $X \subseteq {}^\kappa \kappa$ definable from a κ -sequence of ordinals has the κ -Perfect Set Property.

Recall that a tree $T \subseteq <{}^\kappa \kappa$ is called **Kurepa**, if $||T|| > \kappa$ and $|\mathcal{L}_\alpha(T)| \leq |\alpha|$ for stationary many $\alpha < \kappa$, where $\mathcal{L}_\alpha(T) = T \cap {}^\alpha \kappa$ denotes the α -th level of the tree; it is called **Jech-Kunen**, if $\kappa < ||T|| < 2^\kappa$.

From the analysis of Kurepa trees on ω_1 it is known that the bodies of Kurepa and Jech-Kunen trees are typical examples of κ -thin closed sets [Kun83, Chapter VIII, Lemma 3.4]. The same is true in the generalized setting. For Jech-Kunen trees this is just a simple matter of cardinality; for Kurepa trees, it follows by an argument in [Lüc12, Section 7].

By a result of Hamkins, it is possible to turn the ground model generalized Baire space into a κ -thin closed set. Notice that κ -thin sets constructed in this way differ from the ones discussed above, as in the situation described in Theorem 3.5.12 the closed set ${}^\kappa \kappa \cap V$ is not the body of a Kurepa or a Jech-Kunen tree.

Theorem 3.5.12 ([Ham01]). *Let \mathbb{P} be a forcing notion such that $|\mathbb{P}| < \kappa$. Then, in any forcing extension $V[G]$ by \mathbb{P} , the set ${}^\kappa \kappa \cap V$ is a κ -thin closed set.*

Proof. Work in $V[G]$. By [Ham01, Key Lemma], the tree $T = (<^{\kappa\kappa})^V = <^{\kappa\kappa} \cap V$ does not gain any new branches in the extension of $V[G]$ by \mathbb{Q} , where \mathbb{Q} is such that $V[G] \models \text{“}\mathbb{Q} \text{ is a } < \kappa\text{-closed forcing notion”}$. By Theorem 3.5.8, this means that ${}^{\kappa\kappa} \cap V = [T]$ does not contain any κ -perfect subset. \square

We conclude the section by giving a general method for constructing κ -thin closed sets $X \subseteq {}^{\kappa\kappa}$ on which the κ^+ -Borel hierarchy does not collapse. This construction follows [Mil95, Theorem 14.3].

The following auxiliary lemma is a straightforward generalization of [KM83, Lemma 2].

Lemma 3.5.13. *Let $X \subseteq {}^{\kappa\kappa}$, \mathbb{P} be a forcing notion such that $|\mathbb{P}| \leq \kappa$, and \dot{J} be a \mathbb{P} -name for a set in $\kappa^+ \text{-}\mathbf{\Pi}_\alpha^0(X)$, for some $1 \leq \alpha < \kappa^+$. Then, for any $p \in \mathbb{P}$*

$$K_p(\dot{J}) = \{x \in X \mid p \Vdash \check{x} \in \dot{J}\}$$

is $\kappa^+ \text{-}\mathbf{\Pi}_\alpha^0(X)$ in the ground model.

Proof. We proceed by induction on α . For $\alpha = 1$, suppose that $(x_i)_{i < \kappa}$ is a sequence of points in $K_p(\dot{J})$ (for some $p \in \mathbb{P}$) which converges to $x \in X$. By assumption, \dot{J} is a name for a closed set, so $p \Vdash \check{x} \in \dot{J}$, hence $x \in K_p(\dot{J})$. This shows that $K_p(\dot{J})$ is closed.

In the inductive step $\alpha > 1$, assume the statement is true for all $1 \leq \beta < \alpha$, and that $\dot{J} = \bigcap_{i < \kappa} \dot{J}_i$, where \dot{J}_i is a name for a set in $\kappa^+ \text{-}\mathbf{\Sigma}_{\beta_i}^0(X)$, for $1 \leq \beta_i < \alpha$. For any $p \in \mathbb{P}$ and $x \in X$, we know that

$$p \Vdash \check{x} \in \dot{J} \Leftrightarrow \forall i < \kappa \forall q \leq p \left(q \nVdash \check{x} \notin \dot{J}_i \right).$$

By inductive hypothesis, the set

$$\{x \in X \mid q \nVdash \check{x} \notin \dot{J}_i\} = X \setminus K_q(X \setminus \dot{J}_i)$$

belongs to $\kappa^+ \text{-}\mathbf{\Sigma}_{\beta_i}^0(X)$, and since we assumed $|\mathbb{P}| \leq \kappa$, the set $K_p(\dot{J})$ can be written as an intersection of at most κ -many sets from $\bigcup_{1 \leq \beta < \alpha} \kappa^+ \text{-}\mathbf{\Sigma}_\beta^0(X)$, hence it is $\kappa^+ \text{-}\mathbf{\Pi}_\alpha^0(X)$. \square

Lemma 3.5.14. *Let $X \subseteq {}^{\kappa\kappa}$ be such that $\text{ord}_{\kappa^+}(X) \geq \alpha$ (for some $1 \leq \alpha < \kappa^+$), and let \mathbb{P} be a forcing notion such that $|\mathbb{P}| \leq \kappa$. Then, $\mathbb{P} \Vdash \text{ord}_{\kappa^+}(\check{X}) \geq \alpha$.*

If moreover $\text{ord}_{\kappa^+}(X) = \alpha$, then $\mathbb{P} \Vdash \text{ord}_{\kappa^+}(\check{X}) = \alpha$.

Proof. By Proposition 3.3.4, in the ground model there is a set $A \in \kappa^+ \text{-}\mathbf{\Pi}_\alpha^0(X) \setminus \bigcup_{1 \leq \beta < \alpha} \kappa^+ \text{-}\mathbf{\Pi}_\beta^0(X)$ because we assumed $\text{ord}_{\kappa^+}(X) \geq \alpha$. Towards a contradiction, suppose that there are $1 \leq \beta < \alpha$, a name \dot{J} for a $\kappa^+ \text{-}\mathbf{\Pi}_\beta^0$ subset of X , and $p \in \mathbb{P}$ such that $p \Vdash \check{A} = \dot{J}$. By absoluteness, we know that

$$\{x \in X \mid p \Vdash \check{x} \in \check{A}\} = A.$$

But the set on the left equals $K_p(\dot{J})$ by choice of p , and it belongs to $\kappa^+ \text{-}\mathbf{\Pi}_\beta^0(X)$ by Lemma 3.5.13, contradicting the choice of A .

Assume now that $\text{ord}_{\kappa^+}(X) = \alpha$, and let \dot{J} be any name for a κ^+ -Borel subset of X . Then, for any \mathbb{P} -generic filter G , we have that

$$V[G] \models \dot{J} = \bigcup_{p \in G} \check{K}_p(\dot{J}).$$

Each $K_p(\dot{J})$ is κ^+ -Borel in the ground model because of Lemma 3.5.13, and hence it belongs to $\kappa^+ \text{-}\Sigma_\alpha^0(X)$ because we assumed $\text{ord}_{\kappa^+}(X) = \alpha$. Therefore the interpretation of \dot{J} in $V[G]$ is a $\kappa^+ \text{-}\Sigma_\alpha^0$ subset of X . Since this is true for all \dot{J} and G as above, we have $\mathbb{P} \Vdash \text{ord}_{\kappa^+}(\dot{X}) \leq \alpha$, which together with the first part of the lemma yields $\mathbb{P} \Vdash \text{ord}_{\kappa^+}(\dot{X}) = \alpha$. \square

The most interesting forcing Lemma 3.5.14 applies to is κ -Cohen forcing, which is the only forcing notion of size at most κ that is simultaneously $<\kappa$ -closed.

Finally, combining the results obtained so far we obtain our examples of κ -thin closed sets with non-collapsing κ^+ -Borel hierarchy.

Corollary 3.5.15. *Consistently, there exists a κ -thin closed subset $X \subseteq {}^\kappa\kappa$ such that the κ^+ -Borel hierarchy does not collapse on X .*

Proof. Let $V[G]$ be a forcing extension by a forcing of size smaller than κ . By Theorem 3.5.12, the closed set $X = {}^\kappa\kappa \cap V$ is κ -thin in $V[G]$. Moreover, in the ground model we have that $\text{ord}_{\kappa^+}({}^\kappa\kappa) \geq \alpha$ for every $1 \leq \alpha < \kappa^+$, hence for any such α we have

$$V[G] \models \text{ord}_{\kappa^+}({}^\kappa\kappa \cap V) \geq \alpha$$

by Lemma 3.5.14, that is, the κ^+ -Borel hierarchy on $X = {}^\kappa\kappa \cap V$ does not collapse. \square

3.6 α -forcing

Throughout this section, fix a set $X \subseteq {}^\kappa\kappa$. All κ^+ -Borel sets appearing are to be understood as relatively κ^+ -Borel subsets of X . Our goal will be to modify $\text{ord}_{\kappa^+}(X)$ in a suitable forcing extension to prove Theorem 3.6.19 and Corollary 3.6.20.

For each $\alpha < \omega$, let $T_\alpha = \leq^\alpha \kappa = <^\alpha \kappa \cup \alpha \kappa$ be the canonical well-founded tree with $\text{rk}_{T_\alpha}(\emptyset) = \alpha$. The tree T_α shall serve as a template for the κ^+ -Borel code of a generic $\kappa^+ \text{-}\Pi_\alpha^0$ set: indeed, it is easy to see that a set $B \subseteq X$ is $\kappa^+ \text{-}\Pi_\alpha^0$ if and only if there is some $f: T_\alpha^0 \rightarrow <^\kappa \kappa$ ¹⁴ such that $\langle T_\alpha, f \rangle$ codes B , i.e. $B = \mathcal{I}_{T_\alpha}^f(\emptyset) \cap X$.

Remark 3.6.1. In this section, we restrict our efforts only to finite ordinals, as for limit ordinals α it is necessary to first find an appropriate definition for T_α . Although this is not an issue when $\text{cof}(\alpha) = \kappa$, the advent of limit ordinals of cofinality less than κ introduces non-trivial obstacles that are not present in the classical case. This explains why Miller was able to develop analogues of the results of this section in full generality for all ordinals $\alpha < \omega_1$ (see e.g. [Mil79, Lemma 35 and Lemma 36]), while here we stop already at level ω , which has cofinality strictly smaller than κ . A method for working with ordinals $\alpha \geq \omega$ and nodes whose rank is a limit ordinal with small cofinality is in preparation and will appear in future work by the second author.

We are now equipped to define (a generalization of) Miller's α -forcing.

¹⁴Recall that for a well-founded tree T , T^0 denotes the set of leaves of T , while $T^{>0} = T \setminus T^0$.

Definition 3.6.2. For A, B disjoint subsets of X and a finite ordinal $1 < \alpha < \omega$, let $\mathbb{B}\mathbb{M}_\alpha(A, B, X)$ be the partially ordered set consisting of all pairs $p = \langle f_p, R_p \rangle$ such that:

- (a) $f_p: T_\alpha^0 \rightarrow {}^{<\kappa}\kappa$ is a partial function with $|f_p| < \kappa$;
- (b) $R_p \subseteq T_\alpha^{>0} \times X$ and $|R_p| < \kappa$;
- (c) If $\langle t, x \rangle \in R_p$, then for all $t' \in \text{Succ}_{T_\alpha}(t)$ we have
 - $\langle t', x \rangle \notin R_p$ if $t' \in T_\alpha^{>0}$;
 - $x \notin \mathbf{N}_{f_p(t')}$ if $t' \in T_\alpha^0 \cap \text{dom}(f_p)$.

We also have two constraints involving the parameters A and B :

- (d) $\{x \in X \mid \exists t \in \text{Succ}_{T_\alpha}(\emptyset) (\langle t, x \rangle \in R_p)\} \cap A = \emptyset$;
- (e) $\{x \in X \mid \langle \emptyset, x \rangle \in R_p\} \cap B = \emptyset$.

The ordering is given by $q \leq p$ if and only if $f_q \subseteq f_p$ and $R_q \subseteq R_p$.

Since the space X is fixed, we will often omit it from the notation and write $\mathbb{B}\mathbb{M}_\alpha(A, B)$ instead of $\mathbb{B}\mathbb{M}_\alpha(A, B, X)$. Our main focus are going to be the forcings $\mathbb{B}\mathbb{M}_\alpha(\emptyset, \emptyset)$ and $\mathbb{B}\mathbb{M}_\alpha(A, X \setminus A)$. These two kinds of forcings are fundamentally of a different flavor: while the first one adds a generic $\kappa^+ \text{-}\Pi_\alpha^0$ subset to the space X (see Theorem 3.6.18 for a formalization of the word “generic” in this instance), the latter will instead add a $\kappa^+ \text{-}\Pi_\alpha^0$ code for the set $A \subseteq X$ (see Corollary 3.6.5).

Until further notice, from now on we fix a finite ordinal $1 < \alpha < \omega$ and two disjoint sets $A, B \subseteq X$. If G is $\mathbb{B}\mathbb{M}_\alpha(A, B)$ -generic, then by a density argument $f_G = \bigcup_{p \in G} f_p$ is a total function from T_α^0 to ${}^{<\kappa}\kappa$, thus one can interpret every $t \in T_\alpha$ as a κ^+ -Borel subset $G_t = \mathcal{I}^{f_G}(t)$ of X . The semantic meaning of a pair $\langle t, x \rangle \in R_p$ is to ensure that x is an element of G_t . The next lemmas show that this is sound.

Lemma 3.6.3. For $t \in T_\alpha^{>0}$ and $x \in X$, let $D_{t,x}$ be defined by

$$D_{t,x} = \{p \in \mathbb{B}\mathbb{M}_\alpha(A, B) \mid \langle t, x \rangle \in R_p \vee \exists t' \in \text{Succ}_{T_\alpha}(t) \cap \text{dom}(f_p) (x \in \mathbf{N}_{f_p(t')})\}$$

if $\text{rk}_{T_\alpha}(t) = 1$, and

$$D_{t,x} = \{p \in \mathbb{B}\mathbb{M}_\alpha(A, B) \mid \langle t, x \rangle \in R_p \vee \exists t' \in \text{Succ}_{T_\alpha}(t) (\langle t', x \rangle \in R_p)\}$$

if $\text{rk}_{T_\alpha}(t) > 1$. Then, $D_{t,x}$ is dense in $\mathbb{B}\mathbb{M}_\alpha(A, B)$.

Proof. Given $p \in \mathbb{B}\mathbb{M}_\alpha(A, B)$, we want to find $p' \leq p$ such that $p' \in D_{t,x}$. If $p \in D_{t,x}$ we are done, so assume $p \notin D_{t,x}$ (so that, in particular, $\langle t, x \rangle \notin R_p$). Also, if $x \in A$ and $t = \emptyset$, then $p' = \langle f_p, R_p \cup \{\langle \emptyset, x \rangle\} \rangle$ is a condition in $\mathbb{B}\mathbb{M}_\alpha(A, B)$ satisfying $p' \leq p$ and $p' \in D_{t,x}$. Thus, we can assume that $x \notin A$ or $t \neq \emptyset$.

Since κ is regular, $|f_p| < \kappa$, $|R_p| < \kappa$, and $t \in T_\alpha^{>0}$ has κ -many successors, we can find a $t' \in \text{Succ}(t)$ such that $(\{t'\} \cup \text{Succ}_{T_\alpha}(t')) \cap (\text{dom}(f_p) \cup \text{dom}(R_p)) = \emptyset$ and an $s \in {}^{<\kappa}\kappa$ such that $x \in \mathbf{N}_s$ and $\mathbf{N}_s \cap \{y \mid \langle t, y \rangle \in R_p\} = \emptyset$. But this means that

$$p' = \begin{cases} \langle f_p \cup \{\langle t', s \rangle\}, R_p \rangle & \text{if } \text{rk}_{T_\alpha}(t) = 1, \\ \langle f_p, R_p \cup \{\langle t', x \rangle\} \rangle & \text{if } \text{rk}_{T_\alpha}(t) > 1 \end{cases}$$

is a condition in $\mathbb{B}\mathbb{M}_\alpha(A, B)$ satisfying $p' \leq p$ and $p' \in D_{t,x}$. \square

Lemma 3.6.4. *Let G be $\mathbb{B}\mathbb{M}_\alpha(A, B)$ -generic, and set $f_G = \bigcup_{p \in G} f_p$ and $R_G = \bigcup_{p \in G} R_p$. Then, for each $t \in T_\alpha^{>0}$ and $x \in X$, we have that*

$$V[G] \models x \in G_t \Leftrightarrow \langle t, x \rangle \in R_G.$$

Proof. We proceed by (finite) induction on $\text{rk}_{T_\alpha}(t)$. First notice that, by definition, for every leaf $t' \in T_\alpha^0$ we have $x \in G_{t'} \Leftrightarrow x \in \mathbf{N}_{f_G(t')}$.

We begin with the basic case $\text{rk}_{T_\alpha}(t) = 1$, in which case all $t' \in \text{Succ}_{T_\alpha}(t)$ are in T_α^0 . If $x \in G_t$, then by the definition of the interpretation function \mathcal{I}^{f_G} we have that $x \notin G_{t'} = \mathbf{N}_{f_G(t')}$ for every $t' \in \text{Succ}_{T_\alpha}(t)$. Since $D_{t,x}$ is dense by Lemma 3.6.3, then G must meet $D_{t,x}$, and we can conclude $\langle t, x \rangle \in R_G$. On the other hand, if $\langle t, x \rangle \in R_G$, then for each $p \in G$ and $t' \in \text{Succ}_{T_\alpha}(t) \cap \text{dom}(f_p)$ we have $x \notin \mathbf{N}_{f_p(t')}$, and therefore $x \in G_t = \bigcap_{t' \in \text{Succ}_{T_\alpha}(t)} X \setminus \mathbf{N}_{f_G(t')}$.

Suppose now that $\text{rk}_{T_\alpha}(t) > 1$, and that the statement of the lemma holds for all $t' \in \text{Succ}_{T_\alpha}(t)$. Since the set $D_{t,x}$ from Lemma 3.6.3 is dense, we have

$$\langle t, x \rangle \in R_G \Leftrightarrow \forall t' \in \text{Succ}_{T_\alpha}(t) (\langle t', x \rangle \notin R_G).$$

By inductive hypothesis, this means that

$$\begin{aligned} \langle t, x \rangle \in R_G &\Leftrightarrow \forall t' \in \text{Succ}_{T_\alpha}(t) (\langle t', x \rangle \notin R_G) \\ &\Leftrightarrow \forall t' \in \text{Succ}_{T_\alpha}(t) (x \notin G_{t'}) \\ &\Leftrightarrow x \in G_t. \end{aligned} \quad \square$$

By Lemma 3.6.4 and conditions (d) and (e) in Definition 3.6.2 (which take care of the inclusions $A \subseteq G_\emptyset$ and $G_\emptyset \subseteq X \setminus B$, respectively) we get:

Corollary 3.6.5. *Let G be $\mathbb{B}\mathbb{M}_\alpha(A, B)$ -generic. Then, in the forcing extension $V[G]$ the set G_\emptyset separates A from B , i.e. $A \subseteq G_\emptyset \subseteq X \setminus B$.*

In other words, forcing with $\mathbb{B}\mathbb{M}_\alpha(A, X \setminus A)$ adds a κ^+ - $\mathbf{\Pi}_\alpha^0$ -code $\langle T_\alpha, f_G \rangle$ for the set A , independently of its complexity in the ground model V .

3.6.1 Preservation of cardinals

In the classical case, we are able to iterate any c.c.c. forcing without collapsing cardinals. In the generalized setting, preservation theorems are of a much more limited character, often requiring additional regularity assumptions. Fortunately, our forcing is structurally simple enough that preservation of cardinals is not a difficult issue. Recall Definition 3.1.2.

Lemma 3.6.6. *The forcing $\mathbb{B}\mathbb{M}_\alpha(A, B)$ is $<\kappa$ -closed and well-met.*

Proof. Let $(p_i)_{i < \delta}$ for $\delta < \kappa$ be a decreasing sequence. It is easy to see that $q = \langle f_q, R_q \rangle$ with $f_q = \bigcup_{i < \delta} f_{p_i}$ and $R_q = \bigcup_{i < \delta} R_{p_i}$ is the greatest lower bound of the sequence. Likewise, for two compatible conditions p, q the condition $r = \langle f_p \cup f_q, R_p \cup R_q \rangle$ is their greatest lower bound. \square

To each pair of sets Z and W we can associate the forcing $\text{Fn}(Z, W, <\kappa)$ consisting of all partial functions from Z to W of size smaller than κ , ordered by inclusion.

Lemma 3.6.7. *The forcing $\mathbb{B}\mathbb{M}_\alpha(A, B)$ is κ -linked.*

Proof. We prove this by reducing our forcing to a κ -Cohen forcing. For each $x \in X$ and $p \in \mathbb{B}\mathbb{M}_\alpha(A, B)$, let $h_{x,p} = \{t \in T_\alpha^{>0} \mid \langle t, x \rangle \in R_p\}$, and notice that $|h_{x,p}| < \kappa$. Let $X_p = \{x \in X \mid h_{x,p} \neq \emptyset\}$. Let us also fix a bijection $\psi: \mathcal{P}_{<\kappa}(T_\alpha^{>0}) \rightarrow \kappa$, where $\mathcal{P}_{<\kappa}(T_\alpha^{>0})$ denotes the collection of all subsets of $T_\alpha^{>0}$ of size smaller than κ , and define $g_p: X_p \rightarrow \kappa$ by setting $g_p(x) = \psi(h_{x,p})$ for any $p \in \mathbb{B}\mathbb{M}_\alpha(A, B)$. Then, the map $\varphi(p) = g_p$ assigns to each $p \in \mathbb{B}\mathbb{M}_\alpha(A, B, X)$ a condition in $\text{Fn}(X, \kappa, <\kappa)$.

Given $p = \langle f_p, R_p \rangle$ and $q = \langle f_q, R_q \rangle$, let

$$p \parallel^* q \Leftrightarrow \langle \emptyset, R_p \rangle \parallel \langle \emptyset, R_q \rangle.$$

In other words, the relation \parallel^* only checks if two conditions are compatible at inner nodes of the tree and forgets information about leaves. Note further that $p \parallel^* q$ and $f_p = f_q$ implies $p \parallel q$. It is now easy to see that $\varphi(p) \parallel \varphi(q)$ implies $p \parallel^* q$.¹⁵

The set $\mathcal{F} = \text{Fn}(T_\alpha^0, <\kappa, <\kappa)$ can be construed as the collection of all $<\kappa$ -sized partial assignments of clopen sets to leaves of T_α , and clearly $|\mathcal{F}| \leq \kappa$. Therefore, if we can prove that $\text{Fn}(X, \kappa, <\kappa)$ is κ -linked as witnessed by the equality $\text{Fn}(X, \kappa, <\kappa) = \bigcup_{i < \kappa} \mathbb{P}_i$, then $\mathbb{B}\mathbb{M}_\alpha(A, B)$ is κ -linked as well because it can be written as

$$\mathbb{B}\mathbb{M}_\alpha(A, B) = \bigcup_{i < \kappa, f \in \mathcal{F}} \{p \in \mathbb{B}\mathbb{M}_\alpha(A, B) \mid f_p = f \wedge \varphi(p) \in \mathbb{P}_i\}.$$

Claim 3.6.7.1. If $|Z| \leq 2^\kappa$ and W is a topological space with a dense subset of size $\leq \kappa$, then $\text{Fn}(Z, W, <\kappa)$ is κ -linked.

Proof of the Claim. In [Els11], a generalization of Marczewski's Separability Theorem is proven: if $|I| \leq 2^\kappa$ and X_i has a dense subset of cardinality $\leq \kappa$ for each $i \in I$, then the product space $\prod_{i \in I} X_i$, endowed with the $<\kappa$ -box topology, contains a dense subset of size $\kappa^{<\kappa} = \kappa$.

Let thus $(b_i)_{i < \kappa}$ be a dense subset of ${}^Z W$, seen as the product of $|Z|$ -many copies of W and equipped with the $<\kappa$ -box topology. Since each $f \in \text{Fn}(Z, W, <\kappa)$ corresponds to the basic open set $\mathbf{N}_f = \{x \in {}^Z W \mid f \subseteq x\}$ of ${}^Z W$, two conditions $f_1, f_2 \in \text{Fn}(Z, W, <\kappa)$ are compatible if and only if $\mathbf{N}_{f_1} \cap \mathbf{N}_{f_2} \neq \emptyset$. Therefore we can conclude that $\text{Fn}(Z, W, <\kappa)$ is κ -linked because it can be written as $\text{Fn}(Z, W, <\kappa) = \bigcup_{i < \kappa} D_i$, where for every $i < \kappa$ we let

$$D_i = \{f \in \text{Fn}(Z, W, <\kappa) \mid b_i \in \mathbf{N}_f\}. \quad \square$$

This concludes the proof of Lemma 3.6.7. □

Combining Lemmas 3.6.6 and 3.6.7 with Fact 3.1.3, we finally get:

Corollary 3.6.8. *Any $<\kappa$ -supported iteration of $\mathbb{B}\mathbb{M}_\alpha(A, B)$ -forcings is $<\kappa$ -closed and satisfies the κ^+ -c.c. In particular, no cardinal is collapsed by such an iteration.*

¹⁵The reverse implication is not true, as the reduction is very coarse and forgets a lot of the information contained in p and q .

3.6.2 A model for $\text{ord}_{\kappa^+}(X) = n$

The goal of this section is to show that, consistently, there can be even closed sets $X \subseteq {}^\kappa\kappa$ such that $\text{ord}_{\kappa^+}(X) = n$, for any given natural number $n \geq 2$. This will be achieved using an iteration of α -forcings.

For $n > 2$, the idea is to first force with $\mathbb{B}\mathbb{M}_{n-1}(\emptyset, \emptyset, X)$ to add a “fresh” κ^+ - $\mathbf{\Pi}_{n-1}^0(X)$ subset of X , and then successively, using bookkeeping, force with $\mathbb{B}\mathbb{M}_n(\dot{A}, X \setminus \dot{A}, X)$ to add a κ^+ - $\mathbf{\Pi}_n^0$ -code for every κ^+ -Borel subset of X ; by closure under complements of κ^+ - $\mathbf{Bor}(X)$, this entails κ^+ - $\mathbf{Bor}(X) = \kappa^+$ - $\mathbf{\Delta}_n^0(X)$, so that $\text{ord}_{\kappa^+}(X) \leq n$. To prove that $\text{ord}_{\kappa^+}(X) \geq n$, we will show that the first generic set we added will never become a κ^+ - $\mathbf{\Sigma}_{n-1}^0$ set in the iteration (Theorem 3.6.18); this is the heart and main difficulty of the construction. Notice that by Corollary 3.3.6, we will automatically get that κ^+ - $\mathbf{\Delta}_n^0(X) \setminus (\kappa^+$ - $\mathbf{\Sigma}_{n-1}^0(X) \cup \kappa^+$ - $\mathbf{\Pi}_{n-1}^0(X)) \neq \emptyset$.

For $n = 2$ the construction is much easier and we may skip the first step of forcing with $\mathbb{B}\mathbb{M}_{n-1}(\emptyset, \emptyset, X)$. The reader interested solely in the result for $n = 2$ may skip ahead up to the proof of Theorem 3.6.19. The same result can also be achieved by using Solovay’s almost disjoint forcing in place of $\mathbb{B}\mathbb{M}_2$, see [Lüc12].

Recall that we already fixed an arbitrary set $X \subseteq {}^\kappa\kappa$. Throughout the rest of this section, let us also fix a $<\kappa$ -supported forcing iteration $\overline{\mathbb{B}\mathbb{M}} = (\mathbb{P}_\zeta, \dot{\mathbb{Q}}_\zeta)_{\zeta < \zeta^*}$ such that

- $\mathbb{P}_\zeta \Vdash \dot{\mathbb{Q}}_\zeta = \mathbb{B}\mathbb{M}_{\alpha_\zeta}(\dot{A}_\zeta, \dot{B}_\zeta, \check{X})$.
- $A_0 = B_0 = \emptyset$.
- $1 < \alpha_0 < \omega$ and $\alpha_\zeta = \alpha_0 + 1$ for $0 < \zeta < \zeta^*$.

A condition $p \in \overline{\mathbb{B}\mathbb{M}}$ is thus a tuple $p = (\dot{p}(\zeta))_{\zeta < \zeta^*}$ such that $p \restriction \zeta \Vdash \dot{p}(\zeta) \in \mathbb{B}\mathbb{M}_{\alpha_\zeta}(\dot{A}_\zeta, \dot{B}_\zeta, \check{X})$ and, furthermore, $|\text{supp}(p)| < \kappa$ for

$$\text{supp}(p) = \{\zeta \in \zeta^* \mid \dot{p}(\zeta) \neq \check{\mathbb{1}}\}.$$

For the sake of readability, in what follows we will often (but not always) write $T(\gamma)$ instead of T_{α_γ} .

While in general a coordinate $\dot{p}(\zeta)$ can be just a name for a condition in the forcing $\mathbb{B}\mathbb{M}_{\alpha_\zeta}(\dot{A}_\zeta, \dot{B}_\zeta, \check{X})$, we may without loss of generality work with conditions of a nicer form.

Definition 3.6.9. A condition $p \in \overline{\mathbb{B}\mathbb{M}}$ is **good** if $\dot{p}(\zeta) = \langle \check{f}_{p(\zeta)}, \check{R}_{p(\zeta)} \rangle$ for each $\zeta < \zeta^*$, i.e. all entries of the condition are already decided in the ground model.

Every good condition can be identified with an element of the product forcing $\prod_{\zeta < \zeta^*} \mathbb{B}\mathbb{M}_{\alpha_\zeta}(\emptyset, \emptyset, X)$. However, the converse is not true, as for example if \dot{B}_1 and $x \in X$ are such that $\mathbb{B}\mathbb{M}_{\alpha_0}(A_0, B_0, X) \not\Vdash \dot{x} \in \dot{B}_1$ and $\mathbb{B}\mathbb{M}_{\alpha_0}(A_0, B_0, X) \not\Vdash \dot{x} \notin \dot{B}_1$, then p defined by

$$p(\zeta) = \begin{cases} (\emptyset, \{\langle \emptyset, \dot{x} \rangle\}) & \text{if } \zeta = 1, \\ \check{\mathbb{1}} & \text{otherwise} \end{cases} \quad (3.6.1)$$

does not satisfy $p(0) \Vdash p(1) \in \mathbb{BM}_{\alpha_1}(\dot{A}_1, \dot{B}_1, \dot{X})$, and thus can not be a condition in $\overline{\mathbb{BM}}$.

One of the key points in several proofs of this section will be to check that a sequence $p = (p(\zeta))_{\zeta < \zeta^*}$ we defined explicitly is actually a condition in our forcing. Notice that requiring $p \upharpoonright \zeta \Vdash \dot{p}(\zeta) \in \mathbb{BM}_{\alpha_\zeta}(\dot{A}_\zeta, \dot{B}_\zeta, \dot{X})$ amounts to checking that $p \upharpoonright \zeta \Vdash \check{x} \notin \dot{B}_\zeta$ for any $\langle \emptyset, x \rangle \in R_{p(\zeta)}$, and $p \upharpoonright \zeta \Vdash \check{x} \notin \dot{A}_\zeta$ for any $\langle t, x \rangle \in R_{p(\zeta)}$ with $t \in \text{Succ}_{T(\gamma)}(\emptyset)$.

A standard argument shows that the set of good conditions is dense in $\overline{\mathbb{BM}}$.

Lemma 3.6.10. *For every $p \in \overline{\mathbb{BM}}$ there is $q \in \overline{\mathbb{BM}}$ such that $q \leq p$ and q is good.*

Proof. We prove by induction on γ that good conditions are dense in \mathbb{P}_ζ , for each $\zeta \leq \zeta^*$. For $\zeta = 0$ there is nothing to prove. In the successor step of the induction, we assume that good conditions are dense in \mathbb{P}_ζ , and consider an arbitrary $p \in \mathbb{P}_{\zeta+1}$. We know that there are \mathbb{P}_ζ -names \dot{f} and \dot{R} such that

$$p \upharpoonright \zeta \Vdash \dot{p}(\zeta) = \langle \dot{f}, \dot{R} \rangle \in \mathbb{BM}_{\alpha_\zeta}(\dot{A}, \dot{B}, X) \wedge \dot{f}: T_{\alpha_\zeta}^0 \rightarrow {}^{<\kappa}\kappa \text{ is a partial function} \wedge \\ |\dot{f}| < \kappa \wedge \dot{R} \subseteq T_{\alpha_\zeta}^0 \times \dot{X} \wedge |\dot{R}| < \kappa.$$

Since the iteration is $<\kappa$ -closed, both \dot{f} and \dot{R} can be decided as ground model objects f and R , respectively, by a condition $q \leq p \upharpoonright \zeta$. By inductive hypothesis, we can assume that q is good, hence $q \wedge \langle \dot{f}, \dot{R} \rangle \leq p$ is good as well.

Suppose now that $\zeta \leq \zeta^*$ is a limit. If $\text{cof}(\zeta) \geq \kappa$, every condition in \mathbb{P}_ζ is already in \mathbb{P}_β for some $\beta < \zeta$, hence we are done by the inductive hypothesis. If instead $\text{cof}(\zeta) < \kappa$, let $(\zeta_i)_{i < \text{cof}(\zeta)}$ be a cofinal sequence in ζ and $p \in \mathbb{P}_\zeta$. We inductively define a sequence $(q_i)_{i < \text{cof}(\zeta)}$, where for each $i < j < \text{cof}(\zeta)$

- $q_i \in \mathbb{P}_{\zeta_i}$ is good and $q_i \leq p \upharpoonright \zeta_i$;
- $q_j \upharpoonright \zeta_i \leq q_i$.

(In limit steps of this construction, we make use of the $<\kappa$ -closure of \mathbb{P}_ζ .) Let $q = \bigcup_{i < \text{cof}(\zeta)} q_i$ be the pointwise union of these conditions, it is easy to see that $q \leq p$ and q is good, as desired. \square

Therefore, we shall tacitly assume that all conditions appearing in further constructions are good.

Definition 3.6.11. For a set $H \subseteq X$ define the **rank function**

$$|p|_H = \sup \{ \text{rk}_{T(\gamma)}(t) \mid \gamma < \gamma^* \wedge t \in T(\gamma) \wedge \langle t, x \rangle \in R_{p(\zeta)} \text{ for some } x \in X \setminus H \}.$$

Notice that for every $H \subseteq X$ and $p \in \overline{\mathbb{BM}}$, we have $|p|_H \leq \alpha_0 + 1$. Also, $|p|_H = 0$ if and only if every x appearing in $\bigcup_{\zeta < \zeta^*} R_{p(\zeta)}$ is in H .

Definition 3.6.12. Given $K \subseteq \zeta^*$ and $H \subseteq X$, let

$$\overline{\mathbb{BM}}_{H,K,\leq\beta} = \{ q \in \overline{\mathbb{BM}} \mid \text{supp}(q) \subseteq K, |q|_H \leq \beta \}.$$

The set $\overline{\mathbb{BM}}_{H,K,<\beta} \subseteq \overline{\mathbb{BM}}$ is defined analogously.

We also let $\overline{\mathbb{B}\mathbb{M}}_{H,K} = \overline{\mathbb{B}\mathbb{M}}_{H,K,\leq 0}$ be the set of conditions p with $\text{supp}(p) \subseteq K$ and $|p|_H = 0$.

For every $p \in \overline{\mathbb{B}\mathbb{M}}$, $H \subseteq X$, $K \subseteq \zeta^*$, and $\beta \leq \alpha_0$, define the restriction of p to $(H, K, \leq \beta)$ as the tuple $p \upharpoonright_{H,K,\leq \beta} = (p \upharpoonright_{H,K,\leq \beta}(\zeta))_{\zeta < \zeta^*}$, where

$$p \upharpoonright_{H,K,\leq \beta}(\zeta) = \begin{cases} \langle f_{p(\zeta)}, R_{p(\zeta)} \upharpoonright_{H,\leq \beta} \rangle & \text{if } \zeta \in K \\ \mathbb{1} & \text{if } \zeta \notin K, \end{cases}$$

and for each $\zeta \in K$

$$R_{p(\zeta)} \upharpoonright_{H,\leq \beta} = \{ \langle t, x \rangle \in R_{p(\zeta)} \mid x \in H \vee \text{rk}_{T(\gamma)}(t) \leq \beta \}.$$

Notice that in general $p \upharpoonright_{H,K,\leq \beta}$ is not a condition in $\overline{\mathbb{B}\mathbb{M}}$ (even if p is), since by weakening an initial segment of the condition we can no longer guarantee that $(p \upharpoonright_{H,K,\leq \beta}) \upharpoonright \zeta$ decides any statement of the form $\dot{x} \notin \dot{A}_\zeta$.

Definition 3.6.13. We say that $p \in \overline{\mathbb{B}\mathbb{M}}$ is (H, K) -**restrictable** if for any $\beta < \alpha_0$ we have that $p \upharpoonright_{H,K,\leq \beta}$ is a well-defined condition in $\overline{\mathbb{B}\mathbb{M}}$.

We will just say that p is restrictable when H and K are clear from the context. Notice also that the restriction of any good restrictable condition is still good.

Definition 3.6.14. A pair of sets $H \subseteq X$ and $K \subseteq \zeta^*$ is said to be **appropriate** if the set

$$\{p \in \overline{\mathbb{B}\mathbb{M}} \mid p \text{ is } (H, K)\text{-restrictable}\}$$

is dense in $\overline{\mathbb{B}\mathbb{M}}$. Furthermore, given a $\overline{\mathbb{B}\mathbb{M}}$ -name $\dot{\tau}$ for an element of ${}^\kappa\kappa$, we say that the pair (H, K) is **appropriate for $\dot{\tau}$** if, additionally, for each $i < \kappa$ the set

$$\{p \in \overline{\mathbb{B}\mathbb{M}}_{H,K} \mid p \text{ decides } \dot{\tau} \upharpoonright i\} \quad (3.6.2)$$

is predense¹⁶ in $\overline{\mathbb{B}\mathbb{M}}$.

Lemma 3.6.15. *For every name $\dot{\tau}$ for an element of ${}^\kappa\kappa$ and all sets $H' \subseteq X$ and $K' \subseteq \zeta^*$ of size at most κ , there are $H \supseteq H'$ and $K \supseteq K'$ of size at most κ such that the pair (H, K) is appropriate for $\dot{\tau}$.*

Proof. We recursively define an increasing sequence $(H_i, K_i)_{i < \kappa}$ of sets of size at most κ as follows.

If $i = 0$, for each $j < \kappa$ we choose a maximal antichain C_j that decides $\dot{\tau} \upharpoonright j$. Keeping in mind that $\overline{\mathbb{B}\mathbb{M}}$ has the κ^+ -c.c., we can find $H_0 \supseteq H'$ and $K_0 \supseteq K'$ of size at most κ such that $|p|_{H_0} = 0$ and $\text{supp}(p) \subseteq K_0$ for each $p \in \bigcup_{j < \kappa} C_j$.

For the successor step, suppose that H_i and K_i have already been defined. For each $Y \in [H_i]^{< \kappa}$ and $Z \in [K_i]^{< \kappa}$, let

$$D_{Y,Z} = \{p \in \overline{\mathbb{B}\mathbb{M}} \mid \forall y \in Y \forall \zeta \in Z (p \text{ decides } \check{y} \in \dot{A}_\zeta \text{ and } \check{y} \in \dot{B}_\zeta)\}.$$

¹⁶Recall that a set D is predense if every condition in the forcing is compatible with a condition from D .

Notice that $D_{Y,Z}$ is dense, since $\overline{\mathbb{B}\mathbb{M}}$ is $<\kappa$ -closed. Choose now maximal antichains $C_{Y,Z} \subseteq D_{Y,Z}$, for Y and Z as above. Collect all x and ζ appearing in $\bigcup_{\zeta < \zeta^*} R_{p(\zeta)}$ and $\text{supp}(p)$, respectively, where p varies in the set

$$\bigcup \{C_{Y,Z} \mid Y \in [H_i]^{<\kappa} \wedge Z \in [K_i]^{<\kappa}\},$$

and include them in H_{i+1}, K_{i+1} .

For i limit, we just set $H_i = \bigcup_{j < i} H_j$ and $K_i = \bigcup_{j < i} K_j$.

We claim that $H = H_\kappa$ and $K = K_\kappa$ are as desired. First, notice that $|H|, |K| \leq \kappa$, and that we took care of the predensity of the sets from (3.6.2) in the first step of the construction. It remains to see that the pair (H, K) is appropriate.

Let p be a condition in $\overline{\mathbb{B}\mathbb{M}}$, and let $((\zeta_i, t_i, x_i))_{i < \delta}$ with $\delta < \kappa$ be an enumeration of all triples (ζ, t, x) with $\langle t, x \rangle \in R_{p(\zeta)}$, $\zeta \in \text{supp}(p) \cap K$, $x \in H$, and $t \in \{\emptyset\} \cup \text{Succ}_{T(\gamma)}(\emptyset)$. By construction of H and K , we can find a condition $\bar{p} \in \overline{\mathbb{B}\mathbb{M}}_{H,K}$ with $\bar{p} \parallel p$ such that \bar{p} decides the statements “ $\check{x}_i \in \dot{A}_{\zeta_i}$ ” and “ $\check{x}_i \in \dot{B}_{\zeta_i}$ ”, for all $i < \delta$. Let $q = p \cup \bar{p}$ be the pointwise union of p and \bar{p} . It is easy to see that q is a condition such that $q \leq p, \bar{p}$. We claim that q is (H, K) -restrictable, as wanted.

Fix $\beta < \alpha_0$, and let $q' = q \upharpoonright_{H,K,\leq\beta}$. Note that since $\bar{p} \in \overline{\mathbb{B}\mathbb{M}}_{H,K}$, we have $q' \leq \bar{p}$. We prove by induction that $q' \upharpoonright \zeta$ is a condition in \mathbb{P}_ζ , for each $\zeta \leq \zeta^*$. For ζ limit and $\zeta = 0$, this is trivial. Assume now that the statement holds up to some ζ , and let us show that it is true for $\zeta + 1$ as well. If $\zeta \notin K$, we know that $q'(\zeta) = \mathbb{1}$ and there is nothing to do. If $\zeta \in K$, instead, we need to check that $q' \upharpoonright \zeta \Vdash \check{x} \notin \dot{B}_\zeta$ for any $x \in X$ such that $\langle \emptyset, x \rangle \in R_{q'(\zeta)}$ and $q' \upharpoonright \zeta \Vdash \check{x} \notin \dot{A}_\zeta$ for any $x \in X$ such that $\langle t, x \rangle \in R_{q'(\zeta)}$ for some $t \in \text{Succ}_{T(\gamma)}(\emptyset)$.

Consider $\langle \emptyset, x \rangle \in R_{q'(\zeta)}$. If $\langle \emptyset, x \rangle \in R_{\bar{p}(\zeta)}$, then $\bar{p} \upharpoonright \zeta \Vdash \check{x} \notin \dot{B}_\zeta$, so $q' \leq \bar{p}$ forces this as well. Otherwise, $\langle \emptyset, x \rangle \in R_{p(\zeta)}$. This implies $x \in H$, since

$$\text{rk}_{T(\gamma)}(\emptyset) = \alpha_\zeta = \alpha_0 + 1 > \beta. \quad (3.6.3)$$

Hence (ζ, \emptyset, x) is one of the triples we took care of at the beginning of the proof, and $\bar{p} \upharpoonright \zeta$ decides “ $\check{x} \in \dot{B}_\zeta$ ”. But since \bar{p} is compatible with p , it must be the case that $\bar{p} \upharpoonright \zeta \Vdash \check{x} \notin \dot{B}_\zeta$, and so $q' \upharpoonright \zeta \Vdash \check{x} \notin \dot{B}_\zeta$ as wanted. Any pair $\langle t, x \rangle \in R_{q'(\zeta)}$ with $t \in \text{Succ}_{T(\gamma)}(\emptyset)$ can be treated analogously, with (3.6.3) now reading “ $\text{rk}_{T(\gamma)}(t) = \alpha_\zeta - 1 = \alpha_0 > \beta$ ”.

Thus we can conclude that $q' \upharpoonright \zeta \Vdash q'(\zeta) \in \dot{\mathbb{Q}}_\zeta$ and we are done. \square

Lemma 3.6.16. *Let $0 < \beta < \alpha_0$, $H \subseteq X$, $K \subseteq \zeta^*$, and $p \in \overline{\mathbb{B}\mathbb{M}}$ be (H, K) -restrictable. Then, for every $r \in \overline{\mathbb{B}\mathbb{M}}_{H,K,<\beta}$, we have that if $p \upharpoonright_{H,K,\leq\beta} \parallel r$, then $p \parallel r$.*

Proof. Let $q = p \upharpoonright_{H,K,\leq\beta}$. Fix $r \in \overline{\mathbb{B}\mathbb{M}}_{H,K,<\beta}$, and assume, towards a contradiction, that $q \parallel r$, but $p \perp r$. As in the previous lemma, let $\bar{r} = p \cup r$ be the pointwise union of p and r , and notice that \bar{r} is not a valid condition because we assumed $p \perp r$. Let $\zeta < \zeta^*$ be such that $\bar{r} \upharpoonright \zeta \in \mathbb{P}_\zeta$ but $\bar{r} \upharpoonright \zeta \not\Vdash \check{r}(\zeta) \in \mathbb{Q}_\zeta$, so that in particular $\zeta \in K$. This state of affairs can only happen for one of the following reasons, each of which we are going to exclude:

- $f_{p(\zeta)} \perp f_{r(\zeta)}$. This cannot happen, since $f_{p(\zeta)} = f_{q(\zeta)}$ and $q \parallel r$.

- There are $x \in X$, $t \in T(\gamma)$, and $t' \in \text{Succ}_{T(\gamma)}(t) \cap \text{dom}(f_{r(\zeta)})$ such that $\text{rk}_{T(\gamma)}(t) = 1$ (so that $\text{rk}_{T(\gamma)}(t') = 0$), $\langle t, x \rangle \in R_{p(\zeta)}$, and $x \in \mathbf{N}_{f_{r(\zeta)}(t')}$. This is not possible: since $\text{rk}_{T(\gamma)}(t) = 1$ and $\beta > 0$, we must have $\langle t, x \rangle \in R_{q(\zeta)}$.
- There are $x \in X$, $t \in T(\gamma)$, and $t' \in \text{Succ}_{T(\gamma)}(t) \cap \text{dom}(f_{p(\zeta)})$ such that $\text{rk}_{T(\gamma)}(t) = 1$ (so that $\text{rk}_{T(\gamma)}(t') = 0$), $\langle t, x \rangle \in R_{r(\zeta)}$, and $x \in \mathbf{N}_{f_{p(\zeta)}(t')}$. This is not possible, since $f_{q(\zeta)} = f_{p(\zeta)}$.
- There are $x \in X$, $t \in T(\gamma)$ and $t' \in \text{Succ}_{T(\gamma)}(t)$ such that $\langle t, x \rangle \in R_{r(\zeta)}$ and $\langle t', x \rangle \in R_{p(\zeta)}$. Since $\text{rk}_{T(\gamma)}(t') < \text{rk}_{T(\gamma)}(t)$, we know that $\langle t, x \rangle \in R_{q(\zeta)}$, which is a contradiction.
- There are $x \in X$, $t \in T(\gamma)$, and $t' \in \text{Succ}_{T(\gamma)}(t)$ such that $\langle t, x \rangle \in R_{p(\zeta)}$ and $\langle t', x \rangle \in R_{r(\zeta)}$. If $x \notin H$, it follows that $\beta > \text{rk}_{T(\gamma)}(t')$, and since $\text{rk}_{T(\gamma)}(t) = \text{rk}_{T(\gamma)}(t') + 1$ (because of the specific definition of $T(\gamma)$), we have $\langle t, x \rangle \in R_{q(\zeta)}$. If instead $x \in H$, we also immediately get $\langle t, x \rangle \in R_{q(\zeta)}$. This contradicts $q \parallel r$. \square

Theorem 3.6.17. *Suppose that $\alpha \leq \alpha_0$, $t \in T_\alpha^{>0}$, \dot{f} is a $\overline{\text{BM}}$ -name for an assignment $T_\alpha^0 \rightarrow {}^{<\kappa}\kappa$, $x \in X$, and (H, K) is an appropriate pair for (a coding of) \dot{f} . Then, for every $p \in \overline{\text{BM}}$ such that $p \Vdash \dot{x} \notin \mathcal{I}^{\dot{f}}(t)$ there exists a $q \in \overline{\text{BM}}_{H, K, < \beta}$ such that $q \parallel p$ and $q \Vdash \dot{x} \notin \mathcal{I}^{\dot{f}}(t)$.*

Proof. We proceed by induction on $\text{rk}_{T_\alpha}(t)$. Let

$$p \Vdash \dot{x} \notin \mathcal{I}^{\dot{f}}(t) = \bigcap_{t' \in \text{Succ}_{T_\alpha} t} X \setminus \mathcal{I}^{\dot{f}}(t'),$$

and fix $p' \leq p$ and $t' \in \text{Succ}_{T_\alpha}(t)$ such that $p' \Vdash \dot{x} \in \mathcal{I}^{\dot{f}}(t')$.

Suppose first that $\text{rk}_{T_\alpha}(t) = 1$. Since (H, K) is appropriate for \dot{f} , we can find $q \in \overline{\text{BM}}_{H, K} = \overline{\text{BM}}_{H, K, < 1}$ such that $q \parallel p'$ and q decides $\dot{f}(t')$ as $s \in {}^{<\kappa}\kappa$. Since q is compatible with p' , this implies that $x \in \mathbf{N}_s$, and thus $q \Vdash \dot{x} \in \mathcal{I}^{\dot{f}}(t') \subseteq X \setminus \mathcal{I}^{\dot{f}}(t)$.

Assume now that $\text{rk}_{T_\alpha}(t) > 1$. Since (H, K) is appropriate, we may choose an (H, K) -restrictable $q \leq p'$. We claim that $q \upharpoonright_{H, K, \leq \beta-1} \Vdash \dot{x} \in \mathcal{I}^{\dot{f}}(t')$. If this were not the case, we could find an $r \leq q \upharpoonright_{H, K, \leq \beta-1}$ with $r \Vdash \dot{x} \notin \mathcal{I}^{\dot{f}}(t')$. The inductive hypothesis now yields a $\bar{r} \in \overline{\text{BM}}_{H, K, < \beta-1}$ such that $\bar{r} \parallel r$ and $\bar{r} \Vdash \dot{x} \notin \mathcal{I}^{\dot{f}}(t')$. But $\bar{r} \parallel q \upharpoonright_{H, K, \leq \beta-1}$ implies $\bar{r} \parallel q$ by Lemma 3.6.16, and hence we reach a contradiction. \square

Note that for any name \dot{J} for a set in $\kappa^+ \text{-}\Sigma_\alpha^0(X)$ there exists an $\dot{f}: T_\alpha^0 \rightarrow {}^{<\kappa}\kappa$ as in the above lemma with $\mathbb{1} \Vdash \dot{J} = \dot{X} \setminus \mathcal{I}^{\dot{f}}(\emptyset)$.

The following argument is the heart of the proof, affectionately dubbed “the old switcheroo” by Miller.

Theorem 3.6.18. *Let G be $\overline{\text{BM}}$ -generic, and let G^0 be the projection of G onto the 0-th coordinate, so that G^0 is $\overline{\text{BM}}_{\alpha_0}(\emptyset, \emptyset, X)$ -generic. If $|X| > \kappa$, then*

$$V[G] \models G_\emptyset^0 \in \kappa^+ \text{-}\Pi_{\alpha_0}^0(X) \setminus \kappa^+ \text{-}\Sigma_{\alpha_0}^0(X).$$

Proof. The fact that $V[G] \models G_\emptyset^0 \in \kappa^+ \text{-}\mathbf{\Pi}_{\alpha_0}^0(X)$ easily follows from the fact that $\langle T_{\alpha_0}, f_{C^0} \rangle$ is a $\kappa^+ \text{-}\mathbf{\Pi}_{\alpha_0}^0$ -code for G_\emptyset^0 .

Towards a contradiction, assume that $V[G] \models G_\emptyset^0 \in \kappa^+ \text{-}\mathbf{\Sigma}_{\alpha_0}^0(X)$. Fix a condition $p \in \overline{\mathbb{B}\mathbb{M}}$ and a name \dot{J} for a set in $\kappa^+ \text{-}\mathbf{\Sigma}_{\alpha_0}^0(X)$ such that

$$p \Vdash G_\emptyset^0 = \dot{J}.$$

Then, there exists a $\overline{\mathbb{B}\mathbb{M}}$ -name \dot{f} for an assignment $T_\alpha^0 \rightarrow {}^{<\kappa}\kappa$ such that $p \Vdash \check{X} \setminus \dot{J} = \mathcal{I}^{\dot{f}}(\emptyset)$. Lemma 3.6.15, find an appropriate pair (H, K) for (a coding of) \dot{f} such that $|p|_H = 0$ and $\text{supp}(p) \subseteq K$. For the rest of the proof, fix a $y \in X \setminus H$.

Define $p'(0) = \langle f_{p(0)}, R_{p(0)} \cup \{\langle \emptyset, y \rangle\} \rangle$ and $p'(\zeta) = p(\zeta)$ for $\zeta > 0$, and notice that $p' \in \overline{\mathbb{B}\mathbb{M}}$ is a condition since $A_0 = B_0 = \emptyset$. Since $p' \Vdash \check{y} \in G_\emptyset^0 = \dot{J}$ (i.e. $p' \Vdash \check{y} \notin \mathcal{I}^{\dot{f}}(\emptyset)$), by Theorem 3.6.17 there exists a $q \parallel p'$ such that $|q|_H < \alpha_0$, $\text{supp}(q) \subseteq K$, and $q \Vdash \check{y} \in \dot{J}$. As $q \parallel p$ and $p \in \overline{\mathbb{B}\mathbb{M}}_{H,K}$, we may without loss of generality assume $q \leq p$ (by passing to the pointwise union $q \cup p$ if necessary). Thus we have $q \Vdash G_\emptyset^0 = \dot{J}$, too.

Since $|q|_H < \alpha_0$, we know that $\langle \emptyset, y \rangle \notin R_{q(0)}$, and thus there is $t \in \text{Succ}_{T(0)}(\emptyset)$ such that if q' is defined by $q'(0) = \langle f_{q(0)}, R_{q(0)} \cup \{\langle t, y \rangle\} \rangle$ and $q'(\zeta) = q(\zeta)$ for $\zeta > 0$, then q' is a condition. Note that $q' \Vdash \check{y} \notin G_\emptyset^0$, since $\langle t, y \rangle \in R_{q'(0)}$. However, we also have $q' \Vdash \check{y} \in \dot{J} = G_\emptyset^0$ because $q' \leq p$. This is a contradiction, which concludes our proof. \square

We are now ready to state the main results of this section.

Theorem 3.6.19. *Let $X \subseteq {}^\kappa\kappa$ be such that $|X| > \kappa$, and let $1 < n < \omega$. Then, there exists a $<\kappa$ -closed, κ^+ -c.c. forcing notion \mathbb{P} such that $\mathbb{1}_{\mathbb{P}} \Vdash \text{ord}_{\kappa^+}(\check{X}) = n$.*

Proof. Let $\lambda = 2^\kappa$. For $n > 2$, consider the $<\kappa$ -supported forcing iteration $\mathbb{P} = \langle \mathbb{P}_\zeta, \dot{\mathbb{Q}}_\zeta \mid \zeta < \lambda \rangle$ with $\mathbb{Q}_0 = \mathbb{B}\mathbb{M}_{n-1}(\emptyset, \emptyset, X)$ and $\mathbb{P}_\zeta \Vdash \dot{\mathbb{Q}}_\zeta = \mathbb{B}\mathbb{M}_n(\dot{B}_\zeta, \check{X} \setminus \dot{B}_\zeta, \check{X})$. Using a bookkeeping argument, we can ensure that for every \mathbb{P} -generic G , the family of names \dot{B}_ζ satisfies

$$V[G] \models \forall B \in \kappa^+ \text{-}\mathbf{Bor}(X) \exists \zeta < \lambda (B = \dot{B}_\zeta).$$

By Theorem 3.6.18, we know that $V[G] \models \text{ord}_{\kappa^+}(X) \geq n$, and the rest of the iteration ensures $V[G] \models \text{ord}_{\kappa^+}(X) \leq n$ (see also Corollary 3.6.5).

For $n = 2$, consider instead the $<\kappa$ -supported forcing iteration $\mathbb{P} = \langle \mathbb{P}_\zeta, \dot{\mathbb{Q}}_\zeta \mid \zeta < \lambda \rangle$ with $\mathbb{P}_\zeta \Vdash \dot{\mathbb{Q}}_\zeta = \mathbb{B}\mathbb{M}_2(\dot{B}_\zeta, \check{X} \setminus \dot{B}_\zeta, \check{X})$. Using a bookkeeping argument, ensure that for every \mathbb{P} -generic G

$$V[G] \models \forall B \in \kappa^+ \text{-}\mathbf{Bor}(X) \exists \zeta < \lambda (B = \dot{B}_\zeta).$$

Then, we get that $V[G] \models \text{ord}_{\kappa^+}(X) \leq 2$. On the other hand, every Hausdorff space Y with $\text{ord}_{\kappa^+}(Y) = 1$ is discrete, and since every discrete subspace of ${}^\kappa\kappa$ has size at most κ , we also get $V[G] \models \text{ord}_{\kappa^+}(X) \geq 2$ because we assumed $|X| > \kappa$ (and cardinals are not collapsed by \mathbb{P}). \square

Theorem 3.6.19 can of course be applied, in particular, to κ^+ -Borel sets $X \subseteq {}^\kappa\kappa$. In this case, once we fix a κ^+ -Borel code for it, the κ^+ -Borel set X can also be naturally re-interpreted in $V[G]$ as a set $X^{V[G]}$: it is thus natural to

ask whether the κ^+ -Borel hierarchy is collapsed also on the latter space. The next corollary shows that under suitable assumptions on X , we can combine Theorem 3.6.19 with Corollary 3.5.10 and get that no new element is added to the κ^+ -Borel set at hand, so that $\text{ord}_{\kappa^+}(X^{V[G]}) = n$ as well, for the chosen $1 < n < \omega$. Since the complexity of both X and $X^{V[G]}$ is determined by the very same κ^+ -Borel code, the side effect of the equality $X = X^{V[G]}$ is that X will maintain its κ^+ -Borel complexity in the forcing extension $V[G]$; in particular, if X were closed in the ground model V , then it will stay closed also in $V[G]$. This shows that, consistently, there may be nice Polish-like spaces whose order attains intermediate values strictly between 2 and κ^+ .

Corollary 3.6.20. *Let $X \in \kappa^+\text{-Bor}({}^\kappa\kappa)$ be a κ -thin set such that $|X| > \kappa$, and let $1 < n < \omega$. Then, there is a $<\kappa$ -closed, κ^+ -c.c. forcing extension $V[G]$ of V such that $V[G] \models X^{V[G]} = X \wedge \text{ord}_{\kappa^+}(X) = n$.*

Another natural concern is whether for a given space X with $|X| > \kappa$,¹⁷ the collection $\kappa^+\text{-Bor}(X)$ of κ^+ -Borel sets coincide with the whole powerset $\mathcal{P}(X)$ of X . This is often not the case, for simple cardinality reasons. Indeed, if $w(X) \leq \kappa$ then $|\kappa^+\text{-Bor}(X)| \leq 2^\kappa$ by [AMR22, Lemma 2.3]. Hence, if we further have that $|X| = \lambda$ for some λ with $2^\lambda > 2^\kappa$, then $\kappa^+\text{-Bor}(X) \neq \mathcal{P}(X)$; in particular, this applies to any space X with $w(X) \leq \kappa$ and $|X| = 2^\kappa$, like the generalized Baire space ${}^\kappa\kappa$. However, the situation might be different for spaces X with $\kappa < |X| < 2^\kappa$. By using a longer iteration in Theorem 3.6.19, we can arrange the construction in order to add the requirement

$$V[G] \models \kappa^+\text{-Bor}(X) = \mathcal{P}(X)$$

to its conclusion: it is enough to make sure, using a suitable bookkeeping procedure, that each subset of X appears as a \dot{B}_ζ along the iteration. Thus we obtain:

Proposition 3.6.21. *Consistently, there is a closed subspace $X \subseteq {}^\kappa\kappa$ of size greater than κ and such that $\kappa^+\text{-Bor}(X) = \mathcal{P}(X)$.*

We conclude this section by observing that the our results also provide some information related to Question 5.5.1 from Chapter 5. Indeed, Corollary 5.3.4 shows that when κ is regular, then the closure under κ -limits of the class of continuous functions $\mathcal{M}_1(X, Y)$, under suitable assumptions on X and Y , coincides with $\bigcup_{1 \leq n < \omega} \mathcal{M}_n(X, Y)$. Combined with Proposition 3.3.11 and Corollary 3.6.20, this gives consistent examples of closed spaces $X \subseteq {}^\kappa\kappa$ with $|X| > \kappa$ such that all κ^+ -Borel measurable functions on X can be generated from the continuous functions through κ -limits.

Corollary 3.6.22. *Consistently, there is a κ -Polish space X (equivalently: $X \subseteq {}^\kappa\kappa$ closed) of size greater than κ such that for every spherically complete κ -metrizable space Y , such as $Y = {}^\kappa\kappa$ or $Y = {}^\kappa 2$, the collection of all κ^+ -Borel measurable functions from X to Y coincides with the closure of continuous functions under κ -limits.*

¹⁷Obviously, $\kappa^+\text{-Bor}(X) = \mathcal{P}(X)$ for every Hausdorff space X with $|X| \leq \kappa$.

3.7 The difference hierarchy

The difference hierarchy plays an important role in classical descriptive set theory because, for a Polish space X , it provides a method of constructing “from below” the classes $\Delta_{\xi+1}^0(X)$ from the classes $\Sigma_{\xi}^0(X)$ (see Theorem 1.0.1 and [Kec95, Section 22.E]). We will show that this no longer holds in generalized descriptive set theory (Theorem 3.7.14). Nevertheless, the difference hierarchy remains relevant to study in our generalized setting, both for historical reasons and because of its connections with the Wadge hierarchy (see Section 4.3.2).

In this section, we assume only that κ satisfies $2^{<\kappa} = \kappa$, and thus we allow κ to be either a regular or a singular cardinal.

Recall that every ordinal α can be written uniquely as $\alpha = \gamma + n$, with $n < \omega$ and either $\gamma = 0$ or γ limit. Accordingly, we say that α is even (respectively, odd) if n is even (respectively, odd).

Definition 3.7.1. Let X be a topological space, $\alpha \geq 1$ be an ordinal, and $(C_{\beta})_{\beta < \alpha}$ be a decreasing sequence of subsets of X . We define the set $D_{\alpha}((C_{\beta})_{\beta < \alpha})$ by

$$\begin{cases} \bigcup \{C_{\beta} \setminus C_{\beta+1} \mid \beta \text{ is even and } \beta < \gamma + 2n\} \cup C_{\gamma+2n}, & \text{if } \alpha = \gamma + 2n + 1, \\ \bigcup \{C_{\beta} \setminus C_{\beta+1} \mid \beta \text{ is even and } \beta < \gamma + 2n\}, & \text{if } \alpha = \gamma + 2n. \end{cases}$$

So, for example, $D_1(C_0) = C_0$, $D_2(\langle C_0, C_1 \rangle) = C_0 \setminus C_1$, $D_3(\langle C_0, C_1, C_2 \rangle) = (C_0 \setminus C_1) \cup C_2$, and $D_{\omega}(\langle C_n \rangle_{n < \omega}) = \bigcup_{n < \omega} (C_{2n} \setminus C_{2n+1})$.

Remark 3.7.2. Let $\alpha \geq 1$ be an ordinal and let $(C_{\beta})_{\beta < \alpha}$ be a decreasing sequence of subsets of a topological space X . Then $X \setminus D_{\alpha}((C_{\beta})_{\beta < \alpha}) = D_{\alpha+1}(\langle C'_{\beta} \rangle_{\beta < \alpha+1})$, where $(C'_{\beta})_{\beta < \alpha+1} = X \cap (C_{\beta})_{\beta < \alpha}$.

On a side note, the literature also features a different standard approach to defining differences of sets, which relies on increasing sequences of sets (in contrast to the decreasing sequences used in Definition 3.7.1). Given $\alpha \geq 1$ and an increasing sequence $(A_{\beta})_{\beta < \alpha}$ of subsets of a topological space X , set

$$D_{\alpha}^*((A_{\beta})_{\beta < \alpha}) = \begin{cases} \bigcup \{A_{\beta} \setminus \bigcup_{\beta' < \beta} A_{\beta'} \mid \beta \text{ is even, } \beta < \alpha\} & \text{if } \alpha \text{ is odd,} \\ \bigcup \{A_{\beta} \setminus \bigcup_{\beta' < \beta} A_{\beta'} \mid \beta \text{ is odd, } \beta < \alpha\} & \text{if } \alpha \text{ is even.} \end{cases}$$

These two definitions agree only in part, as the next result illustrates (see also [Fou16, Fact 4.1]).

Fact 3.7.3. Let $\alpha \geq 1$ be an ordinal, and $(C_{\beta})_{\beta < \alpha}$ be a decreasing sequence of subsets of a topological space X . For every $\beta < \alpha$, let $A_{\beta} = X \setminus C_{\beta}$. Then,

$$D_{\alpha}((C_{\beta})_{\beta < \alpha}) = \begin{cases} X \setminus D_{\alpha}^*((A_{\beta})_{\beta < \alpha}) & \text{if } \alpha \text{ is odd,} \\ D_{\alpha}^*((A_{\beta})_{\beta < \alpha}) & \text{if } \alpha \text{ is even.} \end{cases}$$

Given a boldface pointclass $\mathbf{\Gamma}$, a topological space X , and ordinal $\alpha \geq 1$, we let

$$\mathcal{D}_{\alpha}(\mathbf{\Gamma}(X)) = \{D_{\alpha}((C_{\beta})_{\beta < \alpha}) \mid C_{\beta} \in \mathbf{\Gamma}(X) \text{ for all } \beta < \alpha\}.$$

Letting X vary over all topological spaces, it is easy to verify that $\mathcal{D}_\alpha(\Gamma)$ is a boldface pointclass. We also define its dual pointclass $\check{\mathcal{D}}_\alpha(\Gamma)$ and its ambiguous pointclass $\Delta_{\mathcal{D}_\alpha(\Gamma)}$. We set $\mathcal{D}_0(\Gamma(X)) = \{\emptyset\}$ for ease of notation. Note that $\mathcal{D}_1(\Gamma) = \Gamma$, and if Γ is hereditary then $\mathcal{D}_\alpha(\Gamma)$ and $\check{\mathcal{D}}_\alpha(\Gamma)$ are hereditary too.

We call **difference hierarchy over Γ** (or **Γ -difference hierarchy**) on X the hierarchy formed by the classes $\mathcal{D}_\alpha(\Gamma(X))$, $\check{\mathcal{D}}_\alpha(\Gamma(X))$, $\Delta_{\mathcal{D}_\alpha(\Gamma)}(X)$ where α ranges over nonzero ordinals. In this thesis, we will mainly work with $\Gamma = \kappa^+ \text{-}\Pi_1^0$ (see Section 4.3).

One can easily see that the Γ -difference hierarchy on X is increasing, meaning that $\mathcal{D}_\alpha(\Gamma(X)) \subseteq \mathcal{D}_\beta(\Gamma(X))$ for any $1 \leq \alpha \leq \beta$. Together with Remark 3.7.2, we get that for every $1 \leq \alpha < \beta$:

$$\mathcal{D}_\alpha(\Gamma(X)), \check{\mathcal{D}}_\alpha(\Gamma(X)) \subseteq \Delta_{\mathcal{D}_\beta(\Gamma)}(X) \subseteq \mathcal{D}_\beta(\Gamma(X)), \check{\mathcal{D}}_\beta(\Gamma(X)). \quad (3.7.1)$$

If $\gamma = \min\{\alpha \in \text{Ord} \mid \mathcal{D}_\alpha(\Gamma(X)) = \check{\mathcal{D}}_\alpha(\Gamma(X))\}$, we say that the Γ -difference hierarchy on X has length γ .

Remark 3.7.4. Let $\alpha < \kappa^+$, X be a topological space, and Γ be a nonselfdual boldface pointclass closed under finite intersections. If $C \in \mathcal{D}_\alpha(\Gamma(X))$ and $F \in \Gamma(X)$, then $C \cap F \in \mathcal{D}_\alpha(\Gamma(X))$. The proof is an easy induction on $\alpha \geq 1$. The case $\alpha = 1$ is trivial, so assume $\alpha > 1$ and let $C = D_\alpha((C_i)_{i < \alpha})$ for some sequence of sets $(C_i)_{i < \alpha}$ in $\Gamma(X)$. Then, $(C_i \cap F)_{i < \alpha}$ is again a sequence of sets in $\Gamma(X)$ and $D_\alpha((C_i)_{i < \alpha}) \cap F = D_\alpha((C_i \cap F)_{i < \alpha})$.

Remark 3.7.5. Let $\alpha < \kappa^+$ be a limit ordinal, X be a topological space, and Γ be a nonselfdual boldface pointclass closed under intersections of size less than α . If $(C_i)_{i < \alpha}$ is a decreasing sequence of sets in $\Gamma(X)$ such that $\bigcap_{i < \alpha} C_i = \emptyset$, then $D_\alpha((C_i)_{i < \alpha}) \in \check{\mathcal{D}}_\alpha(\Gamma(X))$. Indeed, $X \setminus D_\alpha((C_i)_{i < \alpha}) = D_\alpha((C'_i)_{i < \alpha})$ where $C'_0 = X$, $C'_{i+1} = C_i$ for every $i < \alpha$, and for $i < \alpha$ limit, $C'_i = \bigcap_{j < i} C_j$.

From this point onward (and unless stated otherwise)

all topological spaces are assumed to be regular Hausdorff and of weight at most κ .

Lemma 3.7.6. *Let $(C_i)_{i < \rho}$ a strictly monotone (i.e., decreasing or increasing) sequence of closed subsets of X . Then, $\rho < \kappa^+$.*

This holds similarly for strictly monotone sequences of open sets.

Proof. Let $\{U_j \mid j < \kappa\}$ be a basis for X . For every closed $C \subseteq X$, let $\mathcal{J}(C) = \{j < \kappa \mid U_j \cap C \neq \emptyset\}$. Clearly, $X \setminus C = \bigcup \{U_j \mid j \notin \mathcal{J}(C)\}$ and $C \subseteq C'$ implies $\mathcal{J}(C) \subseteq \mathcal{J}(C')$. Then, the map $C \mapsto \mathcal{J}(C)$ is injective and strictly monotone. Thus for any strictly monotone sequence $(C_i)_{i < \rho}$, $(\mathcal{J}(C_i))_{i < \rho}$ is a strictly monotone sequence of subsets of κ , so obviously $\rho < \kappa^+$. \square

We now turn our attention to the existence of universal sets for the classes appearing in the difference hierarchy.

Proposition 3.7.7. *Let Γ be a hereditary boldface pointclass closed under intersections of length at most κ , and such that $\Gamma(X)$ admits a ${}^\kappa 2$ -universal set. For every $1 < \alpha < \kappa^+$, there are ${}^\kappa 2$ -universal sets for both $\mathcal{D}_\alpha(\Gamma(X))$ and $\check{\mathcal{D}}_\alpha(\Gamma(X))$. Therefore, for every $1 < \alpha < \kappa^+$, there are subsets of ${}^\kappa 2$ that*

are κ -complete for $\mathcal{D}_\alpha(\Gamma)$ and $\check{\mathcal{D}}_\alpha(\Gamma)$, respectively. In contrast, there is no ${}^\kappa 2$ -universal set for $\Delta_{\mathcal{D}_\alpha(\Gamma)}({}^\kappa 2)$.

Moreover, ${}^\kappa 2$ can systematically be replaced by X in all the above statements if ${}^\kappa 2$ embeds into X .

Proof. Fix a ${}^\kappa 2$ -universal set $\mathcal{U} \subseteq {}^\kappa 2 \times X$ for $\Gamma(X)$. Fix $1 < \alpha < \kappa^+$ and let $\langle \cdot, \cdot \rangle: \alpha \times \kappa \rightarrow \kappa$ be the canonical pairing function. For any $\delta < \alpha$, let $f_\delta: {}^\kappa 2 \rightarrow {}^\kappa 2$ be defined by $f_\delta(y)(i) = y(\langle \delta, i \rangle)$ for every $i < \kappa$. Note that each f_δ is continuous. For every $\delta < \alpha$, the set $W_\delta = \{(y, x) \in {}^\kappa 2 \times X \mid (f_\delta(y), x) \in \mathcal{U}\}$ belongs to $\Gamma({}^\kappa 2 \times X)$ because it is the preimage of \mathcal{U} under the continuous function $f_\delta \times \text{id}_X$. For every $\delta < \alpha$, set $V_\delta = \bigcap_{\gamma \leq \delta} W_\gamma$. Then, each V_δ is in $\Gamma({}^\kappa 2 \times X)$ since Γ is closed under intersections of size at most κ .

We set $\mathcal{V}^\alpha = D_\alpha((V_\delta)_{\delta < \alpha})$ and we claim that \mathcal{V}^α is a ${}^\kappa 2$ -universal set for $\mathcal{D}_\alpha(\Gamma(X))$. It is clear that $\mathcal{V}^\alpha \in \mathcal{D}_\alpha(\Gamma({}^\kappa 2 \times X))$. To complete the proof, fix $A \in \mathcal{D}_\alpha(\Gamma(X))$ and let $(A_\delta)_{\delta < \alpha}$ be a decreasing sequence of subsets in $\Gamma(X)$ such that $A = D_\alpha((A_\delta)_{\delta < \alpha})$. Since \mathcal{U} is ${}^\kappa 2$ -universal for $\Gamma(X)$, we can fix $y_\delta \in {}^\kappa 2$ such that $\mathcal{U}_{y_\delta} = A_\delta$ for every $\delta < \alpha$. Set $\bar{y} \in {}^\kappa 2$ such that for every $\delta < \alpha$, $f_\delta(\bar{y}) = y_\delta$, i.e., $\bar{y}(\langle \delta, i \rangle) = y_\delta(i)$ for every $i < \kappa$. It is straightforward to verify that $\mathcal{V}_{\bar{y}}^\alpha = A$. Indeed, for any $x \in {}^\kappa 2$,

$$\begin{aligned} x \in \mathcal{V}_{\bar{y}}^\alpha &\Leftrightarrow (\bar{y}, x) \in \mathcal{V}^\alpha \Leftrightarrow (\bar{y}, x) \in D_\alpha((V_\delta)_{\delta < \alpha}) \\ &\Leftrightarrow x \in D_\alpha((\mathcal{U}_{y_\delta})_{\delta < \alpha}) \Leftrightarrow x \in D_\alpha((A_\delta)_{\delta < \alpha}) = A \end{aligned}$$

where the only non trivial equivalence is the third one and it holds because for every $\delta < \alpha$:

$$\begin{aligned} (\bar{y}, x) \in V_\delta &\Leftrightarrow \forall \gamma \leq \delta ((f_\delta(\bar{y})), x) \in \mathcal{U} \\ &\Leftrightarrow \forall \gamma \leq \delta ((y_\gamma), x) \in \mathcal{U} \\ &\Leftrightarrow x \in \bigcap_{\gamma \leq \delta} \mathcal{U}_{y_\gamma} \\ &\Leftrightarrow x \in \mathcal{U}_{y_\delta}. \end{aligned}$$

For the second part of the statement, when $X = {}^\kappa 2$ apply Lemma 2.6.2 to obtain the existence of the complete sets, or Lemma 2.6.3 to obtain the non-existence of ${}^\kappa 2$ -universal sets for the selfdual classes (relatively to the space ${}^\kappa 2$ itself).

Assume now that there is an embedding $f: {}^\kappa 2 \rightarrow X$. Then Lemma 2.6.1 and the first part ensure that there are X -universal sets for both $\mathcal{D}_\alpha(\Gamma(X))$ and $\check{\mathcal{D}}_\alpha(\Gamma(X))$. As for complete sets, let A' be κ -complete for $\mathcal{D}_\alpha(\Gamma({}^\kappa 2))$, and let $A \in \mathcal{D}_\alpha(\Gamma(X))$ be such that $A \cap f({}^\kappa 2) = f(A')$: then A is κ -complete for $\mathcal{D}_\alpha(\Gamma(X))$. The case of $\check{\mathcal{D}}_\alpha(\Gamma(X))$ is similar. The non-existence of X -universal sets for $\Delta_{\mathcal{D}_\alpha(\Gamma(X))}$ again follows from Lemma 2.6.3. \square

Corollary 3.7.8. *Let Γ be a hereditary boldface pointclass closed under intersections of length at most κ , and such that $\Gamma(X)$ admits a ${}^\kappa 2$ -universal set. Then, for every $1 < \alpha < \kappa^+$, $\mathcal{D}_\alpha(\Gamma(X))$ is non-selfdual.*

Corollary 3.7.9. *Let Γ be a non-selfdual boldface pointclass closed under intersections of length at most κ . If $\Gamma({}^\kappa 2)$ admits a ${}^\kappa 2$ -universal set, then the Γ -difference hierarchy on ${}^\kappa 2$ has length κ^+ .*

Proof. Let γ be the length of the Γ -difference hierarchy on ${}^\kappa 2$. By Lemma 3.7.6, $\gamma \leq \kappa^+$. Towards a contradiction, suppose that $\gamma < \kappa^+$, so that $\mathcal{D}_\gamma(\Gamma)$ is selfdual on ${}^\kappa 2$ because $\mathcal{D}_\gamma(\Gamma) = \check{\mathcal{D}}_\gamma(\Gamma)$. Then the existence of a ${}^\kappa 2$ -universal set for $\mathcal{D}_\gamma(\Gamma)$, granted by Proposition 3.7.7, contradicts Lemma 2.6.3. \square

By Corollary 3.7.9 together with equation (3.7.1), we conclude that for every $1 \leq \alpha < \beta < \kappa^+$, $\mathcal{D}_\alpha(\Gamma(X)) \subsetneq \mathcal{D}_\beta(\Gamma(X))$.

Before concluding this section, we state the following technical lemma, which concerns a basic property of the classes in the κ^+ - Π_1^0 -difference hierarchy in the case where κ is regular. This result will be also used in Chapter 4.

Lemma 3.7.10. *Assume that κ is regular. Let $\{U_i \mid i < \kappa\} \cup \{C\}$ be a partition of a clopen set $Z \subseteq {}^\kappa 2$ such that $C \in \kappa^+$ - $\Pi_1^0(Z)$ and $U_i \in \kappa^+$ - $\Delta_1^0(Z)$ for every $i < \kappa$. Let $A \subseteq {}^\kappa 2$. If $A \cap C = \emptyset$ and there exists an even $\beta < \kappa^+$ such that $A \cap U_i \in \mathcal{D}_\beta(\kappa^+$ - $\Pi_1^0(Z))$ for every $i < \kappa$, then $A \in \mathcal{D}_\beta(\kappa^+$ - $\Pi_1^0(Z))$.*

If $C = \emptyset$, then the result holds for β odd as well.

Proof. Since $A \cap U_i \in \mathcal{D}_\beta(\kappa^+$ - $\Pi_1^0(Z))$ for every $i < \kappa$ and $\mathcal{D}_\beta(\kappa^+$ - $\Pi_1^0)$ is hereditary, $A \in \mathcal{D}_\beta(\kappa^+$ - $\Pi_1^0(U_i))$. Then, there exists a decreasing sequence $\langle F_j^i \mid j < \beta \rangle$ of subsets closed in U_i (hence closed in Z) such that $A \cap U_i = D_\beta((F_j^i)_{j < \beta})$. Set $F_j = \text{cl}(\bigcup_{i < \kappa} F_j^i) \cup C$ for every $j < \beta$.

We claim that for every $j < \beta$, $F_j = \bigcup_{i < \kappa} F_j^i \cup C$. To see this, suppose that $(y_\gamma)_{\gamma < \kappa}$ is a sequence in $\bigcup_{i < \kappa} F_j^i$ which has limit $y \in Z \setminus C$. Then $y \in U_i$ for some $i < \kappa$. By definition of convergence, there exists $\gamma_0 < \kappa$ such that $y_\gamma \in U_i$ for all $\gamma \geq \gamma_0$. Thus, $y_\gamma \in F_j^i$ for all $\alpha \geq \alpha_0$ and since F_j^i is closed, we have that $y \in F_j^i \subseteq \bigcup_{i < \kappa} F_j^i$ as desired.

Finally, we claim that $A \cap Z = D_\beta((F_j)_{j < \beta})$, hence $A \in \mathcal{D}_\beta(\kappa^+$ - $\Pi_1^0(Z))$. To prove this, suppose that $x \in A \cap Z$. Since $A \cap C = \emptyset$ and $\{U_i \mid i < \kappa\}$ is a partition of $Z \setminus C$, $x \in A \cap U_i$ for some $i < \kappa$. If β is even, then $x \in F_j^i \setminus F_{j+1}^i$ for some even $j < \beta$ which implies $x \in F_j$ and $x \in D_\beta((F_j)_{j < \beta})$. If β is odd, then $\beta = \beta' + 1$ some β' , and either $x \in F_j^i \setminus F_{j+1}^i$ for some even $j < \beta$ or $x \in F_{\beta'}^i$; thus either $x \in F_j$ for $j < \beta$ even, or $x \in F_{\beta'}$. In both cases we have $x \in D_\beta((F_j)_{j < \beta})$.

Vice versa, suppose $x \in D_\beta((F_j)_{j < \beta})$. Assume first that β is even. Then, $x \in F_j \setminus F_{j+1}$ for some even $j < \beta$, so in particular $x \in F_j = \bigcup_{i < \kappa} F_j^i \cup C$. Note that $x \notin C$ because β is even and $C \subseteq \bigcap_{j < \beta} F_j$. Thus $x \in \bigcup_{i < \kappa} F_j^i$. Since $F_j^i \subseteq U_i$ for every $i < \kappa$, and $\{U_i \mid i < \kappa\}$ is a partition of ${}^\kappa 2 \setminus C$, $x \in F_j^i$ for some $i < \kappa$. Since j is even, $x \in D_\beta((F_j^i)_{j < \beta}) = A \cap U_i \subseteq A$. Assume now that $\beta = \beta' + 1$ is odd and $C = \emptyset$. Then either $x \in F_j \setminus F_{j+1}$ for some even $j < \beta$ or $x \in F_{\beta'}$; thus either $x \in \bigcup_{i < \kappa} F_j^i$ for $j < \beta$ even, or $x \in \bigcup_{i < \kappa} F_{\beta'}^i$. In both cases (either $j < \beta$ even or $j = \beta'$) we get that there exists $i < \kappa$ such that $x \in F_j^i$, hence $x \in D_\beta((F_j^i)_{j < \beta}) = A \cap U_i$.

Observe that while in the first part of the Claim β needs to be even so that $x \in C$ implies $x \notin D_\beta((F_j)_{j < \beta})$, in the second part the assumption $C = \emptyset$ avoids the issue. \square

3.7.1 Failure of the Hausdorff-Kuratowski theorem

One of the fundamental results in classical descriptive set theory is the Hausdorff-Kuratowski theorem (Theorem 1.0.1), which provides a precise characterization of the ambiguous pointclasses at the successor level of the Borel hierarchy. More precisely, for any Polish space X and any $1 \leq \xi < \omega_1$, $\Delta_{\xi+1}^0(X) = \bigcup_{\alpha < \omega_1} \mathcal{D}_\alpha(\Pi_\xi^0(X))$.

In the next results we prove that, in the generalized setting, the κ^+ - Π_1^0 -difference hierarchy does not always exhaust the class κ^+ - Δ_2^0 . This demonstrates a fundamental divergence between classical and generalized descriptive set theory.

We begin with a lemma that provides sufficient conditions for a set $Z \subseteq X$ to fail to belong to the κ^+ - $\Pi_1^0(X)$ -difference hierarchy. Observe that when $X \subseteq {}^{<\kappa}2$, Proposition 4.3.36 actually yields a stronger result than Lemma 3.7.11. Nonetheless, since Lemma 3.7.11 applies more generally to any (regular Hausdorff) topological space X of weight at most κ , we choose to present it here.

Lemma 3.7.11. *If $Z \subseteq X$ is non-empty, dense and codense in X , then for every $\alpha < \kappa^+$, $Z \notin \mathcal{D}_\alpha(\kappa^+ \text{-}\Pi_1^0(X))$.*

Proof. Let \mathcal{T}_X be the tree of X . We proceed by induction on $\alpha < \kappa^+$, showing that for every $t \in \mathcal{T}_X$, $\mathbf{N}_t^X \cap Z \notin \mathcal{D}_\alpha(\kappa^+ \text{-}\Pi_1^0(X))$. Note that it is enough to work with even α 's, since $\mathcal{D}_\alpha(\kappa^+ \text{-}\Pi_1^0(X)) \subseteq \mathcal{D}_{\alpha+1}(\kappa^+ \text{-}\Pi_1^0(X))$.

There is nothing to prove for the base case $\alpha = 0$, so we assume $\alpha > 0$. We first consider the case $\alpha = 2$. We fix $t \in \mathcal{T}_X$ and we argue by contradiction. Assume that $\mathbf{N}_t^X \cap Z = C_0 \setminus C_1$ for some $C_0, C_1 \in \kappa^+ \text{-}\Pi_1^0(X)$ such that $C_1 \subseteq C_0$. Without loss of generality, we can assume that $C_1 \subseteq C_0 \subseteq \mathbf{N}_t^X$, because if not we can simply consider $C'_0 = C_0 \cap \mathbf{N}_t^X$ and $C'_1 = C_1 \cap \mathbf{N}_t^X$. Note that it must be $C_0 \neq \mathbf{N}_t^X$, otherwise $\mathbf{N}_t^X \cap Z = \mathbf{N}_t^X \setminus C_1 \in \kappa^+ \text{-}\Sigma_1^0(X)$, and since Z is codense in X by hypothesis, it would lead to the impossible conclusion that $\mathbf{N}_t^X \cap Z \cap (X \setminus Z) \neq \emptyset$. However, since Z is dense in X , if $\mathbf{N}_t^X \setminus C_0 \in \kappa^+ \text{-}\Sigma_1^0(X)$ is nonempty, then $(\mathbf{N}_t^X \setminus C_0) \cap Z = (\mathbf{N}_t^X \cap Z) \setminus C_0 \neq \emptyset$, which is again impossible since $\mathbf{N}_t^X \cap Z = C_0 \setminus C_1 \subseteq C_0$.

Assume now that $\alpha > 2$ is a successor ordinal, so that $\alpha = \gamma + 2$ for some (even) ordinal $1 < \gamma < \kappa^+$. We fix $t \in \mathcal{T}_X$ and we argue again by contradiction. Assume that $\mathbf{N}_t^X \cap Z = D_\alpha((C_\beta)_{\beta < \alpha})$ for some decreasing sequence $(C_\beta)_{\beta < \alpha} \subseteq \kappa^+ \text{-}\Pi_1^0(X)$. Since α is even, $\mathbf{N}_t^X \cap Z = \bigcup \{C_\beta \setminus C_{\beta+1} \mid \beta \text{ even}, \beta < \alpha\}$. Without loss of generality, we can assume that $C_\beta \subseteq \mathbf{N}_t^X$ for every $\beta < \alpha$, because if not we can simply consider $C_\beta \cap \mathbf{N}_t^X$. Notice that if $C_\gamma = \mathbf{N}_t^X$, then $C_\beta = \mathbf{N}_t^X$ for every $\beta < \gamma$, and we would get that $\mathbf{N}_t^X \cap Z = \mathbf{N}_t^X \setminus C_{\gamma+1} \in \kappa^+ \text{-}\Sigma_1^0(X)$, leading to the impossible conclusion that $\mathbf{N}_t^X \cap Z \cap (X \setminus Z) \neq \emptyset$ by codensity of Z in X . Therefore, we can assume that $C_\gamma \neq \mathbf{N}_t^X$, and since $C_\gamma \in \kappa^+ \text{-}\Pi_1^0(X)$, it follows that there exists some $u \in {}^{<\kappa}2$, $t \subseteq u$, such that $C_\gamma \cap \mathbf{N}_u^X = \emptyset$. This implies that $(C_\gamma \setminus C_{\gamma+1}) \cap \mathbf{N}_u^X = \emptyset$, hence $\mathbf{N}_u^X \cap Z = \mathbf{N}_u^X \cap \bigcup \{C_\beta \setminus C_{\beta+1} \mid \beta \text{ even}, \beta < \gamma\} = D_\gamma((C_\beta \cap \mathbf{N}_u^X)_{\beta < \gamma}) \in \mathcal{D}_\gamma(\kappa^+ \text{-}\Pi_1^0(X))$, contradicting the inductive hypothesis.

Assume now that α is a limit ordinal. Again, we fix $t \in \mathcal{T}_X$ and argue by contradiction. We assume that $\mathbf{N}_t^X \cap Z = D_\alpha((C_\beta)_{\beta < \alpha}) = \bigcup \{C_\beta \setminus C_{\beta+1} \mid \beta \text{ even}, \beta < \alpha\}$ for some decreasing sequence $(C_\beta)_{\beta < \alpha} \subseteq \kappa^+ \text{-}\Pi_1^0(X)$. Arguing as before, we can assume that $C_\beta \subseteq \mathbf{N}_t^X$ for every $\beta < \alpha$. Note that if

$C_\beta = N_t^X$ for every even $\beta < \alpha$, we would get that $N_t^X \cap Z = \emptyset$, against the hypothesis of density of Z . Therefore, we assume that there exists some even $\gamma < \alpha$ such that $C_\gamma \neq N_t^X$. Since $C_\gamma \in \kappa^+ \text{-}\mathbf{\Pi}_1^0(X)$, there exists $u \in {}^{<\kappa^2}$, $t \subseteq u$, such that $C_\gamma \cap N_u^X = \emptyset$. This implies that $N_u^X \cap Z = N_u^X \cap \bigcup \{C_\beta \setminus C_{\beta+1} \mid \beta \text{ even}, \beta < \gamma\} = D_\gamma((C_\beta \cap N_u^X)_{\beta < \gamma}) \in \mathcal{D}_\gamma(\kappa^+ \text{-}\mathbf{\Pi}_1^0(X))$, contradicting the inductive hypothesis.

We conclude that for every $t \in \mathcal{T}_X$ and for every $\alpha < \kappa^+$, $N_t^X \cap Z \notin \mathcal{D}_\alpha(\kappa^+ \text{-}\mathbf{\Pi}_1^0(X))$. \square

The next example provides a counterexample to the generalized version of the Hausdorff–Kuratowski theorem.

Example 3.7.12. The set $\mathcal{Y} = \{x \in {}^{\kappa^2} \mid |\{\alpha < \kappa \mid x(\alpha) = 1\}| < \aleph_0\}$ is closed and of size κ . We define:

$$\begin{aligned} \mathcal{Y}_0 &= \{x \in {}^{\kappa^2} \mid \exists n < \omega \mid \{\alpha < \kappa \mid x(\alpha) = 1\} = 2n\}, \\ \mathcal{Y}_1 &= \{x \in {}^{\kappa^2} \mid \exists n < \omega \mid \{\alpha < \kappa \mid x(\alpha) = 1\} = 2n + 1\} = \mathcal{Y} \setminus \mathcal{Y}_0. \end{aligned}$$

Clearly, $\mathcal{Y}_0 \in \kappa^+ \text{-}\mathbf{\Delta}_2^0(\mathcal{Y})$ as both \mathcal{Y}_0 and \mathcal{Y}_1 have size κ . Moreover, we have $\mathcal{Y}_0 \notin \mathcal{D}_\alpha(\kappa^+ \text{-}\mathbf{\Pi}_1^0(\mathcal{Y}))$ for every $\alpha < \kappa^+$ by Lemma 3.7.11. In fact, \mathcal{Y}_0 is both dense and codense in \mathcal{Y} . To see this, note that every basic open subset of \mathcal{Y} has the form N_t with $|\{\alpha < \kappa \mid t(\alpha) = 1\}| < \aleph_0$. Given such a t , we can easily construct $y_0, y_1 \in \mathcal{Y}$ with $t \subseteq y_0, y_1$, where $y_0 \in \mathcal{Y}_0$ and $y_1 \in \mathcal{Y}_1$, by extending t with one or two additional 1's (depending on how many 1's already appear in t and whether we want the resulting point to lie in \mathcal{Y}_0 or \mathcal{Y}_1), and then assigning 0 to all of the remaining coordinates.

One can straightforwardly construct further families of sets that yield counterexamples to the same theorem, such as the following.

Example 3.7.13. Let $\lambda < \text{cof}(\kappa)$ be an infinite ordinal. The set $\mathcal{Y}^{(\lambda)} = \{x \in {}^{\kappa^2} \mid \text{ot}(\{\alpha < \kappa \mid x(\alpha) = 1\}) < \lambda\}$ is closed and of size κ . We define:

$$\begin{aligned} \mathcal{Y}_0^{(\lambda)} &= \{x \in {}^{\kappa^2} \mid \text{ot}(\{\alpha < \kappa \mid x(\alpha) = 1\}) \text{ is even}\}, \\ \mathcal{Y}_1^{(\lambda)} &= \{x \in {}^{\kappa^2} \mid \text{ot}(\{\alpha < \kappa \mid x(\alpha) = 1\}) \text{ is odd}\} = \mathcal{Y}^{(\lambda)} \setminus \mathcal{Y}_0^{(\lambda)}. \end{aligned}$$

Theorem 3.7.14. For every $1 \leq \xi < \kappa^+$,

$$\bigcup_{\alpha < \kappa^+} \mathcal{D}_\alpha(\kappa^+ \text{-}\mathbf{\Pi}_\xi^0(X)) \subseteq \kappa^+ \text{-}\mathbf{\Delta}_{\xi+1}^0(X).$$

Moreover, if there is a topological embedding $f: {}^{\kappa^2} \rightarrow X$ with closed range, then there exists a closed set $Y \subseteq X$ such that

$$\bigcup_{\alpha < \kappa^+} \mathcal{D}_\alpha(\kappa^+ \text{-}\mathbf{\Pi}_1^0(Y)) \subsetneq \kappa^+ \text{-}\mathbf{\Delta}_2^0(Y).$$

Proof. For every $\alpha < \kappa^+$, one can easily see that $\mathcal{D}_\alpha(\kappa^+ \text{-}\mathbf{\Pi}_\xi^0(X)) \subseteq \kappa^+ \text{-}\mathbf{\Sigma}_{\xi+1}^0(X)$ using Definition 3.7.1, the fact that $\kappa^+ \text{-}\mathbf{\Sigma}_\xi^0(X) \cup \kappa^+ \text{-}\mathbf{\Pi}_{\xi+1}^0(X) \subseteq \kappa^+ \text{-}\mathbf{\Delta}_{\xi+1}^0(X)$, and the closure of $\kappa^+ \text{-}\mathbf{\Sigma}_{\xi+1}^0(X)$ under unions of length κ (see Proposition 3.3.1). Moreover, by equation (3.7.1), $\check{\mathcal{D}}_\alpha(\kappa^+ \text{-}\mathbf{\Pi}_\xi^0(X)) \subseteq \mathcal{D}_{\alpha+1}(\kappa^+ \text{-}\mathbf{\Pi}_\xi^0(X))$, hence we get $\bigcup_{\alpha < \kappa^+} \mathcal{D}_\alpha(\kappa^+ \text{-}\mathbf{\Pi}_\xi^0(X)) \subseteq \kappa^+ \text{-}\mathbf{\Delta}_{\xi+1}^0(X)$.

Assume now that there is a topological embedding $f: {}^\kappa 2 \rightarrow X$. By Theorem 3.3.9, the κ^+ -Borel hierarchy on X does not collapse. To prove that the inclusion is strict, it is enough to consider $f(\mathcal{Y})$ and $f(\mathcal{Y}_0)$, where \mathcal{Y} and \mathcal{Y}_0 are the sets from Example 3.7.12. Note that $f(\mathcal{Y})$ is closed since \mathcal{Y} and $\text{ran}(f)$ are closed. By Lemma 3.7.11, $\mathcal{Y}_0 \notin \bigcup_{\alpha < \kappa^+} \mathcal{D}_\alpha(\kappa^+ \text{-}\mathbf{\Pi}_1^0(\mathcal{Y}))$, so $f(\mathcal{Y}_0) \notin \bigcup_{\alpha < \kappa^+} \mathcal{D}_\alpha(\kappa^+ \text{-}\mathbf{\Pi}_1^0(f(\mathcal{Y})))$. Moreover, $\mathcal{Y}_0 \in \kappa^+ \text{-}\mathbf{\Delta}_2^0(\mathcal{Y})$, therefore $f(\mathcal{Y}_0) \in \kappa^+ \text{-}\mathbf{\Delta}_2^0(f(\mathcal{Y}))$. \square

Chapter 4

The Wadge hierarchy on the generalized Cantor space

As mentioned in the introduction to this thesis, classical descriptive set theory provides standard tools for measuring the topological complexity of subsets of Polish spaces. A central component of this theory is Wadge theory, which classifies sets through the lens of continuous reducibility. The resulting Wadge hierarchy offers a classification that refines many fundamental hierarchies in descriptive set theory, including the Borel hierarchy and the Hausdorff–Kuratowski difference hierarchy.

We begin with a brief overview. For a more comprehensive and detailed presentation of the concepts introduced below, the reader is referred to Chapter 2. Consider two topological spaces X and Y , along with subsets $A \subseteq X$ and $B \subseteq Y$. A reduction from A to B is any map $f : X \rightarrow Y$ satisfying $f^{-1}(B) = A$. When the reduction f is continuous, we say that A is Wadge reducible to B , or continuously reducible, written $A \leq_W^{X,Y} B$; when f is Lipschitz, we say that A is Lipschitz reducible to B , in symbols $A \leq_L^{X,Y} B$. The strict ordering is denoted by $<_W^{X,Y}$, while equivalence under Wadge reduction is expressed as $\equiv_W^{X,Y}$. When both sets belong to the same space X , we write \leq_W^X , $<_W^X$ and \equiv_W^X . The Wadge reducibility relation is a preorder (being both transitive and reflexive), and \equiv_W^X is an equivalence relation whose equivalence classes are called Wadge degrees. The preorder \leq_W^X induces a partial order on the collection of Wadge degrees, thus generating the Wadge hierarchy on X . All of the preceding discussion applies similarly to Lipschitz reducibility. For a more comprehensive presentation of these notions, and more generally of reducibilities, we refer the reader to Section 2.8.

We also recall from Section 2.12 the alternative, yet equivalent, formulation of the Wadge hierarchy using Wadge classes, as we will employ both perspectives, depending on convenience. A boldface pointclass Γ on a topological space X is a Wadge class when $\Gamma = A \downarrow_X = \{B \subseteq X \mid B \leq_W^X A\}$ for some $A \subseteq X$; any element $A \in \Gamma$ satisfying $\Gamma(X) = A \downarrow_X$ is called X -complete (or κ -complete, when $X = {}^\kappa 2$) for Γ , and we say that A generates $\Gamma(X)$. We also recall that if Γ is a nonselfdual boldface pointclass, then $\text{SLO}_W(\Gamma({}^\kappa 2))$ implies that $\Gamma({}^\kappa 2)$ is a Wadge class (Lemma 2.11.3).

Although continuous preimages were already standard objects in general topology and had appeared in various contexts, it was William W. Wadge

[Wad83] who first carried out a systematic investigation of the structural properties of the relation \leq_W itself on the (classical) Baire space ${}^\omega\omega$. His breakthrough was to reformulate \leq_W in terms of infinite two-player games (see Section 2.10), thereby introducing game-theoretic methods and determinacy assumptions that became standard tools in descriptive set theory.

In subsequent years, many set theorists developed and extended Wadge's results, achieving a comprehensive characterization of the Wadge hierarchy and transforming the theory of Wadge degrees into a sophisticated area of set theory. The study of Wadge theory has attracted considerable interest, as evidenced by [VW78, Lou83, LSR88, Dup01, KM19, AM03, And03, And06, And07, MRS14, CMM22, CMRS25] and numerous other contributions. A variety of generalizations have been explored, including modifications to the reducibility notion, typically replacing continuous functions with other classes such as Lipschitz or Borel maps [AM03, And06, MR09a, MR10a, MR10b], and substituting the Baire space with other topological spaces [MRSS15, Sch18, CMRS25, CM21, CDD25].

However, when moving to generalized descriptive set theory and analyzing the Wadge hierarchy, the available results are scarce, and the overall picture becomes substantially more complex. In this work, we choose to work with the generalized Cantor space ${}^\kappa 2$ for uncountable cardinals κ satisfying $2^{<\kappa} = \kappa$, although analogous results can be obtained by working with ${}^{\text{cof}(\kappa)}\kappa$. In particular, recall that if κ is not weakly compact, these spaces are homeomorphic (see, for example, [AMR22, Proposition 6.6]), and consequently the same results apply verbatim. Instead, the case in which κ is weakly compact exhibits certain minor differences, which we shall address in detail later. The main obstacle when $\kappa > \omega$ is that no higher analogue of Borel determinacy is available in this context; consequently, most of the game-theoretic techniques that made the classical theory so tractable are no longer applicable (see Section 2.9). To contextualize these challenges, we first review what is known in the classical setting.

The Wadge hierarchy on the classical Cantor space

Historically, most of the study of the Wadge hierarchy has been conducted on the classical Baire space ${}^\omega\omega$. Working in ZFC, the Wadge order \leq_W^ω restricted to Borel subsets of ${}^\omega\omega$ is well-founded and satisfies the Semi-Linear Ordering principle $\text{SLO}_W(\mathbf{Bor}({}^\omega\omega))$: for all Borel sets $A, B \subseteq {}^\omega\omega$,

$$A \leq_W^\omega B \quad \text{or} \quad \neg B \leq_W^\omega A.$$

The latter statement is also known as Wadge's Lemma; both of these foundational results were proved in [Wad83] and rely on Borel determinacy, which was established by Martin in [Mar75]. Under the Axiom of Determinacy (AD), this picture extends fully: the Wadge hierarchy on ${}^\omega\omega$ becomes semi-well-ordered, in the sense that \leq_W^ω is well-founded (a result proven by Martin and Monk) and $\text{SLO}_W({}^\omega\omega)$ holds. Analogous statements hold if we replace \leq_W^ω by \leq_L^ω , or if we work over the Cantor space instead of the Baire space [AC13].

The structure of this hierarchy is well understood. We know that the least Wadge degrees are those in the non-selfdual pair $\{\{^\omega\omega\}, \{\emptyset\}\}$, and just above this minimal pair lies a unique selfdual degree containing all non-trivial clopen

subsets of ${}^\omega \omega$. From this point onward, non-selfdual pairs and selfdual degrees alternate throughout the hierarchy.

The Wadge hierarchies on the Cantor and Baire spaces are structurally very similar, with one key difference: in the Cantor space, at every limit level (regardless of its cofinality) there is a non-selfdual pair, whereas in the Baire space, at limit levels of countable cofinality there is a selfdual degree, and at those of uncountable cofinality there is a non-selfdual pair. While most of the classical development of the Wadge hierarchy focuses on the Baire space [Wad83, Sol21, VW77, Ste77], the arguments and game-theoretic techniques transfer directly to the Cantor space (see [And07, Section 2.7] and [AC13, Section 4]).

Set up and main results of the chapter

Throughout this chapter, we fix an uncountable cardinal κ such that $2^{<\kappa} = \kappa$. Regarding the axiomatic framework, our default theory is $\text{ZF} + \text{AC}_\kappa({}^\kappa 2)$, unless explicitly stated otherwise. Moreover, from Section 4.3.1 through Section 4.3.6 (inclusive), we additionally assume the full Axiom of Choice AC.

In our work we concentrate on the generalized Cantor space ${}^\kappa 2$ endowed with the bounded topology τ_b , and whenever $X = {}^\kappa 2$ we will simply write \leq_W , $<_W$, and \equiv_W in place of \leq_W^X , $<_W^X$, and \equiv_W^X . Likewise, whenever we mention the pointclasses $\kappa^+ \text{-}\Sigma_\alpha^0({}^\kappa 2)$, $\kappa^+ \text{-}\Pi_\alpha^0({}^\kappa 2)$, $\kappa^+ \text{-}\Delta_\alpha^0({}^\kappa 2)$, and $\kappa^+ \text{-}\mathbf{Bor}({}^\kappa 2)$, we omit the explicit reference to the space ${}^\kappa 2$, and write instead $\kappa^+ \text{-}\Sigma_\alpha^0$, $\kappa^+ \text{-}\Pi_\alpha^0$, $\kappa^+ \text{-}\Delta_\alpha^0$, and $\kappa^+ \text{-}\mathbf{Bor}$.

The chapter is structured as follows. Section 4.1 begins with an introductory study of the degrees at the bottom of the Wadge hierarchy on ${}^\kappa 2$. After this, the exposition splits into two parts according to the cofinality of κ . We first consider singular cardinals in Section 4.2, and next we turn to regular cardinals in Section 4.3.

The main result of the chapter is that both SLO_W and the well-foundedness of the Wadge hierarchy on ${}^\kappa 2$ fail already for the class $\kappa^+ \text{-}\Delta_2^0$. This failure occurs for both regular and singular κ , although the proofs in the two cases are of a very different nature. We conclude this introduction by outlining the content of the three main sections of the chapter and summarizing their main results.

Section 4.1: The base of the Wadge hierarchy. This short section presents some straightforward facts about the structure of the first three levels at the bottom of the Wadge hierarchy, namely, the non-selfdual pair $\{\{\kappa 2\}, \{\emptyset\}\}$, the selfdual class of proper $\kappa^+ \text{-}\Delta_1^0$ -sets, and the selfdual pair of proper open and proper closed subsets of ${}^\kappa 2$. In particular, we show that $\text{SLO}_W(\kappa^+ \text{-}\Sigma_1^0 \cup \kappa^+ \text{-}\Pi_1^0)$ holds (Corollary 4.1.4).

Section 4.2: The singular case. Assuming that κ is singular, our main result is Theorem 4.2.5, where we establish that $(\mathcal{P}(\text{cof}(\kappa)), \subseteq)$ embeds into the \mathcal{F} -hierarchy on $\kappa^+ \text{-}\Delta_2^0(\text{cof}(\kappa)\kappa)$ for every reducibility $\mathbf{L} \subseteq \mathcal{F} \subseteq \kappa^+ \text{-}\mathbf{Bor}$. This entails that the Wadge hierarchy on $\kappa^+ \text{-}\Delta_2^0$ subsets of ${}^\kappa 2$ is not well-founded, and that $\text{SLO}_W(\kappa^+ \text{-}\Delta_2^0)$ does not hold (see Corollary 4.2.6). In addition, we establish that the Lipschitz hierarchy on $\kappa^+ \text{-}\Delta_1^0$ subsets of ${}^\kappa 2$ is ill-founded, and that SLO_L already fails for clopen sets (Corollary 4.2.8).

Section 4.3: The regular case. We study the Wadge hierarchy on the generalized Cantor space ${}^\kappa 2$, for a regular cardinal κ . In the first part (Sections 4.3.1–4.3.2), we examine the Wadge hierarchy restricted to the sets in the κ^+ - $\mathbf{\Pi}_1^0$ -difference hierarchy on ${}^\kappa 2$ (see Section 3.7 for a discussion on the difference hierarchy in the generalized setting). In the second part (Sections 4.3.3–4.3.7), we instead concentrate on the structure of the Wadge hierarchy for sets that lie above the difference hierarchy with respect to the preorder \leq_W . Along the way, we also establish some results about the Lipschitz hierarchy on ${}^\kappa 2$. In particular, we show that this hierarchy is already ill-founded when restricted to the clopen subsets of ${}^\kappa 2$, and that SLO_L fails for κ^+ - $\mathbf{\Delta}_1^0$ -sets (Corollary 4.3.1).

We now outline the results of each subsection and their respective contents. Recall that we assume AC from Section 4.3.1 through Section 4.3.6.

Sections 4.3.1 and 4.3.2. In these sections, we show that the first κ^+ levels of the Wadge hierarchy are exactly described by the κ^+ - $\mathbf{\Pi}_1^0$ -difference hierarchy, in direct analogy with the classical case.

In Section 4.3.1, we introduce examples of κ -complete sets for the various pointclasses appearing in the κ^+ - $\mathbf{\Pi}_1^0$ -difference hierarchy, in all cases where such sets exist. This also enables us to determine which of these pointclasses are Wadge classes. We carry this out both for the non-selfdual levels of the hierarchy, namely the difference classes $\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$ for $\alpha < \kappa^+$ together with their duals (Definition 4.3.7 and Proposition 4.3.9), and for the selfdual classes appearing at successor stages (Proposition 4.3.11) and at limit stages of cofinality less than κ (Proposition 4.3.13) of the κ^+ - $\mathbf{\Pi}_1^0$ -difference hierarchy.

The situation at limit stages of cofinality κ is more subtle and is treated separately in Section 4.3.2, where we further investigate the Wadge classes of sets in the κ^+ - $\mathbf{\Pi}_1^0$ -difference hierarchy. In particular, unlike the limit levels of cofinality below κ , where we proved that there is always a single selfdual degree, we show that at limit levels of cofinality κ there is a single selfdual degree if κ is not weakly compact (Proposition 4.3.29), whereas if κ is weakly compact we instead obtain a non-selfdual pair (Proposition 4.3.30)—see also Figure 4.1. Recalling that when κ is regular then ${}^\kappa 2$ and ${}^\kappa \kappa$ are homeomorphic if and only if κ is not weakly compact [AMR22, Proposition 6.6], this parallels the situation in the classical setting, where at limit levels of countable cofinality there exists a non-selfdual pair in the Cantor space, while in the Baire space one instead finds a selfdual degree.

The analysis carried out in Sections 4.3.1–4.3.2 thus yields the following description of the Wadge hierarchy on ${}^\kappa 2$:

- (1) \leq_W is a well-founded preorder over the κ^+ - $\mathbf{\Pi}_1^0$ -difference hierarchy on ${}^\kappa 2$ (Theorem 4.3.35).
- (2) In the κ^+ - $\mathbf{\Pi}_1^0$ -difference hierarchy on ${}^\kappa 2$, anti-chains have length at most 2 (Theorem 4.3.15) and are of the form $\{[\mathcal{C}^{(\alpha)}]_W, [\mathcal{U}^{(\alpha)}]_W\}$ for $\alpha < \kappa^+$ (Proposition 4.3.9), where $\mathcal{C}^{(\alpha)}$ is a κ -complete set for $\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$, and $\mathcal{U}^{(\alpha)}$ is κ -complete for $\check{\mathcal{D}}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$ (see Definition 4.3.7). Moreover, for every $\alpha < \kappa^+$, the Wadge classes $\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$ and $\check{\mathcal{D}}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$ are immediately above $\mathbf{\Delta}_{\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)}$ by Corollaries 4.3.20 and 4.3.25.

Therefore, above a selfdual degree there is a non-selfdual pair.

- (3) For every $\alpha < \kappa^+$, the set $\mathcal{U}^{(\alpha)} \oplus \mathcal{C}^{(\alpha)}$ is selfdual and is such that $\mathcal{U}^{(\alpha)}, \mathcal{C}^{(\alpha)} \leq_W \mathcal{U}^{(\alpha)} \oplus \mathcal{C}^{(\alpha)}$ (Lemma 4.3.4 and Proposition 4.3.11). Moreover, $\Delta_{\mathcal{D}_{\alpha+1}(\kappa^+-\Pi_1^0)}$ is the Wadge class strictly above $\mathcal{D}_\alpha(\kappa^+-\Pi_1^0)$ and $\check{\mathcal{D}}_\alpha(\kappa^+-\Pi_1^0)$ by Corollary 4.3.33.

Therefore, above a non-selfdual pair there is a single selfdual degree.

- (4) Consider a limit ordinal $\alpha < \kappa^+$ with $\text{cof}(\alpha) < \kappa$, and fix a sequence $(\alpha_i)_{i < \text{cof}(\alpha)}$. The set $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$ is selfdual and is the supremum of the sets $(\mathcal{C}^{(\alpha_i)})_{i < \text{cof}(\alpha)}$, i.e., $\mathcal{C}^{(\alpha_i)} \leq_W \biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$ for every $i < \text{cof}(\alpha)$, and for any $A \subseteq {}^\kappa 2$, if $\mathcal{C}^{(\alpha_i)} \leq_W A$ for every $i < \text{cof}(\alpha)$, then $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)} \leq_W A$ (Lemma 4.3.5 and Proposition 4.3.13). Moreover, $\Delta_{\mathcal{D}_\alpha(\kappa^+-\Pi_1^0)}$ is the Wadge class strictly above $\bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+-\Pi_1^0)$ by Corollary 4.3.34.

- (5) For every limit ordinal $\alpha < \kappa^+$ with $\text{cof}(\alpha) = \kappa$,
- If κ is not weakly compact, then $\Delta_{\mathcal{D}_\alpha(\kappa^+-\Pi_1^0)}$ is the Wadge class strictly above $\bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+-\Pi_1^0)$ by Proposition 4.3.29, as shown in Figure 4.1a;
 - If κ is weakly compact, then $\bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+-\Pi_1^0) = \Delta_{\mathcal{D}_\alpha(\kappa^+-\Pi_1^0)}$ by Proposition 4.3.30, as shown in Figure 4.1b.

The key reason behind the phenomenon at limit levels of cofinality κ described in (5) is that the space ${}^\kappa 2$ is κ -Lindelöf if and only if κ is weakly compact [MR13, Theorem 5.6]. In particular, if κ is not weakly compact, then ${}^\kappa 2$ is homeomorphic to ${}^\kappa \kappa$, and in the generalized Baire space ${}^\kappa \kappa$ we have that, for every limit ordinal $\alpha < \kappa$, and for every sequence $(\alpha_i)_{i < \kappa}$ cofinal in α , the supremum of the family $\{\mathcal{C}^{(\alpha_i)} \mid i < \kappa\}$ exists and is selfdual (Remark 4.3.14). On the other hand, if κ is weakly compact, then ${}^\kappa 2$ is κ -Lindelöf, and hence such a supremum cannot exist, as shown in Proposition 4.3.30. Note that this is analogous to the classical case of ${}^\omega 2$, which is compact.

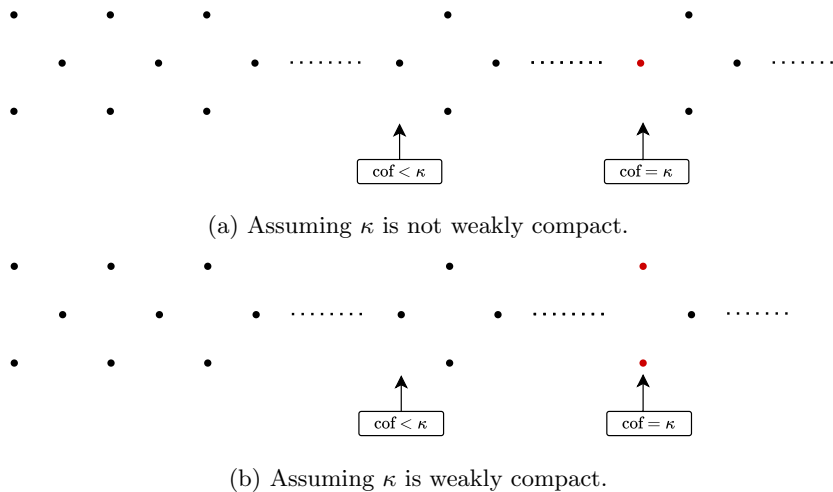


Figure 4.1: The first κ^+ -many levels of the Wadge hierarchy on ${}^\kappa 2$.

Section 4.3.3. In this section we investigate those subsets of ${}^{\kappa}2$ that lie in $\kappa^+-\Delta_2^0$ but strictly above the $\kappa^+-\Pi_1^0$ -difference hierarchy, such as the set \mathcal{Y}_0 from Example 3.7.12 and, more generally, the sets $\mathcal{Y}_0^{(\lambda)}$ (for infinite ordinals $\lambda < \kappa$) introduced in Example 3.7.13. Recall that, unlike in classical descriptive set theory, such sets do exist here because the higher-level Hausdorff–Kuratowski theorem fails (Proposition 4.3.36). We first prove that for every $\omega \leq \lambda < \kappa$, the set $\mathcal{Y}_0^{(\lambda)}$ is non-selfdual (Proposition 4.3.37), and that the family $\{\mathcal{Y}_0^{(\lambda)} \mid \lambda < \kappa \text{ is a limit ordinal}\}$ forms a $<_W$ -chain of length κ contained in $\kappa^+-\Delta_2^0$ but still strictly above the $\kappa^+-\Pi_1^0$ -difference hierarchy (Proposition 4.3.39). Next, we show that one can introduce a difference hierarchy on ${}^{\kappa}2$ over the boldface pointclass generated by \mathcal{Y}_0 , that is, over $\mathcal{Y}_0 \downarrow_{\kappa_2}$; this hierarchy remains inside $\kappa^+-\Delta_2^0$ and has length κ^+ (Corollary 4.3.41). Finally, we establish in Theorem 4.3.48 that the Wadge degree $[\mathcal{Y}_0]_W$ is minimal above the $\kappa^+-\Pi_1^0$ -difference hierarchy.

Section 4.3.4. We introduce, for each $S \subseteq \kappa$, the sets \mathcal{Y}_S (see (4.3.1)), which provide further examples, alongside \mathcal{Y}_0 and $\mathcal{Y}_0^{(\lambda)}$, of sets in $\kappa^+-\Delta_2^0$ that lie strictly above the $\kappa^+-\Pi_1^0$ -difference hierarchy. We first prove that for every $S \subseteq \kappa$, the set \mathcal{Y}_S is non-selfdual (Proposition 4.3.51). We then show that if $S, S' \subseteq \kappa$ are such that $S \Delta S'$ is stationary in κ , then \mathcal{Y}_S and $\mathcal{Y}_{S'}$ are Wadge-incomparable (Proposition 4.3.52). As a consequence, the Wadge hierarchy on $\kappa^+-\Delta_2^0$ contains antichains of size 2^{κ} , and in particular SLO_W fails for $\kappa^+-\Delta_2^0$ -sets (Corollary 4.3.54). On the other hand, in Proposition 4.3.55, we show that if $S, S' \subseteq \kappa$ satisfy that $(S \Delta S') \cap \text{cof}_{\omega}^{\kappa}$ is non-stationary in κ , then \mathcal{Y}_S and $\mathcal{Y}_{S'}$ are Wadge-equivalent. Finally, in Corollary 4.3.58 we prove that the Wadge hierarchy on $\kappa^+-\Delta_2^0$ -subsets of ${}^{\kappa}2$ is ill-founded.

Section 4.3.5 As mentioned in Fact 2.11.2(4), if Γ is a non-selfdual boldface pointclass, then any set that is κ -complete for Γ is $\Gamma({}^{\kappa}2)$ -proper. Under the additional hypothesis $\text{SLO}_W(\Gamma({}^{\kappa}2))$, the converse implication also holds (Lemma 2.11.3). Having established in the previous section that $\text{SLO}_W(\kappa^+-\Delta_2^0)$ does not hold, we now turn to the problem of determining whether it is still the case that every proper $\kappa^+-\Sigma_2^0$ subset of ${}^{\kappa}2$ must be κ -complete for $\kappa^+-\Sigma_2^0$. This turns out to be independent of ZFC. On the one hand, in Theorem 4.3.63 we prove that it is consistent with ZFC+“there is an inaccessible cardinal” that every proper $\kappa^+-\Sigma_2^0$ -set is κ -complete for $\kappa^+-\Sigma_2^0$. On the other hand, assuming $V = L$, we construct a set in $\kappa^+-\Sigma_2^0 \setminus \kappa^+-\Pi_2^0$ that fails to be κ -complete for $\kappa^+-\Sigma_2^0$ (Theorem 4.3.67). We also show that every dense subset of size κ of a strongly κ -perfect subset of ${}^{\kappa}2$ is κ -complete for $\kappa^+-\Sigma_2^0$ (Proposition 4.3.60). Finally, as a side result, Theorem 4.3.68 exhibits an interesting connection between SLO_W and the strongly κ -perfect set property for a boldface pointclass Γ containing all open and closed sets, under the assumption that $\kappa^+-\Pi_1^0$ has the strongly κ -perfect set property.

Section 4.3.6 Given the results of Section 4.3.3 about \mathcal{Y}_0 and, in particular, the result that \mathcal{Y}_0 is minimal above the $\kappa^+-\Pi_1^0$ -difference hierarchy (Theorem 4.3.48), it is natural to ask whether the sets \mathcal{Y}_S studied in Section 4.3.4 are also minimal above the $\kappa^+-\Pi_1^0$ -difference hierarchy. In this section, using some rather technical arguments, we show that this is indeed the case when $\kappa = \omega_1$ (Corollary 4.3.73).

Section 4.3.7. In this section, we analyze the Wadge hierarchy without using the full Axiom of Choice AC, and instead, we work within $\text{ZF} + \text{AC}_\kappa({}^\kappa 2)$. In particular, we establish that if $\mu < \kappa$ is an infinite cardinal with $\kappa = \mu^+$ regular, then $\text{SLO}_W(\kappa^+ - \Delta_2^0)$ fails and the Wadge hierarchy on $\kappa^+ - \Delta_2^0$ -sets is ill-founded (Corollary 4.3.80 and Proposition 4.3.81).

4.1 The base of the Wadge hierarchy

Consider the preorder \leq_W on the generalized Cantor space ${}^\kappa 2$. As already observed in Section 2.8, the bottom of the Wadge hierarchy consists of the non-selfdual pair $\{\{\kappa^2\}, \{\emptyset\}\}$. Right after this initial non-selfdual pair, there occurs a single selfdual degree that contains all non-trivial clopen subsets of ${}^\kappa 2$. Moreover, every other subset of ${}^\kappa 2$ lies above this degree in the Wadge hierarchy.

Fact 4.1.1. *If $A \in \kappa^+ - \Delta_1^0 \setminus \{\emptyset, \kappa^2\}$, then $A \leq_W B$ for all $B \notin \{\emptyset, \kappa^2\}$. Therefore, all non-trivial $\kappa^+ - \Delta_1^0$ -subsets are Wadge-equivalent to each other, and $\kappa^+ - \Delta_1^0 \setminus \{\emptyset, \kappa^2\}$ is the Wadge degree immediately above $\{\{\kappa^2\}, \{\emptyset\}\}$.*

Proof. Fix A and B as in the statement. Since $B \notin \{\emptyset, \kappa^2\}$, we can pick $y_0 \in B$ and $y_1 \in \kappa^2 \setminus B$. Define a function $f: \kappa^2 \rightarrow \kappa^2$ by setting $f(x) = y_0$ if $x \in A$, and $f(x) = y_1$ if $x \notin A$. This map is continuous, and witnesses $A \leq_W B$. \square

Observe that Fact 4.1.1 does not necessarily hold for \leq_L on ${}^\kappa 2$. Indeed, taking the clopen sets $\mathbf{N}_{\langle 0 \rangle}$ and $\mathbf{N}_{\langle 0, 0 \rangle}$, we obtain $\mathbf{N}_{\langle 0, 0 \rangle} \not\leq_L \mathbf{N}_{\langle 0 \rangle}$ as player **I** has an obvious winning strategy in the Lipschitz game $G_L(\mathbf{N}_{\langle 0, 0 \rangle}, \mathbf{N}_{\langle 0 \rangle})$: player **I** opens by playing 0, and then plays $1 - i$ where i is the first move of **II**.

Fact 4.1.2. $\text{SLO}_W(\kappa^+ - \Delta_1^0)$.

Proof. Let $A, B \subseteq \kappa^2$ be clopen sets. If $A = \emptyset$, then either $B \neq \kappa^2$, in which case $A \leq_W B$, or $B = \kappa^2$, in which case $\neg B = \emptyset \leq_W A$. Likewise, if $A = \kappa^2$, then either $B \neq \emptyset$ and thus $A \leq_W B$, or $B = \emptyset$ and hence $\neg B = \kappa^2 \leq_W A$. Therefore, we may restrict to the case $A \neq \emptyset, \kappa^2$. If $B = \emptyset$ or $B = \kappa^2$, then $\neg B \leq_W A$, and if instead $B \neq \emptyset, \kappa^2$, then Fact 4.1.1 yields $A \leq_W B$. \square

Next, we show that the non-selfdual pair $\{\kappa^+ - \Sigma_1^0 \setminus \kappa^+ - \Pi_1^0, \kappa^+ - \Pi_1^0 \setminus \kappa^+ - \Sigma_1^0\}$ sits immediately above the selfdual degree of non-trivial clopen sets, and that every other subset of ${}^\kappa 2$ is strictly above this non-selfdual pair in the Wadge preorder. We emphasize that this argument uses $\text{AC}_\kappa({}^\kappa 2)$.

Fact 4.1.3. *If B is closed and A is not open, or dually if B is open and A is not closed, then $B \leq_L A$. Consequently, every proper closed (respectively, open) subset $A \subseteq \kappa^2$ is κ -Lipschitz-complete, and therefore κ -complete, for $\kappa^+ - \Pi_1^0$ (respectively, $\kappa^+ - \Sigma_1^0$).*

Proof. It suffices to consider the case where B is closed and A is not open. By case assumption, there exists some $x \in A \setminus \text{int}(A)$. Using $\text{AC}_\kappa({}^\kappa 2)$, for every $\alpha < \kappa$ pick $x_\alpha \in \mathbf{N}_{x \upharpoonright \alpha} \setminus A$. Note that such x_α 's exist as x is not in the interior of A . For any non-empty $B \in \kappa^+ - \Pi_1^0$, to show $B \leq_L A$, we describe the following strategy for player **II** in $G_L(B, A)$:

As long as **I** plays in $t \in \mathcal{T}_B$, **II** follows x . If at some round α , Player **I** reaches $t \in \partial\mathcal{T}_B$, then **II** starts following x_α rather than x . \square

As a consequence, we obtain SLO_W for both open and closed sets.

Corollary 4.1.4. $\text{SLO}_W(\kappa^+ - \Sigma_1^0 \cup \kappa^+ - \Pi_1^0)$.

Proof. Immediate from Fact 2.11.3, Fact 4.1.2 and Fact 4.1.3. \square

4.2 The singular case

The arguments developed here generalize [MRS14, Section 6] to the uncountable setting, working under the assumption that κ is a singular cardinal. The main result is Theorem 4.2.5, which in particular establishes that when κ is singular, the Wadge hierarchy on $\kappa^+ - \Delta_2^0$ subsets of ${}^{\text{cof}(\kappa)}\kappa$ is ill-founded and that $\text{SLO}_W(\kappa^+ - \Delta_2^0)$ does not hold. As a consequence, we also derive a corresponding result for the space κ^2 (see Corollary 4.2.6). Moreover, we establish new facts about the Lipschitz hierarchy on κ^2 , proving that it is not well-founded and that SLO_L already fails for $\kappa^+ - \Delta_1^0$ subsets of κ^2 (Corollary 4.2.8). We remind the reader that, throughout this work, the term ‘‘Lipschitz function’’ is used to denote a 0-Lipschitz function in the sense of Definition 2.7.5.

We begin by recalling two results from [MRS14, Section 6], namely Lemma 4.2.1 and Theorem 4.2.2. Theorem 4.2.2 will be applied directly to prove Theorem 4.2.5 in the case $\text{cof}(\kappa) = \omega$. To handle Theorem 4.2.5 when $\text{cof}(\kappa) > \omega$, instead, we first need to generalize Theorem 4.2.2 to the space ${}^\mu 2$ for an arbitrary regular uncountable cardinal μ (this is accomplished in Theorem 4.2.3), and then apply the resulting statement with $\mu = \text{cof}(\kappa)$.

Lemma 4.2.1 was already formulated for an arbitrary infinite cardinal ν in [MRS14]. The only change we need here, relative to that formulation, is to additionally impose that all vertical sections A_x have cardinality ν . This extra requirement is, however, easily accomplished by the construction given in the proof of [MRS14, Lemma 55]. We will use this lemma and this additional condition in the proof of Theorem 4.2.3.

Lemma 4.2.1 ([MRS14, Lemma 55]). *Let ν be an infinite cardinal and X be a set such that $|X| = \nu$. Moreover, let $\mathcal{C} \subseteq [X]^\nu$, and let \mathcal{F} be a collection of functions from X to itself, and suppose that $|\mathcal{C}| = |\mathcal{F}| = \nu$. Then, there is a set $A \subseteq X^2$ such that $A_x \cap C \not\subseteq_{\mathcal{F}} A_y$ for all distinct $x, y \in X$ and all $C \in \mathcal{C}$. Moreover, $|A_x| = \nu$ for all $x \in X$.*

Sketch of the Proof. The proof proceeds exactly as in [MRS14, Lemma 55], the only additional point to check is that $|A_x| = \nu$ holds for every $x \in X$. Fix an arbitrary $x \in X$. By construction, at each stage $\alpha < \nu$ with $h(\alpha) = (C, f, x, y)$ for some $C \in \mathcal{C}$, $f \in \mathcal{F}$, and $y \in X$, we add an element to A_x . By our assumption $|X| = \nu$, there are at least ν many such ordinals α , since for every choice of C and f , each of the ν many possible $y \in X$ yields an α of the required form. Hence, $|A_x| = \nu$. \square

Theorem 4.2.2 ([MRS14, Theorem 56]). *Let $X = (X, d)$ be an uncountable ultrametric Polish space. Then, there is a map $\psi : \mathcal{P}(\omega) \rightarrow \mathcal{P}(X)$ such that for all $a, b \subseteq \omega$,*

- (1) *if $a \subseteq b$, then $\psi(a) \leq_{\mathbb{L}(d)} \psi(b)$;¹*
- (2) *if $\psi(a) \leq_{\mathbf{Bor}(X)} \psi(b)$, then $a \subseteq b$.*

In particular, $(\mathcal{P}(\omega), \subseteq)$ embeds into the \mathcal{F} -hierarchy on X , for every reducibility $\mathbb{L}(d) \subseteq \mathcal{F} \subseteq \mathbf{Bor}(X)$.

As noted also in [MRS14, Remark 57], in order to obtain Lemma 4.2.1 and Theorem 4.2.2 it suffices to assume that X is a well-orderable set, and thus the full use of AC is not actually required.

With the following result, we extend Theorem 4.2.2 from ω to an arbitrary regular uncountable cardinal μ .

Theorem 4.2.3. *Let μ be a regular cardinal, and suppose that ${}^\mu 2$ is well-orderable. Then, there is a map $\psi : \mathcal{P}(\mu) \rightarrow \mathcal{P}({}^\mu 2)$ such that for all $a, b \subseteq \mu$,*

- (1) *if $a \subseteq b$, then $\psi(a) \leq_{\mathbb{L}} \psi(b)$;*
- (2) *if $\psi(a) \leq_{\mu^+ \text{-Bor}} \psi(b)$, then $a \subseteq b$.*

In particular, $(\mathcal{P}(\mu), \subseteq)$ embeds into the \mathcal{F} -hierarchy on ${}^\mu 2$ for every reducibility $\mathbb{L} \subseteq \mathcal{F} \subseteq \mu^+ \text{-Bor}$.

Proof. We apply Lemma 4.2.1 with $\nu = 2^\mu$, $\mathcal{C} = \{A \in \kappa^+ \text{-Bor}({}^\mu 2) \mid |A| = 2^\mu\}$, and \mathcal{F} the collection of all μ^+ -Borel measurable functions from ${}^\mu 2$ to itself; since $w({}^\mu 2) = \mu$, it can be shown that $|\mathcal{C}| = |\mathcal{F}| = 2^\mu = \nu$. In particular, we obtain a μ -sequence of $\leq_{\mu^+ \text{-Bor}}$ -incomparable sets $A_\alpha \subseteq {}^\mu 2$, each of size 2^μ . Notice that $A_\alpha \neq {}^\mu 2$ for every $\alpha < \mu$, as otherwise we would have $A_\alpha \leq_{\mu^+ \text{-Bor}} A_\beta$ for every $\beta < \mu$.

Given $a \subseteq \mu$, set $\psi(a) = \bigcup_{\alpha \in a} (0^{(\alpha)} \cap 1 \cap A_\alpha)$. We want to show that ψ is as required, that is, that it satisfies points (1) and (2) in the statement.

We begin with (1), so suppose that $a, b \subseteq \mu$ are such that $a \subseteq b$. To show that $\psi(a) \leq_{\mathbb{L}} \psi(b)$, we describe a winning strategy for Player **II** in the Lipschitz game $G_{\mathbb{L}}^{\mu, 2}(\psi(a), \psi(b))$:

As long as Player **I** plays 0, Player **II** responds with 0. Suppose that at some stage $\alpha < \mu$ of the game, Player **I** plays 1, so that the sequence he has constructed so far is $0^{(\alpha)} \cap 1$. If $\alpha \in a$, Player **II** answers with 1 as well, and from that point on simply copies the subsequent moves of **I** for the rest of the game. If instead $\alpha \notin a$, then **II** just keeps playing 0's until the game ends. Since $a \subseteq b$, $0^{(\mu)} \notin \psi(a)$, and $0^{(\mu)} \notin \psi(b)$, this strategy ensures that **II** wins.

Next, we want to show (2). Let $a, b \subseteq \mu$ be such that $\psi(a) \leq_{\mu^+ \text{-Bor}} \psi(b)$ and let f be the μ^+ -Bor-measurable function witnessing it. Since f is a reduction, $\psi(a) = f^{-1}(\psi(b)) \subseteq \bigcup_{\beta \in b} f^{-1}(N_{0^{(\beta)}} \cap 1)$. Fix an arbitrary $\alpha \in a$: we want to show that $\alpha \in b$. Notice that $\psi(\{\alpha\}) =$

¹ $\mathbb{L}(d)$ denotes the collection of Lipschitz functions with constant 1, i.e. the functions $f: X \rightarrow X$ such that $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$.

$0^{(\alpha)} \dot{\wedge} 1 \dot{\wedge} A_\alpha \subseteq \left(\bigcup_{\beta \in b} f^{-1}(\mathbf{N}_{0^{(\beta)} \dot{\wedge} 1}) \right) \cap \mathbf{N}_{0^{(\alpha)} \dot{\wedge} 1}$. Since $|0^{(\alpha)} \dot{\wedge} 1 \dot{\wedge} A_\alpha| = 2^\mu$, it follows that $\left| \left(\bigcup_{\beta \in b} f^{-1}(\mathbf{N}_{0^{(\beta)} \dot{\wedge} 1}) \right) \cap \mathbf{N}_{0^{(\alpha)} \dot{\wedge} 1} \right| = 2^\mu$. Since $\text{cof}(2^\mu) > \mu$ ([Jec03, Corollary 5.12]) and $|b| \leq \mu$, there must exist some $\beta \in b$ such that $|f^{-1}(\mathbf{N}_{0^{(\beta)} \dot{\wedge} 1}) \cap \mathbf{N}_{0^{(\alpha)} \dot{\wedge} 1}| = 2^\mu$. Fix such a β , and pick any $\bar{y} \in {}^\mu 2 \setminus A_\beta$. Setting² $C = (f^{-1}(\mathbf{N}_{0^{(\beta)} \dot{\wedge} 1}))_{\lfloor \alpha+1 \rfloor}$, we obtain that C is a μ^+ -Borel subset of ${}^\mu 2$ of cardinality 2^μ . In fact, C is precisely the preimage of $f^{-1}(\mathbf{N}_{0^{(\beta)} \dot{\wedge} 1}) \cap \mathbf{N}_{0^{(\alpha)} \dot{\wedge} 1}$ under the continuous bijection $x \mapsto 0^{(\alpha)} \dot{\wedge} 1 \dot{\wedge} x$, and $f^{-1}(\mathbf{N}_{0^{(\beta)} \dot{\wedge} 1}) \in \mu^+$ -**Bor** because f is μ^+ -**Bor**-measurable. Moreover, the map $g : {}^\mu 2 \rightarrow {}^\mu 2$ defined by

$$g(x) = \begin{cases} f(0^{(\alpha)} \dot{\wedge} 1 \dot{\wedge} x)_{\lfloor \beta+1 \rfloor} & \text{if } x \in C, \\ \bar{y} & \text{otherwise,} \end{cases}$$

witnesses $A_\alpha \cap C \leq_{\mu^+ \text{-Bor}} A_\beta$. Indeed, g is clearly μ^+ -Borel measurable. First, assume $x \in A_\alpha \cap C$. Notice that $x \in A_\alpha$ if and only if $0^{(\alpha)} \dot{\wedge} 1 \dot{\wedge} x \in \psi(a)$. Since $f(\psi(a)) \subseteq \psi(b)$ entails $f(0^{(\alpha)} \dot{\wedge} 1 \dot{\wedge} x) \in \psi(b)$ and $x \in C$ entails $f(0^{(\alpha)} \dot{\wedge} 1 \dot{\wedge} x) \in \mathbf{N}_{0^{(\beta)} \dot{\wedge} 1}$, the two conditions together leads to $f(0^{(\alpha)} \dot{\wedge} 1 \dot{\wedge} x) \in 0^{(\beta)} \dot{\wedge} 1 \dot{\wedge} A_\beta$, hence $g(x) \in A_\beta$.

For the converse direction, let $g(x) \in A_\beta$. Then x must lie in C ; if not, we would have $g(x) = \bar{y} \in {}^\mu 2 \setminus A_\beta$, a contradiction. From this, it follows that $f(0^{(\alpha)} \dot{\wedge} 1 \dot{\wedge} x) \in \mathbf{N}_{0^{(\beta)} \dot{\wedge} 1}$, and in particular that $f(0^{(\alpha)} \dot{\wedge} 1 \dot{\wedge} x) \in 0^{(\beta)} \dot{\wedge} 1 \dot{\wedge} A_\beta \subseteq \psi(b)$. Since $\psi(a) = f^{-1}(\psi(b))$, we conclude that $0^{(\alpha)} \dot{\wedge} 1 \dot{\wedge} x \in \psi(a)$, and thus $x \in A_\alpha$.

By choice of A_α 's (see Lemma 4.2.1), we get that $\alpha = \beta$, hence $\alpha \in b$ as required. \square

Remark 4.2.4. If we additionally assume $\mathbf{V} = \mathbf{L}$ and work under the hypothesis $\mu^{<\mu} = \mu > \omega$, then the map ψ from Theorem 4.2.3 can be chosen to range in the family of μ^+ - Δ_1^0 subsets of ${}^\mu 2$. The proof proceeds by suitably modifying the recursion used in the proof of Theorem 4.2.1 so that, at each stage, membership in each of the sets can be computed in the next admissible set. This yields a generalization of [MRS14, Theorem 63]. We will not pursue these details here, because in Section 4.3 we will establish significantly stronger results; in particular, we will prove an analogous statement already in ZFC for the class μ^+ - Δ_2^0 .

We are now ready to state and prove the main result of this section.

Theorem 4.2.5. *Let κ be singular. Then, there is a map $\varphi : \mathcal{P}(\text{cof}(\kappa)) \rightarrow \kappa^+$ - $\Delta_2^0(\text{cof}(\kappa)_\kappa)$ such that for all $a, b \subseteq \text{cof}(\kappa)$,*

- (1) if $a \subseteq b$, then $\varphi(a) \leq_{\mathbf{L}} \varphi(b)$;
- (2) if $\varphi(a) \leq_{\mathbf{W}} \varphi(b)$, then $a \subseteq b$.

In particular, $(\mathcal{P}(\text{cof}(\kappa)), \subseteq)$ embeds into the \mathcal{F} -hierarchy on κ^+ - $\Delta_2^0(\text{cof}(\kappa)_\kappa)$ for both $\mathcal{F} = \mathbf{L}$ and $\mathcal{F} = \mathbf{W}$.

²Recall from Section 2.2 that for any $z \in {}^\mu 2$ and any ordinal $\gamma < \mu$, we write $z_{\lfloor \gamma \rfloor}$ for the (unique) sequence satisfying $z \upharpoonright \gamma \dot{\wedge} z_{\lfloor \gamma \rfloor} = z$.

Proof. Let $\mu = \text{cof}(\kappa)$, and let $\iota : \mathcal{P}({}^\mu 2) \rightarrow \mathcal{P}({}^\mu \kappa)$ be the inclusion map. We define a function $\varphi : \mathcal{P}(\mu) \rightarrow \mathcal{P}({}^\mu \kappa)$ by setting $\varphi = \iota \circ \psi$, where $\psi : \mathcal{P}(\mu) \rightarrow \mathcal{P}({}^\mu 2)$ is either the function given by Theorem 4.2.2 (with $X = {}^\mu 2$) if $\mu = \omega$, or the function provided by Theorem 4.2.3 when $\mu > \omega$.

Note that for every $a \subseteq \kappa$, $\varphi(a) = \iota(\psi(a)) \in \kappa^+ \text{-}\Delta_2^0({}^\mu \kappa)$ because both $\psi(a)$ and ${}^\mu 2 \setminus \psi(a)$ are in $\mathcal{P}({}^\mu 2)$. Indeed, for any subset $S \subseteq {}^\mu 2$, $S \in \kappa^+ \text{-}\Sigma_2^0({}^\mu \kappa)$ as S can be written as union of 2^μ -many singletons and, since $\mu < \kappa$, $2^\mu \leq 2^{<\kappa} = \kappa$. This shows that $\varphi(a) \in \kappa^+ \text{-}\Sigma_2^0({}^\mu \kappa)$. To see that $\varphi(a) \in \kappa^+ \text{-}\Pi_2^0({}^\mu \kappa)$, note that ${}^\mu \kappa \setminus \varphi(a) = ({}^\mu \kappa \setminus {}^\mu 2) \cup ({}^\mu 2 \setminus \psi(a)) \in \kappa^+ \text{-}\Sigma_2^0({}^\mu \kappa)$ because ${}^\mu 2$ is closed in ${}^\mu \kappa$.

We claim that φ is the function that fulfills the statement, namely, that (1) and (2) in the statement hold. For this, it is enough to show that for any $A, B \subseteq {}^\mu 2$:

- (a) if $A \leq_{\mathbb{L}}^{{}^\mu 2} B$, then $\iota(A) \leq_{\mathbb{L}}^{{}^\mu \kappa} \iota(B)$;
- (b) if $\iota(A) \leq_{\mathbb{W}}^{{}^\mu \kappa} \iota(B)$, then $A \leq_{\mu^+ \text{-Bor}}^{{}^\mu 2} B$.

To prove (a), let τ be a winning strategy for Player **II** in the game $G_{\mathbb{L}}^{\mu, 2}(A, B)$. We define a winning strategy for Player **II** in $G_{\mathbb{L}}^{\mu, \kappa}(\iota(A), \iota(B))$ as follows:

Player **II** follows the moves prescribed by τ as long as Player **I** chooses elements from $\{0, 1\}$. If, at some point in the game, Player **I** plays an element $i \notin \{0, 1\}$, then Player **II** responds by playing the same element i , and from that point on simply continues to mirror all of Player **I**'s moves.

Next, we want to prove (b). Let $f : {}^\mu \kappa \rightarrow {}^\mu \kappa$ be a continuous function reducing $\iota(A)$ to $\iota(B)$ and set $C = f^{-1}({}^\mu 2) \cap {}^\mu 2$. By continuity of f , C is closed in ${}^\mu \kappa$. Moreover, $\iota(A) \subseteq C$ because $\iota(B) \subseteq {}^\mu 2$.

We may assume that $\iota(B) \neq {}^\mu 2$. Indeed, if there were some $b \subseteq \kappa$ with $\iota(\psi(b)) = \iota(B) = {}^\mu 2$, then it would follow that $\psi(b) \leq_{\mathbb{L}} \psi(b')$ for every $\psi(b') \neq \emptyset$. By Theorem 4.2.3(1), this would imply $b = \emptyset$, and hence $\psi(b) = \iota(\psi(b)) = \emptyset$, a contradiction. Fix $z \in {}^\mu 2 \setminus B$ and define $g : {}^\mu 2 \rightarrow {}^\mu 2$ as follows.

$$g(x) = \begin{cases} f(x), & \text{if } x \in C, \\ z & \text{otherwise.} \end{cases}$$

Then, g is a μ^+ -Borel measurable map that reduces A to B . □

Recall that, given our assumptions on κ valid in this section, namely that κ is singular and $2^{<\kappa} = \kappa$, the generalized Cantor space ${}^\kappa 2$ is homeomorphic to the generalized Baire space ${}^{\text{cof}(\kappa)} \kappa$ (see also Section 2.2). Therefore, the result in Theorem 4.2.5 for the reducibility $\mathcal{F} = \mathbb{W}$ also applies to the space ${}^\kappa 2$.

Corollary 4.2.6. *Let κ be a singular cardinal. Then, there is an embedding of $(\mathcal{P}(\text{cof}(\kappa)), \subseteq)$ into the Wadge hierarchy on $\kappa^+ \text{-}\Delta_2^0$ subsets of ${}^\kappa 2$. In particular, $\text{SLO}_{\mathbb{W}}(\kappa^+ \text{-}\Delta_2^0({}^\kappa 2))$ fails, and the Wadge hierarchy on $\kappa^+ \text{-}\Delta_2^0$ is ill-founded.*

The approach employed to derive Corollary 4.2.6, which is well-suited to the context of Wadge reducibility, does not apply to Lipschitz reducibility. To establish an analogue of Theorem 4.2.5 for $\mathcal{F} = \mathbb{L}$ on ${}^\kappa 2$, we therefore employ the alternative argument developed below.

Lemma 4.2.7. *Let ν and λ be arbitrary infinite cardinals such that $\nu < \lambda$. Then, there is an embedding of the Lipschitz hierarchy on ${}^\nu 2$ into the Lipschitz hierarchy on κ^+ - Δ_1^0 subsets of ${}^\lambda 2$.*

Proof. Consider the map ${}^\nu 2 \rightarrow {}^\lambda 2$ sending each $A \subseteq {}^\nu 2$ into $A' = \{x \in {}^\lambda 2 \mid x \upharpoonright \nu \in A\}$. Clearly, $A' \in \kappa^+$ - Δ_1^0 for every $A \subseteq {}^\nu 2$.

We first claim that $A \leq_L^{\nu^2} B$ implies $A' \leq_L^{\lambda^2} B'$. Let σ be a winning strategy for **II** in the game $G_L^{\nu,2}(A, B)$. A winning strategy for **II** in the game $G_L^{\lambda,2}(A', B')$ is obtained by requiring **II** to follow the moves prescribed by σ until stage ν of the play; beyond stage ν , her subsequent moves may be chosen arbitrarily.

Next, we aim to show that $A' \leq_L^{\lambda^2} B'$ implies $A \leq_L^{\nu^2} B$. Let τ be a winning strategy for **II** in the game $G_L^{\lambda,2}(A', B')$. We can define a winning strategy for **II** in $G_L^{\nu,2}(A, B)$ by simply following the moves prescribed by τ for every stage $\alpha < \nu$. It follows immediately from the definition of A' and B' that such strategy is winning for **II** in $G_L^{\nu,2}(A, B)$. \square

Corollary 4.2.8. *Let κ be a singular cardinal. Then, there is an embedding of $(\mathcal{P}(\text{cof}(\kappa)), \subseteq)$ into the Lipschitz hierarchy on κ^+ - Δ_1^0 subsets of κ^2 . In particular, $\text{SLO}_L(\kappa^+$ - $\Delta_1^0(\kappa^2))$ fails, and the Lipschitz hierarchy on κ^+ - Δ_1^0 is ill-founded.*

Proof. Use Lemma 4.2.7 with $\lambda = \kappa$ and $\nu = \text{cof}(\kappa)$, and apply Theorem 4.2.3 with $\mu = \text{cof}(\kappa)$. \square

The proof of Theorem 4.2.5 naturally raises the following question.

Question 4.2.9. Let κ be singular, and let $\mu = \text{cof}(\kappa)$. Is there an embedding of $({}^\mu 2, \leq_W)$ into the \mathcal{F} -hierarchy on Γ over ${}^\mu \kappa$, where Γ is a boldface pointclass that is “sufficiently simple” in the κ^+ -Borel hierarchy, and $L \subseteq \mathcal{F} \subseteq W$?

Observe that the inclusion map $\iota : \mathcal{P}({}^\mu 2) \rightarrow \mathcal{P}({}^\mu \kappa)$ does not yield a positive solution to Question 4.2.9, as shown in Fact 4.2.10.

Fact 4.2.10. *Let κ be singular, and let $\mu = \text{cof}(\kappa)$. Consider the inclusion map $\iota : \mathcal{P}({}^\mu 2) \rightarrow \mathcal{P}({}^\mu \kappa)$. Let $A, B \subseteq {}^\mu 2$ and assume that A is a proper closed set (in ${}^\mu 2$) and that $B \neq \emptyset$ is open (in ${}^\mu 2$). Then, $A \not\leq_W^{\mu^2} B$ but $\iota(A) \leq_W^{\mu \kappa} \iota(B)$.*

Proof. It is immediate to see that $A \not\leq_W^{\mu^2} B$. To see that $\iota(A) \leq_W^{\mu \kappa} \iota(B)$, fix an element $x \in \iota(B)$ and consider the following winning strategy for Player **II** in $G_W^{\mu, \kappa}(A, B)$:

As long as Player **I** plays in $\mathcal{T}_{\iota(A)}$, Player **II** answers enumerating x ; if **I** plays an element outside of $\mathcal{T}_{\iota(A)}$, then **II** plays 2 and then only 0's. \square

However, if we restrict $\iota : \mathcal{P}({}^\mu 2) \rightarrow \mathcal{P}({}^\mu \kappa)$ to those subsets of ${}^\mu 2$ with empty interior, then we do obtain a positive answer to Question 4.2.9 in the case $\mathcal{F} = L$ and $\Gamma = \kappa^+$ - Δ_2^0 .

Proposition 4.2.11. *Let κ be singular, and let $\mu < \kappa$. Then, there is an embedding of the Lipschitz hierarchy on subsets of ${}^\mu 2$ with empty interior into the Lipschitz hierarchy on κ^+ - $\Delta_2^0({}^\mu \kappa)$.*

Proof. Denote by $\bar{\iota}$ this restriction of ι to subsets of ${}^\mu 2$ having empty interior. We already observed in the proof of Theorem 4.2.5 that for every $S \subseteq {}^\mu 2$ we have $S \in \kappa^+ \text{-}\Delta_2^0({}^\mu \kappa)$, hence the range of $\bar{\iota}$ is contained in $\kappa^+ \text{-}\Delta_2^0({}^\mu \kappa)$.

Now fix $A, B \subseteq {}^\mu 2$ with empty interior. We claim that $A \leq_{\mathbb{L}}^{\mu 2} B$ if and only if $\bar{\iota}(A) \leq_{\mathbb{L}}^{\mu \kappa} \bar{\iota}(B)$.

Point (a) in the proof of Theorem 4.2.5 shows that, whenever $A \leq_{\mathbb{L}}^{\mu 2} B$, we have $\bar{\iota}(A) \leq_{\mathbb{L}}^{\mu \kappa} \bar{\iota}(B)$.

Conversely, assume $\bar{\iota}(A) \leq_{\mathbb{L}}^{\mu \kappa} \bar{\iota}(B)$, and let τ be a winning strategy for Player **II** in the game $G_{\mathbb{L}}^{\mu, \kappa}(\bar{\iota}(A), \bar{\iota}(B))$. For every $t \in {}^{<\mu} 2$, pick an element $z_t \in N_t \setminus B$. The existence of such a z_t is guaranteed because we cannot have $N_t \subseteq B$: the set N_t is open in ${}^\mu 2$, whereas B has empty interior in ${}^\mu 2$. Then **II** can use the following strategy to win the game $G_{\mathbb{L}}^{\mu, 2}(A, B)$, and thus $A \leq_{\mathbb{L}}^{\mu 2} B$.

During the play, **II** follows the strategy τ when responding to the moves of **I**, as long as τ prescribes to play elements from $\{0, 1\}$. Assume that at some point τ outputs an element of $\kappa \setminus \{0, 1\}$, and denote by $t \in {}^{<\mu} 2$ the sequence constructed by **II** up to that moment. Then, **II** can continue the play by following z until the game ends. \square

4.3 The regular case

In this section, we turn to the case where κ is a regular cardinal. Accordingly, from now on and throughout the remainder of the chapter, we assume that $\kappa^{<\kappa} = \kappa$.

We first note that the results from Section 4.2 already provide all the necessary information to obtain that the Lipschitz hierarchy on ${}^\kappa 2$ behaves badly also when κ is regular. Indeed, it suffices to invoke Lemma 4.2.7 with $\lambda = \kappa$ and ν any regular cardinal below κ , and then apply Theorem 4.2.3 with $\mu = \nu$ to obtain the following statement.

Corollary 4.3.1. *For any regular $\nu < \kappa$, there is an embedding of $(\mathcal{P}(\nu), \subseteq)$ into the Lipschitz hierarchy on $\kappa^+ \text{-}\Delta_1^0$ subsets of ${}^\kappa 2$. In particular, the Lipschitz hierarchy on $\kappa^+ \text{-}\Delta_1^0$ is ill-founded, and $\text{SLO}_{\mathbb{L}}(\kappa^+ \text{-}\Delta_1^0({}^\kappa 2))$ fails.*

The second part of the statement of Corollary 4.3.1 follows from the first by taking $\nu = \omega$. Notice that we are assuming $2^{<\kappa} = \kappa$, so the set ${}^\nu 2$, and in particular ${}^\omega 2$, is well-orderable.

Corollary 4.3.1 is related to [Gol20, Question 5.25], where Goldberg asked whether, for any ordinal $\delta > \omega$, it is consistent with $\text{ZF} + \text{DC}$ that Lipschitz determinacy holds for all subsets of ${}^\delta 2$. This question appears in the context of his characterization of the Ketonen order on ultrafilters on δ in terms of Lipschitz reductions on ${}^\delta 2$. Corollary 4.3.1 demonstrates that Lipschitz determinacy can fail for clopen subsets of ${}^\kappa 2$ even in the absence of the full Axiom of Choice AC. Observe, however, that Goldberg's question is more delicate, since it assumes only DC, and in particular does not require additional hypotheses such as $2^{<\kappa} = \kappa$, which is used throughout this thesis. To fully answer his question, one would need to show that the failure of Lipschitz determinacy for subsets of ${}^\kappa 2$ also occurs e.g. in models of AD.

We now move to study the Wadge hierarchy on ${}^\kappa 2$. Using again the results from Section 4.2, and in particular Theorem 4.2.3 with $\mu = \kappa$, we already know that $(\mathcal{P}(\kappa), \subseteq)$ embeds into the \mathcal{F} -hierarchy on ${}^\kappa 2$, for every reducibility $\mathbb{L} \subseteq \mathcal{F} \subseteq \kappa^+ \text{-Bor}$. In particular, this shows that the Wadge hierarchy on ${}^\kappa 2$ is not well-founded and that $\text{SLO}_W({}^\kappa 2)$ fails. A natural follow-up question is whether this failure of SLO_W and of wellfoundedness already occurs at the level of simple (i.e. definable) sets, in analogy with Corollary 4.2.6. This is precisely the question we want to address in the upcoming sections.

Throughout the remainder of this chapter, up to the end of Section 4.3.6, we assume the full Axiom of Choice AC and consequently work within the framework of ZFC. Section 4.3.7 is an exception, since in that section we instead assume $\text{ZF} + \text{AC}_\kappa({}^\kappa 2)$.

In Sections 4.3.1–4.3.2, we carry out a level-by-level study of the $\kappa^+ \text{-}\Pi_1^0$ -difference hierarchy from the perspective of Wadge theory. There, we prove that \leq_W is a well-founded preorder on the $\kappa^+ \text{-}\Pi_1^0$ -difference hierarchy over ${}^\kappa 2$ (Theorem 4.3.35), and that within this hierarchy on ${}^\kappa 2$, every anti-chain has length at most 2 and is constituted by a non-selfdual pair (Theorem 4.3.15).

In Sections 4.3.3–4.3.6, we analyze the structure of the Wadge hierarchy on ${}^\kappa 2$ restricted to sets in $\kappa^+ \text{-}\Delta_2^0$ that lie strictly above the $\kappa^+ \text{-}\Pi_1^0$ -difference hierarchy. In particular, we establish that $\text{SLO}_W(\kappa^+ \text{-}\Delta_2^0)$ does not hold (Corollary 4.3.54) and that the Wadge hierarchy on $\kappa^+ \text{-}\Delta_2^0$ -sets is ill-founded (Corollary 4.3.58). In addition, we show that it is independent of ZFC whether every proper $\kappa^+ \text{-}\Sigma_2^0$ subset of ${}^\kappa 2$ is κ -complete for $\kappa^+ \text{-}\Sigma_2^0$ (Theorem 4.3.63 and Theorem 4.3.67).

Lastly, in Section 4.3.7 we re-examine the Wadge hierarchy when dropping the assumption AC.

4.3.1 Canonical complete sets for the difference hierarchy

In Proposition 3.7.7 we have shown the existence of κ -complete sets for $\mathcal{D}_\alpha(\Gamma)$, when Γ is a hereditary boldface pointclass closed under intersections of length at most κ , which admits a ${}^\kappa 2$ -universal set. In this section, we are going to construct some subsets of ${}^\kappa 2$ which are examples of κ -complete sets for $\mathcal{D}_\alpha(\kappa^+ \text{-}\Pi_1^0)$. In addition, we will characterize precisely when the selfdual pointclasses $\Delta_{\mathcal{D}_\alpha(\kappa^+ \text{-}\Pi_1^0)}$ admit a κ -complete set. Similar constructions for complete sets in the classical case are well-known (see e.g. [MR07]). Here, we retrace such analysis in the generalized setting, focusing on the classes appearing in the $\kappa^+ \text{-}\Pi_1^0$ -difference hierarchy.

To begin with, we introduce the following operations and some basic properties they satisfy.

Definition 4.3.2. Let $\gamma \leq \kappa$ be an ordinal, let $C, D \subseteq {}^\kappa 2$, and let $(C_i)_{i < \gamma}$ be a sequence of subsets of ${}^\kappa 2$. We define:

$$\begin{aligned} C^\nabla &= \bigcup_{\beta < \kappa} 0^{(\beta)} \hat{\ } 1 \hat{\ } C, \\ C^\circ &= C^\nabla \cup \{0^{(\kappa)}\}, \\ \biguplus_{i < \gamma} C_i &= \bigcup_{i < \gamma} 0^{(i)} \hat{\ } 1 \hat{\ } C_i, \\ C \oplus D &= (0 \hat{\ } C) \cup (1 \hat{\ } D). \end{aligned}$$

Lemma 4.3.3. *Let $C, D \subseteq {}^{\kappa}2$. The following holds:*

- (1) $C \leq_{\mathbb{L}} C^{\nabla}, C^{\circ}$;
- (2) *If $C \leq_{\mathbb{W}} D$, then $C^{\nabla} \leq_{\mathbb{W}} D^{\nabla}$ and $C^{\circ} \leq_{\mathbb{W}} D^{\circ}$, and similarly for $\leq_{\mathbb{L}}$ instead of $\leq_{\mathbb{W}}$;*
- (3) *If $C \leq_{\mathbb{W}} \neg C$ then $C^{\nabla} \equiv_{\mathbb{W}} \neg C^{\circ}$, and similarly for $\leq_{\mathbb{L}}$ instead of $\leq_{\mathbb{W}}$;*
- (4) $({}^{\kappa}2)^{\nabla}$ is open, and $({}^{\kappa}2)^{\circ}$ is the closure of $({}^{\kappa}2)^{\nabla}$;
- (5) $\emptyset^{\nabla} = \emptyset$ and $\emptyset^{\circ} = \{0^{(\kappa)}\}$.

Proof. Point (1) is given by the function $x \mapsto 1 \wedge x$. Points (4) and (5) are immediate.

To see (2), let τ be a winning strategy for Player **II** in $G_{\mathbb{W}}(C, D)$. A winning strategy for **II** in both $G_{\mathbb{W}}(C^{\nabla}, D^{\nabla})$ and $G_{\mathbb{W}}(C^{\circ}, D^{\circ})$ is the following:

As long as **I** plays 0's, **II** plays 0's. If **I** plays 1, then **II** answers with 1 and then follows τ until the end of the game.

Note that same proof applies when substituting $G_{\mathbb{W}}$ with $G_{\mathbb{L}}$.

For (3), let σ be a winning strategy for Player **II** in $G_{\mathbb{W}}(C, \neg C)$. Then, **II** wins in the game $G_{\mathbb{W}}(C^{\nabla}, \neg C^{\circ})$ as follows:

As long as **I** plays 0, then **II** answers with 0. If at some round **I** plays 1, then **II** plays 1 too and then follows σ .

This shows $C^{\nabla} \leq_{\mathbb{W}} \neg C^{\circ}$; the proof that $\neg C^{\circ} \leq_{\mathbb{W}} C^{\nabla}$ is similar. The same proof applies when substituting $\leq_{\mathbb{W}}$ with $\leq_{\mathbb{L}}$. \square

Observe that because of (2), the $^{\circ}$ and $^{\nabla}$ operations can be defined on Lipschitz and Wadge degrees, that is: $[C]_{\mathbb{W}}^{\nabla} = [C^{\nabla}]_{\mathbb{W}}$, $[C]_{\mathbb{W}}^{\circ} = [C^{\circ}]_{\mathbb{W}}$, $[C]_{\mathbb{L}}^{\nabla} = [C^{\nabla}]_{\mathbb{L}}$, and $[C]_{\mathbb{L}}^{\circ} = [C^{\circ}]_{\mathbb{L}}$.

Lemma 4.3.4. *Let $A, C, D \subseteq {}^{\kappa}2$. Then,*

- (1) $C, D \leq_{\mathbb{L}} C \oplus D$;
- (2) *If $C \leq_{\mathbb{W}} A$ and $D \leq_{\mathbb{W}} A$, then $C \oplus D \leq_{\mathbb{W}} A$.*

Proof. Point (1) is given by the function $x \mapsto 0 \wedge x$ for $C \leq_{\mathbb{L}} C \oplus D$, and by $x \mapsto 1 \wedge x$ for $D \leq_{\mathbb{L}} C \oplus D$. For point (2), assume that $C \leq_{\mathbb{W}} A$ and $D \leq_{\mathbb{W}} A$. A winning strategy for Player **II** in the game $G_{\mathbb{W}}(C \oplus D, A)$ consist of skipping the first turn, and then using the strategy induced by $C \leq_{\mathbb{W}} A$ if **I** had played 0, or the strategy induced by $D \leq_{\mathbb{W}} A$ if **I** had played 1. \square

Lemma 4.3.5. *Let $\gamma \leq \kappa$ be an ordinal, let $(C_i)_{i < \gamma}$ be a sequence of subsets of ${}^{\kappa}2$, and let $A \subseteq {}^{\kappa}2$. The following holds:*

- (1) *For every $i < \gamma$, $C_i \leq_{\mathbb{L}} \biguplus_{i < \gamma} C_i$.*
- (2) *If $\gamma = 2$ and $C_1 \neq {}^{\kappa}2$, then $C_0 \oplus C_1 \equiv_{\mathbb{W}} \biguplus_{i < 2} C_i$.*
- (3) *Assume $\gamma = \kappa$. If $C_i \neq {}^{\kappa}2$ holds for at least one $i < \kappa$, and for every $i, j < \kappa$ there exists $k \geq j$ such that $C_i \leq_{\mathbb{W}} C_k$, then $(\biguplus_{i < \gamma} C_i)^{\nabla} \equiv_{\mathbb{W}} \biguplus_{i < \gamma} C_i$.*

- (4) Assume $\gamma < \kappa$. Then, $\biguplus_{i < \gamma} C_i$ is the supremum of the sets $(C_i)_{i < \gamma}$, i.e., $C_i \leq_W \biguplus_{i < \gamma} C_i$ for every $i < \gamma$, and for $A \neq \kappa^2$, if $C_i \leq_W A$ for every $i < \gamma$, then $\biguplus_{i < \gamma} C_i \leq_W A$.
- (5) Assume $\gamma < \kappa$. If $C_i \neq \emptyset$, κ^2 holds for at least one $i < \gamma$, and for every $i < \gamma$ there exists $j < \gamma$ such that $C_i \leq_W \neg C_j$, then $\biguplus_{i < \gamma} C_i \leq_W \neg(\biguplus_{i < \gamma} C_i)$.

Proof. To see (1), fix $i < \gamma$ and consider $x \mapsto 0^{(i)} \wedge 1 \wedge x$.

For point (2), let $\gamma = 2$ and consider the sets C_0, C_1 . Since $C_1 \neq \kappa^2$, choose some $z \in \neg C_1$. The following strategy is winning for Player **II** in the game $G_L(C_0 \oplus C_1, \biguplus_{i < 2} C_i)$, hence $C_0 \oplus C_1 \leq_L \biguplus_{i < 2} C_i$:

If Player **I** starts by playing 1, then Player **II** responds with $0 \wedge 1$; if instead **I** starts with 0, then **II** responds with 1. After this initial move, Player **II** simply mirrors the subsequent moves of Player **I**.

Conversely, to prove that $\biguplus_{i < 2} C_i \leq_W C_0 \oplus C_1$, we describe a winning strategy for Player **II** in $G_W(\biguplus_{i < 2} C_i, C_0 \oplus C_1)$:

If Player **I** opens the game with 1, then Player **II** replies with 0 and from that point on copies all of Player **I**'s moves until the game ends. If Player **I** starts with 0, then Player **II** answers with 0. After this, if Player **I** plays 0, then Player **II** enumerates the element z until the game concludes; if instead Player **I** plays 1, then Player **II** skips one move and subsequently mirrors Player **I**'s moves for the rest of the game.

We now want to prove (3), so assume $\gamma = \kappa$. By Lemma 4.3.3(1), it is sufficient to show that $(\biguplus_{i < \kappa} C_i)^\nabla \leq_W \biguplus_{i < \kappa} C_i$. A winning strategy for Player **II** in $G_W((\biguplus_{i < \kappa} C_i)^\nabla, \biguplus_{i < \kappa} C_i)$ is as follows.

As long as Player **I** keeps playing 0, Player **II** responds with 0. If at some stage $j' + 1 < \kappa$ Player **I** instead plays 1, then Player **II** still answers with 0. After this point, once more, as long as **I** continues to play 0, Player **II** also plays 0. Suppose that at another stage $j + 1 < \kappa$ (with $j' < j$), Player **I** again plays 1, and let $i < \kappa$ be the unique ordinal satisfying $j' + 1 + i = j$. By hypothesis, there exists some $k \geq j$ such that Player **II** has a winning strategy τ in $G_L(C_i, C_k)$. Let k' be the unique ordinal such that $j + k' = k$. Then, it suffices that **II** plays $0^{(k')} \wedge 1$ and afterwards follows the strategy τ until the game ends.

Next we prove (4), so assume $\gamma < \kappa$. The first part holds by (1) together with the fact that Lipschitz functions are continuous. For the second part, the following strategy for Player **II** is winning in $G_W(\biguplus_{i < \gamma} C_i, A)$:

As long as **I** plays in 0's during the first γ many moves, **II** passes. If **I** ever plays a sequence $t = 0^{(i)} \wedge 1$ for some $i < \gamma$, then player **II** wins by using the winning strategy induced by $C_i \leq_W A$. If **I** plays $0^{(\gamma)}$, then **II** picks some $x \notin A$ and plays initials segments of x for the rest of the run without passing.

Finally, we show (5). Again assume $\gamma < \kappa$. We describe a winning strategy for Player **II** in $G_W(\biguplus_{i < \gamma} C_i, \neg(\biguplus_{i < \gamma} C_i))$:

As long as \mathbf{I} plays in 0's during the first γ many moves, \mathbf{II} passes. If \mathbf{I} ever plays a sequence $t = 0^{(i)} \frown 1$ for some $i < \gamma$, then by hypothesis there is some $j < \gamma$ such that \mathbf{II} has a winning strategy τ in the Wadge game $G_W(C_i, \neg C_j)$, hence player \mathbf{II} wins by playing $0^{(j)} \frown 1$ and then following the strategy τ . If \mathbf{I} plays $0^{(\gamma)}$, then \mathbf{II} picks some $z \in \biguplus_{i < \gamma} C_i$ (this set is non-empty because we assumed $C_i \neq \emptyset$ for at least one $i < \gamma$) and enumerates z until the game ends. \square

Proposition 4.3.6. *Let $\alpha < \kappa^+$ be an ordinal and $C \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$.*

- *If α is odd, $C^\nabla \in \mathcal{D}_{\alpha+1}(\kappa^+ - \mathbf{\Pi}_1^0)$, and $C^\circ \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$.*
- *If α is even, $C^\nabla \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ and $C^\circ \in \mathcal{D}_{\alpha+1}(\kappa^+ - \mathbf{\Pi}_1^0)$.*

Moreover, if $\alpha < \kappa^+$ is a limit ordinal and $(C_i)_{i < \text{cof}(\alpha)}$ is a sequence of sets in $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$, then $\biguplus_{i < \text{cof}(\alpha)} C_i \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$.

Proof. Since $C \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0) \subseteq \mathcal{D}_{\alpha+1}(\kappa^+ - \mathbf{\Pi}_1^0)$, we can write $C = D_\alpha((A_i)_{i < \alpha}) = D_{\alpha+1}((A_i)_{i < \alpha+1})$ with $(A_i)_{i < \alpha}$ decreasing sequence of closed sets and $A_\alpha = \emptyset$. For every $i < \alpha + 1$, we set

$$B_i = A_i^\circ.$$

Note that every B_i is closed as $\kappa^2 \setminus B_i = (\kappa^2 \setminus A_i)^\nabla$ is open, and since $(A_i)_{i < \alpha+1}$ is decreasing, $(B_i)_{i < \alpha+1}$ is decreasing too. Moreover, notice that:

- when α is odd, $0^{(\kappa)} \notin D_{\alpha+1}((B_i)_{i < \alpha+1})$ and $0^{(\kappa)} \in D_\alpha((B_i)_{i < \alpha})$;
- when α is even, $0^{(\kappa)} \in D_{\alpha+1}((B_i)_{i < \alpha+1})$ and $0^{(\kappa)} \notin D_\alpha((B_i)_{i < \alpha})$.

Since for every $\beta < \kappa$,

$$\begin{aligned} D_\alpha((B_i)_{i < \alpha}) \cap \mathbf{N}_{0^{(\beta)} \frown 1} &= 0^{(\beta)} \frown 1 \frown D_\alpha((A_i)_{i < \alpha}) \\ D_{\alpha+1}((B_i)_{i < \alpha+1}) \cap \mathbf{N}_{0^{(\beta)} \frown 1} &= 0^{(\beta)} \frown 1 \frown D_{\alpha+1}((A_i)_{i < \alpha+1}) \end{aligned}$$

we get that:

- when α is odd, $D_{\alpha+1}((B_i)_{i < \alpha+1}) = C^\nabla$ and $D_\alpha((B_i)_{i < \alpha}) = C^\circ$;
- when α is even, $D_{\alpha+1}((B_i)_{i < \alpha+1}) = C^\circ$ and $D_\alpha((B_i)_{i < \alpha}) = C^\nabla$.

Now, assume that $\alpha < \kappa^+$ is a limit ordinal and consider the sets $(C_i)_{i < \text{cof}(\alpha)}$. As $C_i \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ for every $i < \text{cof}(\alpha)$, we can write $C_i = D_\alpha((A_j^i)_{j < \alpha})$ with $(A_j^i)_{j < \alpha}$ decreasing sequence of closed sets.

To see that $\biguplus_{i < \text{cof}(\alpha)} C_i \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$, for every $j < \alpha$, we set

$$F_j = \bigcup_{i < \text{cof}(\alpha)} 0^{(i)} \frown 1 \frown A_j^i \cup \{0^{(\kappa)}\}.$$

Note that every F_j is closed and since $A_{j+1}^i \subseteq A_j^i$ for every $i < \text{cof}(\alpha)$ and $j < \alpha$, then $F_{j+1} \subseteq F_j$. Since for every $i < \text{cof}(\alpha)$,

$$D_\alpha((F_j)_{j < \alpha}) \cap \mathbf{N}_{0^{(i)} \frown 1} = 0^{(i)} \frown 1 \frown D_\alpha((A_j^i)_{j < \alpha}),$$

we get $D_\alpha((F_j)_{j < \alpha}) = \biguplus_{i < \text{cof}(\alpha)} C_i$. \square

Complete sets for non-selfdual classes

In this section, we define subsets of ${}^{\kappa}2$ which are κ -complete sets for the non-selfdual pointclasses $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ and $\check{\mathcal{D}}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$, for all $\alpha < \kappa^+$.

Definition 4.3.7. Let $\mathcal{C}^{(1)}$ be any proper $\kappa^+ - \mathbf{\Pi}_1^0$ -set, e.g. $\mathcal{C}^{(1)} = \{0^{(\kappa)}\}$. Recursively, for every $\alpha < \kappa^+$, we define

$$\mathcal{C}^{(\alpha+1)} = \begin{cases} (\mathcal{C}^{(\alpha)})^\nabla & \text{for } \alpha \text{ odd;} \\ (\mathcal{C}^{(\alpha)})^\circ & \text{for } \alpha \text{ even.} \end{cases}$$

For any limit ordinal α , we fix a strictly increasing cofinal sequence $(\alpha_i)_{i < \text{cof}(\alpha)}$ below α . We define

$$\mathcal{C}^{(\alpha)} = \left(\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)} \right)^\nabla.$$

We also define $\mathcal{U}^{(\alpha)} = {}^{\kappa}2 \setminus \mathcal{C}^{(\alpha)}$ for every $1 \leq \alpha < \kappa^+$.

Moreover, observe that in Definition 4.3.7 the Axiom of Choice AC is used in an essential way, namely for choosing the sequence of complete sets $(\mathcal{C}^{(\alpha_i)})_{i < \text{cof}(\alpha)}$ for every α .

Remark 4.3.8. Let $\alpha < \kappa^+$ be a limit ordinal and let $(A_i)_{i < \alpha}$ be a sequence of closed subsets of ${}^{\kappa}2$. We have $\bigcap_{i < \alpha} A_i = [\bigcap_{i < \alpha} \mathcal{T}_{A_i}]$. However, since the intersection of pruned trees is not necessary pruned, $\mathcal{T}_{\bigcap_{i < \alpha} A_i} \subseteq \bigcap_{i < \alpha} \mathcal{T}_{A_i}$ but reverse inclusion may fail. For example let $(t_i)_{i < 2^\omega}$ be an enumeration of the sequences in ${}^\omega 2 \subseteq {}^{\kappa}2$ and for every $i < 2^\omega$ let $A_i = \mathbf{N}_{\langle 0 \rangle} \setminus \bigcup_{j \leq i} \mathbf{N}_{t_j}$. Then, $\mathcal{T}_{\bigcap_{i < 2^\omega} A_i} = \emptyset$ and $\bigcap_{i < 2^\omega} \mathcal{T}_{A_i} = \{t \in {}^{<\omega}2 \mid t(0) = 0\}$.

We will implicitly use Remark 4.3.8 in our proofs.

Proposition 4.3.9. For every $1 \leq \alpha < \kappa^+$, $\mathcal{C}^{(\alpha)}$ is κ -Lipschitz-complete for $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$, hence $\mathcal{U}^{(\alpha)}$ is κ -Lipschitz-complete for $\check{\mathcal{D}}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$. We conclude that $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ and $\check{\mathcal{D}}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ are Wadge classes.

Proof. The proof is by induction on $1 \leq \alpha < \kappa^+$. The base case $\alpha = 1$ is immediate since any proper $\kappa^+ - \mathbf{\Pi}_1^0$ -set is κ -Lipschitz-complete for $\kappa^+ - \mathbf{\Pi}_1^0$ by Fact 4.1.3, therefore we can move to the case $\alpha > 1$.

First, assume that $\alpha = \beta + 1$ is a successor ordinal. Then, $\mathcal{C}^{(\alpha)} = (\mathcal{C}^{(\beta)})^\nabla$ when α is even, and $\mathcal{C}^{(\alpha)} = (\mathcal{C}^{(\beta)})^\circ$ when α is odd. Since $\mathcal{C}^{(\beta)}$ is κ -Lipschitz-complete for $\mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$ by the induction hypothesis, $\mathcal{C}^{(\alpha)} \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ by Proposition 4.3.6. It remains to prove that $\mathcal{C}^{(\alpha)}$ is κ -Lipschitz-hard for $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$, i.e., to show that for any $A \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$, $A \leq_{\mathbf{L}} \mathcal{C}^{(\alpha)}$. Fix $A = D_{\alpha+1}((A_i)_{i < \alpha+1})$ for some decreasing sequence $(A_i)_{i \leq \beta}$ of closed subsets of ${}^{\kappa}2$. It is sufficient to consider the following strategy for player **II** in $G_{\mathbf{L}}(A, \mathcal{C}^{(\alpha)})$:

As long as **I** plays in \mathcal{T}_{A_β} , **II** plays initial segments of $0^{(\kappa)}$. If **I** ever reaches a minimal $t \notin \mathcal{T}_{A_\beta}$, then $\mathbf{N}_t \cap A \in \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$ and player **II** can answer playing 1 and then proceeds the game using the strategy induced by $A_{|t|} \leq_{\mathbf{L}} \mathcal{C}^{(\beta)}$, which is winning for Player **II** by the inductive hypothesis. We used that $x \in A_{|t|}$ if and only if $t \hat{\ } x \in \mathbf{N}_t \cap A$.

We recall from Definition 3.7.1 that when α is an even ordinal (both successor or limit) $x \in A$ if and only if $x \notin \bigcap_{i < \alpha} A_i$. Therefore in the above strategy, when α is a successor even ordinal, if Player **I** plays a sequence $t \notin \mathcal{T}_{A_\beta} = \mathcal{T}_{\bigcap_{i < \alpha} A_i}$ at some stage of the game, then **II** will not produce the sequence $0^{(\kappa)}$ at the end of the match, as desired according to the definition of $\mathcal{C}^{(\alpha)}$.

Now assume that α is a limit ordinal and fix a sequence $(\alpha_i)_{i < \text{cof}(\alpha)}$ cofinal in α . Then, $\mathcal{C}^{(\alpha)} = (\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla$, and by induction hypothesis we assume that $\mathcal{C}^{(\alpha_i)}$ is κ -Lipschitz-complete for $\mathcal{D}_{\alpha_i}(\kappa^+ - \mathbf{\Pi}_1^0)$ for every $i < \text{cof}(\alpha)$. Then, $\mathcal{C}^{(\alpha_i)} \in \mathcal{D}_{\alpha_i}(\kappa^+ - \mathbf{\Pi}_1^0) \subseteq \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ for every $i < \text{cof}(\alpha)$ and by Proposition 4.3.6, we get that $\mathcal{C}^{(\alpha)} \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$. It remains to show that $\mathcal{C}^{(\alpha)}$ is κ -Lipschitz-hard for $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$. Fix $A \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ and let $A = D_\alpha((A_j)_{j < \alpha})$ for some decreasing sequence $(A_j)_{j < \alpha}$ of closed subsets of κ^2 . The following strategy is winning for player **II** in $G_L(A, \mathcal{C}^{(\alpha)})$:

As long as **I** plays in $\mathcal{T}_{\bigcap_{j < \alpha} A_j}$, **II** plays initial segments of $0^{(\kappa)}$. Assume that **I** reaches a minimal $t \notin \mathcal{T}_{\bigcap_{j < \alpha} A_j}$, so that player **II** has so far played the sequence $0^{(\text{lh}(t))}$. If $t \notin \mathcal{T}_{A_0}$ then **II** can keep playing 0's until the end of the game, as $0^{(\kappa)} \notin \mathcal{C}^{(\alpha)}$. Otherwise, let $1 \leq \beta < \alpha$ such that $t \in \mathcal{T}_{A_\beta} \setminus \mathcal{T}_{A_{\beta+1}}$, and let $\bar{i} < \text{cof}(\alpha)$ be minimal such that $\beta + 1 \leq \alpha_{\bar{i}}$. Then, $D_\alpha((A_j)_{j < \alpha}) \cap N_t = D_{\beta+1}((A_j)_{i < \beta+1}) \in \mathcal{D}_{\beta+1}(\kappa^+ - \mathbf{\Pi}_1^0) \subseteq \mathcal{D}_{\alpha_{\bar{i}}}(\kappa^+ - \mathbf{\Pi}_1^0)$, and by inductive hypothesis, $[D_\alpha((A_i)_{i < \alpha})]_{|t} \leq_L \mathcal{C}^{(\alpha_{\bar{i}})}$. Then, it is enough that **II** follows the strategy induced by $[D_\alpha((A_i)_{i < \alpha})]_{|t} \leq_L 1 \wedge 0^{(\bar{i})} \wedge 1 \wedge \mathcal{C}^{(\alpha_{\bar{i}})}$. \square

Proposition 4.3.10. *For every $\alpha < \kappa^+$, the Lipschitz class generated by $\mathcal{C}^{(\alpha)}$ is independent of the choice of the sequences $(\alpha_i)_{i < \text{cof}(\alpha)}$.*

Proof. Recall that $\mathcal{C}^{(\alpha)}$ is defined using a particular choice of cofinal sequences below each limit ordinal $\gamma \leq \alpha$ and assume that $\bar{\mathcal{C}}^{(\alpha)}$ is defined using a different choice of sequences. However, both sets are κ -Lipschitz-complete for $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ by Proposition 4.3.9. Hence $\mathcal{C}^{(\alpha)} \equiv_L \bar{\mathcal{C}}^{(\alpha)}$. \square

Complete sets for selfdual classes at successor stage and at limit stage of cofinality smaller than κ

In this part, we introduce subsets of κ^2 that serve as κ -complete sets for the selfdual pointclasses $\Delta_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}$, both in the case where $\alpha < \kappa^+$ is a successor ordinal, and in the case where $\alpha < \kappa^+$ is a limit ordinal of cofinality less than κ . The situation where α is a limit ordinal of cofinality κ is more subtle, and it will be addressed in Section 4.3.2.

For every $1 \leq \alpha < \kappa^+$, we recall that $\mathcal{C}^{(\alpha)}$ and $\mathcal{U}^{(\alpha)}$ are the sets from Definition 4.3.7, which are κ -complete for, respectively, $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ and $\check{\mathcal{D}}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ by Proposition 4.3.9.

Proposition 4.3.11. *For every $\alpha > 0$, $\mathcal{U}^{(\alpha)} \oplus \mathcal{C}^{(\alpha)}$ is κ -complete for $\Delta_{\mathcal{D}_{\alpha+1}(\kappa^+ - \mathbf{\Pi}_1^0)}$. Therefore, $\Delta_{\mathcal{D}_{\alpha+1}(\kappa^+ - \mathbf{\Pi}_1^0)}$ is a Wadge class.*

Proof. It is easy to see that $\mathcal{U}^{(\alpha)} \oplus \mathcal{C}^{(\alpha)} \in \Delta_{\mathcal{D}_{\alpha+1}(\kappa^+-\mathbf{\Pi}_1^0)}$. Indeed, for every $0 < \gamma < \kappa^+$, if $A, B \in \mathcal{D}_\gamma(\kappa^+-\mathbf{\Pi}_1^0)$ then $A \oplus B \in \mathcal{D}_\gamma(\kappa^+-\mathbf{\Pi}_1^0)$, and similarly for $\check{\mathcal{D}}_\gamma(\kappa^+-\mathbf{\Pi}_1^0)$. Since $\mathcal{U}^{(\alpha)}, \mathcal{C}^{(\alpha)} \in \mathcal{D}_{\alpha+1}(\kappa^+-\mathbf{\Pi}_1^0) \cap \check{\mathcal{D}}_{\alpha+1}(\kappa^+-\mathbf{\Pi}_1^0)$ by the inclusions in (3.7.1), we are done.

It remains to show that $\mathcal{U}^{(\alpha)} \oplus \mathcal{C}^{(\alpha)}$ is κ -hard for $\Delta_{\mathcal{D}_{\alpha+1}(\kappa^+-\mathbf{\Pi}_1^0)}$.

Let $E \in \Delta_{\mathcal{D}_{\alpha+1}(\kappa^+-\mathbf{\Pi}_1^0)}$. We claim that for every $x \in \kappa^2$ there exists $\beta < \kappa$ such that either $E \cap \mathbf{N}_{x|\beta} \in \mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$ or $E \cap \mathbf{N}_{x|\beta} \in \check{\mathcal{D}}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$.

Assuming the claim is true, a winning strategy for player $\mathbf{\Pi}$ in $G_{\mathbf{W}}(E, \mathcal{U}^{(\alpha)} \oplus \mathcal{C}^{(\alpha)})$ will simply consist of “passing” until stage β and then: if $E \cap \mathbf{N}_{x|\beta} \in \mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$, play 1 followed by the strategy induced by $E_{[x|\beta]} \leq_{\mathbf{W}} \mathcal{C}^{(\alpha)}$; similarly, if $E \cap \mathbf{N}_{x|\beta} \in \check{\mathcal{D}}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$, play 0 followed by the strategy induced by $E_{[x|\beta]} \leq_{\mathbf{W}} \mathcal{U}^{(\alpha)}$.

It remains to prove the claim. By assumption, $E = D_{\alpha+1}((C_i)_{i < \alpha+1})$ and $\kappa^2 \setminus E = D_{\alpha+1}((D_i)_{i < \alpha+1})$ for some decreasing sequences of closed sets $(C_i)_{i < \alpha+1}, (D_i)_{i < \alpha+1}$. Let $x \in \kappa^2$ be fixed.

First, assume that α is odd. Using Definition 3.7.1: either $x \in E$, hence $x \in C_i \setminus C_{i+1}$ for some $i < \alpha$, or $x \notin E$, implying $x \in D_j \setminus D_{j+1}$ for some $j < \alpha$. In the first case, there exists $\beta_i < \kappa$ such that $(\mathbf{N}_{x|\beta_i} \cap E) \cap C_{i+1} = \emptyset$, hence $\mathbf{N}_{x|\beta_i} \cap E \in \mathcal{D}_{i+1}(\kappa^+-\mathbf{\Pi}_1^0) \subseteq \mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$. In the second case, there exists $\beta_j < \kappa$ such that $(\mathbf{N}_{x|\beta_j} \setminus E) \cap D_{j+1} = \emptyset$, thus $\mathbf{N}_{x|\beta_j} \setminus E \in \mathcal{D}_{j+1}(\kappa^+-\mathbf{\Pi}_1^0)$. Then, $E \in \check{\mathcal{D}}_{j+1}(\kappa^+-\mathbf{\Pi}_1^0(\mathbf{N}_{x|\beta_j}))$, hence $\mathbf{N}_{x|\beta_j} \cap E \in \check{\mathcal{D}}_{j+1}(\kappa^+-\mathbf{\Pi}_1^0) \subseteq \check{\mathcal{D}}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$.

Next, assume that α is even. Using Definition 3.7.1 and since $\bigcap_{i < \alpha+1} C_i \subseteq E$ and $\bigcap_{i < \alpha+1} D_i \cap E = \emptyset$, we have three cases. First, assume $x \notin \bigcap_{i < \alpha+1} C_i$ and $x \notin \bigcap_{i < \alpha+1} D_i$. Then, either $x \in E$, hence $x \in C_i \setminus C_{i+1}$ for some $i < \alpha$, or $x \notin E$, implying $x \in D_j \setminus D_{j+1}$ for some $j < \alpha$, and we conclude repeating the argument used in the odd case. Next, assume $x \in \bigcap_{i < \alpha+1} C_i$. Then, it must be $x \notin \bigcap_{i < \alpha+1} D_i$, and so there exists $j < \alpha$ such that $x \in E \setminus D_j$. Let $\beta < \kappa$ such that $(\mathbf{N}_{x|\beta} \cap E) \cap D_j = \emptyset$, then $\mathbf{N}_{x|\beta} \cap E \in \mathcal{D}_j(\kappa^+-\mathbf{\Pi}_1^0) \subseteq \mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$. Similarly, assume that $x \in \bigcap_{i < \alpha+1} D_i$. Then, $x \notin \bigcap_{i < \alpha+1} C_i$, hence $x \notin E \cup C_i$ for some $i < \alpha$. Let $\beta < \kappa$ such that $(\mathbf{N}_{x|\beta} \setminus E) \cap C_i = \emptyset$, we conclude that $\mathbf{N}_{x|\beta} \cap E \in \check{\mathcal{D}}_i(\kappa^+-\mathbf{\Pi}_1^0) \subseteq \check{\mathcal{D}}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$. \square

Proposition 4.3.9 and Lemma 4.3.5(3) shows that when $\alpha < \kappa^+$ is a limit ordinal with $\text{cof}(\alpha) = \kappa$, the set $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$ is κ -(Lipschitz-)complete for $\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$. On the other hand, we show in Proposition 4.3.13 that when $\text{cof}(\alpha) < \kappa$, the set $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$ is selfdual and its Wadge class is immediately below the one of $(\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla$. Indeed, when $\text{cof}(\alpha) < \kappa$, $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$ is κ -complete for $\Delta_{\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)}$.

Lemma 4.3.12. *Let $\alpha < \kappa^+$ be a limit ordinal with $\text{cof}(\alpha) < \kappa$ and let $\langle \alpha_i \mid i < \text{cof}(\alpha) \rangle$ be a sequence cofinal in α . Then,*

- (1) $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$ is selfdual;
- (2) $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)} <_{\mathbf{W}} (\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla$.
- (3) If $B \subseteq \kappa^2$ is such that $B \leq_{\mathbf{W}} (\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla$ and $B \leq_{\mathbf{W}} \neg(\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla$, then $B \leq_{\mathbf{W}} \biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$.

Proof. Note that $\mathcal{C}^{(\alpha_i)}$ is in $\mathcal{D}_{\alpha_i}(\kappa^+ - \Pi_1^0)$ and hence in $\check{\mathcal{D}}_{\alpha_{i+1}}(\kappa^+ - \Pi_1^0)$ by (3.7.1). Moreover, $\mathcal{U}^{(\alpha_{i+1})}$ is κ -Lipschitz-complete for $\check{\mathcal{D}}_{\alpha_{i+1}}(\kappa^+ - \Pi_1^0)$ by Proposition 4.3.9. Fix a Lipschitz reduction f_i witnessing $\mathcal{C}^{(\alpha_i)} \leq_{\mathbb{L}} \mathcal{U}^{(\alpha_{i+1})}$ for each $i < \text{cof}(\alpha)$.

For (1), we invoke Lemma 4.3.5(5). Indeed the sequence $(\mathcal{C}^{(\alpha_i)})_{i < \text{cof}(\alpha)}$ satisfies the condition that for every $i < \text{cof}(\alpha)$ there exists $j < \text{cof}(\alpha)$ with $\mathcal{C}^{(\alpha_i)} \leq_{\mathbb{W}} \neg \mathcal{C}^{(\alpha_j)}$ by simply taking $j = i + 1$.

To show (2), it is sufficient to verify that $(\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla \not\leq_{\mathbb{W}} \biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$. Indeed, $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)} \leq_{\mathbb{W}} (\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla$ holds by Lemma 4.3.3(1).

Towards a contradiction, assume there exists a reduction g of $(\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla$ to $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$. Consider a sequence $(x_\gamma)_{\gamma < \kappa}$ such that each $x_\gamma \in 0^{(\beta)} \dot{\cap} 1 \dot{\cap} 0^{(i)} \dot{\cap} 1 \dot{\cap} \mathcal{C}^{(\alpha_i)}$, with $\gamma = \langle \beta, i \rangle$ for $\beta < \kappa$, $i < \text{cof}(\alpha)$. Since $\lim_{\gamma < \kappa} x_\gamma = 0^{(\kappa)} \in \text{cl}((\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla)$ and g is a continuous reduction, we have $y = \lim_{\gamma < \kappa} g(x_\gamma) \in \text{cl}(\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})$. Then, we must have $y \in \mathbf{N}_{0^{(i)} \dot{\cap} 1}$ for some $i < \text{cof}(\alpha)$, since any $y \in \mathbf{N}_{0^{(\text{cof}(\alpha))}}$ belongs to neither $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$ nor to its boundary. By continuity of g in $0^{(\kappa)}$, there exists $\eta < \kappa$ such that $g(\mathbf{N}_{0^{(\eta)}}) \subseteq \mathbf{N}_{0^{(i)} \dot{\cap} 1}$. Then, $\mathbf{N}_{0^{(\eta)} \dot{\cap} 1 \dot{\cap} 0^{(i+1)} \dot{\cap} 1} \subseteq \mathbf{N}_{0^{(\eta)}}$ and $g \upharpoonright \mathbf{N}_{0^{(\eta)}}$ shows $\mathcal{C}^{(\alpha_{i+1})} \leq_{\mathbb{W}} \mathcal{C}^{(\alpha_i)}$, a contradiction.

Finally, to prove (3), let f be the function witnessing $B \leq_{\mathbb{W}} (\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla$ and g the one witnessing $B \leq_{\mathbb{W}} \neg(\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla$.

Let $h : \kappa^2 \setminus \{0^{(\kappa)}\} \rightarrow \kappa^2$ be defined by $h(x) = x_{\lfloor \beta + 1 \rfloor}$ if $x \in \mathbf{N}_{0^{(\beta)} \dot{\cap} 1}$ for some $\beta < \kappa$. Note that h witness $(\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla \leq_{\mathbb{W}} \biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$ with respect to the space $\kappa^2 \setminus \{0^{(\kappa)}\}$, but since $\text{cof}(\alpha) < \kappa$, h cannot be extended to a continuous function with domain κ^2 by (2).

Moreover, let $k : \kappa^2 \setminus \{0^{(\kappa)}\} \rightarrow \kappa^2$ be defined by $k(x) =$

$$\begin{cases} 0^{(i+1)} \dot{\cap} 1 \dot{\cap} f_i(y) & \text{if } x = 0^{(\beta)} \dot{\cap} 1 \dot{\cap} 0^{(i)} \dot{\cap} 1 \dot{\cap} y \text{ for some } i < \text{cof}(\alpha) \text{ and } \beta < \kappa; \\ x_{\lfloor \beta + 1 \rfloor} & \text{if } x \in \mathbf{N}_{0^{(\beta)} \dot{\cap} 1 \dot{\cap} 0^{(\text{cof}(\alpha))}} \text{ for some } \beta < \kappa. \end{cases}$$

Note that k witness $(\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla \leq_{\mathbb{W}} \neg[\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}]$ with respect to the space $\kappa^2 \setminus \{0^{(\kappa)}\}$, but k cannot be extended to a continuous function with domain κ^2 by (1) and (2).

If $\mathcal{O} \subseteq \kappa^2$ is a set such that $0^{(\kappa)} \notin \text{ran}(f) \upharpoonright \mathcal{O}$, then $h \circ f \upharpoonright \mathcal{O}$ reduces B to $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$. If $\mathcal{O}' \subseteq \kappa^2$ is a set such that $0^{(\kappa)} \notin \text{ran}(g) \upharpoonright \mathcal{O}'$, then $k \circ g \upharpoonright \mathcal{O}'$ reduces B to $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$. Note that $f^{-1}(\{0^{(\kappa)}\})$ and $g^{-1}(\{0^{(\kappa)}\})$ are closed and disjoint, since $f^{-1}(\{0^{(\kappa)}\}) \cap B = \emptyset$ and $g^{-1}(\{0^{(\kappa)}\}) \subseteq B$. By the separation property for $\kappa^+ - \Pi_1^0$ -sets (Theorem 3.3.17), let $\mathcal{O} \in \kappa^+ - \Delta_1^0$ such that $f^{-1}(\{0^{(\kappa)}\}) \cap \mathcal{O} = \emptyset$ and $g^{-1}(\{0^{(\kappa)}\}) \subseteq \mathcal{O}$. Then,

$$(h \circ f) \upharpoonright \mathcal{O} \cup (k \circ g) \upharpoonright \kappa^2 \setminus \mathcal{O}$$

is a continuous reduction of B to $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$. \square

Proposition 4.3.13. *Let $\alpha < \kappa^+$ be a limit ordinal with $\text{cof}(\alpha) < \kappa$ and let $\langle \alpha_i \mid i < \text{cof}(\alpha) \rangle$ be a sequence cofinal in α . Then, the set $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$ is κ -complete for $\Delta_{\mathcal{D}_{\alpha}(\kappa^+ - \Pi_1^0)}$, hence $\Delta_{\mathcal{D}_{\alpha}(\kappa^+ - \Pi_1^0)}$ is a Wadge class.*

Proof. As $\mathcal{C}^{(\alpha_i)} \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ for every $i < \text{cof}(\alpha)$, by Lemma 4.3.6 we get that $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)} \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$. Moreover, $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)} \in \check{\mathcal{D}}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ since it is selfdual by Lemma 4.3.12(1). By Proposition 4.3.9, $(\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla$ is κ -complete for $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ hence its complement is κ -complete for $\check{\mathcal{D}}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$. Then, for any $B \in \Delta_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}$, we have both $B \leq_W (\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla$ and $B \leq_W \neg(\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)})^\nabla$. By Lemma 4.3.12(3), $B \leq_W \biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$. \square

Remark 4.3.14. Working in the generalized Baire space ${}^\kappa\kappa$, the operation \biguplus from Definition 4.3.2 can be reformulated as follows. For an ordinal $\gamma \leq \kappa$ and a sequence $(C_i)_{i < \gamma}$ of subsets of ${}^\kappa\kappa$, define

$$\biguplus_{i < \gamma} C_i = \bigcup_{i < \gamma} i \hat{\ } C_i.$$

By applying the same argument used in Lemma 4.3.5(4), one readily verifies that $\biguplus_{i < \gamma} C_i$ is the supremum of the family $(C_i)_{i < \gamma}$ for every $\gamma \leq \kappa$. This contrasts with the corresponding result for κ^2 , where the restriction $\gamma < \kappa$ was required. By applying arguments analogous to those used in this section, one can show that for every limit ordinal $\alpha < \kappa$ and every sequence $(\alpha_i)_{i < \text{cof}(\alpha)}$ cofinal in α , the set $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$ is κ -complete for $\Delta_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}({}^\kappa\kappa)$.

4.3.2 The Wadge hierarchy within the difference hierarchy

In this section, we continue our analysis of the Wadge classes in the $\kappa^+ - \mathbf{\Pi}_1^0$ -difference hierarchy, showing that SLO_W holds for every set in the $\kappa^+ - \mathbf{\Pi}_1^0$ -difference hierarchy on κ^2 (Theorem 4.3.15), and that the corresponding Wadge hierarchy is well-founded (Theorem 4.3.35). Consequently, the initial κ^+ levels of the Wadge hierarchy are precisely given by the $\kappa^+ - \mathbf{\Pi}_1^0$ -difference hierarchy, in direct analogy with the classical case.

Even though our methods are quite different, it is worth noting that, in the classical setting, [And06, Section 7] provides an analysis of how to propagate SLO_W within individual Δ_2^0 degrees, in a way that parallels the construction developed in the present section.

We now state the main theorem of this section. The majority of what follows is devoted to proving this result, whose proof proceeds by induction and depends on a series of intermediate results that are also involved in the inductive argument.

Theorem 4.3.15. *For every $\alpha < \kappa^+$, $\text{SLO}_W(\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0) \cup \check{\mathcal{D}}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0))$.*

The proof of Theorem 4.3.15 appears on page 133. The overall proof strategy can be summarized as follows.

The argument proceeds by transfinite induction on ordinals $\alpha < \kappa^+$. The base case is provided by Corollary 4.1.4. For each stage $\alpha > 1$, we establish that:

- (a) $\text{SLO}_W(\Delta_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)})$;
- (b) A set is $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ -proper if and only if it is κ -complete for $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$.

These conditions, combined with Lemma 2.11.3, yield the result. As usual, the induction argument divides into the successor and limit cases. Moreover, when α is a limit ordinal, the argument further splits according to the value of $\text{cof}(\alpha)$.

Before diving into the proof, we need the following definition.

Definition 4.3.16. Let $\gamma < \kappa^+$ and $(C_i)_{i \leq \gamma}$ be a decreasing sequence of closed subsets of ${}^\kappa 2$ and let U be a basic open set. Let

$$(*_1)(C_0, C_1, U) : \exists c \in (\partial C_1 \setminus C_0) \cap U$$

and for every $\gamma > 1$,

- when $\gamma = \beta + 1$ is successor ordinal, we define:

$$(*_{\beta+1})((C_i)_{i < \beta+1}, C_{\beta+1}, U): \text{ there exists } x \in (\partial C_\beta \setminus C_{\beta+1}) \cap U \text{ such that for every } \alpha < \kappa \ D_{\beta+1}((C_i)_{i < \beta+1}) \cap \mathbf{N}_{x \upharpoonright \alpha} \notin \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0).$$

- when γ is limit ordinal, we define:

$$(*_\gamma)((C_i)_{i < \gamma}, C_\gamma, U): \text{ there exists } x \in (\partial(\bigcap_{i < \gamma} C_i) \setminus C_\gamma) \cap U \text{ for every } \alpha < \kappa \text{ and for every } \beta < \gamma, \ D_\gamma((C_i)_{i < \gamma}) \cap \mathbf{N}_{x \upharpoonright \alpha} \notin \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0).$$

Observe that for any closed set $C \subseteq {}^\kappa 2$, the condition $(*_1)(C, \emptyset, {}^\kappa 2)$ is satisfied exactly when C is κ -complete for $\kappa^+ - \mathbf{\Pi}_1^0$, that is, when $C \in \kappa^+ - \mathbf{\Pi}_1^0 \setminus \kappa^+ - \mathbf{\Sigma}_1^0$.

Example 4.3.17.

- Consider the space ${}^\kappa 2 \times {}^\kappa 2$. Let $D = \{(0^{(\kappa)}, 0^{(\kappa)})\}$ and $C = {}^\kappa 2 \times \{0^{(\kappa)}\}$. Then, $C = \partial C$ and $D = \partial D$ are closed, $D \subseteq C$ and the point $(0^{(\kappa)}, 0^{(\kappa)})$ witnesses $(*_2)(C, D, \emptyset, {}^\kappa 2)$.

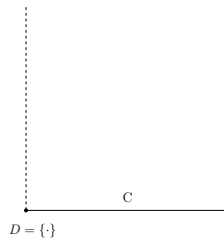
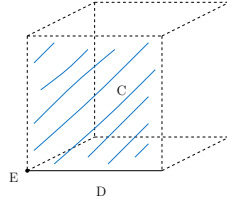


Figure 4.2: D_2

- Consider the space ${}^\kappa 2 \times {}^\kappa 2 \times {}^\kappa 2$. Let $E = \{(0^{(\kappa)}, 0^{(\kappa)}, 0^{(\kappa)})\}$, $D = {}^\kappa 2 \times \{0^{(\kappa)}\} \times \{0^{(\kappa)}\}$ and $C = {}^\kappa 2 \times {}^\kappa 2 \times \{0^{(\kappa)}\}$. Then, $C = \partial C$, $D = \partial D$ and $E = \partial E$ are closed, $E \subseteq D \subseteq C$ and the point $(0^{(\kappa)}, 0^{(\kappa)})$ witnesses $(*_3)(C, D, E, \emptyset, {}^\kappa 2)$.

Figure 4.3: D_3 **Proof of Theorem 4.3.15 — the successor stage** ($\alpha = \beta + 1$)

Assume $\alpha = \beta + 1 < \kappa^+$ is a successor ordinal. The standing inductive hypothesis throughout this section (until Corollary 4.3.20) is that $\text{SLO}_W(\mathcal{D}_\beta(\kappa^+ - \Pi_1^0)) \cup \check{\mathcal{D}}_\beta(\kappa^+ - \Pi_1^0)$ holds.

Proposition 4.3.18. $\text{SLO}_W(\Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)})$.

Proof. We recall that $\Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$ is a Wadge class by Proposition 4.3.11, hence it has a complete set. Let $A, B \in \Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$. If B is κ -complete for $\Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$ then $A \leq_W B$, so assume $B \in \mathcal{D}_\beta(\kappa^+ - \Pi_1^0)$ or $B \in \check{\mathcal{D}}_\beta(\kappa^+ - \Pi_1^0)$. In both cases, if A is κ -complete for $\Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$ then $B \leq_W \kappa^2 \setminus A$, since the class $\Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$ is self-dual. Instead, if also $A \in \mathcal{D}_\beta(\kappa^+ - \Pi_1^0)$ or $A \in \check{\mathcal{D}}_\beta(\kappa^+ - \Pi_1^0)$, then the results follows by $\text{SLO}_W(\mathcal{D}_\beta(\kappa^+ - \Pi_1^0))$ and $\text{SLO}_W(\check{\mathcal{D}}_\beta(\kappa^+ - \Pi_1^0))$. \square

Theorem 4.3.19. Let $(C_i)_{i < \alpha}$ be decreasing sequence of closed subsets of κ^2 . The following conditions are equivalent:

- (1) $(*_\alpha)((C_i)_{i < \alpha}, \emptyset, \kappa^2)$ holds, i.e. there exists $x \in \partial C_\beta$ such that for every $\alpha < \kappa$ $D_\alpha((C_i)_{i < \alpha}) \cap N_{x \upharpoonright \alpha} \notin \check{\mathcal{D}}_\beta(\kappa^+ - \Pi_1^0)$;
- (2) $D_\alpha((C_i)_{i < \alpha})$ is a proper $\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$ -set;
- (3) $D_\alpha((C_i)_{i < \alpha})$ is κ -complete for $\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$.

Proof. (1) \Rightarrow (3) Since $D_\alpha((C_i)_{i < \alpha}) \in \mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$ by definition, it remains to show that it is κ -hard for $\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$. Let $(B_i)_{i < \alpha}$, be any decreasing sequence of closed subsets of κ^2 . To show $D_\alpha((B_i)_{i < \alpha}) \leq_W D_\alpha((C_i)_{i < \alpha})$, consider the following strategy for player \mathbf{II} in $G_W(D_\alpha((B_i)_{i < \alpha}), D_\alpha((C_i)_{i < \alpha}))$:

As long as \mathbf{I} plays in \mathcal{T}_{B_β} , \mathbf{II} plays initial segments of $x \in \partial C_\beta$ given by $(*_\alpha)((C_i)_{i < \alpha}, \emptyset, \kappa^2)$. Now, assume that \mathbf{I} reaches a minimal $t \notin \mathcal{T}_{B_\beta}$ and that player \mathbf{II} has so far played the sequence $x \upharpoonright \gamma$ for some $\gamma \leq |t|$. Since $N_t \cap C_\beta = \emptyset$, $D_\alpha((B_i)_{i < \alpha}) \cap N_t \in \mathcal{D}_\beta(\kappa^+ - \Pi_1^0)$. By $(*_\alpha)$, $D_\alpha((C_i)_{i \leq \beta}) \cap N_{x \upharpoonright \gamma} \notin \check{\mathcal{D}}_\beta(\kappa^+ - \Pi_1^0)$. Recalling that every open set can be written as a union of κ -many clopen sets, let $N_{x \upharpoonright \gamma} \setminus C_\beta = \bigcup_{u \in \mathcal{S}} N_u$ where $\mathcal{S} = \{u \in \kappa^2 \mid N_u \subseteq N_{x \upharpoonright \gamma} \setminus C_\beta\}$; let \mathcal{S}' be the collection of elements $u \in \mathcal{S}$ such that the N_u 's are pairwise disjoint. Let $(u_j)_{j < \kappa}$ be an enumeration of \mathcal{S}' . Note that, for every $j < \kappa$, $x \upharpoonright \gamma \subseteq u_j$ and $D_\alpha((C_i)_{i < \alpha}) \cap N_{u_j} \in \mathcal{D}_\beta(\kappa^+ - \Pi_1^0)$.

We claim that there exists $j_0 < \kappa$ such that $D_\alpha((C_i)_{i < \alpha}) \cap N_{u_{j_0}} \in \mathcal{D}_\beta(\kappa^+ - \Pi_1^0) \setminus \check{\mathcal{D}}_\beta(\kappa^+ - \Pi_1^0)$. Then, $D_\alpha((C_i)_{i < \alpha}) \cap N_{u_{j_0}}$ is κ -complete

for $\mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$ by the induction hypothesis and player **II** wins by continuing the enumeration of the sequence u_j and then using the winning strategy induced by

$$[D_\alpha((B_i)_{i < \alpha})]_{[t]} \leq_W [D_\alpha((C_i)_{i < \alpha})]_{[u_{j_0}]}.$$

It remains to prove the above claim. Assume towards a contradiction that for every $j < \kappa$, $D_\alpha((C_i)_{i < \alpha} \cap \mathbf{N}_{u_j}) \in \check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$. Then, for every $j < \kappa$, $D_\alpha((C_i)_{i < \alpha} \in \check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0(\mathbf{N}_{u_j}))$, hence there exists a decreasing sequence $(F_i^{u_j})_{i < \beta}$ of closed subsets of \mathbf{N}_{u_j} such that $D_\alpha((C_i)_{i < \alpha}) \cap \mathbf{N}_{u_j} = \mathbf{N}_{u_j} \setminus D_\beta((F_i^{u_j})_{i < \beta})$. Since $\mathbf{N}_{u_j} \in \kappa^+ - \mathbf{\Delta}_1^0$, the sets $F_i^{u_j}$'s are closed in κ^2 too. For every $i < \beta$, let

$$F_i^* = \text{cl}\left(\bigcup_{j < \kappa} F_i^{u_j}\right) \cup C_\beta.$$

Note that $F_i^* = \bigcup_{j < \kappa} F_i^{u_j} \cup C_\beta$. To see this, suppose that $(x_\gamma)_{\gamma < \kappa}$ is a sequence in $\bigcup_{j < \kappa} F_i^{u_j}$ which has limit $x \in \kappa^2 \setminus C_\beta$. Then, $x \in \mathbf{N}_{u_j}$ for some $j < \kappa$. By definition of convergence, there exists $\gamma_0 < \kappa$ such that $x_\gamma \in \mathbf{N}_{u_j}$ for all $\gamma \geq \gamma_0$. Thus, $x_\gamma \in F_i^{u_j}$ for all $\gamma \geq \gamma_0$ and since $F_i^{u_j} \in \kappa^+ - \mathbf{\Pi}_1^0$, we have that $x \in F_i^{u_j}$.

Then, for every $j < \kappa$,

$$\mathbf{N}_{u_j} \setminus D_\beta((F_i^{u_j})_{i < \beta}) = \mathbf{N}_{u_j} \setminus D_\beta((F_i^*)_{i < \beta})$$

where the equality holds because $F_i^* \cap \mathbf{N}_{u_j} = F_i^{u_j} \cap \mathbf{N}_{u_j}$ for every $i < \beta$. We conclude that $D_\alpha((C_i)_{i < \alpha} \cap \mathbf{N}_{x \upharpoonright \gamma}) = \mathbf{N}_{x \upharpoonright \gamma} \setminus D_\beta((F_i^*)_{i < \beta}) \in \check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0(\mathbf{N}_{x \upharpoonright \gamma}))$, thus $D_\alpha((C_i)_{i < \alpha} \cap \mathbf{N}_{x \upharpoonright \gamma}) \in \check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$ a contradiction. About the last equality, note that C_β is a subset of both sets if β is even, and it is disjoint from both of them when β is odd instead.

(3) \Rightarrow (2) Follows immediately from Lemma 2.11.2(4).

(2) \Rightarrow (1) Assume $\neg(*_\alpha)((C_i)_{i < \alpha}, \emptyset, \kappa^2)$, i.e., that for every $x \in \partial C_\beta$ there exists $\alpha < \kappa$ such that $D_\alpha((C_i)_{i \leq \beta}) \cap \mathbf{N}_{x \upharpoonright \alpha} \in \check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$. The goal is to prove that $D_\alpha((C_i)_{i < \alpha}) \in \mathbf{\Delta}_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}$. Since $\mathcal{U}^{(\beta)} \oplus \mathcal{C}^{(\beta)} \in \mathbf{\Delta}_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}$ by Proposition 4.3.11, it is enough to show that $D_\alpha((C_i)_{i < \alpha}) \leq_W \mathcal{U}^{(\beta)} \oplus \mathcal{C}^{(\beta)}$. Consider the following winning strategy for player **II** in $G_W((D_\alpha((C_i)_{i < \alpha}), \mathcal{U}^{(\beta)} \oplus \mathcal{C}^{(\beta)})$.

As long as Player **I** plays sequences in $\mathcal{T}_{\partial C_\beta}$, then **II** passes her turn until one of the following two cases occurs:

- (a) **I** reaches a minimal $t \notin \mathcal{T}_{\partial C_\beta}$,
- (b) **I** plays some t such that $D_\alpha((C_i)_{i < \alpha}) \cap \mathbf{N}_t \in \check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$.

Note that a sequence $t \in \kappa^2$ satisfying either (a) or (b) exists – and thus player **II** will not pass for κ -many turns – because if not, then **I** produces a sequence in ∂C_β which witnesses $(*_\alpha)((C_i)_{i < \alpha}, \emptyset, \kappa^2)$, against the hypothesis.

First assume case (a) occurs. Then, Player **II** keeps passing until **I** plays some $u \supseteq t$ such that either $\mathbf{N}_u \subseteq C_\beta$ or $\mathbf{N}_u \cap C_\beta = \emptyset$.

If $N_u \subseteq C_\beta$, then \mathbf{II} enumerates any element inside $\mathcal{U}^{(\beta)} \oplus \mathcal{C}^{(\beta)}$ when β is even, or any element outside of $\mathcal{U}^{(\beta)} \oplus \mathcal{C}^{(\beta)}$ when β is odd. If $N_u \cap C_\beta = \emptyset$, then $N_u \cap D_\alpha((C_i)_{i < \alpha}) \in \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$, hence it is enough that \mathbf{II} plays the sequence $\langle 1 \rangle$ followed by its winning strategy induced by the reduction $[D_\alpha((C_i)_{i < \alpha})]_{|u|} \leq_L \mathcal{C}^{(\beta)}$.

Assume now (b). In this case, Player \mathbf{II} wins by answering with the sequence $\langle 0 \rangle$ and then using the winning strategy induced by the reduction $[D_\alpha((C_i)_{i < \alpha})]_{|t|} \leq_L \mathcal{U}^{(\beta)}$. \square

Corollary 4.3.20. *For every set $A \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$, either A is κ -complete for \mathcal{D}_α , or $A \in \Delta_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}$. Therefore, $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ and $\check{\mathcal{D}}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ are the Wadge classes immediately above $\Delta_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}$.*

This completes the successor case of the induction.

Proof of Theorem 4.3.15 — the limit stage (α limit ordinal)

Let $\alpha < \kappa^+$ be a limit ordinal. The standing inductive hypothesis throughout this section (until Proposition 4.3.32) is that $\text{SLO}_W(\mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0) \cup \check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0))$ holds for every $\beta < \alpha$.

We begin by proving a general result, Lemma 4.3.21, which holds for all limit ordinals. Subsequently, the argument splits according to the cofinality of the ordinal α , and we handle the cases $\text{cof}(\alpha) < \kappa$ and $\text{cof}(\alpha) = \kappa$ separately.

Lemma 4.3.21. *Let $A \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ and $\beta < \alpha$. If $A \notin \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$, then A is κ -hard for $\check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$. Similarly, if $A \notin \check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$, then A is κ -hard for $\mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$.*

Proof. Suppose that $A \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$, and fix a strictly increasing sequence $(\alpha_i)_{i < \text{cof}(\alpha)}$ cofinal in α . Let $\mathcal{C}^{(\alpha)}$ be the set from Definition 4.3.7, recalling that $\mathcal{C}_\alpha = (\bigoplus_{i < \kappa} \mathcal{C}^{(\alpha_i)})^\nabla$. Since $\mathcal{C}^{(\alpha)}$ is κ -complete for $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ by Proposition 4.3.9, there is a continuous reduction f from A to $\mathcal{C}^{(\alpha)}$.

We claim there exists $\{U_j \mid j < \kappa\} \cup \{C\}$ partition of κ^2 such that $U_j \in \kappa^+ - \Delta_1^0$, $C \in \kappa^+ - \mathbf{\Pi}_1^0$ and for every $j < \kappa$ there exists $i(j) < \text{cof}(\alpha)$, $A \cap U_j \in \mathcal{D}_{\alpha_{i(j)}}(\kappa^+ - \mathbf{\Pi}_1^0)$. To see this, let $C = f^{-1}(0^{(\kappa)})$ and notice that:

- if $\text{cof}(\alpha) = \kappa$, let $U_j = f^{-1}(N_{0^{(j)} \smallfrown 1})$ for each $j < \kappa$, so that $i(j) = j$;
- if $\text{cof}(\alpha) < \kappa$, let $U_j = f^{-1}(N_{0^{(j)} \smallfrown 1 \smallfrown 0^{(\ell)} \smallfrown 1})$ where $j = \langle \beta, \ell \rangle$, so that $i(j) = \ell$.

First, assume that $A \cap U_j \notin \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$ for some $j < \kappa$. Recall that $A \cap U_j \in \mathcal{D}_{\alpha_{i(j)}}(\kappa^+ - \mathbf{\Pi}_1^0)$, hence $\beta < \alpha_{i(j)}$. By the induction hypothesis we have $\text{SLO}_W(\mathcal{D}_{\alpha_{i(j)}}(\kappa^+ - \mathbf{\Pi}_1^0))$, that applied to $A \cap U_j$ and to a set κ -complete for $\mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$ implies that $A \cap U_j$ is κ -hard for $\check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$, hence A is κ -hard for $\check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$ and we are done.

From now on, assume $A \cap U_j \in \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$ for all $j < \kappa$. If β is even, then the partition $\{U_j \mid j < \kappa\} \cup \{C\}$ of κ^2 satisfies the assumptions of Lemma 3.7.10, establishing that $A \in \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$. The same occurs if $C = \emptyset$. Since this contradicts the assumption on A , we can assume that β is odd and $C \neq \emptyset$. Note that this argument implies that if β is even then there must be some

$j < \kappa$ such that $A \cap U_j \notin \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$, and therefore the statement holds by the argumentation above. Since β is odd, β is a successor ordinal and we write $\beta = \beta' + 1$ for some even ordinal β' .

Claim 4.3.21.1. There exists $x \in C$ such that for every $\gamma < \kappa$ there exists $\delta \geq \gamma$ $A \cap N_{x \upharpoonright \delta}$ is κ -hard for $\check{\mathcal{D}}_{\beta'}(\kappa^+ - \mathbf{\Pi}_1^0)$.

Proof of the Claim. Towards a contradiction, assume that for every $x \in C$ there exists $\gamma < \kappa$ such that $A \cap N_{x \upharpoonright \delta}$ is not κ -hard for $\check{\mathcal{D}}_{\beta'}(\kappa^+ - \mathbf{\Pi}_1^0)$ for all $\delta \geq \gamma$. Then, for every $x \in C$ let $\gamma_x < \kappa$ be given by the case assumption. Since the statement holds for β' , using that $A \cap N_{x \upharpoonright \gamma_x} \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ and $A \cap N_{x \upharpoonright \gamma_x}$ is not κ -hard for $\check{\mathcal{D}}_{\beta'}(\kappa^+ - \mathbf{\Pi}_1^0)$, we get that $A \cap N_{x \upharpoonright \gamma_x} \in \mathcal{D}_{\beta'}(\kappa^+ - \mathbf{\Pi}_1^0)$. Moreover, $C \subseteq \bigcup_{x \in C} N_{x \upharpoonright \gamma_x}$, but since there are $2^{<\kappa} = \kappa$ many basic open subsets of ${}^\kappa 2$, there exists a family $(t_i)_{i < \kappa}$ in ${}^{<\kappa} 2$ such that $C \subseteq \bigcup_{i < \kappa} N_{t_i}$ and $A \cap N_{t_i} \in \check{\mathcal{D}}_{\beta'}(\kappa^+ - \mathbf{\Pi}_1^0)$ for all $i < \kappa$. We can assume that $N_{t_i} \cap N_{t_j} = \emptyset$ for all $i < j < \kappa$ by recursively splitting up the sets into smaller basic open sets with the same property using Remark 3.7.4. By the separation property for closed sets (Theorem 3.3.17), there exists $\mathcal{O} \in \kappa^+ - \mathbf{\Delta}_1^0$ such that $C \subseteq \mathcal{O} \subsetneq \bigcup_{i < \kappa} N_{t_i}$. Since $A \cap N_{t_i} \cap \mathcal{O} \in \mathcal{D}_{\beta'}(\kappa^+ - \mathbf{\Pi}_1^0)$ by Remark 3.7.4, we apply Lemma 3.7.10 to the partition $\{N_{t_i} \cap \mathcal{O} \mid i < \kappa\}$ of \mathcal{O} to get $A \cap \mathcal{O} \in \mathcal{D}_{\beta'}(\kappa^+ - \mathbf{\Pi}_1^0)$.

On the other hand, $\{U_j \setminus \mathcal{O} \mid j < \kappa\}$ is a partition of ${}^\kappa 2 \setminus \mathcal{O}$ and $A \cap (U_j \setminus \mathcal{O}) \in \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$ for all $j < \kappa$, again using Remark 3.7.4. By Lemma 3.7.10, $A \setminus \mathcal{O} \in \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$. In conclusion, $A \in \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$, resulting to a contradiction. \square

To show that A is κ -hard for $\check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$, let $B = D_\beta((B_\ell)_{\ell < \beta}) \in \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$ for some decreasing sequence of closed sets $(B_\ell)_{\ell < \beta}$. We consider the following strategy for player **II** in $G_W(\neg B, A)$:

As long as **I** plays in $\mathcal{T}_{B_{\beta'}}$, **II** plays initial segments of $x \in C$ given by Claim 4.3.21.1. We recall that C is disjoint from A and that $B_{\beta'}$ is disjoint from $\neg B$ because β is odd (see Definition 3.7.1).

If **I** ever reaches a minimal $t \notin \mathcal{T}_{B_{\beta'}}$, then $\neg B \cap N_t \in \check{\mathcal{D}}_{\beta'}(\kappa^+ - \mathbf{\Pi}_1^0)$. Assume that **II** has so far played the sequence $x \upharpoonright \alpha$, with $\alpha \leq \text{lh}(t)$. By Claim 4.3.21.1, there exists $\delta \geq \alpha$ such that $A \cap N_{x \upharpoonright \delta}$ is κ -hard for $\check{\mathcal{D}}_{\beta'}(\kappa^+ - \mathbf{\Pi}_1^0)$. Then, player **II** wins by continuing the enumeration the sequence x up to level δ , and then by using the winning strategy induced by $(\neg B) \upharpoonright_{[t]} \leq_W A \upharpoonright_{[x \upharpoonright \delta]}$. \square

The remainder of the analysis for limit stages depends on the cofinality of the ordinal α . We treat the two cases separately.

Case α limit ordinal with $\text{cof}(\alpha) < \kappa$. Recall that the standing assumptions throughout this section (and until Proposition 4.3.32) are that α is a limit ordinal and that for every $\beta < \alpha$, $\text{SLO}_W(\mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0) \cup \check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0))$ holds. In this part, and until Corollary 4.3.25, we additionally assume $\text{cof}(\alpha) < \kappa$.

Proposition 4.3.22. $\text{SLO}_W(\Delta_{\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)})$.

Proof. Let $A, B \in \Delta_{\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)}$. If $A, B \in \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+-\mathbf{\Pi}_1^0)$ we are done by the inductive hypothesis. If $A \in \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+-\mathbf{\Pi}_1^0)$ and $B \notin \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+-\mathbf{\Pi}_1^0)$, then by Lemma 4.3.21 B is κ -hard for $\mathcal{D}_\beta(\kappa^+-\mathbf{\Pi}_1^0)$ for every $\beta < \alpha$ and thus $A \leq_W B$. Similarly, $A \notin \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+-\mathbf{\Pi}_1^0)$ and $B \in \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+-\mathbf{\Pi}_1^0)$, then A is κ -hard for $\check{\mathcal{D}}_\beta(\kappa^+-\mathbf{\Pi}_1^0)$ for every $\beta < \alpha$ by Lemma 4.3.21, hence $B \leq_W \neg A$. Finally, assume $A, B \notin \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+-\mathbf{\Pi}_1^0)$. Again using Lemma 4.3.21 we have that A and B are both κ -hard for $\mathcal{D}_\beta(\kappa^+-\mathbf{\Pi}_1^0)$ for every $\beta < \alpha$. Recall that, fixed a sequence $(\alpha_i)_{i < \text{cof}(\alpha)}$ cofinal in α , the set $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$ is κ -complete for $\Delta_{\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)}$ by Proposition 4.3.13. Then, there is a continuous reduction $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)} \leq_W A, B$, implying that both A and B are κ -complete for $\Delta_{\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)}$. We conclude that $A \leq_W B$. \square

Lemma 4.3.23. Let $\gamma < \kappa^+$ be a limit ordinal. Let $(C_i)_{i \leq \gamma}$ be a sequence of closed subsets of κ^2 and U be a basic open set. If $x \in (\partial(\bigcap_{i < \gamma} C_i) \setminus C_\gamma) \cap U$ witnesses that $(*_\gamma)((C_i)_{i < \gamma}, C_\gamma, U)$ holds, then $x \in \partial(D_\gamma((C_i)_{i < \gamma}))$.

Proof. If there were some $\alpha < \kappa$ such that $D_\gamma((C_i)_{i < \gamma}) \cap N_{x \upharpoonright \alpha} = \emptyset$, then we would trivially get that for every $\beta < \gamma$, $D_\gamma((C_i)_{i < \gamma}) \cap N_{x \upharpoonright \alpha} \in \mathcal{D}_\beta(\kappa^+-\mathbf{\Pi}_1^0)$, against $(*_\gamma)((C_i)_{i < \gamma}, C_\gamma, U)$. Similarly, if $N_{x \upharpoonright \alpha} \subseteq D_\gamma((C_i)_{i < \gamma})$ then $D_\gamma((C_i)_{i < \gamma}) \cap N_{x \upharpoonright \alpha} = N_{x \upharpoonright \alpha} \in \mathcal{D}_1(\kappa^+-\mathbf{\Pi}_1^0)$ would be trivially in $\mathcal{D}_\beta(\kappa^+-\mathbf{\Pi}_1^0)$ for every $\beta < \gamma$, again against the hypothesis. \square

Theorem 4.3.24. Let $(C_i)_{i < \alpha}$ be a decreasing sequence of closed subsets of κ^2 . The following conditions are equivalent:

- (1) $(*_\alpha)((C_i)_{i < \alpha}, \emptyset, \kappa^2)$ holds, i.e. there exists $x \in \partial(\bigcap_{i < \alpha} C_i)$ for every $\gamma < \kappa$ and for every $\beta < \alpha$, $D_\alpha((C_i)_{i < \alpha}) \cap N_{x \upharpoonright \alpha} \notin \mathcal{D}_\beta(\kappa^+-\mathbf{\Pi}_1^0)$;
- (2) $D_\alpha((C_i)_{i < \alpha})$ is a proper $\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$ -set;
- (3) $D_\alpha((C_i)_{i < \alpha})$ is κ -complete for $\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$.

Proof. (1) \Rightarrow (3) Since $D_\alpha((C_i)_{i < \alpha}) \in \mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$ by definition, it remains to show that it is κ -hard for $\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$. Let $(B_i)_{i < \alpha}$ be any decreasing sequence of closed subsets of κ^2 . To show $D_\alpha((B_i)_{i < \alpha}) \leq_W D_\alpha((C_i)_{i < \alpha})$, consider the following strategy for player \mathbf{II} in $G_W(D_\alpha((B_i)_{i < \alpha}), D_\alpha((C_i)_{i < \alpha}))$:

As long as \mathbf{I} plays in $\mathcal{T}_{\bigcap_{i < \alpha} B_i}$, \mathbf{II} plays initial segments of $x \in \partial(\bigcap_{i < \alpha} C_i)$ given by $(*_\alpha)((C_i)_{i < \alpha}, \emptyset, \kappa^2)$. If \mathbf{I} reaches a minimal $t \notin \mathcal{T}_{\bigcap_{i < \alpha} B_i}$, assume that player \mathbf{II} has so far played the sequence $x \upharpoonright \gamma$ for some $\gamma \leq |t|$. If $t \notin \mathcal{T}_{B_0}$ then \mathbf{II} can play any extension of $x \upharpoonright \gamma$ which is not in $D_\alpha((C_i)_{i < \alpha})$ and win the game — indeed $x \in \partial(D_\alpha((C_i)_{i < \alpha}))$ by Lemma 4.3.23. Otherwise, let $0 < j < \alpha$ such that $t \in \mathcal{T}_{B_j} \setminus \mathcal{T}_{B_{j+1}}$. Then, $D_\alpha((B_i)_{i < \alpha}) \cap N_t = D_{j+1}((B_i)_{i < j+1}) \in \mathcal{D}_{j+1}(\kappa^+-\mathbf{\Pi}_1^0)$. Using $(*_\alpha)((C_i)_{i < \alpha}, \emptyset, \kappa^2)$, $D_\alpha((C_i)_{i < \alpha}) \cap N_{x \upharpoonright \gamma} \notin \mathcal{D}_{j+2}(\kappa^+-\mathbf{\Pi}_1^0)$, hence $D_\alpha((C_i)_{i < \alpha}) \cap N_{x \upharpoonright \gamma} \notin \check{\mathcal{D}}_{j+1}(\kappa^+-\mathbf{\Pi}_1^0)$ by (3.7.1). Since $D_\alpha((C_i)_{i < \alpha}) \in \mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$, $D_\alpha((C_i)_{i < \alpha}) \cap N_{x \upharpoonright \gamma} \in \mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$ too, and by Lemma 4.3.21 we conclude that $D_\alpha((C_i)_{i < \alpha}) \cap N_{x \upharpoonright \gamma}$ is κ -hard for $\mathcal{D}_{j+1}(\kappa^+-\mathbf{\Pi}_1^0)$. Therefore, \mathbf{II} wins by following the strategy induced by $[D_\alpha((B_i)_{i < \alpha})]_{|t|} \leq_W [D_\alpha((C_i)_{i < \alpha})]_{|x \upharpoonright \gamma}$.

(3) \Rightarrow (2) Follows immediately from Lemma 2.11.2(4).

(2) \Rightarrow (1) Assume $\neg(*_\alpha)((C_i)_{i<\alpha}, \emptyset, \kappa^2)$, i.e., for every $x \in \partial(\bigcap_{i<\alpha} C_i)$ there exist $\gamma < \kappa$ and $\beta < \alpha$ such that $D_\alpha((C_i)_{i<\alpha}) \cap N_{x \upharpoonright \gamma} \in \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$. We want to prove that $D_\alpha((C_i)_{i<\alpha}) \in \Delta_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}$. We distinguish the case $\text{cof}(\alpha) < \kappa$ and $\text{cof}(\alpha) = \kappa$.

Recall that, by assumption, $\text{cof}(\alpha) < \kappa$. By Proposition 4.3.13 the set $\biguplus_{i<\text{cof}(\alpha)} \mathcal{C}_{\alpha_i}$ is κ -complete for $\Delta_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}$, so in particular $\biguplus_{i<\text{cof}(\alpha)} \mathcal{C}_{\alpha_i} \in \Delta_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}$. We want to show that $D_\alpha((C_i)_{i<\alpha})$ is reducible to $\biguplus_{i<\text{cof}(\alpha)} \mathcal{C}_{\alpha_i}$, which implies that $D_\alpha((C_i)_{i<\alpha}) \in \Delta_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}$ as desired. We consider the following winning strategy for player **II** in $G_W((D_\alpha((C_i)_{i<\alpha}), \biguplus_{i<\text{cof}(\alpha)} \mathcal{C}_{\alpha_i})$:

As long as Player **I** plays sequences in $\mathcal{T}_{\partial(\bigcap_{i<\alpha} C_i)}$, then **II** passes its turn until one of the following two cases occurs:

- (a) **I** reaches a minimal $t \notin \mathcal{T}_{\partial(\bigcap_{i<\alpha} C_i)}$,
- (b) **I** plays some t such that $D_\alpha((C_i)_{i<\alpha}) \cap N_t \in \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$ for some $\beta < \alpha$.

Note that a sequence $t \in \kappa^2$ satisfying either (a) or (b) exists – and thus player **II** will not pass for κ -many turns – because if not, then **I** produces a sequence in $\partial(\bigcap_{i<\alpha} C_i)$ which witnesses $(*_\alpha)((C_i)_{i<\alpha}, \emptyset, \kappa^2)$, against the hypothesis.

First assume case (a) occurs. Then, Player **II** keeps passing until **I** plays some $u \supseteq t$ such that either $N_u \subseteq \bigcap_{i<\alpha} C_i$ or $N_u \cap (\bigcap_{i<\alpha} C_i) = \emptyset$. In the first case, since $D_\alpha((C_i)_{i<\alpha}) \cap \bigcap_{i<\alpha} C_i = \emptyset$, Player **II** wins by picking and enumerating any element outside of $\biguplus_{i<\text{cof}(\alpha)} \mathcal{C}_{\alpha_i}$. In the second case, **II** passes again until **I** plays some $v \supseteq u$ such that $N_v \cap C_\beta = \emptyset$ for some $\beta < \alpha$; then, $D_\alpha((C_i)_{i<\alpha}) \cap N_v \in \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$, hence **II** wins by playing the sequence (1) followed by its strategy induced by the reduction $[D_\alpha((C_i)_{i<\alpha})]_{\upharpoonright v} \leq_W \mathcal{C}^{(\beta)}$.

Finally, assume we are in case (b). Again, **II** wins by playing the sequence (1) followed by its strategy induced by $[D_\alpha((C_i)_{i<\alpha})]_{\upharpoonright t} \leq_W \mathcal{C}^{(\beta)}$. \square

The following result is a consequence of Theorem 4.3.24 and it can be viewed as the analogue of Corollary 4.3.20 for the case where α is a limit ordinal of cofinality less than κ (a similar statement also holds when α has cofinality κ , see Corollary 4.3.28).

Corollary 4.3.25. *For every set $A \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$, either A is κ -complete for $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$, or $A \in \Delta_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}$. Therefore, $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ and $\check{\mathcal{D}}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ are the Wadge classes immediately above $\Delta_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}$.*

Case α limit ordinal with $\text{cof}(\alpha) = \kappa$. From this point onward and up to Proposition 4.3.32, we assume that $\text{cof}(\alpha) = \kappa$. Recall that the standing assumptions throughout this section (and until Proposition 4.3.32) are that α is a limit ordinal and that $\text{SLOW}(\mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0) \cup \check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0))$ holds for every $\beta < \alpha$.

Before proceeding with the inductive proof of Theorem 4.3.15, we need to complete the analysis of Section 4.3.1 regarding the structure of the κ^+ - $\mathbf{\Pi}_1^0$ -Difference hierarchy at limit stages of cofinality κ . In particular, we will investigate whether $\Delta_{\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)}$ is a Wadge class when α is a limit ordinal of cofinality κ . The picture changes depending on whether the cardinal κ is weakly compact or not:

- if κ is not weakly compact, then $\Delta_{\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)}$ is a Wadge class (Proposition 4.3.29);
- if κ is weakly compact, then $\Delta_{\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)} = \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+-\mathbf{\Pi}_1^0)$ (Proposition 4.3.30).

Lemma 4.3.26. *Let $(\alpha_\ell)_{\ell < \kappa}$ be a sequence cofinal in α . Let $A \in \mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$.*

- (1) *Either A is κ -complete for $\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$, or there exists a partition $\{W_\ell \mid \ell < \kappa\}$ of κ^2 such that $W_\ell \in \kappa^+-\Delta_1^0$ and $A \cap W_\ell \in \mathcal{D}_{\alpha_\ell}(\kappa^+-\mathbf{\Pi}_1^0)$ for every $\ell < \kappa$. In the latter case, $A \in \Delta_{\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)}$.*
- (2) *If there exists a partition $\{W_\ell \mid \ell < \kappa\}$ of κ^2 such that $W_\ell \in \kappa^+-\Delta_1^0$ and $A \cap W_\ell$ is κ -hard for $\mathcal{D}_{\alpha_\ell}(\kappa^+-\mathbf{\Pi}_1^0)$ for every $\ell < \kappa$, then A is κ -hard for $\Delta_{\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)}$.*

Proof. First, we prove (1). Since $\text{cof}(\alpha) = \kappa$, fixed a sequence $(\alpha_i)_{i < \kappa}$ cofinal in α , $\mathcal{C}^{(\alpha)} = \biguplus_{i < \kappa} \mathcal{C}^{(\alpha_i)}$ is κ -complete for $\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$ by Proposition 4.3.9 and Lemma 4.3.5(3). Since $A \in \mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$, there is a continuous reduction f from A to $\mathcal{C}^{(\alpha)}$. Let $C = f^{-1}(\{0^{(\kappa)}\})$, and recall that $A \cap C = \emptyset$. Moreover, for every $i < \kappa$, let $U_i = f^{-1}(\mathbf{N}_{0^{(i)} \smallfrown 1})$. Then, $\{U_i \mid i < \kappa\}$ is a clopen partition of $\kappa^2 \setminus C$ such that $A \cap U_i \in \mathcal{D}_{\alpha_i}(\kappa^+-\mathbf{\Pi}_1^0)$ for every $i < \kappa$.

If $C = \emptyset$, then $\{U_i \mid i < \kappa\}$ is the desired partition and we are done. In particular, since $A \cap U_i \in \check{\mathcal{D}}_{\alpha_i}(\kappa^+-\mathbf{\Pi}_1^0)$ for every $i < \kappa$, then $A \in \check{\mathcal{D}}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$ by Lemma 3.7.10. This shows that $A \in \Delta_{\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)}$, hence it not κ -complete for $\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$.

Assume now that $C \neq \emptyset$. We consider the following two complementary cases.

- There exists some $x \in C$ such that for every $i, j < \kappa$ $A \cap \mathbf{N}_{x \upharpoonright j}$ is κ -hard for $\mathcal{D}_{\alpha_i}(\kappa^+-\mathbf{\Pi}_1^0)$.
- For all $x \in C$ there exists $i, j < \kappa$ such that $A \cap \mathbf{N}_{x \upharpoonright j}$ is not κ -hard for $\mathcal{D}_{\alpha_i}(\kappa^+-\mathbf{\Pi}_1^0)$.

First, assume we are in case (i) and we want to prove that A is κ -complete for $\mathcal{D}_\alpha(\kappa^+-\mathbf{\Pi}_1^0)$. It is enough to exhibit a winning strategy for player \mathbf{II} in $G_W(\mathcal{C}^{(\alpha)}, A)$:

As long as \mathbf{I} plays in 0's, \mathbf{II} plays initial segments of $x \in C$ given by (i). If \mathbf{I} ever plays a sequence $u = 0^{(i)} \smallfrown 1$ for some $i < \kappa$, and we assume that \mathbf{II} has so far played the sequence $t = x \upharpoonright \text{lh}(u)$, then $A \cap \mathbf{N}_t$ is κ -hard for $\mathcal{D}_{\alpha_i}(\kappa^+-\mathbf{\Pi}_1^0)$ by (i). Then, Player \mathbf{II} wins the game using the winning strategy induced by $\mathcal{C}^{(\alpha)} \leq_W A \upharpoonright_{[t]}$.

Now suppose that (ii) holds. For every $x \in C$, let $i_x, j_x < \kappa$ be given by case assumption. Then, $A \cap N_{x \upharpoonright j_x}$ is not κ -hard for $\mathcal{D}_{\alpha_{i_x}}(\kappa^+ - \Pi_1^0)$, and since $A \cap N_{x \upharpoonright j_x} \in \mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$, by Lemma 4.3.21 we get that $A \cap N_{x \upharpoonright j_x} \in \check{\mathcal{D}}_{\alpha_{i_x}}(\kappa^+ - \Pi_1^0) \subseteq \mathcal{D}_{\alpha_{i_x+1}}(\kappa^+ - \Pi_1^0)$. Since $C \subseteq \bigcup_{x \in C} N_{x \upharpoonright j_x}$, but there are $2^{<\kappa} = \kappa$ many basic open subsets of ${}^\kappa 2$, there exists a family $(t_l)_{l < \kappa}$ in ${}^{<\kappa} 2$ such that $C \subseteq \bigcup_{l < \kappa} N_{t_l}$. Then, $\mathcal{A} = \{U_i \mid i < \kappa\} \cup \{N_{t_l} \mid l < \kappa\}$ is a clopen covering of ${}^\kappa 2$ such that for every $O \in \mathcal{A}$, $A \cap O \in \mathcal{D}_\beta(\kappa^+ - \Pi_1^0)$ for some $\beta < \kappa$.

We can refine \mathcal{A} to a clopen partition \mathcal{A}_0 of size κ such that every piece has the same property. Let $(O_i)_{i < \kappa}$ be an enumeration of \mathcal{A} . Let $O'_0 = O_0$ and $O'_i = O_i \setminus \bigcup_{j < i} O_j$ for every $i < \kappa$. Clearly, each $O'_i \in \kappa^+ - \Delta_1^0$. Moreover, by Remark 3.7.4, if $A \cap O_i \in \mathcal{D}_{\beta_i}(\kappa^+ - \Pi_1^0)$ for some $\beta_i < \kappa$ then $A \cap O'_i \in \mathcal{D}_{\beta_i}(\kappa^+ - \Pi_1^0)$ too.

For every $i < \kappa$, we set $W_i = \bigcup \{O \in \mathcal{A}_0 \mid A \cap O \in \mathcal{D}_{\alpha_i}(\kappa^+ - \Pi_1^0) \setminus \bigcup_{j < i} \mathcal{D}_{\alpha_j}(\kappa^+ - \Pi_1^0)\}$. For every $i < \kappa$, $W_i \in \kappa^+ - \Delta_1^0$ and $A \cap W_i \in \mathcal{D}_{\alpha_i}(\kappa^+ - \Pi_1^0)$, therefore $\{W_i \mid i < \kappa\}$ is the required partition. Finally, since $A \cap W_i \in \check{\mathcal{D}}_\alpha(\kappa^+ - \Pi_1^0)$ for every $i < \kappa$, we have $A \in \check{\mathcal{D}}_\alpha(\kappa^+ - \Pi_1^0)$ by Lemma 3.7.10, hence $A \in \Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$.

Point (2) follows from (1). To see that A is κ -hard for $\Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$, consider any set $B \in \Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$; we must show that $B \leq_W A$. Since B is not κ -complete for $\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$, point (1) guarantees the existence of a clopen partition $\{W'_\ell \mid \ell < \kappa\}$ such that $B \cap W'_\ell \in \mathcal{D}_{\alpha_\ell}(\kappa^+ - \Pi_1^0)$ for each $\ell < \kappa$. Consequently, $B \cap W'_\ell \leq_W A \cap W_\ell$ holds for every $\ell < \kappa$, and therefore $B \leq_W A$. \square

Corollary 4.3.27. *For every $A \in \mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$, A is a proper $\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$ -set if and only if A is κ -complete for $\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$.*

The next result follows immediately from Corollary 4.3.27.

Corollary 4.3.28. *For every set $A \in \mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$, either A is κ -complete for $\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$, or $A \in \Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$. Therefore, $\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$ and $\check{\mathcal{D}}_\alpha(\kappa^+ - \Pi_1^0)$ are the Wadge classes immediately above $\Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$.*

Proposition 4.3.29. *For every $A \in \Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$, either A is κ -complete for $\Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$ or $A \in \mathcal{D}_\beta(\kappa^+ - \Pi_1^0)$ for some $\beta < \alpha$.*

Proof. Since $A \in \Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$ by assumption, A is not κ -complete for $\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$. By Lemma 4.3.26 (1), there exists a partition $\{W_l \mid l < \kappa\}$ of ${}^\kappa 2$ such that for every $l < \kappa$, $W_l \in \kappa^+ - \Delta_1^0$ and $A \cap W_l \in \mathcal{D}_{\alpha_l}(\kappa^+ - \Pi_1^0)$.

If there exists $j < \kappa$ such that, for every $\ell < \kappa$, $A \cap W_\ell \in \mathcal{D}_{\alpha_j}(\kappa^+ - \Pi_1^0)$, then by Lemma 3.7.10 we get that $A \in \mathcal{D}_{\alpha_j}(\kappa^+ - \Pi_1^0) \subseteq \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+ - \Pi_1^0)$.

Assume that for every $j < \kappa$ there is some $\ell(j) < \kappa$ such that $A \cap W_{\ell(j)} \notin \mathcal{D}_{\alpha_j}(\kappa^+ - \Pi_1^0)$. We claim that for each $j < \kappa$, there exist unboundedly many $\ell(j) < \kappa$ such that $A \cap W_{\ell(j)} \notin \mathcal{D}_{\alpha_j}(\kappa^+ - \Pi_1^0)$. To see this, suppose that there exists a bound $\ell' < \kappa$. By our case assumption, we can pick some $i(\ell') < \kappa$ such that $A \cap W_{i(\ell')} \notin \mathcal{D}_{\alpha_{\ell'}}(\kappa^+ - \Pi_1^0)$. Since $A \cap W_{i(\ell')} \in \mathcal{D}_{\alpha_{i(\ell')}}(\kappa^+ - \Pi_1^0)$, we have $\ell' < i(\ell')$, contradicting that ℓ' is a bound. Therefore, we can choose $\{\ell(j) \mid j < \kappa\}$ such that the $\ell(j) \neq \ell(j')$ for every $j \neq j'$ and $A \cap W_{\ell(j)} \notin \mathcal{D}_{\alpha_j}(\kappa^+ - \Pi_1^0)$ for every $j < \kappa$. Let

$$U_0 = W_{\ell(0)} \cup \bigcup \{W_l \mid l \neq \ell(j) \text{ for all } j < \kappa\}$$

and $U_j = W_{\ell(j)}$ for every $j < \kappa$. Then, $\{U_j \mid j < \kappa\}$ is a partition of ${}^\kappa 2$ such that for all $j < \kappa$, $U_j \in \kappa^+ - \mathbf{\Delta}_1^0$ and $A \cap U_j \notin \mathcal{D}_{\alpha_j}(\kappa^+ - \mathbf{\Pi}_1^0)$. In particular for $j = 0$ we get that $A \cap U_0 \notin \mathcal{D}_{\alpha_0}(\kappa^+ - \mathbf{\Pi}_1^0)$ because $A \cap W_{\ell(0)} \notin \mathcal{D}_{\alpha_0}(\kappa^+ - \mathbf{\Pi}_1^0)$ and $W_{\ell(0)} \subseteq U_0$ is clopen. Since each $A \cap U_j \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$, by Lemma 4.3.21 we get that each $A \cap U_j$ is κ -hard for $\check{\mathcal{D}}_{\alpha_j}(\kappa^+ - \mathbf{\Pi}_1^0)$. We conclude that A is κ -complete for $\mathbf{\Delta}_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)}$ by Lemma 4.3.26 (2). \square

Proposition 4.3.30. *Let κ be weakly compact. Then, for every $A \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$, either A is κ -complete for $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ or $A \in \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$. Therefore, $\mathbf{\Delta}_{\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)} = \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$.*

Proof. Recall by [MR13, Theorem 5.6] that if κ is weakly compact then the space ${}^\kappa 2$ is κ -compact, hence any partition of ${}^\kappa 2$ in non-empty clopen sets has size smaller than κ .

Suppose that $A \notin \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$. We want to show that A is κ -complete for $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$. Fix a sequence $(\alpha_i)_{i < \kappa}$ cofinal in α . Since $A \notin \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$, A is κ -hard for $\check{\mathcal{D}}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$ for every $\beta < \alpha$ by Lemma 4.3.21, hence in particular A is κ -hard for $\mathcal{D}_{\alpha_i}(\kappa^+ - \mathbf{\Pi}_1^0)$ for every $i < \kappa$.

We assume $\text{cof}(\alpha) = \kappa$, so the canonical κ -complete for $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ set is $\mathcal{C}^{(\alpha)} = \biguplus_{i < \kappa} \mathcal{C}^{(\alpha_i)}$ by Proposition 4.3.9 and Lemma 4.3.5(3). Since $A \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$, there is a continuous reduction $f: {}^\kappa 2 \rightarrow {}^\kappa 2$ from A to $\mathcal{C}^{(\alpha)}$. Let $C = f^{-1}(\{0^{(\kappa)}\})$, and for every $i < \kappa$, let $U_i = f^{-1}(\mathbf{N}_{0^{(i)} \smallfrown 1})$. Clearly, $A \cap U_i \in \mathcal{D}_{\alpha_i}(\kappa^+ - \mathbf{\Pi}_1^0)$ for every $i < \kappa$ since $\mathbf{N}_{0^{(i)} \smallfrown 1}$ is the cone containing $\mathcal{C}^{(\alpha_i)}$.

Claim 4.3.30.1. $C \neq \emptyset$.

Proof of the Claim. Towards a contradiction, assume that $C = \emptyset$. Then, $\{U_i \mid i < \kappa\}$ is a clopen partition of ${}^\kappa 2$ and, since κ is weakly compact, there exists some $i_0 < \kappa$ such that $U_i = \emptyset$ for all $i_0 \leq i < \kappa$. Let $\bar{\alpha} = \sup_{i < i_0} \alpha_i$. We have that $\bar{\alpha} < \kappa$ since $i_0 < \kappa$. Then, $\{U_i \mid i < i_0\}$ is a clopen partition of ${}^\kappa 2$ such that $A \cap U_i \in \mathcal{D}_{\bar{\alpha}}(\kappa^+ - \mathbf{\Pi}_1^0)$ for every $i < i_0$. By Lemma 3.7.10, we have $A \in \mathcal{D}_{\bar{\alpha}}(\kappa^+ - \mathbf{\Pi}_1^0)$, in contradiction with the assumption $A \notin \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0)$. \square

Claim 4.3.30.2. There exists $x \in C$ such that for every $j < \kappa$ and every $i < \kappa$ $A \cap \mathbf{N}_{x \smallfrown j}$ is κ -hard for $\mathcal{D}_{\alpha_i}(\kappa^+ - \mathbf{\Pi}_1^0)$.

Proof of the Claim. Towards a contradiction, suppose that for all $x \in C$ there exists $t_x \in \mathcal{T}_C$ and there exists some $i_x < \kappa$ such that $A \cap \mathbf{N}_{t_x}$ is not κ -hard for $\mathcal{D}_{\alpha_{i_x}}(\kappa^+ - \mathbf{\Pi}_1^0)$. Clearly, for every $x \in C$, the sequence $t_x \in \mathcal{T}_C$ is of the form $t_x = x \smallfrown j$ for some $j < \kappa$. Since C is closed and $\{\mathbf{N}_{t_x} \mid x \in C\}$ is a clopen covering of C , then there exists some $\ell_0 < \kappa$ and a sequence $(t_\ell)_{\ell < \ell_0}$ such that $C \subseteq \bigcup_{\ell < \ell_0} \mathbf{N}_{t_\ell}$ and for every $\ell < \ell_0$, $t_\ell = t_x$ and $i_\ell = i_x$ for some $x \in C$. Let $\gamma = \sup_{\ell < \ell_0} \alpha_{i_\ell} < \kappa$. Then, $A \cap \mathbf{N}_{t_\ell} \in \mathcal{D}_\gamma(\kappa^+ - \mathbf{\Pi}_1^0)$ for every $\ell < \ell_0$.

Now, $\mathcal{A} = \{U_i \mid i < \kappa\} \cup \{\mathbf{N}_{t_\ell} \mid \ell < \ell_0\}$ is a clopen covering of ${}^\kappa 2$. Again by weak compactness, there exists a subcovering of size smaller than κ . In particular, there exists $i_0 < \kappa$ such that $\mathcal{A}_0 = \{U_i \mid i < i_0\} \cup \{\mathbf{N}_{t_\ell} \mid \ell < \ell_0\}$ covers ${}^\kappa 2$. Let $\gamma' = \sup_{i < i_0} \alpha_i < \kappa$. Then, for every $i < i_0$, $A \cap U_i \in \mathcal{D}_{\gamma'}(\kappa^+ - \mathbf{\Pi}_1^0)$. Let $\bar{\alpha} = \max\{\gamma, \gamma'\}$. Then, for every $O \in \mathcal{A}_0$, $A \cap O \in \mathcal{D}_{\bar{\alpha}}(\kappa^+ - \mathbf{\Pi}_1^0)$. Finally, since $|\mathcal{A}_0| < \kappa$ and using Remark 3.7.4, we can refine \mathcal{A}_0 to a clopen partition \mathcal{A}_{00} of ${}^\kappa 2$ such that $A \cap U \in \mathcal{D}_{\bar{\alpha}}(\kappa^+ - \mathbf{\Pi}_1^0)$ for every

$U \in \mathcal{A}_{00}$. By Lemma 3.7.10, A is in $\mathcal{D}_{\bar{\alpha}}(\kappa^+ - \mathbf{\Pi}_1^0)$, in contradiction with the assumption $A \notin \bigcup_{\beta < \alpha} \mathcal{D}_{\beta}(\kappa^+ - \mathbf{\Pi}_1^0)$. \square

To show $\mathcal{C}^{(\alpha)} \leq_L A$, we consider the following strategy for player \mathbf{II} in $G_L(\mathcal{C}^{(\alpha)}, A)$:

As long as \mathbf{I} plays 0's, \mathbf{II} plays initial segments of $x \in C$ given by Claim 4.3.30.2 — recall that C is disjoint from A . If \mathbf{I} ever plays a sequence $t = 0^{(i)} \frown 1$ for some $i < \kappa$, and we assume that \mathbf{II} has so far played the sequence $x \upharpoonright \text{lh}(t)$, then by Claim 4.3.30.2 $A \cap N_{x \upharpoonright \text{lh}(t)}$ is κ -hard for $\mathcal{D}_{\alpha_i}(\kappa^+ - \mathbf{\Pi}_1^0)$. Then, player \mathbf{II} wins by continuing the enumeration of the sequence u and then using the winning strategy induced by $\mathcal{C}^{(\alpha_i)} \leq_L A_{\lfloor x \upharpoonright \text{lh}(t) \rfloor}$. \square

Corollary 4.3.31. *Let κ be weakly compact. Then, there exist no κ -complete set for $\Delta_{\mathcal{D}_{\alpha}(\kappa^+ - \mathbf{\Pi}_1^0)}$.*

Proof. Towards a contradiction, assume that A is κ -complete set for $\Delta_{\mathcal{D}_{\alpha}(\kappa^+ - \mathbf{\Pi}_1^0)}$. Then, $A \in \Delta_{\mathcal{D}_{\alpha}(\kappa^+ - \mathbf{\Pi}_1^0)}$ and by Proposition 4.3.30 there exists $\beta < \alpha$ such that $A \in \mathcal{D}_{\beta}(\kappa^+ - \mathbf{\Pi}_1^0)$. It is sufficient to consider any $B \in \mathcal{D}_{\beta+1}(\kappa^+ - \mathbf{\Pi}_1^0) \setminus \mathcal{D}_{\beta}(\kappa^+ - \mathbf{\Pi}_1^0)$ to see that $B \in \Delta_{\mathcal{D}_{\alpha}(\kappa^+ - \mathbf{\Pi}_1^0)}$ but $B \not\leq_W A$, against the assumption that A is κ -hard for $\Delta_{\mathcal{D}_{\alpha}(\kappa^+ - \mathbf{\Pi}_1^0)}$. \square

Note that if $\text{cof}(\alpha) = \kappa$ is weakly compact and $A \in \mathcal{D}_{\alpha}(\kappa^+ - \mathbf{\Pi}_1^0)$ satisfies the conditions of Lemma 4.3.26(2), then A is a proper $\mathcal{D}_{\alpha}(\kappa^+ - \mathbf{\Pi}_1^0)$ -set. Indeed, if it were $A \in \Delta_{\mathcal{D}_{\alpha}(\kappa^+ - \mathbf{\Pi}_1^0)}$ then A would be κ -complete for $\Delta_{\mathcal{D}_{\alpha}(\kappa^+ - \mathbf{\Pi}_1^0)}$, against Corollary 4.3.31.

Proposition 4.3.32. $\text{SLO}_W(\Delta_{\mathcal{D}_{\alpha}(\kappa^+ - \mathbf{\Pi}_1^0)})$.

Proof. Let $A, B \in \Delta_{\mathcal{D}_{\alpha}(\kappa^+ - \mathbf{\Pi}_1^0)}$. If κ is weakly compact, the result follows by inductive hypothesis since $\bigcup_{\beta < \alpha} \mathcal{D}_{\beta}(\kappa^+ - \mathbf{\Pi}_1^0) = \Delta_{\mathcal{D}_{\alpha}(\kappa^+ - \mathbf{\Pi}_1^0)}$ by Proposition 4.3.30.

Assume now that κ is not weakly compact. If $A, B \in \bigcup_{\beta < \alpha} \mathcal{D}_{\beta}(\kappa^+ - \mathbf{\Pi}_1^0)$ we are done by the inductive hypothesis. If $A \in \bigcup_{\beta < \alpha} \mathcal{D}_{\beta}(\kappa^+ - \mathbf{\Pi}_1^0)$ and $B \notin \bigcup_{\beta < \alpha} \mathcal{D}_{\beta}(\kappa^+ - \mathbf{\Pi}_1^0)$, then by Lemma 4.3.21 B is κ -hard for $\mathcal{D}_{\beta}(\kappa^+ - \mathbf{\Pi}_1^0)$ for every $\beta < \alpha$ and thus $A \leq_W B$. Similarly, $A \notin \bigcup_{\beta < \alpha} \mathcal{D}_{\beta}(\kappa^+ - \mathbf{\Pi}_1^0)$ and $B \in \bigcup_{\beta < \alpha} \mathcal{D}_{\beta}(\kappa^+ - \mathbf{\Pi}_1^0)$, then A is κ -hard for $\check{\mathcal{D}}_{\beta}(\kappa^+ - \mathbf{\Pi}_1^0)$ for every $\beta < \alpha$ by Lemma 4.3.21, hence $B \leq_W \neg A$. Finally, assume $A, B \notin \bigcup_{\beta < \alpha} \mathcal{D}_{\beta}(\kappa^+ - \mathbf{\Pi}_1^0)$. Then, A and B are both κ -complete for $\Delta_{\mathcal{D}_{\alpha}(\kappa^+ - \mathbf{\Pi}_1^0)}$ by Proposition 4.3.29, hence $A \leq_W B$. \square

This concludes the limit case of the induction, and hence concludes the proof of Theorem 4.3.15.

Proof of Theorem 4.3.15, and its consequences

Combining together the results obtained in the previous sections, we finally get the proof of Theorem 4.3.15.

Proof of Theorem 4.3.15. The proof proceeds by induction on $\alpha < \kappa^+$. The base case is Corollary 4.1.4. At each stage $\alpha > 1$, we show:

- (a) $\text{SLOW}(\Delta_{\mathcal{D}_\alpha(\kappa^+-\Pi_1^0)})$;
- (b) A set is $\mathcal{D}_\alpha(\kappa^+-\Pi_1^0)$ -proper if and only if it is κ -complete for $\mathcal{D}_\alpha(\kappa^+-\Pi_1^0)$.

These conditions, combined with Lemma 2.11.3, yield the result. As usual, the induction argument divides into the successor and limit cases. When α is a successor ordinal, Point (a) is established in Proposition 4.3.18, and Point (b) is established in Theorem 4.3.19. When α is a limit ordinal, the argument splits according to the value of $\text{cof}(\alpha)$: when $\text{cof}(\alpha) < \kappa$, Point (a) is proved in Proposition 4.3.22, while for the the proof of Point (b) we use Theorem 4.3.24; when $\text{cof}(\alpha) = \kappa$, Point (a) is Proposition 4.3.32 and Point (b) is Corollary 4.3.27. \square

The next two results are corollaries of Theorem 4.3.15. We show that the Wadge class $\Delta_{\mathcal{D}_{\alpha+1}(\kappa^+-\Pi_1^0)}$ (see Proposition 4.3.11) is the least above $\mathcal{D}_\alpha(\kappa^+-\Pi_1^0)$ and $\check{\mathcal{D}}_\alpha(\kappa^+-\Pi_1^0)$. Similarly, we shows that when α has cofinality smaller than κ , the Wadge class $\Delta_{\mathcal{D}_\alpha(\kappa^+-\Pi_1^0)}$ (see Proposition 4.3.13) is the least above $\bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+-\Pi_1^0) = \bigcup_{\beta < \alpha} \check{\mathcal{D}}_\beta(\kappa^+-\Pi_1^0)$.

Corollary 4.3.33. *For every $1 \leq \alpha < \kappa^+$ and for every $A \in \Delta_{\mathcal{D}_{\alpha+1}(\kappa^+-\Pi_1^0)}$, either A is κ -complete for $\Delta_{\mathcal{D}_{\alpha+1}(\kappa^+-\Pi_1^0)}$ or, $A \in \mathcal{D}_\alpha(\kappa^+-\Pi_1^0)$ or, $A \in \check{\mathcal{D}}_\alpha(\kappa^+-\Pi_1^0)$.*

Proof. We recall that $\mathcal{U}^{(\alpha)} \uplus \mathcal{C}^{(\alpha)}$ is κ -complete for $\Delta_{\mathcal{D}_{\alpha+1}(\kappa^+-\Pi_1^0)}$ by Proposition 4.3.11. Assume that A is not κ -complete for $\Delta_{\mathcal{D}_{\alpha+1}(\kappa^+-\Pi_1^0)}$. Then, $\mathcal{U}^{(\alpha)} \uplus \mathcal{C}^{(\alpha)} \not\leq_W A$, which means that either $\mathcal{U}^{(\alpha)} \not\leq_W A$ or $\mathcal{C}^{(\alpha)} \not\leq_W A$ by Lemma 4.3.4. Using $\text{SLOW}(\mathcal{D}_{\alpha+1}(\kappa^+-\Pi_1^0) \cup \check{\mathcal{D}}_{\alpha+1}(\kappa^+-\Pi_1^0))$ (Theorem 4.3.15) and the fact that $\mathcal{U}^{(\alpha)} = \kappa^2 \setminus \mathcal{C}^{(\alpha)}$, we get that either $A \leq_W \mathcal{C}^{(\alpha)}$ or $A \leq_W \mathcal{U}^{(\alpha)}$. \square

Corollary 4.3.34. *Let $\alpha < \kappa^+$ be a limit ordinal such that $\text{cof}(\alpha) < \kappa$. For every $A \in \Delta_{\mathcal{D}_\alpha(\kappa^+-\Pi_1^0)}$, either A is κ -complete for $\Delta_{\mathcal{D}_\alpha(\kappa^+-\Pi_1^0)}$ or, $A \in \mathcal{D}_\beta(\kappa^+-\Pi_1^0)$ for some $\beta < \alpha$.*

Proof. Fix a sequence $\langle \alpha_i \mid i < \text{cof}(\alpha) \rangle$ cofinal in α . We recall that the set $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)}$ is κ -complete for $\Delta_{\mathcal{D}_\alpha(\kappa^+-\Pi_1^0)}$ by Proposition 4.3.13. Assume that A is not κ -complete for $\Delta_{\mathcal{D}_\alpha(\kappa^+-\Pi_1^0)}$. Then, $\biguplus_{i < \text{cof}(\alpha)} \mathcal{C}^{(\alpha_i)} \not\leq_W A$, and by Lemma 4.3.5 there exists $i < \text{cof}(\alpha)$ such that $\mathcal{C}^{(\alpha_i)} \not\leq_W A$. Using $\text{SLOW}(\mathcal{D}_\alpha(\kappa^+-\Pi_1^0) \cup \check{\mathcal{D}}_\alpha(\kappa^+-\Pi_1^0))$ (Theorem 4.3.15) and the fact that $\mathcal{C}^{(\alpha_i)} = \kappa^2 \setminus \mathcal{U}^{(\alpha_i)}$, we get that $A \leq_W \mathcal{U}^{(\alpha_i)}$, hence $A \in \check{\mathcal{D}}_{\alpha_i}(\kappa^+-\Pi_1^0) \subseteq \mathcal{D}_{\alpha_i+1}(\kappa^+-\Pi_1^0) \subseteq \bigcup_{\beta < \alpha} \mathcal{D}_\beta(\kappa^+-\Pi_1^0)$. \square

Having established that the first κ^+ levels of the Wadge hierarchy on κ^2 are precisely filled by the $\kappa^+-\Pi_1^0$ -difference hierarchy, we now finish by proving the well-foundedness of the Wadge hierarchy when restricted to the sets in the $\kappa^+-\Pi_1^0$ -difference hierarchy.

Theorem 4.3.35. *The Wadge hierarchy restricted to the sets in the κ^+ - Π_1^0 -difference hierarchy on ${}^{\kappa}2$ is well-founded.*

Proof. Assume towards a contradiction that there exists a sequence of sets $(A_i)_{i < \omega}$ in $\bigcup_{\alpha < \kappa^+} \mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$ such that $A_{i+1} <_W A_i$ for every $i < \omega$. By the analysis of the Wadge hierarchy on the κ^+ - Π_1^0 -difference hierarchy developed in this section (in particular Theorem 4.3.15), for each $i < \omega$ there is some $\alpha < \kappa^+$ such that either $A_i \in \mathcal{D}_\alpha(\kappa^+ - \Pi_1^0) \setminus \check{\mathcal{D}}_\alpha(\kappa^+ - \Pi_1^0)$, or $A_i \in \check{\mathcal{D}}_\alpha(\kappa^+ - \Pi_1^0) \setminus \mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$, or A_i is κ -complete for $\Delta_{\mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)}$. For each $i < \omega$, let $\alpha_i < \kappa^+$ be such an ordinal.

Now fix $i < \omega$. Since $A_{i+1} <_W A_i$, by applying $\text{SLO}(\mathcal{D}_{\alpha_i}(\kappa^+ - \Pi_1^0) \cup \check{\mathcal{D}}_{\alpha_i}(\kappa^+ - \Pi_1^0))$ we obtain $A_{i+1} \leq_W A_i, \neg A_i$. Hence $A_{i+1} \in \Delta_{\mathcal{D}_{\alpha_i}(\kappa^+ - \Pi_1^0)}$, and therefore $\alpha_{i+1} \leq \alpha_i$. Observe that if $\alpha_{i+1} = \alpha_i$, then by the definition of α_i , the set A_{i+1} is κ -complete for $\Delta_{\mathcal{D}_{\alpha_i}(\kappa^+ - \Pi_1^0)}$.

If $A_i \in \Delta_{\mathcal{D}_{\alpha_i}(\kappa^+ - \Pi_1^0)}$, then in fact $\alpha_{i+1} < \alpha_i$; indeed, if $\alpha_{i+1} = \alpha_i$, then $A_i <_W A_{i+1}$, contradicting our assumption. If instead $A_i \in \mathcal{D}_{\alpha_i}(\kappa^+ - \Pi_1^0) \setminus \check{\mathcal{D}}_{\alpha_i}(\kappa^+ - \Pi_1^0)$, then $\alpha_{i+1} = \alpha_i$ implies that $A_{i+1} \in \Delta_{\mathcal{D}_{\alpha_i}(\kappa^+ - \Pi_1^0)}$. In this case we must have $\alpha_{i+2} < \alpha_{i+1}$, since otherwise we would get $A_{i+1} <_W A_{i+2}$, again contradicting the hypothesis.

Thus, the sequence $(\alpha_i)_{i < \omega}$ is non-increasing, and from it we can extract a strictly decreasing infinite sequence of ordinals as follows: set $\alpha'_0 = \alpha_0$. For each $i < \omega$, if $\alpha_{i+1} < \alpha'_i$ let $\alpha'_{i+1} = \alpha_{i+1}$, while if $\alpha_{i+1} = \alpha'_i$ set $\alpha'_{i+1} = \alpha_{i+2}$. This defines an infinite strictly decreasing sequence of ordinals, a contradiction. \square

4.3.3 A minimal set above the difference hierarchy

Consider the set \mathcal{Y}_0 from Example 3.7.12 and, more in general, for each infinite ordinal $\lambda < \kappa$, the set $\mathcal{Y}_0^{(\lambda)}$ from Example 3.7.13. Our aim is to study how these sets fit into the Wadge hierarchy. In the previous section we obtained a full description of the first κ^+ levels of this hierarchy, which coincide with the κ^+ - Π_1^0 -difference hierarchy. Furthermore, Lemma 3.7.11 established that for every $\alpha < \kappa^+$, $\mathcal{Y}_0, \mathcal{Y}_0^{(\lambda)} \notin \mathcal{D}_\alpha(\kappa^+ - \Pi_1^0)$. Since both of these sets lie in $\kappa^+ - \Delta_2^0$, they provide the counterexamples witnessing the failure of the Hausdorff–Kuratowski theorem (Theorem 3.7.14).

In this section, we first show that $\mathcal{Y}_0^{(\lambda)}$ lies strictly above the κ^+ - Π_1^0 -difference hierarchy. This is done in Proposition 4.3.36, that strengthens Lemma 3.7.11 in the specific context where $X \subseteq {}^{\kappa}2$. (Recall that Lemma 3.7.11 was proved in the more general setting of regular Hausdorff spaces of weight at most κ .) In fact, in the case $X \subseteq {}^{\kappa}2$, Lemma 3.7.11 follows directly from Proposition 4.3.36 by taking $Z \subseteq X = Y$: towards a contradiction, assume there exists $\alpha < \kappa^+$ such that $Z \in \mathcal{D}_\alpha(\kappa^+ - \Pi_1^0(Y))$ and take a set $A \in \mathcal{D}_{\alpha+1}(\kappa^+ - \Pi_1^0(Y)) \setminus \mathcal{D}_\alpha(\kappa^+ - \Pi_1^0(Y))$; by Proposition 4.3.36, $A \leq_L^Y Z$, hence $A \in \mathcal{D}_\alpha(\kappa^+ - \Pi_1^0(Y))$, a contradiction.

Proposition 4.3.36. *Let $X, Y, Z \subseteq {}^{\kappa}2$ be non-empty sets and assume that $Y \cap Z$ is dense and codense in Y . Then, for every $\alpha < \kappa^+$ and every $A \in \mathcal{D}_\alpha(\kappa^+ - \Pi_1^0(X))$,*

$$A \leq_L^{X, Y} Z.$$

Proof. The proof goes by induction on $\alpha < \kappa^+$, simultaneously for all such sets X, Y, Z . Let $A = D_\alpha((A_i)_{i < \alpha})$ for some decreasing sequence $(A_i)_{i < \alpha}$ of $\kappa^+ - \mathbf{\Pi}_1^0$ -subsets of X . The goal is to exhibit a winning strategy for Player **II** in the Lipschitz game $G_L^{X,Y}(A, Z)$. We can assume that the sequence in κ^2 produced by Player **I** at the end of the game is an element of X , and hence that all along the game he always plays in \mathcal{T}_X , because if not loses straight away by game rules.

When $\alpha = 1$, $A \in \kappa^+ - \mathbf{\Pi}_1^0(X)$ and the strategy goes as follows:

As long as **I** plays sequences in \mathcal{T}_A , then **II** answers by playing initial sequences of any element $y \in Y \cap Z$. If **I** ever reaches $t \in \partial\mathcal{T}_A$, since **II** has so far played $y \upharpoonright \gamma$ where $\gamma = \text{lh}(t) - 1$ if $\text{lh}(t)$ is a successor ordinal, and $\gamma = \text{lh}(t)$ if $\text{lh}(t)$ is limit, then **II** can answer by picking any element in $\mathbf{N}_{y \upharpoonright \gamma}^Y \cap (Y \setminus Z)$ by codensity of $Y \cap Z$ in Y and following it until the end of the game.

Now, let $\alpha = \beta + 1$ be a successor ordinal. Without loss of generality we can assume that α is even, since $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0(X)) \subseteq \mathcal{D}_{\alpha+1}(\kappa^+ - \mathbf{\Pi}_1^0(X))$.

As long as **I** plays sequences in \mathcal{T}_{A_β} , then **II** answers by playing initial sequences of any element $x \in Y \setminus Z$. If **I** ever reaches $t \in \partial\mathcal{T}_{A_\beta}$, then $\mathbf{N}_t^X \cap A_\beta = \emptyset$, and **II** has so far played $x \upharpoonright \gamma$, where $\gamma = \text{lh}(t) - 1$ if $\text{lh}(t)$ is a successor ordinal, and $\gamma = \text{lh}(t)$ if $\text{lh}(t)$ is limit. Since $A \cap \mathbf{N}_t^X \in \mathcal{D}_\beta(\kappa^+ - \mathbf{\Pi}_1^0(X))$, by inductive hypothesis there exists a winning strategy for **II** in $G_L^{X[t], Y[x \upharpoonright \gamma]}(A[t], Z[x \upharpoonright \gamma])$, because $Y[x \upharpoonright \gamma] \cap Z[x \upharpoonright \gamma]$ dense and codense in $Y[x \upharpoonright \gamma]$.

Finally, let α be a limit ordinal. Recall that $\mathcal{T}_{\bigcap_{i < \alpha} A_i} \subseteq \bigcap_{i < \alpha} \mathcal{T}_{A_i}$ holds by Remark 4.3.8.

As long as **I** plays sequences in $\bigcap_{i < \alpha} \mathcal{T}_{A_i}$, then **II** answers by playing initial segments of any element $y \in Y \setminus Z$. If **I** ever reaches a minimal $t \notin \bigcap_{i < \alpha} \mathcal{T}_{A_i}$, then we have two cases: either $t \notin \mathcal{T}_{A_0}$, and then **II** can carry on playing initial segments of y , or there exists some minimal $0 < j < \alpha$ such that $t \notin \mathcal{T}_{A_j}$ and thus $\mathbf{N}_t^X \cap A_j = \emptyset$. In the latter case, **II** has so far played $x \upharpoonright \gamma$, where $\gamma = \text{lh}(t) - 1$ if $\text{lh}(t)$ is a successor ordinal and $\gamma = \text{lh}(t)$ if $\text{lh}(t)$ is limit. Since $A \cap \mathbf{N}_t^X \in \mathcal{D}_j(\kappa^+ - \mathbf{\Pi}_1^0(X))$, by inductive hypothesis there exists a winning strategy for **II** in $G_L^{X[t], Y[x \upharpoonright \gamma]}(A[t], Z[x \upharpoonright \gamma])$, because $Y[x \upharpoonright \gamma] \cap Z[x \upharpoonright \gamma]$ dense and codense in $Y[x \upharpoonright \gamma]$. \square

In the next result, we show that for every infinite ordinal $\lambda < \kappa$, the set $\mathcal{Y}_0^{(\lambda)}$, and hence the Wadge class it generates, are non-selfdual. Recall that when $\lambda = \omega$, we have $\mathcal{Y}_0^{(\lambda)} = \mathcal{Y}_0$.

Proposition 4.3.37. *For every $\omega \leq \lambda < \kappa$, the Wadge class $\mathcal{Y}_0^{(\lambda)} \downarrow_{\kappa^2}$ is non-selfdual.*

Proof. It suffices to prove that Player **I** has a winning strategy in the Wadge game $G_W(\mathcal{Y}_0^{(\lambda)}, \neg\mathcal{Y}_0^{(\lambda)})$. Since the Wadge game is zero-sum, it is impossible for both players to have winning strategies. Hence, once we know that **I** has

a winning strategy, it follows that **II** does not, and therefore $\mathcal{Y}_0^{(\lambda)} \not\leq_W \neg\mathcal{Y}_0^{(\lambda)}$ (see also Section 2.10). A winning strategy for **I** in $G_W(\mathcal{Y}_0^{(\lambda)}, \neg\mathcal{Y}_0^{(\lambda)})$ is the following.

Player **I** starts the game by choosing 0. After this initial move, **I** plays 1 if and only if Player **II** plays 1. In other words, whenever Player **II** chooses 0 or passes, **I** responds with 0, and whenever **II** chooses 1, **I** responds with 1. At every limit stage of the game, player **I** starts again by playing 0.

Let $x \in {}^\kappa 2$ denote the element chosen by **I**, and $y \in {}^\kappa 2$ the element chosen by **II**, at the conclusion of the game in which **I** follows the above strategy. Then, $x \in \mathcal{Y}_0^{(\lambda)} \Leftrightarrow y \in \mathcal{Y}_0^{(\lambda)} \Leftrightarrow y \notin \neg\mathcal{Y}_0^{(\lambda)}$. \square

The next proposition shows that, on any non-trivial closed subset of ${}^\kappa 2$, every subset that is both dense and codense lies above \mathcal{Y}_0 in the Wadge hierarchy (and even in the Lipschitz hierarchy). Furthermore, if the subset is $< \lambda$ -closed, this statement generalizes to $\mathcal{Y}_0^{(\lambda)}$.

Proposition 4.3.38. ³For every closed, non-empty $A \subseteq {}^\kappa 2$, if $A_0 \subseteq A$ is dense and codense in A , then $\mathcal{Y}_0 \leq_L^{\mathcal{Y}, A} A_0$ and $\mathcal{Y}_0 \leq_L A_0$.

Moreover, for every $\omega \leq \lambda < \kappa$, if A is $< \lambda$ -closed, then $\mathcal{Y}_0^{(\lambda)} \leq_L^{\mathcal{Y}^{(\lambda)}, A} A_0$ and $\mathcal{Y}_0^{(\lambda)} \leq_L A_0$.

Proof. Note that player **I**'s moves are in $\mathcal{T}_{\mathcal{Y}^{(\lambda)}}$ if and only if he plays sequences with $< \lambda$ -many 1's.

First, assume that $\lambda = \omega$. We describe a winning strategy for Player **II** in $G_L^{\mathcal{Y}, A}(\mathcal{Y}_0, A_0)$. We may assume that the sequence in ${}^\kappa 2$ produced by Player **I** at the end of the game belongs to $\mathcal{Y}^{(\lambda)}$. Hence, throughout the game he must always play within $\mathcal{T}_{\mathcal{Y}^{(\lambda)}}$; otherwise, he would immediately lose. Fix $x_0 \in A_0$.

At the beginning of the game, as long as **I** plays 0's, **II** answers by enumerating x_0 . If **I** ever plays 1 at some stage $\alpha_0 < \kappa$, then by codensity of A_0 in A there exists $x_1 \in N_{x_0 \upharpoonright \alpha_0} \cap (A \setminus A_0)$, hence **II** answers by playing $x_1(\alpha_0)$. Then, as long as **I** plays 0's, **II** follows x_1 .

Now assume that **I** plays the $i + 1$ -th 1 at stage α_i of the game, for some $i < \lambda$. Then, in the previous move, **II** was following the sequence $x_i \in A$ —indeed, $x_i \in A_0$ if i is even and $x_i \in A \setminus A_0$ if i is odd. If i is even, by codensity of A_0 in A there exists $x_{i+1} \in N_{x_i \upharpoonright \alpha_i} \cap (A \setminus A_0)$; if i is odd, by density of A_0 in A , there exists $x_{i+1} \in N_{x_i \upharpoonright \alpha_i} \cap A_0$. Then, **II** answers by playing $x_{i+1}(\alpha_i)$. Then, as long as **I** plays 0's, **II** follows x_{i+1} .

Since **I** can play only $< \lambda$ -many 1's and $\lambda < \text{cof}(\kappa) = \kappa$, there exists a stage $\beta < \kappa$ such that from that moment on **I** will play only 0's, hence **II** will stabilize either inside or outside of A_0 .

³The statement of Proposition 4.3.38 remains valid even without assuming that κ is regular; it suffices to require instead that $\lambda < \text{cof}(\kappa)$.

The same argument proves that **II** also has a winning strategy in $G_L(\mathcal{Y}_0, A_0)$. The only additional remark is the following.

Suppose that at some stage of the game **I** plays a minimal $t \notin \mathcal{T}_{\mathcal{Y}^{(\lambda)}}$, and that up to this point **II** has played a sequence $u \in {}^{<\kappa}2$. Then, either $u \notin \mathcal{T}_A$, in which case any subsequent move by **II** leads to a win, or $u \in \mathcal{T}_A$. In the latter case, by density of $A \setminus A_0$ in A , there is some $x \in N_u \cap (A \setminus A_0)$, and therefore **II** can secure victory by simply playing along x for the rest of the game.

Now, for the second part of the statement, let $\omega \leq \lambda < \kappa$ and assume that A is $<\lambda$ -closed. The winning strategies for Player **II** in $G_L^{\mathcal{Y}^{(\lambda)}, A}(\mathcal{Y}_0^{(\lambda)}, A_0)$ and in $G_L(\mathcal{Y}_0^{(\lambda)}, A_0)$ are the same described in the first part of the proof, but with the difference that when $\lambda > \omega$ also the limit case must be considered.

Assume that $i < \lambda$ is a limit ordinal and that **I** has played i -many 1's at some stage α_i of the game. Note that $x_j \upharpoonright \alpha_j \subseteq x_{j+1} \upharpoonright \alpha_{j+1}$ for every $j < i$. Then, $t_i = \bigcup_{j < i} x_j \upharpoonright \alpha_j \in {}^{<\kappa}2$. Since A is $<\lambda$ -closed, and $x_i \upharpoonright \alpha_i \in \mathcal{T}_A$ for every $i < \lambda$, $t_i \in \mathcal{T}_A$. Since A_0 is dense in A , there exists $x_i \in N_{t_i} \cap A_0$. Then, **II** answers by playing $x_i(\alpha_i)$ and next, as long as **I** plays 0's, **II** follows x_i . \square

The next result shows the existence of a $<_W$ -chain of length κ in $\kappa^+ - \Delta_2^0$, for example the one given by the sets $\{\mathcal{Y}_0^{(\lambda)} \mid \lambda < \kappa \text{ is a limit ordinal}\}$.

Proposition 4.3.39. *For every pair of ordinals $\omega \leq \lambda < \lambda' < \kappa$, we have $\mathcal{Y}_0^{(\lambda)} <_W \mathcal{Y}_0^{(\lambda')}$.*

Proof. To see that $\mathcal{Y}_0^{(\lambda)} \leq_W \mathcal{Y}_0^{(\lambda')}$, consider the following winning strategy per Player **II** in the game $G_W(\mathcal{Y}_0^{(\lambda)}, \mathcal{Y}_0^{(\lambda')})$.

Player **II** imitates Player **I**'s moves as long as **I** plays within $\mathcal{T}_{\mathcal{Y}^{(\lambda)}}$. Suppose that at some stage of the game **I** plays a minimal node $t \notin \mathcal{T}_{\mathcal{Y}^{(\lambda)}}$. Then, up to that stage, **II** has produced a sequence $u \in {}^{<\kappa}2$ containing exactly λ many 1's. Since $\lambda < \lambda'$, if λ is odd then $u \notin \mathcal{T}_{\mathcal{Y}_0^{(\lambda)'}}$, and thus **II** can secure a win by playing only 0's for the remainder of the game. If instead λ is even, then $u \in \mathcal{T}_{\mathcal{Y}_0^{(\lambda)'}}$, so it is enough for **II** to next play $1^{(\gamma)}$, where γ is the unique ordinal satisfying $\lambda + \gamma = \lambda'$, and then to continue with 0's until the end.

Similarly to the argument in the proof of Proposition 4.3.37, to prove that $\mathcal{Y}_0^{(\lambda')} \not\leq_W \mathcal{Y}_0^{(\lambda)}$, it suffices to describe a winning strategy for Player **I** in the Wadge game $G_W(\mathcal{Y}_0^{(\lambda')}, \mathcal{Y}_0^{(\lambda)})$. We recall that every limit ordinal $\alpha < \kappa$ is of the form $\alpha = \omega \cdot \beta$ for some ordinal $\beta < \kappa$.

Player **I** begins the game by playing 1. After this first move, as long as **II** keeps playing inside $\mathcal{T}_{\mathcal{Y}^{(\lambda)}}$, Player **I** follows this rule: whenever **II** plays 1, Player **I** also plays 1; if **II** plays 0 or passes, then **I** plays 0. Moreover, at each limit stage α of the game (again, provided that **II** keeps playing in $\mathcal{T}_{\mathcal{Y}^{(\lambda)}}$) we proceed as follows:

- if $\alpha = \omega \cdot \beta$ with $\beta = \beta' + 1$ a successor ordinal, then Player **I** looks at the sequence $u \in {}^\omega 2$ produced by Player **II** between stages $\omega \cdot \beta'$ and $\omega \cdot \beta$. If the set of positions where u takes the value 1 has order type ω , then Player **I** starts again by playing 1; otherwise, **I** plays 0.
- if $\alpha = \omega \cdot \beta$ with β a limit ordinal, then Player **I** looks at the sequence $u \in {}^\alpha 2$ played so far by **II**. If the order type of the 1's in u is a limit ordinal, then Player **I** again starts over by playing 1; otherwise, **I** plays 0.

If, at any point in the game, Player **II** plays a minimal node $u \notin \mathcal{T}_{\mathcal{Y}^{(\lambda)}}$, that is, a node for which the order type of the 1's in u is exactly λ , then up to that stage Player **I** has produced a sequence whose 1's also have order type λ . Since $\lambda < \lambda'$, if λ is even then it suffices for **I** to keep playing 0 for the rest of the game, whereas if λ is odd, **I** plays one more 1 and then continues with 0's until the end.

The strategy just defined guarantees that, at the end of the game, if **II** has produced a sequence $y \in \mathcal{Y}^{(\lambda)}$, then the order type of the 1's in the resulting sequence x played by **I** is $\gamma + 1$, where γ is the order type of the 1's in y . Hence the outcome is winning for Player **I**. Conversely, if **II** ends up playing a sequence $y \notin \mathcal{Y}^{(\lambda)}$, then **I** produces a sequence $x \in \mathcal{Y}_0^{(\lambda')}$, and therefore wins in this case as well. \square

The following proposition states that every subset of κ^2 that is Wadge-reducible to \mathcal{Y}_0 is in fact also Lipschitz-reducible to it. This will allow us to conclude in Corollary 4.3.41 that there exists a difference hierarchy of length κ^+ over the Wadge class generated by \mathcal{Y}_0 .

Proposition 4.3.40. *For every $A \subseteq \kappa^2$, $A \leq_W \mathcal{Y}_0$ if and only if $A \leq_L \mathcal{Y}_0$.*

Proof. Clearly, if $A \leq_L \mathcal{Y}_0$, then also $A \leq_W \mathcal{Y}_0$. For the converse, suppose Player **II** has a winning strategy τ_W in $G_W(A, \mathcal{Y}_0)$. We then define a strategy τ_L for Player **II** in $G_L(A, \mathcal{Y}_0)$ by having **II** mimic the moves she makes in $G_W(A, \mathcal{Y}_0)$, with the extra rule that whenever **II** would pass in $G_W(A, \mathcal{Y}_0)$, she instead plays 0 in $G_L(A, \mathcal{Y}_0)$.

By construction, for the same sequence $x \in \kappa^2$ played by **I**, the order type of the set of 1's in the sequence $y_W \in \kappa^2$ produced by **II** when following τ_W in $G_W(A, \mathcal{Y}_0)$ coincides with the order type of the set of 1's in the sequence $y_L \in \kappa^2$ produced by **II** when following τ_L in $G_L(A, \mathcal{Y}_0)$. Since τ_W is a winning strategy, it follows that τ_L is also winning. \square

Corollary 4.3.41. *Let $\Gamma(\kappa^2) = \mathcal{Y}_0 \downarrow_{\kappa^2}$. Then, the difference hierarchy over Γ on κ^2 has length κ^+ . Moreover, for every $\alpha < \kappa^+$, $\mathcal{D}_\alpha(\Gamma(\kappa^2)) \subseteq \kappa^+ - \Delta_2^0$.*

Proof. For the first part of the statement, we proved in Proposition 4.3.37 that $\mathcal{Y}_0 \downarrow_{\kappa^2}$ is non-selfdual. Then, by Propositions 4.3.40 and 2.13.3, $\Gamma(\kappa^2)$ has κ^2 -universal sets, and the result follows by Proposition 3.7.7.

For the second part, notice that if $A \in \kappa^+ - \Delta_2^0$ and $B \in \mathcal{D}_\alpha(A \downarrow_{\kappa^2})$ for some ordinal $\alpha < \kappa^+$, then $B \in \kappa^+ - \Delta_2^0$. Indeed, since $\kappa^+ - \Sigma_2^0$ is closed under unions of size κ , it is immediate to see that $B \in \kappa^+ - \Sigma_2^0$ and, by Fact 3.7.2, that $\kappa^2 \setminus B \in \kappa^+ - \Sigma_2^0$ too. \square

We now introduce the definition of the Hausdorff–Kuratowski derivative. This is motivated by the argument used in the proof of [Kec95, Theorem 22.27].

Definition 4.3.42. For any closed set $X \subseteq {}^\kappa 2$ and for any set $A \subseteq {}^\kappa 2$, let

$$\partial_X(A) = \text{cl}_X(A \cap X) \cap \text{cl}_X(X \setminus A)$$

be the boundary of $A \cap X$ in X , i.e., the set of all $x \in X$ such that x is a limit point of both $X \cap A$ and of $X \setminus A$. Equivalently,

$$\partial_X(A) = X \setminus \bigcup \{N_t \mid N_t \cap A = \emptyset \text{ or } N_t \subseteq A\}.$$

Clearly, $\partial_X(A)$ is closed.

Definition 4.3.43. For any closed set $X \subseteq {}^\kappa 2$ and any $A \subseteq {}^\kappa 2$, we define the *A-Hausdorff-Kuratowski derivative* (or *A-HK derivative*) of X as follows.

$$\begin{aligned} X_0^{(A)} &= X, \\ X_{\alpha+1}^{(A)} &= \partial_{X_\alpha^{(A)}}(A), \\ X_\alpha^{(A)} &= \bigcap_{\beta < \alpha} X_\beta^{(A)} \text{ if } \alpha \text{ is limit.} \end{aligned}$$

Let $\gamma < \kappa^+$ such that $X_\gamma^{(A)} = X_{\gamma+1}^{(A)}$, we call the least such γ the *A-HK rank* of X , and we say that $X_\gamma^{(A)}$ is the *A-HK kernel* of X . If $X_\gamma^{(A)} = \emptyset$, we say X is *A-HK scattered*.

Observe that for every $A \subseteq {}^\kappa 2$, the *A-HK rank* $\gamma < \kappa^+$ of X exists by Lemma 3.7.6, since $X \subseteq {}^\kappa 2$ and therefore $w(X) \leq \kappa$.

In our context, Definition 4.3.43 is particularly helpful because, given a set A that is Wadge-reducible to \mathcal{Y}_0 , determining whether ${}^\kappa 2$ is *A-HK-scattered* enables us to decide whether A lies in the κ^+ - $\mathbf{\Pi}_1^0$ -difference hierarchy (Proposition 4.3.45), or whether instead it is Lipschitz equivalent to \mathcal{Y}_0 (Proposition 4.3.47). This allows us to deduce in Theorem 4.3.48 the minimality above the κ^+ - $\mathbf{\Pi}_1^0$ -difference hierarchy of both the Wadge degree $[\mathcal{Y}_0]_W$ and the Lipschitz degree $[\mathcal{Y}_0]_L$.

Lemma 4.3.44. *Let $A, B \subseteq {}^\kappa 2$.*

- (1) *If ${}^\kappa 2$ is B -HK scattered and $A \leq_W B$, then ${}^\kappa 2$ is A -HK scattered.*
- (2) *If ${}^\kappa 2$ is A -HK scattered then ${}^\kappa 2$ is $({}^\kappa 2 \setminus A)$ -HK scattered.*

Proof. To see (1), let $f: {}^\kappa 2 \rightarrow {}^\kappa 2$ be continuous such that $A = f^{-1}(B)$. Then, $f(A) \subseteq B$ and $f({}^\kappa 2 \setminus A) \subseteq {}^\kappa 2 \setminus B$. We claim that for any closed sets $X, Y \subseteq {}^\kappa 2$ such that $f(X) \subseteq Y$, $f(\partial_X(A)) \subseteq \partial_Y(B)$. Indeed, using the continuity of f ,

$$\begin{aligned} f(\partial_X(A)) &\subseteq f(\text{cl}_X(A \cap X)) \cap f(\text{cl}_X(X \setminus A)) \\ &\subseteq \text{cl}_Y(f(A \cap X)) \cap \text{cl}_Y(f(X \setminus A)) \\ &\subseteq \text{cl}_Y(B \cap Y) \cap \text{cl}_Y(Y \setminus B) = \partial_Y(B). \end{aligned}$$

It follows by induction on $\alpha < \kappa^+$ that $f(X_\alpha^{(A)}) \subseteq X_\alpha^{(B)}$, therefore if ${}^\kappa 2$ is B -HK scattered and the B -HK rank is α , then $X_\alpha^{(B)} = \emptyset$ and $X_\alpha^{(A)} \subseteq f^{-1}(X_\alpha^{(B)}) = \emptyset$, so ${}^\kappa 2$ is A -HK scattered too (with A -HK rank $\leq \alpha$).

(2) holds because for any closed set $X \subseteq {}^\kappa 2$, $\partial_X(A) = \partial_X({}^\kappa 2 \setminus A)$, hence the A -HK kernel and the $({}^\kappa 2 \setminus A)$ -HK kernel of ${}^\kappa 2$ coincide. \square

Proposition 4.3.45. *Let $X \subseteq {}^\kappa 2$ be closed, and let $A \subseteq X$. If X is A -HK scattered, then $A \in \mathcal{D}_\alpha(\kappa^+ \text{-}\Pi_1^0)$ for some $\alpha < \kappa^+$.*

Proof. Let α be the A -HK rank of X . For every $i < \alpha$ we define

$$\begin{aligned} F_i &= X_i^{(A)}; \\ H_i &= \text{cl}_X(F_i \setminus A). \end{aligned}$$

Note that $F_i, H_i \in \kappa^+ \text{-}\Pi_1^0$ and that $F_i \supseteq H_i \supseteq F_{i+1}$ for every $i < \alpha$. In particular, the second inclusion holds since $F_{i+1} = \text{cl}_X(F_i \cap A) \cap \text{cl}_X(F_i \setminus A) \subseteq \text{cl}_X(F_i \setminus A) = H_i$.

We claim that $A = \bigcup_{i < \alpha} (F_i \setminus H_i)$. To see this, first let $x \in A$. Since X is A -HK scattered, there exists $i < \alpha$ such that $x \in F_i \setminus F_{i+1}$. If $x \in H_i$, then $x \in \text{cl}_X(F_i \setminus A) \cap (F_i \cap A) \subseteq F_{i+1}$, which is a contradiction. So $x \in F_i \setminus H_i$. Conversely, assume that $x \in F_i \setminus H_i$ for some $i < \alpha$. If $x \notin A$, then $x \in F_i \setminus A \subseteq H_i$, a contradiction. Thus, $x \in A$.

Since $\bigcup_{i < \alpha} (F_i \setminus H_i) \in \mathcal{D}_{2,\alpha}(\kappa^+ \text{-}\Pi_1^0)$, we conclude that $A \in \mathcal{D}_{2,\alpha}(\kappa^+ \text{-}\Pi_1^0)$. \square

Lemma 4.3.46. *Let $X \subseteq {}^\kappa 2$ be a closed set, and let $A \subseteq {}^\kappa 2$ be any set. Suppose that X is not A -HK scattered, and let $K \neq \emptyset$ be the A -HK kernel of X . Then, there exist $b \in K \cap A$ and $b' \in K \setminus A$.*

Proof. The set $A \cap K$ is dense and co-dense in K , because $\partial_K(A) = K$ implies $\text{cl}_K(A \cap K) = \text{cl}_K(K \setminus A) = K$. Since $K \neq \emptyset$, pick any $y \in K$. Then, for every $\alpha < \kappa$ there exist $b \in \mathbf{N}_{y \upharpoonright \alpha} \cap K \cap A$ and $b' \in \mathbf{N}_{y \upharpoonright \alpha} \cap (K \setminus A)$. \square

Proposition 4.3.47. *Let $A \subseteq {}^\kappa 2$ such that $A \leq_W \mathcal{Y}_0$. If ${}^\kappa 2$ is not A -HK scattered, then $\mathcal{Y}_0 \leq_L A$.*

Proof. Since $A \leq_W \mathcal{Y}_0$, by Proposition 4.3.40 there exists a Lipschitz function $f: {}^\kappa 2 \rightarrow {}^\kappa 2$ witnessing $A = f^{-1}(\mathcal{Y}_0)$. Our goal is to construct a Lipschitz function $h: {}^\kappa 2 \rightarrow {}^\kappa 2$ which reduces \mathcal{Y}_0 to A .

Let K be the A -HK kernel of ${}^\kappa 2$. Since ${}^\kappa 2$ is not A -HK scattered, then $K \neq \emptyset$. Note that, since f witnesses $A \leq_L \mathcal{Y}_0$, for every $b \in A$, $f(b) \in \mathcal{Y}_0$ hence there exists some $\gamma < \kappa$ such that $f(b)(\delta) = 0$ for every $\delta \geq \gamma$.

Claim 4.3.47.1. Let $b \in K \cap A$ and let $\gamma < \kappa$ such that $f(b)(\delta) = 0$ for every $\delta \geq \gamma$. Then, for every $\alpha \geq \gamma$ and for every $c \in \mathbf{N}_{b \upharpoonright \alpha} \cap (K \setminus A)$ the following conditions are satisfied:

- (a) $f(c) \in \mathbf{N}_{f(b) \upharpoonright \gamma}$,
- (b) $f(c)(\beta) = 1$ for some $\beta > \gamma$.

Proof of the Claim. Let b, γ, α be as in the statement. Note that since $b \in K$, by density of $K \setminus A$ in K , $\mathbf{N}_{b \upharpoonright \alpha} \cap (K \setminus A) \neq \emptyset$. Since $\alpha \geq \gamma$ and f is Lipschitz, any $c \in \mathbf{N}_{b \upharpoonright \alpha} \cap (K \setminus A)$ satisfies condition (a). For condition (b), note that $f(b) = f(b) \upharpoonright \gamma \cap 0^{(\kappa)}$ by definition of γ , and $f(b) \in \mathcal{Y}_0$ since $b \in A$. On other hand, $c \notin A$ hence $f(c) \notin \mathcal{Y}_0$. Using (a), we have that there is some $\beta > \gamma$ such that $f(c)(\beta) = 1$. \square

We first construct $b^s \in 2^\kappa$ and $\gamma^s < \kappa$ for $s \in \kappa^{<\omega}$ by induction on $\text{lh}(s) < \omega$. We will write $\gamma^{<s \hat{\ } (0)} = \gamma^s + 1$ and $\gamma^{<s \hat{\ } \langle \beta \rangle} = \sup_{\alpha < \beta} (\gamma^{s \hat{\ } \langle \alpha \rangle} + 1)$ for $s \in \kappa^{<\omega}$ and $\beta < \kappa$.

- If $s = \emptyset$, find $b^\emptyset \in K \cap A$ by Lemma 4.3.46. Let γ^\emptyset be the largest γ such that $f(b^\emptyset)(\gamma) = 1$, if this exists, and 0 otherwise.
- If $s = \langle \beta \rangle$ for some $\beta < \kappa$, find $b^{(\beta)} \in \mathbf{N}_{b^\emptyset \upharpoonright \gamma^{<\beta}} \cap (K \setminus A)$ by the Claim 4.3.47.1 (applied to $b = b^\emptyset$, $\gamma = \gamma^\emptyset$ and $\alpha = \gamma^{<\beta}$). Let γ be the least ordinal $\geq \gamma^{<\beta}$ such that $f(b^{(\beta)})(\gamma) = 1$, and set $\gamma^{(\beta)} = \gamma$. Note that such γ exists by Claim 4.3.47.1(b).
- Suppose that $s = \langle \beta_0, \beta_1, \dots, \beta_n \rangle$ for some $\beta_0, \dots, \beta_n < \kappa$.
 - If n is odd, find $b^s \in \mathbf{N}_{b^{s \upharpoonright n} \upharpoonright \gamma^{<s}} \cap (K \cap A)$, which exists by density of $K \cap A$ in K : indeed, since $b^{s \upharpoonright n} \in K \setminus A$, $\mathbf{N}_{b^{s \upharpoonright n} \upharpoonright \gamma^{<s}} \cap (K \cap A) \neq \emptyset$. Let γ^s be largest ordinal with $f(b^s)(\gamma^s) = 1$. Then, $\gamma^s \geq \gamma^{<s}$.
 - If n is even, find $b^s \in \mathbf{N}_{b^{s \upharpoonright n} \upharpoonright \gamma^{<s}} \cap (K \setminus A)$ by the Claim 4.3.47.1 (applied to $b = b^{s \upharpoonright n} \in K \cap A$, $\gamma = \gamma^{s \upharpoonright n}$ and $\alpha = \gamma^{<s}$). Let γ^s be the least ordinal $\geq \gamma^{<s}$ such that $f(b^s)(\gamma^s) = 1$. Note that such γ exists by condition Claim 4.3.47.1(b).

We now define $\varphi : \mathcal{T}_\mathcal{Y} \rightarrow {}^{<\kappa}2$ for every $t \in \mathcal{T}_\mathcal{Y}$. This is done by induction on $\text{lh}(s_t)$, where $s_t = \langle \beta_0, \beta_1, \dots, \beta_n \rangle \in \kappa^{<\omega}$ is the sequences that determines all and only the values 1 in t , i.e., $t(\beta_i) = 1$ for every $i \leq n$ and $t(\delta) = 0$ for every $\delta < \text{lh}(t)$, $\delta \notin \{\beta_i \mid i \leq n\}$. If $s_t = \emptyset$, we let

$$\varphi(t) = \begin{cases} b^\emptyset \upharpoonright \gamma^\emptyset & \text{if } t = \emptyset \\ b^\emptyset \upharpoonright \gamma^{<\beta} & \text{if } t = 0^{(\beta)} \text{ for some } \beta < \kappa; \end{cases}$$

if $s_t = \langle \beta_0 \rangle$, we let

$$\varphi(t) = \begin{cases} b^{(\beta_0)} \upharpoonright \gamma^{(\beta_0)} & \text{if } t = 0^{(\beta_0)} \hat{\ } 1 \\ b^{(\beta_0)} \upharpoonright \gamma^{<\langle \beta_0, \beta \rangle} & \text{if } t = 0^{(\beta_0)} \hat{\ } 1 \hat{\ } 0^{(\beta)}. \end{cases}$$

Note that the sequences $\{\varphi(0^{(\delta)} \hat{\ } 1) = b^{(\delta)} \upharpoonright \gamma^{(\delta)} \mid \delta < \kappa\}$ are pairwise incompatible and each branches off b^\emptyset at at some level $\alpha_\delta < \kappa$ with $\gamma^{<\langle \delta \rangle} \leq \alpha_\delta < \gamma^{(\delta)}$, and since $\gamma^{<\langle \delta \rangle} < \gamma^{(\delta)} < \gamma^{<\langle \delta+1 \rangle}$ for every $\delta < \kappa$, then $\langle \alpha_\delta \mid \delta < \kappa \rangle$ is a strictly increasing sequence of ordinals.

If $s_t = \langle \beta_0, \beta_1, \dots, \beta_n \rangle$,

$$\varphi(t) = \begin{cases} b^{s_t} \upharpoonright \gamma^{s_t} & \text{if } t = 0^{(\beta_0)} \hat{\ } 1 \hat{\ } \dots \hat{\ } 0^{(\beta_n)} \hat{\ } 1 \\ b^{s_t} \upharpoonright \gamma^{<s_t \hat{\ } \langle \beta \rangle} & \text{if } t = 0^{(\beta_0)} \hat{\ } 1 \hat{\ } \dots \hat{\ } 0^{(\beta_n)} \hat{\ } 1 \hat{\ } 0^{(\beta)}. \end{cases}$$

Note that the sequences in $\{\varphi(0^{(\beta_0)} \frown 1 \frown \dots \frown 0^{(\beta_{n-1})} \frown 1 \frown 0^{(\delta)} \frown 1) \mid \delta < \kappa\}$ are pairwise incompatible and each of them branches off $b^{s_t \upharpoonright n}$ at some level $\alpha_\delta < \kappa$ with $\gamma^{< s_t \frown (\delta)} \leq \alpha_\delta < \gamma^{s_t \frown (\delta)}$, and since $\gamma^{< s_t \frown (\delta)} < \gamma^{s_t \frown (\delta)} < \gamma^{< s_t \frown (\delta+1)}$ for every $\delta < \kappa$, then $\langle \alpha_\delta \mid \delta < \kappa \rangle$ is a strictly increasing sequence of ordinals.

Recall that for any $u \in {}^{<\kappa}2$, either $u \in \mathcal{T}_y$ or $u = t \frown v$ for some $t \in \partial\mathcal{T}_y$ and $v \in {}^{<\kappa}2$; in this case, clearly $t \upharpoonright i \in \mathcal{T}_y$ for every $i < \text{lh}(s)$. Then, we define $\varphi' : {}^{<\kappa}2 \rightarrow {}^{<\kappa}2$ as follows:

- $\varphi'(u) = \varphi(u)$ if $u \in \mathcal{T}_y$;
- $\varphi'(u) = \bigcup_{i < \text{lh}(t)} \varphi(t \upharpoonright i) \frown v$ if $u = t \frown v$ for some $t \in \partial\mathcal{T}_y$ and $v \in {}^{<\kappa}2$.

Let $h : \kappa^2 \rightarrow \kappa^2$ be the function induced by φ' . Note that by construction (and the definition of b^s, γ^s for $s \in {}^{<\omega}\kappa$) φ is monotone and $\text{lh}(t) \leq \text{lh}(\varphi(t))$ for every $t \in \mathcal{T}_y$. Then, it is easy to see that φ' is Lipschitz, in fact strictly increasing, hence h is Lipschitz. It remains to show that h is a reduction of \mathcal{Y}_0 to A .

Claim 4.3.47.2. $\mathcal{Y}_0 = h^{-1}(A)$.

Proof of the Claim. Note that for every $s \in \kappa^{<\omega}$, $\sup_{\beta < \kappa} \gamma^{< s \frown (\beta)} = \kappa$.

First, assume that $x \in \mathcal{Y}_0$. Then, let $\delta < \kappa$ be the largest ordinal with $x(\delta) = 1$ if this exists, and 0 otherwise. Let $t = x \upharpoonright \delta + 1 \in \mathcal{T}_y$. Then, $x = t \frown 0^{(\kappa)}$ by definition of t , and s_t has even length since $x \in \mathcal{Y}_0$.

If $s_t = \emptyset$, then $x = 0^{(\kappa)}$ and $h(x) = \bigcup_{\beta < \kappa} \varphi(0^{(\beta)}) = \bigcup_{\beta < \kappa} (b^\emptyset \upharpoonright \gamma^{< (\beta)}) = b^\emptyset \in A$. Assume now that $s_t = \langle \beta_0, \beta_1, \dots, \beta_n \rangle \in \kappa^{<\omega}$. Then, $\varphi(t) = b^{s_t} \upharpoonright \gamma^{s_t}$ and $b^{s_t} \in A$ by construction, because n is odd. Then, $h(x) = \bigcup_{\beta < \kappa} (b^{s_t} \upharpoonright \gamma^{< s_t \frown (\beta)}) = b^{s_t} \in A$.

Assume now that $x \notin \mathcal{Y}_0$. Then, either $x \in \mathcal{Y}_1$ or $x \notin \mathcal{Y}$. If $x \in \mathcal{Y}_1$, let $\delta < \kappa$ be the largest ordinal with $x(\delta) = 1$ and let $t = x \upharpoonright \delta + 1 \in \mathcal{T}_y$. Then, $x = t \frown 0^{(\kappa)}$ and s_t has odd length. If $s_t = \langle \beta_0, \beta_1, \dots, \beta_n \rangle \in \kappa^{<\omega}$, then $\varphi(t) = b^{s_t} \upharpoonright \gamma^{s_t}$ and $b^{s_t} \notin A$ by construction, because n is even. Then, $h(x) = \bigcup_{\beta < \kappa} (b^{s_t} \upharpoonright \gamma^{< s_t \frown (\beta)}) = b^{s_t} \notin A$.

Finally, assume that $x \notin \mathcal{Y}$ and let $t \in \partial\mathcal{T}_y$ be the (unique) sequence such that $t \subseteq x$. Then, t takes value 1 ω -many times: let $\bar{s}_t = \langle \beta_n \mid n < \omega \rangle \in \omega\kappa$ be the sequences that determines all and only the values 1 in t .

For every $n < \omega$, $\varphi(t \upharpoonright \beta_n + 1) = b^{\bar{s}_t \upharpoonright n+1} \upharpoonright \gamma^{\bar{s}_t \upharpoonright n+1}$, and

$$\varphi'(t) = \bigcup_{n < \omega} b^{\bar{s}_t \upharpoonright n+1} \upharpoonright \gamma^{< \bar{s}_t \upharpoonright n+2}.$$

By definition of $\gamma^{\bar{s}_t \upharpoonright n+1}$, for every $n < \omega$, $f(b^{\bar{s}_t \upharpoonright n+1})(\gamma^{\bar{s}_t \upharpoonright n+1}) = 1$. Since $\gamma^{\bar{s}_t \upharpoonright n+1} < \gamma^{< \bar{s}_t \upharpoonright n+2}$, this means that for every $z \in \kappa^2$ such that $\varphi'(t) \subseteq z$, the sequence $f(z)$ has value 1 in at least ω -many entries.

Therefore, by construction $f(\mathbf{N}_{\varphi'(t)}) \cap \mathcal{Y}_0 = \emptyset$, hence $\mathbf{N}_{\varphi'(t)} \cap A = \emptyset$. Indeed, $\mathbf{N}_{\varphi'(t)} \subseteq f^{-1}(f(\mathbf{N}_{\varphi'(t)}))$ and $A = f^{-1}(\mathcal{Y}_0)$ by hypothesis. \square

Since h is Lipschitz, this shows $\mathcal{Y}_0 \leq_L A$. \square

Theorem 4.3.48. *Let $A \subseteq \kappa^2$ such that $A \leq_W \mathcal{Y}_0$. Then, either $A \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ for some $\alpha < \kappa^+$, or $\mathcal{Y}_0 \equiv_L A$. Therefore, the Lipschitz degree $[\mathcal{Y}_0]_L$ (hence, the Wadge degree $[\mathcal{Y}_0]_W$) is minimal above the $\kappa^+ - \mathbf{\Pi}_1^0$ -Difference hierarchy.*

Proof. Since $A \leq_W \mathcal{Y}_0$, then $A \leq_L \mathcal{Y}_0$ by Proposition 4.3.40. If κ^2 is A -HK scattered, $A \in \mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ for some $\alpha < \kappa$ by Proposition 4.3.45. Otherwise, $\mathcal{Y}_0 \leq_L A$ by Proposition 4.3.47. \square

Recall that SLO_W holds for the classes in the $\kappa^+ - \mathbf{\Pi}_1^0$ -difference hierarchy by Theorem 4.3.15, and that the Wadge hierarchy is well-founded when restricted to the sets in the $\kappa^+ - \mathbf{\Pi}_1^0$ -difference hierarchy by Corollary 4.3.35. Since \mathcal{Y}_0 is minimal above the $\kappa^+ - \mathbf{\Pi}_1^0$ -difference hierarchy by Theorem 4.3.48, SLO_W holds for $\Gamma \cup \check{\Gamma}$ where Γ is Wadge class $\mathcal{Y}_0 \downarrow_{\kappa^2}$. Moreover, the Wadge hierarchy on the $\Gamma \cup \check{\Gamma}$ -sets is well-founded.

Based on the result in Corollary 4.3.41, the following question arises.

Question 4.3.49. Let $\Gamma = \mathcal{Y}_0 \downarrow_{\kappa^2}$. Does SLO_W hold for the difference hierarchy over Γ ? Is the Wadge hierarchy on $\bigcup_{\alpha < \kappa^+} \mathcal{D}_\alpha(\Gamma)$ well-founded?

4.3.4 The failure of SLO_W in $\kappa^+ - \mathbf{\Delta}_2^0$

Let us return, once again, to the set $\mathcal{Y} = \{x \in \kappa^2 \mid |\{\alpha < \kappa \mid x(\alpha) = 1\}| < \aleph_0\}$ from Example 3.7.12. This time, we use it to define a new family of sets, namely the sets \mathcal{Y}_S for $S \subseteq \kappa$, which serve as the prospective witnesses for the failure of SLO_W for $\kappa^+ - \mathbf{\Delta}_2^0$.

Given any $S \subseteq \kappa$, we define:

$$\mathcal{Y}_S = \mathcal{Y}_0 \cup \bigcup \{N_t \mid t \in \partial\mathcal{T}_\mathcal{Y}, \text{lh}(t) \in S\}. \quad (4.3.1)$$

Clearly, $\mathcal{Y}_S \in \kappa^+ - \mathbf{\Delta}_2^0$. However, by Proposition 4.3.36 (applied with $Z = \mathcal{Y}_S$ and $X = Y = \mathcal{Y}$), \mathcal{Y}_S does not lie in the $\kappa^+ - \mathbf{\Pi}_1^0(\mathcal{Y})$ -difference hierarchy, since $\mathcal{Y} \cap \mathcal{Y}_S$ is both dense and codense in \mathcal{Y} . Furthermore, because $\mathcal{Y} \subseteq \kappa^2$ and the pointclass $\mathcal{D}_\alpha(\kappa^+ - \mathbf{\Pi}_1^0)$ is hereditary (see Section 3.7), it follows that \mathcal{Y}_S does not belong to the $\kappa^+ - \mathbf{\Pi}_1^0$ -difference hierarchy.

Remark 4.3.50. The set \mathcal{Y}_S only depends on the set of limit ordinals of countable cofinality in S , i.e., for any two sets $S, T \subseteq \kappa$ such that $S \cap \text{cof}_\omega^\kappa = T \cap \text{cof}_\omega^\kappa$, we have $\mathcal{Y}_S = \mathcal{Y}_T$. The reason is that $\partial\mathcal{T}_\mathcal{Y}$ only contains nodes $t \in {}^{<\kappa}2$ with $\text{cof}(\text{lh}(t)) = \omega$.

First of all, notice that for every $S \subseteq \kappa$ the set \mathcal{Y}_S does not reduce to its complement, thus it is non-selfdual.

Proposition 4.3.51. *For any $S \subseteq \kappa$, the Wadge class $\mathcal{Y}_S \downarrow_{\kappa^2}$ is non-selfdual.*

Proof. Similarly to the argument in the proof of Proposition 4.3.37, to prove that $\mathcal{Y}_S \not\leq_W \neg\mathcal{Y}_S$, it suffices to describe a winning strategy for Player **I** in the Wadge game $G_W(\mathcal{Y}_S, \neg\mathcal{Y}_S)$.

Player **I** starts the game by choosing 0. After this initial move, **I** plays 1 if and only if Player **II** plays 1. At every limit stage, player **I** starts again by playing 0.

Let $x \in \kappa^2$ denote the element chosen by **I**, and $y \in \kappa^2$ the element chosen by **II**, at the conclusion of the game in which **I** follows the above strategy. Then, $\text{ot}(\{\alpha < \kappa \mid x(\alpha) = 1\}) = \text{ot}(\{\alpha < \kappa \mid y(\alpha) = 1\})$. In particular, if at any (limit) stage in the game Player **II** plays a minimal node $u \in \partial\mathcal{T}_\mathcal{Y}$, that is, a

node for which the order type of the 1's in u is exactly ω , then also Player **I** has produced a sequence in $\partial\mathcal{T}_y$. We conclude that $x \in \mathcal{Y}_S \Leftrightarrow y \in \mathcal{Y}_S \Leftrightarrow y \notin \neg\mathcal{Y}_S$. \square

The following proposition provides a sufficient condition on two sets $S, S' \subseteq \kappa$ ensuring that \mathcal{Y}_S and $\mathcal{Y}_{S'}$ are Wadge-incomparable. As a consequence (Corollary 4.3.54), the Wadge hierarchy on $\kappa^+ \text{-}\Delta_2^0$ contains antichains of cardinality 2^κ . In particular, this shows that SLO_W does not hold for $\kappa^+ \text{-}\Delta_2^0$ -sets.

We write $\mathcal{M} \preceq \mathcal{N}$ if \mathcal{M} is an elementary submodel of \mathcal{N} .

Proposition 4.3.52. *Let $S, S' \subseteq \kappa$ be such that $S \Delta S'$ is stationary in κ . Then, \mathcal{Y}_S and $\mathcal{Y}_{S'}$ are W-incomparable.*

Proof. Towards a contradiction, assume there exists a continuous function $f: \kappa^2 \rightarrow \kappa^2$ that witnesses $\mathcal{Y}_S \leq_W \mathcal{Y}_{S'}$.

For a fixed uncountable (sufficiently large) regular θ , let

$$C_\theta^* = \{\gamma < \kappa \mid \mathcal{M} \preceq H_\theta, |\mathcal{M}| < \kappa, \kappa \in \mathcal{M}, \mathcal{M} \cap \kappa = \gamma\}$$

Claim 4.3.52.1. C_θ^* contains a closed and unbounded subset C_θ of κ .

Proof of the Claim. We construct a continuous elementary chain $(\mathcal{M}_i)_{i < \kappa}$ of elementary substructures of size $< \kappa$ such that $\mathcal{M}_i \preceq H_\theta$ for $i < \kappa$. To do this, fix a countable set of Skolem functions for H_θ . Given a set $X \subseteq H_\theta$, denote by $\mathcal{H}^{H_\theta}(X)$ the Skolem hull of X , i.e., the closure of X under the Skolem functions fixed above. Set $\mathcal{M}_0 = \mathcal{H}^{H_\theta}(\emptyset)$, $\mathcal{M}_{i+1} = \mathcal{H}^{H_\theta}(\mathcal{M}_i \cup \{\alpha_i\})$, and for limits j set $\mathcal{M}_j = \bigcup_{i < j} \mathcal{M}_i$. Let $\alpha_i = \mathcal{M}_i \cap \kappa$ for each $i < \kappa$. Then, $\alpha_j = \sup_{i < j} \alpha_i$ for all limits $j < \kappa$, so $\{\alpha_i \mid i < \kappa\}$ is closed and unbounded in κ . \square

Fix a set C_θ as in Claim 4.3.52.1 and let $\gamma \in (S \Delta S') \cap C_\theta$. We can assume that $\gamma \in (S \setminus S')$, as the other case is proven similarly. Fix $\mathcal{M} \preceq H_\theta$ of size $< \kappa$ such that $\mathcal{M} \cap \kappa = \gamma$ and $S, S', f \in \mathcal{M}$. By Remark 4.3.50, we can assume without loss of generality that $S \cup S' \subseteq \text{cof}_\omega^\kappa$. Since $\text{cof}(\gamma) = \omega$, we fix a strictly increasing sequence $(\gamma_n)_{n < \omega}$ of ordinals cofinal in γ .

We construct a sequence $(x_n)_{n \in \omega}$ in $\kappa^2 \cap \mathcal{M}$ and two strictly increasing sequences $(\alpha_n)_{n \in \omega}$ and $(\beta_n)_{n \in \omega}$ of ordinals in $\mathcal{M} \cap \kappa$ such that:

- (a) $x_{n+1}(\alpha) = 1$ for a unique $\alpha \in [\alpha_n, \alpha_{n+1})$;
- (b) $x_n \upharpoonright \alpha_n \subseteq x_{n+1}$;
- (c) $f(x) \upharpoonright \beta_n = f(x_n) \upharpoonright \beta_n$ for all $x \in \mathbf{N}_{x_n \upharpoonright \alpha_n}$;
- (d) $f(x_{n+1})(\beta) = 1$ for at least one and at most finitely many $\beta \in [\beta_n, \beta_{n+1})$;
- (e) $\alpha_n, \beta_n \geq \gamma_n$.

Let $a = \bigcup_{n < \omega} x_n \upharpoonright \alpha_n$ and $b = \bigcup_{n < \omega} f(x_n) \upharpoonright \beta_n$. Note that $\sup_{n < \omega} \alpha_n = \sup_{n < \omega} \beta_n = \gamma$ by (e) and by the choice of \mathcal{M} , hence $\text{lh}(a) = \text{lh}(b) = \gamma$. Using that $\gamma \in S \setminus S'$, $\mathbf{N}_a \subseteq \mathcal{Y}_S$ by (a) and (b), $f(\mathbf{N}_a) \subseteq \mathbf{N}_b$ by (c), and $\mathbf{N}_b \cap \mathcal{Y}_{S'} = \emptyset$ by (d) and (e). This contradicts that f is a reduction of \mathcal{Y}_S to $\mathcal{Y}_{S'}$.

For every $x \in \kappa^2$, we denote by $\#_f(x) = \{\beta < \kappa \mid f(x)(\beta) = 1\}$. We set $x_0 = 0^{(\kappa)}$. Since $x_0 \in \mathcal{Y}_S$, $f(x_0) \in \mathcal{Y}_{S'}$.

Claim 4.3.52.2. $|\#_f(x_0)| < \aleph_0$.

Proof of the Claim. Towards a contradiction, suppose that $|\#_f(x_0)| = \mu \geq \aleph_0$ and let $(\beta_i)_{i < \mu}$ be a strictly increasing enumeration of $\#_f(x_0)$. Let $\bar{\beta} = \sup_{i < \omega} \beta_i$ and $t = f(x_0) \upharpoonright \bar{\beta}$. Then, $\bar{\beta} \in S'$ and $\mathbf{N}_t \subseteq \mathcal{Y}_{S'}$. Using the continuity of f , let $\delta < \kappa$ such that $f[\mathbf{N}_{x_0 \upharpoonright \delta}] \subseteq \mathbf{N}_t$. Consider $x = (x_0 \upharpoonright \delta) \wedge 1 \wedge 0^{(\kappa)}$. Since $|\{\beta < \kappa \mid x(\beta) = 1\}| = 1$, $x \notin \mathcal{Y}_S$ and therefore $f(x) \notin \mathcal{Y}_{S'}$ by case assumption. However, $f(x) \in \mathbf{N}_t \subseteq \mathcal{Y}_{S'}$ by continuity. A contradiction. \square

By Claim 4.3.52.2, $\max(\#_f(x_0))$ exists. Let $\beta_0 \in \mathcal{M}$ such that $\beta_0 \geq \max\{\gamma_0, \max(\#_f(x_0)) + 1\}$ and, by continuity of f , let $\alpha_0 \geq \gamma_0$ such that $\alpha_0 \in \mathcal{M}$ and $f(x) \upharpoonright \beta_0 = f(x_0) \upharpoonright \beta_0$ for every $x \in \mathbf{N}_{x_0 \upharpoonright \alpha_0}$. We can find such $\alpha_0, \beta_0 \in \mathcal{M}$, since $\gamma_0, x_0 \in \mathcal{M}$ and $\mathcal{M} \preceq H_\theta$. Conditions (c) and (e) are clearly satisfied, while the other ones do not apply at this stage.

Suppose that $x_i, \alpha_i, \beta_i \in \mathcal{M}$ have been defined for all $i \leq n$. Let $x_{n+1} = x_n \upharpoonright \alpha_n \wedge 1 \wedge 0^{(\kappa)}$. Since $|\{\beta < \kappa \mid (x_n \upharpoonright \alpha_n)(\beta) = 1\}| = n$ by (a) and (b), we have that $x_{n+1} \in \mathcal{Y}_S$ if and only if n is odd.

Claim 4.3.52.3. $|\#_f(x_{n+1})| < \aleph_0$.

Proof of the Claim. Towards a contradiction, suppose that $|\#_f(x_{n+1})| = \mu \geq \aleph_0$ and let $(\beta_i)_{i < \mu}$ be a strictly increasing enumeration of $\#_f(x_{n+1})$. Let $\bar{\beta} = \sup_{i < \omega} \beta_i$ and $t = f(x_{n+1}) \upharpoonright \bar{\beta}$.

First, assume that n is even. Then, $x_{n+1} \notin \mathcal{Y}_S$ and $f(x_{n+1}) \notin \mathcal{Y}_{S'}$, hence $\bar{\beta} \notin S'$ and $\mathbf{N}_t \cap \mathcal{Y}_{S'} = \emptyset$. By continuity of f , let $\delta < \kappa$ such that $f[\mathbf{N}_{x_{n+1} \upharpoonright \delta}] \subseteq \mathbf{N}_t$. Pick $\varepsilon > \max\{\alpha_n + 1, \delta\}$ and let $x = (x_{n+1} \upharpoonright \varepsilon) \wedge 1 \wedge 0^{(\kappa)}$. Since $|\{\beta < \kappa \mid x(\beta) = 1\}| = n + 2$ and n is even, $x \in \mathcal{Y}_S$ and therefore $f(x) \in \mathcal{Y}_{S'}$ by case assumption. However, $f(x) \in \mathbf{N}_t$ and $\mathbf{N}_t \cap \mathcal{Y}_{S'} = \emptyset$. A contradiction.

Next, assume that n is odd. Then, $x_{n+1} \in \mathcal{Y}_S$ and $f(x_{n+1}) \in \mathcal{Y}_{S'}$, hence $\bar{\beta} \in S'$ and $\mathbf{N}_t \subseteq \mathcal{Y}_{S'}$. By continuity of f let $\delta < \kappa$ such that $f[\mathbf{N}_{x_{n+1} \upharpoonright \delta}] \subseteq \mathbf{N}_t$. Pick $\varepsilon > \max\{\alpha_n + 1, \delta\}$ and let $x = (x_{n+1} \upharpoonright \varepsilon) \wedge 1 \wedge 0^{(\kappa)}$. Since $|\{\beta < \kappa \mid x(\beta) = 1\}| = n + 2$ and n is odd, $x \notin \mathcal{Y}_S$ and therefore $f(x) \notin \mathcal{Y}_{S'}$ by case assumption. However, $f(x) \in \mathbf{N}_t \subseteq \mathcal{Y}_{S'}$. A contradiction. \square

By Claim 4.3.52.3, $\max(\#_f(x_{n+1}))$ exists. Let $\beta_{n+1} \in \mathcal{M}$ such that $\beta_{n+1} \geq \max\{\gamma_{n+1}, \max(\#_f(x_{n+1})) + 1\}$. Note that $\beta_n < \beta_{n+1}$. By continuity of f , let $\alpha_{n+1} > \max\{\alpha_n, \gamma_{n+1}\}$ such that $\alpha_{n+1} \in \mathcal{M}$ and $f(x) \upharpoonright \beta_{n+1} = f(x_{n+1}) \upharpoonright \beta_{n+1}$ for every $x \in \mathbf{N}_{x_{n+1} \upharpoonright \alpha_{n+1}}$. We can find such $\alpha_{n+1}, \beta_{n+1} \in \mathcal{M}$, since $\gamma_{n+1}, x_{n+1} \in \mathcal{M}$ and $\mathcal{M} \preceq H_\theta$. Conditions (c) and (e) are clearly satisfied. Moreover, (a) and (b) hold true by definition of x_{n+1} . Finally, to see that (d) is satisfied we need Claim 4.3.52.3. Indeed, it follows immediately that $f(x_{n+1})(\beta) = 1$ for at most finitely many $\beta \in [\beta_n, \beta_{n+1})$. Moreover, using conditions (a)-(c), the definition of $\mathcal{Y}_S, \mathcal{Y}_{S'}$, and the fact that f is a reduction of \mathcal{Y}_S to $\mathcal{Y}_{S'}$, we have that $f(x_n) \in \mathcal{Y}_0$ if n is even, and $f(x_n) \in \mathcal{Y}_1$ if n is odd. Therefore, $f(x_{n+1})(\beta) = 1$ for at least one $\beta \in [\beta_n, \beta_{n+1})$ because $f(x_{n+1}) \in \mathcal{Y}_1$ if n is even, and $f(x_{n+1}) \in \mathcal{Y}_0$ if n is odd.

The case $\mathcal{Y}_{S'} \not\leq_W \mathcal{Y}_S$ is proven similarly. \square

Corollary 4.3.53. *Let $S \subseteq \kappa$ be such that $S \cap \text{cof}_\omega^\kappa$ is stationary in κ . Then, \mathcal{Y}_S and \mathcal{Y}_0 are W -incomparable.*

Proof. Apply Proposition 4.3.52 with $S' = \emptyset$. \square

Corollary 4.3.54. *In the Wadge hierarchy on κ^+ - Δ_2^0 subsets of κ^2 there are antichains of size 2^κ . In particular, the SLOW fails for κ^+ - Δ_2^0 -sets.*

Proof. Let $S \subseteq \kappa$ be stationary and costationary in κ . By Proposition 4.3.52, \mathcal{Y}_S , $\mathcal{Y}_{\kappa \setminus S}$ and \mathcal{Y}_0 are \mathbb{W} -incomparable and, since they all are κ^+ - Δ_2^0 -sets, we conclude that the Wadge semi-linear ordering principle for κ^+ - Δ_2^0 sets fails.

Moreover, cof_ω^κ is the union of κ -many disjoint sets $(S_i)_{i < \kappa}$ stationary in κ by a Theorem of Solovay ([Jec03, Lemma 8.8]). For each subset I of κ , let $S_I = \bigcup_{i \in I} S_i$. Then, the sets \mathcal{Y}_{S_I} for $I \subseteq \kappa$ are pairwise \mathbb{W} -incomparable since $S_I \Delta S_{I'}$ is stationary for every $I \neq I'$. \square

In contrast to Proposition 4.3.52, the next result gives a sufficient condition on two sets $S, S' \subseteq \kappa$ guaranteeing that \mathcal{Y}_S and $\mathcal{Y}_{S'}$ are Wadge-equivalent. Note that the following proposition does not, in fact, make use of AC.

Proposition 4.3.55. *Let $S, S' \subseteq \kappa$ be such that $(S \Delta S') \cap \text{cof}_\omega^\kappa$ is non-stationary in κ . Then, $\mathcal{Y}_S \equiv_{\mathbb{W}} \mathcal{Y}_{S'}$.*

Proof. Before starting the proof, note that for any $s \in \mathcal{T}_{\mathcal{Y}}$ there is a canonical choice of elements $z_s, z'_s \in \kappa^2$ extending s such that $z_s \in \mathcal{Y}_0 \subseteq \mathcal{Y}_{S'}$ and $z'_s \in \mathcal{Y}_1 \subseteq \mathcal{Y} \setminus \mathcal{Y}_{S'}$. Indeed, since $\text{ot}(\{\alpha < \text{lh}(s) \mid s(\alpha) = 1\}) = n$ for some $n < \omega$, if n is even it is sufficient to set $z_s = s \hat{\ } 0^{(\kappa)}$ and $z'_s = s \hat{\ } 1 \hat{\ } 0^{(\kappa)}$, and if n is odd it is sufficient to set $z_s = s \hat{\ } 1 \hat{\ } 0^{(\kappa)}$ and $z'_s = s \hat{\ } 0^{(\kappa)}$.

Let $C \subseteq \kappa$ be a closed and unbounded set disjoint from $(S \Delta S') \cap \text{cof}_\omega^\kappa$. By Remark 4.3.50 we can assume without loss of generality that $S \cup S' \subseteq \text{cof}_\omega^\kappa$ and that $0 \in C$.

To show that $\mathcal{Y}_S \leq_{\mathbb{W}} \mathcal{Y}_{S'}$, we define a monotone function $\varphi : <\kappa^2 \rightarrow <\kappa^2$ as follows. If $t \in \mathcal{T}_{\mathcal{Y}}$, set $\varphi(t) = r_t$ where $r_t \subseteq t$ is maximal such that $\text{lh}(r_t) \in C$. Note that such r_t exists because C is closed, and that $r_t = t$ if $\text{lh}(t) \in C$, and $r_t \subsetneq t$ otherwise.

Next, consider $t \notin \mathcal{T}_{\mathcal{Y}}$, and let $u_t \in \partial \mathcal{T}_{\mathcal{Y}}$ and $v_t \in <\kappa^2$ such that $t = u_t \hat{\ } v_t$.

- (a) If $\text{lh}(u_t) \in S \setminus S'$, then $\text{lh}(u_t) \notin C$. Let $r_t \subsetneq u_t$ be maximal such that $\text{lh}(r_t) \in C$. Note that for every $r_t \subseteq v \subsetneq u_t$, $\varphi(v) = r_t$. Since $r_t \in \mathcal{T}_{\mathcal{Y}}$, let $y_{u_t} = z_{r_t} \in \mathcal{Y}_{S'}$, and set $\varphi(t) = y_{u_t} \upharpoonright \text{lh}(t)$.
- (b) If $\text{lh}(u_t) \notin S \Delta S'$, set $\varphi(t) = t$.
- (c) If $\text{lh}(u_t) \in S' \setminus S$, then $\text{lh}(u_t) \notin C$. Let $r_t \subsetneq u_t$ be maximal such that $\text{lh}(r_t) \in C$. Let $y_{u_t} = z'_{r_t} \notin \mathcal{Y}_{S'}$, and set $\varphi(t) = y_{u_t} \upharpoonright \text{lh}(t)$.

First, we show that the function φ is monotone. Let $s, t \in <\kappa^2$ such that $s \subseteq t$.

First assume that $s, t \in \mathcal{T}_{\mathcal{Y}}$. Note that $\varphi(s) \subseteq s$ and $\varphi(t) \subseteq t$ hold by definition of φ . If $\text{lh}(t) \in C$, then $\varphi(s) \subseteq s \subseteq t = \varphi(t)$; if $\text{lh}(t) \notin C$, by maximality of r_t we have $s \subseteq r_t$, so $\varphi(s) \subseteq s \subseteq r_t = \varphi(t)$.

Next, suppose $s, t \notin \mathcal{T}_{\mathcal{Y}}$ and consider $u_s, u_t \in \partial \mathcal{T}_{\mathcal{Y}}$. Since $s \subseteq t$ we have $u_s = u_t$. If $\text{lh}(u_s) \notin S \Delta S'$ it is immediate that $\varphi(s) = s \subseteq t = \varphi(t)$, otherwise $\varphi(s) = y_{u_s} \upharpoonright \text{lh}(s) \subseteq y_{u_t} \upharpoonright \text{lh}(t) = \varphi(t)$.

Finally, suppose that $s \in \mathcal{T}_{\mathcal{Y}}$ and $t \notin \mathcal{T}_{\mathcal{Y}}$. Again we consider $u_t \in \partial \mathcal{T}_{\mathcal{Y}}$. If $\text{lh}(u_t) \notin S \Delta S'$ then $\varphi(s) \subseteq s \subseteq t = \varphi(t)$, otherwise, consider r_t and notice that by its maximality we have $s \subseteq r_t$, so $\varphi(s) \subseteq s \subseteq r_t \subsetneq \varphi(t)$ as desired.

Next, to see that φ is continuous, first observe that $\{N_u \mid u \in \partial\mathcal{T}_Y\}$ is a clopen partition of $\kappa^2 \setminus \mathcal{Y}$. Let $x \in \kappa^2$. If $x \in \mathcal{Y}$, using that φ is monotone and C is unbounded, we have that $\bigcup_{i < \kappa} \varphi(x \upharpoonright i) = \bigcup_{i \in C} \varphi(x \upharpoonright i) = \bigcup_{i \in C} x \upharpoonright i$. If $x \in N_u$ for some $u \in \partial\mathcal{T}$, then $\varphi(x \upharpoonright i) = y_u \upharpoonright i$ for every $i < \kappa$. In both cases, it is clear that $\text{lh}(\bigcup_{i < \kappa} \varphi(x \upharpoonright i)) = \kappa$.

Let $f = f_\varphi: \kappa^2 \rightarrow \kappa^2$ be the function induced by φ . Then, f is continuous. It remains to show that $\mathcal{Y}_S = f^{-1}(\mathcal{Y}_{S'})$. To see this, suppose that $x \in \kappa^2$.

First, assume that $x \in \mathcal{Y}_S$. If $x \in \mathcal{Y}_0$, then $x \upharpoonright i \in \mathcal{T}_Y$ for every $i < \kappa$ and since for every $i < \kappa$ we have $\varphi(x \upharpoonright i) \subseteq x \upharpoonright i$ and φ is continuous, $f(x) = x \in \mathcal{Y}_0 \subseteq \mathcal{Y}_{S'}$. If $x \notin \mathcal{Y}_0$, then $x \upharpoonright i \in \partial\mathcal{T}_Y$ for some $i \in S$ and by (a) we get that $f(x) = y_{u_{x \upharpoonright i}} \in \mathcal{Y}_{S'}$.

Assume now that $x \notin \mathcal{Y}_S$. If $x \in \mathcal{Y}_1$, then $x \upharpoonright i \in \mathcal{T}_Y$ for every $i < \kappa$ and again we have $f(x) = x \in \mathcal{Y}_1$, thus $f(x) \notin \mathcal{Y}_S$. If $x \notin \mathcal{Y}_1$, then $x \upharpoonright i \in \partial\mathcal{T}_Y$ for some $i < \kappa$. Clearly $i \notin S$ because $x \notin \mathcal{Y}_S$. Then, either $i \in S' \setminus S$ and by (c) we get that $f(x) = y_{u_{x \upharpoonright i}} \notin \mathcal{Y}_{S'}$, or $i \notin S \cup S'$ and by (b) we get that $f(x) = x \notin \mathcal{Y}_{S'}$.

The case $\mathcal{Y}_{S'} \leq_W \mathcal{Y}_S$ is carried out similarly, allowing us to conclude that $\mathcal{Y}_S \equiv_W \mathcal{Y}_{S'}$. \square

Corollary 4.3.56. *Let $S \subseteq \kappa$ such that $S \cap \text{cof}_\omega^\kappa$ is non-stationary in κ . Then, $\mathcal{Y}_S \equiv_W \mathcal{Y}_0$.*

Proof. Apply Proposition 4.3.55 with $S' = \emptyset$. \square

In light of the results of Section 4.3.3 concerning \mathcal{Y}_0 , where we established that it is minimal above the κ^+ - Π_1^0 -difference hierarchy (Theorem 4.3.48), it is natural to ask whether the sets \mathcal{Y}_S considered in this section are likewise minimal above the κ^+ - Π_1^0 -difference hierarchy. In Section 4.3.6, using some rather technical arguments, we demonstrate that this is indeed the case when $\kappa = \omega_1$.

We now introduce a technical result from which we will derive, as a corollary, the ill-foundedness of the Wadge hierarchy on κ^+ - Δ_2^0 . Note that the proof does not use AC, as $\text{AC}_\kappa(\kappa^2)$ suffices.

Proposition 4.3.57. *Let $2 < \alpha \leq \kappa$, and Γ be a boldface pointclass closed under unions⁴ of size $\leq \alpha$ and such that $\Gamma \subseteq \kappa^+$ - Δ_1^0 . Assume there exists a family $\{Y_i \mid i < \alpha\} \subseteq \Gamma(\kappa^2)$ such that for every $i < \alpha$, $Y_i \notin \{\emptyset, \kappa^2\}$, and for every $i, j < \alpha$, $i \neq j$ implies that for every $t \in {}^{<\kappa^2}$, $Y_i \cap N_t \not\leq_W Y_j$, unless $Y_i \cap N_t = \emptyset$ or $Y_i \cap N_t = N_t$. Then, $(\mathcal{P}(\alpha), \subseteq)$ embeds into the Wadge hierarchy on $\Gamma(\kappa^2)$ -sets.*

The conclusion of Proposition 4.3.57 fails $\alpha = 1$ by taking $\Gamma = \{\emptyset\}$ and for $\alpha = 2$ by taking $\Gamma = \{\emptyset, \kappa^2\}$, since there is no family of sets $\{Y_i \mid i < \alpha\}$ as required in the statement.

Proof. Observe that the requirements imposed on $\{Y_i \mid i < \alpha\}$ already ensure that $\{Y_i \mid i < \alpha\}$ forms an antichain (it suffices to take $t = \emptyset$, since $Y_i \notin \{\emptyset, \kappa^2\}$ for all $i < \alpha$).

⁴The proof shows that Proposition 4.3.57 holds assuming that Γ is a boldface pointclass closed under disjoint unions of size $\leq \alpha$ over clopen partitions. It is easy to see that κ^+ - Σ_β^0 , κ^+ - Π_β^0 and κ^+ - Δ_β^0 have this property for all $1 \leq \beta < \kappa^+$.

They also guarantee that $Y_i \notin \kappa^+-\Delta_1^0$ for every $i < \alpha$. In fact, assume towards a contradiction that there exists $i_0 < \alpha$ with $Y_{i_0} \in \kappa^+-\Delta_1^0$, and let $j \neq i_0$. Choose $y \in Y_j$ and $z \notin Y_j$; then the map $f: \kappa^2 \rightarrow \kappa^2$ given by $f(x) = y$ if $x \in Y_{i_0}$ and $f(x) = z$ otherwise is a continuous reduction of Y_{i_0} to Y_j , contradicting our assumptions.

Similarly, we obtain that $Y_i \notin \kappa^+-\Sigma_1^0$ for all $i < \alpha$. Suppose instead that there is some $i_0 < \alpha$ with $Y_{i_0} \in \kappa^+-\Sigma_1^0$, and choose two distinct indices $j, j' \neq i_0$. Since $Y_{i_0} \not\leq_W Y_j$ and $Y_{i_0} \not\leq_W Y_{j'}$, Fact 4.1.3 implies that both Y_j and $Y_{j'}$ are closed. But then we would have $Y_j \leq_W Y_{j'}$, contradicting our hypothesis.

For every $I \subseteq \alpha$, let $Y_I = \bigcup_{i \in I} 0^{(i)} \cap 1 \cap Y_i$. Using that $Y_i \in \Gamma(\kappa^2)$, we get that $t \cap Y_i \in \Gamma$ for every $t \in {}^{<\kappa^2}$ and since Γ is closed under unions of size at most α , $Y_I \in \Gamma(\kappa^2)$ for every $I \in \mathcal{P}(\alpha)$.

Claim 4.3.57.1. If $I \subseteq I' \subseteq \alpha$, then $Y_I \leq_W Y_{I'}$.

Proof of the Claim. It suffices to consider the following winning strategy for **II** in $G_W(Y_I, Y_{I'})$:

As long as Player **I** plays 0, Player **II** also plays 0. Suppose that at some stage $i + 1$ of the game, Player **I** plays 1; then, Player **II** responds with a 1, and next proceeds as follows: if $i \in I \cup (\alpha \setminus I')$, then **II** keeps mirroring **I**'s moves; if $i \in I' \setminus I$, then **II** selects some $y_i \in 0^{(i)} \cap 1 \cap (\kappa^2 \setminus Y_i)$ (which is possible because $Y_i \neq \kappa^2$) and follows this element until the game ends. \square

Claim 4.3.57.2. If $I \not\subseteq I'$ are subsets of α , then $Y_I \not\leq_W Y_{I'}$.

Proof of the Claim. Suppose, towards a contradiction, that Player **II** has a winning strategy in $G_W(Y_I, Y_{I'})$. Consider the particular play in which **I** first plays $0^{(i)} \cap 1$ for some $i \in I \setminus I'$, and then continues with some $y \in \partial Y_i \cap Y_i$. Note that $\partial Y_i \cap Y_i \neq \emptyset$ because $Y_i \notin \kappa^+-\Sigma_1^0$. Since **II** is assumed to have a winning strategy in this game, in this run she must at some point play a sequence of the form $0^{(j)} \cap 1$ for some $j \in I'$; indeed, if either $j \notin I'$ or **II** only ever played 0's, then the resulting point would lie outside $Y_{I'}$, and **II** would lose. Note in particular that $i \neq j$. Let $t \in {}^{<\kappa^2}$ be the initial segment of y that **I** has played up to the moment when **II produces the sequence $0^{(j)} \cap 1$ (if $j < i$, simply take $t = \emptyset$). Then the supposed winning strategy for **II** in $G_W(Y_I, Y_{I'})$ yields a winning strategy for **II** in the game $G_W(Y_i \cap \mathbf{N}_t, Y_j)$.**

We conclude that $Y_i \cap \mathbf{N}_t \leq_W Y_j$ contradicts our hypotheses on Y_i and Y_j : indeed, $Y_i \cap \mathbf{N}_t$ is neither \emptyset nor all of \mathbf{N}_t , since $y \in \partial Y_i \cap \mathbf{N}_t = \partial(Y_i \cap \mathbf{N}_t) \neq \emptyset$. \square

\square

Corollary 4.3.58. *The Wadge hierarchy on $\kappa^+-\Delta_2^0$ subsets of κ^2 is ill-founded.*

Proof. Using Corollary 4.3.54, let $(S_i)_{i < \omega}$ be a sequence of subsets of κ such that for all $i, j < \kappa$, if $i \neq j$ then $S_i \triangle S_j$ is stationary. By Remark 4.3.50 we can assume without loss of generality that $S_i \subseteq \text{cof}_\omega^\kappa$ for every $i < \omega$.

By Definition of \mathcal{Y}_{S_i} (see (4.3.1)), $\mathcal{Y}_{S_i} \notin \{\emptyset, \kappa^2\}$ for every $i < \omega$. Moreover, one can easily verify that $\mathcal{Y}_{S_i} \cap \mathbf{N}_t \notin \{\emptyset, \mathbf{N}_t\}$ if and only if $t \in \mathcal{T}_y$.

We want to show that for any two subsets $S, S' \subseteq \kappa$ such that $S\Delta S'$ is stationary, and for every $t \in \mathcal{T}_y$, we have $\mathcal{Y}_S \cap \mathbf{N}_t \not\leq_W \mathcal{Y}_{S'}$. Then, the desired conclusion follows by applying Proposition 4.3.57 to $\alpha = \omega < \kappa$, $Y_i = \mathcal{Y}_{S_i}$, and $\Gamma = \kappa^+ \text{-}\Delta_2^0$ (we recall that $\kappa^+ \text{-}\Delta_2^0$ is closed under countable unions by [ACRP25, Proposition 4.1]).

Claim 4.3.58.1. For every stationary $S \subseteq \kappa$ and for any $t \in \mathcal{T}_y$, $\mathcal{Y}_S \cap \mathbf{N}_t \equiv_W \mathcal{Y}_S$.

Proof of the Claim. To begin with, observe that $\mathcal{Y}_S \cap \mathbf{N}_t \equiv_W \mathcal{Y}_{S_{[\text{lh}(t)]}}$. We recall that

$$\mathcal{Y}_{S_{[\text{lh}(t)]}} = \mathcal{Y}_0 \cup \bigcup \{ \mathbf{N}_s \mid s \in \partial \mathcal{T}_y, \text{lh}(t) + \text{lh}(s) \in S \}.$$

Indeed, $\mathcal{Y}_{S_{[\text{lh}(t)]}} \leq_W \mathcal{Y}_S \cap \mathbf{N}_t$ as witnessed by the function $x \mapsto t \hat{\ } x$ if t has an even number of 1s and by $x \mapsto t \hat{\ } 1 \hat{\ } x$ if t has an odd number of 1s. Moreover, fix $y \notin \mathcal{Y}_{S_{[\text{lh}(t)]}}$, then $\mathcal{Y}_S \cap \mathbf{N}_t \leq_W \mathcal{Y}_{S_{[\text{lh}(t)]}}$. If t has an even number of 1s this is witnessed by the function $f: {}^{<\kappa^2} \rightarrow {}^{<\kappa^2}$ defined by

$$f(x) = \begin{cases} x_{[\text{lh}(t)]} & \text{if } x \in \mathbf{N}_t \\ y & \text{otherwise,} \end{cases}$$

whereas if t has an odd number of 1s, then the reduction is witnessed by the same function where we substitute $x_{[\text{lh}(t)]}$ for $1 \hat{\ } x_{[\text{lh}(t)]}$.

Next, notice that \mathcal{Y}_S and $\mathcal{Y}_{S_{[\gamma]}}$ are in the same Wadge class for any $\gamma < \kappa$ by Proposition 4.3.55. This holds since $S\Delta S_{[\gamma]} \subseteq \gamma \cdot \omega$, since for any $\delta < \kappa$ we have $\delta \in S_{[\gamma]} \Leftrightarrow \gamma + \delta \in S$ and hence for any $\delta \geq \gamma \cdot \omega$, we have $\delta \in S \Leftrightarrow \delta \in S_{[\gamma]}$. We conclude that $\mathcal{Y}_S \cap \mathbf{N}_t \equiv_W \mathcal{Y}_{S_{[\text{lh}(t)]}} \equiv_W \mathcal{Y}_S$. \square

Now, let S and S' are subsets of κ with $S\Delta S'$ stationary. If there was some $t \in \mathcal{T}_y$ such that $\mathcal{Y}_S \cap \mathbf{N}_t \leq_W \mathcal{Y}_{S'}$, then we would have $\mathcal{Y}_S \leq_W \mathcal{Y}_{S'}$, since $\mathcal{Y}_S \cap \mathbf{N}_t \equiv_W \mathcal{Y}_S$ by Claim 4.3.58.1, in contradiction with Proposition 4.3.52. \square

4.3.5 Complete $\kappa^+ \text{-}\Sigma_2^0$ -sets

Fact 2.11.2(4) shows that if Γ is a non-selfdual boldface pointclass, then every κ -complete set for Γ is $\Gamma(\kappa^2)$ -proper. Moreover, assuming $\text{SLO}_W(\Gamma(\kappa^2))$, the converse holds as well (Lemma 2.11.3). Since Corollary 4.3.54 establishes that $\text{SLO}_W(\kappa^+ \text{-}\Delta_2^0)$ fails, we now investigate whether it nonetheless follows that every proper $\kappa^+ \text{-}\Sigma_2^0$ subset of κ^2 must be κ -complete for $\kappa^+ \text{-}\Sigma_2^0$. We will see that this statement is independent of ZFC.

We begin this section by stating the higher-level counterpart of a classical result asserting that, in a perfect Polish space X , every countable dense subset $Q \subseteq X$ is complete for Σ_2^0 , and therefore it is a proper $\Sigma_2^0(X)$ -set (see, for instance, [Kec95, Exercise 8.7] and [Kec95, Exercise 21.17]).

Remark 4.3.59. If $X \subseteq \kappa^2$ is a strongly κ -perfect set and $Q \subseteq X$ has size κ , then Q is codense in X . This is true because $|X \setminus Q| = 2^\kappa$ and for any open $U \subseteq X$, $|U| = 2^\kappa$ by [AMRS23, Lemma 3.2], so $(X \setminus Q) \cap U \neq \emptyset$.

Proposition 4.3.60. *Let $X \subseteq \kappa^2$ be a strongly κ -perfect set. If $Q \subseteq X$ is dense in X , with $|Q| = \kappa$, then Q is κ -Lipschitz-complete for κ^+ - Σ_2^0 .*

Proof. Note that by Remark 4.3.59, Q is codense in X . Let $\mathcal{T}_X \subseteq {}^{<\kappa}2$ be a κ -perfect tree such that $[\mathcal{T}_X] = X$. Let $(q_\alpha)_{\alpha < \kappa}$ be an enumeration of Q without repetitions and take any $A \in \kappa^+$ - Σ_2^0 . We write $A = \bigcup_{\alpha < \kappa} X_\alpha$ for $X_\alpha \in \kappa^+$ - Π_1^0 , and for every $\alpha < \kappa$ we let $\mathcal{T}_\alpha = \mathcal{T}_{X_\alpha}$. To show that $A \leq_{\mathbb{L}} Q$, we want to identify a winning strategy for Player **II** in $G_{\mathbb{L}}(A, Q)$. In order to do so, we also construct a sequence $\langle r_i, \alpha_i, \beta_i, \gamma_i \mid i < \lambda \rangle$ for some $\lambda < \kappa$ with $r_i \in Q$ and $\alpha_i, \beta_i, \gamma_i < \kappa$ such that $\langle \alpha_i \mid i < \kappa \rangle, \langle \beta_i \mid i < \kappa \rangle$ are strictly increasing and continuous with $\alpha_i \leq \beta_i \leq \alpha_j$ for any $i < j < \lambda$, and moreover, $\alpha_i < \beta_i$ when i is a successor ordinal.

At the beginning of the game, **I** will play some $\langle i \rangle$ with $i < \kappa$. If $\langle i \rangle \notin \bigcup_{\alpha < \kappa} \mathcal{T}_\alpha$ then, since $|X| = 2^\kappa$, **II** can pick any $y \in X \setminus Q$ and follow it until the end of the game. Otherwise, let $\langle i \rangle \in \bigcup_{\alpha < \kappa} \mathcal{T}_\alpha$ and γ_0 be the least such that $\langle i \rangle \in \mathcal{T}_{\gamma_0}$. Then, **II** picks any $r_0 \in Q$ such that $r_0(0) \neq q_0(0)$ and answers by playing the first digit $r_0(0)$. We set $\alpha_0 = \beta_0 = 0$. Now, as long as **I** plays nodes $t \in \mathcal{T}_{\gamma_0}$, player **II** will continue enumerating digits of r_0 . If at some stage $\alpha_1 < \kappa$ player **I** exits \mathcal{T}_{γ_0} , then let $\gamma_1 > \gamma_0$ be minimal such that $t \in \mathcal{T}_{\gamma_1}$. Then, consider $r_0 \upharpoonright \alpha_1 \in \mathcal{T}_X$ the sequence constructed by **II** so far. Since \mathcal{T}_X is splitting, there exists some $\beta_1 > \alpha_1$ and some $s \in \mathcal{T}_X$ with $\text{lh}(s) > \beta_1$ such that $r_0 \upharpoonright \alpha_1 \subseteq s$ and $s(\beta_1) \neq q_1(\beta_1)$. By density of Q in X , there exists $r_1 \in Q$ such that $s \subseteq r_1$. Then, Player **II** will play $r_1(\delta)$ for $\alpha_1 \leq \delta < \beta_1$, independently of the moves of player **I**. At level β_1 , if **I** has played $t \notin \bigcup_{\alpha < \kappa} \mathcal{T}_\alpha$, then by codensity of Q in X and by strongly κ -perfectness of X , **II** can play any sequence $r \supseteq r_1 \upharpoonright \text{lh}(t)$ such that $r \in X \setminus Q$ and follow it until the end game. Otherwise, player **II** will repeat the process described above.

Now, suppose $r_i \in Q, \alpha_i, \beta_i, \gamma_i < \kappa$ have been defined, so that the game is at stage β_i and **II** has played the sequence $r_i \upharpoonright \beta_i$. Now, as long as **I** plays nodes $t \in \mathcal{T}_{\gamma_i}$, player **II** will continue enumerating r_i . If at some stage $\alpha_{i+1} < \kappa$ player **I** exits \mathcal{T}_{γ_i} , then let $\gamma_{i+1} > \gamma_i$ be minimal such that $t \in \mathcal{T}_{\gamma_{i+1}}$. Then, consider $r_i \upharpoonright \alpha_{i+1} \in \mathcal{T}_X$ the sequence constructed by **II** so far. Since \mathcal{T}_X is splitting, there exists some $\beta_{i+1} > \alpha_{i+1}$ and some $s \in \mathcal{T}_X$ with $\text{lh}(s) > \beta_{i+1}$, such that $r_i \upharpoonright \alpha_{i+1} \subseteq s$ and $s(\beta_{i+1}) \neq q_{i+1}(\beta_{i+1})$. By density of Q in X , there exists $r_{i+1} \in Q$ such that $s \subseteq r_{i+1}$. Then, Player **II** will be playing $r_{i+1}(\delta)$ for $\alpha_{i+1} \leq \delta < \beta_{i+1}$, independently of the moves of player **I**. At level β_{i+1} , if **I** has played $t \notin \bigcup_{\alpha < \kappa} \mathcal{T}_\alpha$, then by codensity of Q in X and by strongly κ -perfectness of X , **II** can play any sequence $r \supseteq r_{i+1} \upharpoonright \text{lh}(t)$ such that $r \in X \setminus Q$. Otherwise, the strategy for player **II** is defined as above.

Finally, assume that i is a limit ordinal, and suppose $r_j \in Q$, and $\alpha_j, \beta_j, \gamma_j < \kappa$ have been defined for every $j < i$. Then, we set $\alpha_i = \sup_{j < i} \alpha_j = \beta_i = \sup_{j < i} \beta_j$. Consider $s = \bigcup_{j < i} r_j \upharpoonright \beta_j$. $s \in \mathcal{T}_X$ because \mathcal{T}_X is $< \kappa$ -closed. If **I** has played $t \notin \bigcup_{\alpha < \kappa} \mathcal{T}_\alpha$, then by codensity of Q in X and by strongly κ -perfectness of X , **II** can

play any sequence $r \supseteq s \upharpoonright \text{lh}(t)$ such that $r \in X \setminus Q$. Otherwise, let $\gamma_i \geq \sup_{j < i} \gamma_j$ be the least such that $t \in \mathcal{T}_{\gamma_i}$. By density of Q in X , there exists $r_i \in Q$ such that $s \subseteq r_i$.

Let $x \in {}^\kappa 2$ and $r = \bigcup_{i < \kappa} r_i \upharpoonright \beta_i \in X$ be the sequences built by **I** and **II** respectively at the end of the game. Notice that there are two cases for player **I**: either he has played $x \notin A$ and then $r \in X \setminus Q$, or there exists some $\alpha < \kappa$ such that $x \in [\mathcal{T}_\alpha]$ and then $r \in Q$. So, in both cases, **II** has a winning strategy.

To see that at the end of the game Player **I** must either get a sequence $x \notin A$ or there exists some $\alpha < \kappa$ such that $x \in [\mathcal{T}_\alpha]$, assume towards a contradiction that **I** keeps moving between the trees \mathcal{T}_α 's switching κ -many times along the game. Then, the strategy for player **II** we described would lead **II** to build a sequence $r \in Q$ such that $r \neq q_\alpha$ for every $\alpha < \kappa$, a contradiction. \square

Remark 4.3.61. By applying arguments analogous to those used in Proposition 4.3.60, one can also establish the following: if $X \subseteq {}^\kappa 2$ is superclosed and $Q \in \kappa^+ \text{-}\Sigma_2^0(X)$ is both dense and codense in X , then Q is κ -complete for $\kappa^+ \text{-}\Sigma_2^0$. Observe that this latter statement is actually stronger than Proposition 4.3.60. Indeed, every strongly κ -perfect set is, in particular, superclosed, and by Remark 4.3.59 any dense subset of X of cardinality κ is automatically codense in X .

The following proposition is not essential but may still be of interest.

Proposition 4.3.62. *Let $X \subseteq {}^\kappa 2$ be closed, κ -Baire and without isolated points. For every $B \subseteq X$, if B is dense and κ -meager, then $B \notin \kappa^+ \text{-}\Delta_2^0$.*

Proof. Towards a contradiction, assume that $B \in \kappa^+ \text{-}\Pi_2^0$. Then, $B \in \kappa^+ \text{-}\Pi_2^0(X)$, so $B = \bigcap_{\alpha < \kappa} U_\alpha$ for some open dense sets $U_\alpha \subseteq X$. For every $\alpha < \kappa$, $X \setminus U_\alpha$ is closed nowhere dense and $X \setminus B = \bigcup_{\alpha < \kappa} (X \setminus U_\alpha)$. On the other hand, since B is κ -meager, there are some nowhere dense sets E_β for $\beta < \kappa$ such that $B = \bigcup_{\beta < \kappa} E_\beta$. We conclude that $X = B \cup X \setminus B$ is the union of κ -many nowhere dense sets, hence X is κ -meager in itself, a contradiction. \square

The following results shows that it is consistent with ZFC that every proper $\kappa^+ \text{-}\Sigma_2^0$ -set is κ -complete for $\kappa^+ \text{-}\Sigma_2^0$.

Theorem 4.3.63. *It is consistent with ZFC+ “there is an inaccessible cardinal” that every proper $\kappa^+ \text{-}\Sigma_2^0$ -set is κ -complete for $\kappa^+ \text{-}\Sigma_2^0$.*

Proof. Let $\mathbb{R}_\kappa = \{x \in {}^\kappa 2 \mid \forall \alpha < \kappa \exists \beta \geq \alpha (x(\beta) \neq 0)\}$ and $\mathbb{Q}_\kappa = {}^\kappa 2 \setminus \mathbb{R}_\kappa$. Given $X, Y \subseteq {}^\kappa \kappa$ such that $X \cap Y = \emptyset$, a generalization of the Kechris-Louveau-Woodin dichotomy by Schlicht and Sziraki [SS23, Theorem 1.7] says that either:

- (1) there is $A \in \kappa^+ \text{-}\Sigma_2^0({}^\kappa \kappa)$ such that $X \subseteq A$ and $Y \cap A = \emptyset$, or
- (2) there is a homeomorphism f from ${}^\kappa 2$ onto a closed subset of ${}^\kappa \kappa$ such that $f(\mathbb{R}_\kappa) \subseteq X$ and $f(\mathbb{Q}_\kappa) \subseteq Y$.

Apply the above dichotomy to any $Y \in \kappa^+ \text{-}\Sigma_2^0$ and to $X = {}^\kappa 2 \setminus Y$. In case (1), we get a set $A \in \kappa^+ \text{-}\Sigma_2^0({}^\kappa \kappa)$ such that $X \subseteq A$ and $Y \cap A = \emptyset$, then $A' = A \cap {}^\kappa 2 \in \kappa^+ \text{-}\Sigma_2^0$ and we still have that $X \subseteq A'$ and $Y \cap A' = \emptyset$, allowing us to conclude that $Y \in \kappa^+ \text{-}\Delta_2^0$. In case (2), we get a continuous function $f: {}^\kappa 2 \rightarrow {}^\kappa 2$ such that $\mathbb{Q}_\kappa = f^{-1}(Y)$. Since \mathbb{Q}_κ is dense in ${}^\kappa 2$ and $|\mathbb{Q}_\kappa| = \kappa$, by Proposition 4.3.60, Y is κ -complete for $\kappa^+ \text{-}\Sigma_2^0$.

In conclusion, we have that for every $Y \in \kappa^+ \text{-}\Sigma_2^0$, if $Y \in \kappa^+ \text{-}\Sigma_2^0 \setminus \kappa^+ \text{-}\Pi_2^0$ then Y is κ -complete for $\kappa^+ \text{-}\Sigma_2^0$. \square

We now aim to exhibit a model of ZFC in which there is a proper $\kappa^+ \text{-}\Sigma_2^0$ -set that fails to be κ -complete for $\kappa^+ \text{-}\Sigma_2^0$. The next proposition is an initial result aiming at providing insights into what such a set ought to be (see also Corollary 4.3.65).

Proposition 4.3.64. *Let $X \subseteq {}^\kappa 2$ be a closed set. If there exists $A \subseteq X$ such that A is a κ -complete set for $\kappa^+ \text{-}\Sigma_2^0$, then X contains a strongly κ -perfect subset.*

Proof. Let $Q \subseteq {}^\kappa 2$ be dense in ${}^\kappa 2$ and such that $|Q| = \kappa$. By Lemma 4.3.60, Q is κ -complete set for $\kappa^+ \text{-}\Sigma_2^0$, hence $Q \leq_W A$. Moreover, Q is codense in ${}^\kappa 2$ by Remark 4.3.59. Let $f: {}^\kappa 2 \rightarrow {}^\kappa 2$ be a continuous function such that $Q = f^{-1}(A)$. Let also $\varphi_f: <{}^\kappa 2 \rightarrow <{}^\kappa 2$ be the approximation of f . We claim that there is a κ -perfect tree $\mathcal{T} \subseteq <{}^\kappa 2$ such that $[\mathcal{T}] \subseteq Q$ and $f \upharpoonright [\mathcal{T}]$ is injective. This is enough to prove that A , and consequently X , has a strongly κ -perfect subset.

We construct a sequence $(u_s)_{s \in <{}^\kappa 2} \subseteq \mathcal{T}_Q$ by recursion on $\text{lh}(s) < \kappa$ as follows. Set $u_\emptyset = \emptyset$. Next, assume $u_s \in \mathcal{T}_Q$ is defined. Since Q is dense and codense in ${}^\kappa 2$, there exist $x_0, x_1 \in {}^\kappa 2$, extensions of u_s , such that $x_0 \in Q$ and $x_1 \notin Q$. Then, $f(x_0) \in A$ and $f(x_1) \notin A$, thus there exists $\alpha < \kappa$, $\alpha > \text{lh}(u_s)$, such that $f(N_{x_0 \upharpoonright \alpha}) \cap f(N_{x_1 \upharpoonright \alpha}) = \emptyset$. Using again that Q is dense, let $x'_1 \in N_{x_1 \upharpoonright \alpha} \cap Q$. Again, $f(N_{x_0 \upharpoonright \alpha}) \cap f(N_{x'_1 \upharpoonright \alpha+1}) = \emptyset$. Let $u_{s \smallfrown 0} = x_0 \upharpoonright \alpha$ and $u_{s \smallfrown 1} = x'_1 \upharpoonright \alpha+1$. Clearly, $u_{s \smallfrown 0}, u_{s \smallfrown 1} \in \mathcal{T}_Q$. If $\text{lh}(s)$ is a limit ordinal, using density of Q , let $x \in N_{\bigcup_{\alpha < \text{lh}(s)} u_{s \upharpoonright \alpha}} \cap Q$ and set $u_s = x \upharpoonright \text{lh}(u_s) \in \mathcal{T}_Q$.

Let $\mathcal{T} = \{t \in <{}^\kappa 2 \mid \exists s \in <{}^\kappa 2 (t \subseteq u_s)\}$. By construction, $\mathcal{T} \subseteq \mathcal{T}_Q$ is $< \kappa$ -closed and splitting, and thus it is strongly κ -perfect. Furthermore, $f \upharpoonright [\mathcal{T}]$ is injective: indeed, for any distinct $x, y \in [\mathcal{T}]$, let $\alpha < \kappa$ be the least ordinal such that $x(\alpha) \neq y(\alpha)$. Then there exist $s_0, s_1 \in <{}^\kappa 2$ with $\text{lh}(s_0) = \text{lh}(s_1)$ such that $x \upharpoonright (\alpha+1) \subseteq u_{s_0}$, $y \upharpoonright (\alpha+1) \subseteq u_{s_1}$, and $u_{s_0} \perp u_{s_1}$. By construction we have $f(N_{u_{s_0}}) \cap f(N_{u_{s_1}}) = \emptyset$, and therefore $f(x) \neq f(y)$. \square

Corollary 4.3.65. *Let $X \subseteq {}^\kappa 2$ be closed and κ -Baire such that X does not contain a strongly κ -perfect subset. If $B \in \kappa^+ \text{-}\Sigma_2^0(X)$ is dense and codense in X , then $B \in \kappa^+ \text{-}\Sigma_2^0 \setminus \kappa^+ \text{-}\Pi_2^0$ but B is not κ -complete for $\kappa^+ \text{-}\Sigma_2^0$.*

Proof. Note that if $B \subseteq X$ is both dense and codense in X , then $B \notin \kappa^+ \text{-}\Delta_2^0$. In fact, it is straightforward to verify that a space is κ -Baire precisely when the intersection of any κ -sized family of dense $\kappa^+ \text{-}\Pi_2^0$ sets is dense. Consequently, if we had $B \in \kappa^+ \text{-}\Delta_2^0$, then both B and $X \setminus B$ would be dense $\kappa^+ \text{-}\Pi_2^0$ sets, but their intersection would be empty, contradicting the assumption that X is κ -Baire.

If B were κ -complete for $\kappa^+ \text{-}\Sigma_2^0$, then by Proposition 4.3.64 the set X would necessarily contain a strongly κ -perfect subset, which contradicts our assumption. \square

By Corollary 4.3.65, the remaining problem is to determine whether there exists a set $X \subseteq {}^\kappa 2$ that is closed and κ -Baire, contains no strongly κ -perfect subset, and admits a subset $B \in \kappa^+ \text{-}\Sigma_2^0(X)$ that is both dense and codense in X . In Theorem 4.3.67 we establish that such sets do exist under the assumption $\mathbf{V} = \mathbf{L}$. Before formulating the theorem, we need some preliminary results concerning trees and topological embeddings.

Consider the function

$$h: {}^\kappa \kappa \rightarrow {}^\kappa 2: x \mapsto \langle 0^{(x(0))} \frown 1 \frown 0^{(x(1))} \frown 1 \frown \dots \frown 0^{(x(\alpha))} \frown 1 \dots \rangle. \quad (4.3.2)$$

It is easy to see that h is a topological embedding as h is bijective, open and continuous on its image. Moreover, $\text{ran}(h)$ is dense in ${}^\kappa 2$ since for every $s \in <{}^\kappa 2$, $s \frown 1^{(\kappa)} \in \text{ran}(h)$.

Lemma 4.3.66. *Let $h: {}^\kappa \kappa \rightarrow {}^\kappa 2$ be topological embedding defined in (4.3.2). Let $\mathcal{T}' \subseteq <{}^\kappa \kappa$ be a tree and set $\mathcal{T} = \mathcal{T}_{h([\mathcal{T}'])} \subseteq <{}^\kappa 2$. Then,*

- (1) *If \mathcal{T}' is pruned then \mathcal{T} is pruned;*
- (2) *If \mathcal{T}' is κ -splitting then \mathcal{T} is splitting;*
- (3) *If \mathcal{T} contains a κ -perfect subtree, then \mathcal{T}' contains a κ -perfect subtree;*
- (4) *if $[\mathcal{T}']$ is κ -Baire, then $[\mathcal{T}] = \text{cl}(h([\mathcal{T}']))$ is κ -Baire.*

Proof. Let $\varphi = \varphi_h: <{}^\kappa \kappa \rightarrow <{}^\kappa 2$ be an approximation function for h . Since h is a homeomorphism on its image, φ is monotone by Proposition 2.7.2, and it preserves incompatibility, i.e., for every $s, t \in <{}^\kappa \kappa$, $s \perp t$ implies $\varphi(s) \perp \varphi(t)$. Therefore, the function φ is such that for every $s, t \in <{}^\kappa \kappa$, $s \subseteq t$ if and only if $\varphi(s) \subseteq \varphi(t)$. Moreover, $\text{ran}(\varphi)$ is dense in $<{}^\kappa 2$, that is, for every $s \in <{}^\kappa 2$ there exists $t \in \text{ran}(\varphi)$ such that $s \subseteq t$ (it is sufficient to take $t = s \frown 1$).

(1) Let $z \in \mathcal{T}$. Since $\text{ran}(\varphi) \cap \mathcal{T}$ is dense in \mathcal{T} , there exists $s \in \text{ran}(\varphi) \cap \mathcal{T}$ such that $z \subseteq s$. Let $t \in \mathcal{T}'$ such that $\varphi(t) = s$. Since \mathcal{T}' is pruned, there exists $x \in [\mathcal{T}']$ such that $t \subseteq x$; in particular, for every $\alpha < \kappa$, $x \upharpoonright \alpha \in \mathcal{T}'$. Then, for every $\alpha < \kappa$, $\varphi(x \upharpoonright \alpha) \in \mathcal{T}$, and for every $\text{lh}(t) \leq \alpha < \kappa$ we also have $z \subseteq s \subseteq \varphi(x \upharpoonright \alpha)$. Then, $z \subseteq \bigcup_{\alpha < \kappa} \varphi(x \upharpoonright \alpha) = h(x)$, so it remains to prove that $h(x) \in [\mathcal{T}]$. This is true since for every $\beta < \kappa$ there exists $\alpha < \kappa$ such that $h(x) \upharpoonright \beta \subseteq \varphi(x \upharpoonright \alpha) \in \mathcal{T}$.

(2) Let $z \in \mathcal{T}$, and consider $s \in \text{ran}(\varphi) \cap \mathcal{T}$ such that $z \subseteq s$. Let $t \in \mathcal{T}'$ such that $\varphi(t) = s$. Since \mathcal{T}' is splitting, there exists $u, v \in \mathcal{T}'$ extending t such that $u \perp v$. Then, $z \subseteq s \subseteq \varphi(u), \varphi(v)$, and $\varphi(u) \perp \varphi(v)$. Thus z is splitting.

(3) Let $\mathcal{S} \subseteq \mathcal{T}$ be a κ -perfect subtree. We claim that $\mathcal{S}' = \varphi^{-1}(\mathcal{S})$ is a κ -perfect subtree of \mathcal{T}' . Clearly, $\varphi^{-1}(\mathcal{S}) \subseteq \mathcal{T}'$. Moreover, $\varphi^{-1}(\mathcal{S})$ is closed under initial segments since \mathcal{S} is.

To see that $\varphi^{-1}(\mathcal{S})$ is splitting, let $z \in \varphi^{-1}(\mathcal{S})$. Then, $\varphi(z) \in \mathcal{S}$ and since \mathcal{S} is splitting, there exists $u, v \in \mathcal{S}$ extending t such that $u \perp v$. We have that $z \subseteq \varphi^{-1}(u)$, $z \subseteq \varphi^{-1}(v)$, and $\varphi^{-1}(u) \perp \varphi^{-1}(v)$.

To see that $\varphi^{-1}(\mathcal{S})$ is $<\kappa$ -closed, let $\gamma < \kappa$ be a limit ordinal and let $(t_i)_{i < \gamma}$ be a strictly increasing sequence in $\varphi^{-1}(\mathcal{S})$. Then, $(\varphi(t_i))_{i < \gamma}$ is in \mathcal{S} and it is

strictly increasing, and since \mathcal{S} is $< \kappa$ -closed, $s = \bigcup_{i < \gamma} f(t_i) \in \mathcal{S}$. Using that $\text{ran}(\varphi) \cap \mathcal{S}$ is dense in \mathcal{S} , pick $u \supseteq s$ such that $u \in \text{ran}(\varphi) \cap \mathcal{S}$. Then, $u = \varphi(t)$ for some $t \in \varphi^{-1}(\mathcal{S})$. We claim that $\bigcup_{i < \gamma} t_i \subseteq t$: if not, then there exists $i < \gamma$ such that either $t \perp t_i$, but then $u \perp \varphi(t_i)$, a contradiction, or $t \subseteq t_i$, but then $u \subseteq \varphi(t_i)$, again a contradiction.

(4) It sufficient to observe that if $[\mathcal{T}']$ is κ -Baire then $h([\mathcal{T}'])$ is κ -Baire because h is homeomorphism hence it preserves open sets, density and intersections. Since every set which has a dense κ -Baire subset is again κ -Baire by Remark 2.3.2, $[\mathcal{T}]$ is κ -Baire too. \square

Given a tree $\mathcal{T} \subseteq {}^{<\kappa}2$, recall that $[\mathcal{T}]$ has no isolated points if and only if \mathcal{T} is splitting and $< \kappa$ -closed (hence, superclosed).

Theorem 4.3.67. *Assume $\mathsf{V} = \mathsf{L}$. Then, there exists $B \subseteq {}^\kappa 2$ such that $B \in \kappa^+ \text{-}\Sigma_2^0 \setminus \kappa^+ \text{-}\Pi_2^0$ and B is not κ -complete for $\kappa^+ \text{-}\Sigma_2^0$.*

Proof. By Corollary 4.3.65, it suffices to find a set $X \subseteq {}^\kappa 2$ that is closed and κ -Baire, contains no strongly κ -perfect subset, and has a subset $B \subseteq X$ that is both dense and codense in X .

By [LMRS16, Propositions 7.1 and 7.2] in L there exists a pruned tree $\mathcal{T}' \subseteq {}^{<\kappa}\kappa$ such that every node in \mathcal{T}' is κ -splitting, \mathcal{T}' does not contain a κ -perfect subtree and $[\mathcal{T}']$ is κ -Baire. Let $h: {}^\kappa \kappa \rightarrow {}^\kappa 2$ be the topological embedding defined in (4.3.2), and set $\mathcal{T} = \mathcal{T}_{h([\mathcal{T}'])} \subseteq {}^{<\kappa}2$. By Lemma 4.3.66, we have that the tree \mathcal{T} is pruned, splitting, it does not contain a κ -perfect subtree and its body is κ -Baire. Therefore, $X = [\mathcal{T}]$ is closed, κ -Baire, and contains no strongly κ -perfect subset. Moreover, the following property holds.

Claim 4.3.67.1. For every $t \in \mathcal{T}$, $|\mathbf{N}_t \cap [\mathcal{T}]| \geq \kappa$.

Proof of the Claim. Let $t \in \mathcal{T}$. Since $\text{ran}(\varphi) \cap \mathcal{T}$ is dense in \mathcal{T} , there exists $u \in \text{ran}(\varphi) \cap \mathcal{T}$ such that $t \subseteq u$. Let $v \in \mathcal{T}'$ such that $\varphi(v) = u$. Using that v is κ -splitting, let $\{v_i \mid i < \kappa\} \subseteq \mathcal{T}'$ be a set of incompatible extensions of v . Since φ is monotone and preserves incompatibility, $\{\varphi(v_i) \mid i < \kappa\}$ is a set of incompatible extensions of u in \mathcal{T} . Using that \mathcal{T} is pruned, for every $i < \kappa$ there exists $x_i \in [\mathcal{T}]$ such that $\varphi(v_i) \subseteq x_i$. We conclude that for every $i < \kappa$, $x_i \in \mathbf{N}_t \cap [\mathcal{T}]$. Moreover, the x_i 's are pairwise distinct since the $\varphi(v_i)$'s are pairwise incompatible. \square

Using $2^{<\kappa} = \kappa$, fix an enumeration $(t_i)_{i < \kappa}$ of the nodes of \mathcal{T} . Proceed by recursion on $i < \kappa$ to choose, for each i , distinct branches $x_i, y_i \in [\mathcal{T}]$ that extend t_i and are different from all previously chosen branches. This can be done because \mathcal{T} is both splitting and pruned, and, using Claim 4.3.67.1, because the set $\{x_j, y_j \mid j < \kappa\}$ constructed at any stage has cardinality strictly less than κ . We define the set $B = \{x_i \mid i < \kappa\}$. By construction, $B \subseteq [\mathcal{T}]$ is dense and codense in $[\mathcal{T}]$. \square

We say that a subset A of ${}^\kappa 2$ has the **strongly κ -perfect set property** if either $|A| \leq \kappa$ or A contains a strongly κ -perfect subset. Recall that when κ is regular, it is consistent that $\kappa^+ \text{-}\Pi_1^0$ has the strongly κ -perfect set property, as shown in [Sch17].

The next theorem is the generalized analogue of a theorem of Wadge (see [And07, Theorem 24]) concerning SLO_W and the Perfect Set Property. We

include it here because its proof relies on the same method used in the proof of Proposition 4.3.64.

Theorem 4.3.68. *Suppose that $\kappa^+ \text{-}\mathbf{\Pi}_1^0$ satisfies the strongly κ -Perfect Set Property. Let $\Gamma \subseteq {}^\kappa 2$ be a boldface pointclass with $\kappa^+ \text{-}\Sigma_2^0 \cup \kappa^+ \text{-}\mathbf{\Pi}_2^0 \subseteq \Gamma$. If Γ fails to have the κ -Perfect Set Property, then $\text{SLO}_W(\Gamma)$ does not hold.*

Proof. We argue by contraposition: assume that $\text{SLO}_W(\Gamma)$ holds, and then prove that Γ satisfies the κ -Perfect Set Property.

Let $A \in \Gamma$ and $G = \{x \in {}^\kappa 2 \mid \forall \alpha < \kappa \exists \beta \geq \alpha (x(\beta) = 0)\}$. The set ${}^\kappa 2 \setminus G = \{x \in {}^\kappa 2 \mid \exists \alpha < \kappa \forall \beta \geq \alpha (x(\beta) = 1)\}$ is dense in ${}^\kappa 2$ and it is of size $2^{<\kappa} = \kappa$. Since ${}^\kappa 2$ is strongly κ -perfect, ${}^\kappa 2 \setminus G$ is κ -complete for $\kappa^+ \text{-}\Sigma_2^0$ by Proposition 4.3.60. As a consequence, G is κ -complete for $\kappa^+ \text{-}\mathbf{\Pi}_2^0$. By $\text{SLO}_W(\Gamma)$, either

- (1) $A \leq_W \neg G$, or else
- (2) $G \leq_W A$.

First, suppose (1) holds. Since $G \in \kappa^+ \text{-}\mathbf{\Pi}_2^0$, then $A \in \kappa^+ \text{-}\Sigma_2^0$, therefore $A = \bigcup_{\alpha < \kappa} C_\alpha$ where $C_\alpha \in \kappa^+ \text{-}\mathbf{\Pi}_1^0$ for every $\alpha < \kappa$. Since $\kappa^+ \text{-}\mathbf{\Pi}_1^0$ has the strongly κ -Perfect Set Property, then for every $\alpha < \kappa$, either $|C_\alpha| \leq \kappa$ or C_α contains a strongly κ -perfect set. If some C_α contains a strongly κ -perfect set, then so does A . Otherwise, each C_α is of size κ hence $|A| = \kappa$.

Suppose now (2) holds, and let $f : {}^\kappa 2 \rightarrow {}^\kappa 2$ be a continuous function such that $G = f^{-1}(A)$. Let also $\varphi_f : {}^{<\kappa} 2 \rightarrow {}^{<\kappa} 2$ be the approximation of f .

We construct a sequence $(u_s)_{s \in {}^{<\kappa} 2} \subseteq \mathcal{T}_G$ by recursion on $\text{lh}(s) < \kappa$ as follows. We set $u_\emptyset = \emptyset$. Assume u_s is defined, since $u_s \cap 0^{(\kappa)} \in G$ and $u_s \cap 1^{(\kappa)} \notin G$, $f(u_s \cap 0^{(\kappa)}) \in A$ and $f(u_s \cap 1^{(\kappa)}) \notin A$, hence there are $i, j < \kappa$ such that $\varphi_f(u_s \cap 0^{(i)}) \perp \varphi_f(u_s \cap 1^{(j)})$. Let $u_{s \cap 0} = u_s \cap 0^{(i)}$ and $u_{s \cap 1} = u_s \cap 1^{(j)} \cap 0$. Clearly, $u_{s \cap 0}, u_{s \cap 1} \in \mathcal{T}_G$. If $\text{lh}(s)$ is a limit ordinal, we define $u_s = \bigcup_{\alpha < \text{lh}(s)} u_{s \upharpoonright \alpha}$, noticing that $u_s \in \mathcal{T}_G$ by definition of $u_{s \upharpoonright \alpha}$ for every $\alpha < \text{lh}(s)$.

For every $s \in {}^{<\kappa} 2$, let $v_s = \varphi_f(u_s)$. Since f reduces G to A , $(v_s)_{s \in {}^{<\kappa} 2} \subseteq \mathcal{T}_A$. Since φ is monotone, for every $s \in {}^{<\kappa} 2$, $v_s \subseteq v_{s \cap 0}, v_{s \cap 1}$ and $v_{s \cap 0} \perp v_{s \cap 1}$. Let $\mathcal{V} = \{v \in {}^{<\kappa} 2 \mid \exists s \in {}^{<\kappa} 2 (v \subseteq v_s)\}$. By construction, $\mathcal{V} \subseteq \mathcal{T}_A$ is $< \kappa$ -closed and splitting, hence $[\mathcal{V}]$ is a strongly κ -perfect subset of A . \square

4.3.6 Minimality in the case of $\omega_1 2$

In this section, we focus exclusively on the case $\kappa = \omega_1$. The results we establish here are rather technical. The ultimate aim of the section is to show that the sets \mathcal{Y}_S introduced in Section 4.3.4 are minimal above the $\kappa^+ \text{-}\mathbf{\Pi}_1^0$ -difference hierarchy (Corollary 4.3.73).

We begin with a result showing that stationary costationary subsets S of ω_1 exhibit a form of self-similarity: one can find clubs C that are arbitrarily thin such that S is isomorphic to $S \cap C$ via the strictly order-preserving enumeration of C . This proof will be useful later, as a similar idea will appear in the proofs of Lemma 4.3.71 and Theorem 4.3.72.

Proposition 4.3.69. *Let $S \subseteq \omega_1$ be a stationary and costationary subset of ω_1 and $(\beta_i)_{i < \omega_1}$ a continuous strictly increasing sequence below ω_1 . Then, there exists a continuous strictly increasing sequence $(\alpha_i)_{i < \omega_1}$ below ω_1 such that:*

- (1) $\alpha_i \geq \beta_i$ for every $i < \omega_1$;
 (2) $i \in S \Leftrightarrow \alpha_i \in S$ for all $i < \omega_1$.

Proof. We recursively define a decreasing sequence of closed unbounded sets $(C_j)_{j < \omega_1}$ as follows. Let $C_0 = \{\delta < \omega_1 \mid \beta_\delta = \delta > 0\}$. For every $j < \omega_1$ we set

$$C_{j+1} = \{\delta < \omega_1 \mid \delta \cap (C_j \cap S) \text{ and } \delta \cap (C_j \setminus S) \text{ are unbounded in } \delta\},$$

$$C_j = \bigcap_{j' < j} C_{j'} \text{ if } j \text{ is a limit ordinal.}$$

Observe that, for every $j < \omega_1$, C_j is closed and unbounded in ω_1 . This is clear when $j = 0$ and also when j is a limit. In the successor case $j + 1$, note that $C_j \cap S$ and $C_j \setminus S$ are both unbounded since S is stationary and costationary in κ , hence their closures $\text{cl}(C_j \cap S)$ and $\text{cl}(C_j \setminus S)$ in ω_1 are closed and unbounded. C_{j+1} is the intersection of these two sets.

We let $C = \Delta_{j < \omega_1} C_j$. Then, C is closed and unbounded. We let $(\delta_\gamma)_{\gamma < \omega_1}$ be a strictly increasing enumeration of C . Note that $\delta_\gamma \in C_{\delta_\gamma}$ for every $\gamma < \omega_1$, indeed if $\delta \in C$, then $\delta \in C_j$ for every $j < \delta$, hence $\delta \in C_\delta$ by continuity.

Claim 4.3.69.1. Every $\delta \in C_j$ is a multiple of ω^j , i.e., $\delta = \omega^j \cdot \eta$ for some $\eta < \omega_1$. In particular, for every $\gamma < \omega_1$ we have $\omega^{\delta_\gamma} = \delta_\gamma$.

Proof. The proof is by induction on $j < \omega_1$. The claim is trivially true for $j = 0$. and, for $j = 1$, note that $C_1 \subseteq \{\delta < \omega_1 \mid \delta \text{ limit}\}$, hence C_1 contains only multiples of ω . For the successor step, consider $j + 1$. By the induction hypothesis, C_j only contains multiples of ω^j . Note that every $\delta \in C_{j+1}$ is a limit of elements of C_j . If $\delta \in C_{j+1}$ then $\delta \in C_j$, so $\delta = \omega^j \cdot \eta$ for some $\eta < \omega_1$. It suffices to show that η is a limit ordinal. Then, $\eta = \omega \cdot \xi$ for some $\xi < \omega_1$, so $\delta = \omega^j \cdot \eta = \omega^{j+1} \cdot \xi$. To see this, suppose that $\eta = \xi + 1$ is a successor ordinal. Then, $\delta = \omega^j \cdot (\xi + 1) = \omega^j \cdot \xi + \omega^j$, but then δ would not be a limit of multiples of ω^j , contradicting that $\delta \in C_{j+1}$. Finally, suppose that j is a limit ordinal and $\delta \in C_j$. By the induction hypothesis we have that δ is a multiple of $\omega^{j'}$ for every $j' < j$ because $C_j \subseteq C_{j'}$. To see that δ is a multiple of ω^j , let ω^ℓ be the least summand in the Cantor normal form of δ . If $\ell < j$ then δ is not a multiple of $\omega^{\ell+1}$ against our assumption, therefore $\ell \geq j$ and we get that δ is a multiple of ω^j as required. \square

By Claim 4.3.69.1, the least ordinal $\delta \in C_j$ is at least ω^j .

Claim 4.3.69.2. Let $\bar{S} \subseteq \kappa$. For every $j < \omega_1$, $\delta \in C_j$, and for every increasing sequence $(\zeta_i)_{i < \omega^j}$ below δ , there exists a strictly increasing continuous cofinal map $i \mapsto \alpha_i$ from ω^j to δ such that for every $i < \omega^j$:

- $i \in \bar{S} \Leftrightarrow \alpha_i \in S$;
- $\alpha_i \geq \zeta_i$.

Proof of the Claim. The proof is by induction on $j < \omega_1$. The case $j = 0$ is trivial, as it is sufficient to choose $\alpha_0 \geq \zeta_0$ such that $0 \in \bar{S} \Leftrightarrow \alpha_0 \in S$, which is possible since S is stationary and costationary in ω_1 .

Successor case. Suppose that $j = k + 1$. Let $(\xi_n)_{n < \omega}$ be a strictly increasing sequence cofinal in δ . Since $\delta \in C_j$, both $C_k \cap S$ and $C_k \setminus S$ are

unbounded in δ . We can thus find a strictly increasing sequence $(\varepsilon_n)_{n \in \omega}$ such that $\varepsilon_n \geq \max\{\xi_n, \zeta_{\omega^k \cdot (n+1)}\}$, $\varepsilon_n \in C_k$, and $\omega^k \cdot (n+1) \in \bar{S} \Leftrightarrow \varepsilon_n \in S$. Note that this implies that $(\varepsilon_n)_{n \in \omega}$ is cofinal in δ . We now proceed by induction on $n < \omega$.

For $n = 0$, we apply the induction hypothesis to \bar{S} , $k < \omega_1$, $\varepsilon_0 \in C_k$, and to the sequence $(\zeta_i)_{i < \omega^k}$. We get a strictly increasing continuous sequence $(\alpha_i)_{i < \omega^k}$ cofinal in ε_0 such that, for all $i < \omega^k$, $\alpha_i \geq \zeta_i$ and $i \in \bar{S} \Leftrightarrow \alpha_i \in S$.

Now, assume $n \geq 1$ and let $(\alpha_i)_{i < \omega^k \cdot n}$ be the sequence defined so far. We apply the induction hypothesis to $\bar{S}_{[\omega^k \cdot n]}$, $k < \omega_1$, $\varepsilon_n \in C_k$, and to the sequence $(\zeta'_i)_{i < \omega^k}$, where $\zeta'_i = \max\{\zeta_{\omega^k \cdot n + i}, \varepsilon_{n-1}\}$. We find a strictly increasing continuous sequence $(\alpha'_i)_{i < \omega^k}$ cofinal in ε_n such that for all $i < \omega^k$, we have $\alpha'_i \geq \zeta'_i$ and $i \in \bar{S}_{[\omega^k \cdot n]} \Leftrightarrow \alpha'_i \in S$. We now define $\alpha_{\omega^k \cdot n} = \varepsilon_{n-1}$ and $\alpha_{\omega^k \cdot n + i} = \alpha'_i$ for $0 < i < \omega^k$. Then, for every $i < \omega^k$ $\alpha_i \geq \zeta_i$ and $i \in \bar{S} \Leftrightarrow \alpha_i \in S$.

Limit case. Let $(j_n)_{n < \omega}$ and $(\xi_n)_{n < \omega}$ be strictly increasing sequences cofinal in j and δ , respectively. Since $\delta \in C_j = \bigcap_{n < \omega} C_{j_n}$, for each $n < \omega$ both $C_{j_n} \cap S$ and $C_{j_n} \setminus S$ are unbounded in δ . We can thus find a strictly increasing sequence $(\varepsilon_n)_{n \in \omega}$ such that $\varepsilon_n \geq \max\{\xi_n, \zeta_{\omega^{j_n}}\}$, $\varepsilon_n \in C_{j_n}$ and $\omega^{j_n} \in \bar{S} \Leftrightarrow \varepsilon_n \in S$. Note that this implies that $(\varepsilon_n)_{n \in \omega}$ is cofinal in δ . We now proceed by induction on $n < \omega$.

When $n = 0$, we apply the induction hypothesis to \bar{S} , $j_0 < \omega_1$, $\varepsilon_0 \in C_{j_0}$, and to the sequence $(\zeta_i)_{i < \omega^{j_0}}$. We get a strictly increasing continuous sequence $(\alpha_i)_{i < \omega^{j_0}}$ cofinal in ε_0 such that, for all $i < \omega^{j_0}$, we have $\alpha_i \geq \zeta_i$ and $i \in \bar{S} \Leftrightarrow \alpha_i \in S$.

Now, assume $n \geq 1$ and let and let $(\alpha_i)_{i < \omega^{j_{n-1}}}$ be the sequence defined so far. We apply the induction hypothesis to $\bar{S}_{[\omega^{j_{n-1}}]}$, $j_n < \omega_1$, $\varepsilon_n \in C_{j_n}$, and to the sequence $(\zeta'_i)_{i < \omega^{j_n}}$, where $\zeta'_i = \max\{\zeta_{\omega^{j_{n-1}} + i}, \varepsilon_{n-1}\}$. We get a strictly increasing continuous sequence $(\alpha'_i)_{i < \omega^{j_n}}$ cofinal in ε_n such that, for all $i < \omega^{j_n}$, we have $\alpha'_i \geq \zeta'_i$ and $i \in \bar{S}_{[\omega^{j_{n-1}}]} \Leftrightarrow \alpha'_i \in S$. We now define $\alpha_{\omega^{j_{n-1}}} = \varepsilon_{n-1}$ and $\alpha_{\omega^{j_{n-1}} + i} = \alpha'_i$ for $0 < i < \omega^{j_n}$. Then, for every $i < \omega^{j_n}$, we have $\alpha_i \geq \zeta_i$ and $i \in \bar{S} \Leftrightarrow \alpha_i \in S$. \square

We are now ready to define $(\alpha_i)_{i < \omega_1}$. Note that for every $\gamma < \omega_1$ we have $\omega^{\delta_\gamma} = \delta_\gamma$ by Claim 4.3.69.1.

First, we set $\alpha_{\delta_\gamma} = \delta_\gamma$ for every $\gamma < \omega_1$. Then, (2) is trivially satisfied and (1) is true because $\beta_{\delta_\gamma} = \delta_\gamma = \alpha_{\delta_\gamma}$ by definition of $C_0 \subseteq C = (\delta_\gamma)_{\gamma < \omega_1}$.

For every $\gamma < \omega_1$, we have $\beta_i < \delta_\gamma$ for every $i < \delta_\gamma$ because $(\beta_i)_{i < \delta_\gamma}$ is continuous strictly increasing and $\beta_{\delta_\gamma} = \delta_\gamma$.

Next, we define the map $(\alpha_i)_{i < \delta_0}$. We apply Claim 4.3.69.2 to $\bar{S} = S$, $j = \delta = \delta_0$ and to the sequence $(\beta_i)_{i < \delta_0}$. We get a continuous strictly increasing cofinal map $i \mapsto \alpha_i$ from δ_0 to δ_0 such that, for every $i < \delta_0$, (1) and (2) are satisfied.

Next, we define the map separately in each interval between successive elements of C , that is, we define $(\alpha_i)_{\delta_\gamma < i < \delta_{\gamma+1}}$ for every $\gamma < \omega_1$.

Fix $\gamma < \omega_1$. We apply Claim 4.3.69.2 to $\bar{S} = S$, $j = \delta = \delta_{\gamma+1}$ and to the sequence $(\beta_i)_{i < \delta_{\gamma+1}}$. We get a continuous strictly increasing cofinal map $i \mapsto \bar{\alpha}_i$ from $\delta_{\gamma+1}$ to $\delta_{\gamma+1}$ such that, for every $i < \delta_{\gamma+1}$, $i \in S \Leftrightarrow \bar{\alpha}_i \in S$ and $\bar{\alpha}_i \geq \beta_i$. Moreover, this map satisfies $\delta_\gamma < \bar{\alpha}_i < \delta_{\gamma+1}$ for $\delta_\gamma < i < \delta_{\gamma+1}$, since $\bar{\alpha}_{\delta_\gamma} \geq \beta_{\delta_\gamma} = \delta_\gamma$. We set $\alpha_i = \bar{\alpha}_i$ for $\delta_\gamma < i < \delta_{\gamma+1}$. Then, $(\alpha_i)_{i < \delta_{\gamma+1}}$ is continuous, cofinal and strictly increasing. \square

Remark 4.3.70. If $\kappa = \omega_2$, then Proposition 4.3.69 does not hold for all stationary subsets of ω_2 . Indeed, consider the set $S = \{\omega_1\} \cup \text{cof}_{\omega}^{\omega_2}$ and an arbitrary strictly increasing sequence of ordinals $(\beta_i)_{i < \omega_2}$ with $\beta_0 \geq \omega_1$, and assume that the statement of the proposition holds, that is, there exists a continuous strictly increasing cofinal map $i \mapsto \alpha_i$ from ω_2 to ω_2 such that for every $i < \omega_2$:

1. $\alpha_i \geq \beta_i$;
2. $i \in S \Leftrightarrow \alpha_i \in S$.

In particular $\omega_1 \in S \Leftrightarrow \alpha_{\omega_1} \in S$. We have $\omega_1 \in S$ by the definition of S , but $\alpha_{\omega_1} > \alpha_0 \geq \beta_0 \geq \omega_1$. Since $\text{cof}(\alpha_{\omega_1}) = \omega_1$, we have $\alpha_{\omega_1} \notin S$, a contradiction.

Next, we prove the minimality of \mathcal{Y}_S above the difference hierarchy for arbitrary subsets S of ω_1 . Note that the construction cannot be carried out level by level as in the proof of Proposition 4.3.47, since one cannot anticipate from below whether an ordinal is in S . Instead, we will fix the height of values of the function on a club and construct the function in between by induction. Each step is done in a countable submodel of some H_θ , allowing us to uniformly bound the values.

We use the notation $\mathcal{M} \preceq \mathcal{N}$ if \mathcal{M} is an elementary submodel of \mathcal{N} . We recall from Section 2.7 that a monotone function $h: {}^{<\kappa}2 \rightarrow {}^{<\kappa}2$ is continuous* if it is continuous and for all increasing sequences $(t_i)_{i < \lambda}$ in ${}^{<\kappa}2$ with $\lambda < \kappa$, $h(\bigcup_{i < \lambda} t_i) = \bigcup_{i < \lambda} h(t_i)$. For any $\mathcal{T} \subseteq {}^{<\kappa}2$ and any strictly monotone function $h: \mathcal{T} \rightarrow {}^{<\kappa}2$, we let h^∂ denote the unique continuous* extension of h to the boundary $\partial\mathcal{T}$.

We now introduce two conditions that will be used in Lemma 4.3.71.

- (A) Suppose S is a stationary and costationary subset of ω_1 . Suppose that $A \leq_W \mathcal{Y}_S$ via a continuous function f induced by a continuous function $\varphi: {}^{<\omega_1}2 \rightarrow {}^{<\omega_1}2$. By Remark 2.7.4 we can assume that φ is continuous*. Fix a regular cardinal $\theta > \omega_1$ and a wellorder $<_{H_\theta}$ of H_θ . We consider H_θ as a structure with \in and $<_{H_\theta}$. Fix ϵ and a continuous \preceq -increasing sequence $(\mathcal{M}_i)_{i < \omega_1}$ of countable elementary substructures of H_θ such that for $\gamma_i = \mathcal{M}_i \cap \omega_1$:
- a) $A, S, f, \varphi \in \mathcal{M}_0$
 - b) $(\mathcal{M}_j)_{j \leq i} \in \mathcal{M}_{i+1}$ for all $i < \omega_1$.
- (B) Let C_0 be the set of all γ_i for $i < \omega_1$. For every $i < \omega_1$, let $C_{i+1} = \{\delta < \omega_1 \mid \delta \cap C_i \cap S \text{ and } \delta \cap C_i \setminus S \text{ are unbounded in } \delta\}$, and $C_i = \bigcap_{j < i} C_j$ if i is a limit ordinal. Finally, let $C = \Delta_{i < \omega_1} C_i$ and $(\delta_\alpha)_{\alpha < \omega_1}$ the strictly increasing enumeration of C .

It follows from Condition (A) that $(\gamma_i)_{i < \omega_1}$ is strictly monotone. For instance, one may choose \mathcal{M}_{i+1} to be the $<_{H_\theta}$ -least elementary substructure of H_θ with these properties and take unions at limits.

Note that in Condition (B) we constructed a closed and unbounded set C from a decreasing sequence of closed unbounded sets $(C_i)_{i < \omega_1}$. The sets C_i defined there are indeed closed and unbounded, because both S and its complement are stationary. Note that the least element above 0 of C_α is at least α , since C_{i+1} removes the least element above 0 of C_i . Moreover, we have

that $\delta_\alpha \in C_{\delta_\alpha}$ for every $\alpha < \omega_1$, indeed if $\delta \in C$, then $\delta \in C_i$ for every $i < \delta$, hence $\delta \in C_\delta$ by continuity of the sequence $(C_i)_{i < \omega_1}$.

We adopt the following notation for the next proof. Let K be the A-HK kernel of ${}^{\omega_1}2$ (see Definition 4.3.43). Let \mathcal{T}_y^* denote the set consisting of \emptyset together with all $t \in \mathcal{T}_y$ that end with 1. For each $t \in {}^{<\omega_1}2$, let t^* be the longest initial segment of t that belongs to \mathcal{T}_y^* .

Lemma 4.3.71. *Suppose that $\omega_1, S, A, f, \varphi$ and $(\mathcal{M}_i)_{i < \omega_1}$ satisfy the above Conditions (A) and (B). Suppose that $\alpha, \varepsilon < \omega_1$ are limit ordinals with $\varepsilon \in C_\alpha$, $r \in \mathcal{T}_y$. Let $i < \omega_1$ such that $\varepsilon = \gamma_i$. Suppose that $u \in \mathcal{T}_K \cap {}^{<\varepsilon}2$ with $u \in \mathcal{M}_i$ and $\bar{S} \in \mathcal{M}_i$ is a subset of α . Then, there exists a continuous* strictly monotone map⁵ $h: (\mathcal{T}_y)_{[r]} \cap {}^{\leq \alpha}2 \rightarrow {}^{\leq \varepsilon}2$ with the following properties:*

- (1) $h(\emptyset) = u$.
- (2) (a) h^∂ maps $(\mathcal{T}_y)_{[r]} \cap {}^{\alpha}2$ to ${}^\varepsilon 2$ and $\partial(\mathcal{T}_y)_{[r]} \cap {}^{\alpha}2$ to ${}^\varepsilon 2$.
 (b) φh^∂ maps $(\mathcal{T}_y)_{[r]} \cap {}^{\alpha}2$ to $\mathcal{T}_y \cap {}^\varepsilon 2$ and $\partial(\mathcal{T}_y)_{[r]} \cap {}^{\alpha}2$ to $\partial\mathcal{T}_y \cap {}^\varepsilon 2$.
- (3) For any $v \in (\mathcal{T}_y)_{[r]} \cap {}^{\leq \alpha}2$:
 (a) $\varphi h(v) \in \mathcal{T}_y \cap {}^{\leq \varepsilon}2$.
 (b) The number of 1s in $\varphi h(v)$ has the same parity as the sum of the numbers of 1s in $\varphi(u)$ and v .
 (c) The number of 1s in $\varphi h(v)$ above $\varphi(u)$ is greater than or equal to the number of 1s in v .
- (4) For any $w \in \partial(\mathcal{T}_y)_{[r]} \cap {}^{<\alpha}2$, $\varphi h^\partial(w) \in \partial\mathcal{T}_y \cap {}^{<\varepsilon}2$. If moreover $w \in {}^{<\alpha}2$ has limit length, then

$$\text{lh}(w) \in \bar{S} \Leftrightarrow \text{lh}(\varphi h^\partial(w)) \in S.$$

There further exists a family $(b_t)_{t \in \mathcal{T}_y^* \cap {}^{<\alpha}2}$ in K such that for all $t \in \mathcal{T}_y \cap {}^{\leq \alpha}2$, $b_{t^*} \in \mathcal{M}_i$ ⁶ and:

- (I) b_{t^*} extends $h(t)$.
- (II) $f(b_{t^*})$ has no 1s above ε .

Suppose that in addition, some $b \in {}^{\omega_1}2$ extending u is given such that $f(b)$ has no 1s above $\text{lh}(u)$, $b \in K \cap A$ if the number of 1s in $\varphi(u)$ is even, $b \in K \setminus A$ otherwise, and $b \in \mathcal{M}_i$. Then, there exists $(b_t)_{t \in \mathcal{T}_y^* \cap {}^{<\alpha}2}$ with $b_\emptyset = b$ in addition to the above properties.⁷

Proof. The proof works by induction on limit ordinals $\alpha < \omega_1$. The statement is clear for $\alpha = 0$.

⁵We recall that $(\mathcal{T}_y)_{[r]} = \{t \mid r \hat{\ } t \in \mathcal{T}_y\}$. Here, r only appears in $(\mathcal{T}_y)_{[r]}$ and this tree equals \mathcal{T}_y , but we have kept the notation $(\mathcal{T}_y)_{[r]}$ to clarify how the lemma is applied.

⁶Note that for $\alpha = \varepsilon$, the family $(b_t)_{t \in \mathcal{T}_y^* \cap {}^{<\alpha}2}$ cannot be an element of \mathcal{M}_i , since its domain is $\mathcal{T}_y^* \cap {}^{<\alpha}2$ and ε is the height of $\mathcal{M}_i \cap \omega_1$.

⁷In fact, we construct a family of b_t such that the set of splitting nodes of their union is isomorphic to $\mathcal{T}_y \cap {}^{<\alpha}2$ and the range of h forms a tree with the same splitting nodes. But the weaker condition stated here suffices for the inductive application of the lemma.

Claim 4.3.71.1. $f(x) \in \mathcal{Y}$ for all $x \in K$.

Proof. Otherwise there would exist some $\alpha < \omega_1$ such that for all y extending $x \upharpoonright \alpha$, $f(y)$ has infinitely many 1s with the first limit of 1s at some $\beta < \omega_1$. If $\beta \in S$, take some $y \in K \setminus A$. This cannot be since $f(y) \notin \mathcal{Y}_S$. If $\beta \notin S$, take some $y \in K \cap A$. This cannot be since $f(y) \in \mathcal{Y}_S$. \square

Claim 4.3.71.2. For every $b \in K \cap A$, let $\gamma < \omega_1$ such that $f(b)(\delta) = 0$ for every $\delta \geq \gamma$. There exists some $\gamma \leq \gamma' < \omega_1$ such that for every $c \in \mathbf{N}_{b \upharpoonright \gamma'} \cap (K \setminus A)$ the following conditions are satisfied:

- (i) $f(c) \in \mathbf{N}_{f(b) \upharpoonright \gamma}$,
- (ii) $f(c)(\beta) = 1$ for some $\beta > \gamma$.

Proof. Let b, γ be as in the statement. Since f is continuous, there exists some $\gamma' \geq \gamma$ such that for every $c \in \mathbf{N}_{b \upharpoonright \gamma'}$, $f(c) \in \mathbf{N}_{f(b) \upharpoonright \gamma}$. Then, any $c \in \mathbf{N}_{b \upharpoonright \gamma'} \cap (K \setminus A)$ satisfies condition (i). For condition (ii), note that $f(c) \notin \mathcal{Y}_0$ since $c \notin A$. Hence $f(b) \neq f(c)$. Since $f(b) = (f(b) \upharpoonright \gamma) \frown 0^{(\omega_1)}$ by definition of γ , $f(c)$ must have an additional 1 at or above γ . \square

Base case. Suppose that $\alpha = \omega$. Let $(\varepsilon_n)_{n < \omega}$ be a strictly increasing cofinal sequence of limit ordinals below ε . The construction of $h \upharpoonright < \omega^2$ will proceed in ω steps. We will assume that every step of the construction takes place in \mathcal{M}_i so that all ordinals are below ε . Finally, h is extended continuously to ω^2 .

We construct b_t for $t \in \mathcal{T}_y^* \cap < \alpha^2$ and $h(t)$ for $t \in (\mathcal{T}_y)_{[r]} \cap < \alpha^2$ by induction on $k(t)$, where $k(t)$ denotes the number of 1s in t . We assume that the number of 1s in $\varphi(u)$ is even. The construction for the odd case is similar.

The first step defines $h(t)$ for all $t \in < \omega^2$ with $k(t) = 0$ and in addition, those t with an extra 1 at the end.

First let $h(\emptyset) = u$. Pick any $b_\emptyset \in K \cap A$ extending u . Then, $f(b_\emptyset) \in \mathcal{Y}_0$. Instead, if some b as in the statement is given, then we will choose $b_\emptyset = b$.

Next, pick by induction a strictly increasing sequence of ordinals $(\alpha_\emptyset^n)_{n < \omega}$ and a family $(b_{0^{(n)} \frown 1})_{n < \omega}$ in K . Since φ is continuous, we can pick α_\emptyset^0 such that:

- (i) $\varphi(b_\emptyset \upharpoonright \alpha_\emptyset^0)$ includes the last 1 of $f(b_\emptyset)$,
- (ii) $\alpha_\emptyset^0 \geq \varepsilon_0$ and $\text{lh}(\varphi(b_\emptyset \upharpoonright \alpha_\emptyset^0)) \geq \varepsilon_0$,

In each step $n \geq 0$, we choose first $b_{0^{(n)} \frown 1}$ and then α_\emptyset^{n+1} such that:

- (iii) $b_{0^{(n)} \frown 1} \in K \setminus A$ and $b_\emptyset \wedge b_{0^{(n)} \frown 1} \supseteq u$,
- (iv) $(b_\emptyset \wedge b_{0^{(n)} \frown 1})_{n < \omega}$ is strictly increasing, and $\text{lh}(b_\emptyset \wedge b_{0^{(n)} \frown 1}) \geq \alpha_\emptyset^n$,
- (v) $\alpha_\emptyset^{n+1} \geq \alpha_\emptyset^n + 2$,
- (vi) $\alpha_\emptyset^{n+1} \geq \varepsilon_{n+1}$,
- (vii) $\text{lh}(\varphi(b_\emptyset \upharpoonright \alpha_\emptyset^{n+1})) \geq \varepsilon_{n+1}$,
- (viii) $\varphi(b_{0^{(n)} \frown 1} \upharpoonright \alpha_\emptyset^{n+1})$ includes the last 1 of $f(b_{0^{(n)} \frown 1})$.

Condition (iii) can be achieved since $b_\emptyset \supseteq u$. Choosing $b_{0^{(n)} \smallfrown 1}$ as in (iii) and (iv) works by Claim 4.3.71.2. (v)-(viii) hold if α_\emptyset^{n+1} is chosen sufficiently large, where (vii) and (viii) use continuity of φ .

We can assume that $\text{lh}(b_\emptyset \wedge b_{0^{(n)} \smallfrown 1}) = \alpha_\emptyset^n$ in (iv) by increasing α_\emptyset^n in the n th stage of the induction, if necessary.

Define $h(0^{(n)}) = b_\emptyset \upharpoonright \alpha_\emptyset^n$ and $h(0^{(n)} \smallfrown 1) = b_{0^{(n)} \smallfrown 1} \upharpoonright \alpha_\emptyset^{n+1}$ for $n < \omega$. h is strictly monotone, since $\alpha_\emptyset^n + 1 < \alpha_\emptyset^{n+1}$ for all $n < \omega$.

(1) is clear since we chose $h(\emptyset) = u$. (2) is only relevant later.

To see that (3) (a) holds, note that $\varphi h(0^{(n)}), \varphi h(0^{(n)} \smallfrown 1) \in {}^{<\varepsilon}2$, since the construction steps take place in \mathcal{M}_i and $\mathcal{M}_i \cap \omega_1 = \varepsilon$. Moreover, these sequences are in \mathcal{T}_Y , since they are initial segments of $f(b_\emptyset)$ and $f(b_{0^{(n)} \smallfrown 1})$, respectively. Both values are in Y by Claim 4.3.71.1.

We now show (3) (b). To see this for $0^{(n)}$, note that $\varphi h(0^{(n)}) = \varphi(b_\emptyset \upharpoonright \alpha_\emptyset^n)$ has the same 1s as $f(b_\emptyset)$ by (i). Note that $f(b_\emptyset) \in \mathcal{Y}_0$ by Claim 4.3.71.1, so it has an even number of 1s just like $0^{(n)}$, since we assume that $\varphi(u)$ has an even number of 1s. To see this for $0^{(n)} \smallfrown 1$, note that $\varphi h(0^{(n)} \smallfrown 1) = \varphi(b_{0^{(n)} \smallfrown 1} \upharpoonright \alpha_\emptyset^{n+1})$ has the same 1s as $f(b_{0^{(n)} \smallfrown 1})$ by (viii). Recall that $b_{0^{(n)} \smallfrown 1} \in K \setminus A$ by (iii) and hence $f(b_{0^{(n)} \smallfrown 1}) \in \mathcal{Y}_1$ by Claim 4.3.71.1, so it has an odd number of 1s just like $0^{(n)} \smallfrown 1$. (3) (c) is clear.

(4) is void since there is no limit ordinal below ω .

(I) states that $h(0^{(n)}) \subseteq b_\emptyset$ and $h(0^{(n)} \smallfrown 1) \subseteq b_{0^{(n)} \smallfrown 1}$. This holds by the definition of h . (II) holds since $f(b_\emptyset), f(b_{0^{(n)} \smallfrown 1}) \in \mathcal{M}_i$ and $\mathcal{M}_i \cap \omega_1 = \varepsilon$.

The remaining steps are similar. Suppose that $t \in {}^{<\omega}2$ ends with a 1 and b_t and $h(t) \subseteq b_t$ are defined.

In each of the following steps, pick by induction a strictly increasing sequence $(\alpha_t^n)_{n < \omega}$ and a family $(b_{t \smallfrown 0^{(n)} \smallfrown 1})_{n < \omega}$ in K . Since φ is continuous, we first pick α_t^0 such that:

- (i) $\varphi(b_t \upharpoonright \alpha_t^0)$ includes the last 1 of $f(b_t)$,
- (ii) $\alpha_t^0 \geq \varepsilon_{k(t)}$ and $\text{lh}(\varphi(b_t \upharpoonright \alpha_t^0)) \geq \varepsilon_{k(t)}$,

For any $t \in (\mathcal{T}_Y)_{[r]}$, let t^- denote the largest proper initial segment of t in \mathcal{T}_Y^* . In each step $n \geq 0$, we choose first $b_{t \smallfrown 0^{(n)} \smallfrown 1} \in K$ and then α_t^{n+1} such that:

- (iii) $b_{t \smallfrown 0^{(n)} \smallfrown 1} \in K \setminus A$ if and only if $k(t)$ is even, and $b_{t \smallfrown 0^{(n)} \smallfrown 1} \wedge b_t \supseteq b_t \wedge b_{t^-}$.
- (iv) $(b_t \wedge b_{t \smallfrown 0^{(n)} \smallfrown 1})_{n < \omega}$ is strictly monotone, and $\text{lh}(b_t \wedge b_{t \smallfrown 0^{(n)} \smallfrown 1}) \geq \alpha_t^n$,
- (v) $\alpha_t^{n+1} \geq \alpha_t^n + 2$,
- (vi) $\alpha_t^{n+1} \geq \varepsilon_{n+1+k(t)}$,
- (vii) $\text{lh}(\varphi(b_t \upharpoonright \alpha_t^{n+1})) \geq \varepsilon_{n+1+k(t)}$,
- (viii) $\text{lh}(\varphi(b_{t \smallfrown 0^{(n)} \smallfrown 1} \upharpoonright \alpha_t^{n+1})) \geq \varepsilon_{k(t)}$,
- (ix) $\varphi(b_{t \smallfrown 0^{(n)} \smallfrown 1} \upharpoonright \alpha_t^{n+1})$ includes the last 1 of $f(b_{t \smallfrown 0^{(n)} \smallfrown 1})$.

This works as in the previous step, and we can assume that $\text{lh}(b_t \wedge b_{t \smallfrown 0^{(n)} \smallfrown 1}) = \alpha_t^n$ in (iv) by increasing α_t^n , if necessary.

We use throughout the proof that f reduces A to \mathcal{Y}_S . It follows from Claim 4.3.71.1 and (iii) that $f(b_{t \smallfrown 0^{(n)} \smallfrown 1}) \in \mathcal{Y}_0$ if $k(t)$ is odd, since $b_{t \smallfrown 0^{(n)} \smallfrown 1} \in K \cap A$

and $f(b_{t \frown 0^{(n)} \frown 1}) \in Y \cap \mathcal{Y}_S = \mathcal{Y}_0$. Similarly, $f(b_{t \frown 0^{(n)} \frown 1}) \in \mathcal{Y}_1$ if $k(t)$ is even, since $b_{t \frown 0^{(n)} \frown 1} \in K \setminus A$ and $f(b_{t \frown 0^{(n)} \frown 1}) \in Y \setminus \mathcal{Y}_S = \mathcal{Y}_1$.

Define $h(t \frown 0^{(n)}) = b_t \upharpoonright \alpha_t^n$ and $h(t \frown 0^{(n)} \frown 1) = b_{t \frown 0^{(n)} \frown 1} \upharpoonright \alpha_t^{n+1}$ for each $n < \omega$. h is strictly monotone by (iii), (iv) and (v).

(1) is clear and (2) is only relevant later.

To see that (3) (a) holds, note that $\varphi h(t \frown 0^{(n)})$, $\varphi h(t \frown 0^{(n)} \frown 1) \in {}^{<\varepsilon}2$, since the construction steps take place in \mathcal{M}_i and $\mathcal{M}_i \cap \omega_1 = \varepsilon$. Moreover, these sequences are in \mathcal{T}_Y , since they are initial segments of $f(b_t)$ and $f(b_{t \frown 0^{(n)} \frown 1})$, respectively. Both values are in Y by Claim 4.3.71.1.

We now show (3) (b). To see this for $t \frown 0^{(n)}$, note that $\varphi h(t \frown 0^{(n)}) = \varphi(b_t \upharpoonright \alpha_t^n)$ has the same 1s as $f(b_t)$ by (i). First suppose that $k(t)$ is even. Then, $f(b_t) \in \mathcal{Y}_0$ by Claim 4.3.71.1 and (iii) for b_t , so it has an even number of 1s just like the sum of the parities of $t \frown 0^{(n)}$ and $\varphi(u)$, since we assume $\varphi(u)$ has an even number of 1s. If $k(t)$ is odd, then $f(b_t) \in \mathcal{Y}_1$ and the remaining argument is similar.

To see this for $t \frown 0^{(n)} \frown 1$, note that $\varphi h(t \frown 0^{(n)} \frown 1) = \varphi(b_{t \frown 0^{(n)} \frown 1} \upharpoonright \alpha_t^{n+1})$ has the same 1s as $f(b_{t \frown 0^{(n)} \frown 1})$ by (ix). First suppose that $k(t)$ is even. Then, $f(b_{t \frown 0^{(n)} \frown 1}) \in \mathcal{Y}_1$ by Claim 4.3.71.1 and (iii) for $b_{t \frown 0^{(n)} \frown 1}$, so it has an odd number of 1s just like the sum of the parities of $t \frown 0^{(n)} \frown 1$ and $\varphi(u)$, since we assume $\varphi(u)$ has an even number of 1s. If $k(t)$ is odd, then $f(b_{t \frown 0^{(n)} \frown 1}) \in \mathcal{Y}_0$ and the remaining argument is similar.

For (3) (c), we have by the induction hypothesis that $\varphi h(t)$ above $\varphi(u)$ has at least as many 1s as t . It suffices to show that $\varphi h(t \frown 0^{(n)} \frown 1)$ has at least one more 1 compared to $\varphi h(t)$. Since φh is monotone, it suffices to show that their parities are different. Recall that $\varphi h(t)$ has the same 1s as $f(b_t)$ and $\varphi h(t \frown 0^{(n)} \frown 1)$ has the same 1s as $f(b_{t \frown 0^{(n)} \frown 1})$. If $k(t)$ is even, then $f(b_t) \in \mathcal{Y}_0$ and $f(b_{t \frown 0^{(n)} \frown 1}) \in \mathcal{Y}_1$ as we showed above, so the parities are different. The argument is similar if $k(t)$ is odd.

(4) is void since there are no limits below ω .

Since we assume that t ends with a 1, we have $t^* = t$. (I) states that $h(t \frown 0^{(n)}) \subseteq b_t$ and $h(t \frown 0^{(n)} \frown 1) \subseteq b_{t \frown 0^{(n)} \frown 1}$. This holds by the definition of h . (II) holds since $f(b_t), f(b_{t \frown 0^{(n)} \frown 1}) \in \mathcal{M}_i$ and $\mathcal{M}_i \cap \omega_1 = \varepsilon$.

Finally, we define h as the unique continuous* extension of the previous maps. Since we already checked the conditions for finite sequences, it only remains to verify (2) for h .

For (2), first suppose that $t \in (\mathcal{T}_Y)_{\lfloor r \rfloor} \cap \omega_2$. Since $h(t^* \frown 0^{(n)})$ has length $\alpha_{t^*}^n$ and we chose $\alpha_{t^*}^n \geq \varepsilon_{n+k(t^*)} \geq \varepsilon_n$ by (vi), $h(t)$ has length ε . Since the length of $\varphi h(t^* \frown 0^{(n)}) = \varphi(b_{t^*} \upharpoonright \alpha_{t^*}^n)$ is at least $\varepsilon_{n+k(t^*)} \geq \varepsilon_n$ by (vii), $\varphi h(t)$ has length ε .

Now suppose that $t \in \partial \mathcal{T}_Y \cap \omega_2$. Then, t has unboundedly many initial segments of the form $v \frown 0^{(n)} \frown 1$. For each such v , we have $h(v \frown 0^{(n)} \frown 1) = b_{v \frown 0^{(n)} \frown 1} \upharpoonright \alpha_v^{n+1}$. Since $\alpha_v^{n+1} \geq \varepsilon_{n+1+k(v)} \geq \varepsilon_{n+1}$ by (vi), $h(t)$ has length ε . Note that the length of $\varphi h(v \frown 0^{(n)} \frown 1) = \varphi(b_{v \frown 0^{(n)} \frown 1} \upharpoonright \alpha_v^{n+1})$ is at least $\varepsilon_{k(v)}$ by (viii). The values of $k(v)$ are unbounded and it follows that $\varphi h(t)$ has length ε .

This completes the argument in the case that $\varphi(u)$ has an even number of 1s. If it has an odd number of 1s, then we take elements of $K \cap A$ in (iii) in both $\alpha = 0$ and $\alpha = \omega$ above and switch the parity of $k(t)$ and \mathcal{Y}_0 by \mathcal{Y}_1 in the verification of the conditions. Apart from that, the argument remains the same.

Successor case. Suppose that $\alpha = \lambda + \omega$ for some limit $\lambda > 0$. Since ε is in C_α , it is a limit point of C_λ . Recall that $\varepsilon = \gamma_i$ is the height of $\mathcal{M}_i \cap \omega_1$. Then, i is a limit of $j < i$ such that the height of $\mathcal{M}_j \cap \omega_1$ is in C_λ . Since the sequence $(\mathcal{M}_j)_{j < \omega_1}$ is continuous, $\mathcal{M}_i = \bigcup_{j < i} \mathcal{M}_j$. Since $\bar{S}, u, b \in \mathcal{M}_i$, we can pick some $j_0 < i$ such that $\bar{S}, u, b \in \mathcal{M}_{j_0}$. Let ξ_0 be the height of $\mathcal{M}_{j_0} \cap \omega_1$. If $\lambda \in \bar{S}$, find some $\xi \in C_\lambda \cap S \cap \varepsilon$ with $\xi \geq \xi_0$. If $\lambda \notin \bar{S}$, find some $\xi \in (C_\lambda \setminus S) \cap \varepsilon$ with $\xi \geq \xi_0$. This works since $\varepsilon \in C_\alpha$.

We apply the induction hypothesis to $\lambda, \xi, r, u, S, \bar{S}$, and to b if it is given. We can apply it since for $j < \omega_1$ such that the height of $\mathcal{M}_j \cap \omega_1$ is ξ , we have $S, A, f \in \mathcal{M}_0 \subseteq \mathcal{M}_j$, and $\bar{S}, u, b \in \mathcal{M}_j$ since $j \geq j_0$.

Thus, a function $\bar{h}: (\mathcal{T}_Y)_{[r]} \cap \leq^{\lambda_2} \rightarrow \leq^{\xi_2}$ and a family $(\bar{b}_t)_{t \in \mathcal{T}_Y^* \cap < \lambda_2}$ with (1)-(4) and (I)-(II) exists. This is a statement in H_{ω_2} . We have $\bar{h} \in H_{\omega_2}$, since the domain of \bar{h} is countable, and $(\bar{b}_t)_{t \in \mathcal{T}_Y^* \cap < \lambda_2} \in H_{\omega_2}$. Since $\mathcal{M}_i \preceq H_{\omega_2}$, such \bar{h} and $(\bar{b}_t)_{t \in \mathcal{T}_Y^* \cap < \lambda_2}$ exist in \mathcal{M}_i and the required properties hold in \mathcal{M}_i . Note that the parameters of this statement are $\lambda, \xi, u, S, \bar{S}, A, f, \varphi, \mathcal{M}_j$ and b if it is given. We have $\mathcal{M}_j \in \mathcal{M}_{j+1} \subseteq \mathcal{M}_i$ by our assumption on the models. Note that $\xi \in \mathcal{M}_i$, since ξ is the height of $\mathcal{M}_j \cap \omega_1$, and $\lambda \leq \xi$, since $\xi \in C_\lambda$ and $\lambda > 0$. Since $\mathcal{M}_i \cap \omega_1$ is transitive, $\lambda \in \mathcal{M}_i$. Using $\mathcal{M}_i \preceq H_\theta$ again, \bar{h} and $(\bar{b}_t)_{t \in \mathcal{T}_Y^* \cap < \lambda_2}$ satisfy (1)-(4) and (I)-(II) (in the set-theoretic universe V).

We then have (3) for all $v \in \leq^{\lambda_2}$ and (4) for all $w \in <^{\lambda_2}$. Furthermore, (4) holds for all $w \in \lambda_2$ by (2) for \bar{h}^∂ and the choice of ξ .

Next, we apply the induction hypothesis for ω and ε separately to $r \hat{\ } t$ for each $t \in (\mathcal{T}_Y)_{[r]} \cap \lambda_2$,⁸ and to $u_t = \bar{h}(t), \bar{b} = \bar{b}_{t^*}, S$ and $\bar{S}_{[\lambda]}$. We can apply it to u_t , since $\bar{h}, t \in \mathcal{M}_i$ and hence $u_t \in \mathcal{M}_i$. Moreover $u_t \in T_K$, since $u_t \subseteq \bar{b}_{t^*} \in K$ by (I). We can apply it to $\bar{S}_{[\lambda]} \in \mathcal{M}_i$, since $\lambda \in \mathcal{M}_i$. We now show that we can apply it to \bar{b} . We have $\bar{b} = \bar{b}_{t^*} \in \mathcal{M}_j \subseteq \mathcal{M}_i$ by the induction hypothesis. For the remaining properties, we use (I) and (II) of the induction hypothesis. We have $u_t = \bar{h}(t) \subseteq \bar{b}_{t^*} = \bar{b}$ by (I) and $\varphi \bar{h}(t) = f(\bar{b}) \upharpoonright \xi$ by (2) for \bar{h} . Moreover, $f(\bar{b})$ has no 1s above ξ by (II) for \bar{h} and \bar{b} . Therefore, $\varphi \bar{h}(t)$ and $f(\bar{b})$ have the same 1s, and $f(\bar{b})$ has no 1 above $\text{lh}(u_t)$, using that $\text{lh}(u_t) = \xi$ by (2). Recall that $\bar{b} \in K$ and hence $f(\bar{b}) \in Y$ by Claim 4.3.71.1. If $\bar{b} \in K \cap A$, then $f(\bar{b}) \in \mathcal{Y}_0$ and $\varphi \bar{h}(t) = \varphi(u_t)$ has a finite even number of 1s. Similarly, if $\bar{b} \in K \setminus A$ then $\varphi \bar{h}(t) = \varphi(u_t)$ has a finite odd number of 1s.

These applications of the lemma yield maps $h_t: (\mathcal{T}_Y)_{[r \hat{\ } t]} \cap \leq^{\omega_2} \rightarrow \leq^{\varepsilon_2}$ for each $t \in (\mathcal{T}_Y)_{[r]} \cap \lambda_2$. We combine these with \bar{h} to $h: (\mathcal{T}_Y)_{[r]} \cap \leq^{\alpha_2} \rightarrow \leq^{\varepsilon_2}$ by letting $h(t) = \bar{h}(t)$ for $t \in \leq^{\lambda_2}$ and $h(t \hat{\ } v) = h_t(v)$ for $t \in (\mathcal{T}_Y)_{[r]} \cap \lambda_2$ and $v \in \leq^{\omega_2}$.

The map h is strictly monotone, since all maps in the construction are strictly monotone, and they agree on the values $h(t) = u_t$ for all $t \in \mathcal{T}_Y \cap \lambda_2$.

(1) holds for h since it holds for \bar{h} . To verify (2) for h^∂ , take any $t \hat{\ } v \in \alpha_2$ in $(\mathcal{T}_Y)_{[r]}$ or $\partial(\mathcal{T}_Y)_{[r]}$, where $t \in \lambda_2$ and $v \in \omega_2$. Then, $h^\partial(t \hat{\ } v) = h_t^\partial(v), \varphi h^\partial(t \hat{\ } v) = \varphi h_t^\partial(v) \in \varepsilon_2$ by (2) for h_t^∂ . Now v has finitely many 1's if and only if the same holds for $\varphi h^\partial(t \hat{\ } v) = \varphi h_t^\partial(v)$ by (2) of the claim for h_t^∂ .

(3)(a) for h holds by (3)(a) for h_t for all $t \in (\mathcal{T}_Y)_{[r]} \cap \lambda_2$. For (3)(b), take any $t \hat{\ } v \in (\mathcal{T}_Y)_{[r]} \cap \leq^{\alpha_2}$, where $t \in \lambda_2$ and $v \in \leq^{\omega_2}$. The parity of the number

⁸ t plays the role of r in the lemma. Recall that r may be omitted since $(\mathcal{T}_Y)_{[r]} = \mathcal{T}_Y$ and hence t is not relevant for the maps given by the lemma. However, it is used when h is defined from the maps h_t .

of 1s in $\varphi h(t) = \varphi(u_t)$ is the same as that of the sums of the numbers in t and in $\varphi(u)$ by (3)(b) for \bar{h} . The parity of the number of 1s in $\varphi h(t \hat{\wedge} v) = \varphi h_t(v)$ equals the parity of the sum of the numbers of 1s in $\varphi(u_t)$ and v by (3)(b) for h_t , hence it equals the parity of the sums of the numbers of 1s in $t \hat{\wedge} v$ and u , as required. For (3)(c), the number of 1s in $\varphi h(t) = \varphi(u_t)$ is greater or equal to the number of 1s of t by (3)(c) for \bar{h} , and the number of 1s in $\varphi h(t \hat{\wedge} v) = \varphi_t(u)$ above $\varphi h(t) = \varphi(u_t)$ is greater or equal to the number of 1s of $t \hat{\wedge} v$ above t by (3)(c) for h_t . By adding up the numbers of 1s, we obtain (3)(c) for h .

The first part of (4) h follows from (3)(c) for h . We already checked the second part up to λ and it is void between λ and α . Hence (4) holds.

Next we verify (I) and (II). Let $(b_s^t)_{s \in \mathcal{T}_y^* \cap < \omega_2}$ denote the family obtained for each $t \in (\mathcal{T}_y)_{[r]} \cap \lambda_2$. Note that $b_\emptyset^t = \bar{b}_{t^*}$, since the induction hypothesis was applied to \bar{b}_{t^*} . We define $(b_w)_{w \in \mathcal{T}_y^* \cap < \alpha_2}$ by letting $b_w = \bar{b}_w$ for $w \in \mathcal{T}_y^* \cap \leq \lambda_2$ and $b_{t \hat{\wedge} s} = b_s^t$ for $w = t \hat{\wedge} s$ with $t \in (\mathcal{T}_y)_{[r]} \cap \lambda_2$ and $s \in \mathcal{T}_y^* \cap < \omega_2$, if s has at least one 1. To verify (I) for h and $(b_w)_{w \in \mathcal{T}_y^* \cap < \alpha_2}$, note that we have (I) for \bar{h} and $(b_w)_{w \in \mathcal{T}_y^* \cap < \lambda_2}$, and for h_t and $(b_s^t)_{s \in \mathcal{T}_y^* \cap < \omega_2}$ for any $t \in (\mathcal{T}_y)_{[r]} \cap \lambda_2$, by the induction hypothesis. Take any $w = t \hat{\wedge} s$ with $t \in (\mathcal{T}_y)_{[r]} \cap \lambda_2$ and $s \in (\mathcal{T}_y)_{[r \hat{\wedge} t]} \cap < \omega_2$. If s has at least one 1, then s^* has at least one 1 and hence $w^* = t \hat{\wedge} s^*$. Therefore $h(w) = h(t \hat{\wedge} s) = h_t(s) \subseteq b_s^t = b_{t \hat{\wedge} s^*} = b_{w^*}$. If s has no 1, then $s = 0^{(n)}$ for some $n < \omega$ and $w^* = t^*$. Then, $h(w) = h(t \hat{\wedge} s) = h_t(s) \subseteq b_s^t = b_\emptyset^t = \bar{b}_{t^*} = b_{t^*} = b_{w^*}$. For (II), note that $b_{t^*} = \bar{b}_{t^*}$, so $f(\bar{b}_{t^*})$ has no 1 above $\xi \leq \varepsilon$ for any $t \in (\mathcal{T}_y)_{[r]} \cap \leq \lambda_2$ by the induction hypothesis. Moreover, $b_{(t \hat{\wedge} s)^*} = b_s^t$ for any $t \in (\mathcal{T}_y)_{[r]} \cap \lambda_2$ and $s \in (\mathcal{T}_y)_{[r \hat{\wedge} t]} \cap \leq \omega_2$, as argued in the proof of (I), so $f(b_{(t \hat{\wedge} s)^*})$ has no 1 above ε by the induction hypothesis.

Limit case. Suppose that $\alpha = \sup_{n < \omega} \alpha_n$, where $(\alpha_n)_{n < \omega}$ is a strictly increasing sequence of limit ordinals.

Since $\varepsilon \in C_\alpha$, we have $\varepsilon \in C_{\alpha_{n+1}}$ for all $n < \omega$. Hence both $C_{\alpha_n} \cap S$ and $C_{\alpha_n} \setminus S$ are unbounded below ε . This allows us to inductively construct a strictly increasing cofinal sequence $(\varepsilon_n)_{n < \omega}$ below ε such that $\varepsilon_n \in C_{\alpha_n}$ and $\alpha_n \in \bar{S} \Leftrightarrow \varepsilon_n \in S$ for all $n < \omega$. Let $i_n < \omega_1$ be such that the height of $\mathcal{M}_{i_n} \cap \omega_1$ is ε_n for each $n < \omega$. We can always assume that ε_0 is large enough that in addition we have $\bar{S}, u, b \in \mathcal{M}_{i_0}$.

We will construct an increasing sequence of maps $h_n: (\mathcal{T}_y)_{[r]} \cap \leq \alpha_n 2 \rightarrow \leq \varepsilon_n 2$ and a family $(b_t)_{t \in \mathcal{T}_y^* \cap < \alpha_n 2}$ such that h_n and $(b_t)_{t \in \mathcal{T}_y^* \cap < \alpha_n 2}$ satisfy (1)-(4) and (I)-(II) for all $n < \omega$ and in addition $h_n \in \mathcal{M}_{i_{n+1}}$ and $(b_t)_{t \in \mathcal{T}_y^* \cap < \alpha_n 2}$ in $\mathcal{M}_{i_{n+1}}$.

In the first step, we apply the induction hypothesis to $\alpha_0, \varepsilon_0, r, u, S, \bar{S}$, and in addition to b if it is given. Recall that the height of $\mathcal{M}_{i_0} \cap \omega_1$ is ε_0 and hence $\bar{S}, u, b \in \mathcal{M}_{i_0}$. This yields a map $h_0: (\mathcal{T}_y)_{[r]} \cap \leq \alpha_0 2 \rightarrow \leq \varepsilon_0 2$ and a family $(b_t)_{t \in \mathcal{T}_y^* \cap < \alpha_0 2}$ satisfying (1)-(4) and (I)-(II).

By an argument using elementarity of \mathcal{M}_{i_1} just as in the successor case, we can choose h_0 and $(b_t)_{t \in \mathcal{T}_y^* \cap < \alpha_0 2}$ in \mathcal{M}_{i_1} . The parameters of this statement are $\alpha_0, \varepsilon_0, u, S, \bar{S}, A, f, \varphi, \mathcal{M}_{i_0}$ and b if it is given. We have $S, A, f, \varphi \in \mathcal{M}_0 \subseteq \mathcal{M}_{i_1}$ and $\bar{S}, u, b \in \mathcal{M}_{i_0} \subseteq \mathcal{M}_{i_1}$. Note that $\varepsilon_0 \in \mathcal{M}_{i_1}$ since ε_0 is the height of $\mathcal{M}_{i_0} \cap \omega_1$, and $\alpha_0 \leq \varepsilon_0$ since $\varepsilon_0 \in C_{\alpha_0}$. Since $\mathcal{M}_{i_1} \cap \omega_1$ is transitive, $\alpha_0 \in \mathcal{M}_{i_1}$.

In successor steps, suppose that the map $h_n: (\mathcal{T}_y)_{[r]} \cap \leq \alpha_n 2 \rightarrow \leq \varepsilon_n 2$ and the family $(b_t)_{t \in \mathcal{T}_y^* \cap < \alpha_n 2}$ are defined as in the statement of the lemma satisfying

(1)-(4) and (I)-(II) such that h_n satisfies the induction hypothesis $h_n \in \mathcal{M}_{i_{n+1}}$. We apply the induction hypothesis for $\alpha_{n+1} - \alpha_n$,⁹ ε_{n+1} , S and $\tilde{S}_{[\alpha_n]}$ separately to $r \hat{\ } t$ for each $t \in (\mathcal{T}_Y)_{[r]} \cap \alpha_n 2$, and to $u_t^n = h_n(t)$ and b_{t^*} .

We first claim that $\alpha_n \in \mathcal{M}_{i_{n+1}}$. Indeed, we have $\varepsilon_n \in \mathcal{M}_{i_{n+1}}$, since ε_n is the height of $\mathcal{M}_{i_n} \cap \omega_1$. As $\mathcal{M}_{i_{n+1}} \cap \omega_1$ is transitive, it suffices to check that $\alpha_n \leq \varepsilon_n$, but this holds as $\varepsilon_n \in C_{\alpha_n}$ and $\varepsilon_n > 0$.

We can apply the induction hypothesis to $\alpha_{n+1} - \alpha_n$ and ε_{n+1} , since $\alpha_{n+1} - \alpha_n \leq \alpha_{n+1}$ and thus $\varepsilon_{n+1} \in C_{\alpha_{n+1}} \subseteq C_{\alpha_{n+1} - \alpha_n}$. We can apply it to $u_t^n = h_n(t)$, since $h_n \in \mathcal{M}_{i_{n+1}}$ by the induction hypothesis and $t \in (\mathcal{T}_Y)_{[r]}$ has length α_n and hence $t \in \mathcal{M}_{i_{n+1}}$. Moreover $u_t^n \in T_K$, since $u_t^n = h_n(t) \subseteq b_{t^*} \in K$ by (I). We can apply it to $\tilde{S}_{[\alpha_n]} \in \mathcal{M}_{i_{n+1}}$, since $\tilde{S} \in \mathcal{M}_{i_0}$ and $\alpha_n \in \mathcal{M}_{i_{n+1}}$.

We now show that we can apply the induction hypothesis to b_{t^*} . We have $b_{t^*} \in \mathcal{M}_{i_n} \subseteq \mathcal{M}_{i_{n+1}}$ by the induction hypothesis. For the remaining properties, we use (I) and (II) of the induction hypothesis. We have $h_n(t) \subseteq b_{t^*}$ by (I) for h_n and $\varphi h_n(t) = f(b_{t^*}) \upharpoonright \varepsilon_n$ by (2) for h_n . Moreover, $f(b_{t^*})$ has no 1s above ε_n by (II) for h_n and b_{t^*} . So $\varphi h_n(t)$ and $f(b_{t^*})$ have the same 1s and $f(b_{t^*})$ has no 1 above $\text{lh}(u_t^n)$ using that $\text{lh}(u_t) = \varepsilon_n$ by (2). Recall that $b_{t^*} \in K$ and hence $f(b_{t^*}) \in Y$ by Claim 4.3.71.1. If $b_{t^*} \in K \cap A$, then $f(b_{t^*}) \in \mathcal{Y}_0$ and $\varphi h_n(t) = \varphi(u_t^n)$ has a finite even number of 1s. Similarly, if $b_{t^*} \in K \setminus A$ then $\varphi h_n(t) = \varphi(u_t^n)$ has a finite odd number of 1s.

In each application of the lemma, we obtain a map $h_t^n: (\mathcal{T}_Y)_{[r \hat{\ } t]} \cap \leq (\alpha_{n+1} - \alpha_n) 2 \rightarrow \leq \varepsilon_{n+1} 2$ and a family $(b_s^t)_{s \in \mathcal{T}_Y^* \cap < (\alpha_{n+1} - \alpha_n) 2}$ satisfying (1)-(4) and (I)-(III). By an argument using elementarity of $\mathcal{M}_{i_{n+2}}$ similar as in the successor case, we can choose the families of all h_t^n and $(b_s^t)_{s \in \mathcal{T}_Y^* \cap < (\alpha_{n+1} - \alpha_n) 2}$ as families in $t \in \alpha_n 2$ to be elements of $\mathcal{M}_{i_{n+2}}$. This uses that the parameters of this statement, namely $\alpha_n, \alpha_{n+1} - \alpha_n, \varepsilon_{n+1}, S, \tilde{S}_{[\alpha_n]}, A, f, \varphi, \mathcal{M}_{i_{n+1}}, h_n, (b_t)_{t \in \mathcal{T}_Y^* \cap < \alpha_n 2}$, are elements of $\mathcal{M}_{i_{n+2}}$. Then, the family of all u_t^n is also in $\mathcal{M}_{i_{n+2}}$, since it can be defined from h_n and α_n .

We have to check that all the parameters belong to $\mathcal{M}_{i_{n+2}}$. We have $S, A, f, \varphi \in \mathcal{M}_0 \subseteq \mathcal{M}_{i_{n+2}}$ and we have shown previously that $\alpha_n, \tilde{S}_{[\alpha_n]} \in \mathcal{M}_{i_{n+1}} \subseteq \mathcal{M}_{i_{n+2}}$. We have $\varepsilon_{n+1} \in \mathcal{M}_{i_{n+2}}$, since ε_{n+1} is the height of $\mathcal{M}_{i_{n+1}} \cap \omega_1$. We claim that $\alpha_{n+1} - \alpha_n \in \mathcal{M}_{i_{n+2}}$. The argument is essentially the same as for $\alpha_n \in \mathcal{M}_{i_{n+1}}$ above: as $\varepsilon_{n+1} \in \mathcal{M}_{i_{n+2}}$, $\mathcal{M}_{i_{n+2}} \cap \omega_1$ is transitive and $\alpha_{n+1} - \alpha_n \leq \alpha_{n+1}$, it suffices to check that $\alpha_{n+1} \leq \varepsilon_{n+1}$, but this holds as $\varepsilon_{n+1} \in C_{\alpha_{n+1}}$ and $\varepsilon_{n+1} > 0$. Finally, $\mathcal{M}_{i_{n+1}} \in \mathcal{M}_{i_{n+2}}$ since the models form an \in -chain, and h_n and $(b_t)_{t \in \mathcal{T}_Y^* \cap < \alpha_n 2}$ are in $\mathcal{M}_{i_{n+1}} \subseteq \mathcal{M}_{i_{n+2}}$ by the induction hypothesis.

We combine h_n and the maps h_t^n to $h_{n+1}: (\mathcal{T}_Y)_{[r]} \cap \leq \alpha_{n+1} 2 \rightarrow \leq \varepsilon_{n+1} 2$ by letting $h_{n+1}(t) = h_n(t)$ for $t \in \mathcal{T}_Y \cap \leq \alpha_n 2$ and $h_{n+1}(t \hat{\ } s) = h_t^n(s)$ for $t \in \alpha_n 2$ and $s \in \leq (\alpha_{n+1} - \alpha_n) 2$. Then, $h_{n+1} \in \mathcal{M}_{i_{n+2}}$. Moreover, h_{n+1} is strictly monotone since all maps in its definition are strictly monotone, and they agree on the values $h_{n+1}(t) = u_t^n$ for all $t \in (\mathcal{T}_Y)_{[r]} \cap \alpha_n 2$.

The verification of the conditions for h_n is similar to the case $\lambda + \omega$. (2) holds for h_{n+1}^∂ , since h_{n+1}^∂ maps $(\mathcal{T}_Y)_{[r]} \cap \alpha_{n+1} 2$ to $\varepsilon_{n+1} 2$ and $\partial(\mathcal{T}_Y)_{[r]} \cap \alpha_{n+1} 2$ to $\varepsilon_{n+1} 2$, and φh_{n+1}^∂ maps $(\mathcal{T}_Y)_{[r]} \cap \alpha_{n+1} 2$ to $\mathcal{T}_Y \cap \varepsilon_{n+1} 2$ and $\partial(\mathcal{T}_Y)_{[r]} \cap \alpha_{n+1} 2$ to $\partial \mathcal{T}_Y \cap \varepsilon_{n+1} 2$.

(3)(a) holds for h_{n+1} by (3)(a) for h_n and h_t^n for all $t \in (\mathcal{T}_Y)_{[r]} \cap \alpha_n 2$, using that $\varepsilon_{n+1} = \varepsilon_n + \varepsilon_{n+1}$. For (3)(b), note that this holds for all $t \in$

⁹By $\gamma - \alpha$, where $\alpha \leq \gamma$ are ordinals, we mean the unique ordinal β such that $\alpha + \beta = \gamma$.

$(\mathcal{T}_y)_{[r]} \leq^{\alpha_n 2}$ by (3)(b) for h_n . Now take any $t \wedge s \in \leq^{\alpha_{n+1} 2}$, where $t \in \alpha_n 2$ and $s \in \leq^{(\alpha_{n+1} - \alpha_n) 2}$. Note that the parity of the number of 1s in $\varphi h_n(t) = \varphi(u_t^n)$ is the same as that of the sums of the numbers in t and in $\varphi(u)$ by (3)(b) for h_n . The parity of the number of 1s in $\varphi h_{n+1}(t \wedge s) = \varphi h_t^n(s)$ equals the parity of the sum of the numbers of 1s in $\varphi(u_t^n)$ and s by (3)(b) for h_t^n , hence it equals the parity of the sums of numbers of 1s in $t \wedge s$ and $\varphi(u)$ as required. For (3)(c), the number of 1s in $\varphi h_n(t) = \varphi(u_t^n)$ above $\varphi(u)$ is greater or equal to the number of 1s of t by (3)(c) for h_n , and the number of 1s in $\varphi h_{n+1}(t \wedge s) = \varphi(h_t^n(s))$ above $\varphi(u_t^n)$ is greater or equal to the number of 1s of s by (3)(c) for h_t^n . By adding up the numbers of 1s in $\varphi(u_t^n)$ above $\varphi(u)$ and in $\varphi h_{n+1}(t \wedge s)$ above $\varphi(u_t^n)$ on the one hand, and those in t and s on the other hand, we obtain (3)(c) for h_{n+1} .

We next want to verify (4) for h_{n+1}^∂ . For the first part of (4), we claim that $\varphi h_{n+1}^\partial(w) \in \partial \mathcal{T}_y \cap <^{\varepsilon 2}$ for any $w \in \partial(\mathcal{T}_y)_{[r]} \cap <^{\alpha_{n+1} 2}$. Indeed, it is true when $w \in <^{\alpha_n 2}$ by (4) for h_n . Furthermore, it holds for all $w \in \alpha_n 2$ by (2) for h_n^∂ . Finally, if $w \in <^{\alpha_{n+1} 2} \setminus \leq^{\alpha_n 2}$, then it holds by (4) for h_t^n .

For the second part of (4), suppose that $w \in \partial(\mathcal{T}_y)_{[r]} \cap <^{\alpha_{n+1} 2}$ has limit length. If $w \in <^{\alpha_n 2}$, then $\text{lh}(w) \in \bar{S}$ if and only if $\text{lh}(\varphi h_{n+1}^\partial(w)) = \text{lh}(\varphi h_n^\partial(w)) \in S$ by (4) for h_n . If $w \in \alpha_n 2$, by (2) for h_n^∂ and the choice of ε_n , we get $\alpha_n \in \bar{S}$ if and only if $\varepsilon_n \in S$. Now suppose that $w = t \wedge s$ for some $t \in \alpha_n 2$ and $s \in <^{(\alpha_{n+1} - \alpha_n) 2}$. Then, $\text{lh}(t \wedge s) \in \bar{S}$ if and only if $\text{lh}(s) \in \bar{S}_{[\alpha_n]}$. Moreover, $\text{lh}(s) \in \bar{S}_{[\alpha_n]}$ if and only if $\text{lh}(\varphi(h_t^n)^\partial(s)) \in S$ for any $s \in <^{(\alpha_{n+1} - \alpha_n) 2}$ by (4) for $(h_t^n)^\partial$, using that $\varphi h_{n+1}^\partial(t \wedge s) = \varphi(h_t^n)^\partial(s)$.

We next verify (I) and (II) for h_{n+1} . We define $(b_w)_{w \in \mathcal{T}_y^* \cap <^{\alpha_{n+1} 2}}$ as follows. For $w \in \mathcal{T}_y^* \cap <^{\alpha_n 2}$, b_w is already defined. For $w \in \mathcal{T}_y^* \cap <^{\alpha_{n+1} 2} \setminus <^{\alpha_n 2}$, $w = t \wedge s$ for some $t \in \mathcal{T}_y \cap \alpha_n 2$ and $s \in \mathcal{T}_y^* \cap <^{(\alpha_{n+1} - \alpha_n) 2}$ with at least one 1, let $b_{t \wedge s} = b_s^t$.

To verify (I) for h_{n+1} and $(b_w)_{w \in \mathcal{T}_y^* \cap <^{\alpha_{n+1} 2}}$, note that (I) holds for each h_t^n and $(b_s^t)_{s \in \mathcal{T}_y^* \cap <^{(\alpha_{n+1} - \alpha_n) 2}}$. Take any $t \wedge s$ with $t \in \mathcal{T}_y \cap \alpha_n 2$ and $s \in \mathcal{T}_y \cap <^{(\alpha_{n+1} - \alpha_n) 2}$. If s has at least one 1, then s^* has at least one 1 and hence $h_{n+1}(t \wedge s) = h_t^n(s) \subseteq b_s^t = b_{t \wedge s^*}$. Since $(t \wedge s)^* = t \wedge s^*$, $h_{n+1}(t \wedge s) \subseteq b_{(t \wedge s)^*}$ as required. If s has no 1, then $s = 0^{(\gamma)}$ for some $\gamma < \alpha_{n+1} - \alpha_n$ and $h_{n+1}(t \wedge s) = h_t^n(s) \subseteq b_s^t = b_\emptyset^t = b_{t^*}$. Since $(t \wedge s)^* = t^*$, we have $h_{n+1}(t \wedge s) \subseteq b_{(t \wedge s)^*}$.

Since (II) holds for $(b_w)_{w \in \mathcal{T}_y^* \cap <^{\alpha_n 2}}$ and for $(b_s^t)_{s \in \mathcal{T}_y^* \cap <^{(\alpha_{n+1} - \alpha_n) 2}}$ for each $t \in \mathcal{T}_y \cap \alpha_n 2$, (II) holds for $(b_w)_{w \in \mathcal{T}_y^* \cap <^{\alpha_{n+1} 2}}$.

Finally, let $h: (\mathcal{T}_y)_{[r]} \cap \leq^{\alpha 2} \rightarrow \leq^{\varepsilon 2}$ be the unique continuous* extension to $\leq^{\alpha 2}$ of the union of the maps h_n for all $n < \omega$.

(1) is obvious. To see that (2) holds for h^∂ , note that h^∂ maps sequences v of length α in \mathcal{T}_y and in $\partial \mathcal{T}_y$ to sequences of length ε , since $h(v \upharpoonright \alpha_n) = h_n(v \upharpoonright \alpha_n)$ has length ε_n for each $n < \omega$. We next show that φh maps $\mathcal{T}_y \cap \alpha 2$ to $\mathcal{T}_y \cap \varepsilon 2$. Suppose that $v \in \mathcal{T}_y \cap \alpha 2$ with its last 1 below α_n . By (I) and (2) for h_j , we have $\varphi h(v \upharpoonright \alpha_j) = \varphi h_j(v \upharpoonright \alpha_j) = f(b_{(v \upharpoonright \alpha_j)^*}) \upharpoonright \varepsilon_j = f(b_{v^*}) \upharpoonright \varepsilon_j$ for all $j \geq n$. Since φ and h are continuous*, $\varphi h(v) = f(b_{v^*}) \upharpoonright \varepsilon$. By Claim 4.3.71.1, $f(b_{v^*}) \in Y$ and hence $\varphi h(v) \in \mathcal{T}_y \cap \varepsilon 2$. Now suppose that $v \in \partial \mathcal{T}_y \cap \alpha 2$. By (3)(c) for h_0 , the number of 1s of $\varphi h(v \upharpoonright \alpha_0)$ above $\varphi(u)$ is at least that of $v \upharpoonright \alpha_0$. By (3)(c) for h_n , the number of 1s of $\varphi h(v \upharpoonright \alpha_{n+1})$ above

$$\varphi(u_{v \upharpoonright \alpha_n}^n) = \varphi(h_n(v \upharpoonright \alpha_n)) = \varphi(h(v \upharpoonright \alpha_n))$$

is at least that of $v \upharpoonright \alpha_{n+1}$. Hence the number of 1s of $\varphi h^\partial(v)$ above $\varphi(u)$ is at least that of v and therefore $\varphi h^\partial(v) \in \partial\mathcal{T}_Y \cap \varepsilon^2$.

(3)(a) for h and all $v \in {}^\alpha 2$ follows from (2) for h , and since φ and h are strictly monotone, (3)(a) holds for h and all $v \in \leq^\alpha 2$. The remaining parts of (3) and (4) hold since each h_n satisfies (3) and (4) and h is a continuous* extension of the union of the maps h_n .

To see that (I) holds, suppose that $t \in \mathcal{T}_Y \cap \leq^\alpha 2$. If $\text{lh}(t) < \alpha_n$ for some $n < \omega$, then $h(t) = h_n(t) \subseteq b_{t^*}$ by (I) for h_n . Now suppose that $\text{lh}(t) = \alpha$ and the last 1 of t appears below α_k . For any n with $k \leq n < \omega$, we have $h(t \upharpoonright \alpha_n) = h_n(t \upharpoonright \alpha_n) \subseteq b_{(t \upharpoonright \alpha_n)^*} = b_{t^*}$ by (I) for h_n . Since h is strictly monotone, we have $h(t) \subseteq b_{t^*}$. This shows (I) for h .

For (II), suppose that $t \in \mathcal{T}_Y \cap \leq^\alpha 2$. Then, $\text{lh}(t^*) < \alpha_n$ for some $n < \omega$, $t^* = (t \upharpoonright \alpha_n)^*$ and $b_{t^*} = b_{(t \upharpoonright \alpha_n)^*}$. Hence $f(b_{t^*})$ has no 1s above ε_n by (II) for h_n . \square

Theorem 4.3.72. *Let $S \subseteq \omega_1$ and let $A \subseteq {}^{\omega_1} 2$ be such that $A \leq_W \mathcal{Y}_S$. If ${}^{\omega_1} 2$ is not A -HK scattered, then $\mathcal{Y}_S \leq_W A$.*

Proof. Let K be the A -HK kernel of ${}^{\omega_1} 2$, obtained at the end of the sequence of iterated Hausdorff-Kuratowski derivatives as in Definition 4.3.43. Since ${}^{\omega_1} 2$ is not A -HK scattered, we have $K \neq \emptyset$. Let $f: {}^{\omega_1} 2 \rightarrow {}^{\omega_1} 2$ be a continuous function witnessing $A \leq_W \mathcal{Y}_S$, and let $\varphi: <{}^{\omega_1} 2 \rightarrow <{}^{\omega_1} 2$ be a continuous* approximation for f (using Remark 2.7.4). We can assume that $\varphi(\emptyset) = \emptyset$ by changing the value of $\varphi(\emptyset)$ if necessary. Take an elementary chain $(\mathcal{M}_i)_{i < \omega_1}$ in H_θ , a sequence $(\gamma_i)_{i < \omega_1}$, a club C in ω_1 and its enumeration $(\delta_i)_{i < \omega_1}$ such that Conditions (A) and (B) hold for A , S , f , φ , $(\mathcal{M}_i)_{i < \omega_1}$, $(\gamma_i)_{i < \omega_1}$, $(\delta_i)_{i < \omega_1}$ and C .

It suffices to construct a strictly continuous* function $h: \mathcal{T}_Y \rightarrow <{}^{\omega_1} 2$ such that:

- (i) For any $t \in <{}^{\omega_1} 2$ with finitely many 1s, $\varphi h(t)$ has finitely many 1s and the parity is the same.
- (ii) For any $t \in <{}^{\omega_1} 2$ with cofinally many 1s with order type ω , $\varphi h^\partial(t)$ has cofinally many 1s with order type ω and

$$\text{lh}(t) \in S \Leftrightarrow \text{lh}(\varphi h^\partial(t)) \in S.$$

We construct h by applying Lemma 4.3.71 to the intervals between successive elements of C . More precisely, we construct a coherent sequence of maps $h_i: \mathcal{T}_Y \cap \leq^{\delta_i} 2 \rightarrow \leq^{\delta_i} 2$ and families $(b_t)_{t \in \mathcal{T}_Y^* \cap <^{\delta_i} 2}$ by induction on $i < \omega_1$ to satisfy the properties (1)-(4) and (I)-(II) in Lemma 4.3.71 for $\alpha = \varepsilon = \delta_i$, h_i and b_t 's are in $\mathcal{M}_{j_{i+1}}$ and moreover, (i) and (ii) hold for all $t \in \leq^{\delta_i} 2$.

Each δ_i where $i < \omega_1$ equals γ_j for some $j < \omega_1$ and hence δ_i is the height of $\mathcal{M}_j \cap \omega_1$. Let $j_i < \omega_1$ be such that $\mathcal{M}_{j_i} \cap \omega_1$ has height δ_i . Since $\mathcal{M}_{j_i} \preceq H_\theta$ and H_θ is equipped with a wellorder $<_{H_\theta}$, \mathcal{M}_{j_i} comes with a wellorder $<_{\mathcal{M}_{j_i}}$.

Base case. In the first step, apply Lemma 4.3.71 to $r = u = \emptyset$, $\alpha = \varepsilon = \delta_0$, S and $\bar{S} = S$. We can apply the lemma to $u = \emptyset$ since $K \neq \emptyset$ and to α and ε since $\delta_0 \in C_{\delta_0}$. We thus obtain a map $h_0: \mathcal{T}_Y \cap \leq^{\delta_0} 2 \rightarrow \leq^{\delta_0} 2$ and a family $(b_t)_{t \in \mathcal{T}_Y \cap <^{\delta_0} 2}$ with the required properties. In particular, $b_t \in \mathcal{M}_{j_0}$ for

all $t \in \mathcal{T}_Y \cap <^{\delta_0}2$. By an argument using elementarity of \mathcal{M}_{j_1} just as in the successor case in the proof of Lemma 4.3.71, we can choose h_0 and $(b_t)_{t \in \mathcal{T}_Y \cap <^{\delta_0}2}$ in \mathcal{M}_{j_1} . Choose the $<_{\mathcal{M}_{j_1}}$ -least ones. (i) holds for all $t \in \leq^{\delta_0}2$ and (ii) holds for all $t \in <^{\delta_0}2$ by the conditions (3) and (4) of Lemma 4.3.71 for h_0 . (ii) holds for all $t \in \delta_0 2$, since $\varphi h^\partial(t) \in \delta_0 2$ for all $t \in \delta_0 2$ by (2).

Successor case. For successors $i + 1$, apply Lemma 4.3.71 separately to each $t \in \mathcal{T}_Y \cap \delta_i 2$ with $r = t$, $u_t^i = h_i(t)$, b_{t^*} , $\alpha = \varepsilon = \delta_{i+1}$, S and $\bar{S} = S_{\lfloor \delta_i \rfloor}$. Note that $u_t^i \in T_K$ since $u_t^i \subseteq b_{t^*}$ by condition (I) in Lemma 4.3.71. We can apply the lemma to u_t^i and b_{t^*} by (I) and (II) for h_i precisely as in the successor case in the proof of Lemma 4.3.71, using that $u_t^i \in \mathcal{M}_{j_{i+1}}$ since $h_i \in \mathcal{M}_{j_{i+1}}$ and $b_{t^*} \in \mathcal{M}_{j_i} \subseteq \mathcal{M}_{j_{i+1}}$.

We can apply the lemma to \bar{S} , since $\delta_i \in \mathcal{M}_{j_{i+1}}$ and hence $\bar{S} = S_{\lfloor \delta_i \rfloor} \in \mathcal{M}_{j_{i+1}}$. We thus obtain a map $h_{i+1}: \mathcal{T}_Y \cap \leq^{\delta_{i+1}}2 \rightarrow \leq^{\delta_{i+1}}2$ and a family $(b_t)_{t \in \mathcal{T}_Y^* \cap <^{\delta_{i+1}}2}$ with the required properties. By an argument using elementarity of $\mathcal{M}_{j_{i+2}}$ just as in the successor case of the proof of Lemma 4.3.71, we can choose h_{i+1} and $(b_t)_{t \in \mathcal{T}_Y^* \cap <^{\delta_{i+1}}2}$ in $\mathcal{M}_{j_{i+2}}$. Choose the $<_{\mathcal{M}_{j_{i+2}}}$ -least ones.

The argument for (i) and (ii) is as in the base case. (i) holds for all $t \in \leq^{\delta_{i+1}}2$ and (ii) holds for all $t \in <^{\delta_{i+1}}2$ by the conditions (3) and (4) of Lemma 4.3.71 for h_{i+1} . (ii) holds since $\varphi h^\partial(t) \in \delta_{i+1} 2$ for all $t \in \delta_{i+1} 2$ by (2).

Limit case. At limits i , let h_i be the unique continuous* extension of the union of the maps h_l for all $l < i$ to $\mathcal{T}_Y \cap \leq^{\delta_i}2$ and $(b_t)_{t \in \mathcal{T}_Y^* \cap <^{\delta_i}2}$ the union of all previous families. We can verify precisely as in the limit case of Lemma 4.3.71 that h_i and $(b_t)_{t \in \mathcal{T}_Y^* \cap <^{\delta_i}2}$ have the required properties. All properties except (2) hold since h_i is a continuous* extension of the union of h_l for $l < i$. To see that the first part of (2) holds for h^∂ at length δ_i , note that h_l^∂ maps sequences of length δ_l in \mathcal{T}_Y to sequences of length δ_l by (2) for h_l . Hence h^∂ maps both $\mathcal{T}_Y \cap \delta_i 2$ and $\partial \mathcal{T}_Y \cap \delta_i 2$ to $\delta_i 2$. Then, the second part of (2) holds by (3) (c). Since we chose the least ones in each successor step and we have $(\mathcal{M}_{j_i})_{l < i} \in \mathcal{M}_{j_{i+1}}$, h_i and $(b_t)_{t \in \mathcal{T}_Y^* \cap <^{\delta_i}2}$ are elements of $\mathcal{M}_{j_{i+1}}$, as required.

(i) holds for all $t \in <^{\delta_i}2$ since it holds for all $t \in \leq^{\delta_i}2$ and all $l < i$. To see that (i) holds for all $t \in \delta_i 2$, we show that h maps $\mathcal{T}_Y \cap \delta_i 2$ to $\mathcal{T}_Y \cap \delta_i 2$. Suppose that $t \in \mathcal{T}_Y \cap \delta_i 2$ with its last 1 below δ_l for some $l < i$. By (I), we have $\varphi h(t \upharpoonright \delta_j) = f(b_{t^*}) \upharpoonright \delta_j$ for all $l \leq j < i$. Since φ and h are continuous*, $\varphi h(t) = f(b_{t^*}) \upharpoonright \delta_i \in \mathcal{T}_Y \cap \delta_i 2$ and hence $\varphi h(t) \in \mathcal{T}_Y$.

(ii) holds for all $t \in <^{\delta_i}2$ since it holds for all $t \in \leq^{\delta_i}2$, where $l < i$. The first part of (ii) holds for all $t \in \delta_i 2$ by (2) for h_i . The second part of (ii) holds by (4) for h_i .

Finally, let $h: \mathcal{T}_Y \rightarrow <^{\omega_1}2$ be the union of the function h_i for $i < \omega_1$. Then, h satisfies (i) and (ii). We can extend h to an arbitrary strictly monotone function $\bar{h}: <^{\omega_1}2 \rightarrow <^{\omega_1}2$. (i) and (ii) for h imply that the function induced by $\varphi \bar{h}$ reduces \mathcal{Y}_S to \mathcal{Y}_S , and since f reduces A to \mathcal{Y}_S , the function induced by \bar{h} reduces \mathcal{Y}_S to A . \square

Corollary 4.3.73. *Let $S \subseteq \omega_1$ and $A \subseteq {}^{\omega_1}2$ such that $A \leq_W \mathcal{Y}_S$. Then, either $A \in \mathcal{D}_\alpha(\omega_2 - \mathbf{\Pi}_1^0({}^{\omega_1}2))$ for some $\alpha < \omega_2$, or $\mathcal{Y}_S \equiv_W A$. Therefore, the Wadge degree $[\mathcal{Y}_S]_W$ is minimal above the $\omega_2 - \mathbf{\Pi}_1^0({}^{\omega_1}2)$ -Difference hierarchy.*

Proof. It follows from Proposition 4.3.45 applied to $\kappa = \omega_1$ and Proposition 4.3.72. \square

Note that the two cases in Theorem 4.3.72 are mutually exclusive by Lemma 4.3.44, since for any subset S of ω_1 , the kernel of \mathcal{Y}_S is \mathcal{Y} .

4.3.7 Without the full Axiom of Choice

In this section, we do not assume the full Axiom of Choice AC. Therefore, unless explicitly indicated otherwise, all of our arguments are in $\text{ZF} + \text{AC}_\kappa({}^\kappa 2)$.

Let $C_0 = \{0^{(\kappa)}\}$. Recall that any proper closed subset C of ${}^\kappa 2$ is κ -complete for κ^+ - Π_1^0 by Fact 4.1.3, and that $\text{AC}_\kappa({}^\kappa 2)$ is necessary in the argument of the proof. This implies that for any proper closed $C \subseteq {}^\kappa 2$, $C_0 \leq_W C$. In contrast, the next proposition shows that assuming $\kappa = \omega_1$ and AD, there exists a proper closed set $F \subseteq {}^{\omega_1} 2$ such that $C_0 \not\leq_W F$ and $F \not\leq_W \neg C_0$. We recall that AD implies that there exist no ω_1 -sequences of distinct reals.

Proposition 4.3.74. *Assume $\text{ZF} +$ “there exist no ω_1 -sequences of distinct reals”. Then, $\text{SLO}_W(\omega_2\text{-}\Pi_1^0({}^{\omega_1} 2))$ fails.*

Proof. Let $C_0 = \{0^{(\omega_1)}\}$. It is enough to find $F \in \omega_2\text{-}\Pi_1^0({}^{\omega_1} 2) \setminus \omega_2\text{-}\Sigma_1^0({}^{\omega_1} 2)$ such that $C_0 \not\leq_W F$. Indeed, if F is not open, then we immediately know that there cannot be any continuous reduction of F to $\neg C_0$. For every $\alpha < \omega_1$, let

$$WO_\alpha = \{x \in {}^{\omega_1} 2 \mid x \text{ codes a well-order } R_x \subseteq \omega \times \omega \text{ of order type } \alpha\}.$$

We define

$$F = \{0^{(\omega_1)}\} \cup \bigcup_{\alpha < \omega_1} \{N_{0^{(\alpha)} \smallfrown 1 \smallfrown z} \mid z \in {}^{\omega_1} 2 \setminus WO_\alpha\}.$$

We have that $F \in \omega_2\text{-}\Pi_1^0({}^{\omega_1} 2) \setminus \omega_2\text{-}\Sigma_1^0({}^{\omega_1} 2)$. Indeed, it is closed since the only limit point of F is $0^{(\omega_1)} \in F$, and it is not open because for every $\alpha < \omega_1$ $N_{0^{(\alpha)}} \not\subseteq F$.

Claim 4.3.74.1. Let $(y_\alpha)_{\alpha < \omega_1}$ be a sequence in ${}^{\omega_1} 2 \setminus F$ converging to $0^{(\omega_1)}$. Then, there exists an ω_1 -sequence of distinct reals.

Proof of the Claim. For every $\alpha < \omega_1$, since $y_\alpha \notin F$, there exist $\gamma_\alpha < \omega_1$ and $z_\alpha \in WO_{\gamma_\alpha}$ such that $y_\alpha \in N_{0^{(\gamma_\alpha)} \smallfrown 1 \smallfrown z_\alpha}$. Since ω_1 is regular, there exists a strictly increasing sequence $(\alpha_i)_{i < \omega_1}$ cofinal in ω_1 such that $(\gamma_{\alpha_i})_{i < \omega_1}$ is strictly increasing. Clearly, the subsequence $(y_{\alpha_i})_{i < \omega_1}$ converges to $0^{(\omega_1)}$. Since $(\gamma_{\alpha_i})_{i < \omega_1}$ is strictly increasing and cofinal in ω_1 , $(z_{\alpha_i})_{i < \omega_1}$ is a sequence of distinct reals. \square

Now, assume towards a contradiction that there exists a continuous reduction $f : {}^{\omega_1} 2 \rightarrow {}^{\omega_1} 2$ witnessing $C_0 \leq_W F$. Then, $f(0^{(\omega_1)}) = 0^{(\omega_1)}$, because $0^{(\omega_1)} \in \partial C_0$, hence $f(0^{(\omega_1)}) \in \partial F = \{0^{(\omega_1)}\}$. Take a sequence $(x_\alpha)_{\alpha < \omega_1}$ converging to $0^{(\omega_1)}$, e.g., let $x_\alpha = 0^{(\alpha)} \smallfrown 1 \smallfrown 0^{(\omega_1)}$ for every $\alpha < \omega_1$. Since f is continuous, the sequence $(f(x_\alpha))_{\alpha < \omega_1}$ converges to $f(0^{(\omega_1)}) = 0^{(\omega_1)}$. Note that, since $C_0 = \{0^{(\omega_1)}\}$, $x_\alpha \notin C_0$ for every $\alpha < \omega_1$, and since f is a reduction, $f(x_\alpha) \notin F$ for every $\alpha < \omega_1$. By Claim 4.3.74.1, there exists an ω_1 -sequence of distinct reals. A contradiction. \square

Remark 4.3.75. It is a standard fact that in ZF the Perfect Set Property for ${}^\omega \omega$ implies there is no ω_1 -sequence of pairwise distinct real numbers [Kan09, Proposition 11.4]. Suppose, for a contradiction, that there is an injective function $f: \omega_1 \rightarrow {}^\omega \omega$. Then $\text{ran}(f)$ is uncountable, and therefore contains a subset \mathcal{C} homeomorphic to the Cantor space ${}^\omega 2$. In particular, $\mathcal{C} \subseteq \text{ran}(f)$ is well-ordered, and hence so is ${}^\omega \omega$, since it is in bijection with \mathcal{C} . But from a well-ordering of the reals one can construct a Bernstein set, which fails to have the perfect set property. This is a contradiction.

Corollary 4.3.76. *Assume ZF + $\text{AC}_\omega(\mathbb{R})$. Then, $\text{SLO}_W(\omega_2\text{-}\Pi_1^0({}^{\omega_1}2))$ and $\text{SLO}_W({}^\omega 2)$ cannot both be true.*

Proof. We argue towards a contradiction, assuming both $\text{SLO}_W(\omega_2\text{-}\Pi_1^0({}^{\omega_1}2))$ and $\text{SLO}_W({}^\omega 2)$. We have that $\text{SLO}_W(\omega_2\text{-}\Pi_1^0({}^{\omega_1}2))$ implies that there exists an ω_1 -sequence of distinct reals by Proposition 4.3.74 and that $\text{SLO}_W({}^\omega 2) \Rightarrow \text{SLO}_W({}^\omega \omega)$ by Lemma 2.11.1. Since $\text{SLO}_W({}^\omega \omega)$ yields the Perfect Set Property for ${}^\omega \omega$ by a theorem of Wadge (see, for instance, [And07, Theorem 24] for a proof that relies only on $\text{AC}_\omega(\mathbb{R})$), Remark 4.3.75 then implies that there cannot exist an ω_1 -sequence of pairwise distinct real numbers. This is a contradiction. \square

To move toward showing the failure of SLO_W , we first observe that Proposition 4.3.52 remains valid even without assuming AC, as demonstrated by the following result.

Proposition 4.3.77. *Let κ be regular and $S, S' \subseteq \kappa$ be such that $S \triangle S'$ is stationary in κ . Then, \mathcal{Y}_S and $\mathcal{Y}_{S'}$ are W-incomparable.*

Proof. Suppose towards a contradiction that $f: {}^\kappa 2 \rightarrow {}^\kappa 2$ is a continuous function witnessing $\mathcal{Y}_S \leq_W \mathcal{Y}_{S'}$ and $\varphi_f: {}^{<\kappa} 2 \rightarrow {}^{<\kappa} 2$ is an approximation of f . We consider the inner model $\mathcal{M} = L[S, S', \varphi_f]$ of ZFC and write $f^\mathcal{M} = f \cap \mathcal{M}$. The following statements hold in \mathcal{M} by Π_1 -downward absoluteness:

1. φ_f is an approximation of the function $f^\mathcal{M}$.
2. $f^\mathcal{M}$ is a reduction from \mathcal{Y}_S to $\mathcal{Y}_{S'}$.
3. κ is regular.
4. $S \triangle S'$ is a stationary subset of κ .

Therefore, \mathcal{M} satisfies the statement that there is a continuous reduction from \mathcal{Y}_S to $\mathcal{Y}_{S'}$. But this would contradict Proposition 4.3.52, since \mathcal{M} is a model of ZFC. \square

Lemma 4.3.78. *Suppose that $\mu < \kappa$ is an infinite cardinal such that $\kappa = \mu^+$ is regular and there exists a κ -sequence of distinct elements of ${}^\mu 2$. Then, there is a function F that sends every stationary subset S of κ to two disjoint stationary subsets of S .*

Proof. Given $S \subseteq \kappa$, an S -club in κ is by definition a set of the form $C \cap S$, where $C \subseteq \kappa$ is closed unbounded in κ . Let S be a stationary subset of κ and let $j : S \rightarrow {}^\mu 2$ be an injective function. We set $S_t = j^{-1}(\mathbf{N}_t)$ for $t \in {}^{<\mu} 2$.

We select a sequence $t^* \in {}^{\leq \mu} 2$ as follows. Set $t_0 = \emptyset$. Then, proceed by induction on $i < \mu$. Assuming that t_i has been constructed, if $S_{t_i \frown 0}$ and $S_{t_i \frown 1}$ are both stationary then set $t_{i+1} = t_i$, otherwise, there exists $a \in \{0, 1\}$ such that $S_{t_i \frown a}$ contains an S -club, and in this case set $t_{i+1} = t_i \frown a$. If i is a limit ordinal, set $t_i = \bigcup_{j < i} t_j$. Finally, let $t^* = \bigcup_{i < \mu} t_i$.

We claim that $t^* \in {}^{<\mu} 2$, meaning that there exists some $i < \mu$ such that $t_i = t^*$. Then, $S_{t_i \frown 0}, S_{t_i \frown 1} \subseteq S$ are stationary in κ and disjoint, and we conclude by setting $F(S) = (S_{t_i \frown 0}, S_{t_i \frown 1})$. Indeed, if $t^* \in {}^\mu 2$, then $S_{t^*} = \bigcap_{i < \mu} S_{t_i}$ and since each S_{t_i} contains an S -club by construction, then S_{t^*} contains an S -club too, i.e. the intersection of all the S -clubs in the μ -steps above. However, $S_{t^*} = \bigcap_{i < \mu} S_{t_i} = j^{-1}(\{t^*\})$ contains at most one point since j is injective, a contradiction. \square

Corollary 4.3.79. *Suppose that $\mu < \kappa$ is an infinite cardinal such that $\kappa = \mu^+$ is regular and there exists a κ -sequence of distinct elements of ${}^\mu 2$. Then, there function F that sends every stationary subset S of κ to a sequence $(S_n)_{n < \omega}$ of disjoint stationary subsets of S .*

Proof. We write $F(T) = (U_T, V_T)$ for stationary subsets T of κ , where F is a function as in Lemma 4.3.78. We construct $(U_i)_{i < \omega}$ and $(V_i)_{i < \omega}$ as follows. Let $U_0 = U_S$ and $V_0 = V_S$. If S_0, \dots, S_n have been constructed, let $U_{n+1} = U_{V_{S_n}}$ and $V_{n+1} = U_{V_{S_n}}$. \square

Corollary 4.3.80. *Suppose that $\mu < \kappa$ is an infinite cardinal such that $\kappa = \mu^+$ is regular and there exists a κ -sequence of distinct elements of ${}^\mu 2$. Then, the Wadge hierarchy on $\kappa^+ - \Delta_2^0$ -sets is illfounded.*

Proof. Let $(S'_n)_{n < \omega}$ be a sequence of disjoint stationary subsets of κ obtained by applying Corollary 4.3.79 to $S = \text{cof}_\omega^\kappa$. Then, there is $(S_i)_{i < 2^\omega}$ sequence of subsets of κ such that $S_i \triangle S_j$ is stationary for every $j \neq i$. Indeed for each $I \subseteq \omega$ let $S_I = \bigcup_{n \in I} S'_n$, then $S_I \triangle S_{I'}$ is stationary for every $I \neq I'$. Consider the sequence $(S_i)_{i < \omega}$ and repeat the argument in Corollary 4.3.58, referring to Proposition 4.3.77 instead of Proposition 4.3.52. In this process, note that every result we appeal to in these arguments, whenever a choice principle is required, relies solely on $\text{AC}_\kappa({}^\kappa 2)$. \square

Proposition 4.3.81. *Suppose that $\mu < \kappa$ is an infinite cardinal such that $\kappa = \mu^+$ is regular. Then, $\text{SLO}_W(\kappa^+ - \Delta_2^0)$ fails.*

Proof. We assume $\text{SLO}_W(\kappa^+ - \Pi_1^0)$, otherwise the statement is trivially true. The proof is an adaptation of the argument in the proof of Proposition 4.3.74.

Let $(X_\alpha)_{\alpha < \kappa}$ be a sequence of non-empty pairwise disjoint subsets of ${}^\mu 2$. For example, for every $\alpha < \kappa$, let

$$WO_\alpha = \{x \in {}^\mu 2 \mid x \text{ codes a well-order } R_x \subseteq \mu \times \mu \text{ of order type } \mu + \alpha\}.$$

Let $C_0 = \{0^{(\kappa)}\}$ and

$$F = \{0^{(\kappa)}\} \cup \bigcup_{\alpha < \kappa} \{\mathbf{N}_{0^{(\alpha)} \frown 1 \frown z} \mid z \in {}^\mu 2 \setminus X_\alpha\}.$$

We have that $F \in \kappa^+ - \Pi_1^0 \setminus \kappa^+ - \Sigma_1^0$. Since $F \not\leq_W \neg C_0$, by $\text{SLO}_W(\kappa^+ - \Pi_1^0)$ we have that $C_0 \leq_W F$. Let $f : \kappa^2 \rightarrow \kappa^2$ be a continuous reduction witnessing $C_0 \leq_W F$. Then, $f(0^{(\kappa)}) = 0^{(\kappa)}$, because $0^{(\kappa)} \in \partial C_0$, hence $f(0^{(\kappa)}) \in \partial F = \{0^{(\kappa)}\}$.

Claim 4.3.81.1. Let $(y_\alpha)_{\alpha < \kappa}$ be a sequence in $\kappa^2 \setminus F$ converging to $0^{(\kappa)}$. Then, there exists a κ -sequence of distinct elements of ${}^\mu 2$.

Proof of the Claim. For every $\alpha < \kappa$, since $y_\alpha \notin F$, there exist $\gamma_\alpha < \kappa$ and $z_\alpha \in X_{\gamma_\alpha}$ such that $y_\alpha \in \mathbf{N}_{0^{(\gamma_\alpha)} \cap 1 \cap z_\alpha}$. Since κ is regular, there exists a strictly increasing sequence $(\alpha_i)_{i < \kappa}$ cofinal in κ such that $(\gamma_{\alpha_i})_{i < \kappa}$ is strictly increasing. Clearly, the subsequence $(y_{\alpha_i})_{i < \kappa}$ converges to $0^{(\kappa)}$. Since $(\gamma_{\alpha_i})_{i < \kappa}$ is strictly increasing and cofinal in κ , $(z_{\alpha_i})_{i < \kappa}$ is a sequence of distinct elements of ${}^\mu 2$. \square

Take any sequence $(x_\alpha)_{\alpha < \kappa}$ converging to $0^{(\kappa)}$. e.g. $x_\alpha = 0^{(\alpha)} \cap 1 \cap 0^{(\kappa)}$ for every $\alpha < \kappa$. Since f is continuous, the sequence $(f(x_\alpha))_{\alpha < \kappa}$ converges to $f(0^{(\kappa)}) = 0^{(\kappa)}$. Note that, since $C_0 = \{0^{(\kappa)}\}$, $x_\alpha \notin C_0$ for every $\alpha < \kappa$, and since f is a reduction, $f(x_\alpha) \notin F$ for every $\alpha < \kappa$. By Claim 4.3.81.1 applied to $(y_\alpha)_{\alpha < \kappa} = (f(x_\alpha))_{\alpha < \kappa}$, there exists a κ -sequence of distinct elements of ${}^\mu 2$. By Corollary 4.3.80, the Wadge hierarchy on $\kappa^+ - \Delta_2^0$ -sets is illfounded. Moreover, using Lemma 4.3.78, let $S, S' \subseteq \text{cof}_\omega^\kappa$ such that S, S' are stationary in κ and $S \cap S' = \emptyset$. By Proposition 4.3.77, $\mathcal{Y}_S, \mathcal{Y}_{S'}$ and \mathcal{Y}_0 are W -incomparable, hence $\text{SLO}_W(\kappa^+ - \Delta_2^0)$ fails. \square

It remains to investigate whether Proposition 4.3.81 can be extended to regular limit cardinals.

Chapter 5

Generalized Baire class functions

Throughout this chapter we work in ZFC and, unless otherwise specified, assume that κ is an uncountable cardinal satisfying $2^{<\kappa} = \kappa$. We also let $\mu = \text{cof}(\kappa)$. (Although some of the results would work with any infinite regular cardinal μ .) Moreover, all topological spaces are tacitly assumed to be regular and Hausdorff, unless otherwise specified.

The goal of the chapter is to study definable functions between μ -metrizable spaces of weight at most κ (see Section 2.4), focusing in particular on κ^+ -Borel measurable functions and their stratifications.¹

In classical descriptive set theory, two of the most fundamental results concerning Borel functions between separable metrizable spaces, due to Lebesgue, Hausdorff, and Banach, are the following ones.

Theorem 5.0.1 (See e.g. [Kec95, Theorem 11.6, or Theorems 24.3 and 24.10]). *Let X and Y be separable metrizable spaces, and further assume that either X is zero-dimensional or $Y = \mathbb{R}$. Then, the class of Borel functions from X to Y coincides with the closure under pointwise limits of the class of continuous functions.*

This can be refined by considering the Baire hierarchy, which is recursively defined by stipulating that a function $f: X \rightarrow Y$ is of Baire class 0 if it is continuous, while it is of Baire class $\xi > 0$ if it can be written as a pointwise limit of functions of lower Baire classes.

Theorem 5.0.2 (See e.g. [Kec95, Theorems 24.3 and 24.10]). *Let X and Y be separable metrizable spaces, and further assume that either X is zero-dimensional or $Y = \mathbb{R}$. Let $\xi < \omega_1$. Then, f is a Baire class ξ function if and only if it is $\Sigma_{\xi+1}^0$ -measurable.*

Notice that the extra hypotheses² on the spaces X and Y in Theorems 5.0.1 and 5.0.2 cannot be avoided. For instance, there are arbitrarily complex Borel

¹After completing this work, we were informed that the same kind of problems (but restricted to regular cardinals) were tackled in [Nob18] using completely different methods. Unfortunately, the proof of [Nob18, Theorem 4.12], which is the main result in this direction from that source, is flawed, and the definition of κ -Baire class ξ functions given there cannot work as expected. We will come back to this issue in Section 5.5.

²Slight variations are possible. For example, one could let Y be an interval in \mathbb{R} , or \mathbb{R}^n , or \mathbb{C}^n , and so on.

functions between the real line \mathbb{R} and the Baire space ${}^\omega\omega$, but the closure under pointwise limits of the class of continuous functions between such spaces reduces to the collection of constant functions.

Moving to generalized descriptive set theory, Borel and Σ_ξ^0 -measurable functions admit straightforward generalizations: κ^+ -Borel measurable and κ^+ - Σ_ξ^0 -measurable functions. We recall that a function $f: X \rightarrow Y$ is $\mathbf{\Gamma}$ -measurable if $f^{-1}(U) \in \mathbf{\Gamma}(X)$ for every open $U \subseteq Y$. In particular, f is κ^+ -**Borel measurable** if $f^{-1}(U) \in \kappa^+\text{-Bor}(X)$ for all $U \in \kappa^+\text{-}\Sigma_1^0(Y)$ (equivalently, for all $U \in \kappa^+\text{-Bor}(Y)$). If Y has weight at most κ , then for every κ^+ -Borel function $f: X \rightarrow Y$ there is $1 \leq \xi < \kappa^+$ such that $f^{-1}(U) \in \kappa^+\text{-}\Sigma_\xi^0(X)$ for all open sets $U \subseteq Y$: in this case, we say that f is κ^+ - Σ_ξ^0 -**measurable**. We denote the collection of all κ^+ - Σ_ξ^0 -measurable functions from X to Y by $\mathcal{M}_\xi(X, Y)$, and to simplify the notation we also write $\mathcal{M}_{<\xi}(X, Y)$ instead of $\bigcup_{1 \leq \xi' < \xi} \mathcal{M}_{\xi'}(X, Y)$. In a similar fashion, one can also define κ^+ - Δ_ξ^0 -measurable functions and alike.

As we will see in Section 5.1, the analysis of pointwise limits is instead more surprising. To simplify the present discussion, let us temporarily assume that κ is regular and work with κ -metrizable spaces. Although the topology on such spaces is completely determined by κ -limits, this kind of limits are no longer sufficient to generate the collection of κ^+ -Borel measurable functions. In fact, when $\kappa > \omega$ the closure under κ -limits of the class of continuous functions is precisely the collection of all functions which are κ^+ - Σ_n^0 -measurable for some finite $n \geq 1$ (Corollary 5.3.4). This forces us to consider other well-studied kinds of pointwise limits, i.e. limits over directed sets (Section 2.5), and eventually get the following result.

Theorem 5.0.3. *Let X and Y be κ -metrizable spaces of weight at most κ . Then, the class of κ^+ -Borel measurable functions between X and Y coincides with the closure under pointwise D -limits of the class of continuous functions, where D varies among all directed sets of size at most κ .*

The analogue of this theorem in the classical setting $\kappa = \omega$ holds as well, but it is subsumed by the stronger Theorem 5.0.1, which drastically reduces the limits to be employed to a single sequential limit. Although this is no longer possible in the generalized setup $\kappa > \omega$, we observe that it is enough to use sequential limits together with limits over the partial order Fin_κ of finite subsets of κ (Theorem 5.1.5), or even just a single kind of non-sequential limit, namely, $\widehat{\text{Fin}}_\kappa$ -limits (Theorem 5.1.8). Notice also that in Theorem 5.0.3 there is no additional hypothesis on the spaces involved: this difference from Theorem 5.0.1 is only apparent, though, as κ -metrizability implies zero-dimensionality when $\kappa > \omega$ (see Theorem 2.4.1).

Getting a higher analogue of Theorem 5.0.2 is an even more delicate matter, addressed in Section 5.3. It turns out that there are serious obstacles at limit levels of cofinality smaller than κ . Nevertheless, we managed to find a reasonable definition of generalized Baire class ξ functions (Definition 5.3.12) and, using an argument quite different from the classical one, prove the following result (see Theorem 5.3.14).

Theorem 5.0.4. *Let X and Y be κ -metrizable spaces of weight at most κ , and assume that Y is spherically complete. Let $\xi < \kappa^+$. Then, $f: X \rightarrow Y$ is a κ -Baire class ξ function if and only if it is κ^+ - $\Sigma_{\xi+1}^0$ -measurable.*

Although somewhat technical for the problematic levels ξ , our definition of κ -Baire class ξ functions is close to optimal, as shown by the various counterexamples presented in Section 5.3.1. The additional spherically completeness hypothesis on Y can be avoided using slight variations of such definition (see the discussion after Theorem 5.3.14).

The case of singular cardinals κ needs further adjustments, but one can still get results along the lines of Theorems 5.0.3 and 5.0.4 (see Section 5.3.2 and, in particular, Theorem 5.3.16).

Along the way, we prove other structural results concerning κ^+ - Σ_ξ^0 -measurable functions, among which it is worth mentioning the following ones (which are the counterparts in the generalized context of some classical results coming from [MR09b]):

- (a) A characterization of κ -Baire class 1 functions in terms of limits of surprisingly simple Lipschitz functions, called κ -full functions (Section 5.2).
- (b) A characterization of κ^+ - Σ_ξ^0 -measurable functions in terms of *uniform* limits of simpler functions (Section 5.4).

5.1 Generalized Borel functions as limits of continuous functions

We recall that we denote by \mathcal{M}_κ the collection of all μ -metrizable spaces of weight at most κ (see Section 2.4).

Let Γ be a boldface pointclass. Let $X, Y \in \mathcal{M}_\kappa$, and let \mathcal{F} be some set of functions between X and Y . A function $f: X \rightarrow Y$ is **locally in \mathcal{F} on a Γ -partition** $(A_\alpha)_{\alpha < \nu}$ of X if for each $\alpha < \nu$ there is $f_\alpha \in \mathcal{F}$ such that $f \upharpoonright A_\alpha = f_\alpha \upharpoonright A_\alpha$ for every $\alpha < \nu$. We will often consider functions which are locally constant on a Γ -partition.

Lemma 5.1.1. *Let $X, Y \in \mathcal{M}_\kappa$, and let $\xi < \kappa^+$ be a limit ordinal. If $f: X \rightarrow Y$ is locally constant on a finite κ^+ - Δ_ξ^0 -partition of X , then*

$$f \in \text{cof}(\xi)\text{-lim } \mathcal{M}_{<\xi}(X, Y).$$

Proof. Let $n \in \omega$ and $(A_j)_{j \leq n}$ be a finite κ^+ - Δ_ξ^0 -partition of X such that f is constant with value y_j on each A_j . Using Lemma 3.3.16(2) if $\mu > \omega$ or [DMR25, Proposition 4.2.1] if $\mu = \omega$, we can find a sequence of ordinals $(\xi_i)_{i < \text{cof}(\xi)}$ cofinal in ξ and sets $B_i^j \in \kappa^+$ - $\Delta_{\xi_i}^0(X)$ such that $A_j = \bigcup_{i < \text{cof}(\xi)} B_i^j$ for every $j \leq n$, and moreover $B_i^j \subseteq B_{i'}^j$ for every $i \leq i' < \text{cof}(\xi)$. Fix $\bar{y} \in Y$, and for $i < \text{cof}(\xi)$ let

$$f_i(x) = \begin{cases} y_j, & \text{if } x \in B_i^j \text{ for some } j \leq n, \\ \bar{y}, & \text{otherwise.} \end{cases}$$

Notice that f is well-defined because $B_i^j \subseteq A_j$ and $A_j \cap A_{j'} = \emptyset$ for every $j \neq j'$. Since $X \setminus \bigcup_{j \leq n} B_i^j \in \kappa^+$ - $\Delta_{\xi_i}^0(X)$, each f_i is constant on a finite κ^+ - $\Delta_{\xi_i}^0$ -partition, and hence κ^+ - $\Sigma_{\xi_i}^0$ -measurable. It remains to show that $f = \lim_{i < \text{cof}(\xi)} f_i$. Given $x \in X$, let $j \leq n$ and $i < \text{cof}(\xi)$ be such that $x \in B_i^j$. Since the sequence $(B_i^j)_{i < \text{cof}(\xi)}$ is increasing, $x \in B_{i'}^j$ for every $i' \geq i$, and thus $f_{i'}(x) = y_j = f(x)$. \square

Remark 5.1.2. For later use (see Definition 5.3.8), we note that the family of functions $(f_i)_{i < \text{cof}(\xi)}$ from the proof of Lemma 5.1.1 satisfies a stronger form of convergence to f . More precisely, setting $X_i = \bigcup_{j \leq n} B_i^j$ we get a covering $(X_i)_{i < \text{cof}(\xi)}$ of X such that for every $i \leq i' < \text{cof}(\xi)$:

- $X_i \in \kappa^+ \text{-}\Sigma_\xi^0(X)$;
- $X_i \subseteq X_{i'}$;
- $f_{i'}(x) = f(x)$ for every $x \in X_i$.

We now consider limits over the directed set $\text{Fin}_\kappa = ([\kappa]^{< \aleph_0}, \subseteq)$ of finite subsets of κ , ordered by inclusion.

Proposition 5.1.3. *Let $X, Y \subseteq {}^\mu \kappa$, and fix an ordinal $1 \leq \xi < \kappa^+$. Then, every $f \in \mathcal{M}_{\xi+1}(X, Y)$ can be written as*

$$f = \lim_{d \in \text{Fin}_\kappa} f_d,$$

where each $f_d: X \rightarrow Y$ is locally constant on a finite $\kappa^+ \text{-}\Delta_\xi^0$ -partition of X .

Proof. Let \mathcal{T}_Y be the tree of Y . For each $s \in \mathcal{T}_Y$, let $N_s(Y) = N_s \cap Y$ and fix any $y_s \in N_s(Y)$. Let $\{P_\alpha^s \mid \alpha < \kappa\}$ be a family of non-empty $\kappa^+ \text{-}\Pi_\xi^0(X)$ -sets such that $f^{-1}(N_s(Y)) = \bigcup_{\alpha < \kappa} P_\alpha^s$.

Since $|\mathcal{T}_Y| \leq \kappa$, we can clearly work with the directed set $D = ([\mathcal{T}_Y \times \kappa]^{< \aleph_0}, \subseteq)$ instead of Fin_κ . Fix a non-empty $d \in D$. Let $S_d = \{s \in \mathcal{T}_Y \mid \exists \alpha < \kappa [(s, \alpha) \in d]\}$, and let β_0, \dots, β_k enumerate the set $\{\text{lh}(s) \mid s \in S_d\}$ in increasing order, for the appropriate $k \in \omega$. To simplify the notation, for $s \in \mathcal{T}_Y$ we let $\text{lev}(s) = j$ if and only if $\text{lh}(s) = \beta_j$, and for every $i \leq k$ we let

$$s|_i = \begin{cases} s \upharpoonright \beta_i & \text{if } \text{lh}(s) \geq \beta_i \\ s & \text{otherwise.} \end{cases}$$

Finally, for any $j \leq k$ we let $S_d^j = S_d^{j,0} \cup S_d^{j,1}$, where

$$\begin{aligned} S_d^{j,0} &= \{s|_j \mid s \in S_d \wedge \text{lev}(s) \geq j\} \quad \text{and} \\ S_d^{j,1} &= \{s \in S_d \mid \text{lev}(s) < j \wedge s \text{ maximal in } S_d\}. \end{aligned}$$

Clearly, S_d^j consists of pairwise incomparable sequences because of the maximality requirement in the definition of $S_d^{j,1}$. Moreover, S_d^j is finite because so is d . Notice also that if $s \in S_d^j$, then $s|_i \in S_d^i$ for every $i \leq j$. Finally, by definition S_d^k coincides with the set of all maximal elements of S_d .

We build a collection of finite $\kappa^+ \text{-}\Delta_\xi^0$ -partitions $\mathcal{C}_j = \{C_s^j \mid s \in S_d^j\}$ of X satisfying the following two conditions:

- (1) For every $s \in S_d^{j,0}$, $P_\alpha^t \subseteq C_s^j$ for every $(t, \alpha) \in d$ with $t \supseteq s$.
- (2) For every $0 < j \leq k$ and $s \in S_d^j$, $C_s^j \subseteq C_{s|_{j-1}}^{j-1}$.

In particular, \mathcal{C}_j refines \mathcal{C}_{j-1} . Notice that condition (2) is equivalent to: $C_s^j \subseteq C_{s|_i}^i$ for every $i \leq j \leq k$ and $s \in S_d^j$.

The construction is by recursion on $j \leq k$. If $j = 0$, then $S_d^j = S_d^{j,0} = \{s|_0 \mid s \in S_d\}$. For every $s \in S_d^{j,0}$, let $P_s = \bigcup \{P_\alpha^t \mid (t, \alpha) \in d \wedge t \supseteq s\}$. The finite family $\{P_s \mid s \in S_d^{j,0}\}$ consists of pairwise disjoint κ^+ - Π_ξ^0 -sets because $P_\alpha^t \subseteq f^{-1}(\mathbf{N}_t(Y)) \subseteq f^{-1}(\mathbf{N}_{t|_0}(Y))$ and κ^+ - $\Pi_\xi^0(X)$ is closed under finite unions. Using Corollary 3.3.18, let $\mathcal{C}_0 = \{C_s^0 \mid s \in S_d^{j,0} = S_d^j\}$ be any (finite) κ^+ - Δ_ξ^0 -partition of X separating the sets P_s from each other. It is clear that (1) holds by construction, while (2) needs not be checked in this case.

Assume now that $j > 0$, and that \mathcal{C}_{j-1} has already been defined. Fix any $s \in S_d^j$. We distinguish two cases. If $s \in S_d^{j,1}$, then $s = s|_{j-1} \in S_d^{j-1}$ and we can set $C_s^j = C_s^{j-1}$. With this choice, (2) is trivially satisfied. The remaining case is when $s \in S_d^{j,0}$. Let $\hat{S}_d^j = \{s|_{j-1} \mid s \in S_d^{j,0}\}$, and notice that $\hat{S}_d^j \subseteq S_d^{j-1,0} \subseteq S_d^{j-1}$. For each $\hat{s} \in \hat{S}_d^j$, we repeat the argument from the basic case $j = 0$ but working within $C_{\hat{s}}^{j-1}$ and considering only those $s \in S_d^{j,0}$ such that $s|_{j-1} = \hat{s}$. More precisely, for each such s let $P_s = \bigcup \{P_\alpha^t \mid (t, \alpha) \in d \wedge t \supseteq s\}$. By (1) applied to \hat{s} , we get $P_s \subseteq C_{\hat{s}}^{j-1}$. Moreover, the sets P_s are pairwise disjoint and they belong to κ^+ - $\Pi_\xi^0(X)$. So by Corollary 3.3.18 we can find a κ^+ - Δ_ξ^0 -partition $\{C_s^j \mid s \in S_d^{j,0}, s|_{j-1} = \hat{s}\}$ of $C_{\hat{s}}^{j-1}$ separating the sets P_s from each other. It is clear that both (1) and (2) are satisfied by construction.

For each $d \in D$, let $f_d: X \rightarrow Y$ be the unique function which is locally constant on the finite κ^+ - Δ_ξ^0 -partition \mathcal{C}_k and assumes value y_s on each $C_s^k \in \mathcal{C}_k$.

Claim 5.1.3.1. For every $(s, \alpha) \in d$ and $x \in P_\alpha^s$ there is $t \in \mathcal{T}_Y$ such that $t \supseteq s$ and $f_d(x) = y_t$.

Proof of the Claim. Let t be the unique sequence in S_d^k such that $x \in C_t^k$, so that $f_d(x) = y_t$. Let $j = \min\{\text{lev}(s), \text{lev}(t)\}$. Notice that $x \in P_\alpha^s \subseteq C_{s|_j}^j$ by (1). Also, $x \in C_{t|_j}^j$ by (2). Therefore $s|_j = t|_j$ because $C_{s|_j}^j \cap C_{t|_j}^j \neq \emptyset$. It follows that s and t are compatible, and indeed $s \subseteq t$ because t , being an element of S_d^k , is maximal in S_d . \square

To conclude the proof, we just need to show that $\lim_{d \in D} f_d = f$. Fix $x \in X$ and any $s \in \mathcal{T}_Y$ such that $f(x) \in \mathbf{N}_s(Y)$. Let $\alpha < \kappa$ be such that $x \in P_\alpha^s$, and set $d = \{(s, \alpha)\}$. Then, Claim 5.1.3.1 entails that for every $d' \supseteq d$ there is $t \supseteq s$ such that $f_{d'}(x) = y_t \in \mathbf{N}_t(Y)$, and since $\mathbf{N}_t(Y) \subseteq \mathbf{N}_s(Y)$ we are done. \square

Corollary 5.1.4. Let $X, Y \in \mathcal{M}_\kappa$, and further assume that $\dim(X) = \dim(Y) = 0$ if $\mu = \omega$. For every limit ordinal $\xi < \kappa^+$,

$$\mathcal{M}_{\xi+1}(X, Y) \subseteq \text{Fin}_\kappa\text{-lim}(\text{cof}(\xi)\text{-lim } \mathcal{M}_{<\xi}(X, Y)).$$

Proof. By Theorems 2.4.1 and 2.4.4, we can assume that $X, Y \subseteq {}^\mu\kappa$. Therefore it is enough to combine Proposition 5.1.3 with Lemma 5.1.1. \square

Given a family \mathcal{D} of directed sets and a collection of functions \mathcal{F} between topological spaces X and Y , we say that \mathcal{F} is closed under \mathcal{D} -limits if for every $D \in \mathcal{D}$ and every family $(f_d)_{d \in D}$ of functions from \mathcal{F} we have $\lim_{d \in D} f_d \in \mathcal{F}$

(whenever such limit exists). The \mathcal{D} -closure of \mathcal{F} is the smallest collection of functions which contains \mathcal{F} and is closed under \mathcal{D} -limits.

We consider the following families of directed sets:

- $\mathcal{D}_0 = \{\lambda \leq \kappa \mid \lambda \text{ regular}\} \cup \{\text{Fin}_\kappa\}$
- $\mathcal{D}_\kappa = \{D \mid |D| \leq \kappa\}$

Clearly, $\mathcal{D}_0 \subsetneq \mathcal{D}_\kappa$.

Theorem 5.1.5. *Let $X, Y \in \mathcal{M}_\kappa$, and assume that $\dim(X) = 0$ if $\mu = \omega$. For every function $f: X \rightarrow Y$, the following are equivalent:*

- (1) f is κ^+ -Borel measurable;
- (2) f is in the \mathcal{D}_0 -closure of the collection of all continuous functions;
- (3) f is in the \mathcal{D}_κ -closure of the collection of all continuous functions.

Clearly, in Theorem 5.1.5 we can replace \mathcal{D}_0 and \mathcal{D}_κ with any intermediate $\mathcal{D}_0 \subseteq \mathcal{D} \subseteq \mathcal{D}_\kappa$.

Proof. (1) \Rightarrow (2) First assume that either $\mu > \omega$, or else $\mu = \omega$ and $\dim(Y) = 0$. Then, we may assume, without loss of generality, that $X, Y \subseteq {}^\mu\kappa$ by Theorems 2.4.1 and 2.4.4. We show by induction on $\xi < \kappa^+$ that $\mathcal{M}_{\xi+1}(X, Y)$ is contained in the \mathcal{D}_0 -closure of $\mathcal{M}_1(X, Y)$. The basic case $\xi = 0$ is trivial, so assume that $\xi > 0$ and fix any $f \in \mathcal{M}_{\xi+1}(X, Y)$. If ξ is a limit ordinal, then since $\mathcal{M}_{<\xi}(X, Y) = \bigcup_{\xi' < \xi} \mathcal{M}_{\xi'+1}(X, Y)$ and $\text{cof}(\xi) \in \mathcal{D}_0$ (because $\text{cof}(\xi) \leq |\xi| \leq \kappa$ is regular), it is enough to use Corollary 5.1.4 and the inductive hypothesis. If instead $\xi = \xi' + 1$ is a successor ordinal, then we use Proposition 5.1.3 and the inductive hypothesis applied to ξ' : this works because if a function is locally constant on a finite κ^+ - Δ_ξ^0 -partition of X , then it is trivially κ^+ - $\Sigma_{\xi'+1}^0$ -measurable.

It remains to consider the case where $\mu = \omega$ but $\dim(Y) \neq 0$. First we prove the result for all κ^+ -Borel measurable functions with finite range, which are precisely the functions which are locally constant on a finite κ^+ -**Bor**-partition.

Claim 5.1.5.1. *If $f: X \rightarrow Y$ is locally constant on a finite κ^+ -**Bor**-partition, then f is in the \mathcal{D}_0 -closure of $\mathcal{M}_1(X, Y)$.*

Proof of the Claim. Let $Z = f(X)$. When construed as a function from X to Z , the map f is still κ^+ -Borel measurable. Since Z is finite, and hence discrete, then $Z \in \mathcal{M}_\kappa$ and $\dim(Z) = 0$, therefore we already know from what we proved above that f is in the \mathcal{D}_0 -closure of $\mathcal{M}_1(X, Z)$ (as computed among functions from X to Z). But since Z is finite, the latter coincides with the \mathcal{D}_0 -closure of $\mathcal{M}_1(X, Z)$ when viewed as a collection of functions from X to Y : since $\mathcal{M}_1(X, Z) \subseteq \mathcal{M}_1(X, Y)$, we are done. \square

Consider now an arbitrary κ^+ -Borel measurable function $f: X \rightarrow Y$. Let τ be the topology of Y . By [DMR25, Corollary 4.3.6], there is a topology $\tau' \supseteq \tau$ on Y such that $Y' = (Y, \tau') \in \mathcal{M}_\kappa$, $\dim(Y') = 0$, and κ^+ -**Bor**(Y') = κ^+ -**Bor**(Y). It follows that f , viewed as a function from X to Y' , is still κ^+ -Borel measurable, and thus it is κ^+ - $\Sigma_{\xi+1}^0$ -measurable for some $\xi < \kappa^+$. Since both X and Y' can now be construed as subspaces of ${}^\mu\kappa$ by Theorem 2.4.4, we

can apply Proposition 5.1.3 and get a family of functions $f_d: X \rightarrow Y'$ which are locally constant on a finite κ^+ -**Bor**-partition and such that $f = \lim_{d \in \text{Fin}_\kappa} f_d$, where the limit is computed with respect to Y' . Since $\tau' \supseteq \tau$, we still have $f = \lim_{d \in \text{Fin}_\kappa} f_d$ if the limit is computed with respect to Y , and clearly the functions f_d remain locally constant on the same finite κ^+ -**Bor**-partition if we step back from Y' to Y . Therefore we are done by Claim 5.1.5.1.

(2) \Rightarrow (3) Trivial, as $\mathcal{D}_0 \subseteq \mathcal{D}_\kappa$.

(3) \Rightarrow (1) Let $f = \lim_{d \in D} f_d$ with $(f_d)_{d \in D}$ a family of κ^+ -Borel measurable functions. Assume first that $\mu > \omega$, so that Y is zero-dimensional. Then, for every clopen $U \subseteq Y$,

$$f^{-1}(U) = \bigcup_{d \in D} \bigcap_{d' \geq d} f_{d'}^{-1}(U). \quad (5.1.1)$$

Since $|D| \leq \kappa$ and $f_{d'}^{-1}(U) \in \kappa^+$ -**Bor**(X), this proves that f is κ^+ -Borel measurable too. If instead $\mu = \omega$, given any open set $U \subseteq Y$ we consider an open covering $\{U_n \mid n \in \omega\}$ of U such that $\text{cl}(U_n) \subseteq U$ for every $n \in \omega$. Then,

$$f^{-1}(U) = \bigcup_{n \in \omega} \bigcup_{d \in D} \bigcap_{d' \geq d} f_{d'}^{-1}(\text{cl}(U_n)), \quad (5.1.2)$$

hence $f^{-1}(U) \in \kappa^+$ -**Bor**(X) again. \square

One might wonder whether the class \mathcal{D}_0 can be further reduced, still getting an analogue of Theorem 5.1.5. For example, in the classical setting, which would correspond to the case $\kappa = \omega$, it is enough to consider ω -limits (Theorem 5.0.1). This is no longer true in the uncountable setup. For example, the following proposition implies that if κ has uncountable cofinality, then $\mathcal{M}_{<\omega}(X, Y)$, which is a proper subclass of all κ^+ -Borel measurable functions if the κ^+ -Borel hierarchy on X does not collapse and $|Y| \geq 2$, is already closed under κ -limits, and thus it contains the \mathcal{D} -closure of $\mathcal{M}_1(X, Y)$ for $\mathcal{D} = \{\kappa\}$. (See also Corollary 5.3.4.) A similar phenomenon occurs if $\text{cof}(\kappa) = \omega$: in this case, Proposition 5.1.6 yields that $\mathcal{M}_{\omega_1}(X, Y)$ is closed under κ -limits, and therefore such limits are again not enough to generate all κ^+ -Borel measurable functions (if X and Y are large enough).

Proposition 5.1.6. *Let $X, Y \in \mathcal{M}_\kappa$. Let $\lambda \leq \kappa$ and $\xi < \kappa^+$ be such that ξ is limit and $\text{cof}(\xi) \neq \text{cof}(\lambda)$. Then, $\mathcal{M}_{<\xi}(X, Y)$ is closed under λ -limits.*

Proof. Assume first that $\text{cof}(\xi) < \text{cof}(\lambda)$. Suppose that $f = \lim_{\alpha < \lambda} f_\alpha$ for some sequence $(f_\alpha)_{\alpha < \lambda}$ of functions in $\mathcal{M}_{<\xi}(X, Y)$. Let $(\xi_i)_{i < \text{cof}(\xi)}$ be a sequence of ordinals cofinal in ξ . Then, for every $\alpha < \lambda$ there exists $i < \text{cof}(\xi)$ such that f_α is κ^+ - $\Sigma_{\xi_i}^0$ -measurable. Since $\text{cof}(\xi) < \text{cof}(\lambda)$, there exists some $\bar{i} < \text{cof}(\xi)$ such that

$$A = \{\alpha < \lambda \mid f_\alpha \text{ is } \kappa^+ \text{-}\Sigma_{\xi_{\bar{i}}}^0 \text{-measurable}\}$$

is unbounded in λ , so that $f = \lim_{\alpha \in A} f_\alpha$. Being a limit of κ^+ - $\Sigma_{\xi_{\bar{i}}}^0$ -measurable functions over an index set of size at most κ , we get that f is κ^+ - $\Sigma_{\xi_{\bar{i}+1}}^0$ -measurable (by the computations (5.1.1) and (5.1.2) in Theorem 5.1.5), and thus $f \in \mathcal{M}_{<\xi}(X, Y)$ because ξ is limit.

Assume now that $\text{cof}(\lambda) < \text{cof}(\xi)$. By Remark 2.5.1, if f is a λ -limit of functions in $\mathcal{M}_{<\xi}(X, Y)$, then $f = \lim_{\alpha < \text{cof}(\lambda)} f_\alpha$ for some sequence $(f_\alpha)_{\alpha < \text{cof}(\lambda)}$ of functions in the same class $\mathcal{M}_{<\xi}(X, Y)$. By the assumption on $\text{cof}(\lambda)$,

there is $1 \leq \xi' < \xi$ such that $f_\alpha \in \mathcal{M}_{\xi'}(X, Y)$ for all $\alpha < \text{cof}(\lambda)$. Using again the computations (5.1.1) and (5.1.2) in Theorem 5.1.5, we get $f \in \mathcal{M}_{\xi'+1}(X, Y) \subseteq \mathcal{M}_{<\xi}(X, Y)$, as desired. \square

On the other hand, short sequential limits do not suffice either. Indeed, if κ is regular, then $\kappa^+-\mathbf{\Delta}_2^0(X)$ is a κ -algebra. Therefore the class of $\kappa^+-\mathbf{\Delta}_2^0$ -measurable functions, which is contained in $\mathcal{M}_2(X, Y)$, is closed under D -limits for all directed sets D with $|D| < \kappa$, and thus it already contains the \mathcal{D} -closure of $\mathcal{M}_1(X, Y)$ for $\mathcal{D} = \{\lambda < \kappa \mid \lambda \text{ regular}\}$. It is open whether the class of κ^+ -Borel measurable functions can be realized as the \mathcal{D} -closure of $\mathcal{M}_1(X, Y)$ for $\mathcal{D} = \{\lambda \leq \kappa \mid \lambda \text{ regular}\}$ or $\mathcal{D} = \{\text{Fin}_\kappa\}$ (see Section 5.5 for more on the matter).

We conclude this section by showing that there is a variant of Fin_κ which is rich enough to generate the whole class of κ^+ -Borel measurable functions by itself. The idea is to still consider finite subsets of κ , but labeling each of their elements with an ordinal number. More precisely, for every $d \in {}^\kappa\kappa$ let $\text{supp}(d) = \{i < \kappa \mid d(i) \neq 0\}$ be the support of d , and let

$$\widehat{\text{Fin}}_\kappa = \{d \in {}^\kappa\kappa \mid \text{supp}(d) \text{ is finite and } d(i) < 2 + i \text{ for all } i < \kappa\},$$

be ordered pointwise, that is, for all $d, d' \in \widehat{\text{Fin}}_\kappa$ set

$$d \leq d' \iff d(i) \leq d'(i) \text{ for all } i < \kappa.$$

Clearly $\widehat{\text{Fin}}_\kappa \in \mathcal{D}_\kappa$, and $d \leq d' \Rightarrow \text{supp}(d) \subseteq \text{supp}(d')$.

Lemma 5.1.7. *For every $D \in \mathcal{D}_0$, there is a surjection $\iota: \widehat{\text{Fin}}_\kappa \rightarrow D$ which is order-preserving, i.e. $d \leq d' \Rightarrow \iota(d) \leq \iota(d')$ for every $d, d' \in \widehat{\text{Fin}}_\kappa$.*

Proof. If $D = \text{Fin}_\kappa$, then we let $\iota(d) = \text{supp}(d)$. If $D = \lambda$ for some regular $\lambda < \kappa$, we let $\iota(d) = d(\lambda)$. Finally, if $D = \kappa$ then we let $\iota(d) = \max \text{supp}(d)$. It is easy to check that in all three cases ι is as required. \square

The map ι from Lemma 5.1.7 allows us to simulate any D -limit with a $\widehat{\text{Fin}}_\kappa$ -limit, for every $D \in \mathcal{D}_0$. Indeed, if $f = \lim_{d \in D} f_d$, then $f = \lim_{d \in \widehat{\text{Fin}}_\kappa} \hat{f}_d$ once we set $\hat{f}_d = f_{\iota(d)}$. Combining this with Theorem 5.1.5 we then get:

Theorem 5.1.8. *Let $X, Y \in \mathcal{M}_\kappa$, and assume that $\dim(X) = 0$ if $\mu = \omega$. For every function $f: X \rightarrow Y$, the following are equivalent:*

- (1) f is κ^+ -Borel measurable;
- (2) f is in the $\widehat{\text{Fin}}_\kappa$ -closure of the collection of all continuous functions.

5.2 Generalized Baire class 1 functions and κ -full functions

The following definitions and results generalize [MR09b, Section 2] to the uncountable setup.

Definition 5.2.1. Let (X, d) be a \mathbb{G} -ultrametric space. A set $A \subseteq X$ is **full** if there is some $\rho \in \mathbb{G}^+$ such that $B_d(x, \rho) \subseteq A$ for every $x \in A$. When we want to single out a specific ρ witnessing that A is full, we say that A is full **with constant** ρ .

Obviously, if $\rho' \in \mathbb{G}^+$ is smaller than ρ , then every set $A \subseteq X$ which is full with constant ρ is also full with constant ρ' . If $X = {}^\mu\kappa$, then $B_d(x, r_\alpha) = N_{x \upharpoonright \alpha}$, and thus a set $A \subseteq {}^\mu\kappa$ is full with constant r_α if and only if $N_{x \upharpoonright \alpha} \subseteq A$ for every $x \in A$.

Proposition 5.2.2. Let (X, d) be a \mathbb{G} -ultrametric space. For every $\rho \in \mathbb{G}^+$, the collection of all full subsets of X with constant ρ is a complete algebra consisting of clopen sets. Therefore, the collection of all full subsets of X (with any constant) is a μ -subalgebra of its clopen sets.

Proof. It is obvious that the collection of full sets with constant ρ is closed under arbitrary unions and consists of open sets, so it is enough to show that it is also closed under complements. Let $A \subseteq X$ be full with constant ρ , and let $y \in X \setminus A$. Assume towards a contradiction that $B_d(y, \rho) \cap A \neq \emptyset$, as witnessed by x . Then, $B_d(x, \rho) = B_d(y, \rho)$ and hence $y \in A$ by fullness, a contradiction.

Finally, let $(A_\alpha)_{\alpha < \nu}$ for $\nu < \mu$ be a family of full sets, and let $\rho_\alpha \in \mathbb{G}^+$ be such that A_α is full with constant ρ_α . Since μ is regular and \mathbb{G} has degree μ , there is $\rho \in \mathbb{G}^+$ such that $\rho \leq \rho_\alpha$ for all $\alpha < \nu$. Then, each A_α is full with constant ρ , and thus so is $\bigcup_{\alpha < \nu} A_\alpha$. \square

Definition 5.2.3. Let (X, d) be a \mathbb{G} -ultrametric space, Y be any set, and ν be a cardinal. A function $f: X \rightarrow Y$ is called ν -**full (with constant** $\rho \in \mathbb{G}^+$) if $|\text{ran}(f)| \leq \nu$, and for every $y \in \text{ran}(f)$ its preimage $f^{-1}(y)$ is full (with constant ρ). The function f is $< \nu$ -**full (with constant** $\rho \in \mathbb{G}^+$) if it is ν' -full (with constant ρ) for some $\nu' < \nu$. Finally, f is **full (with constant** $\rho \in \mathbb{G}^+$) if it is ν -full (with constant ρ) for some cardinal ν or, equivalently, if $f^{-1}(y)$ is full (with constant ρ) for all $y \in \text{ran}(f)$.

Equivalently, f is ν -full (with constant ρ) if it is locally constant on a partition of X consisting of at most ν -many full sets (with constant ρ). Note also that if f is $< \nu$ -full for some $\nu \leq \mu$, then there is $\rho \in \mathbb{G}^+$ such that f is $< \nu$ -full with constant ρ .

As in the classical setting, full functions are intimately related to Lipschitz functions, where we say that a map f between two \mathbb{G} -metric spaces (X, d_X) and (Y, d_Y) is **Lipschitz (with constant** $R \in \mathbb{G}^+$) if for all $x, y \in X$

$$d_Y(f(x), f(y)) \leq R \cdot d_X(x, y).$$

(This makes sense because we assumed that \mathbb{G} is a field.)

Lemma 5.2.4. Let (X, d_X) be a \mathbb{G} -ultrametric space, and Y be a topological space. If $f: X \rightarrow Y$ is full, then it is continuous.

Moreover, if (Y, d_Y) is a \mathbb{G} -metric space, f has bounded³ range, and there is some $\rho \in \mathbb{G}^+$ such that f is full with constant ρ , then f is Lipschitz.

³A subset A of a \mathbb{G} -metric space (X, d) is **bounded** if there is $r \in \mathbb{G}^+$ such that $\text{diam}(A) \leq r$, i.e. $d(x, y) \leq r$ for all $x, y \in A$.

The second part of the lemma applies to any $< \mu$ -full function, and also to any full function with constant ρ whenever Y has bounded diameter.

Proof. The first part of the lemma is obvious, so we only prove the second one. Let $r' \in \mathbb{G}^+$ be such that $\text{diam}(\text{ran}(f)) \leq r'$, and set $R = r' \cdot \rho^{-1}$. Pick any $x, x' \in X$. If $d_X(x, x') < \rho$, then $d_Y(f(x), f(x')) = 0 \leq R \cdot d_X(x, x')$ because f is full with constant ρ . If instead $d_X(x, x') \geq \rho$, then

$$d_Y(f(x), f(x')) \leq r' = r' \cdot \rho^{-1} \cdot \rho \leq R \cdot d_X(x, x').$$

Thus f is Lipschitz with constant R . □

Lemma 5.2.5. *Let (X, d_X) and (Z, d_Z) be \mathbb{G} -ultrametric spaces, and let Y be any set. Let $f: X \rightarrow Y$ be a ν -full function, for some cardinal ν . If $h: Z \rightarrow X$ is a Lipschitz function, then $f \circ h: Z \rightarrow Y$ is ν -full. Moreover, if f were ν -full with constant $\rho \in \mathbb{G}^+$, then there is $\rho' \in \mathbb{G}^+$ such that $f \circ h$ is ν -full with constant ρ' . The same is true if we replace ν -full with $< \nu$ -full.*

Proof. Suppose that h is Lipschitz with constant $R \in \mathbb{G}^+$. It is enough to show that the preimage via h of a full set $A \subseteq X$ with constant $\rho \in \mathbb{G}^+$ is a full set with constant $\rho' = \rho \cdot R^{-1}$. Indeed, let $z \in Z$ be such that $h(z) \in A$ and let $z' \in Z$ be such that $d_Z(z, z') < \rho'$. Then, $d_X(h(z), h(z')) \leq R \cdot d_Z(z, z') < R \cdot \rho \cdot R^{-1} = \rho$, hence $h(z') \in A$. This shows that $B_{d_Z}(z, \rho') \subseteq h^{-1}(A)$. □

We now come to the problem of finding the “right” generalization of the classical notion of a Baire class 1 function. When we move to cardinals $\kappa > \omega$ and consider functions $f: X \rightarrow Y$ between two spaces $X, Y \in \mathcal{M}_\kappa$, we have two options: either we only consider κ -limits of continuous functions (i.e. the class $\kappa\text{-lim } \mathcal{M}_1(X, Y)$), or, in view of Theorem 5.1.5, we allow limits over arbitrary directed sets of size at most κ (i.e. we consider $\mathcal{D}_\kappa\text{-lim } \mathcal{M}_1(X, Y)$). In this thesis, the former are dubbed **κ -Baire class 1 functions**, while the latter are called **weak κ -Baire class 1 functions**. We are going to show that if κ is regular and Y is spherically complete, then the two notions (as well as all the intermediate ones) coincide, and if moreover X is a \mathbb{G} -ultrametric space, then this is the same as considering the class of κ -limits of κ -full functions.

Proposition 5.2.6. *Let κ be regular, and let $X, Y \subseteq {}^\kappa\kappa$ with Y superclosed. For every κ^+ - Σ_2^0 -measurable function $f: X \rightarrow Y$ there is a sequence of functions $(f_\alpha)_{\alpha < \kappa}$ such that $f = \lim_{\alpha < \kappa} f_\alpha$ and $f_\alpha: X \rightarrow Y$ is κ -full with constant r_α .*

If moreover we assume κ to be strong limit (hence inaccessible), then each f_α can be taken to be $< \kappa$ -full with constant r_α .

Proof. For $i, \ell < \kappa$, set $\mathcal{N}_i^\ell = \{y \in Y \mid y(i) = \ell\}$. Since \mathcal{N}_i^ℓ is clopen, $f^{-1}(\mathcal{N}_i^\ell) \in \kappa^+\text{-}\Sigma_2^0(X)$. Let $F_{j,i}^\ell$ be closed subsets of X such that $f^{-1}(\mathcal{N}_i^\ell) = \bigcup_{j < \kappa} F_{j,i}^\ell$. For every $i, j, \ell < \kappa$, let $\mathcal{T}_{j,i}^\ell$ be the pruned tree of $F_{j,i}^\ell$, and let \mathcal{T}_X be the pruned tree of X . Note that $\bigcup_{\ell < \kappa} f^{-1}(\mathcal{N}_i^\ell) = X$ for every $i < \kappa$, and therefore

$$\bigcup_{j, \ell < \kappa} \mathcal{T}_{j,i}^\ell = \mathcal{T}_X. \tag{5.2.1}$$

For every $s \in \mathcal{T}_X$ and $i < \kappa$, let (j_i^s, ℓ_i^s) be the least (with respect to the Gödel ordering) pair $(j, \ell) \in \kappa \times \kappa$ such that $s \in \mathcal{T}_{j,i}^\ell$. Notice that for every $x \in X$

and every $i < \kappa$ there is $\xi_i^x < \kappa$ such that $\ell_i^{x \upharpoonright \alpha} = \ell_i^{x \upharpoonright \xi_i^x}$ and $j_i^{x \upharpoonright \alpha} = j_i^{x \upharpoonright \xi_i^x}$, for all $\xi_i^x \leq \alpha < \kappa$. (Otherwise $x \notin \mathcal{T}_{j,i}^\ell$ for any $j, \ell < \kappa$, contradicting (5.2.1).) In particular, $\ell_i^{x \upharpoonright \alpha} = f(x)(i)$ for every $\alpha \geq \xi_i^x$.

Let \mathcal{T}_Y be the superclosed pruned tree of Y , and for every sequence $t \in \mathcal{T}_Y$ pick some $y_t \in Y$ such that $t \subseteq y_t$. Let $\varphi': \mathcal{T}_X \rightarrow {}^{<\kappa}\kappa$ be such that $\text{lh}(\varphi'(s)) = \text{lh}(s)$ and $\varphi'(s)(i) = \min\{\ell_i^s, \text{lh}(s)\}$ for every $i < \text{lh}(s)$. We define the map $\varphi: \mathcal{T}_X \rightarrow \mathcal{T}_Y$ by letting $\varphi(s) \subseteq \varphi'(s)$ be the longest sequence still in \mathcal{T}_Y , namely,

$$\varphi(s) = \bigcup \{v \in \mathcal{T}_Y \mid v \subseteq \varphi'(s)\}.$$

Note that $\varphi(s) \in \mathcal{T}_Y$ since \mathcal{T}_Y is superclosed. Finally, for every $\alpha < \kappa$, let $f_\alpha: X \rightarrow Y$ be defined by

$$f_\alpha(x) = y_{\varphi(x \upharpoonright \alpha)}.$$

Notice that $\varphi'(s) \in {}^{\text{lh}(s)}(\text{lh}(s) + 1)$, thus f_α attains at most $|\alpha(\alpha + 1)|$ -many values. This means that $|\text{ran}(f_\alpha)| \leq \kappa$, and furthermore $|\text{ran}(f_\alpha)| < \kappa$ when κ is strong limit. Moreover, if $x \upharpoonright \alpha = y \upharpoonright \alpha$, then $f_\alpha(x) = f_\alpha(y)$. Therefore, f_α is κ -full (or even $< \kappa$ -full, if κ is strong limit) with constant r_α .

It remains to show that $f = \lim_{\alpha < \kappa} f_\alpha$. To this aim, fix an element $x \in A$: we need to prove that for every $\gamma < \kappa$ there exists $\bar{\alpha} < \kappa$ such that $f_\alpha(x) \upharpoonright \gamma = f(x) \upharpoonright \gamma$ for every $\alpha \geq \bar{\alpha}$. Set

$$\bar{\alpha} = \sup\{\gamma, \xi_i^x, f(x)(i) \mid i < \gamma\}.$$

Note that $\bar{\alpha} < \kappa$ because κ is regular. Fix any $i < \gamma$ and $\alpha \geq \bar{\alpha}$. Since $\bar{\alpha} \geq \xi_i^x$, then $\ell_i^{x \upharpoonright \alpha} = f(x)(i)$. Since $\bar{\alpha} \geq f(x)(i) = \ell_i^{x \upharpoonright \alpha}$, then $\varphi'(x \upharpoonright \alpha)(i) = \ell_i^{x \upharpoonright \alpha}$; this implies that $\varphi'(x \upharpoonright \alpha) \upharpoonright \gamma = f(x) \upharpoonright \gamma \in \mathcal{T}_Y$, hence $\varphi(x \upharpoonright \alpha)(i) = \varphi'(x \upharpoonright \alpha)(i)$. Finally, since $\bar{\alpha} \geq \gamma$, then $f_\alpha(x)(i) = \varphi(x \upharpoonright \alpha)(i)$. It follows that $f_\alpha(x) \upharpoonright \gamma = f(x) \upharpoonright \gamma$, as desired. \square

In the uncountable regular case, the following is the analogue of [MR09b, Corollary 2.16].

Theorem 5.2.7. *Let κ be regular, $X, Y \in \mathcal{M}_\kappa$, and suppose that Y is spherically complete. For every $f: X \rightarrow Y$, the following are equivalent:*

- (1) f is a κ -Baire class 1 function;
- (2) f is a weak κ -Baire class 1 function;
- (3) f is κ^+ - Σ_2^0 -measurable.

If X is a \mathbb{G} -ultrametric space, then the above conditions are also equivalent to

- (4) $f = \lim_{\alpha < \kappa} f_\alpha$ with f_α a κ -full function with constant ρ_α , for some $\rho_\alpha \in \mathbb{G}^+$.

In case κ is strong limit (and X is \mathbb{G} -ultrametric), item (4) can be replaced by

- (4') f is a κ -limit of $< \kappa$ -full functions,

and if Y is a \mathbb{G} -metric space we can further add:

- (5) f is a κ -limit of Lipschitz functions.

Notice that because of Lemma 5.2.4, condition (5) can be added to the list also when X is \mathbb{G} -ultrametric and Y is a \mathbb{G} -metric space with bounded diameter, independently of whether κ is strong limit or not.

Proof. The implication (1) \Rightarrow (2) is obvious. To see (2) \Rightarrow (3), suppose that f is the D -limit of a family $(f_d)_{d \in D}$ of continuous functions $f_d: X \rightarrow Y$, for some $D \in \mathcal{D}_\kappa$. Since Y is zero-dimensional, it is enough to check that $f^{-1}(U) \in \kappa^+ \text{-}\Sigma_2^0(X)$ for any clopen set $U \subseteq Y$, which is the case because $f^{-1}(U) = \bigcup_{d \in D} \bigcap_{d' \geq d} f_{d'}^{-1}(U)$. To prove (3) \Rightarrow (1), first notice that by Theorem 2.4.1 we can assume that $X, Y \subseteq {}^\kappa\kappa$ with Y superclosed: applying Proposition 5.2.6 and using Lemma 5.2.4, we get the desired result.

Suppose now that (X, d) is a \mathbb{G} -ultrametric space. Without loss of generality, we can again assume that Y is a superclosed subset of ${}^\kappa\kappa$. By Proposition 2.4.2 there is a topological embedding $h: X \rightarrow {}^\kappa\kappa$ which is Lipschitz. Let $X' = \text{ran}(h)$. Since the map $f \circ h^{-1}$ is $\kappa^+ \text{-}\Sigma_2^0$ -measurable, by Proposition 5.2.6, it is the κ -limit of a sequence $(f'_\alpha)_{\alpha < \kappa}$ of κ -full (or $< \kappa$ -full, if κ is strong limit) functions $f'_\alpha: X' \rightarrow Y$ with constant r_α . Then, by Lemma 5.2.5 each $f_\alpha = f'_\alpha \circ h: X \rightarrow Y$ is κ -full (or $< \kappa$ -full) with constant ρ_α for some $\rho_\alpha \in \mathbb{G}^+$, and $f = \lim_{\alpha < \kappa} f_\alpha$. This proves (3) \Rightarrow (4), and also (3) \Rightarrow (4') if κ is strong limit. Lemma 5.2.4 yields (4) \Rightarrow (1) and (4') \Rightarrow (1).

Finally, assume that κ is strong limit, X is \mathbb{G} -ultrametric, and Y is a \mathbb{G} -metric space. Every $< \kappa$ -full function from X to Y has bounded range because the order of the field \mathbb{G} has cofinality κ , therefore (4') \Rightarrow (5) by Lemma 5.2.4 again. Finally, (5) \Rightarrow (1) because Lipschitz functions are obviously continuous. \square

5.3 Level-by-level analysis of generalized Borel functions

The goal of this section is to isolate the “right” notion of Baire class ξ function in the generalized context. More in detail, we seek a stratification of κ^+ -Borel measurable functions defined in terms of suitable limit operators, which starts from continuous functions and is “compatible” with $\kappa^+ \text{-}\Sigma_\xi^0$ -measurability in the sense of Theorem 5.0.2. We first deal with regular cardinals, as in this case we can exploit the results of Section 5.2 through a suitable change-of-topology argument. After that, we briefly discuss what needs to be modified in the singular case in order to get analogous definitions and results.

5.3.1 The regular case

The results from Section 5.2 allow us to obtain structural information on the stratification in terms of measurability of κ^+ -Borel functions. Indeed, using also Lemma 5.3.1 we can prove that if ξ is a successor ordinal or a limit ordinal of cofinality κ , then $\kappa^+ \text{-}\Sigma_{\xi+1}^0$ -measurable functions can be characterized as κ -limits of functions from lower levels (see Theorem 5.3.3).

Lemma 5.3.1. *Let κ be regular, $X \in \mathcal{M}_\kappa$, and $1 \leq \xi < \kappa^+$. For every family $(A_i)_{i < \kappa}$ of $\kappa^+ \text{-}\Sigma_\xi^0$ -subsets of X there exist $D \subseteq {}^\kappa\kappa$ and a bijection $h: D \rightarrow X$ such that:*

- (1) h is continuous;

(2) for every $s \in {}^{<\kappa}\kappa$, one of the following holds, depending on the given ξ :

Case 1 if $\xi = \xi' + 1 > 1$ is successor, then $h(\mathbf{N}_s \cap D) \in \kappa^+ - \mathbf{\Pi}_{\xi'}^0$;

Case 2 if ξ is limit with $\text{cof}(\xi) = \kappa$, then $h(\mathbf{N}_s \cap D) \in \kappa^+ - \mathbf{\Pi}_{\nu(\text{lh}(s))}^0$, for some fixed non-decreasing function $\nu: \kappa \rightarrow \xi$;

Case 3 if ξ is limit with $\text{cof}(\xi) < \kappa$, then $h(\mathbf{N}_s \cap D) \in \kappa^+ - \mathbf{\Pi}_{\xi}^0$;

(3) $h^{-1}(A_i) \in \kappa^+ - \mathbf{\Sigma}_1^0(D)$ for every $i < \kappa$.

Proof. Without loss of generality, we can assume that $X \subseteq {}^{\kappa}\kappa$ by Theorem 2.4.1. Let $(C_\gamma)_{\gamma < \kappa}$ be a family of sets in $\bigcup_{\xi' < \xi} \kappa^+ - \mathbf{\Pi}_{\xi'}^0$ such that $A_i = \bigcup_{\gamma \in K_i} C_\gamma$ for some $K_i \subseteq \kappa$.

We construct a family $\{B_s \mid s \in {}^{<\kappa}\kappa\}$ of subsets of X such that for every $\alpha, \gamma < \kappa$ and $s \in {}^{<\kappa}\kappa$:

- (i) $B_s \subseteq B_t$ whenever $t \subseteq s$;
- (ii) $\{B_t \mid \text{lh}(t) = \alpha\}$ is a disjoint covering of X ;
- (iii) $B_s \subseteq \mathbf{N}_{t_s} \cap X$ for some $t_s \in {}^{<\kappa}\kappa$ with $\text{lh}(t_s) = \text{lh}(s)$;
- (iv) $B_s \in \kappa^+ - \mathbf{\Pi}_{\xi'}^0$ if $\xi = \xi' + 1 > 1$ is a successor ordinal; $B_s \in \kappa^+ - \mathbf{\Pi}_{\nu(\text{lh}(s))}^0$ for a suitable non-decreasing map $\nu: \kappa \rightarrow \xi$ if ξ is limit ordinal of cofinality κ ; $B_s \in \kappa^+ - \mathbf{\Pi}_{\xi}^0$ if ξ is a limit ordinal of cofinality smaller than κ ;
- (v) $C_\gamma = \bigcup_{t \in K} B_t$ for some $K \subseteq \gamma + 1$.

Consider the induced map $h: D \subseteq {}^{\kappa}\kappa \rightarrow X$, $x \mapsto h(x) \in \bigcap_{i < \kappa} B_{x \upharpoonright i}$, where $D = \{x \in {}^{\kappa}\kappa \mid \bigcap_{i < \kappa} B_{x \upharpoonright i} \neq \emptyset\}$. Note that h is well-defined and continuous (so that (1) is satisfied) by (i) and (iii). Condition (ii) entails that h is a bijection and that $h(\mathbf{N}_s \cap D) = B_s$, so that (2) is satisfied as well by (iv). Finally, observe that by (v) and (ii) applied to $\alpha = \gamma + 1$,

$$\begin{aligned} h^{-1}(C_\gamma) &= \bigcup_{t \in K} h^{-1}(B_t) = \bigcup_{t \in K} (\mathbf{N}_t \cap D) \\ h^{-1}(X \setminus C_\gamma) &= \bigcup_{t \in \gamma + 1 \setminus K} h^{-1}(B_t) = \bigcup_{t \in \gamma + 1 \setminus K} (\mathbf{N}_t \cap D). \end{aligned}$$

Hence $h^{-1}(C_\gamma)$ is clopen for every $\gamma < \kappa$, so that (3) follows by the choice of the sets C_γ .

The construction of the family $\{B_s \mid s \in {}^{<\kappa}\kappa\}$ is by induction on $\alpha = \text{lh}(s) < \kappa$; when ξ is a limit ordinal of cofinality κ , along the recursive process we also define the value $\nu(\alpha)$ of the function $\nu: \kappa \rightarrow \xi$ on input α . We set $B_\emptyset = X$ and, if ξ is limit of cofinality κ , $\nu(0) = 1$.

Suppose first that $\alpha = \gamma + 1$ is a successor ordinal. Let $\bar{\xi} < \xi$ be such that $C_\gamma \in \kappa^+ - \mathbf{\Pi}_{\bar{\xi}}^0$. By Lemma 3.3.16 (applied to $X \setminus C_\gamma$), there are $J \leq \kappa$ and a $\kappa^+ - \mathbf{\Pi}_{\bar{\xi}}^0$ -partition $\mathcal{C}_\gamma = \{C_\gamma^{(j)} \mid j < J\}$ of X such that $C_\gamma^{(0)} = C_\gamma$. For each $t \in {}^\alpha\kappa$, $j < J$, and $s \in {}^\gamma\kappa$, let $P_{t,j,s} = \mathbf{N}_t \cap C_\gamma^{(j)} \cap B_s$. By inductive hypothesis, $\{B_s \mid s \in {}^\gamma\kappa\}$ is a disjoint covering of X , hence so is

$$\mathcal{P} = \{P_{t,j,s} \mid t \in {}^\alpha\kappa, j < J, s \in {}^\gamma\kappa\}.$$

For every $s \in {}^\gamma\kappa$, fix a bijection $f_s: \kappa \rightarrow {}^\alpha\kappa \times J \times \{s\}$ and set $B_{s \smallfrown \beta} = P_{f_s(\beta)}$ for all $\beta < \kappa$. We now check that all of (i)–(v) are satisfied. Condition (i) is

obvious, while condition (ii) follows from the fact that $\{B_t \mid \text{lh}(t) = \alpha\} = \mathcal{P}$. Condition (iii) holds by construction, as it is enough to let $t_{s \frown \beta}$ be the first coordinate of $f_s(\beta)$, for every $s \in \gamma\kappa$ and $\beta < \kappa$. We now check (iv). If $\xi = \xi' + 1 > 1$ is a successor ordinal, then by inductive hypothesis $B_s \in \kappa^+ \text{-}\mathbf{\Pi}_{\xi'}^0$ for every $s \in \gamma\kappa$, and since $C_\gamma^{(j)} \in \kappa^+ \text{-}\mathbf{\Pi}_{\xi}^0$ and $\bar{\xi} \leq \xi'$, we conclude $B_{s \frown \beta} \in \kappa^+ \text{-}\mathbf{\Pi}_{\xi'}^0$. If ξ is a limit ordinal of cofinality κ , set $\nu(\alpha) = \max\{\nu(\gamma), \bar{\xi}\} < \xi$. By inductive hypothesis again, $B_s \in \kappa^+ \text{-}\mathbf{\Pi}_{\nu(\gamma)}^0 \subseteq \kappa^+ \text{-}\mathbf{\Pi}_{\nu(\alpha)}^0$ for every $s \in \gamma\kappa$, and $C_\gamma^{(j)} \in \kappa^+ \text{-}\mathbf{\Pi}_{\xi}^0 \subseteq \kappa^+ \text{-}\mathbf{\Pi}_{\nu(\alpha)}^0$: hence $B_{s \frown \beta} \in \kappa^+ \text{-}\mathbf{\Pi}_{\nu(\alpha)}^0$. If instead ξ is a limit ordinal of cofinality smaller than κ , then by inductive hypothesis $B_s \in \kappa^+ \text{-}\mathbf{\Pi}_{\xi}^0$ and $C_\gamma^{(j)} \in \kappa^+ \text{-}\mathbf{\Pi}_{\xi}^0 \subseteq \kappa^+ \text{-}\mathbf{\Pi}_{\xi}^0$, hence $B_{s \frown \beta} \in \kappa^+ \text{-}\mathbf{\Pi}_{\xi}^0$. Finally, using that \mathcal{P} is a disjoint covering of X , it is easy to verify that (v) holds by taking $K = \{s \frown \beta \mid s \in \gamma\kappa \wedge B_{s \frown \beta} \subseteq C_\gamma^{(0)}\}$.

Now let $\alpha > 0$ be a limit ordinal. For every $s \in {}^\alpha\kappa$, set $B_s = \bigcap_{\beta < \alpha} B_{s \upharpoonright \beta}$. Then (i) is satisfied by construction, while (ii) holds because by inductive hypothesis it is satisfied at all levels $\beta < \alpha$. Condition (iii) is satisfied by setting $t_s = \bigcup_{\beta < \alpha} t_{s \upharpoonright \beta}$. To prove (iv) we again consider various cases. If $\xi = \xi' + 1$ is a successor ordinal, then by inductive hypothesis $B_{s \upharpoonright \beta} \in \kappa^+ \text{-}\mathbf{\Pi}_{\xi'}^0$ for every $\beta < \alpha$, hence $B_s = \bigcap_{\beta < \alpha} B_{s \upharpoonright \beta} \in \kappa^+ \text{-}\mathbf{\Pi}_{\xi'}^0$ as well. If ξ is a limit ordinal of cofinality κ , then we set $\nu(\alpha) = \sup_{\beta < \alpha} \nu(\beta)$: since $\text{cof}(\xi) = \kappa$, we still have $\nu(\alpha) < \xi$. Then, by inductive hypothesis $B_{s \upharpoonright \beta} \in \kappa^+ \text{-}\mathbf{\Pi}_{\nu(\beta)}^0 \subseteq \kappa^+ \text{-}\mathbf{\Pi}_{\nu(\alpha)}^0$, hence $B_s = \bigcap_{\beta < \alpha} B_{s \upharpoonright \beta} \in \kappa^+ \text{-}\mathbf{\Pi}_{\nu(\alpha)}^0$ too. Finally, if ξ is a limit ordinal with $\text{cof}(\xi) < \kappa$, then $B_{s \upharpoonright \beta} \in \kappa^+ \text{-}\mathbf{\Pi}_{\xi}^0$ for every $\beta < \alpha$, and thus $B_s = \bigcap_{\beta < \alpha} B_{s \upharpoonright \beta} \in \kappa^+ \text{-}\mathbf{\Pi}_{\xi}^0$ as well. Condition (v) is relevant only at successor stages, and hence our proof is complete. \square

Corollary 5.3.2. *Let κ be regular, $(X, \tau) \in \mathcal{M}_\kappa$, and $1 \leq \xi < \kappa^+$. Given a family $(A_i)_{i < \kappa}$ of $\kappa^+ \text{-}\mathbf{\Sigma}_\xi^0(\tau)$ subsets of X , there is a topology τ' on X such that:*

- (1) $(X, \tau') \in \mathcal{M}_\kappa$;
- (2) $\tau \subseteq \tau' \subseteq \kappa^+ \text{-}\mathbf{\Sigma}_\xi^0(\tau)$ if ξ is either a successor ordinal or a limit ordinal with cofinality κ , and $\tau \subseteq \tau' \subseteq \kappa^+ \text{-}\mathbf{\Sigma}_{\xi+1}^0(\tau)$ otherwise;
- (3) $A_i \in \kappa^+ \text{-}\mathbf{\Sigma}_1^0(\tau')$ for every $i < \kappa$.

Proof. Let $h: D \rightarrow X$ be as in Lemma 5.3.1. Then, it is enough to let τ' be the pushforward via h of the topology induced by ${}^\kappa\kappa$ on D . \square

Theorem 5.3.3. *Let κ be regular, $X, Y \in \mathcal{M}_\kappa$, and suppose that Y is spherically complete. Let $1 \leq \xi < \kappa^+$ be either a successor ordinal or a limit ordinal of cofinality κ . Then*

$$\mathcal{M}_{\xi+1}(X, Y) = \kappa\text{-lim } \mathcal{M}_\xi(X, Y).$$

If ξ is limit, then we further have

$$\mathcal{M}_{\xi+1}(X, Y) = \kappa\text{-lim } \mathcal{M}_{<\xi}(X, Y).$$

Proof. For the right-to-left inclusions, consider $f = \lim_{\alpha < \kappa} f_\alpha$ with $f_\alpha \in \mathcal{M}_\xi(X, Y)$ or $f_\alpha \in \mathcal{M}_{<\xi}(X, Y)$, depending on whether ξ is successor or limit. Fix a clopen basis $(U_i)_{i < \kappa}$ for Y : it is enough to show that $f^{-1}(U_i) \in \kappa^+ \text{-}\Sigma_{\xi+1}^0(X)$ for every $i < \kappa$. As U_i is clopen, we have

$$f^{-1}(U_i) = \bigcup_{\bar{\alpha} < \kappa} \bigcap_{\bar{\alpha} \leq \alpha < \kappa} f_\alpha^{-1}(U_i),$$

and since in any case $f_\alpha^{-1}(U_i) \in \kappa^+ \text{-}\Delta_\xi^0(X)$, the result easily follows.

We now deal with the inclusion from left to right. Let τ be the topology of X . The proof is by induction on $1 \leq \xi < \kappa^+$. The base case $\xi = 1$ is Theorem 5.2.7, therefore we can move to the case $\xi > 1$.

Assume first that ξ is a successor ordinal: in this case, we want to show that f is a κ -limit of $\kappa^+ \text{-}\Sigma_\xi^0$ -measurable functions $f_\alpha: X \rightarrow Y$. As $f \in \mathcal{M}_{\xi+1}(X, Y)$, for every open set $U \subseteq Y$ we can write $f^{-1}(U) = \bigcup_{i < \kappa} A_i$ with $A_i \in \kappa^+ \text{-}\Pi_\xi^0(X)$. Applying Corollary 5.3.2 to the family $(X \setminus A_i)_{i < \kappa}$, we obtain a topology τ' on X such that $(X, \tau') \in \mathcal{M}_\kappa$, $\tau \subseteq \tau' \subseteq \kappa^+ \text{-}\Sigma_\xi^0(\tau)$, and $A_i \in \kappa^+ \text{-}\Pi_1^0(\tau')$ for every $i < \kappa$. Then, f is $\kappa^+ \text{-}\Sigma_2^0(\tau')$ -measurable: applying Theorem 5.2.7, we get that $f = \lim_{\alpha < \kappa} f_\alpha$ with each $f_\alpha: (X, \tau') \rightarrow Y$ continuous. Since $\tau' \subseteq \kappa^+ \text{-}\Sigma_\xi^0(\tau)$, the functions f_α are $\kappa^+ \text{-}\Sigma_\xi^0$ -measurable as maps between (X, τ) and Y , hence we are done.

Now assume that ξ is a limit ordinal with $\text{cof}(\xi) = \kappa$: we need to show that f is a κ -limit of a sequence of functions $f_\alpha: X \rightarrow Y$, where each f_α is $\kappa^+ \text{-}\Sigma_{\xi_\alpha}^0$ -measurable for some $\xi_\alpha < \xi$. Without loss of generality, we may assume that Y is a superclosed subspace of ${}^\kappa\kappa$ (Theorem 2.4.1). Fix a basis $(U_j)_{j < \kappa}$ of Y , and let $\{A_i \mid i < \kappa\} \subseteq \kappa^+ \text{-}\Sigma_\xi^0(X)$ be such that for every $j < \kappa$, $f^{-1}(U_j) = \bigcup_{i \in K_j} (X \setminus A_i)$ for some $K_j \subseteq \kappa$. By Lemma 5.3.1, there are a non-decreasing function $\nu: \kappa \rightarrow \xi$, a subspace $D \subseteq {}^\kappa\kappa$, and a continuous bijection $h: D \rightarrow X$ such that $h^{-1}(A_i) \in \kappa^+ \text{-}\Sigma_1^0(D)$ for every $i < \kappa$, and moreover $h(\mathcal{N}_s \cap D) \in \kappa^+ \text{-}\Pi_{\nu(\text{lh}(s))}^0$ for every $s \in {}^{<\kappa}\kappa$. By the choice of the sets A_i , the function $f \circ h: D \rightarrow Y$ is thus $\kappa^+ \text{-}\Sigma_2^0(D)$ -measurable. By Proposition 5.2.6, there is a family of functions $(g_\alpha)_{\alpha < \kappa}$ such that $f \circ h = \lim_{\alpha < \kappa} g_\alpha$ and each $g_\alpha: D \rightarrow Y$ is κ -full with constant r_α : set $f_\alpha = g_\alpha \circ h^{-1}$. Clearly, $f = \lim_{\alpha < \kappa} f_\alpha$. Since g_α is κ -full with constant r_α , the range of g_α consists of (at most) κ -many points $\{y_i \mid i < \kappa\} \subseteq Y$ such that for every $i < \kappa$

$$g_\alpha^{-1}(y_i) = \bigcup \{ \mathcal{N}_t \cap D \mid t \in L_i \},$$

for some $L_i \subseteq {}^\alpha\kappa$. By the choice of h , it follows that $f_\alpha^{-1}(y_i) = h(g_\alpha^{-1}(y_i)) \in \kappa^+ \text{-}\Sigma_{\nu(\alpha)+1}^0(X)$. Since $\text{ran}(f_\alpha) = \text{ran}(g_\alpha)$ has at most κ -many elements, f_α is thus $\kappa^+ \text{-}\Sigma_{\xi_\alpha}^0$ -measurable for $\xi_\alpha = \nu(\alpha) + 1 < \xi$. \square

Corollary 5.3.4. *Let κ be regular, $X, Y \in \mathcal{M}_\kappa$, and suppose that Y is spherically complete. Then, the closure under κ -limits of the collection of all continuous functions is precisely $\mathcal{M}_{<\omega}(X, Y)$.*

More generally, for every $1 \leq \xi < \kappa^+$ which is either a successor ordinal, or a limit ordinal of cofinality κ , the closure of $\mathcal{M}_\xi(X, Y)$ under κ -limits is $\mathcal{M}_{<\xi+\omega}(X, Y)$.

Proof. Exploiting Theorem 5.3.3, an easy induction shows that $\mathcal{M}_{<\xi+\omega}(X, Y) = \bigcup_{n \in \omega} \mathcal{M}_{\xi+n}(X, Y)$ is contained in the closure under κ -limits of $\mathcal{M}_{\xi}(X, Y)$. Since $\text{cof}(\xi + \omega) = \omega < \kappa = \text{cof}(\kappa)$, the reverse inclusion follows from Proposition 5.1.6. \square

Theorem 5.3.3 tells us that, in order to find an appropriate definition of generalized Baire class functions, we can use κ -limits at all successor stages and at all limit levels with cofinality κ . When ξ is a limit ordinal of cofinality smaller than κ , instead, functions in $\mathcal{M}_{\xi+1}$ can no longer be characterized as κ -limits of functions in $\mathcal{M}_{<\xi}(X, Y)$ because of Proposition 5.1.6, so a different approach is in order. In this situation, Corollary 5.1.4 suggests to use limits of the form

$$\lim_{\text{Fin}_\kappa} \circ \lim_{\text{cof}(\xi)}.$$

However, this would be an overdoing: as shown in the next proposition, the inclusion in Corollary 5.1.4 is in most cases proper, and thus the resulting class would be larger than expected.

Proposition 5.3.5. *Let κ be regular, $X, Y \in \mathcal{M}_\kappa$, and let $\xi < \kappa^+$ be a limit ordinal. Suppose that $A \in \kappa^+ \text{-}\Sigma_{\xi+1}^0(X) \setminus \kappa^+ \text{-}\Pi_{\xi+1}^0(X)$ and that there are at least two distinct points $y_0, y_1 \in Y$. Let $f: X \rightarrow Y$ be the “characteristic function” of A defined by $f(x) = y_0$ if $x \in A$, and $f(x) = y_1$ otherwise. Then, $f \notin \mathcal{M}_{\xi+1}(X, Y)$, yet $f \in \text{Fin}_\kappa\text{-lim}(\text{cof}(\xi)\text{-lim } \mathcal{M}_{<\xi}(X, Y))$. In particular,*

$$\mathcal{M}_{\xi+1}(X, Y) \subsetneq \text{Fin}_\kappa\text{-lim}(\text{cof}(\xi)\text{-lim } \mathcal{M}_{<\xi}(X, Y)).$$

Proof. We first prove a claim which might be of independent interest.

Claim 5.3.5.1. Let $A_0, \dots, A_n \in \kappa^+ \text{-}\Pi_\xi^0(X)$, and let $g: X \rightarrow Y$ be such that $g \upharpoonright (X \setminus \bigcup_{j \leq n} A_j)$ and $g \upharpoonright A_j$, for $j \leq n$, are all constant. Then, $g = \lim_{\alpha < \text{cof}(\xi)} g_\alpha$ for some sequence of function $(g_\alpha)_{\alpha < \text{cof}(\xi)}$ with $g_\alpha \in \mathcal{M}_{<\xi}(X, Y)$.

Proof of the Claim. Since g is constant on each A_j , we can assume without loss of generality that $A_j \cap A_i = \emptyset$ for every $j < i \leq n$. Let \bar{y} be the value of g on $X \setminus \bigcup_{j \leq n} A_j$, and y_j be the value of g on A_j , for every $j \leq n$. Let $(\xi_\alpha)_{\alpha < \text{cof}(\xi)}$ be a strictly increasing sequence of ordinals cofinal in ξ . By Lemma 3.3.16(2), each $X \setminus A_j$ can be written as $X \setminus A_j = \bigcup_{\alpha < \text{cof}(\xi)} B_\alpha^j$ where $B_\alpha^j \subseteq B_{\alpha'}^j$, if $\alpha \leq \alpha' < \text{cof}(\xi)$ and $B_\alpha^j \in \kappa^+ \text{-}\Delta_{\xi_{\alpha+2}}^0(X)$. For every $\alpha < \text{cof}(\xi)$, let $g_\alpha: X \rightarrow Y$ be defined by

$$g_\alpha(x) = \begin{cases} y_j, & \text{if } x \notin \bigcap_{j \leq n} B_\alpha^j \text{ and } j \leq n \text{ is smallest such that } x \notin B_\alpha^j, \\ \bar{y}, & \text{otherwise.} \end{cases}$$

Each g_α is $\kappa^+ \text{-}\Sigma_{\xi_{\alpha+2}}^0$ -measurable because it is locally constant on a $\kappa^+ \text{-}\Delta_{\xi_{\alpha+2}}^0$ -partition, and hence it belongs to $\mathcal{M}_{<\xi}(X, Y)$. Pick any $x \in X$. If $x \notin \bigcup_{j \leq n} A_j$, then for every $j \leq n$ there is $\alpha_j < \text{cof}(\xi)$ such that $x \in B_{\alpha_j}^j$. Set $\bar{\alpha} = \max\{\alpha_j \mid j \leq n\}$: then for every $\bar{\alpha} \leq \alpha < \text{cof}(\xi)$ we have $x \in \bigcap_{j \leq n} B_\alpha^j$, and hence $g_\alpha(x) = \bar{y} = g(x)$. If instead $x \in \bigcup_{j \leq n} A_j$, then there is a unique $\bar{j} \leq n$ such that $x \in A_{\bar{j}}$ because the sets A_j are assumed to be pairwise disjoint. For each $j < \bar{j}$, let $\alpha_j < \text{cof}(\xi)$ be such that $x \in B_{\alpha_j}^j$, and set $\bar{\alpha} = \max\{\alpha_j \mid j < \bar{j}\}$. Then, by construction $g_\alpha(x) = y_j = g(x)$ for every $\bar{\alpha} \leq \alpha < \text{cof}(\xi)$. This shows that $g = \lim_{\alpha < \text{cof}(\xi)} g_\alpha$ and concludes our proof. \square

By Claim 5.3.5.1, it suffices to show that $f = \lim_{d \in \text{Fin}_\kappa} f_d$, where each $f_d: X \rightarrow Y$ is locally constant on both A_d and $X \setminus A_d$ for some $A_d \in \kappa^+ \text{-}\mathbf{\Pi}_\xi^0(X)$. To this aim, let $\{A_\alpha \mid \alpha < \kappa\} \subseteq \kappa^+ \text{-}\mathbf{\Pi}_\xi^0(X)$ be such that $A = \bigcup_{\alpha < \kappa} A_\alpha$. For each $d \in \text{Fin}_\kappa$, let $A_d = \bigcup\{A_\alpha \mid \alpha \in d\}$, and then let $f_d(x) = y_0$ if $x \in A_d$ and $f_d(x) = y_1$ otherwise. If $x \notin A$, then $f_d(x) = y_1 = f(x)$ for all $d \in \text{Fin}_\kappa$. If instead $x \in A$, then $f_d(x) = y_0 = f(x)$ for all $d \supseteq \{\alpha\}$, where $\alpha < \kappa$ is any ordinal such that $x \in A_\alpha$. This shows that $f = \lim_{d \in \text{Fin}_\kappa} f_d$, as desired. \square

The core problem with using the double limit $\lim_{\text{Fin}_\kappa} \circ \lim_{\text{cof}(\xi)}$ at limit levels is actually the fact that even when $\text{cof}(\xi) < \kappa$, the class $\mathcal{M}_\xi(X, Y)$ is not closed under $\text{cof}(\xi)$ -limits. More in detail, let κ be regular and $X, Y \in \mathcal{M}_\kappa$. Then, for every $1 \leq \xi < \kappa^+$, the class $\kappa^+ \text{-}\mathbf{\Delta}_{\xi+1}^0(X)$ is a κ -algebra. It easily follows that if D is a directed set of size smaller than κ and $f \in D\text{-}\lim \mathcal{M}_\xi(X, Y)$, then f is $\kappa^+ \text{-}\mathbf{\Delta}_{\xi+1}^0$ -measurable. However, this is optimal. To see this, suppose that there is $A \in \kappa^+ \text{-}\mathbf{\Pi}_\xi^0(X) \setminus \kappa^+ \text{-}\mathbf{\Sigma}_\xi^0(X)$ and that Y contains at least two distinct points y_0 and y_1 . Let $f: X \rightarrow Y$ be defined by $f(x) = y_0$ if $x \in A$ and $f(x) = y_1$ otherwise. Then, by Claim 5.3.5.1 the function f is a $\text{cof}(\xi)$ -limit of functions in $\mathcal{M}_{<\xi}(X, Y) \subseteq \mathcal{M}_\xi(X, Y)$, yet it is not $\kappa^+ \text{-}\mathbf{\Sigma}_\xi^0$ -measurable by the choice of A .

To overcome this difficulty, we have to perform a deeper analysis of limits over directed sets of small size. Let (D, \leq) be a directed set. A covering $(X_d)_{d \in D}$ of a set X is (D) -**increasing** if $X_d \subseteq X_{d'}$ for every $d, d' \in D$ such that $d \leq d'$.

Definition 5.3.6. Let (D, \leq) be a directed set, and let f and $(f_d)_{d \in D}$ be functions between topological spaces X and Y . Then, $f = \widehat{\lim}_{d \in D} f_d$ if there exists a D -increasing covering $(X_d)_{d \in D}$ of X such that $f_{d'} \upharpoonright X_d = f \upharpoonright X_d$ for every $d, d' \in D$ with $d' \geq d$.

Notice that if $f = \widehat{\lim}_{d \in D} f_d$, then there is a *canonical* increasing covering $(X_d)_{d \in D}$ of X witnessing that, namely,

$$X_d = \{x \in X \mid f_{d'}(x) = f(x) \text{ for all } d' \geq d\}. \quad (5.3.1)$$

Obviously, $f = \widehat{\lim}_{d \in D} f_d$ implies $f = \lim_{d \in D} f_d$. The reverse is not true in general. For example, assume that κ is regular and consider $D = \kappa$, ordered as usual. Let $f: {}^\kappa\kappa \rightarrow {}^\kappa\kappa$ be constant with value $0^{(\kappa)}$ and, for every $\alpha < \kappa$, let $f_\alpha: {}^\kappa\kappa \rightarrow {}^\kappa\kappa$ be constant with value $0^{(\alpha)} \cap 1^{(\kappa)}$. Then, $f = \lim_{\alpha < \kappa} f_\alpha$, but $f \neq \widehat{\lim}_{\alpha < \kappa} f_\alpha$ for the simple reason that $f(x) \neq f_\alpha(x)$ for every $\alpha < \kappa$ and $x \in {}^\kappa\kappa$.

In contrast, we are now going to show that if D has small size, then the two notions of limit coincide. Let $\mathcal{D}_{<\kappa}$ be the collection of all directed sets of size strictly smaller than κ . Clearly, $\mathcal{D}_{<\kappa}$ is a proper subclass of \mathcal{D}_κ .

Lemma 5.3.7. Let κ be regular, $X, Y \in \mathcal{M}_\kappa$, and let $D \in \mathcal{D}_{<\kappa}$. Let $(f_d)_{d \in D}$ be a family of functions from X to Y . Then, for every $f: X \rightarrow Y$, the following are equivalent:

- (1) $f = \lim_{d \in D} f_d$;
- (2) for every $x \in X$ there exists $d \in D$ such that $f_{d'}(x) = f(x)$ for every $d' \geq d$;

$$(3) f = \widehat{\lim}_{d \in D} f_d.$$

Proof. By Theorem 2.4.1, without loss of generality we can assume $X, Y \subseteq {}^\kappa\kappa$.

(1) \Rightarrow (2) Fix $x \in X$. For every $\beta < \kappa$ there exists $d_\beta \in D$ such that $f_d(x) \upharpoonright \beta = f(x) \upharpoonright \beta$, for all $d \geq d_\beta$. For every $d \in D$, let $G_d = \{\beta < \kappa \mid d_\beta = d\}$. Since $(G_d)_{d \in D}$ is a covering of κ in $|D|$ -many pieces and κ is regular, there is $\bar{d} \in D$ such that $|G_{\bar{d}}| = \kappa$, so that $G_{\bar{d}}$ is unbounded in κ . It follows that for every $d' \geq \bar{d}$ there are arbitrarily large $\beta < \kappa$ such that $f_{d'}(x) \upharpoonright \beta = f(x) \upharpoonright \beta$, and thus $f_{d'}(x) = f(x)$.

(2) \Rightarrow (3) For $d \in D$, let X_d be as in equation (5.3.1). By definition $X_d \subseteq X_{d'}$ whenever $d \leq d'$, and (2) ensures that $\bigcup_{d \in D} X_d = X$. It is then easy to verify that $(X_d)_{d \in D}$ witnesses $f = \widehat{\lim}_{d \in D} f_d$.

(3) \Rightarrow (1) Obvious. \square

Although they are equivalent when $D \in \mathcal{D}_{<\kappa}$, the advantage of using the limit operator $\widehat{\lim}_{d \in D}$ instead of $\lim_{d \in D}$ is that we can naturally impose a definability condition on the increasing coverings involved in its definition. Lemma 5.3.10 and Theorem 5.3.11 show that this move fixes the problems encountered when dealing with limit levels of cofinality smaller than κ .

Definition 5.3.8. Let (D, \leq) be a directed set, $1 \leq \xi < \kappa^+$ be an ordinal, and let f and $(f_d)_{d \in D}$ be functions between topological spaces X and Y . Then, $f = \widehat{\lim}_{d \in D}^\xi f_d$ if there exists a D -increasing covering $(X_d)_{d \in D}$ of X witnessing $f = \widehat{\lim}_{d \in D} f_d$ such that $X_d \in \kappa^+ \text{-}\Sigma_\xi^0(X)$ for every $d \in D$.

Remark 5.3.9. It can be shown that if $f = \widehat{\lim}_{d \in D} f_d$ and $f_d \in \mathcal{M}_\xi(X, Y)$ for every $d \in D$, then the canonical increasing covering $(X_d)_{d \in D}$ from equation (5.3.1) is such that $X_d \in \kappa^+ \text{-}\Pi_\xi^0(X)$ for all $d \in D$, because $X_d = \bigcap_{d', d'' \geq d} \{x \in X \mid f_{d'}(x) = f_{d''}(x)\}$. Thus in Definition 5.3.8 we are adopting the weakest possible definability requirement.

In contrast to the other results of this section, which all require κ to be regular, the next two results hold unconditionally, and indeed they will be used in Section 5.3.2 too.

Lemma 5.3.10. Let $X, Y \in \mathcal{M}_\kappa$ and $D \in \mathcal{D}_\kappa$. For every $1 \leq \xi < \kappa^+$, if $f \in D\text{-}\widehat{\lim}^\xi \mathcal{M}_\xi(X, Y)$, then $f \in \mathcal{M}_\xi(X, Y)$.

Proof. Let $f = \widehat{\lim}_{d \in D}^\xi f_d$, as witnessed by the D -increasing covering $(X_d)_{d \in D}$. Then, for every open set $U \subseteq Y$ we have

$$f^{-1}(U) = \bigcup_{d \in D} (f_d^{-1}(U) \cap X_d),$$

so that $f^{-1}(U) \in \kappa^+ \text{-}\Sigma_\xi^0(X)$ because $X_d \in \kappa^+ \text{-}\Sigma_\xi^0(X)$. \square

The proof of the next result is essentially the same of Corollary 5.1.4: we only have to check that, thanks to Lemma 5.3.10, the definability condition added in Definition 5.3.8 allows us to obtain an equality instead of an inclusion.

Theorem 5.3.11. Let $X, Y \in \mathcal{M}_\kappa$, and further assume that $\dim(X) = \dim(Y) = 0$ if $\mu = \omega$. For every limit ordinal $1 \leq \xi < \kappa^+$,

$$\mathcal{M}_{\xi+1}(X, Y) = \text{Fin}_\kappa\text{-lim} \left(\text{cof}(\xi)\text{-}\widehat{\lim}^\xi \mathcal{M}_{<\xi}(X, Y) \right).$$

Proof. Without loss of generality, we can assume $X, Y \subseteq {}^\mu\kappa$ by Theorems 2.4.1 and 2.4.4. The inclusion from left to right follows again from Proposition 5.1.3 and Lemma 5.1.1, together with Remark 5.1.2.

For the reverse inclusion, observe that $\text{cof}(\xi)\text{-}\widehat{\text{lim}}^\xi \mathcal{M}_{<\xi}(X, Y) \subseteq \mathcal{M}_\xi(X, Y)$ by Lemma 5.3.10, so it is enough to check that $\text{Fin}_\kappa\text{-lim } \mathcal{M}_\xi(X, Y) \subseteq \mathcal{M}_{\xi+1}(X, Y)$. Since $|\text{Fin}_\kappa| = \kappa$ this can easily be obtained from the same argument used in the implication (3) \Rightarrow (1) of Theorem 5.1.5, the only difference being that now the functions f_d are assumed to be κ^+ - Σ_ξ^0 -measurable. \square

We are now ready to discuss some simple and natural generalizations to the uncountable setting of the Baire stratification of Borel functions, with the goal of obtaining an analogue of Theorem 5.0.2. Recall that in the classical case $\kappa = \omega$, the class of Baire class ξ functions is defined by taking continuous functions if $\xi = 0$, and (ω) -limits of functions of lower Baire class if $\xi > 0$. As discussed at length, when moving to a regular cardinal we can naturally use κ -limits at successor stages and at limit levels with cofinality κ , while in the remaining cases we have to use a different limit operator. This leads us to the following first proposal. To simplify the notation, we write $\mathcal{B}_{<\xi}(X, Y)$ instead of $\bigcup_{\xi' < \xi} \mathcal{B}_{\xi'}(X, Y)$.

Definition 5.3.12. Let κ be regular, and let $X, Y \in \mathcal{M}_\kappa$. For every $\xi < \kappa^+$, we recursively define the collection $\mathcal{B}_\xi(X, Y)$ of κ -**Baire class ξ functions** as follows:

- (1) $\mathcal{B}_0(X, Y)$ is the collection of all continuous functions $f: X \rightarrow Y$.
- (2) If ξ is either a successor ordinal or a limit ordinal with $\text{cof}(\xi) = \kappa$, then

$$\mathcal{B}_\xi(X, Y) = \kappa\text{-lim } \mathcal{B}_{<\xi}(X, Y).$$

- (3) If ξ is a limit ordinal with $\text{cof}(\xi) < \kappa$, then

$$\mathcal{B}_\xi(X, Y) = \text{Fin}_\kappa\text{-lim} \left(\text{cof}(\xi)\text{-}\widehat{\text{lim}}^\xi \mathcal{B}_{<\xi}(X, Y) \right).$$

Definition 5.3.12 is designed to both be as close as possible to the classical definition and to minimize the kind of limit operators involved, at the price of having to consider different cases depending on the ordinal $\xi < \kappa^+$. In the opposite direction, one could desire to have a uniform definition which maximizes the class of limit operators that can be employed. This leads us to the idea of allowing the use of the special double-limits from part (3) of Definition 5.3.12 at every level of the hierarchy, and to replace Fin_κ and $\text{cof}(\xi)$ with any pair of directed sets in \mathcal{D}_κ . We again write $\widehat{\mathcal{B}}_{<\xi}(X, Y)$ instead of $\bigcup_{\xi' < \xi} \widehat{\mathcal{B}}_{\xi'}(X, Y)$.

Definition 5.3.13. Let κ be regular, and let $X, Y \in \mathcal{M}_\kappa$. For every $\xi < \kappa^+$, we recursively define the classes of functions $\widehat{\mathcal{B}}_\xi(X, Y)$ as follows:

- (1) $\widehat{\mathcal{B}}_0(X, Y)$ is the collection of all continuous functions $f: X \rightarrow Y$.
- (2) If $\xi > 0$, then

$$\widehat{\mathcal{B}}_\xi(X, Y) = \mathcal{D}_\kappa\text{-lim} \left(\mathcal{D}_\kappa\text{-}\widehat{\text{lim}}^\xi \widehat{\mathcal{B}}_{<\xi}(X, Y) \right).$$

We are now going to show that by imposing weak conditions on the spaces X and Y , both definitions do the job, and they are in fact equivalent to each other. We remark that the two options are the extremes of a whole range of intermediate possibilities, which however would give rise to the very same classes of functions because of the following theorem.

Theorem 5.3.14. *Let κ be regular, $X, Y \in \mathcal{M}_\kappa$, and suppose that Y is spherically complete. Then, for every $\xi < \kappa^+$,*

$$\mathcal{B}_\xi(X, Y) = \widehat{\mathcal{B}}_\xi(X, Y) = \mathcal{M}_{\xi+1}(X, Y).$$

Proof. By induction on $\xi < \kappa^+$. The equality $\mathcal{B}_\xi(X, Y) = \mathcal{M}_{\xi+1}(X, Y)$ follows from Theorems 5.3.3 and 5.3.11, while the inclusion $\mathcal{B}_\xi(X, Y) \subseteq \widehat{\mathcal{B}}_\xi(X, Y)$ is obvious. Finally, to prove $\widehat{\mathcal{B}}_\xi(X, Y) \subseteq \mathcal{M}_{\xi+1}(X, Y)$ it is enough to use Lemma 5.3.10 and the usual computation yielding $D\text{-lim } \mathcal{M}_\xi(X, Y) \subseteq \mathcal{M}_{\xi+1}(X, Y)$ for every $D \in \mathcal{D}_\kappa$. \square

The assumption that Y be spherically complete in Theorem 5.3.14 is very mild, as every κ -Cauchy-complete⁴ $X \in \mathcal{M}_\kappa$ can easily be turned into a spherically complete space by adding at most κ -many points. However, if one wants to dispense from such assumption and work with arbitrary spaces in \mathcal{M}_κ , it is enough to modify Definition 5.3.12 as follows:

- when ξ is a successor ordinal, κ -limits have to be replaced with Fin_κ -limits;
- at all limit levels ξ we use the double limit $\lim_{\text{Fin}_\kappa} \circ \widehat{\lim}_{\text{cof}(\xi)}^\xi$, independently of the cofinality of ξ .

This works because even after such modifications, in the first part of the proof of Theorem 5.3.14 we can still prove the inclusion $\mathcal{M}_{\xi+1}(X, Y) \subseteq \mathcal{B}_\xi(X, Y)$ by using Proposition 5.1.3 and (a suitable variant of) Corollary 5.1.4. This alternative approach turns out to work also in the singular case, as briefly discussed in the next section.

5.3.2 The singular case

We now move to the case where $\mu = \text{cof}(\kappa) < \kappa$. Compared to the the regular case, there are two major differences:

- (1) κ -limits are no longer relevant, as κ -limits and μ -limits are equivalent by Remark 2.5.1.
- (2) When ξ is limit we do not have to distinguish cases depending on the cofinality of ξ : since κ is singular, $\text{cof}(\xi) < \kappa$ for every limit $\xi < \kappa^+$.

This naturally brings us to the following variation of Definition 5.3.12.

Definition 5.3.15. Let κ be singular, and let $X, Y \in \mathcal{M}_\kappa$. For every $\xi < \kappa^+$, we recursively define the collection $\mathcal{B}_\xi(X, Y)$ of κ -**Baire class ξ functions** as follows:

⁴These spaces are one of the natural generalizations to the uncountable setting of Polish spaces, and they are indeed called \mathbb{G} -Polish spaces in the specialized literature (see e.g. [AMRS23]).

(1) $\mathcal{B}_0(X, Y)$ is the collection of all continuous functions $f: X \rightarrow Y$.

(2) If $\xi = \xi' + 1$ is a successor ordinal, then

$$\mathcal{B}_\xi(X, Y) = \text{Fin}_\kappa\text{-lim } \mathcal{B}_{<\xi}(X, Y) = \text{Fin}_\kappa\text{-lim } \mathcal{B}_{\xi'}(X, Y).$$

(3) If ξ is a limit ordinal, then

$$\mathcal{B}_\xi(X, Y) = \text{Fin}_\kappa\text{-lim } \left(\text{cof}(\xi)\text{-}\widehat{\text{lim}}^\xi \mathcal{B}_{<\xi}(X, Y) \right).$$

Moreover, it still makes sense to consider the alternative approach taken in Definition 5.3.13: in this case, the definition of the classes $\widehat{\mathcal{B}}_\xi(X, Y)$ needs not to be updated. As in the regular case, it turns out that both definitions allow us to generalize Theorem 5.0.2.

Theorem 5.3.16. *Let κ be singular, $X, Y \in \mathcal{M}_\kappa$, and further assume that $\dim(X) = \dim(Y) = 0$ if $\mu = \omega$. Then, for every $\xi < \kappa^+$,*

$$\mathcal{B}_\xi(X, Y) = \widehat{\mathcal{B}}_\xi(X, Y) = \mathcal{M}_{\xi+1}(X, Y).$$

Proof. We argue as in the proof of Theorem 5.3.14, with the following exception. Instead of showing that $\mathcal{B}_\xi(X, Y) = \mathcal{M}_{\xi+1}(X, Y)$, we just prove the inclusion $\mathcal{M}_{\xi+1}(X, Y) \subseteq \mathcal{B}_\xi(X, Y)$ using Proposition 5.1.3 (in the successor case) and a suitable variant of Corollary 5.1.4 (in the limit case) instead of Theorems 5.3.3 and 5.3.11. The rest of the proof goes unchanged. \square

Remark 5.3.17. It is natural to wonder whether, at least in the successor case, we can use sequential limits of arbitrary length below κ (instead of Fin_κ -limits) to generate the next class $\mathcal{M}_{\xi+1}(X, Y)$. The answer is in general negative. For example, suppose that X contains a copy of ${}^\omega 2$ and that $|Y| \geq 2$. Let \mathcal{D} be the collection of all cardinals $\lambda \leq \kappa$. Then, $\mathcal{D}\text{-lim } \mathcal{M}_\xi(X, Y) \subsetneq \mathcal{M}_{\xi+1}(X, Y)$. To see this, notice that by Remark 2.5.1 and the fact that κ is singular, we have $\mathcal{D}\text{-lim } \mathcal{M}_\xi(X, Y) = \mathcal{D}'\text{-lim } \mathcal{M}_\xi(X, Y)$ for $\mathcal{D}' = \{\lambda < \kappa \mid \lambda \text{ regular}\}$. Let $f = \lim_{\alpha < \lambda} f_\alpha$ for some regular $\lambda < \kappa$ and some sequence $(f_\alpha)_{\alpha < \lambda}$ of functions from $\mathcal{M}_\xi(X, Y)$. Let \mathcal{A} be the smallest κ -algebra containing $\kappa^+\text{-}\Sigma_\xi^0(X)$. The usual computation⁵ (see (5.1.1) and (5.1.2) in Theorem 5.1.5) gives that $f^{-1}(U) \in \mathcal{A}$ for every open $U \subseteq Y$. On the other hand, in Chapter 3 it is shown that when κ is singular, $\mathcal{A} \subsetneq \kappa^+\text{-}\Delta_{\xi+1}^0(X)$. Fix a set $C \in \kappa^+\text{-}\Delta_{\xi+1}^0(X) \setminus \mathcal{A}$ and distinct points $y_0, y_1 \in Y$. Consider the function $g: X \rightarrow Y$ defined by $g(x) = y_0$ if $x \in C$, and $g(x) = y_1$ otherwise. Then, $g \in \mathcal{M}_{\xi+1}(X, Y)$, yet $g^{-1}(U) \notin \mathcal{A}$ for every open set $U \subseteq Y$ containing exactly one of y_0 and y_1 . Therefore $g \in \mathcal{M}_{\xi+1}(X, Y) \setminus \mathcal{D}\text{-lim } \mathcal{M}_\xi(X, Y)$.

5.4 Uniform limits

This short section aims at generalizing the results from [MR09b, Section 3] to the uncountable setting. The theme is the characterization $\kappa^+\text{-}\Sigma_\xi^0$ -measurable functions in terms of *uniform* limits of simpler functions. Although somewhat

⁵When $\mu > \omega$, we are also using that every open subset of $Y \in \mathcal{M}_\kappa$ can be written as a union of μ -many clopen sets.

unrelated to the previous sections, these results provide additional information on the structure of κ^+ - Σ_ξ^0 -measurable functions with respect to another kind of limit, and thus we believe that they fit well with the overall topic of the chapter.

Recall that in Section 2.4 we fixed a field \mathbb{G} with degree $\mu = \text{cof}(\kappa)$ and a strictly decreasing sequence $(r_\alpha)_{\alpha < \mu}$ coinital in \mathbb{G}^+ . By the choice we made, $r_{\alpha+1} + r_{\alpha+1} \leq r_\alpha$ for every $\alpha < \mu$. Let $\widehat{\mathcal{M}}_\kappa$ be the collection of all \mathbb{G} -metric spaces of weight at most κ . Given a directed set D , a topological space $X \in \widehat{\mathcal{M}}_\kappa$, and a \mathbb{G} -metric space $(Y, d_Y) \in \widehat{\mathcal{M}}_\kappa$, we say that a function $f: X \rightarrow Y$ is the **uniform D -limit** of a family $(f_d)_{d \in D}$ of functions from X to Y , and we write $f = \text{ulim}_{d \in D} f_d$, if for every $\epsilon \in \mathbb{G}^+$ there is $d \in D$ such that $d_Y(f_{d'}(x), f(x)) < \epsilon$ for every $x \in X$ and $d' \geq d$. It is not hard to see that one could restrict the attention to directed sets of size at most μ : if $f = \text{ulim}_{d \in D} f_d$, then there is $D' \subseteq D$ with $|D'| = \min\{\mu, |D|\}$ such that $f = \text{ulim}_{d \in D'} f_d$.

As in the classical setting, it is easy to check that continuous functions are closed under uniform κ -limits. We generalize this to higher levels and to all directed sets D .

Lemma 5.4.1. *Let D be any directed set, $X \in \widehat{\mathcal{M}}_\kappa$, $(Y, d_Y) \in \widehat{\mathcal{M}}_\kappa$, and $1 \leq \xi < \kappa^+$. Then, $\mathcal{M}_\xi(X, Y)$ is closed under uniform D -limits.*

Proof. Suppose that $f = \text{ulim}_{d \in D} f_d$ with $f_d \in \mathcal{M}_\xi(X, Y)$ for every $d \in D$. We need to show that given any $y \in Y$ and $\epsilon \in \mathbb{G}^+$, the preimage of the open ball $B_{d_Y}(y, \epsilon)$ belongs to κ^+ - $\Sigma_\xi^0(X)$. Let $\beta < \mu$ be such that $r_\beta < \epsilon$, and for each $\alpha < \mu$ set $\epsilon_\alpha = \epsilon - r_{\beta+\alpha}$. Then, $0 < \epsilon_\alpha < \epsilon$, and for every $\epsilon' < \epsilon$ there is $\alpha < \mu$ such that $\epsilon' \leq \epsilon_\alpha$.

By choice of $(f_d)_{d \in D}$, for every $\alpha < \mu$ there is $d_\alpha \in D$ such that $d_Y(f_{d_\alpha}(x), f(x)) < r_{\beta+\alpha}$ for every $x \in X$ and $d \geq d_\alpha$. We claim that

$$f^{-1}(B_{d_Y}(y, \epsilon)) = \bigcup_{\alpha < \mu} f_{d_\alpha}^{-1}(B_{d_Y}(y, \epsilon_\alpha)),$$

from which the result clearly follows.

For the inclusion from right to left, fix $\alpha < \mu$ and a point $x \in f_{d_\alpha}^{-1}(B_{d_Y}(y, \epsilon_\alpha))$. By the triangle inequality and the choice of d_α ,

$$d_Y(y, f(x)) \leq d_Y(y, f_{d_\alpha}(x)) + d_Y(f_{d_\alpha}(x), f(x)) < \epsilon_\alpha + r_{\beta+\alpha} = \epsilon,$$

so that $x \in f^{-1}(B_{d_Y}(y, \epsilon))$.

Conversely, given $x \in f^{-1}(B_{d_Y}(y, \epsilon))$ let $\epsilon' = d_Y(y, f(x)) < \epsilon$. Let $\alpha < \mu$ be such that $r_{\beta+\alpha} < \epsilon - \epsilon'$, so that $\epsilon' + r_{\beta+\alpha+1} + r_{\beta+\alpha+1} \leq \epsilon' + r_{\beta+\alpha} \leq \epsilon$ and hence $\epsilon' + r_{\beta+\alpha+1} \leq \epsilon - r_{\beta+\alpha+1} = \epsilon_{\alpha+1}$. Then, by the triangular inequality and the choice of d_α

$$d_Y(y, f_{d_{\alpha+1}}(x)) \leq d_Y(y, f(x)) + d_Y(f(x), f_{d_{\alpha+1}}(x)) < \epsilon' + r_{\beta+\alpha+1} \leq \epsilon_{\alpha+1}.$$

Therefore $x \in f_{d_{\alpha+1}}^{-1}(B_{d_Y}(y, \epsilon_{\alpha+1}))$ and we are done. \square

Let X and Y be topological spaces, and let $1 \leq \xi < \kappa^+$. A κ^+ - Δ_ξ^0 -**function** is a map $f: X \rightarrow Y$ such that $f^{-1}(A) \in \kappa^+$ - $\Sigma_\xi^0(X)$, for every $A \in \kappa^+$ - $\Sigma_\xi^0(Y)$. The set of κ^+ - Δ_ξ^0 -functions is denoted by $\Delta_\xi(X, Y)$. It is easy to check that

κ^+ - Δ_ξ^0 -functions provide an alternative stratification of κ^+ -Borel measurable functions: indeed, $\Delta_\xi(X, Y) \subseteq \mathcal{M}_\xi(X, Y) \subseteq \Delta_{\xi, \omega}(X, Y)$ for every $1 \leq \xi < \kappa^+$. Moreover, under mild assumptions on the spaces X and Y the set $\Delta_\xi(X, Y)$ is a proper subclass of $\mathcal{M}_\xi(X, Y)$. Notice also that if a function is locally in $\Delta_\xi(X, Y)$ on a κ^+ - Σ_ξ^0 -partition of X of size at most κ , then it is a κ^+ - Δ_ξ^0 -function itself. We now have all the tools to mimic the proof of [MR09b, Theorem 3.5] and get the following characterization of $\mathcal{M}_\xi(X, Y)$ in terms of uniform μ -limits of simpler functions.

Theorem 5.4.2. *Let $X \in \mathcal{M}_\kappa$, $(Y, d_Y) \in \widehat{\mathcal{M}}_\kappa$, and $1 \leq \xi < \kappa^+$. For every function $f: X \rightarrow Y$, the following are equivalent:*

- (1) $f \in \mathcal{M}_\xi(X, Y)$;
- (2) $f = \text{ulim}_{\alpha < \mu} f_\alpha$, where each f_α is locally constant on a κ^+ - Σ_ξ^0 -partition of X of size at most κ ;
- (3) $f = \text{ulim}_{\alpha < \mu} f_\alpha$ with $f_\alpha \in \Delta_\xi(X, Y)$ for all $\alpha < \mu$.

Moreover, in part (2) locally constant functions can be replaced by locally Lipschitz or even just locally continuous maps.

Proof. The implication (2) \Rightarrow (3) follows from the fact that every constant (or Lipschitz, or continuous) function belongs to $\Delta_\xi(X, Y)$. Moreover, if (3) holds then $f_\alpha \in \mathcal{M}_\xi(X, Y)$ for all $\alpha < \mu$, and thus $f \in \mathcal{M}_\xi(X, Y)$ by Lemma 5.4.1, so that (1) is satisfied. It remains to prove (1) \Rightarrow (2).

Let $\{y_i \mid i < \kappa\}$ be a dense subset of Y , and for each $\alpha < \mu$ and $i < \kappa$ let $U_\alpha^i = B_{d_Y}(y_i, r_\alpha)$. Fix $\alpha < \mu$. The family $\{U_\alpha^i \mid i < \kappa\}$ is an open covering of Y . Since $f \in \mathcal{M}_\xi(X, Y)$, each set $A_\alpha^i = f^{-1}(U_\alpha^i)$ belongs to κ^+ - $\Sigma_\xi^0(X)$. By the κ -reduction property (Theorem 3.3.17), we can find a disjoint κ^+ - Σ_ξ^0 -covering $\mathcal{B}_\alpha = \{B_\alpha^i \mid i < \kappa\}$ of X such that $B_\alpha^i \subseteq A_\alpha^i$ for every $i < \kappa$. Thus the non-empty elements of \mathcal{B}_α form a κ^+ - Σ_ξ^0 -partition of X . Let f_α be defined on an arbitrary $x \in X$ by letting $f_\alpha(x) = y_i$, where $i < \kappa$ is such that $x \in B_\alpha^i$. Then, f_α is locally constant on the above mentioned κ^+ - Σ_ξ^0 -partition, and $f = \text{ulim}_{\alpha < \mu} f_\alpha$ by the choice of the open sets U_α^i . \square

Of course, by (the analogue for uniform limits of) Remark 2.5.1 we can replace uniform μ -limits with uniform κ -limits in both (2) and (3) of Theorem 5.4.2.

Corollary 5.4.3. *Let $X \in \mathcal{M}_\kappa$, $(Y, d_Y) \in \widehat{\mathcal{M}}_\kappa$, and $1 \leq \xi < \kappa^+$. Then*

$$\mathcal{M}_\xi(X, Y) = \mathcal{D}_\kappa\text{-ulim } \Delta_\xi(X, Y).$$

Proof. The left-to-right inclusion follows from Theorem 5.4.2, while the reverse inclusion holds by $\Delta_\xi(X, Y) \subseteq \mathcal{M}_\xi(X, Y)$ and Lemma 5.4.1. \square

By virtue of Theorems 5.3.14 and 5.3.16, all the results of this section also hold for the collection $\mathcal{B}_\xi(X, Y)$ of κ -Baire class ξ functions, for every $\xi < \kappa^+$ (under the appropriate hypotheses on X and Y). In particular, while $\mathcal{B}_\xi(X, Y)$ is far from being closed under \mathcal{D}_κ -limits, which indeed generate the next class $\mathcal{B}_{\xi+1}(X, Y)$, it is instead closed under *uniform* \mathcal{D}_κ -limits.

5.5 Final remarks and open questions

Theorem 5.1.5, Theorem 5.3.14, and Theorem 5.3.16 provide a quite satisfactory answer to our initial problem. However, as discussed at the end of Section 5.1, it would be interesting to understand if it is possible to further reduce the number of limits needed to perform this kind of analysis. In particular, in the regular case it is natural to wonder whether sequential limits (of any length) suffice. An attempt in this direction was made in [Nob18], where κ -Baire class ξ functions are defined as the collection of all λ -limits, for $\lambda \leq \kappa$, of functions with κ -Baire class smaller than ξ . The setup is that of a regular cardinal κ and of functions between some fixed subset X of ${}^\kappa\kappa$ and the whole generalized Baire space $Y = {}^\kappa\kappa$, thus this definition is precisely what we are considering in the present discussion. After providing a game characterization of the classes $\mathcal{M}_{\xi+1}(X, Y)$, in [Nob18, Theorem 4.12] it is claimed that all κ^+ - $\Sigma_{\xi+1}^0$ -measurable functions are of κ -Baire class ξ according to the previous definition. The proof is by induction on $\xi < \kappa^+$, but as observed it has to fail at the first limit level with small cofinality, i.e. at stage $\xi = \omega$: since the functions of finite κ -Baire class are those in $\mathcal{M}_{<\omega}(X, Y)$, any λ -limit of such functions must be in $\mathcal{M}_{<\omega}(X, Y)$ itself if $\lambda = \kappa$ (Proposition 5.1.6), or it has to be κ^+ - $\Delta_{\xi+1}^0$ -measurable if $\lambda < \kappa$ (because κ^+ - $\Delta_{\xi+1}^0$ is a κ -algebra). In other words: when confining ourselves to sequential limits, in order to capture κ^+ - $\Sigma_{\xi+1}^0$ -measurable functions for ξ limit with $\text{cof}(\xi) < \kappa$ it is necessary to use (at least) double limits. Despite the failure of the naïve approach undertaken in [Nob18], using nested sequential limits it might still be possible to answer affirmatively the following questions, which to the best of our knowledge are still open.

Question 5.5.1. Let κ be regular, and $X, Y \in \mathcal{M}_\kappa$. Does the collection of all κ^+ -Borel measurable functions from X to Y coincide with the closure of continuous functions under λ -limits, where λ varies among regular cardinals up to κ ? If $\xi < \kappa^+$ is limit with $\text{cof}(\xi) < \kappa$, is it true that $\mathcal{M}_{\xi+1}(X, Y) = \kappa\text{-lim} \left(\text{cof}(\xi)\text{-}\widehat{\text{lim}}^\xi \mathcal{M}_{<\xi}(X, Y) \right)$? Can we at least have $\mathcal{M}_{\xi+1}(X, Y) \subseteq \kappa\text{-lim} (\text{cof}(\xi)\text{-lim} \mathcal{M}_{<\xi}(X, Y))$?

A positive answer to these questions would provide optimal results. As a partial contribution in this direction, we notice the following.

Proposition 5.5.2. Let $X, Y \in \mathcal{M}_\kappa$, and let $\xi < \kappa^+$ be a limit ordinal with $\text{cof}(\xi) < \kappa$.

- (1) If $f: X \rightarrow Y$ is locally constant on a κ^+ - Σ_ξ^0 -partition of X of size at most κ , then $f \in \text{cof}(\xi)\text{-}\widehat{\text{lim}}^\xi \mathcal{M}_{<\xi}(X, Y)$.
- (2) If $f: X \rightarrow Y$ is κ^+ - Σ_ξ^0 -measurable, then $f \in \kappa\text{-ulim} \left(\text{cof}(\xi)\text{-}\widehat{\text{lim}}^\xi \mathcal{M}_{<\xi}(X, Y) \right)$.

Proof. (1) The proof is similar to that of Claim 5.3.5.1. Let $(A_j)_{j < J}$, for some $J \leq \kappa$, be a κ^+ - Σ_ξ^0 -partition of X such that f is locally constant on it. By Lemma 3.3.16 (if $\mu > \omega$) and [DMR25, Proposition 4.2.1] (if $\mu = \omega$), there are a strictly increasing sequence $(\xi_\alpha)_{\alpha < \text{cof}(\xi)}$ cofinal in ξ and sets $B_\alpha^j \in \kappa^+$ - $\Delta_{\xi_\alpha}^0(X)$

such that $\alpha \leq \alpha' \Rightarrow B_\alpha^j \subseteq B_{\alpha'}^j$ and $A_j = \bigcup_{\alpha < \text{cof}(\xi)} B_\alpha^j$, for every $j < J$. Fix any $\bar{y} \in Y$ and define $g_\alpha: X \rightarrow Y$ by setting

$$g_\alpha(x) = \begin{cases} f(x) & \text{if } x \in \bigcup_{j < J} B_\alpha^j \\ \bar{y} & \text{otherwise.} \end{cases}$$

Then, g_α is κ^+ - $\Sigma_{\xi_{\alpha+1}}^0$ -measurable, and $f = \widehat{\lim}_{\alpha < \text{cof}(\xi)}^\xi g_\alpha$, as witnessed by the increasing covering $(X_\alpha)_{\alpha < \text{cof}(\xi)}$ of X given by $X_\alpha = \bigcup_{j < J} B_\alpha^j$.

(2) Use part (1) and Theorem 5.4.2. □

We also observe that by Theorem 5.4.2, to deal with the case of arbitrary functions in $\mathcal{M}_{\xi+1}(X, Y)$ for $\xi < \kappa^+$ limit with $\text{cof}(\xi) < \kappa$, it is enough to solve the problem for functions which are locally constant on a κ^+ - $\Sigma_{\xi+1}^0$ -partition of size at most κ . Arguing as in Section 5.3 (and using in particular Claim 5.3.5.1), such functions can be obtained through a limit operator of the form $\lim_\kappa \circ \lim_{\text{cof}(\xi)}$ if the κ^+ - $\Sigma_{\xi+1}^0$ -partition is finite or, more generally, of size smaller than $\text{cof}(\xi)$. Unfortunately, although the remaining gap might seem small, we cannot yet close it.

Also Corollary 3.6.22 provides some information related to Question 5.5.1. Indeed, Corollary 5.3.4 shows that when κ is regular, then the closure under κ -limits of the class of continuous functions $\mathcal{M}_1(X, Y)$, under suitable assumptions on X and Y , coincides with $\bigcup_{1 \leq n < \omega} \mathcal{M}_n(X, Y)$. Combined with Proposition 3.3.11 and Corollary 3.6.20, this gives consistent examples of closed spaces $X \subseteq {}^\kappa\kappa$ with $|X| > \kappa$ such that all κ^+ -Borel measurable functions on X can be generated from the continuous functions through κ -limits.

In the case of a singular cardinal κ , instead, it seems hard to conjecture that sequential limits can suffice, as only short limits are available because κ -limits are equivalent to μ -limits by Remark 2.5.1, where as usual $\mu = \text{cof}(\kappa) < \kappa$. (See also Remark 5.3.17.) However, it makes sense to ask whether Fin_κ -limits are enough to describe the whole structure of κ^+ -Borel measurable functions and its stratification given by κ -Baire class ξ functions. The question is relevant also in the regular case.

Question 5.5.3. Let $X, Y \in \mathcal{M}_\kappa$. Does the closure under Fin_κ -limits of $\mathcal{M}_1(X, Y)$ coincide with the collection of all κ^+ -Borel measurable functions? In particular, is it true that $\mathcal{M}_{\xi+1}(X, Y) = \text{Fin}_\kappa\text{-lim } \mathcal{M}_{<\xi}(X, Y)$ when $\xi < \kappa^+$ is a limit ordinal?

Finally, recall that in Theorem 5.1.8 we proved that there is a single directed set in \mathcal{D}_κ , namely $\widehat{\text{Fin}}_\kappa$, which can generate the whole class of κ^+ -Borel measurable functions, globally. This naturally raises the question of whether it can also give a level-by-level result. As usual, the answer is positive for successor levels and, if κ is regular, also for limit levels of cofinality κ . However, the remaining cases are still unclear.

Question 5.5.4. Let $X, Y \in \mathcal{M}_\kappa$, and let $\xi < \kappa^+$ be a limit ordinal. Is it true that $\mathcal{M}_{\xi+1}(X, Y) = \widehat{\text{Fin}}_\kappa\text{-lim } \mathcal{M}_{<\xi}(X, Y)$?

Chapter 6

Characterizing classes of functions via games

The Lipschitz and Wadge games introduced in Section 2.10 are specific instances of reduction games. These games simultaneously encode two distinct types of information: first, the existence of a reduction between the corresponding payoff sets, and second, that such a reduction is either Lipschitz or continuous. These two aspects can in fact be separated, and the same game-theoretic frameworks can be employed to characterize Lipschitz and continuous functions *per se*, independently of any particular pair of sets between which a reduction is realized. Thus, in contrast to earlier game formulations that primarily emphasized *sets*, the focus here is shifted toward *functions*. In the context of classical descriptive set theory, this function-oriented approach was developed in [Sem07, MR11, Car14].

In this chapter, we work in ZFC, and we adopt the same approach to analyze and characterize specific classes of partial functions (continuous, Lipschitz, Baire class 1, and piecewise continuous) via generalized reduction games. Each of these games can be viewed as a variant of the generalized Gale–Stewart game.

We begin by generalizing the basic definition of Gale–Stewart games (Section 2.9 or [Kec95, section 20]) to cardinals ν, μ with $\nu \geq 2$ and $\mu \geq \omega$, i.e. we define a two-person, perfect-information game on ν of length μ .

Let $f: {}^\mu\nu \rightarrow {}^\mu\nu$ be a partial function. The game $G_\nu^\mu(f)$ is played as follows. There are two players, **I** and **II**, which take turns in picking ordinals $x_i < \nu$ and $y_i < \nu$ (respectively) in the i -th round, with **I** starting first, for μ -many rounds. At limit levels, Player **I** starts again.

$$\begin{array}{c} \mathbf{I} \\ \hline x_0 \quad x_1 \quad \cdots \quad x_\omega \quad \cdots \\ \hline \mathbf{II} \\ y_0 \quad y_1 \quad \cdots \quad y_\omega \quad \cdots \end{array}$$

Once the sequences $x = (x_i)_{i < \mu}$ and $y = (y_i)_{i < \mu}$ of length μ are selected, Player **II** wins if $x \notin \text{dom}(f)$ or $f(x) = y$, and Player **I** wins otherwise.

A strategy for Player **II** is defined as a monotone function $\sigma: {}^{<\mu}\nu \setminus \{\emptyset\} \rightarrow {}^{<\mu}\nu$ such that $\text{lh}(\sigma(s)) = \text{lh}(s)$ and $y_i = \sigma((x_j)_{j \leq i})(i)$ for every $i < \mu$. Similarly, we define a strategy for Player **I** as a function $\tau: {}^{<\mu}\nu \rightarrow {}^{<\mu}\nu$ such that $\text{lh}(\tau(s)) = \text{lh}(s) + 1$ and $x_i = \tau((y_j)_{j < i})(i)$ for every $1 < i < \mu$. We say that a strategy is winning if it guarantees victory for the corresponding player against

any strategy of the opponent. Clearly, it cannot happen that both **I** and **II** have a winning strategy. We say that the game $G_\nu^\mu(f)$ is determined if one of the two players has a winning strategy. When $\mu = \omega$, we have the classical Gale-Stewart game $G_\nu^\omega(f) = G(f)$ for $f : {}^\omega X \rightarrow {}^\omega X$ with $X = \nu$.

When the space ${}^\mu\nu$ is clear from the context, we drop it from all the above notation. As usual, we fix each player's gender for purposes of convenience: Player **II** is regarded as feminine and thus referred to as "she", while Player **I** is regarded as masculine and is titled "he".

In this chapter, we will consider several generalized reduction games that characterize certain classes of partial functions on the generalized Baire space (continuous, Lipschitz, Baire class 1, piecewise continuous), and which can all be considered as variants of the Gale-Stewart game defined above. Let ν, μ be cardinals such that $\nu \geq 2$ and $\mu \geq \omega$. In each game, there will be a set $A \subseteq {}^\mu\nu$ and a function $f : A \rightarrow {}^\mu\nu$. The idea is to slightly modify the rules of the game and equip player **II** with a more powerful extended alphabet. One way to do this, inspired by [MR11] and [Sem07], is the following.

A **generalized reduction game** is a tuple $\odot_\nu^\mu = (f, M_\odot, R_\odot, \iota_\odot)$, where \odot is a symbol identifying the game, such that:

- $f : {}^\mu\nu \rightarrow {}^\mu\nu$ is a partial function,
- M_\odot is a set of cardinality at most μ disjoint from ν , called the set of moves,
- $R_\odot \subseteq {}^\mu(\nu \cup M_\odot)$ is called the set of rules for Player **II**, and
- $\iota_\odot : R_\odot \rightarrow {}^\mu\nu$ is called the interpretation function.

The components serve the following purposes: M_\odot encodes alternative moves for Player **II** (such as "pass", "backtrack", "erase", etc.); R_\odot encodes the rules that Player **II** must follow to have any chance of victory; and ι_\odot recovers from Player **II**'s complete play the element in ${}^\mu\nu$ needed to check the winning condition.

The game rules are the following: Player **I** plays elements from ν , while Player **II** plays elements from $\nu \cup M_\odot$. After μ turns, Player **I** produces $x \in {}^\mu\nu$ and Player **II** produces $y \in {}^\mu(\nu \cup M_\odot)$ (called Player **II**'s complete play). The game outcome depends on the following two conditions:

- (1) If $x \notin \text{dom}(f)$, then Player **I** loses;
- (2) If $x \in \text{dom}(f)$ but $y \notin R_\odot$, then Player **II** loses.

If τ is a strategy for Player **I** and y is Player **II**'s complete play in a run of the game \odot_ν^μ , then $\tau * y$ denotes the element in ${}^\mu\nu$ that Player **I** enumerates while following τ against y . Conversely, if σ is a strategy for Player **II** and $x \in {}^\mu\nu$, then $x * \sigma$ denotes the complete play produced by Player **II** when Player **I** enumerates x and Player **II** follows σ . (The expressions $\tau * t$ and $s * \sigma$ are defined analogously for $t \in {}^{<\mu}(\nu \cup M_\odot)$ and non-empty $s \in {}^{<\mu}\nu$.)

A strategy τ for Player **I** is legal if $\tau * y \in \text{dom}(f)$ for every $y \in {}^\mu(\nu \cup M_\odot)$. A strategy σ for Player **II** is legal if $x * \sigma \in R_\odot$ for every $x \in \text{dom}(f)$. We denote the collection of legal strategies for Player **II** in \odot by LS_\odot .

The winning conditions for the game $\odot(f)$ are the following: when neither condition (1) nor (2) occurs, Player **II** wins if and only if $f(x) = \iota_\odot(y)$. In this case, $\iota_\odot(y)$ is called Player **II**'s play.

A strategy is winning in $\odot(f)$ if it is legal and guarantees victory for the corresponding Player against any legal strategy of the opponent. In more details, a strategy $\sigma \in LS_\odot$ for Player **II** is a monotone function $\sigma : <^\mu \nu \setminus \{\emptyset\} \rightarrow <^\mu(\nu \cup M_\odot)$ and it is winning for **II** if either $x \notin \text{dom}(f)$ or $\bar{\sigma}(x) = f(x)$, where we define $\bar{\sigma}(x) = \iota_\odot(\bigcup_{i < \mu} \sigma(x \upharpoonright i))$.

Another convenient approach to extend the definition of Gale-Stewart game in order to obtain (the same) different classes of functions is inspired by [Car14], and consists in modifying the rules of the game allowing Player **II** to play at each round any element in $<^\mu \nu$ (instead of just an ordinal $< \nu$). A strategy for Player **II** in such a game is a function $\sigma : <^\mu \nu \setminus \{\emptyset\} \rightarrow <^\mu \nu$ — note that it does not need to be monotone anymore. Example of the latter games are found in Definitions 6.1.2, 6.2.2, 6.3.2, 6.4.2. To distinguish these games from the previous ones, we will add the suffix “-seq” to the name of the game (e.g. generalized Wadge-seq game).

In the next sections, we will define the games in full generality for cardinals ν, μ with $\nu \geq 2$ and $\mu \geq \omega$. The results connecting game strategies and classes of functions (Theorems 6.1.3, 6.2.3, 6.3.3, 6.4.3) sometimes need further assumption on the cardinal μ .

6.1 Generalized Wadge game

In classical descriptive set theory, the Wadge game was introduced by Wadge in [Wad72] and later in [Wad83]. Here we generalize it to our new setting. We consider the cardinals ν, μ with $\nu \geq 2$ and $\mu \geq \omega$.

In this game, Player **I** plays elements of ν , while Player **II** plays elements of $\nu \cup \{\mathbf{P}\}$, where the token **P** represents “pass”. This gives Player **II** the additional option to skip her turn at any point during the game, subject to the constraint that at the end of the run she must have produced a sequence in ${}^\mu \nu$.

Definition 6.1.1. Given a partial function $f : {}^\mu \nu \rightarrow {}^\mu \nu$, the **generalized Wadge game** is $W_\nu^\mu(f) = (f, M_W, R_W, \iota_W)$, where

- $M_W = \{\mathbf{P}\}$,
- $R_W = \{y \in {}^\mu(\nu \cup \{\mathbf{P}\}) \mid \forall i \exists j \geq i (y(j) \neq \mathbf{P})\}$,

and the interpretation function is defined as follows. We first recursively define ι_W on sequences of length $< \mu$, that is, $\iota_W : <^\mu(\nu \cup \{\mathbf{P}\}) \rightarrow <^\mu \nu$. We set $\iota_W(\emptyset) = \emptyset$,

$$\iota_W(s \frown z) = \begin{cases} \iota_W(s) & \text{if } z = \mathbf{P}, \\ \iota_W(s) \frown z & \text{otherwise,} \end{cases}$$

and $\iota_W(s) = \bigcup_{i < \text{lh}(s)} \iota_W(s \upharpoonright i)$ if $\text{lh}(s)$ is a limit ordinal. Note that ι_W is monotone. Then, we extend the interpretation function to sequences in ${}^\mu(\nu \cup \{\mathbf{P}\})$ by setting $\iota_W(y) = \bigcup_{i < \mu} \iota_W(y \upharpoonright i) \in {}^\mu \nu$.

We recall that, given a strategy $\sigma : {}^{<\mu}\nu \setminus \{\emptyset\} \rightarrow {}^{<\mu}(\nu \cup \{\mathbf{P}\})$ for Player **II**, we let $\bar{\sigma}(x) = \iota_{\mathbb{W}}(\bigcup_{i < \mu} \sigma(x \upharpoonright i))$ and σ is winning if $\bar{\sigma}(x) = f(x)$ for all $x \in A$. Note that $\bar{\sigma}(x) = \bigcup_{i < \mu} \iota_{\mathbb{W}}(\sigma(x \upharpoonright i))$.

With the following definition, we present a slightly different but equivalent version of the Wadge game.

Definition 6.1.2. Let $A \subseteq {}^\mu\nu$ and $f : A \rightarrow {}^\mu\nu$. In the **generalized Wadge-seq game** $\mathbb{W}_\nu^\mu(f)$, player **I** plays ordinals $x_i < \nu$ and player **II** plays sequences $t_i \in {}^{<\mu}\nu$ such that $\bigcup_{j < i} t_j \subseteq t_i$. After μ -many rounds, player **I** has produced $x = \langle x_i \mid i < \mu \rangle$ and player **II** has produced $y = \bigcup_{i < \mu} t_i \in {}^{\leq \mu}\nu$. Player **II** wins the game if $x \notin A$ or $y = f(x)$.

As before, note that it is implicit in the rules that, in order to have a chance at winning, at the end of the game, Player **II** must provide a sequence of length μ whenever Player **I** plays $x \in A$, that is, for every $x \in A$ and for every $\alpha < \mu$ there must be some $i < \mu$ such that $\text{lh}(t_i) \geq \alpha$.

A strategy for Player **II** in $\mathbb{W}_\nu^\mu(f)$ is a function $\tau : {}^{<\mu}\nu \setminus \{\emptyset\} \rightarrow {}^{<\mu}\nu$ such that $p \subseteq q$ implies $\tau(p) \subseteq \tau(q)$, and it is winning if for all $x \in A$, $\bigcup_{i < \mu} \tau(x \upharpoonright i) = f(x)$.

Theorem 6.1.3. Let ν, μ be cardinals with $\nu \geq 2$ and $\mu \geq \omega$. Let A be an arbitrary subset of ${}^\mu\nu$ and $f : A \rightarrow {}^\mu\nu$ be a function. Then, the following are equivalent:

- (1) f is continuous;
- (2) Player **II** has a winning strategy in $\mathbb{W}_\nu^\mu(f)$;
- (3) Player **II** has a winning strategy in $\mathbb{W}_\nu^\mu(f)$.

Proof. (1) \Rightarrow (2). Let f be continuous. We recursively define σ as follows. We set $\sigma(\emptyset) = \emptyset$, for every $s \in {}^{<\mu}\nu$

$$\sigma(s \frown \alpha) = \begin{cases} \sigma(s) \frown \beta & \text{if } f(N_s \frown \alpha) \subseteq N_{\iota_{\mathbb{W}}(\sigma(s)) \frown \beta} \\ \sigma(s) \frown \mathbf{P} & \text{otherwise,} \end{cases}$$

and we set $\sigma(s) = \bigcup_{i < \text{lh}(s)} \sigma(s \upharpoonright i)$ if $\text{lh}(s)$ is a limit ordinal. By continuity of f , σ is well-defined and winning for Player **II** in $\mathbb{W}_\nu^\mu(f)$. Indeed, fix $x \in A$. By continuity of f , for every $\gamma < \mu$ there exists $i < \mu$ $f(N_x \upharpoonright i) \subseteq N_{f(x) \upharpoonright \gamma}$, and it follows from the definition of σ that $f(x) \upharpoonright \gamma \supseteq \iota_{\mathbb{W}}(\sigma(x \upharpoonright i))$, thus $f(x) = \bigcup_{i < \mu} \iota_{\mathbb{W}}(\sigma(x \upharpoonright i)) = \bar{\sigma}(x)$.

(2) \Rightarrow (3) Assume that σ is a winning strategy for Player **II** in $\mathbb{W}_\nu^\mu(f)$. Then the function $\tau : {}^{<\mu}\nu \rightarrow {}^{<\mu}\nu$ such that $\tau(s) = \iota_{\mathbb{W}}(\sigma(s))$ is a winning strategy for Player **II** in $\mathbb{W}_\nu^\mu(f)$. Indeed, $s \subseteq t$ implies $\tau(s) \subseteq \tau(t)$ because both σ and $\iota_{\mathbb{W}}$ are monotone, and for all $x \in A$, $\bigcup_{i < \mu} \tau(x \upharpoonright i) = \bigcup_{i < \mu} \iota_{\mathbb{W}}(\sigma(x \upharpoonright i)) = \bar{\sigma}(x) = f(x)$.

(3) \Rightarrow (1) Now assume that τ is a winning strategy for **II** in $\mathbb{W}_\nu^\mu(f)$ and take any $t \in {}^{<\mu}\nu$. Then, $f^{-1}(N_t) = \bigcup \{N_s \cap A \mid t \subseteq \tau(s)\}$ is open and therefore f is continuous. \square

6.2 Generalized Lipschitz games

We consider again the cardinals ν, μ with $\mu \geq \omega$ and $\nu \geq 2$. Let $\alpha < \mu$ be an ordinal. For $\alpha \geq \omega$, let $\llbracket \alpha \rrbracket = \max\{\gamma \leq \alpha, \mid \gamma \text{ limit}\}$, and let $\llbracket \alpha \rrbracket = \omega$ otherwise. Then, for every $\alpha \geq \omega$, $\alpha = \llbracket \alpha \rrbracket + \delta$ for some $\delta < \llbracket \alpha \rrbracket$.

The α -Lipschitz game is a variant of the Wadge game (cf. Definition 6.1.1) where Player **II** has the following additional rules.

- Player **II** can pass at most α -many times, and the pass count resets at every stage $\llbracket \alpha \rrbracket \cdot \beta$ for $\beta < \mu$;
- Player **II** must produce a sequence whose interpretation has length $\llbracket \alpha \rrbracket$ every time she plays from stage $\llbracket \alpha \rrbracket \cdot \beta$ to stage $\llbracket \alpha \rrbracket \cdot (\beta + 1)$, for $\beta < \mu$, i.e. the complete play y must satisfy:

$$\forall \beta > 0 \forall i < \llbracket \alpha \rrbracket \cdot \beta \exists j (i \leq j < \llbracket \alpha \rrbracket \cdot \beta) (y(j) \neq \text{P}). \quad (6.2.1)$$

We will see that these games characterize the α -Lipschitz functions as defined in Definition 2.7.5. In classical descriptive set theory, the k -Lipschitz game for $k < \omega$ was defined in [MR11].

Definition 6.2.1. Given a partial function $f: {}^\mu\nu \rightarrow {}^\mu\nu$, and an ordinal $\alpha < \mu$, the **generalized α -Lipschitz game** is $(L_\alpha)_\nu^\mu(f) = (f, M_{L_\alpha}, R_{L_\alpha}, \iota_{L_\alpha})$, where

- $M_{L_\alpha} = \{\text{P}\}$,
- $R_{L_\alpha} = \{y \in {}^\mu(\nu \cup \{\text{P}\}) \mid \forall i \text{ot}(\{j < i \mid y(j) \neq \text{P}\}) + \alpha \geq i \wedge y \text{ satisfies (6.2.1)}\}$,
- $\iota_{L_\alpha} = \iota_W$.

We also define the following equivalent version of the Lipschitz game (cf. Definition 6.1.2).

Definition 6.2.2. Let $A \subseteq {}^\mu\nu$ and $f: A \rightarrow {}^\mu\nu$. In the **generalized α -Lipschitz-seq game** $(\mathbb{L}_\alpha)_\nu^\mu(f)$, player **I** plays ordinals $x_i < \nu$ and player **II** plays sequences $t_i \in <{}^\mu\nu$ such that $\bigcup_{j < i} t_j \subseteq t_i$ and $i \leq \text{lh}(t_i) + \alpha$. After μ -many rounds, player **I** has produced $x = \langle x_i \mid i < \mu \rangle$ and player **II** has produced $y = \bigcup_{i < \mu} t_i$. Player **II** wins the game if $x \notin A$ or $y = f(x)$.

Theorem 6.2.3. Let ν, μ be cardinals with $\nu \geq 2$ and $\mu \geq \omega$. Let A be an arbitrary subset of ${}^\mu\nu$ and $f: A \rightarrow {}^\mu\nu$ be a function. Then, for any ordinal $\alpha < \mu$, the following are equivalent:

- (1) f is α -Lipschitz;
- (2) Player **II** has a winning strategy in $(L_\alpha)_\nu^\mu(f)$;
- (3) Player **II** has a winning strategy in $(\mathbb{L}_\alpha)_\nu^\mu(f)$.

Proof. (1) \Rightarrow (2) Let $f = f_\varphi$ for some α -Lipschitz* function $\varphi : \mathcal{T}_A \rightarrow <^\mu \nu$ (see Remark 2.7.4). For every $s \in <^\mu \nu$, let $\beta_s = \sup\{\beta \mid \llbracket \alpha \rrbracket \cdot \beta \leq \text{lh}(s)\}$. We recursively define the strategy σ for Player **II** in $(\mathbf{L}_\alpha)_\nu^\mu(f)$ as follows. Set $\sigma(\emptyset) = \emptyset$.

If $\text{lh}(s) = 0$ or $\text{lh}(s) = \llbracket \alpha \rrbracket \cdot \beta_s$, set

$$\sigma(s \frown \beta) = \begin{cases} \sigma(s) \frown \mathbf{P} & \text{if } \text{lh}(s \frown \beta) > \text{lh}(\varphi(s \frown \beta)), \\ \sigma(s) \frown \varphi(s \frown \beta)(\text{lh}(\varphi(s))) & \text{otherwise;} \end{cases}$$

otherwise, set $\sigma(s \frown \beta) =$

$$\begin{cases} \sigma(s) \frown \mathbf{P} & \text{if } \text{lh}(s \frown \beta) > \text{lh}(\varphi(s \frown \beta)) + \text{ot}(\{\llbracket \alpha \rrbracket \cdot \beta_s \leq j < \text{lh}(s) \mid \sigma(s)(j) = \mathbf{P}\}), \\ \sigma(s) \frown \varphi(s \frown \beta)(\text{lh}(\varphi(s))) & \text{otherwise.} \end{cases}$$

Finally, if $\text{lh}(s)$ is a limit ordinal, set $\sigma(s) = \bigcup_{i < \text{lh}(s)} \sigma(s \upharpoonright i)$. By definition, σ is monotone. Fix $x \in A$ and let $y = \bigcup_{i < \mu} \sigma(x \upharpoonright i)$. Note that since $\text{lh}(s) = \text{lh}(\sigma(s))$ for every $s \in <^\mu \nu$, we have $y \upharpoonright i = \sigma(x \upharpoonright i)$ and $\text{lh}(\iota_{\mathbf{W}}(\sigma(x \upharpoonright i))) \leq i$ for every $i < \mu$.

Claim 6.2.3.1. For every $i < \mu$, $\iota_{\mathbf{W}}(\sigma(x \upharpoonright i)) = \varphi(x \upharpoonright i)$.

Proof of the Claim. We proceed by induction on $1 \leq i < \mu$, using the definition of σ . If $i = 1$, then either $\varphi(x \upharpoonright 1) = \emptyset$ hence $\sigma(x \upharpoonright 1) = \emptyset = \iota_{\mathbf{W}}(\sigma(x \upharpoonright 1))$, or $\text{lh}(\varphi(x \upharpoonright 1)) = 1$ hence $\iota_{\mathbf{W}}(\sigma(x \upharpoonright 1)) = \sigma(x \upharpoonright 1) = \varphi(x \upharpoonright 1)$.

If $i = j + 1$ is a successor ordinal, we have two cases. If $\sigma(x \upharpoonright i) = \sigma(x \upharpoonright j) \frown \mathbf{P}$, then $i > \text{lh}(\varphi(x \upharpoonright i))$ and since φ is monotone, this implies that $\varphi(x \upharpoonright i) = \varphi(x \upharpoonright j)$. Therefore, using the induction hypothesis, $\iota_{\mathbf{W}}(\sigma(x \upharpoonright i)) = \iota_{\mathbf{W}}(\sigma(x \upharpoonright j)) = \varphi(x \upharpoonright j) = \varphi(x \upharpoonright i)$. Otherwise, if $\sigma(x \upharpoonright i) = \sigma(x \upharpoonright j) \frown \varphi(x \upharpoonright i)(j)$, again using the induction hypothesis we get $\iota_{\mathbf{W}}(\sigma(x \upharpoonright i)) = \iota_{\mathbf{W}}(\sigma(x \upharpoonright j) \frown \varphi(x \upharpoonright i)(j)) = \varphi(x \upharpoonright j) \frown \varphi(x \upharpoonright i)(j) = \varphi(x \upharpoonright i)$.

Finally, if i is a limit ordinal, then $\iota_{\mathbf{W}}(\sigma(x \upharpoonright i)) = \iota_{\mathbf{W}}(\bigcup_{j < i} \sigma(x \upharpoonright j)) = \bigcup_{j < i} \iota_{\mathbf{W}}(\sigma(x \upharpoonright j)) = \bigcup_{j < i} \varphi(x \upharpoonright j)$ by induction hypothesis. Since φ is Lipschitz*, $\varphi(x \upharpoonright i) = \bigcup_{j < i} \varphi(x \upharpoonright j)$. \square

Since φ is α -Lipschitz, for every $i < \mu$, $i \leq \text{lh}(\varphi(x \upharpoonright i)) + \alpha$, hence $i \leq \text{ot}(\{j < i \mid y(j) \neq \mathbf{P}\}) + \alpha$, because by Claim 6.2.3.1:

$$\text{ot}(\{j < i \mid y(j) \neq \mathbf{P}\}) = \text{lh}(\iota_{\mathbf{W}}(\sigma(x \upharpoonright i))) = \text{lh}(\varphi(x \upharpoonright i)).$$

Moreover, it follows again from Claim 6.2.3.1 that $\iota_{\mathbf{W}}(y) = \bigcup_{i < \mu} \iota_{\mathbf{W}}(\sigma(x \upharpoonright i)) = \bigcup_{i < \mu} \varphi(x \upharpoonright i) = f(x)$.

It remains to verify that (6.2.1) is satisfied by y . To see this, assume, towards a contradiction, that there exists $\beta > 0$ and $i < \llbracket \alpha \rrbracket \cdot \beta$ such that for every j , if $i \leq j < \llbracket \alpha \rrbracket \cdot \beta$, then $y(j) = \mathbf{P}$. Then, for any such j , we have $\iota_{\mathbf{W}}(y \upharpoonright j) = \iota_{\mathbf{W}}(y \upharpoonright i)$.

First, assume that $\alpha \geq \omega$. Let $\delta < \llbracket \alpha \rrbracket$ such that $\alpha = \llbracket \alpha \rrbracket + \delta$. By Claim 6.2.3.1,

$$\llbracket \alpha \rrbracket \cdot \beta + \delta + 1 \leq \text{lh}(\iota_{\mathbf{W}}(y \upharpoonright \llbracket \alpha \rrbracket \cdot \beta + \delta + 1)) + \llbracket \alpha \rrbracket + \delta. \quad (6.2.2)$$

Since $\llbracket \alpha \rrbracket \cdot \beta$ is a limit, by definition of σ we get

$$\iota_{\mathbb{W}}(y \upharpoonright \llbracket \alpha \rrbracket \cdot \beta) = \bigcup_{i' < \llbracket \alpha \rrbracket \cdot \beta} \iota_{\mathbb{W}}(y \upharpoonright i') = \iota_{\mathbb{W}}(y \upharpoonright i),$$

and using that for every $i' < \mu$, $i' = \text{lh}(y \upharpoonright i')$ and $\text{lh}(\iota_{\mathbb{W}}(y \upharpoonright i')) \leq i'$, we have

$$\begin{aligned} \text{lh}(\iota_{\mathbb{W}}(y \upharpoonright \llbracket \alpha \rrbracket \cdot \beta + \delta + 1)) &\leq \text{lh}(\iota_{\mathbb{W}}(y \upharpoonright i)) + \delta + 1 \\ &\leq i + \delta + 1. \end{aligned}$$

Since $\delta < \llbracket \alpha \rrbracket$ and $\llbracket \alpha \rrbracket$ is limit, Equation (6.2.2) becomes

$$\begin{aligned} \llbracket \alpha \rrbracket \cdot \beta + \delta + 1 &\leq i + \delta + 1 + \llbracket \alpha \rrbracket + \delta \\ &= i + \llbracket \alpha \rrbracket + \delta. \end{aligned}$$

We recall that $i < \llbracket \alpha \rrbracket \cdot \beta$. If $i < \llbracket \alpha \rrbracket$ then Equation (6.2.2) implies $\llbracket \alpha \rrbracket \cdot \beta + \delta + 1 \leq \llbracket \alpha \rrbracket + \delta$, which is false. Suppose now that $\llbracket \alpha \rrbracket \leq i < \llbracket \alpha \rrbracket \cdot \beta$. This implies that $\beta > 1$ and $i = \llbracket \alpha \rrbracket \cdot \beta' + \xi$ for some $1 \leq \beta' < \beta$ and $\xi < \llbracket \alpha \rrbracket$. Thus $i + \llbracket \alpha \rrbracket = \llbracket \alpha \rrbracket \cdot \beta' + \xi + \llbracket \alpha \rrbracket = \llbracket \alpha \rrbracket \cdot \beta' + \llbracket \alpha \rrbracket = \llbracket \alpha \rrbracket \cdot (\beta' + 1) \leq \llbracket \alpha \rrbracket \cdot \beta$. By Equation (6.2.2), we conclude that $\llbracket \alpha \rrbracket \cdot \beta + \delta + 1 \leq \llbracket \alpha \rrbracket \cdot \beta + \delta$, again a contradiction.

Finally, suppose that $\alpha = n < \omega$, hence $\llbracket \alpha \rrbracket = \omega$. Recall that $\beta > 0$ and $i < \omega \cdot \beta$ are such that for every $i \leq j < \omega \cdot \beta$, $y(j) = \mathbf{P}$. Using Claim 6.2.3.1 and the same argument conducted above for α infinite, we get

$$\omega \cdot \beta \leq \text{lh}(\iota_{\mathbb{W}}(y \upharpoonright \omega \cdot \beta)) + n \leq i + n, \quad (6.2.3)$$

which is immediately false for $i < \omega$. If $\omega \leq i < \omega \cdot \beta$, then $\beta > 1$ and $i = \omega \cdot \beta' + m$ for some $1 \leq \beta' < \beta$ and $m < \omega$, thus $i + n = \omega \cdot \beta' + m + n < \omega \cdot \beta$ and (6.2.3) is false again.

(2) \Rightarrow (3) Assume that σ is a winning strategy for Player **II** in $(\mathbb{L}_\alpha)_\nu^\mu(f)$ and recall that for every $x \in A$, $\bar{\sigma}(x) = \bigcup_{i < \mu} \iota_{\mathbb{W}}(\sigma(x \upharpoonright i))$. Then, the function $\tau : {}^{<\mu}\nu \rightarrow {}^{<\mu}\nu$ such that $\tau(s) = \iota_{\mathbb{W}}(\sigma(s))$ is a winning strategy for Player **II** in $(\mathbb{L}_\alpha)_\nu^\mu(f)$. Indeed, $s \subseteq t$ implies $\tau(s) \subseteq \tau(t)$ and for all $x \in A$, $\bigcup_{i < \mu} \tau(x \upharpoonright i) = \bar{\sigma}(x) = f(x)$. Moreover, the rules $R_{\mathbb{L}_\alpha}$ for Player **II** in $(\mathbb{L}_\alpha)_\nu^\mu(f)$ guarantee that $i \leq \text{lh}(\tau(x \upharpoonright i)) + \alpha$ as desired, because $\text{ot}(\{j < i \mid \sigma(x \upharpoonright i)(j) \neq \mathbf{P}\}) = \text{lh}(\iota_{\mathbb{W}}(\sigma(x \upharpoonright i))) = \text{lh}(\tau(x \upharpoonright i))$.

(3) \Rightarrow (1) Now assume that τ is a winning strategy for **II** in $(\mathbb{L}_\alpha)_\nu^\mu(f)$. For every $x \in A$ and $i < \mu$, $\tau(x \upharpoonright i) = t_i^x$ for some $t_i^x \in {}^{<\mu}\nu$ with $i \leq \text{lh}(t_i^x) + \alpha$. Fix $x \in A$. Then, it is enough to set $\varphi(x \upharpoonright i) = t_i^x$ for every $i < \mu$. Clearly φ is α -Lipschitz and $f_\varphi = f$. \square

6.3 Generalized Eraser game

Another interesting game in the classical descriptive set theory literature is due to Duparc [Dup01] and it is called the eraser game. In this game, Player **II** has the option of erasing her previous moves. We define the generalized version of this game, considering as usual the cardinals ν, μ with $\mu \geq \omega$ and $\nu \geq 2$. Player **II** plays elements in $\nu \cup \{\mathbf{E}_i \mid i < \mu\}$, where each token \mathbf{E}_i means “erase back until you get a sequence of length i ”, and this option allows Player **II** to erase a block of moves he previously made in ν . However, Player **II** must use the tokens \mathbf{E}_i 's so that at the end of each run **II** has indeed played an element of ${}^\mu\nu$.

Definition 6.3.1. Fix a set $A \subseteq {}^\mu\nu$ and a function $f : A \rightarrow {}^\mu\nu$. The **generalized Eraser game** is $\mathbb{E}_\nu^\mu(f) = (f, M_E, R_E, \iota_E)$, where

- $M_E = \{E_i \mid i < \mu\}$,

The function $\iota_E : <^\mu(\nu \cup \{E_i \mid i < \mu\}) \rightarrow <^\mu\nu$ is defined by recursion by setting $\iota_E(\emptyset) = \emptyset$,

$$\iota_E(s \hat{\ } z) = \begin{cases} \iota_E(s) \hat{\ } z & \text{if } z \in \nu, \\ \iota_E(s) \upharpoonright i & \text{if } z = E_i \text{ for some } i < \mu \text{ and } \text{lh}(\iota_E(s)) > i, \\ \iota_E(s) & \text{if } z = E_i \text{ for some } i < \mu \text{ and } \text{lh}(\iota_E(s)) \leq i, \end{cases}$$

and, if $\text{lh}(s)$ is a limit ordinal, $\iota_E(s)$ is set to be the longest sequence $t \in \leq^{\text{lh}(s)}\nu$ such that:

$$\forall \alpha < \text{lh}(t) \exists i < \text{lh}(s) \forall j \geq i (\iota_E(s \upharpoonright j) \upharpoonright \alpha = t \upharpoonright \alpha).$$

- $R_E = \{y \in {}^\mu(\nu \cup \{E_i \mid i < \mu\}) \mid \forall \alpha \exists i \forall j \geq i (\text{lh}(\iota_E(y \upharpoonright j)) \geq \alpha)\}$.

Finally, we extend the interpretation function to sequence $y \in R_E$. We set $\iota_E(y) = z$ for some $z \in {}^\mu\nu$ if:

$$\forall \alpha < \mu \exists i < \mu \forall j \geq i (\iota_E(y \upharpoonright j) \upharpoonright \alpha = z \upharpoonright \alpha).$$

We recall that a tree $\mathcal{T} \subseteq <^\mu\nu$ is $< \lambda$ -splitting (for a cardinal λ) if every node is extended by a $< \lambda$ -splitting node, that is, if for every $t \in \mathcal{T}$ there exists $s \in \mathcal{T}$ such that $t \subseteq s$ and $|\{\alpha < \nu \mid s \hat{\ } \alpha \in \mathcal{T}\}| < \lambda$. For any sequence $(s_i)_{i < \mu} \subseteq \leq^\mu\nu$, the tree generated by $(s_i)_{i < \mu}$ is:

$$\mathcal{T}_{(s_i)_{i < \mu}} = \{v \in <^\mu\nu \mid \exists i (v \subseteq s_i)\}.$$

Definition 6.3.2. We consider a subset $A \subseteq {}^\mu\nu$ and $f : A \rightarrow {}^\mu\nu$. The **generalized Eraser-seq game** $\mathbb{E}_\nu^\mu(f)$ is a two-player infinite alternating game of length μ , where at each turn $i < \mu$ Player **I** plays ordinals $x_i < \nu$ and Player **II** plays sequences $s_i \in <^\mu\nu$. Let $x = \langle x_i \mid i < \mu \rangle \in {}^\mu\nu$ be the sequence produced by Player **I** at the end of the game. Player **II** wins the game if either $x \notin A$ or the tree $\mathcal{T}_{(s_i)_{i < \mu}}$ is $< \mu$ -splitting and $f(x)$ is its unique branch of length μ .

Note that when μ is regular, the winning condition for player **II** in Definition 6.3.2 is equivalent to asking that for every $\gamma < \mu$ there exists $i < \mu$ such that, for all $j \geq i$, $s_j \upharpoonright \gamma = s_i \upharpoonright \gamma = f(x) \upharpoonright \gamma$.

A strategy $\tau : <^\mu\nu \setminus \{\emptyset\} \rightarrow <^\mu\nu$ for Player **II** in $\mathbb{E}_\nu^\mu(f)$ is winning if for all $x \in A$, $\mathcal{T}_{(\tau(x \upharpoonright i))_{i < \mu}}$ is $< \mu$ -splitting and $f(x)$ is its unique branch of length μ .

Theorem 6.3.3. Let ν, μ be cardinals with $\nu \geq 2$ and $\mu \geq \omega$, and assume that μ is regular. Let A be an arbitrary subset of ${}^\mu\nu$ and $f : A \rightarrow {}^\mu\nu$ be a function. Then, the following are equivalent:

- (1) f is the μ -limit of continuous functions;
- (2) Player **II** has a winning strategy in $\mathbb{E}_\nu^\mu(f)$;
- (3) Player **II** has a winning strategy in $\mathbb{E}_\nu^\mu(f)$.

Proof. (1) \Rightarrow (3) Let $f = \lim_{i < \mu} f_i$, with $(f_i)_{i < \mu}$ sequence of continuous functions. Since every $f_i : A \rightarrow {}^\mu\nu$ is continuous, using Theorem 6.1.3, we know that for every $i < \mu$ there exists τ_i winning strategy for Player **II** in $\mathbb{W}_\nu^\mu(f_i)$. The strategy τ for Player **II** consists in following τ_0 until it gives a sequence of length 1 and then, recursively, following τ_i until it gives a sequence of length $i + 1$. More precisely, for every $s \in {}^{<\mu}\nu$ define $\tau(s) = \tau_{\alpha_s}(s)$ where $\alpha_s < \mu$ is an ordinal defined by recursion on $\text{lh}(s)$ as follows. Set $\alpha_\emptyset = 0$,

$$\alpha_s \frown \alpha = \begin{cases} \alpha_s & \text{if } \text{lh}((\tau_{\alpha_s}(s \frown \alpha))) \leq \alpha_s, \\ \alpha_s + 1 & \text{otherwise,} \end{cases}$$

and if $\text{lh}(s)$ is limit, $\alpha_s = \sup_{i < \text{lh}(s)} \alpha_{s \upharpoonright i}$. Note that $\alpha_s \leq \text{lh}(s)$ for every $s \in {}^{<\mu}\nu$.

For every $x \in A$ and for every $i < \mu$ there is some $j \leq i$ such that $\tau(x \upharpoonright i) = \tau_j(x \upharpoonright i) = f_j(x) \upharpoonright j \subsetneq f_j(x)$ hence $\mathcal{T}_{(\tau(x \upharpoonright i))_{i < \mu}} \subseteq \mathcal{T}_{(f_i(x))_{i < \mu}}$, and since the latter is $< \mu$ -splitting —because $(f_i(x))_{i < \mu}$ converges to $f(x)$ — $\mathcal{T}_{(\tau(x \upharpoonright i))_{i < \mu}}$ is $< \mu$ -splitting as well.

Now, we want to prove that $f(x)$ is a branch in $\mathcal{T}_{(\tau(x \upharpoonright i))_{i < \mu}}$, that is, that for every $\gamma < \mu$, $f(x) \upharpoonright \gamma \in \mathcal{T}_{(\tau(x \upharpoonright i))_{i < \mu}}$. To do that, we will show that for every $\gamma < \mu$ there exists i such that $f(x) \upharpoonright \gamma \subseteq \tau(x \upharpoonright i)$. Fix $\gamma < \mu$ and let $j' < \mu$ be such that $f_{j'}(x) \upharpoonright \gamma = f(x) \upharpoonright \gamma$. Let $j = \max\{j', \gamma\}$, then there exists i' for all $i \geq i'$ $f(x) \upharpoonright \gamma \subseteq \tau_j(x \upharpoonright i)$. Set $i_0 = \min\{i \geq i' \mid \alpha_{x \upharpoonright i} = j\}$. We get $f(x) \upharpoonright \gamma \subseteq \tau_j(x \upharpoonright i_0) = \tau(x \upharpoonright i_0)$ as desired.

Finally, it remains to show that $f(x)$ is unique. Let y be a cofinal branch in $\mathcal{T}_{(\tau(x \upharpoonright i))_{i < \mu}}$. For every $\alpha < \mu$, there exists $i_\alpha < \mu$ such that $y \upharpoonright \alpha \subseteq \tau(x \upharpoonright i_\alpha)$. Let $j_\alpha \leq i_\alpha$ such that $\tau(x \upharpoonright i_\alpha) = \tau_{j_\alpha}(x \upharpoonright i_\alpha) = f_{j_\alpha}(x) \upharpoonright j_\alpha$. Then, $y \upharpoonright \alpha \subseteq f_{j_\alpha}(x) \upharpoonright j_\alpha$. Note that this implies that $\alpha \leq j_\alpha$ and that $(j_\alpha)_{\alpha < \mu}$ is strictly increasing. Since $f = \lim_{i < \mu} f_i$, for every $\alpha < \mu$ there exists a minimal $j'_\alpha < \mu$, for every $j \geq j'_\alpha$, $f_j(x) \upharpoonright \alpha = f(x) \upharpoonright \alpha$. We claim that $j'_\alpha \leq j_\alpha$ for every $\alpha < \mu$, and then we conclude that $y \upharpoonright \alpha = f_{j_\alpha}(x) \upharpoonright \alpha = f(x) \upharpoonright \alpha$, hence $y = f(x)$. Indeed, towards a contradiction, assume there exists $\alpha < \mu$ such that $\alpha \leq j_\alpha < j'_\alpha$. By minimality of j'_α , we get $y \upharpoonright \alpha \perp f(x) \upharpoonright \alpha$. Let $\beta > \alpha$ such that $j'_\alpha \leq j_\beta$. Then, $y \upharpoonright \beta \subseteq f_{j_\beta}(x) \upharpoonright j_\beta$, hence $y \upharpoonright \alpha \subseteq f_{j_\beta}(x) \upharpoonright \alpha = f(x) \upharpoonright \alpha$, a contradiction.

(3) \Rightarrow (2) Let $\tau : {}^{<\mu}\nu \setminus \{\emptyset\} \rightarrow {}^{<\mu}\nu$ be a winning strategy for player **II** in the eraser game $\mathbb{E}_\nu^\mu(f)$. We will define $\sigma : {}^{<\mu}\nu \rightarrow {}^{<\mu}(\nu \cup \{\mathbf{E}_i \mid i < \mu\})$ winning strategy for player **II** in $\mathbb{E}_\nu^\mu(f)$ by recursion on $\text{lh}(s)$. We set $\sigma(\emptyset) = \emptyset$, and for every $s \in {}^{<\mu}\nu$:

$$\sigma(s \frown \beta) = \begin{cases} \sigma(s) \frown \tau(s \frown \beta)(\text{lh}(\iota_{\mathbf{E}}(\sigma(s)))) & \text{if } \iota_{\mathbf{E}}(\sigma(s)) \subseteq \tau(s \frown \beta), \\ \sigma(s) \frown \mathbf{E}_i & \text{otherwise,} \end{cases}$$

where $i = \min\{\alpha < \mu \mid \iota_{\mathbf{E}}(\sigma(s))(\alpha) \neq \tau(s \frown \beta)(\alpha)\}$, and if $\text{lh}(s)$ is a limit ordinal, $\sigma(s) = \bigcup_{j < \text{lh}(s)} \sigma(s \upharpoonright j)$. Clearly, σ is monotone and $\text{lh}(\sigma(s)) = \text{lh}(s)$. Fix $x \in A$. Since τ is winning for **II** and μ is regular, for every $\gamma < \kappa$ there exists $i_\gamma < \kappa$ such that, for all $j \geq i_\gamma$, $\tau(x \upharpoonright j) \upharpoonright \gamma = f(x) \upharpoonright \gamma$.

Without loss of generality we can assume that $i_\gamma = i' + 1$ is a successor ordinal (if not, consider $i_\gamma + 1$). Let $\alpha_\gamma = \min\{\alpha < \mu \mid \iota_{\mathbf{E}}(\sigma(x \upharpoonright i'))(\alpha) \neq \tau(x \upharpoonright i_\gamma)(\alpha)\}$. Then, $\text{lh}(\iota_{\mathbf{E}}(\sigma(x \upharpoonright i_\gamma))) = \alpha_\gamma$. If $\alpha_\gamma \geq \gamma$, set $j_\gamma = i_\gamma$, otherwise we need the following Claim.

Claim 6.3.3.1. If $\alpha_\gamma < \gamma$, there exists $j_\gamma > i_\gamma$ such that $\text{lh}(\iota_{\mathbb{E}}(\sigma(x \upharpoonright j_\gamma))) = \gamma$ and, for every j such that $i_\gamma \leq j < j_\gamma$, $\iota_{\mathbb{E}}(\sigma(x \upharpoonright j)) \subseteq \tau(x \upharpoonright j) \upharpoonright \gamma$.

Proof of the Claim. In order to prove the statement of the claim, we need to prove that for every $\beta < \mu$ such that $\alpha_\gamma + \beta \leq \gamma$,

$$\text{lh}(\iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma + \beta))) = \alpha_\gamma + \beta. \quad (6.3.1)$$

The proof of (6.3.1) is by induction on $\beta \geq 0$, and it is true for $\beta = 0$ by assumption.

Using that τ is winning, by minimality of α_γ and since $\alpha_\gamma < \gamma$, $\iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma)) \subseteq \tau(x \upharpoonright i_\gamma) \upharpoonright \gamma = \tau(x \upharpoonright i_\gamma + 1) \upharpoonright \gamma$. Then, by definition of σ , $\text{lh}(\iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma + 1))) = \text{lh}(\iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma))) + 1 = \alpha_\gamma + 1$. This proves (6.3.1) for $\beta \leq 1$. If $\alpha_\gamma + 1 = \gamma$, we let $j_\gamma = i_\gamma + 1$ and we are done. Otherwise, if $\alpha_\gamma + 1 < \gamma$, we iterate the procedure until we find $\beta < \mu$ such that $\text{lh}(\iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma + \beta))) = \gamma$.

Assume that $\beta = \beta' + 1$ is a successor ordinal and that $\alpha_\gamma + \beta' < \gamma$. By induction hypothesis, $\text{lh}(\iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma + \beta'))) = \alpha_\gamma + \beta'$. By minimality of α_γ and by definition of σ , $\iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma + \beta')) \subseteq \tau(x \upharpoonright i_\gamma + \beta') \upharpoonright \gamma = \tau(x \upharpoonright i_\gamma + \beta) \upharpoonright \gamma$ because τ is winning. Then, $\text{lh}(\iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma + \beta))) = \text{lh}(\iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma + \beta'))) + 1 = \alpha_\gamma + \beta$. This proves (6.3.1) for β successor ordinal. If $\alpha_\gamma + \beta = \gamma$ we let $j_\gamma = i_\gamma + \beta$ and we are done. Otherwise, if $\alpha_\gamma + \beta < \gamma$, we continue the proof and the induction on β .

Assume that β is a limit ordinal, and that $\alpha_\gamma + \beta' < \gamma$ for every $\beta' < \beta$. By induction hypothesis and the definition of σ , $\text{lh}(\iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma + \beta))) = \sup_{\beta' < \beta} \text{lh}(\iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma + \beta'))) = \sup_{\beta' < \beta} \alpha_\gamma + \beta' = \alpha_\gamma + \beta$. This proves (6.3.1) for β limit ordinal. If $\alpha_\gamma + \beta = \gamma$ we let $j_\gamma = i_\gamma + \beta$ and we are done. Otherwise, if $\alpha_\gamma + \beta < \gamma$, we continue the proof and the induction on β .

Finally, towards a contradiction, suppose that for every $\beta < \mu$, $\text{lh}(\iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma + \beta))) < \gamma$. This means that $\bigcup_{i < \mu} \iota_{\mathbb{E}}(\sigma(x \upharpoonright i)) = \bigcup_{\beta < \mu} \iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma + \beta)) \subseteq \tau(x \upharpoonright i_\gamma) \upharpoonright \gamma$, hence $\text{lh}(\iota_{\mathbb{E}}(\bigcup_{i < \mu} \sigma(x \upharpoonright i))) = \text{lh}(\bigcup_{i < \mu} \iota_{\mathbb{E}}(\sigma(x \upharpoonright i))) \leq \gamma < \mu$. However, it also implies that (6.3.1) holds for every $\beta < \mu$, thus $\text{lh}(\iota_{\mathbb{E}}(\bigcup_{i < \mu} \sigma(x \upharpoonright i))) = \text{lh}(\bigcup_{\beta < \mu} \iota_{\mathbb{E}}(\sigma(x \upharpoonright i_\gamma + \beta))) = \sup_{\beta < \mu} \alpha_\gamma + \beta = \mu$. A contradiction. \square

If $\alpha_\gamma < \gamma$, let $j_\gamma > i_\gamma$ as in Claim 6.3.3.1, so that for every j such that $i_\gamma < j < j_\gamma$, we have $\iota_{\mathbb{E}}(\sigma(x \upharpoonright j)) \subseteq \tau(x \upharpoonright j) \upharpoonright \gamma$. Recall that we set $j_\gamma = i_\gamma$ if $\alpha_\gamma \geq \gamma$. Then, for every $j \geq j_\gamma$,

$$\iota_{\mathbb{E}}(\sigma(x \upharpoonright j_\gamma)) \upharpoonright \gamma = \iota_{\mathbb{E}}(\sigma(x \upharpoonright j)) \upharpoonright \gamma = \tau(x \upharpoonright j) \upharpoonright \gamma.$$

Indeed, since $i_\gamma \leq j_\gamma$ and $\iota_{\mathbb{E}}(\sigma(x \upharpoonright j_\gamma)) \upharpoonright \gamma = \tau(x \upharpoonright j) \upharpoonright \gamma$, if there exists $j > j_\gamma$ such that $\iota_{\mathbb{E}}(\sigma(x \upharpoonright j)) \not\subseteq \tau(x \upharpoonright j + 1)$, then $\min\{\alpha < \mu \mid \iota_{\mathbb{E}}(\sigma(x \upharpoonright j))(\alpha) \neq \tau(x \upharpoonright j + 1)(\alpha)\} > \gamma$. Hence $\sigma(x \upharpoonright j + 1) = \sigma(x \upharpoonright j) \wedge \mathbb{E}_i$ for some $i > \gamma$.

Therefore, we proved that for every $\gamma < \mu$ there exists $i (= j_\gamma) < \mu$ such that for every $j \geq i$, $\iota_{\mathbb{E}}(\sigma(x \upharpoonright j)) \upharpoonright \gamma = \tau(x \upharpoonright j) \upharpoonright \gamma = f(x) \upharpoonright \gamma$. We conclude that, $\sigma(x) \in R_{\mathbb{E}}$ and $\bar{\sigma}(x) = \bigcup_{i < \mu} \iota_{\mathbb{E}}(\sigma(x \upharpoonright i)) = f(x)$.

(2) \Rightarrow (1) Let σ be a winning strategy for player **II** in the eraser game $E_\nu^\mu(f)$, hence for every $x \in A$, $\bar{\sigma}(x) = f(x)$. We set $f_i(x) = \iota_{\mathbb{E}}(\sigma(x \upharpoonright i)) \wedge 0^{(\mu)}$. The functions f_i 's are continuous as they are locally constant. Finally, since τ is winning for **II**, it is immediate to see that for all $x \in A$, $(f_i(x))_{i < \mu}$ converges to $f(x)$. Indeed, recalling that $\bar{\tau}(x) = \bigcup_{i < \mu} \iota_{\mathbb{E}}(\tau(x \upharpoonright i))$, for every $\gamma < \mu$ there

exists $i < \mu$ $\iota_E(\sigma(x \upharpoonright i)) = f(x) \upharpoonright \gamma$, hence for every $\gamma < \mu$ there exists $i < \mu$ for all $j \geq i$ $\iota_E(\sigma(x \upharpoonright j)) \upharpoonright \gamma = f(x) \upharpoonright \gamma$, and since $\iota_E(\sigma(x \upharpoonright j)) \upharpoonright \gamma = f_j(x) \upharpoonright \gamma$, we are done. \square

Assume that κ is an uncountable cardinal satisfying $2^{<\kappa} = \kappa$. If κ is regular, by Theorem 5.2.7, given any $A \subseteq {}^\kappa\kappa$ and $f : A \rightarrow {}^\kappa\kappa$, f is κ^+ - Σ_2^0 -measurable if and only if f is κ -Baire class 1, that is, $f \in \kappa\text{-lim } \mathcal{M}_1(A, {}^\kappa\kappa)$. Therefore, Theorem 6.3.3 applied to $\nu = \mu = \kappa$ shows that both versions of the generalized eraser games characterize the class of κ^+ - Σ_2^0 -measurable functions. The same is not true when κ is singular (see Remark 5.3.17 and Section 5.3.2).

Corollary 6.3.4. *Assume that κ is a regular uncountable cardinal satisfying $2^{<\kappa} = \kappa$. Let A be an arbitrary subset of ${}^\kappa\kappa$ and $f : A \rightarrow {}^\kappa\kappa$ be a function. Then, the following are equivalent:*

- (1) f is a κ -Baire class 1 function;
- (2) player **II** has a winning strategy in $E_\kappa^\kappa(f)$;
- (3) player **II** has a winning strategy in $\mathbb{E}_\kappa^\kappa(f)$;
- (4) f is κ^+ - Σ_2^0 -measurable.

6.4 Generalized Backtrack game

We now define the generalized version of the classical van Wesep's backtrack game [Wes77].

We consider the cardinals ν, μ with $\nu \geq 2$ and $\mu \geq \omega$. In the backtrack game, Player **II** plays elements of $\nu \cup \{\mathbf{P}, \mathbf{B}\}$, where the token **P** represents “pass” as in the Wadge game, and **B** represents “backtrack”. The backtrack move allows Player **II** to erase her entire output and begin playing a new sequence. However, she may use this option only $< \mu$ -times in each run, which ensures that she produces a sequence in ${}^\mu\nu$ at the end of the game — as long as she does not pass too often.

Definition 6.4.1. Fix a set $A \subseteq {}^\mu\nu$ and a function $f : A \rightarrow {}^\mu\nu$. The **generalized Backtrack game** $\mathbb{B}_\nu^\mu(f) = (f, M_{\mathbf{B}}, R_{\mathbf{B}}, \iota_{\mathbf{B}})$, where

- $M_{\mathbf{B}} = \{\mathbf{P}, \mathbf{B}\}$,
- $R_{\mathbf{B}} = \{y \in {}^\mu(\nu \cup \{\mathbf{P}, \mathbf{B}\}) \mid \forall i \exists j \geq i (y(j) \neq \mathbf{P}) \wedge \exists i \forall j \geq i (y(j) \neq \mathbf{B})\}$,

and interpretation function $\iota_{\mathbf{B}} : {}^\mu(\nu \cup \{\mathbf{P}, \mathbf{B}\}) \rightarrow {}^{\leq\mu}\nu$ is defined by $\iota_{\mathbf{B}}(y) = \iota_{\mathbf{W}}((y(j))_{j \geq i})$ where $i = \min\{k < \mu \mid \forall j \geq k (y(j) \neq \mathbf{B})\}$.

Definition 6.4.2. We consider a subset $A \subseteq {}^\mu\nu$ and $f : A \rightarrow {}^\mu\nu$. The **generalized Backtrack-seq game** $\mathbb{B}_\nu^\mu(f)$ is a two-player infinite alternating game of length μ , where at each turn $i < \mu$ Player **I** plays ordinals $x_i < \nu$ and Player **II** plays sequences $s_i \in {}^{<\mu}\nu$. After μ -many rounds, Player **I** has produced $x = (x_i)_{i < \mu} \in {}^\mu\nu$. Player **II** wins the game if the tree $\mathcal{T}_{(s_i)_{i < \mu}}$ is $< \mu$ -splitting, it satisfies the condition

$$\exists \alpha < \mu \forall \beta \geq \alpha \exists ! s \in \mathcal{T}_{(s_i)_{i < \mu}} (\text{lh}(s) = \beta) \quad (6.4.1)$$

and $f(x)$ is its unique branch of length μ .

A strategy for Player **II** in $\mathbb{B}_\nu^\mu(f)$ is a function $\tau : {}^{<\mu}\nu \setminus \{\emptyset\} \rightarrow {}^{<\mu}\nu$ and it is winning for Player **II** if for all $x \in A$, $\mathcal{T}_{(\tau(x \upharpoonright i))_{i < \mu}}$ is $<\mu$ -splitting, $f(x)$ is its unique branch of length μ , and condition (6.4.1) is satisfied. Note that this means that there exists $i < \mu$ such that:

- (a) for every $j, j' \geq i$, if $j \leq j'$ then $\tau(x \upharpoonright j) \subseteq \tau(x \upharpoonright j') \subseteq f(x)$;
- (b) for every $\beta \geq \alpha$ there exists $j \geq i$ such that $f(x) \upharpoonright \beta = \tau(x \upharpoonright j) \upharpoonright \beta$.

Indeed, given $\alpha < \mu$ as in (6.4.1), it is sufficient to take $i = \min\{i' < \mu \mid \text{lh}(\tau(x \upharpoonright i')) \geq \alpha\}$.

The following theorem is the generalized counterpart of a classical Theorem due to Andretta [And06, Theorem 20].

Theorem 6.4.3. *Let ν, μ be cardinals with $\nu \geq 2$ and $\mu \geq \omega$, and assume that μ is regular. Let A be an arbitrary subset of ${}^\mu\nu$ and $f : A \rightarrow {}^\mu\nu$ be a function. Then, the following are equivalent:*

- (1) *there is a partition $\{D_i \mid i < \mu\}$ of closed subsets of A such that each $f \upharpoonright D_i$ is continuous;*
- (2) *Player **II** has a winning strategy in $\mathbb{B}_\nu^\mu(f)$;*
- (3) *Player **II** has a winning strategy in $\mathbb{B}_\nu^\mu(f)$.*

Proof. (1) \Rightarrow (3) For every $i < \mu$, let \mathcal{T}_i the tree induced by the closed set D_i and let $\tau_i : \mathcal{T}_i \rightarrow {}^{<\mu}\nu$ be a winning strategy for Player **II** in $\mathbb{W}_\nu^\mu(f \upharpoonright D_i)$. For every $s \in {}^{<\mu}\nu$, let $i_s = \min\{i < \mu \mid N_s \cap D_i \neq \emptyset\}$. Then, we define $\tau : {}^{<\mu}\nu \setminus \{\emptyset\} \rightarrow {}^{<\mu}\nu$ as follows. For every $s \in {}^{<\mu}\nu$, we set

$$\tau(s \frown \alpha) = \begin{cases} \tau_{i_s}(s \frown \alpha) & \text{if } s \frown \alpha \in \mathcal{T}_{i_s}, \\ \emptyset & \text{otherwise;} \end{cases}$$

if $\text{lh}(s)$ is limit, let $\bar{i}_s = \sup\{i_{s \upharpoonright j} \mid j < \text{lh}(s)\}$, then we set

$$\tau(s) = \begin{cases} \tau_{i_s}(s) & \text{if } i_s = \bar{i}_s, \\ \emptyset & \text{otherwise, i.e. } i_s > \bar{i}_s. \end{cases}$$

It is easy to see that τ is a strategy for Player **II** in $\mathbb{B}_\nu^\mu(f)$. To see it is winning, fix $x \in A$ and let $i < \mu$ such that $x \in D_i$, then there exists $j < \mu$ minimal such that, for every $j' \geq j$, $i_{x \upharpoonright j'} = i$, hence $\tau(x \upharpoonright j') = \tau_i(x \upharpoonright j')$. Since τ_i is monotone, condition (6.4.1) is satisfied, and since τ_i is winning for Player **II**, $f \upharpoonright D_i(x) = f(x)$ is the unique branch of $\mathcal{T}_{(\tau(x \upharpoonright i))_{i < \mu}}$. Finally, $\mathcal{T}_{(\tau(x \upharpoonright i))_{i < \mu}}$ is $<\mu$ -splitting because $\{D_i \mid i < \mu\}$ is a partition of closed subsets of A .

(3) \Rightarrow (2) Let $\tau : {}^{<\mu}\nu \setminus \{\emptyset\} \rightarrow {}^{<\mu}\nu$ be a winning strategy for player **II** in $\mathbb{B}_\nu^\mu(f)$. We will define $\sigma : {}^{<\mu}\nu \rightarrow {}^{<\mu}(\nu \cup \{\mathbf{P}, \mathbf{B}\})$ winning strategy for player **II** in $\mathbb{B}_\nu^\mu(f)$ by recursion on $\text{lh}(s)$. We set $\sigma(\emptyset) = \emptyset$, and for every $s \in {}^{<\mu}\nu$:

$$\sigma(s \frown \alpha) = \begin{cases} \sigma(s) \frown \tau(s \frown \alpha)(\text{lh}(\iota_{\mathbf{B}}(\sigma(s)))) & \text{if } \iota_{\mathbf{B}}(\sigma(s)) \subseteq \tau(s \frown \alpha), \\ \sigma(s) \frown \mathbf{B} & \text{otherwise,} \end{cases}$$

and if $\text{lh}(s)$ is a limit ordinal, we set $\sigma(s) = \bigcup_{i < \text{lh}(s)} \sigma(s \upharpoonright i)$. Clearly σ is monotone and $\text{lh}(\sigma(s)) = \text{lh}(s)$. Fix $x \in A$ and let $y = \bigcup_{j < \mu} \sigma(x \upharpoonright j)$. Since τ is winning for **II**, there exists $\alpha < \mu$ as in condition (6.4.1), and $i = \min\{j < \mu \mid \text{lh}(\tau(x \upharpoonright j)) \geq \alpha\}$ which satisfies both conditions (a) and (b).

Claim 6.4.3.1. For every $j \geq i$, $\iota_{\mathbf{B}}(\sigma(x \upharpoonright j)) \subseteq \tau(x \upharpoonright j + 1)$.

Moreover, $(\text{lh}(\iota_{\mathbf{B}}(\sigma(x \upharpoonright j))))_{i \leq j < \mu}$ is strictly increasing.

Proof. We prove by induction on $\beta \geq 0$ that $\iota_{\mathbf{B}}(\sigma(x \upharpoonright i + \beta)) \subseteq \tau(x \upharpoonright i + \beta + 1)$.

For the base case $\beta = 0$, without loss of generality we can assume that $i = i' + 1$ is a successor ordinal (if not, simply consider $i + 1$ instead of i). Let $\gamma_{-1} = \text{lh}(\iota_{\mathbf{B}}(\sigma(x \upharpoonright i')))$, and note that $\gamma_{-1} \leq i' < i$. If $\iota_{\mathbf{B}}(\sigma(x \upharpoonright i')) \subseteq \tau(x \upharpoonright i)$, using condition (a), $\iota_{\mathbf{B}}(\sigma(x \upharpoonright i)) = \iota_{\mathbf{B}}(\sigma(x \upharpoonright i')) \wedge \tau(x \upharpoonright i)(\gamma_{-1}) \subseteq \tau(x \upharpoonright \gamma_{-1} + 1) \subseteq \tau(x \upharpoonright i)$. Otherwise, $\iota_{\mathbf{B}}(\sigma(x \upharpoonright i)) = \emptyset \subseteq \tau(x \upharpoonright i)$. Let $\gamma_0 = \text{lh}(\iota_{\mathbf{B}}(\sigma(x \upharpoonright i)))$.

Assume that $\beta = \beta' + 1$ is a successor ordinal. Then, $\gamma_{\beta'} = \text{lh}(\iota_{\mathbf{B}}(\sigma(x \upharpoonright i + \beta')))$, and note that $\gamma_{\beta'} \leq i + \beta' < i + \beta$. By induction hypothesis $\iota_{\mathbf{B}}(\sigma(x \upharpoonright i + \beta')) \subseteq \tau(x \upharpoonright i + \beta)$, hence $\iota_{\mathbf{B}}(\sigma(x \upharpoonright i + \beta)) = \iota_{\mathbf{B}}(\sigma(x \upharpoonright i + \beta')) \wedge \tau(x \upharpoonright i + \beta)(\gamma_{\beta'}) = \tau(x \upharpoonright \gamma_{\beta'} + 1) \subseteq \tau(x \upharpoonright i + \beta + 1)$ by definition of σ and (a). Let $\gamma_{\beta} = \text{lh}(\iota_{\mathbf{B}}(\sigma(x \upharpoonright i + \beta)))$ and note that $\gamma_{\beta} = \gamma_{\beta'} + 1$.

Finally, suppose that β is a limit ordinal. Again using condition (a) and the definition of σ we get $\iota_{\mathbf{B}}(\sigma(x \upharpoonright i + \beta)) = \bigcup_{\beta' < \beta} \iota_{\mathbf{B}}(\sigma(x \upharpoonright i + \beta')) \subseteq \bigcup_{\beta' < \beta} \tau(x \upharpoonright i + \beta' + 1) \subseteq \tau(x \upharpoonright i + \beta + 1)$. Moreover, let $\gamma_{\beta} = \text{lh}(\iota_{\mathbf{B}}(\sigma(x \upharpoonright i + \beta)))$ and note that $\gamma_{\beta} = \sup_{\beta' < \beta} \gamma_{\beta'}$. \square

By Claim 6.4.3.1 and definition of σ , for every $j \geq i$, $y(j) \neq \mathbf{B}$, therefore $y \in R_{\mathbf{B}}$ (note that no symbol \mathbf{P} is involved in the definition of σ). Moreover, for every $j \geq i$ let $\gamma_j = \text{lh}(\iota_{\mathbf{B}}(\sigma(x \upharpoonright j)))$: again by Claim 6.4.3.1, $(\gamma_j)_{i \leq j < \mu}$ is strictly increasing hence $\sup_{i \leq j < \mu} \gamma_j = \mu$.

Now, by condition (a), for every $\beta \geq \alpha$ there exists $j_{\beta} \geq i$ such that $f(x) \upharpoonright \beta = \tau(x \upharpoonright j_{\beta}) \upharpoonright \beta$. Let $j'_{\beta} \geq i$ such that $\gamma_{j'_{\beta}} \geq \beta$, and let $J = \max\{j_{\beta}, j'_{\beta}\}$. By condition (b), for every $j' \geq J$, $f(x) \upharpoonright \beta = \tau(x \upharpoonright j') \upharpoonright \beta$. Therefore,

$$\iota_{\mathbf{B}}(\sigma(x \upharpoonright J)) \upharpoonright \beta = \tau(x \upharpoonright J + 1) \upharpoonright \beta = f(x) \upharpoonright \beta.$$

This proves that $\iota_{\mathbf{B}}(y) = \bigcup_{j > i} \iota_{\mathbf{B}}(\sigma(x \upharpoonright j)) = f(x)$ as desired.

(2) \Rightarrow (1) Let $\sigma : {}^{<\mu}\nu \setminus \{\emptyset\} \rightarrow {}^{<\mu}\nu$ be a winning strategy for Player \mathbf{II} in $\mathbb{W}_{\nu}^{\mu}(f)$. For every $x \in A$, let $y_x = \bigcup_{i < \mu} \sigma(x \upharpoonright i)$. We define for every $i < \mu$:

$$D_0 = \bigcap_{j < \mu} \{x \in A \mid y_x(j) \neq \mathbf{B}\},$$

$$D_{i+1} = \{x \in A \mid y_x(i) = \mathbf{B}\} \cap \bigcap_{i < j < \mu} \{x \in A \mid y_x(j) \neq \mathbf{B}\}, \text{ and}$$

$$D_i = \bigcap_{j < i} \bigcup_{j \leq j' < i} \{x \in A \mid y_x(j') = \mathbf{B}\} \cap \bigcap_{i \leq j < \mu} \{x \in A \mid y_x(j) \neq \mathbf{B}\}, \text{ if } i \text{ is limit ordinal.}$$

Since the algebra of clopen subsets of ${}^{\mu}\nu$ is a μ -algebra and $i < \mu$, it is easy to see that each D_i is closed. Moreover, $D_i \cap D_j = \emptyset$ for every $i < j < \mu$, and since σ is winning for player \mathbf{II} , also $A = \bigcup_{i < \mu} D_i$.

It remains to show that each $f \upharpoonright D_i$ is continuous. Fix $i < \mu$. The idea is that, for every $x \in D_i$ enumerated by Player \mathbf{I} , Player \mathbf{II} doesn't play any more \mathbf{B} after i -many turns: from that moment on, σ becomes just a Wadge strategy in $\mathbb{W}_{\nu}^{\mu}(f \upharpoonright D_i)$, and since σ is winning for Player \mathbf{II} , $f \upharpoonright D_i$ is continuous. Formally, we define $\sigma_i : \mathcal{T}_{D_i} \rightarrow {}^{<\mu}\nu$ as

$$\sigma_i(s) = \begin{cases} \sigma(s) & \text{if } \text{lh}(s) \geq i \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly, σ_i is monotone and for every $x \in D_i$, $\bigcup_{\alpha < \mu} \sigma_i(x \upharpoonright \alpha) = f(x)$. \square

Chapter 7

On Borel sets in ideal topologies

In most works in generalized descriptive set theory, the generalized Cantor space and the generalized Baire space are equipped with the bounded topology under the assumption that $\kappa^{<\kappa} = \kappa$, yielding interesting results on their κ^+ -Borel and κ -analytic subsets (see e.g. [MV93, FHK14, HK18, LS15, ACRP25]). As already mentioned in the introduction to this thesis, the bounded topology is particularly natural due to its applications in model theory and infinitary logics [Vau75, MV93, SV00, She01, SV02, She04, DV11, FHK14, MMR21, Mor23b].

However, meaningful variants of this classical framework have emerged in the literature. One approach is to drop or weaken the cardinal assumption: either working with regular κ while permitting $\kappa^{<\kappa} > \kappa$ [HMV25], or adopting the weaker assumption $2^{<\kappa} = \kappa$ and allowing κ to be singular, as in [AMR22, DMR25, ACRP25, MRP25] as well as in the previous chapters of this thesis. Another direction involves changing the topology itself. While papers on infinitary logics predominantly use the bounded topology, works in general topology [Kur66, Ü82, Kra04, Ili12, CK13] typically employ either the product or the box topology. Recent developments have shown that the product topology can be essential in certain contexts in generalized descriptive set theory, as demonstrated in [AMR22, Sections 12 and 13] and [DMR25, Sections 5 and 6].

This brings us to the framework that will be central in this chapter. In [HKSJ22], Holy, Koelbing, Schlicht, and Wohofsky introduced the *ideal topology* $\tau_{\mathcal{I}}$ on ${}^{\kappa}2$ and ${}^{\kappa}\kappa$, where κ is a regular uncountable cardinal and \mathcal{I} is an ideal extending the bounded ideal (see Definition 7.1.3). This topology refines the bounded topology and provides a more general framework that encompasses the bounded topology as a special case. While [HKSJ22] studied many properties of ideal topologies and initiated the investigation of \mathcal{I} -Borel sets, their analysis of the \mathcal{I} -Borel hierarchy remained incomplete, and they left open fundamental questions about its structure. Moreover, the notion of analytic sets was not addressed in the ideal topology context. Our work provides a comprehensive treatment of both topics, resolving the main open question from [HKSJ22].

We now describe the content of the chapter in detail. Section 7.1 introduces the ideal topology on ${}^{\kappa}2$ and ${}^{\kappa}\kappa$ and establishes its basic properties (Lemma 7.1.5). In Section 7.2, we characterize continuous functions on these spaces via monotone and domain-increasing functions (Definition 7.2.1).

Section 7.3 provides a comprehensive description of the κ^+ -Borel hierarchy on spaces equipped with the ideal topology $\tau_{\mathcal{I}}$. We provide a rigorous treatment of all relevant definitions and key foundational results about \mathcal{I} -Borel sets. This includes natural features of the \mathcal{I} -Borel hierarchy, like the property of being increasing (Corollary 7.3.5), the closure properties of the pointclasses $\mathcal{I}\text{-}\Sigma_{\alpha}^0$, $\mathcal{I}\text{-}\Pi_{\alpha}^0$ and $\mathcal{I}\text{-}\Delta_{\alpha}^0$ appearing in it (Proposition 7.3.11) and the length of the hierarchy.

In [HKS^W22], the following is highlighted as a major open problem:

Question (Question 1, [HKS^W22]). Do the \mathcal{I} -Borel sets form a strict hierarchy of length κ^+ ?

We answer this question affirmatively in Theorem 7.3.9, establishing that the \mathcal{I} -Borel hierarchy does not collapse for any ideal \mathcal{I} extending the bounded ideal.

Section 7.4 introduces and analyzes the class of \mathcal{I} -analytic sets. Surprisingly, this class behaves quite differently from its classical counterpart. We prove in Theorem 7.4.2 that, when the ideal \mathcal{I} contains an unbounded set, the class of \mathcal{I} -analytic sets coincides with the entire powerset of the space. This dramatic collapse occurs for all reasonable definitions of \mathcal{I} -analyticity (Corollary 7.4.3 and Proposition 7.4.5), revealing a fundamental distinction between the bounded topology and ideal topology settings.

Throughout this chapter, we work in ZFC and we assume that κ is an uncountable regular cardinal.

7.1 Ideals and ideal topologies

An ideal \mathcal{I} on κ is κ -**complete** if for every $\mathcal{J} \subseteq \mathcal{I}$, $|\mathcal{J}| < \kappa$ implies $\bigcup \mathcal{J} \in \mathcal{I}$, and it is proper if $\mathcal{I} \neq \mathcal{P}(\kappa)$. Some well known κ -complete proper ideals are the collection of bounded subsets of κ , denoted by “ b ”, and NS_{κ} , the collection of non-stationary subsets of κ . A set $\mathcal{B} \subseteq \mathcal{I}$ is a **basis for the ideal \mathcal{I}** if for every $D \in \mathcal{I}$ there exists $D' \in \mathcal{B}$ with $D \subseteq D'$. For example, $\{\alpha \mid \alpha < \kappa\}$ is a basis (of size κ) for the bounded ideal. Note that there are proper ideals other than the bounded ideal which have a basis of size κ , e.g., the ideal generated by the bounded ideal together with a set A , where A is an unbounded subset of κ with unbounded complement (see Example 7.2.4). Given \mathcal{I} and \mathcal{J} ideals on κ , we say that \mathcal{I} is \mathcal{J} -**tall** if for all $A \subseteq \kappa$ such that $A \notin \mathcal{J}$, there is $B \in \mathcal{I} \setminus \mathcal{J}$ such that $B \subseteq A$. We say that \mathcal{I} is **tall** if it is b -tall, and **stationary tall** if it is NS_{κ} -tall.

From this point onward (and unless stated otherwise), we assume that

\mathcal{I} is a κ -complete proper ideal on κ that extends the ideal of bounded sets.

Definition 7.1.1. An ideal \mathcal{I} has the **approximation property** if there is a strictly increasing continuous¹ sequence $(B_{\alpha})_{\alpha < \kappa}$ in \mathcal{I} such that $B_0 = \emptyset$, and for all $B \in \mathcal{I}$ there is $\alpha < \kappa$ such that $B \subsetneq B_{\alpha}$. We call $(B_{\alpha})_{\alpha < \kappa}$ an approximation sequence of \mathcal{I} .

¹A sequence $(s_{\alpha})_{\alpha < \kappa} \subseteq \kappa$ is strictly increasing if for every $\alpha < \beta < \kappa$, $s_{\alpha} \subsetneq s_{\beta}$, and it is continuous if for every limit ordinal $\alpha < \kappa$, $s_{\alpha} = \bigcup_{\beta < \alpha} s_{\beta}$.

Lemma 7.1.2. *An ideal \mathcal{I} has the approximation property if and only if there is a partition $\{C_\alpha \mid \alpha < \kappa\}$ of κ such that $\{D_\alpha \mid \alpha < \kappa\}$ is a basis where \mathcal{I} and $D_\alpha = \bigcup_{\beta < \alpha} C_\beta$.*

Proof. If $(B_\alpha)_{\alpha < \kappa}$ is an approximation sequence of \mathcal{I} , then it is a basis for \mathcal{I} . Moreover, let $C_0 = B_1$ and, for all $0 < \alpha < \kappa$, let $C_\alpha = B_{\alpha+1} \setminus \bigcup_{\beta < \alpha} C_\beta$. Clearly, $D_\alpha = B_{\alpha+1}$, thus $\{D_\alpha \mid \alpha < \kappa\}$ is a basis for \mathcal{I} .

Vice versa, let $\{C_\alpha \mid \alpha < \kappa\}$ be a partition of κ such that $\{D_\alpha \mid \alpha < \kappa\}$ is a basis for \mathcal{I} and $D_\alpha = \bigcup_{\beta < \alpha} C_\beta$. Set $B_\alpha = \bigcup_{\beta < \alpha} D_\beta$ for every $\alpha < \kappa$. Clearly $(B_\alpha)_{\alpha < \kappa}$ is strictly increasing, and since $\{D_\alpha \mid \alpha < \kappa\}$ is a basis, for all $B \in \mathcal{I}$ there is $\alpha \in \kappa$ such that $B \subseteq D_\alpha \subseteq B_{\alpha+1}$, therefore \mathcal{I} has the approximation property. \square

It follows from Lemma 7.1.2 that \mathcal{I} has the approximation property if and only if it admits a basis of size κ .

Let $\nu \in \{2, \kappa\}$ and consider the set ${}^\kappa\nu = \{x \mid x: \kappa \rightarrow \nu\}$. For any set $I \subseteq \mathcal{P}(\kappa)$, let $\text{Fn}_I({}^\kappa\nu) = \{f \mid f: D \rightarrow \nu \text{ is a function, and } D \in I\}$. For $f \in \text{Fn}_I({}^\kappa\nu)$, we define

$$\mathbf{N}_f({}^\kappa\nu) = \{x \in {}^\kappa\nu \mid f \subseteq x\}.$$

Definition 7.1.3. Let $\nu \in \{2, \kappa\}$. The \mathcal{I} -topology $\tau_{\mathcal{I}}$ on ${}^\kappa\nu$ is the topology generated by the collection

$$\mathcal{B}_{\mathcal{I}}({}^\kappa\nu) = \{\mathbf{N}_f({}^\kappa\nu) \mid f \in \text{Fn}_{\mathcal{I}}({}^\kappa\nu)\}.$$

The family $\mathcal{B}_{\mathcal{I}}({}^\kappa\nu)$ is the **canonical basis** for $\tau_{\mathcal{I}}$ and its elements are called the (canonical) **basic \mathcal{I} -open sets** or **\mathcal{I} -cones**. The elements of $\tau_{\mathcal{I}}$ are also called **\mathcal{I} -open sets**, and similarly we will address closed (clopen) sets with respect to $\tau_{\mathcal{I}}$ as **\mathcal{I} -closed (\mathcal{I} -clopen)** sets. An **ideal topology** is an \mathcal{I} -topology for an ideal \mathcal{I} .

When the space is clear from the context, we drop it from all the above notation.

Clearly, if $\mathcal{I} \subseteq \mathcal{I}'$ then $\tau_{\mathcal{I}} \subseteq \tau_{\mathcal{I}'}$. It is immediate that the NS_κ -topology refines the bounded topology. While the \mathcal{I} -topology coincides with the bounded topology when $\mathcal{I} = b$, it is strictly finer than τ_b whenever $\mathcal{I} \supsetneq b$. Indeed, for any unbounded set $D \in \mathcal{I}$ and any function $f: D \rightarrow 2$, the set \mathbf{N}_f is \mathcal{I} -open and b -closed, but not b -open. As already mentioned in the introduction, when $\mathcal{I} = b$ we will refer to $({}^\kappa\kappa, \tau_b)$ and $({}^\kappa 2, \tau_b)$ as the generalized Baire space and the generalized Cantor space, respectively.

The following result is an easy observation about the size of the basic \mathcal{I} -open sets, which will be useful later on.

Lemma 7.1.4. *Let $\nu \in \{2, \kappa\}$. For every $f \in \text{Fn}_{\mathcal{I}}({}^\kappa\nu)$, $|\mathbf{N}_f({}^\kappa\nu)| > \kappa$.*

Proof. Let $f \in \text{Fn}_{\mathcal{I}}({}^\kappa\nu)$ with $D = \text{dom}(f)$. Then, $|\mathbf{N}_f({}^\kappa\nu)| = 2^{|\kappa \setminus D|}$, so it is sufficient to show that $\kappa \setminus D$ is unbounded. Towards a contradiction, suppose that $|\kappa \setminus D| < \kappa$. Then, there exists $\gamma < \kappa$ such that $\kappa \setminus D \subseteq \gamma$. Since \mathcal{I} extends the bounded ideal, $D \cup \gamma \in \mathcal{I}$. On the other hand, $D \cup \gamma = \kappa$ hence \mathcal{I} is not proper, a contradiction. \square

When $\mathcal{I} = \text{NS}_\kappa$ and $\nu = 2$, Lemma 7.1.4 holds in a stronger form since any \mathcal{I} -cone with the induced topology is homeomorphic to the space $({}^\kappa 2, \tau_{\mathcal{I}})$ by [HKS22, Lemma 1.6].

We recall that for a topological space (X, τ) , the density character is the least possible cardinality of a dense subset of X , and the weight is the least possible cardinality of an open basis. Moreover, (X, τ) is κ -compact (or κ -Lindelöf) if every τ -open covering of X has a subcovering of size smaller than κ .

We say that a subset $A \subseteq {}^\kappa \nu$ is \mathcal{I} -dense if it is dense with respect to $\tau_{\mathcal{I}}$, i.e., if for every \mathcal{I} -open set $U \subseteq {}^\kappa \nu$, $A \cap U \neq \emptyset$.

The next lemma summarizes some properties of the ideal topology on ${}^\kappa 2$ and ${}^\kappa \kappa$. Some of these properties already appeared in [HKS22], but we include them for the reader's convenience.

Lemma 7.1.5. *Consider the spaces $({}^\kappa 2, \tau_{\mathcal{I}})$ and $({}^\kappa \kappa, \tau_{\mathcal{I}})$. The following properties hold.*

- (1) *The topology $\tau_{\mathcal{I}}$ is perfect, regular Hausdorff, and zero-dimensional.*
- (2) *The topology $\tau_{\mathcal{I}}$ is closed under intersections of length at most α (for some ordinal α) if and only if $\alpha < \kappa$. Therefore, the collection of all \mathcal{I} -clopen subsets of ${}^\kappa \nu$ is a κ -algebra.*
- (3) *For all $x \in {}^\kappa \nu$, any \mathcal{I} -open neighborhood basis of x has size at least κ . Moreover, each point has an \mathcal{I} -open neighborhood basis of size κ if and only if \mathcal{I} has a basis of size κ .*

Moreover, if \mathcal{I} contains an unbounded subset of κ :

- (4) *The topology $\tau_{\mathcal{I}}$ is not compact nor κ -compact.*
- (5) *The topology $\tau_{\mathcal{I}}$ has weight 2^κ , and $|\tau_{\mathcal{I}}| = 2^{2^\kappa}$.*
- (6) *The topology $\tau_{\mathcal{I}}$ has density character 2^κ .*

Proof. Let $\nu \in \{2, \kappa\}$.

(1) It is easy to check that the sets in $\mathcal{B}_{\mathcal{I}}$ are clopen, hence the topology $\tau_{\mathcal{I}}$ is zero-dimensional and regular. Moreover, it is perfect and Hausdorff, because $b \subseteq \mathcal{I}$ and τ_b is perfect and Hausdorff.

(2) Suppose $\alpha < \kappa$, let $(U_\beta)_{\beta < \alpha}$ be a sequence of \mathcal{I} -open sets, and let $V = \bigcap_{\beta < \alpha} U_\beta$. To show that V is \mathcal{I} -open, for every $x \in V$ we construct $f \in \text{Fn}_{\mathcal{I}}$ such that $x \in \mathbf{N}_f \subseteq V$. For every $\beta < \alpha$, let $D_\beta \in \mathcal{I}$ be such that $\mathbf{N}_{x \upharpoonright D_\beta} \subseteq U_\beta$. Then, $D = \bigcup_{\beta < \alpha} D_\beta \in \mathcal{I}$ by κ -completeness of \mathcal{I} , and $\mathbf{N}_{x \upharpoonright D} \subseteq V$.

Suppose $\alpha \geq \kappa$, and let $(D_\beta)_{\beta < \alpha}$ be an increasing sequence of elements of \mathcal{I} such that $\bigcup_{\beta < \alpha} D_\beta = \kappa \notin \mathcal{I}$. Set $V_\beta = \mathbf{N}_{0^{(\kappa)} \upharpoonright D_\beta}$. Then, $\bigcap_{\beta < \alpha} V_\beta = \{0^{(\kappa)}\}$ is an \mathcal{I} -closed set and it is not \mathcal{I} -open.

(3) Let $x \in {}^\kappa \nu$ and $\mathcal{U}_{\mathcal{I}}$ be an arbitrary \mathcal{I} -open neighborhood basis of x . We want to construct a map $\mathcal{U}_{\mathcal{I}} \rightarrow \text{Fn}_{\mathcal{I}}$ that sends each $U \in \mathcal{U}_{\mathcal{I}}$ to $f_U \subseteq x$ such that $\mathbf{N}_{f_U} \subseteq U$, yielding the neighborhood basis $\mathcal{U}' = \{\mathbf{N}_{f_U} \mid U \in \mathcal{U}_{\mathcal{I}}\}$. To do that, for every $U \in \mathcal{U}_{\mathcal{I}}$ choose $D_U \in \mathcal{I}$ such that $\mathbf{N}_{x \upharpoonright D_U} \subseteq U$ and consider $\mathcal{U}' = \{\mathbf{N}_{x \upharpoonright D_U} \mid U \in \mathcal{U}_{\mathcal{I}}\}$. Notice that the canonical map $\mathcal{U}_{\mathcal{I}} \rightarrow \mathcal{U}'$ is a surjection,

hence for the first part it is enough to show that $|\mathcal{U}'| \geq \kappa$. Suppose, towards a contradiction, that $|\mathcal{U}'| < \kappa$. Since \mathcal{I} is κ -complete, $\bar{D} = \bigcup_{U \in \mathcal{U}_x} D_U \in \mathcal{I}$, so the \mathcal{I} -open neighborhood $\mathbf{N}_{x \upharpoonright \bar{D}}$ does not contain any element of the open neighborhood basis \mathcal{U}' , a contradiction.

Next, we take care of the second part the statement. For the implication from left to right, we show that $\{D_U \mid U \in \mathcal{U}_x\}$ is a basis for \mathcal{I} , so that $|\mathcal{U}_x| = \kappa$ implies that \mathcal{I} has a basis of size κ . Let $D \in \mathcal{I}$. Clearly, $\mathbf{N}_{x \upharpoonright D}$ is an \mathcal{I} -open neighborhood of x , so there exists $U \in \mathcal{U}_x$ such that $x \in U \subseteq \mathbf{N}_{x \upharpoonright D}$. Thus $\mathbf{N}_{x \upharpoonright D_U} \subseteq U \subseteq \mathbf{N}_{x \upharpoonright D}$ and $D \subseteq D_U$. For the other direction, let \mathcal{B} be a basis for \mathcal{I} . It is enough to show that $\{\mathbf{N}_{x \upharpoonright D} \mid D \in \mathcal{B}\}$ is a neighborhood basis of x . Let V be an \mathcal{I} -open set such that $x \in V$. There exists $f \in \text{Fn}_{\mathcal{I}}$ such that $x \in \mathbf{N}_f \subseteq V$. Let $\text{dom}(f) = D \in \mathcal{I}$, so there exists $D' \in \mathcal{B}$ such that $D \subseteq D'$. Therefore $x \in \mathbf{N}_{x \upharpoonright D'} \subseteq \mathbf{N}_{x \upharpoonright D} = \mathbf{N}_f$ and $\{\mathbf{N}_{x \upharpoonright D} \mid D \in \mathcal{B}\}$ is a neighborhood basis of x .

Suppose now that \mathcal{I} contains an unbounded subset D of κ . Consider the collection of functions $\mathcal{F}_D = \{f \mid f : D \rightarrow \nu\}$, and let $\mathcal{C}_D = \{\mathbf{N}_f \mid f \in \mathcal{F}_D\}$. Note that \mathcal{C}_D is an \mathcal{I} -clopen partition of ${}^\kappa\nu$. Clearly, the sets in \mathcal{C}_D are \mathcal{I} -clopen and pairwise disjoint. Moreover, for every $x \in {}^\kappa\nu$, $x \upharpoonright D = f$ for some $f \in \mathcal{F}_D$, hence $x \in \mathbf{N}_f$.

(4) Since \mathcal{C}_D is a \mathcal{I} -clopen partition of ${}^\kappa\nu$ of size 2^κ , no subcover of size $< \kappa$ can be extracted.

(5) Let \mathcal{B} be an arbitrary basis for $\tau_{\mathcal{I}}$. Since \mathcal{C}_D is a partition, the map

$$\mathcal{B} \rightarrow \mathcal{F} : B \mapsto \begin{cases} \emptyset & \text{if } \forall f \in \mathcal{F}_D (B \not\subseteq \mathbf{N}_f) \\ f & \text{if } B \subseteq \mathbf{N}_f \end{cases}$$

is a well-defined surjection. Since $|\mathcal{F}_D| = 2^\kappa$, $\mathcal{B} \geq 2^\kappa$. On the other hand, $\{\mathbf{N}_f \mid f \in \text{Fn}_{\mathcal{I}}\}$ is a basis of size 2^κ , hence $\tau_{\mathcal{I}}$ has weight 2^κ .

Clearly, $|\tau_{\mathcal{I}}| \leq |\mathcal{P}(\mathcal{B})| = 2^{2^\kappa}$ as witnessed by the injective map $\tau_{\mathcal{I}} \rightarrow \mathcal{P}(\mathcal{B}) : U \mapsto \{B \in \mathcal{B} \mid B \subseteq U\}$. For the other inequality, for all $X \subseteq \mathcal{F}_D$ define $U_X = \bigcup_{f \in X} \mathbf{N}_f$. Notice that if $X, Y \subseteq \mathcal{F}_D$, then $X \neq Y$ implies $U_X \neq U_Y$. Thus, the map $\mathcal{P}(\mathcal{F}_D) \rightarrow \tau_{\mathcal{I}} : X \mapsto U_X$ is injective and we get $|\tau_{\mathcal{I}}| \geq 2^{2^\kappa}$.

(6) Let $E \subseteq {}^\kappa\nu$ be an \mathcal{I} -dense set. Since \mathcal{C}_D is a partition of ${}^\kappa\nu$, the surjection $E \rightarrow \mathcal{F} : x \mapsto f$, where $f \in \mathcal{F}_D$ is (unique) such that $x \in \mathbf{N}_f$, witnesses $|E| \geq 2^\kappa$. Moreover, for every $f \in \text{Fn}_{\mathcal{I}}$ define the function f^0 as follows:

$$f^0(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma \in \text{dom}(f) \\ 0 & \text{otherwise.} \end{cases}$$

The set $F^0 = \{f^0 \mid f \in \text{Fn}_{\mathcal{I}}\} \subseteq {}^\kappa\nu$ is \mathcal{I} -dense and has cardinality $|\text{Fn}_{\mathcal{I}}| = 2^\kappa$. \square

The next result, although not essential, is nevertheless of independent interest.

Proposition 7.1.6. *Let $\nu \in \{2, \kappa\}$. The space $({}^\kappa\nu \times {}^\kappa\kappa, \tau_{\mathcal{I}} \times \tau_{\mathcal{I}})$ is homeomorphic to $({}^\kappa\kappa, \tau_{\mathcal{I}})$.*

Proof. Fix a bijection $\varphi : \nu \times \kappa \rightarrow \kappa$, and for every $\alpha < \kappa$, let $(\alpha)_1 < \nu$ and $(\alpha)_2 < \kappa$ such that $\varphi((\alpha)_1, (\alpha)_2) = \alpha$. Let $\Phi : {}^\kappa\nu \times {}^\kappa\kappa \rightarrow {}^\kappa\kappa$ be the function

defined by $\Phi(x, y)(\alpha) = \varphi(x(\alpha), y(\alpha))$ for every $\alpha < \kappa$. We have that Φ is bijective because φ is a bijection. We claim that Φ is an homeomorphism. To see that Φ is continuous, note that for any $f \in \text{Fn}_{\mathcal{I}}(\kappa\kappa)$, if $D = \text{dom}(f)$,

$$\Phi^{-1}(\mathbf{N}_f) = \{(x, y) \mid \forall \alpha \in D (x(\alpha) = (f(\alpha))_1 \wedge y(\alpha) = (f(\alpha))_2)\} = \mathbf{N}_{f_1} \times \mathbf{N}_{f_2}.$$

where $f_1, f_2 \in \text{Fn}_{\mathcal{I}}$ are defined by $f_1(\alpha) = (f(\alpha))_1$ and $f_2(\alpha) = (f(\alpha))_2$ for every $\alpha \in D$. To see that Φ is open, for any $f, g \in \text{Fn}_{\mathcal{I}}$, if $D_f = \text{dom}(f)$ and $D_g = \text{dom}(g)$, notice that $\Phi(\mathbf{N}_f \times \mathbf{N}_g) = \mathbf{N}_h$, where $h : D_f \cap D_g \rightarrow \kappa$ is defined by $h(\alpha) = \varphi(f(\alpha), g(\alpha))$ for every $\alpha \in D_f \cap D_g$. Clearly, $D_f \cap D_g \in \mathcal{I}$. \square

7.2 The \mathcal{I} -continuous functions

Given $\nu, \mu \in \{2, \kappa\}$, a function $\Phi : \kappa\nu \rightarrow \kappa\mu$ is \mathcal{I} -**continuous** if it is continuous with respect to $\tau_{\mathcal{I}}$, i.e., $f : (\kappa\nu, \tau_{\mathcal{I}}) \rightarrow (\kappa\mu, \tau_{\mathcal{I}})$ is continuous.

Throughout this section, let $\nu \in \{2, \kappa\}$. Similarly to classical setting (see [Kec95, Proposition 2.6]) and to the generalized setting with respect to the bounded topology τ_b (see Section 2.7), every \mathcal{I} -continuous function can be represented by a monotone and domain-increasing function $\varphi : \text{Fn}_{\mathcal{I}}(\kappa\nu) \rightarrow \text{Fn}_{\mathcal{I}}(\kappa\nu)$. However, when dealing with ideals \mathcal{I} strictly extending the bounded ideal, the definition is rather different, as limits over direct sets are necessary (see Section 2.5 for definitions). We will consider \mathcal{D} -limits where $\mathcal{D} = (\mathcal{I}, \subseteq)$.

Definition 7.2.1. Let $\varphi : \text{Fn}_{\mathcal{I}}(\kappa\nu) \rightarrow \text{Fn}_{\mathcal{I}}(\kappa\nu)$ be a function. We say that φ is **monotone** if $f \subseteq g$ implies $\varphi(f) \subseteq \varphi(g)$ for all $f, g \in \text{Fn}_{\mathcal{I}}(\kappa\nu)$.

We say that φ is **continuous** if it is monotone and for all $x \in \kappa\nu$ and $D \in \mathcal{I}$ there exists $E \in \mathcal{I}$ such that $D \subseteq \text{dom}(\varphi(x \upharpoonright E))$.

Let $\varphi : \text{Fn}_{\mathcal{I}}(\kappa\nu) \rightarrow \text{Fn}_{\mathcal{I}}(\kappa\nu)$ be a continuous function. The function induced by φ is $\varphi^* : \kappa\nu \rightarrow \kappa\nu$ defined for every $x \in \kappa\nu$ by

$$\varphi^*(x) = \lim_{D \in \mathcal{I}} \varphi(x \upharpoonright D). \tag{7.2.1}$$

Note that since φ is continuous, the limit in (7.2.1) exists because for all $D \in \mathcal{I}$ there exists $E \in \mathcal{I}$ such that

$$\varphi(x \upharpoonright E') \upharpoonright D = \varphi(x \upharpoonright E) \upharpoonright D$$

holds for all $E \subseteq E'$ with $E' \in \mathcal{I}$. Moreover, such limit is unique since $\kappa\nu$ is Hausdorff by Lemma 7.1.5(1), and therefore φ^* is well-defined.

Notice that if φ is continuous, then $\lim_{D \in \mathcal{I}} \varphi(x \upharpoonright D) = \lim_{D \in \mathcal{B}} \varphi(x \upharpoonright D)$ for any basis \mathcal{B} for \mathcal{I} . In particular, when \mathcal{I} is the bounded ideal, $\varphi^*(x) = \lim_{\alpha < \kappa} \varphi(x \upharpoonright \alpha)$ is the usual definition also adopted in Section 2.7.

Proposition 7.2.2. Let $\varphi : \text{Fn}_{\mathcal{I}}(\kappa\nu) \rightarrow \text{Fn}_{\mathcal{I}}(\kappa\nu)$ be a continuous function. Then, $\varphi^* : \kappa\nu \rightarrow \kappa\nu$ is \mathcal{I} -continuous.

Proof. It is sufficient to check that for every $f \in \text{Fn}_{\mathcal{I}}$,

$$(\varphi^*)^{-1}(\mathbf{N}_f) = \bigcup \{\mathbf{N}_g \mid f \subseteq \varphi(g)\}.$$

Since $\bigcup \{\mathbf{N}_g \mid f \subseteq \varphi(g)\}$ is \mathcal{I} -open for every $f \in \text{Fn}_{\mathcal{I}}$, φ^* is \mathcal{I} -continuous.

First, assume that $x \in \bigcup\{\mathbf{N}_g \mid f \subseteq \varphi(g)\}$ and let $D = \text{dom}(f)$. Our goal is to show that $\varphi^*(x) \upharpoonright D = f$. Let $g \in \text{Fn}_{\mathcal{I}}$ such that $g \subseteq x$ and $f \subseteq \varphi(g)$. If $E = \text{dom}(g)$, $g = x \upharpoonright E$ and $f = \varphi(x \upharpoonright E) \upharpoonright D$. Since φ is monotone, for every $E' \in \mathcal{I}$ such that $E \subseteq E'$ we have $\varphi(x \upharpoonright E) \upharpoonright D = \varphi(x \upharpoonright E') \upharpoonright D$. By definition of φ^* , given $\mathbf{N}_{\varphi^*(x) \upharpoonright D}$ there exists $F \in \mathcal{I}$ for every $F' \in \mathcal{I}$ with $F \subseteq F'$, $\varphi^*(x) \upharpoonright D = \varphi(x \upharpoonright F') \upharpoonright D$. Let $G = E \cup F \in \mathcal{I}$. Then, for every $G' \in \mathcal{I}$ such that $G \subseteq G'$ we have

$$\varphi^*(x) \upharpoonright D = \varphi(x \upharpoonright G') \upharpoonright D = f.$$

Hence $x \in (\varphi^*)^{-1}(\mathbf{N}_f)$.

Next, assume that $x \in (\varphi^*)^{-1}(\mathbf{N}_f)$. Then, $\varphi^*(x) \in \mathbf{N}_f$ and, if $D = \text{dom}(f)$, $\varphi^*(x) \upharpoonright D = f$. Since φ is continuous, there exists $E \in \mathcal{I}$ such that $D \subseteq \text{dom}(\varphi(x \upharpoonright E))$. Therefore, by definition of φ^* , $\varphi(x \upharpoonright E) \upharpoonright D = \varphi^*(x) \upharpoonright D$. Let $g = x \upharpoonright E$ to get $x \in \bigcup\{\mathbf{N}_g \mid f \subseteq \varphi(g)\}$. \square

Proposition 7.2.3. *Let $\Phi: {}^\kappa\nu \rightarrow {}^\kappa\nu$ be an \mathcal{I} -continuous function. Then, there is a continuous $\varphi: \text{Fn}_{\mathcal{I}}({}^\kappa\nu) \rightarrow \text{Fn}_{\mathcal{I}}({}^\kappa\nu)$ such that $\varphi^* = \Phi$.*

Proof. For every $g \in \text{Fn}_{\mathcal{I}}$, we define $X_g = \{f \in \text{Fn}_{\mathcal{I}} \mid \Phi(\mathbf{N}_g) \subseteq \mathbf{N}_f\}$. Note that X_g is a lattice and every chain \mathcal{C} in X_g has an upper bound (it is sufficient to take $G = \bigcup_{f \in \mathcal{C}} f$, since $\Phi(\mathbf{N}_g) \subseteq \bigcap_{f \in \mathcal{C}} \mathbf{N}_f = \mathbf{N}_G$). By Zorn's lemma, X_g has at least one maximal element. We now show that this maximal element is unique. Assume that $f, f' \in X_g$ are both maximal. Since $\Phi(\mathbf{N}_g) \subseteq \mathbf{N}_f \cap \mathbf{N}_{f'}$, f and f' must be compatible. Therefore, in order not to violate the maximality of either one, we must have $f = f'$. For every $g \in \text{Fn}_{\mathcal{I}}$, let $\varphi(g)$ be the (unique) maximal $f \in X_g$.

Let us show that the function φ is monotone. Assume that $g \subseteq g' \in \text{Fn}_{\mathcal{I}}$ and let $E = \text{dom}(g)$ and $E' = \text{dom}(g')$. Since $g \subseteq g'$, $\mathbf{N}_{g'} \subseteq \mathbf{N}_g$ and $\Phi(\mathbf{N}_{g'}) \subseteq \Phi(\mathbf{N}_g)$. From the definition of Φ we know that $\Phi(\mathbf{N}_g) \subseteq \mathbf{N}_{\varphi(g)}$. Therefore $\Phi(\mathbf{N}_{g'}) \subseteq \mathbf{N}_{\varphi(g)}$ and $\varphi(g) \in X_{g'}$. Thus $\varphi(g) \subseteq \varphi(g')$, by maximality of $\varphi(g')$.

It remains to prove that φ is continuous and that for every $x \in {}^\kappa\nu$, $\varphi^*(x) = \Phi(x)$. Fix $x \in {}^\kappa\nu$. By the \mathcal{I} -continuity of Φ , for every $D \in \mathcal{I}$ there exists $D' \in \mathcal{I}$ such that $\Phi(\mathbf{N}_{x \upharpoonright D'}) \subseteq \mathbf{N}_{\Phi(x) \upharpoonright D}$. Let $E = D \cup D' \in \mathcal{I}$, so $\mathbf{N}_{x \upharpoonright E} \subseteq \mathbf{N}_{x \upharpoonright D'}$ and $\Phi(\mathbf{N}_{x \upharpoonright E}) \subseteq \Phi(\mathbf{N}_{x \upharpoonright D'})$. Then, $\Phi(\mathbf{N}_{x \upharpoonright E}) \subseteq \mathbf{N}_{\Phi(x) \upharpoonright D}$ and $\Phi(x) \upharpoonright D \in X_{x \upharpoonright E}$. By maximality of $\varphi(x \upharpoonright E)$, $\Phi(x) \upharpoonright D \subseteq \varphi(x \upharpoonright E)$. Thus, we have proved that for every $D \in \mathcal{I}$ there exists $E \in \mathcal{I}$ such that $\Phi(x) \upharpoonright D = \varphi(x \upharpoonright E) \upharpoonright D$. Finally, $\varphi^*(x) = \Phi(x)$ due to the uniqueness of the limit. \square

One could be tempted to only require that φ is monotone and that $\bigcup_{D \in \mathcal{I}} \varphi(x \upharpoonright D) \in {}^\kappa\nu$ for every $x \in {}^\kappa\nu$. However, the function $\varphi^*(x) = \bigcup_{D \in \mathcal{I}} \varphi(x \upharpoonright D)$ may fail to be continuous, as the next example shows.

For all $A \subseteq \kappa$, let $\beta^{(A)}$ be the constant function with value β and domain A .

Example 7.2.4. Let $D \subseteq \kappa$ be the subset of odd² ordinals, and let \mathcal{I} the ideal generated by the bounded ideal together with D . Note that $E \in \mathcal{I}$ if and only if $E \setminus D$ is bounded.

Define $\varphi: \text{Fn}_{\mathcal{I}}({}^\kappa\nu) \rightarrow \text{Fn}_{\mathcal{I}}({}^\kappa\nu)$ as follows. For any $f \in \text{Fn}_{\mathcal{I}}({}^\kappa\nu)$, and for every $\alpha \in \text{dom}(f) \setminus D$, set:

²Recall that every ordinal α can be written uniquely as $\alpha = \gamma + n$, with $n < \omega$ and either $\gamma = 0$ or γ limit. Accordingly, we say that γ is even (respectively, odd) if n is even (respectively, odd).

- $\varphi(f)(\alpha) = f(\alpha)$,
- $\varphi(f)(\alpha + 1) = f(\alpha)$.

Note that $\text{dom}(\varphi(f)) = (\text{dom}(f) \setminus D) \cup \{\alpha + 1 \mid \alpha \in \text{dom}(f) \setminus D\}$, and since $\text{dom}(f) \in \mathcal{I}$, we get that $\text{dom}(f) \setminus D$ is bounded. Therefore, for every $f \in \text{Fn}_{\mathcal{I}}({}^\kappa\nu)$, $\text{dom}(\varphi(f))$ is bounded.

It is easy to verify that φ is monotone and that for every $x \in {}^\kappa\nu$, $\varphi^*(x) = \bigcup_{D \in \mathcal{I}} \varphi(x \upharpoonright D) \in {}^\kappa\nu$. To see this, suppose, towards a contradiction, that there are $E, E' \in \mathcal{I}$ and $\alpha \in \text{dom}(\varphi(x \upharpoonright E)) \cap \text{dom}(\varphi(x \upharpoonright E'))$ such that $\varphi(x \upharpoonright E)(\alpha) \neq \varphi(x \upharpoonright E')(\alpha)$. Then, either $\alpha \notin D$ or $\alpha \in D$. If $\alpha \notin D$, then by the definition of φ , $\varphi(x \upharpoonright E)(\alpha) = x(\alpha) = \varphi(x \upharpoonright E')(\alpha)$, a contradiction. If $\alpha \in D$, then there is $\beta \notin D$ such that $\alpha = \beta + 1$ and $\beta \in \text{dom}(\varphi(x \upharpoonright E)) \cap \text{dom}(\varphi(x \upharpoonright E'))$. By the definition of φ , $\varphi(x \upharpoonright E)(\alpha) = x(\beta) = \varphi(x \upharpoonright E')(\alpha)$, a contradiction.

However, φ^* is not \mathcal{I} -continuous. To see this, let $f = 0^{(D)}$. We will show that $(\varphi^*)^{-1}(\mathbf{N}_f)$ is not open.

Notice that $\varphi^*(0^{(\kappa)}) = \bigcup_{E \in \mathcal{B}} \varphi(0^{(\kappa)} \upharpoonright E) = 0^{(\kappa)}$. Thus, $0^{(\kappa)} \in (\varphi^*)^{-1}(\mathbf{N}_f)$. It remains to verify that for all $E \in \mathcal{I}$, $\mathbf{N}_{0^{(\kappa)} \upharpoonright E} \not\subseteq (\varphi^*)^{-1}(\mathbf{N}_f)$.

Fix $E \in \mathcal{I}$ and let $x = 0^{(E)} \cup 1^{(\kappa \setminus E)}$. Since $0^{(E)} = 0^{(\kappa)} \upharpoonright E$, $x \in \mathbf{N}_{0^{(\kappa)} \upharpoonright E}$. We have that $\kappa \setminus (D \cup E) \neq \emptyset$ (otherwise $\kappa = E \cup D \in \mathcal{I}$ goes against \mathcal{I} being proper) so let $\alpha \in \kappa \setminus (D \cup E)$. Since $\alpha \in \text{dom}(x \upharpoonright E \cup \alpha + 1) \setminus D$, $\alpha + 1 \in \text{dom}(\varphi(x \upharpoonright E \cup \alpha + 1))$ and

$$\varphi(x \upharpoonright E \cup \alpha + 1)(\alpha + 1) = (x \upharpoonright E \cup \alpha + 1)(\alpha) = x(\alpha) = 1.$$

Then, $\varphi^*(x)(\alpha + 1) = \varphi(x \upharpoonright E \cup \alpha + 1)(\alpha + 1) = 1$. Finally, notice that $\alpha + 1 \in D$, thus $f(\alpha + 1) = 0 \neq 1 = \varphi^*(x)(\alpha + 1)$. So $x \notin (\varphi^*)^{-1}(\mathbf{N}_f)$.

7.3 The \mathcal{I} -Borel sets and their hierarchy

We recall that a κ^+ -algebra on a set X is a family of subsets of X closed under the operations of complementation and unions of length at most κ . When X is a topological space, the κ^+ -algebra generated by the topology of X , denoted by $\kappa^+\text{-Bor}(X)$, is the smallest κ^+ -algebra on X containing all its open sets.

Let $\nu \in \{2, \kappa\}$, and consider the topological space $({}^\kappa\nu, \tau_{\mathcal{I}})$. We denote by $\mathcal{I}\text{-Bor}({}^\kappa\nu)$ the κ^+ -algebra generated by $\tau_{\mathcal{I}}$, i.e., $\mathcal{I}\text{-Bor}({}^\kappa\nu) = \kappa^+\text{-Bor}({}^\kappa\nu, \tau_{\mathcal{I}})$, and we call its elements \mathcal{I} -Borel sets. When $\mathcal{I} = b$, $\mathcal{I}\text{-Bor}({}^\kappa\nu) = \kappa^+\text{-Bor}({}^\kappa\nu, \tau_b)$ are the usual generalized Borel sets extensively studied in Chapter 3 and in [MV93, FHK14, ACRP25].

As already observed in Chapter 3, for every topological space X , the κ^+ -algebra generated by any topology on X can be stratified in a hierarchy (called the κ^+ -Borel hierarchy) formed by the classes $\kappa^+\text{-}\Sigma_\alpha^0(X)$, $\kappa^+\text{-}\Pi_\alpha^0(X)$, and $\kappa^+\text{-}\Delta_\alpha^0(X)$, where α ranges over non-zero ordinals. The pointclasses $\kappa^+\text{-Bor}$, $\kappa^+\text{-}\Sigma_\alpha^0$, $\kappa^+\text{-}\Pi_\alpha^0$ and $\kappa^+\text{-}\Delta_\alpha^0$ are boldface, and $\kappa^+\text{-Bor}$, $\kappa^+\text{-}\Sigma_\alpha^0$, $\kappa^+\text{-}\Pi_\alpha^0$ are also hereditary. In our case study $X = ({}^\kappa\nu, \tau_{\mathcal{I}})$, the hierarchy of \mathcal{I} -Borel sets is defined as follows, and we call it the \mathcal{I} -Borel hierarchy.

Definition 7.3.1. The following classes are defined by recursion on the ordinal $\alpha \geq 1$:

$$\begin{aligned} \mathcal{I}\text{-}\Sigma_1^0(\kappa\nu) &= \{U \subseteq \kappa\nu \mid U \text{ is } \mathcal{I}\text{-open}\} & \mathcal{I}\text{-}\Pi_1^0(\kappa\nu) &= \{C \subseteq \kappa\nu \mid C \text{ is } \mathcal{I}\text{-closed}\} \\ \mathcal{I}\text{-}\Sigma_\alpha^0(\kappa\nu) &= \left\{ \bigcup_{\gamma < \kappa} A_\gamma \mid A_\gamma \in \bigcup_{1 \leq \beta < \alpha} \mathcal{I}\text{-}\Pi_\beta^0(\kappa\nu) \right\} & \mathcal{I}\text{-}\Pi_\alpha^0(\kappa\nu) &= \{X \setminus A \mid A \in \mathcal{I}\text{-}\Sigma_\alpha^0(\kappa\nu)\}. \end{aligned}$$

We also set $\mathcal{I}\text{-}\Delta_\alpha^0(\kappa\nu) = \mathcal{I}\text{-}\Sigma_\alpha^0(\kappa\nu) \cap \mathcal{I}\text{-}\Pi_\alpha^0(\kappa\nu)$.

Note that, to be coherent with Chapter 3 and with the usual notation of generalized Borel hierarchy, one should write the classes in Definition 7.3.1 as $\kappa^+\text{-}\Sigma_\alpha^0(\kappa\nu, \tau_{\mathcal{I}})$, $\kappa^+\text{-}\Pi_\alpha^0(\kappa\nu, \tau_{\mathcal{I}})$, and $\kappa^+\text{-}\Delta_\alpha^0(\kappa\nu, \tau_{\mathcal{I}})$. However, for simplicity of notation, we decide to adopt the present one. When the space is clear from the context, we drop it from all the notation above.

It is easy to check that

$$\mathcal{I}\text{-}\mathbf{Bor}(\kappa\nu) = \bigcup_{1 \leq \alpha < \kappa^+} \mathcal{I}\text{-}\Sigma_\alpha^0(\kappa\nu) = \bigcup_{1 \leq \alpha < \kappa^+} \mathcal{I}\text{-}\Pi_\alpha^0(\kappa\nu) = \bigcup_{1 \leq \alpha < \kappa^+} \mathcal{I}\text{-}\Delta_\alpha^0(\kappa\nu). \quad (7.3.1)$$

Since the ideal topology $\tau_{\mathcal{I}}$ is finer than the bounded topology τ_b , a straightforward induction shows the following relations between the classes in the \mathcal{I} -Borel and the κ^+ -Borel hierarchy.

Proposition 7.3.2. *For every $\alpha < \kappa^+$, $\kappa^+\text{-}\Sigma_\alpha^0(\kappa\nu, \tau_b) \subseteq \mathcal{I}\text{-}\Sigma_\alpha^0(\kappa\nu)$, hence $\kappa^+\text{-}\Pi_\alpha^0(\kappa\nu, \tau_b) \subseteq \mathcal{I}\text{-}\Pi_\alpha^0(\kappa\nu)$ and $\kappa^+\text{-}\Delta_\alpha^0(\kappa\nu, \tau_b) \subseteq \mathcal{I}\text{-}\Delta_\alpha^0(\kappa\nu)$. Therefore, $\kappa^+\text{-}\mathbf{Bor}(\kappa\nu, \tau_b) \subseteq \mathcal{I}\text{-}\mathbf{Bor}(\kappa\nu)$.*

Proof. We show that $\kappa^+\text{-}\Sigma_\alpha^0(\kappa\nu, \tau_b) \subseteq \mathcal{I}\text{-}\Sigma_\alpha^0(\kappa\nu)$ by induction on $\alpha < \kappa^+$. Since $b \subseteq \mathcal{I}$, $\tau_b \subseteq \tau_{\mathcal{I}}$. Assume $\alpha > 1$ and that $\kappa^+\text{-}\Sigma_\beta^0(\kappa\nu, \tau_b) \subseteq \mathcal{I}\text{-}\Sigma_\beta^0(\kappa\nu)$ for every $\beta < \alpha$. If $A \in \kappa^+\text{-}\Sigma_\alpha^0(\kappa\nu, \tau_b)$, then $A = \bigcup_{i < \kappa} A_i$ with $A_i \in \bigcup_{\beta < \alpha} \kappa^+\text{-}\Pi_\beta^0(\kappa\nu, \tau_b)$. Since $\kappa^+\text{-}\Pi_\beta^0(\kappa\nu, \tau_b) \subseteq \mathcal{I}\text{-}\Pi_\beta^0(\kappa\nu)$, by the induction hypothesis, $A_i \in \bigcup_{\beta < \alpha} \mathcal{I}\text{-}\Pi_\beta^0(\kappa\nu)$ for every $i < \kappa$. Hence $A \in \mathcal{I}\text{-}\Sigma_\alpha^0(\kappa\nu)$. \square

Recall that $\mathcal{B}_{\mathcal{I}}$ denotes the canonical basis for $\tau_{\mathcal{I}}$. The following remark shows that $\mathcal{B}_{\mathcal{I}}(\kappa\nu) \subseteq \kappa^+\text{-}\Pi_1^0(\kappa\nu, \tau_b)$.

Remark 7.3.3. Let $f \in \text{Fn}_{\mathcal{I}}$ and let $D = \text{dom}(f)$. Then, $\mathbf{N}_f = \bigcap_{\alpha \in D} \mathcal{N}_\alpha^f$, where $\mathcal{N}_\alpha^f = \{x \in \kappa\nu \mid x(\alpha) = f(\alpha)\}$. Since $\mathcal{N}_\alpha^f \in \kappa^+\text{-}\Delta_1^0(\kappa\nu, \tau_b)$ and $|D| \leq \kappa$, $\mathbf{N}_f \in \kappa^+\text{-}\Pi_1^0(\kappa\nu, \tau_b)$. Thus for every $f \in \text{Fn}_{\mathcal{I}}$, $\mathbf{N}_f \in \kappa^+\text{-}\Pi_1^0(\kappa\nu, \tau_b)$.

We say that the \mathcal{I} -Borel hierarchy is **increasing** if for every $1 \leq \alpha < \beta$ we have $\mathcal{I}\text{-}\Sigma_\alpha^0(\kappa\nu) \subseteq \mathcal{I}\text{-}\Sigma_\beta^0(\kappa\nu)$. In this respect, the only problematic case is when $\alpha = 1$ and $\beta = 2$, as already noticed more in general in [AMR22, Lemma 2.2] for any topological space (X, τ) . Indeed, $\mathcal{I}\text{-}\Sigma_\alpha^0(\kappa\nu) \subseteq \mathcal{I}\text{-}\Sigma_\beta^0(\kappa\nu)$ for every $2 \leq \alpha < \beta$, and $\mathcal{I}\text{-}\Sigma_1^0(\kappa\nu) \subseteq \mathcal{I}\text{-}\Pi_2^0(\kappa\nu) \subseteq \mathcal{I}\text{-}\Sigma_3^0(\kappa\nu)$. In particular, the \mathcal{I} -Borel hierarchy is increasing if and only if $\mathcal{I}\text{-}\Sigma_1^0(\kappa\nu) \subseteq \mathcal{I}\text{-}\Sigma_2^0(\kappa\nu)$.

It was already observed in [HKS22] that $\mathcal{I}\text{-}\Sigma_1^0(\kappa\nu) \subseteq \mathcal{I}\text{-}\Sigma_2^0(\kappa\nu)$ when \mathcal{I} has a basis of size κ .

Proposition 7.3.4. [HKS22, Proposition 1.5] *Assume that \mathcal{I} has a basis of size κ . Then, every \mathcal{I} -open subset of $\kappa\nu$ is the union of κ -many \mathcal{I} -clopen sets, therefore $\mathcal{I}\text{-}\Sigma_1^0(\kappa\nu) \subseteq \mathcal{I}\text{-}\Sigma_2^0(\kappa\nu)$.*

Proof. Let $\mathcal{B} = \{D_i \mid i < \kappa\}$ be a basis for \mathcal{I} and let $U \subseteq {}^\kappa\nu$ be \mathcal{I} -open. For every $x \in U$, there exists $B_x \in \mathcal{B}$ such that $\mathbf{N}_{x \upharpoonright B_x} \subseteq U$. Clearly $U = \bigcup_{x \in U} \mathbf{N}_{x \upharpoonright B_x}$. Now, for every $i < \kappa$, let

$$O_i = \bigcup \{\mathbf{N}_{x \upharpoonright B_x} \mid x \in U \wedge B_x = D_i\}.$$

If $y \notin O_i$, then $\mathbf{N}_{y \upharpoonright D_i} \cap O_i = \emptyset$. Thus O_i is \mathcal{I} -clopen and $U = \bigcup_{i < \kappa} O_i$. \square

Corollary 7.3.5. *Assume that \mathcal{I} has a basis of size κ . Then, the \mathcal{I} -Borel hierarchy on ${}^\kappa\nu$ is increasing.*

As soon as we remove the assumption on the size of the basis of \mathcal{I} , the \mathcal{I} -Borel hierarchy may not be increasing. For example, if $\mathcal{I} = \mathbf{NS}_\kappa$ and we consider the space ${}^\kappa 2$, the set $ub_\kappa = \{x \in {}^\kappa 2 \mid x = \chi_A^3 \text{ for some } A \subseteq \kappa \text{ unbounded}\}$ is \mathcal{I} -open but not $\mathcal{I}\text{-}\Sigma_2^0({}^\kappa 2)$ (see [HKS22, Corollary 3.9 and Theorem 3.10]).

Note however that certain inclusions hold regardless of whether the \mathcal{I} -Borel hierarchy is increasing or not. In particular, for all ordinals $\alpha \leq \beta$ we have $\mathcal{I}\text{-}\Sigma_\alpha^0({}^\kappa\nu) \cup \mathcal{I}\text{-}\Pi_\alpha^0({}^\kappa\nu) \subseteq \mathcal{I}\text{-}\Sigma_\beta^0({}^\kappa\nu) \cup \mathcal{I}\text{-}\Pi_\beta^0({}^\kappa\nu)$ and $\mathcal{I}\text{-}\Delta_\alpha^0({}^\kappa\nu) \subseteq \mathcal{I}\text{-}\Delta_\beta^0({}^\kappa\nu)$.

Remark 7.3.6. The set $ub_\kappa \subseteq {}^\kappa 2$ is \mathcal{I} -open if and only if \mathcal{I} is tall by [HKS22, Corollary 3.9]. Since $ub_\kappa \notin \mathcal{I}\text{-}\Sigma_2^0({}^\kappa 2)$ for every ideal \mathcal{I} by [HKS22, Theorem 3.10], Proposition 7.3.4 implies that no ideal \mathcal{I} can simultaneously be tall and have a basis of cardinality κ . Consequently, an ideal has the approximation property if and only if it is not tall.

We now address the question of whether the \mathcal{I} -Borel hierarchy collapses, as a fundamental parameter measuring the behavior of the \mathcal{I} -Borel hierarchy is its length. From (7.3.1), we know that an upper bound for its length is κ^+ , but it remained unknown until now whether the \mathcal{I} -Borel hierarchy on ${}^\kappa\nu$ could be strictly shorter. We say that the \mathcal{I} -Borel hierarchy on ${}^\kappa\nu$ **collapses** if

$$\min \{\alpha \in \text{Ord} \mid \mathcal{I}\text{-}\Sigma_\alpha^0({}^\kappa\nu) = \mathcal{I}\text{-}\mathbf{Bor}({}^\kappa\nu)\} < \kappa^+.$$

In the next results we show that for *any* ideal \mathcal{I} , the \mathcal{I} -Borel hierarchy has length κ^+ , that is, $\mathcal{I}\text{-}\Sigma_\alpha^0({}^\kappa\nu) \subsetneq \mathcal{I}\text{-}\Sigma_\beta^0({}^\kappa\nu)$ for all $1 \leq \alpha < \beta < \kappa^+$, and therefore it does not collapse. This answers [HKS22, Question 1].

In the generalized context with $\mathcal{I} = b$, the non-collapse of the κ^+ -Borel hierarchy on $({}^\kappa 2, \tau_b)$ was first established in [AMR22, Proposition 4.19] for arbitrary infinite cardinals κ , without assuming $2^{<\kappa} = \kappa$. As shown in Theorem 7.3.7, this implies that the κ^+ -Borel hierarchy does not collapse on any space containing a homeomorphic copy of $({}^\kappa 2, \tau_b)$. This result is well-known, but we include the proof for the reader's convenience.

Theorem 7.3.7. *For any topological space X containing an homeomorphic copy of $({}^\kappa 2, \tau_b)$, the κ^+ -Borel hierarchy on X does not collapse.*

Proof. Let $f : {}^\kappa 2 \rightarrow X$ be a topological embedding and $Y = \text{ran}(f) \subseteq X$. Suppose, towards a contradiction, that $\kappa^+\text{-}\Sigma_\alpha^0(X) = \kappa^+\text{-}\mathbf{Bor}(X)$ for some $\alpha < \kappa^+$. Since the pointclasses $\kappa^+\text{-}\mathbf{Bor}$ and $\kappa^+\text{-}\Sigma_\alpha^0$ are boldface and hereditary,

$$\begin{aligned} \kappa^+\text{-}\Sigma_\alpha^0(Y) &= \{A \cap Y \mid A \in \kappa^+\text{-}\Sigma_\alpha^0(X)\}, \\ \kappa^+\text{-}\mathbf{Bor}(Y) &= \{A \cap Y \mid A \in \kappa^+\text{-}\mathbf{Bor}(X)\}. \end{aligned}$$

³For any $A \subseteq \kappa$, $\chi_A : \kappa \rightarrow 2$ is the function satisfying $\chi_A(\alpha) = 1 \Leftrightarrow \alpha \in A$.

Thus, we get $\kappa^+ \text{-}\Sigma_\alpha^0(Y) = \kappa^+ \text{-}\mathbf{Bor}(Y)$ and the κ^+ -Borel hierarchy on $({}^\kappa 2, \tau_b)$ collapses, a contradiction. \square

Before stating the main theorem of the section, we need the following result from [HKS^W22], which is Mycielski's theorem for ideal topologies.

Lemma 7.3.8. [HKS^W22, Corollary 1.4] *The intersection of κ -many \mathcal{I} -open \mathcal{I} -dense subsets of ${}^\kappa 2$ contains a closed set homeomorphic to $({}^\kappa 2, \tau_b)$.*

Theorem 7.3.9. *The \mathcal{I} -Borel hierarchy on ${}^\kappa 2$ and on ${}^\kappa \kappa$ does not collapse, therefore it has length κ^+ in both spaces.*

Proof. First, we show that the \mathcal{I} -Borel hierarchy on ${}^\kappa 2$ has length κ^+ . By Proposition 7.3.8 and Theorem 7.3.7, it is enough to show that there are κ -many \mathcal{I} -open \mathcal{I} -dense subsets of ${}^\kappa 2$.

Claim. For any $X \subseteq {}^\kappa 2$, if $|X| \leq \kappa$ then ${}^\kappa 2 \setminus X$ is \mathcal{I} -dense in ${}^\kappa 2$.

Proof of the Claim. Suppose, towards a contradiction, that there exists an \mathcal{I} -open set $U \subseteq {}^\kappa 2$ such that $U \cap ({}^\kappa 2 \setminus X) = \emptyset$. Then, $U \subseteq X$, and since $|X| \leq \kappa$, also $|U| \leq \kappa$, in contradiction with Lemma 7.1.4. \square

Let $(C_\alpha)_{\alpha < \kappa}$ be an injective sequence of \mathcal{I} -closed subsets of ${}^\kappa 2$ such that $|C_\alpha| \leq \kappa$ for every $\alpha < \kappa$. For example, let $(\gamma_\alpha)_{\alpha < \kappa}$ be an enumeration of the limit ordinals below κ and let $C_\alpha = \{x \in {}^\kappa 2 \mid |\{\beta < \kappa \mid x(\beta) = 1\}| < \gamma_\alpha\}$. Each C_α is b -closed (thus \mathcal{I} -closed) and of size κ . Then, $\{{}^\kappa 2 \setminus C_\alpha \mid \alpha < \kappa\}$ is a collection of κ -many distinct \mathcal{I} -open sets, which are also \mathcal{I} -dense by the Claim above.

Next, consider the \mathcal{I} -Borel hierarchy on ${}^\kappa \kappa$. Suppose, towards a contradiction, that there exists $\alpha < \kappa^+$ such that $\mathcal{I}\text{-}\Sigma_\alpha^0({}^\kappa \kappa) = \mathcal{I}\text{-}\mathbf{Bor}({}^\kappa \kappa)$. Given $\nu \in \{2, \kappa\}$, we recall that in our notation $\mathcal{I}\text{-}\Sigma_\alpha^0({}^\kappa \nu) = \kappa^+ \text{-}\Sigma_\alpha^0({}^\kappa \nu, \tau_{\mathcal{I}})$ and $\mathcal{I}\text{-}\mathbf{Bor}({}^\kappa \kappa) = \kappa^+ \text{-}\mathbf{Bor}({}^\kappa \kappa, \tau_{\mathcal{I}})$. Since $({}^\kappa 2, \tau_{\mathcal{I}})$ is a subspace of $({}^\kappa \kappa, \tau_{\mathcal{I}})$, and the classes $\kappa^+ \text{-}\Sigma_\alpha^0$ and $\kappa^+ \text{-}\mathbf{Bor}$ are hereditarily boldface:

$$\begin{aligned} \kappa^+ \text{-}\Sigma_\alpha^0({}^\kappa 2, \tau_{\mathcal{I}}) &= \{A \cap {}^\kappa 2 \mid A \in \kappa^+ \text{-}\Sigma_\alpha^0({}^\kappa \kappa, \tau_{\mathcal{I}})\}, \\ \kappa^+ \text{-}\mathbf{Bor}({}^\kappa 2, \tau_{\mathcal{I}}) &= \{A \cap {}^\kappa 2 \mid A \in \kappa^+ \text{-}\mathbf{Bor}({}^\kappa \kappa, \tau_{\mathcal{I}})\}. \end{aligned}$$

Therefore, $\mathcal{I}\text{-}\Sigma_\alpha^0({}^\kappa 2) = \kappa^+ \text{-}\Sigma_\alpha^0({}^\kappa 2, \tau_{\mathcal{I}}) = \kappa^+ \text{-}\mathbf{Bor}({}^\kappa 2, \tau_{\mathcal{I}}) = \mathcal{I}\text{-}\mathbf{Bor}({}^\kappa 2)$. A contradiction, since we proved that the \mathcal{I} -Borel hierarchy on ${}^\kappa 2$ does not collapse. \square

Another relevant point is whether the notion of \mathcal{I} -Borelness is non-trivial, that is, whether $\mathcal{I}\text{-}\mathbf{Bor}({}^\kappa \nu) \neq \mathcal{P}({}^\kappa \nu)$. In some cases, cardinality considerations are useful, for example if $\mathcal{I} = b$ and $\kappa^{< \kappa} = \kappa$ then $|\kappa^+ \text{-}\mathbf{Bor}({}^\kappa \nu)| = 2^{(\nu^{< \kappa})} < |\mathcal{P}({}^\kappa \nu)|$. However, this approach does not work when we drop the assumption $2^{< \kappa} = \kappa$, or when dealing with ideal topologies. Indeed, if \mathcal{I} contains an unbounded subset of κ , by Lemma 7.1.5(5)

$$2^{2^\kappa} = |\tau_{\mathcal{I}}| \leq |\mathcal{I}\text{-}\mathbf{Bor}({}^\kappa \nu)| \leq |\mathcal{P}({}^\kappa \nu)| = 2^{2^\kappa},$$

therefore $|\mathcal{I}\text{-}\mathbf{Bor}({}^\kappa \nu)| = |\tau_{\mathcal{I}}| = |\mathcal{P}({}^\kappa \nu)|$.

If $2^{< \kappa} = \kappa$, for any ideal \mathcal{I} there exists a subset of ${}^\kappa 2$ which is not \mathcal{I} -Borel ([HKS^W22, Observation 3.24]). If \mathcal{I} is not stationarily tall, the same is true

even when $2^{<\kappa} > \kappa$ (see [HKS22, Corollary 3.23]). It remains open whether there is non- \mathcal{I} -Borel set if $2^{<\kappa} > \kappa$ and \mathcal{I} is stationarily tall. This question also appears in [HKS22].

In classical descriptive set theory, universal sets play a crucial role in proving the non-collapse of the Borel hierarchy on uncountable Polish spaces. When moving to the uncountable setting with $\kappa > \omega$ and considering $({}^\kappa\nu, \tau_b)$, the situation is as follows. Under the assumption $2^{<\kappa} = \kappa$, there exist ${}^\kappa 2$ -universal sets for both $\kappa^+ \text{-}\Sigma_\alpha^0({}^\kappa\nu, \tau_b)$ and $\kappa^+ \text{-}\Pi_\alpha^0({}^\kappa\nu, \tau_b)$ for each $1 \leq \alpha < \kappa^+$ (see Chapter 3). On the other hand, if the condition $2^{<\kappa} = \kappa$ fails, then neither $\kappa^+ \text{-}\Sigma_\alpha^0({}^\kappa 2, \tau_b)$ nor $\kappa^+ \text{-}\Pi_\alpha^0({}^\kappa 2, \tau_b)$ admits a ${}^\kappa 2$ -universal set, for any $1 \leq \alpha < \kappa^+$ ([AMR22, Corollary 4.16]). The result below shows that when \mathcal{I} contains an unbounded subset of κ , there are no universal sets for $\mathcal{I}\text{-}\Sigma_\alpha^0({}^\kappa\nu)$ nor $\mathcal{I}\text{-}\Pi_\alpha^0({}^\kappa\nu)$, regardless of whether $2^{<\kappa} = \kappa$ holds or not.

Proposition 7.3.10. *Assume that \mathcal{I} contains an unbounded subset of κ . Then, for $1 \leq \alpha < \kappa^+$, neither $\mathcal{I}\text{-}\Sigma_\alpha^0({}^\kappa\nu)$ nor $\mathcal{I}\text{-}\Pi_\alpha^0({}^\kappa\nu)$ have a ${}^\kappa 2$ -universal set.*

Proof. By Point (5) in Lemma 7.1.5, $\mathcal{I}\text{-}\Sigma_\alpha^0({}^\kappa\nu)$ has size greater than 2^{2^κ} , while $\{\mathcal{U}_y \mid y \in {}^\kappa 2\}$ has size at most 2^κ for all $\mathcal{U} \subseteq {}^\kappa 2 \times {}^\kappa\nu$. The result for $\mathcal{I}\text{-}\Pi_\alpha^0({}^\kappa\nu)$ follows by taking complements. \square

We continue with the closure properties of the classes in the \mathcal{I} -Borel hierarchy on ${}^\kappa\nu$. For the proof of the next result, we direct the reader to Chapter 3. Although in that chapter we worked with (regular Hausdorff) topological spaces of weight at most κ and assumed $2^{<\kappa} = \kappa$, the argument in fact goes through without imposing any restriction on the weight. Since the \mathcal{I} -Borel hierarchy may not satisfy the inclusion $\mathcal{I}\text{-}\Sigma_1^0({}^\kappa\nu) \subseteq \mathcal{I}\text{-}\Sigma_2^0({}^\kappa\nu)$, the only case requiring verification is $\alpha = 1$: by Lemma 7.1.5(2), the class $\mathcal{I}\text{-}\Sigma_1^0({}^\kappa\nu)$ is closed under arbitrary unions and under intersections of fewer than κ sets.

Proposition 7.3.11. [Proposition 3.3.1] *Let $\nu \in \{2, \kappa\}$, and $2^{<\kappa} = \kappa$. Given any $1 \leq \alpha < \kappa^+$, let $\hat{\alpha} = \kappa$ if α is a successor ordinal, and $\hat{\alpha} = \text{cof}(\alpha)$ if α is limit. Then,*

- (1) $\mathcal{I}\text{-}\Sigma_\alpha^0({}^\kappa\nu)$ is closed under unions of length κ and intersections of size smaller than $\hat{\alpha}$;
- (2) $\mathcal{I}\text{-}\Pi_\alpha^0({}^\kappa\nu)$ is closed under intersections of length κ and unions of size smaller than $\hat{\alpha}$;
- (3) $\mathcal{I}\text{-}\Delta_\alpha^0({}^\kappa\nu)$ is closed under complements and both unions and intersections of size smaller than $\hat{\alpha}$, that is, $\mathcal{I}\text{-}\Delta_\alpha^0({}^\kappa\nu)$ is an $\hat{\alpha}$ -algebra.

We refer the reader to Proposition 3.3.2 for a detailed discussion of the optimality of the closure properties presented in Proposition 7.3.11.

We conclude this section by pointing out a connection with [HMV25]. When one drops the assumption $\kappa^{<\kappa} = \kappa$ (equivalently, that κ is regular and $2^{<\kappa} = \kappa$), there are two main approaches in the literature. The first maintains the condition $2^{<\kappa} = \kappa$ but allows κ to be singular, as in Chapter 3 of this thesis and [AMR22, DMR25, ACRP25, MRP25]; this ensures that the spaces still have weight κ . The second approach, developed in [HMV25], keeps κ regular while allowing $\kappa^{<\kappa} > \kappa$. This constitutes a different setup, as the spaces no longer

have weight κ . In that context, the authors identify new connections with the model theory of uncountable structures by formulating an alternative notion of Borel sets, tailored to their setting and better suited for model-theoretic applications. They begin by defining what they call a *basic κ -open set*, namely a set of the form $[f] = \{x \in {}^\kappa\nu \mid f \subseteq x\}$, where $f: X \rightarrow \nu$ and $|X| < \kappa$. As usual, $\nu \in \{2, \kappa\}$.

Definition 7.3.12. The class of κ -Borel sets of ${}^\kappa\nu$ is the smallest class containing the basic κ -open sets and closed under complements and unions of size at most κ .

In line with Definition 7.3.12, one could attempt to develop the theory of Borel sets in ideal topologies using the following definition instead of the one adopted in this thesis (cf. Section 3.2.1).

Definition 7.3.13. The class of **basic \mathcal{I} -Borel** of ${}^\kappa\nu$ is the smallest class containing the basic \mathcal{I} -open sets and closed under complements and unions of size at most κ .

As the next result shows, for any ideal \mathcal{I} the class of basic \mathcal{I} -Borel sets coincides with the class defined in Definition 7.3.12.

Fact 7.3.14. For any ideal \mathcal{I} , the class of basic \mathcal{I} -Borel sets of ${}^\kappa\nu$ coincides with the class of κ -Borel sets of ${}^\kappa\nu$. As a consequence, the class of basic \mathcal{I} -Borel is independent of the ideal \mathcal{I} .

Proof. As $b \subseteq \mathcal{I}$, any basic κ -open set is a basic \mathcal{I} -open set, therefore every κ -Borel set is a basic \mathcal{I} -Borel set. For the other inclusion, it is sufficient to show that all basic \mathcal{I} -open sets are κ -Borel sets. As observed in Remark 7.3.3, for every $f \in \text{Fn}_{\mathcal{I}}$, $\mathbf{N}_f = \bigcap_{\alpha \in D} \mathcal{N}_\alpha^f$, where $D = \text{dom}(f)$ and $\mathcal{N}_\alpha^f = \{x \in {}^\kappa\nu \mid x(\alpha) = f(\alpha)\}$. Since each \mathcal{N}_α^f is a basic κ -open sets and $|D| \leq \kappa$, \mathbf{N}_f is κ -Borel. \square

7.4 The \mathcal{I} -Analytic sets

In this section we introduce the notion of generalized analytic set for the spaces ${}^\kappa 2$ and ${}^\kappa \kappa$ endowed with the ideal topology $\tau_{\mathcal{I}}$. However, we immediately observe that this notion is trivial, because the class under consideration actually coincides with the entire powerset (Corollary 7.4.3). Moreover, the same phenomenon occurs for the generalized versions of all the equivalent characterizations of analytic sets from classical descriptive set theory (Corollary 7.4.4). As usual, fix $\nu \in \{2, \kappa\}$.

Given $\mu \in \{2, \kappa\}$, a function $\Phi: {}^\kappa\nu \rightarrow {}^\kappa\mu$ is **\mathcal{I} -Borel** if $\Phi^{-1}(U) \in \mathcal{I}\text{-Bor}({}^\kappa\nu)$ for every \mathcal{I} -open set $U \subseteq {}^\kappa\mu$. We say that a set $B \subseteq {}^\kappa\mu$ is an \mathcal{I} -continuous image of $A \subseteq {}^\kappa\nu$ if there is an \mathcal{I} -continuous function $\Phi: {}^\kappa\nu \rightarrow {}^\kappa\mu$ such that $\Phi(A) = B$. If Φ is \mathcal{I} -Borel, then we say that B is an \mathcal{I} -Borel image of A .

Definition 7.4.1. A set $A \subseteq {}^\kappa\nu$ is called **\mathcal{I} -analytic** ($\mathcal{I}\text{-}\Sigma_1^1({}^\kappa\nu)$) if it is either empty or an \mathcal{I} -continuous image of an \mathcal{I} -closed subset of ${}^\kappa\kappa$.

When $\mathcal{I} = b$, $\mathcal{I}\text{-}\Sigma_1^1({}^\kappa\nu)$ is the usual class of κ -analytic sets as in [FHK14, LS15, AMR22].

Theorem 7.4.2. *If \mathcal{I} contains an unbounded subset of κ , then the class $\mathcal{I}\text{-}\Sigma_1^1(\kappa\nu)$ is closed under unions of size at most 2^κ .*

Proof. Let $(A_\alpha)_{\alpha < 2^\kappa}$ be a sequence of $\mathcal{I}\text{-}\Sigma_1^1$ -subsets of ${}^\kappa\nu$. For every $\alpha < 2^\kappa$, let $C_\alpha \subseteq {}^\kappa\kappa$ be \mathcal{I} -closed and $\Phi_\alpha: {}^\kappa\kappa \rightarrow {}^\kappa\nu$ be \mathcal{I} -continuous such that $\Phi_\alpha(C_\alpha) = A_\alpha$. Since \mathcal{I} contains an unbounded set, there are $U_0, U_1 \subseteq \kappa$ unbounded such that $U_0, U_1 \in \mathcal{I}$ and $U_0 \cap U_1 = \emptyset$. Fix the bijections

$$\begin{aligned} \theta: U_1 \times \kappa &\rightarrow 2^\kappa \\ \delta: U_0 &\rightarrow \kappa \end{aligned}$$

and define $\Delta: {}^\kappa\kappa \rightarrow {}^\kappa\kappa$ such that $\Delta(x)(\alpha) = x(\delta^{-1}(\alpha))$ for all $\alpha < \kappa$. Given $x \in {}^\kappa\kappa$, notice that for every $y \in \mathbf{N}_{x \upharpoonright U_0}$, $\Delta(y) = \Delta(x)$ and, for every $z \in \mathbf{N}_{x \upharpoonright U_1}$, $\theta(z \upharpoonright U_1) = \theta(x \upharpoonright U_1)$. Finally, let

$$\Phi: {}^\kappa\kappa \rightarrow {}^\kappa\nu: x \mapsto \Phi_{\theta(x \upharpoonright U_1)}(\Delta(x)),$$

and define

$$C = \{x \in {}^\kappa\kappa \mid \Delta(x) \in C_{\theta(x \upharpoonright U_1)}\}.$$

We claim that Φ and C witness $\bigcup_{\alpha < 2^\kappa} A_\alpha \in \mathcal{I}\text{-}\Sigma_1^1(\kappa\nu)$.

Notice that for every $x \in {}^\kappa\kappa$ and for every $z \in \mathbf{N}_{x \upharpoonright U_0 \cup U_1}$, $\Phi(z) = \Phi_{\theta(x \upharpoonright U_1)}(\Delta(x)) = \Phi(x)$, therefore $\Phi(\mathbf{N}_{x \upharpoonright U_0 \cup U_1}) = \{\Phi(x)\}$.

The function Φ is \mathcal{I} -continuous since for every $x \in {}^\kappa\kappa$, and for every $D \in \mathcal{I}$, the set $\mathbf{N}_{x \upharpoonright U_0 \cup U_1}$ is \mathcal{I} -open (because $U_0 \cup U_1 \in \mathcal{I}$) and it satisfies $\Phi(\mathbf{N}_{x \upharpoonright U_0 \cup U_1}) \subseteq \mathbf{N}_{\Phi(x) \upharpoonright D}$. To see that C is \mathcal{I} -closed we show that for every $x \notin C$, $\mathbf{N}_{x \upharpoonright U_0 \cup U_1} \subseteq {}^\kappa\kappa \setminus C$. This is immediate, as for every $y \in \mathbf{N}_{x \upharpoonright U_0 \cup U_1}$, $\Delta(y) = \Delta(x)$ and $\theta(y \upharpoonright U_1) = \theta(x \upharpoonright U_1)$, therefore $\Delta(y) \notin C_{\theta(y \upharpoonright U_1)}$. The following claim concludes the proof.

Claim. $\Phi(C) = \bigcup_{\alpha < 2^\kappa} A_\alpha$.

Proof of the Claim. If $y \in \bigcup_{\alpha < 2^\kappa} A_\alpha$, then there is some $\alpha < 2^\kappa$ and some $x \in C_\alpha$ such that $y = \Phi_\alpha(x)$. Let $z \in {}^\kappa\kappa$ such that $\theta(z \upharpoonright U_1) = \alpha$ and $\Delta(z) = x$. Then,

$$\Phi(z) = \Phi_{\theta(z \upharpoonright U_1)}(\Delta(z)) = \Phi_\alpha(x) = y.$$

Finally, since $x \in C_\alpha$, $\Delta(x) \in C_{\theta(x \upharpoonright U_1)}$. Thus $x \in C$ and $y \in \Phi(C)$.

For the other direction, let $y \in C$. From the definition of C , $\Delta(y) \in C_{\theta(y \upharpoonright U_1)}$. Let $\alpha = \theta(y \upharpoonright U_1)$ and $x = \Delta(y)$. Then, $x \in C_\alpha$ and $\Phi_\alpha(x) \in A_\alpha$. Therefore, $\Phi(y) = \Phi_{\theta(y \upharpoonright U_1)}(\Delta(y)) = \Phi_\alpha(x) \in A_\alpha$. \square

\square

Corollary 7.4.3. *If \mathcal{I} contains an unbounded subset of κ , then $\mathcal{I}\text{-}\Sigma_1^1(\kappa\nu) = \mathcal{P}({}^\kappa\nu)$.*

The generalized Cantor and Baire spaces (equipped with the bounded topology, hence $\mathcal{I} = b$) have $2^{(2^{<\kappa})}$ -many κ -analytic subsets, so the hypothesis $2^{<\kappa} = \kappa$ guarantees that the class $\mathcal{I}\text{-}\Sigma_1^1$ is strictly smaller than the full powerset. In contrast, when $\kappa^{<\kappa} > \kappa$, [HMV25, Lemma 2.1] shows that $\mathcal{I}\text{-}\Sigma_1^1(\kappa 2) = \mathcal{P}(\kappa 2)$ occurs even if \mathcal{I} is the bounded ideal.

Since in classical descriptive set theory the notion of analytic set can be reformulated giving several equivalent definitions [Kec95, Section 14.A], one may wonder if the generalized counterparts of any of the other definitions might be non-trivial. This is not the case, as the next result shows.

Corollary 7.4.4. *If \mathcal{I} contains an unbounded subset of κ , the following are equivalent:*

- (1) $A \in \mathcal{P}({}^\kappa\nu)$
- (2) A is \mathcal{I} -analytic;
- (3) $A = p(C)$ for some \mathcal{I} -closed $C \subseteq {}^\kappa\nu \times {}^\kappa\kappa$;
- (4) A is an \mathcal{I} -continuous image of an \mathcal{I} -Borel subset of ${}^\kappa\kappa$;
- (5) $A = p(B)$ for some \mathcal{I} -Borel $B \subseteq {}^\kappa\nu \times {}^\kappa\kappa$;
- (6) A is an \mathcal{I} -Borel image of an \mathcal{I} -Borel subset of ${}^\kappa\kappa$.

Proof. Trivially, (2)-(6) implies (1). Clearly (2) implies (4). (3) implies (5), since every \mathcal{I} -closed set is \mathcal{I} -Borel. (4) implies (6), since every \mathcal{I} -continuous function is \mathcal{I} -Borel. By Corollary 7.4.3, (1) implies (2). It remains to show that (2) implies (3). Let $A \subseteq {}^\kappa\nu$ be \mathcal{I} -analytic and let $\Phi: C \rightarrow {}^\kappa 2$ be an \mathcal{I} -continuous surjection onto A for some \mathcal{I} -closed $C \subseteq {}^\kappa\kappa$. Since $({}^\kappa\kappa, \tau_{\mathcal{I}})$ is an Hausdorff space by Lemma 7.1.5(1), this implies that $\text{Graph}(\Phi) = \{(x, y) \in C \times {}^\kappa\nu \mid \Phi(x) = y\}$ is \mathcal{I} -closed in $C \times {}^\kappa\nu$, hence also \mathcal{I} -closed in ${}^\kappa\kappa \times {}^\kappa\nu$. Clearly, $A = p(\{(y, x) \in {}^\kappa\nu \times {}^\kappa\kappa \mid (x, y) \in \text{Graph}(\Phi)\})$. \square

Only one possible definition of \mathcal{I} -analytic set is missing from Corollary 7.4.4. In classical descriptive set theory (hence $\kappa = \omega$), it is well-known (see [Kec95, Section 14-15]) that a non-empty set $A \subseteq {}^\omega 2$ is:

- a continuous image of ${}^\omega\omega$ if and only if A is analytic;
- an injective continuous image of some closed $C \subseteq {}^\omega\omega$ if and only if A is Borel.

Moreover, in the generalized setting $\kappa^{<\kappa} = \kappa > \omega$, the class of continuous injective images of closed subsets of $({}^\kappa\kappa, \tau_b)$ consistently coincide with the class of κ -analytic sets [LS15]. It is therefore natural to ask whether the generalization of at least one of these two definitions may give a non-trivial notion of \mathcal{I} -analytic set. The answer is negative again.

Proposition 7.4.5. *Assume that \mathcal{I} contains an unbounded subset of κ . Then,*

- (1) *Every non-empty $A \subseteq {}^\kappa\nu$ is an \mathcal{I} -continuous image of ${}^\kappa\kappa$.*
- (2) *Every $A \subseteq {}^\kappa\nu$ is an injective \mathcal{I} -continuous image of some \mathcal{I} -closed $C \subseteq {}^\kappa\kappa$.*

Proof. Let $U \in \mathcal{I}$ be unbounded and fix a bijection $\theta : {}^U\kappa \rightarrow {}^U\nu$. Then,

$$\Phi : {}^\kappa\kappa \rightarrow {}^\kappa\nu : x \mapsto \theta(x \upharpoonright U)$$

is such that for every non-empty $A \subseteq {}^\kappa\nu$, $\Phi^{-1}(A)$ is \mathcal{I} -clopen, because

$$\Phi^{-1}(A) = \bigcup \{N_f \mid f \in {}^U\kappa, \theta(f) \in A\}$$

and

$${}^\kappa\kappa \setminus \Phi^{-1}(A) = \bigcup \{N_f \mid f \in {}^U\kappa, \theta(f) \notin A\}$$

are both \mathcal{I} -open. In particular, $\Phi : {}^\kappa\kappa \rightarrow {}^\kappa\nu$ is \mathcal{I} -continuous.

To show (1), we define the surjective function $\Psi : {}^\kappa\kappa \rightarrow A$ by setting:

$$\Psi(x) = \begin{cases} \Phi(x) & \text{if } x \in \Phi^{-1}(A) \\ x_0 & \text{if } x \notin \Phi^{-1}(A) \end{cases}$$

for some $x_0 \in A$.

For (2), consider the set

$$C = \{x \in {}^\kappa\kappa \mid x(\alpha) = 0 \text{ for all } \alpha \in \kappa \setminus U, \theta(x \upharpoonright U) \in A\}.$$

Since $|\kappa \setminus U| \leq \kappa$ and $\{x \in {}^\kappa\kappa \mid x(\alpha) \neq 0\} \in \kappa^+ \text{-}\mathbf{\Delta}_1^0 \subseteq \mathcal{I}\text{-}\mathbf{\Delta}_1^0$,

$${}^\kappa\kappa \setminus C = \bigcup_{\alpha \in \kappa \setminus U} \{x \in {}^\kappa\kappa \mid x(\alpha) \neq 0\} \cup \bigcup \{N_f \mid \theta(f) \notin A\}$$

is \mathcal{I} -open. Thus, C is \mathcal{I} -closed. Then, $\Phi \upharpoonright C$ is injective by construction (since θ is bijective) and it is onto A . \square

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