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On the role of stubbornness in Nash equilibrium problems

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***Abstract*— We consider Nash equilibrium (NE) seeking in partial-decision information regimes where the data circulating across the agents' communication network are affected by the agents' stubborn behavior. Unlike traditional linear averaging approaches, the communication protocol is here based on a nonlinear, state-dependent dynamics that captures the agents' biases, which hinders the direct application of standard techniques. Building on the explicit characterization of the convergence rate of the opinion dynamics resulting from the proposed scheme, and leveraging operator theory, we embed the communication pattern into two novel NE seeking algorithms for which we derive rigorous convergence guarantees.**

***Index Terms*— Nash equilibrium problems, network dynamics, optimization algoriothms**

I. Introduction

CIRCULATING information among agents is a key aspect for the distributed computation of Nash equilibria (NE) in games. For this reason, available works study how communication protocols affect the equilibrium seeking process, including scenarios where the communication fails [1], [2], it is time-varying or event-triggered [3]–[5], or experiences delays [4]. Depending on the information available to each single agent, two settings are usually considered in a game-theoretic framework: *full-decision information* [6], [7], where each agent has direct access to the decision variables of all its opponents, and *partial-decision information*, where agents share data only with their neighbors and lack direct knowledge on others [5], [8], [9]. In this second scenario, communication between agents and averaging schemes are widely employed to reconstruct the missing information [8], [9]. Although time-varying and averaging dynamics are reasonable for multi-agent systems where mere computations are required, such mechanisms implicitly assume that i) agents fully trust each other, and ii) the information process is independent of their current opinions. When dealing with not fully rational agents (e.g., humans), communication is instead strongly influenced by existing beliefs and self-preservation mechanisms. Attempts have been made to model human behavior in the context of finite games and full knowledge of the players' strategies [10] or within evolutionary game theory, focusing on the strategies' evolution rather than on the players' optimization problem [11]. On the other hand, few works have considered communication protocols different from the standard averaging scheme. For instance, [12], [13] derived a gossip-based mechanism involving only two agents at a time, while other methods build a belief of the opponents' strategies instead of retrieving the actual decision variables [14]. We propose here a *nonlinear*, *state-dependent* multi-agent communication dynamic accounting for the agents' potential stubborn attitude. We previously considered this dynamic in generalized Nash equilibrium problems (GNEPs) featuring coupling constraints [15], which however prevented us from fully exploiting the underlying communication scheme. Thus, we remove the shared constraints here to enable for a different analysis and the possibility to consider other types of iterative equilibrium seeking algorithms. Our proofing technique also differs from [12], [13], and the corresponding schemes in full-decision information setting [6], [7].

Inspired by the polar opinion dynamics in [15], [16], we propose a discrete-time communication protocol to capture agents' susceptibility with respect to (w.r.t.) the current state. We merge such dynamics with the NE seeking mechanism in partial-decision information and design a variant of two classic iterative schemes: the forward-backward (FB) [17] and the relaxed FB (RFB) algorithms [18] *with stubborn agents* (FBwS and RFBwS, respectively). These consist in alternating a (pseudo)gradient step with a communication one, according to our proposed state-dependent dynamics, to retrieve the decision variables of the other participants. Building on our crucial result on the convergence rate of the communication protocol, a novelty compared to [15], [16], we show that the algorithms converge to an NE, despite the potentially biased behavior of the agents. Our contribution is therefore twofold: i) we propose a nonlinear state-dependent communication dynamics that introduces stubborn behaviors in the NE seeking, and ii) provide two new distributed iterative schemes to reach an equilibrium. While the FBwS represents an extension of the classic FB scheme to the polar case, the RFBwS is novel in itself: besides [15], this is the first time that the RFB scheme is used in a partial-decision information setup. Finally, we validate our theoretical findings on a traffic model, showing how changing the drivers' level of stubbornness may ultimately shape the equilibrium reached.

Notation: We use Standing Assumption to postulate conditions that hold throughout the paper, while Assumption is used

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only in the section where it is introduced.

For a closed set $C \subseteq \mathbb{R}^n$, $\mathrm{proj}_C : \mathbb{R}^n \to C$ denotes the projection onto C, $\mathrm{proj}_C(x) = \mathrm{argmin}_{y \in C} \|y - x\|$. A mapping $F : \mathrm{dom}\, F \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is ℓ-Lipschitz continuous if, for some $\ell > 0$, $\|F(x) - F(y)\| \leq \ell \|x - y\|$, $\forall x, y \in \mathrm{dom}(F)$; (strictly) monotone if for all $x, y \in \mathrm{dom}(F)$, $x \neq y$, $\langle F(x) - F(y), x - y\rangle \geq (>) 0$; paramonotone if it is monotone and for all $x, y \in \mathrm{dom}(F)$ $\langle F(x) - F(y), x - y\rangle = 0 \Rightarrow F(x) = F(y)$; β-cocoercive if, for all $x, y \in \mathrm{dom}(F)$ and for some $\beta > 0$, $\langle F(x) - F(y), x - y\rangle \geq \beta \|F(x) - F(y)\|^2$.

An (undirected) graph $\mathcal{G} = (\mathcal{I}, \mathcal{E})$ consists of a node set $\mathcal{I}$ and an edge set $\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{I}\}$, where edge $(i, j) \in \mathcal{E} \iff (j, i) \in \mathcal{E}$ means that i and j can exchange information. A path is a sequence of distinct vertices, with consecutive ones connected by an edge. Let $N = |\mathcal{I}|$, $W \in \mathbb{R}^{N \times N}$ is the (weighted) adjacency matrix of $\mathcal{G}$, such that $w_{ij} > 0 \iff (i, j) \in \mathcal{E}$ and $w_{ij} = 0$ otherwise, with $w_{ii} = 0$ for all $i \in \mathcal{I}$. Let $d_i = \sum_{j=1}^N w_{i,j}$ be the degree of i and $d_{\max} = \max_{i \in \mathcal{I}} d_i$. For a connected graph (with a path between every pair of vertices) the Laplaciam matrix $L = \mathrm{diag}\{d_1, \ldots, d_N\} - W$ has $\mathrm{null}(L) = \{\kappa \mathbf{1}_N \mid \kappa \in \mathbb{R}\}$ and eigenvalues $0 = s_1(L) < s_2(L) \leq \cdots \leq s_N(L) \leq 2d_{\max}$.

II. Nash equilibrium problem description

We consider a Nash equilibrium problem (NEP) involving N agents, $\mathcal{I} = \{1, \ldots, N\}$, each aiming at minimizing its own cost function $J_i : \mathbb{R}^n \to \mathbb{R}$ subject to local constraints, yielding the following coupled optimization problems:

$$\forall i \in \mathcal{I} : \min_{x_i \in \Omega_i} J_i(x_i, \boldsymbol{x}_{-i}). \quad (1)$$

The cost function of each agent $i \in \mathcal{I}$ depends on its own constrained decision variable $x_i \in \Omega_i \subseteq \mathbb{R}^{n_i}$, and on the decision variables of the other agents, $\boldsymbol{x}_{-i} = \mathrm{col}((x_j)_{j \in \mathcal{I}\setminus\{i\}}) \in \Omega_{-i}$, with $\Omega_{-i} := \prod_{j \in \mathcal{I}\setminus\{i\}} \Omega_j \subseteq \mathbb{R}^{n - n_i}$, $n = \sum_{i \in \mathcal{I}} n_i$. We indicate the collective decision variable, taking values in $\Omega := \prod_{i \in \mathcal{I}} \Omega_i \subseteq \mathbb{R}^n$, as $\boldsymbol{x} = \mathrm{col}((x_i)_{i \in \mathcal{I}})$ or $\boldsymbol{x} = (x_i, \boldsymbol{x}_{-i})$.

Standing Assumption 1. *For all $i \in \mathcal{I}$, Ω_i is convex and bounded, and for all $\boldsymbol{x}_{-i} \in \Omega_{-i}$, $x_i \mapsto J_i(x_i, \boldsymbol{x}_{-i})$ is a convex and continuously differentiable function.*

The first part of Asm. 1 will be relevant in §III, while the second part is standard for NEPs. Together, they guarantee existence of a solution, i.e., an NE [6], [7], [12], [19].

Definition 1. *A collective strategy $\boldsymbol{x}^\star \in \Omega$ is an NE of the NEP in* (1) *if, for all $i \in \mathcal{I}$, $J_i(x_i^\star, \boldsymbol{x}_{-i}^\star) \leq \inf_{y_i \in \Omega_i} J_i(y_i, \boldsymbol{x}_{-i}^\star)$.*

From [20, Prop. 1.4.2], an NE can be characterized as the solution to the variational inequality (VI) associated with the NEP in (1). Specifically, any $\boldsymbol{x}^\star \in \Omega$ so that:

$$\langle F(\boldsymbol{x}^\star), \boldsymbol{y} - \boldsymbol{x}^\star\rangle \geq 0 \text{ for all } \boldsymbol{y} \in \Omega, \quad (2)$$

also meets Definition 1. In (2), the so-called *pseudogradient mapping* $F : \mathbb{R}^n \to \mathbb{R}^n$ is formally defined as $F(\boldsymbol{x}) = \mathrm{col}((F_i(\boldsymbol{x})_{i \in \mathcal{I}})) := \mathrm{col}(\nabla_{x_i} J_i(x_i, \boldsymbol{x}_{-i}))_{i \in \mathcal{I}}$, and we indicate the solution set to the VI in (2) as $\mathrm{SOL}(F, \Omega)$.

We consider a partial-decision information regime, where the agents cannot access all the decision variables of the opponents, but they keep an estimate of such variables. In particular, $\hat{x}_{i,j} \in \mathbb{R}^{n_j}$ is the estimate that agent $i \in \mathcal{I}$ keeps of the opponent j's decision variable, $j \in \mathcal{I}$, which are then collected in $\hat{x}_i = \mathrm{col}((\hat{x}_{i,j})_{j \in \mathcal{I}}) \in \mathbb{R}^n$, for each agent, and globally in $\hat{\boldsymbol{x}} = \mathrm{col}((\hat{x}_i)_{i \in \mathcal{I}}) \in \mathbb{R}^{nN}$. Note that $\hat{x}_{i,i} = x_i$. To share information, we assume the agents are connected over a communication graph $\mathcal{G} = (\mathcal{I}, \mathcal{E})$ characterized as follows:

Standing Assumption 2. *$\mathcal{G}$ is undirected and connected.*

To build each $\hat{x}_i$ so that the estimates actually match the strategies of the opponents of agent i, the simplest communication dynamic is for the agents to average their opinion with their neighbors'. For non-rational agents with human-like thinking, however, it is not realistic to fully trust their neighbors. Inspired by [16], where possible stubborn agents are modeled via a state-dependent susceptibility diagonal matrix $A : \mathbb{R}^{nN} \to [0, 1]^{nN \times nN}$, $A(\hat{\boldsymbol{x}}) = \mathrm{diag}(a_i(\hat{\boldsymbol{x}}_i)_{i \in \mathcal{I}})$, with $a_i : \mathbb{R}^n \to [0, 1]^n$, we investigate different communication patterns that better fit the behavior of non-rational agents. In fact, we consider a nonlinear, state-dependent mechanism that yields the following discrete-time communication protocol:

$$\hat{\boldsymbol{x}}^+ = \hat{\boldsymbol{x}} - \kappa A(\hat{\boldsymbol{x}})(L \otimes I_n)\hat{\boldsymbol{x}} =: (I_{nN} - \kappa A(\hat{\boldsymbol{x}}) L_n)\hat{\boldsymbol{x}}, \quad (3)$$

where $L \in \mathbb{R}^{N \times N}$ denotes the Laplacian matrix associated to $\mathcal{G}$ and $\kappa > 0$ is some discretization step.

III. Polar opinions on the strategies estimate

Inspired by the analysis conducted in [15], we consider (3) as the discretized version of the dynamic proposed in [16]. Such a communication protocol allows us to include agents with different levels of stubbornness via a state-dependent diagonal matrix $A(\hat{\boldsymbol{x}}) \in \mathrm{diag}([0, 1]^{nN})$ that can in principle lead to polarized behaviors of the agents. Consistently with [16], we refer to it as *polar opinion dynamics*. The entries of the i-th diagonal block, $A_i(\hat{x}_i)$, can also be interpreted as the susceptibility to persuasion of agent i. In particular, $a_{ij}(\cdot) = [a_i(\cdot)]_j = [A_i(\cdot)]_{jj}$ refers to the j-th entry of the estimate vector and indicates the behavioral attitude of agent i w.r.t. the current estimate of the decision variable of agent j. According to (3), if in the limit $a_{ij} = 0$ the agent is completely stubborn and does not revise its estimates of the decision variable, while intermediate values $a_{ij} \in (0, 1)$ capture bounded trust and biases. Suitable examples for $A(\cdot)$ can be found in [16], where it has been shown that matrix functions $(I_{nN} - \mathrm{diag}(\hat{\boldsymbol{x}})^2)$, $\frac{1}{2}(I_{nN} - \mathrm{diag}(\hat{\boldsymbol{x}}))$, and $\mathrm{diag}(\hat{\boldsymbol{x}})^2$ capture different behavioral attitudes, such as stubborn extremist (SE), stubborn positive (SP), and stubborn neutral (SN) agents, respectively.

The model in (3) consists of two main terms: the averaging component L_n that steers the agents toward consensus, and the susceptibility term $A(\cdot)$ that modulates such process. Under some assumptions we proved in [15, Lem. 1] that the dynamics in (3) is well-defined and converges to a consensus.

Standing Assumption 3. *Function $A : \mathbb{R}^{nN} \to [0, 1]^{nN \times nN}$ is so that $\sigma = \min_{i,j,\hat{\boldsymbol{x}} \in \Omega^N} a_{ij}(\hat{x}_i) > 0$ and $\kappa < 1/2d_{\max}$.*

We prove next the convergence of (3) toward a consensus point under the postulated assumptions. Unlike [15, Lem. 1], our proof will be based on operator theory arguments.

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Lemma 1. *The domain Ω^N is invariant under dynamics in* (3)*, and the sequence $\{\hat{\boldsymbol{x}}^k\}_{k\in\mathbb{N}}$, generated by* (3) *from initial condition $\hat{\boldsymbol{x}}^0 \in \Omega^N$, converges to some $\bar{\boldsymbol{x}} \in \mathrm{null}(L_n)$.*

Proof. From [15, Lem. 2], $(A(\cdot)L_n)(\cdot)$ is a $(1/2d_{\max})$-cocoercive operator, hence we have that $(I - \tau A(\cdot)L)(\cdot)$ is an averaged operator for any $\tau \in (0, 1/d_{\max})$. Therefore, in view of Stand. Asm. 3, the associated Picard-Banach iteration, which corresponds to (3) with τ in place of κ, converges to a fixed-point (if one does exist) for any $\tau \in (0, 1/d_{\max})$. From [21, Prop. 1] [22], we have convergence to some $\bar{x}$ so that $A(\bar{x})L\bar{x} = 0$, which in case $A(\bar{x})$ does not have zero entries it is a consensus vector, since $L\bar{x} = 0$. ■

Lem. 1 says that, under some conditions if $\hat{\boldsymbol{x}}^0 \in \Omega^N$, the estimates $\hat{\boldsymbol{x}}^k$ stay in Ω^N for all $k \geq 0$, eventually converging to a consensus. As a consequence of [15, Lem. 2], $(A(\cdot)L_n)(\cdot)$ is a Lipschitz continuous operator [17]. However, the key aspect of (3) for our analysis is not only that our communication dynamics converges, but also that this will happen relatively fast. In fact, by introducing vectors $z \in \mathrm{null}(L)$ and $\boldsymbol{z} = 1_N \otimes z \in \mathrm{null}(L_n)$, the following result is crucial for convergence to an NE, as we will see in §IV.

Theorem 1. *Dynamics in* (3) *is so that $\sum_{k\in\mathbb{N}} \left\|\hat{\boldsymbol{x}}^k - \boldsymbol{z}\right\|^2 < \infty$.*

Proof. Consider a generic agent $i \in \mathcal{I}$ and its estimate of the decision variable ℓ. According to (3), $\hat{x}^{k+1}_{i,\ell} = (1 - \kappa a_{i\ell}(\hat{x}^k_i)d_i)\hat{x}^k_{i,\ell} + \kappa a_{i\ell}(\hat{x}^k_i)\sum_{j\in\mathcal{I}\setminus\{i\}} W_{ij}\hat{x}^k_{j,\ell}$. Let $M^k_\ell := \arg\max_{i\in\mathcal{I}} \hat{x}^k_{i,\ell}$ and $m^k_\ell := \arg\min_{i\in\mathcal{I}} \hat{x}^k_{i,\ell}$. Consequently, $\hat{x}^k_{M,\ell} := \hat{x}^k_{M^k_\ell,\ell}$ and $\hat{x}^k_{m,\ell} := \hat{x}^k_{m^k_\ell,\ell}$. Let $D := \mathrm{diam}\,\mathcal{G}$ and $\mathcal{V}_s$ be the set of nodes reachable from m^k_ℓ via a path of length s, including self-loops. So, $\mathcal{V}_s \subseteq \mathcal{V}_{s+1}$ and $\mathcal{V}_D = \mathcal{I}$. Being (3) a convex combination (since $\kappa a_{i\ell}(\hat{x}^k_i)d_i < 1$), it holds $\hat{x}^{k+1}_{j,\ell} \leq \hat{x}^k_{M,\ell}$ for all $j \notin \mathcal{V}_1$; moreover, using (3) and Stand. Asm. 3, we bound $x^{k+1}_{j,\ell} \leq (1-\kappa\sigma)\hat{x}^k_{M,\ell} + \kappa\sigma\hat{x}^k_{m,\ell}$ for all $j \in \mathcal{V}_1$. Using this argument recursively for the sets from $\mathcal{V}_2$ to $\mathcal{V}_D$, we ultimately bound $\hat{x}^{k+D}_{i,\ell} \leq (1 - \kappa^D\sigma^D)\hat{x}^k_{M,\ell} + \kappa^D\sigma^D\hat{x}^k_{m,\ell}$ for all $i \in \mathcal{V}_D$. Being $\mathcal{V}_D = \mathcal{I}$, this implies

$$\hat{x}^{k+D}_{M,\ell} \leq (1 - \kappa^D\sigma^D)\hat{x}^k_{M,\ell} + \kappa^D\sigma^D\hat{x}^k_{m,\ell}. \tag{4}$$

Following a similar argument, we obtain

$$\hat{x}^{k+D}_{m,\ell} \geq (1 - \kappa^D\sigma^D)\hat{x}^k_{m,\ell} + \kappa^D\sigma^D\hat{x}^k_{M,\ell}. \tag{5}$$

From (4) and (5), we bound $\hat{x}^{k+D}_{M,\ell} - \hat{x}^{k+D}_{m,\ell} \leq (1 - 2\kappa^D\sigma^D)(\hat{x}^k_{M,\ell} - \hat{x}^k_{m,\ell})$, which is well defined, being $\kappa\sigma \leq 1/2d < 1/2$. Then, combining this with the fact that $\hat{x}^k_{M,\ell}$ and $\hat{x}^k_{m,\ell}$ are monotonically non-increasing and non-decreasing, respectively, at the k-th iteration we bound

$$\hat{x}^k_{M,\ell} - \hat{x}^k_{m,\ell} \leq (1 - 2\kappa^D\sigma^D)^{\lfloor \frac{k}{D} \rfloor} K_\ell, \tag{6}$$

where $K_\ell := \max_{\zeta\in\cup_{i\in\mathcal{I}}\Omega_i} \zeta_\ell - \min_{\xi\in\cup_{i\in\mathcal{I}}\Omega_i} \xi_\ell$ is a constant in view of Stand. Asm. 1.(i), and clearly it holds $K_\ell \geq \hat{x}^0_{M,\ell} - \hat{x}^0_{m,\ell}$. Due to Lem. 1, $\bar{x}_\ell \in [\hat{x}^k_{m,\ell}, \hat{x}^k_{M,\ell}]$, for all $k \geq 0$. Consistently, using the triangular inequality and then the latter observation, we can bound: $\|\hat{\boldsymbol{x}}^k - \boldsymbol{z}\|^2 \leq \sum_{\ell=1}^n N^2(\hat{x}^k_{M,\ell} - \hat{x}^k_{M,\ell})^2$. Finally, by making use of (6) and letting $K := \max_{\ell\in\{1,\dots,n\}} K_\ell$, we further bound:

$$\sum_{k=1}^{\infty} \|\hat{\boldsymbol{x}}^k - \boldsymbol{z}\|^2 \leq n\sum_{k=1}^{\infty} N^2K^2(1 - 2\kappa^D\gamma^D)^{2\lfloor \frac{k}{D} \rfloor}$$
$$= nN^2K^2D\sum_{s=1}^{\infty}(1 - 2\kappa^D\gamma^D)^{2s}$$
$$\leq \frac{nN^2K^2D(1 - 2\kappa^D\gamma^D)^2}{4\kappa^D\gamma^D(1 - \kappa^D\gamma^D)} < \infty, \tag{7}$$

where Stand. Asm. 3 yields $2\kappa\sigma < 1$ and Stand. Asm. 2 implies $D \leq N - 1 < \infty$, yielding the claim. ■

Theorem 1 suggests that, despite the state-dependent stubbornness of the agents, the discrete-time polar opinion dynamics in (3) converges fast enough to a consensus. Moreover, (7) provides interesting insights on how the different model parameters shape such rate of convergence. In particular, note how densely connected networks (i.e., with smaller D) may offer guarantees of faster convergence.

IV. Nash Equilibrium Seeking Algorithms

To compute an NE we propose two distributed algorithms and include the communication step for the agents to share their opinions on the opponents' strategies. The core of the schemes is reported in Algorithms 1–2. In both cases, agents perform a gradient step using local estimates on the other agents' decision variables, differently from [15]. Next, we state the common assumptions, while the algorithm-specific ones will be introduced in the dedicated subsections.

Standing Assumption 4. *For all $i \in \mathcal{I}$, (i) $x_i \mapsto \nabla_{x_i} J_i(x_i, \boldsymbol{x}_{-i})$ is σ_i-Lipschitz continuous, and $\boldsymbol{x}_{-i} \mapsto \nabla_{x_i} J_i(x_i, \boldsymbol{x}_{-i})$ is ℓ_i-Lipschitz continuous, and (ii) there exists $C > 0$ such that $\|F_i(\boldsymbol{x})\| \leq C$, for all $\boldsymbol{x} \in \Omega$.*

As per Stand. Asm. 4, F is ρ-Lipschitz continuous, with $\rho = \sqrt{2\sum_{i\in\mathcal{I}}(\ell_i^2 + \sigma_i^2)}$—see [12, Eq. (18)]. We make the a standard assumption on the stepsize sequence [12], [13].

Standing Assumption 5. *For all $i \in \mathcal{I}$, the stepsize sequences satisfy $\sum_{k=0}^\infty \alpha_i^k = \infty$ and $\sum_{k=0}^\infty (\alpha_i^k)^2 < \infty$.*

A. FB algorithm with stubborn agents

Algorithm 1 consists of a standard FB iteration [17], paired with a communication round involving the susceptibility to persuasion matrix $A(\cdot)$. Following this iterative scheme, the agents share their decision variables with some neighbors only, i.e., in the set $\mathcal{N}_i := \{j \in \mathcal{I} \mid (i,j) \in \mathcal{E}\}$, and then use the biased estimates to evaluate the local gradient $F_i(\cdot)$.

Assumption 6. *The mapping $F(\cdot)$ is strictly monotone.*

From Asm. 6, the VI in (2) has a unique solution, hence the game in (1) has only one equilibrium [20, Th. 2.3.3]. We now prove convergence of Algorithm 1 to such an NE.

Theorem 2. *Under Asm. 6, Algorithm 1 produces a sequence $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ convergent to the NE of the game in* (1)*.*

Proof. Let $\boldsymbol{x}^\star$ be the NE of (1) and let $\boldsymbol{z}_{-i} = \mathrm{col}((z_j)_{j\in\mathcal{I}\setminus\{i\}})$, according to as in Theorem 1. Then, since the projection is nonexpansive, we have:

$$\|x_i^{k+1} - x_i^\star\|^2 \leq \|x_i^k - x_i^\star - \alpha_i^k(F_i(x_i^k, \hat{\boldsymbol{x}}^{k+1}_{i,-i}) + F_i(x_i^\star, \boldsymbol{x}^\star_{-i}))\|^2$$
$$\leq \|x_i^k - x_i^\star\|^2 + (\alpha_i^k)^2\|F_i(x_i^k, \hat{\boldsymbol{x}}^{k+1}_{i,-i}) - F_i(x_i^k, \boldsymbol{z}_{-i})\|^2$$
$$+ (\alpha_i^k)^2\|F_i(x_i^k, \boldsymbol{z}_{-i}) - F_i(x_i^k, \boldsymbol{x}^k_{-i})\|^2$$

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Algorithm 1: FB with stubborn agents (FBwS)

Initialization: $x_i^0 \in \Omega_i$
Iteration $k \in \mathbb{N}_0$**:** Agent i receives $\hat{\boldsymbol{x}}_j^k$ from each $j \in \mathcal{N}_i$, then updates

$$\hat{x}_i^{k+1} = \hat{x}_i^k - \kappa A_i(\hat{x}_i^k) \textstyle\sum_{j\in\mathcal{N}_i} \left(\hat{x}_i^k - w_{ij}\hat{x}_j^k\right)$$
$$x_i^{k+1} = \mathrm{proj}_{\Omega_i}(x_i^k - \alpha_i^k F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}))$$
$$\hat{x}_{i,i}^{k+1} = x_i^{k+1}$$

Algorithm 2: RFB with stubborn agents (RFBwS)

Initialization: $x_i^0, \bar{x}_i^{-1} \in \Omega_i$
Iteration $k \in \mathbb{N}_0$**:** Agent i receives $\hat{\boldsymbol{x}}_j^k$ from each $j \in \mathcal{N}_i$, then updates

$$\hat{x}_i^{k+1} = \hat{x}_i^k - \kappa A_i(\hat{x}_i^k) \textstyle\sum_{j\in\mathcal{N}_i} \left(\hat{x}_i^k - w_{ij}\hat{x}_j^k\right)$$
$$\bar{x}_i^k = (1-\delta)x_i^k + \delta \bar{x}_i^{k-1}$$
$$x_i^{k+1} = \mathrm{proj}_{\Omega_i}(\bar{x}_i^k - \alpha_i^k F(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}))$$
$$\hat{x}_{i,i}^{k+1} = x_i^{k+1}$$

$$\begin{aligned}
&+ (\alpha_i^k)^2 \|F_i(x_i^\star, \boldsymbol{x}_{-i}^\star) - F_i(x_i^k, \boldsymbol{x}_{-i}^k)\|^2 \\
&+ 2\alpha_i^k \langle F_i(x_i^k, \boldsymbol{z}_{-i}) - F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}), x_i^k - x_i^\star\rangle \\
&+ 2\alpha_i^k \langle F_i(x_i^k, \boldsymbol{x}_{-i}^k) - F_i(x_i^k, \boldsymbol{z}_{-i}), x_i^k - x_i^\star\rangle \\
&+ 2\alpha_i^k \langle F_i(x_i^\star, \boldsymbol{x}_{-i}^\star) - F_i(x_i^k, \boldsymbol{x}_{-i}^k), x_i^k - x_i^\star\rangle,
\end{aligned}$$

where we have added and subtracted $F_i(x_i^k, \boldsymbol{x}_{-i}^k)$ and $F_i(x_i^k, \boldsymbol{z}_{-i})$ into the norm of the first inequality. By Young's inequality, $2\alpha_i^k \langle F_i(x_i^k, \boldsymbol{z}_{-i}) - F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}), x_i^k - x_i^\star\rangle \leq \|F_i(x_i^k, \boldsymbol{z}_{-i}) - F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1})\|^2 + (\alpha_i^k)^2\|x_i^k - x_i^\star\|^2$ and $2\alpha_i^k\langle F_i(x_i^k, \boldsymbol{x}_{-i}^k) - F_i(x_i^k, \boldsymbol{z}_{-i}), x_i^k - x_i^\star\rangle \leq \|F_i(x_i^k, \boldsymbol{z}_{-i}) - F_i(x_i^k, \boldsymbol{x}_{-i}^k)\|^2 + (\alpha_i^k)^2\|x_i^k - x_i^\star\|^2$. Then, using Stand. Asm. 4 with $\ell_{\max} = \max_{i\in\mathcal{I}} \ell_i$ and $\alpha_{\max}^k = \max_{i\in\mathcal{I}} \alpha_i^k$, and summing over the agents, we have:

$$\begin{aligned}
&\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^\star\|^2 \leq \left(1 + 2(\alpha_{\max}^k)^2\right) \|\boldsymbol{x}^k - \boldsymbol{x}^\star\|^2 \\
&- 2\alpha_{\max}^k \langle F(\boldsymbol{x}^\star) - F(\boldsymbol{x}^k), \boldsymbol{x}^\star - \boldsymbol{x}^k\rangle + 6NC^2(\alpha_{\max}^k)^2 \\
&+ \ell_{\max}^2 \textstyle\sum_{i\in\mathcal{I}} \left(\|\boldsymbol{z}_{-i} - \boldsymbol{x}_{-i}^k\|^2 + \|\boldsymbol{z}_{-i} - \hat{\boldsymbol{x}}_{-i}^{k+1}\|^2\right).
\end{aligned} \tag{8}$$

Due to Theorem 1 and Stand. Asm. 5, $\ell_{\max}^2 \sum_{i\in\mathcal{I}} \left(\|\boldsymbol{z}_{-i} - \boldsymbol{x}_{-i}^k\|^2 + \|\boldsymbol{z}_{-i} - \hat{\boldsymbol{x}}_{-i}^{k+1}\|^2\right) + 6NC^2(\alpha_{\max}^k)^2$ in (8) act as the vanishing ε^k term of [23, Lem. 3.6], and the RHS collectively meets its assumptions. By [23, Lem. 3.6] $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ is bounded, so it admits a cluster point $\boldsymbol{y} \in \Omega$. Since the inner product is negative, by [23, Lem. 3.6] it is summable, so $\langle F(\boldsymbol{x}^\star) - F(\boldsymbol{y}), \boldsymbol{x}^\star - \boldsymbol{y}\rangle = 0$, i.e., $\boldsymbol{y}$ is a solution to the VI, and $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ converges to the NE of (1). ■

Theorem 2 shows that, despite a state-dependent dynamic could, in principle, compromise the convergence because of the stubborn agents, this is not the case.

Remark 1. *The proof of Theorem 2 partly relies on common tools for the convergence of FB schemes (non-expansivity, strict monotonicity [6], [17]), however, it is thanks to Theorem 1 that convergence of Algorithm 1 can be established, as it ensures that the extra terms in* (8) *vanish fast enough.*

B. Relaxed FB algorithm with stubborn agents

In the spirit of [7], [18], we derive a variant of Algorithm 1 by including a relaxation step, which leads to the instructions reported in Algorithm 2. While such a relaxation step tends to slow convergence, it also allows us to obtain convergence to an NE under weaker assumptions than Algorithm 1.

Assumption 7. *The mapping F is paramonotone.*

Remark 2. *Paramonotonicity is implied by strict monotonicity, but it is stronger than mere monotonicity. For our convergence analysis we will need the so-called* cut property*, a consequence of paramonotonicity, formalized as follows:*

$$\left.\begin{array}{r} \boldsymbol{x}^\star \in \mathrm{SOL}(F,\Omega) \\ \boldsymbol{y} \in \Omega,\ \textit{with}\ \boldsymbol{y} \neq \boldsymbol{x}^\star \\ \langle F(\boldsymbol{y}), \boldsymbol{y} - \boldsymbol{x}^\star\rangle = 0 \end{array}\right\} \Longrightarrow \boldsymbol{y} \in \mathrm{SOL}(F,\Omega). \tag{9}$$

Given a solution $\boldsymbol{x}^\star$, the relation in (9) *allows one to verify whether some $\boldsymbol{y}$ is also a solution by comparing it with $\boldsymbol{x}^\star$ only, rather than with all the other points in Ω.*

The NEP in (1) then features multiple equilibria. Next, we show convergence of Algorithm 2 to one of them.

Theorem 3. *Let $\delta \in (0,1)$. Under Asm. 7, Algorithm 2 produces a sequence $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ convergent to an NE of* (1)*.*

Proof. Let $\boldsymbol{x}^\star$ be an NE of (1) and let $\boldsymbol{z}_{-i} = \mathrm{col}((z_j)_{j\in\mathcal{I}\setminus\{i\}})$ according to Theorem 1. The fact that the projection is firmly quasi-nonexpansive yields the following:

$$\begin{aligned}
\|x_i^{k+1} - x_i^\star\|^2 &\leq \|x_i^\star - \bar{x}_i^k + \alpha_i^k F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1})\|^2 \\
&\qquad - \|\bar{x}_i^k - \alpha_i^k F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}) - x_i^{k+1}\|^2 \\
&\leq \|x_i^\star - \bar{x}_i^k\|^2 - \|x_i^{k+1} - \bar{x}_i^k\|^2 \\
&\quad + (\alpha_i^k)^2 \|F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}) - F(x_i^k, \boldsymbol{z}_{-i})\|^2 \\
&\quad + (\alpha_i^k)^2 \|F(x_i^k, \boldsymbol{z}_{-i}) - F(x_i^k, \boldsymbol{x}_{-i}^k)\|^2 \\
&\quad + (\alpha_i^k)^2 \|F(x_i^k, \boldsymbol{x}_{-i}^k)\|^2 - (\alpha_i^k)^2 \|F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1})\|^2 \\
&\quad + 2\alpha_i^k \langle F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}), x_i^{k+1} - \bar{x}_i^k\rangle \\
&\quad + 2\alpha_i^k \langle F(x_i^k, \boldsymbol{x}_{-i}^k), x_i^\star - \bar{x}_i^k\rangle \\
&\quad + 2\alpha_i^k \langle F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}) - F(x_i^k, \boldsymbol{z}_{-i}), x_i^\star - \bar{x}_i^k\rangle \\
&\quad + 2\alpha_i^k \langle F(x_i^k, \boldsymbol{z}_{-i}) - F(x_i^k, \boldsymbol{x}_{-i}^k), x_i^\star - \bar{x}_i^k\rangle,
\end{aligned}$$

where we have added and subtracted $F_i(x_i^k, \boldsymbol{x}_{-i}^k)$ and $F_i(x_i^k, \boldsymbol{z}_{-i})$. Using Lem. [7, Lem. 6.(ii–iii)], we rewrite

$$\begin{aligned}
&\tfrac{1}{1-\delta}\|\bar{x}_i^{k+1} - x_i^\star\|^2 \leq \tfrac{1}{1-\delta}\|\bar{x}_i^k - x_i^\star\|^2 - (\delta+1)\|x_i^{k+1} - \bar{x}_i^k\|^2 \\
&\quad + 2\alpha_i^k \langle F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}), x_i^{k+1} - \bar{x}_i^k\rangle \\
&\quad + 2\alpha_i^k \langle F(x_i^k, \boldsymbol{x}_{-i}^k), x_i^\star - \bar{x}_i^k\rangle \\
&\quad + (\alpha_i^k)^2 \|F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}) - F(x_i^k, \boldsymbol{z}_{-i})\|^2 \\
&\quad + (\alpha_i^k)^2 \|F(x_i^k, \boldsymbol{z}_{-i}) - F(x_i^k, \boldsymbol{x}_{-i}^k)\|^2 \\
&\quad + (\alpha_i^k)^2 \|F(x_i^k, \boldsymbol{x}_{-i}^k)\|^2 - (\alpha_i^k)^2 \|F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1})\|^2 \\
&\quad + 2\alpha_i^k \langle F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}) - F(x_i^k, \boldsymbol{z}_{-i}), x_i^\star - \bar{x}_i^k\rangle \\
&\quad + 2\alpha_i^k \langle F(x_i^k, \boldsymbol{z}_{-i}) - F(x_i^k, \boldsymbol{x}_{-i}^k), x_i^\star - \bar{x}_i^k\rangle.
\end{aligned} \tag{10}$$

By applying Young's inequality to the inner products above we obtain: $2\alpha_i^k\langle F(x_i^k, \boldsymbol{x}_{-i}^k), x_i^\star - \bar{x}_i^k\rangle \leq 2\alpha_i^k\langle F(x_i^k, \boldsymbol{x}_{-i}^k), x_i^\star - x_i^k\rangle + (\alpha_i^k)^2\|F(x_i^k, \boldsymbol{x}_{-i}^k)\|^2 + \|x_i^k - \bar{x}_i^k\|^2$, where we have added and subtracted x_i^k in the RHS, $2\alpha_i^k\langle F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}), x_i^{k+1} - \bar{x}_i^k\rangle \leq$

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$(\alpha_i^k)^2\|F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1})\|^2 + \|x_i^{k+1} - \bar{x}_i^k\|^2$, $2\alpha_i^k\langle F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}) - F(x_i^k, \boldsymbol{z}_{-i}), x_i^\star - \bar{x}_i^k\rangle \leq (\alpha_i^k)^2\|x_i^\star - \bar{x}_i^k\|^2\|F_i(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^{k+1}) - F(x_i^k, \boldsymbol{z}_{-i})\|^2$, and $2\alpha_i^k\langle F(x_i^k, \boldsymbol{z}_{-i}) - F(x_i^k, \boldsymbol{x}_{-i}^k), x_i^\star - \bar{x}_i^k\rangle \leq (\alpha_i^k)^2\|x_i^\star - \bar{x}_i^k\|^2 + \|F(x_i^k, \boldsymbol{z}_{-i}) - F(x_i^k, \boldsymbol{x}_{-i}^k)\|^2$. By using [7, Lem. 6.(i)], summing over all the agents and using Stand. Asm. 4 with $\ell_{\max} = \max_{i\in\mathcal{I}} \ell_i$ and $\alpha_{\max}^k = \max_{i\in\mathcal{I}} \alpha_i^k$, (10) reads

$$\begin{aligned}&\tfrac{1}{1-\delta}\|\bar{\boldsymbol{x}}^{k+1} - \boldsymbol{x}^\star\|^2 + \delta\|\boldsymbol{x}^{k+1} - \bar{\boldsymbol{x}}^k\|^2\\ &\leq \left(\tfrac{1}{1-\delta} + 2(\alpha^k)^2\right)\|\bar{\boldsymbol{x}}^k - \boldsymbol{x}^\star\|^2 + \delta^2\|\boldsymbol{x}^k - \bar{\boldsymbol{x}}^{k-1}\|^2\\ &\quad + \ell_{\max}^2 \textstyle\sum_{i\in\mathcal{I}} \left(\|\hat{\boldsymbol{x}}_{-i}^{k+1} - \boldsymbol{z}_{-i}\|^2 + \|\boldsymbol{z}_{-i} - \boldsymbol{x}_{-i}^k\|^2\right)\\ &\quad + 2\alpha_{\max}^k\langle F(\boldsymbol{x}^k), \boldsymbol{x}^\star - \boldsymbol{x}^k\rangle + 5NC^2(\alpha_{\max}^k)^2.\end{aligned} \tag{11}$$

Similar to Theorem 2, and due to Theorem 1, $\varepsilon^k = \ell_{\max}^2 \sum_{i\in\mathcal{I}}(\|\boldsymbol{z}_{-i} - \boldsymbol{x}_{-i}^k\|^2 + \|\boldsymbol{z}_{-i} - \hat{\boldsymbol{x}}_{-i}^{k+1}\|^2) + 5NC^2(\alpha_{\max}^k)^2$ is a summable sequence, and since $\delta \in (0,1)$, we can apply [23, Lem. 3.6]. We conclude that $\{\bar{\boldsymbol{x}}^k\}_{k\in\mathbb{N}}$ and $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ are bounded sequences and admit cluster points $\bar{\boldsymbol{y}}$ and $\boldsymbol{y}$, respectively, in the compact set Ω. Since $\bar{\boldsymbol{x}}^k = (1-\delta)\boldsymbol{x}^k + \delta\bar{\boldsymbol{x}}^{k-1}$, by taking the limit we obtain that $\bar{\boldsymbol{y}} = \boldsymbol{y}$. Moreover, since $\langle F(\boldsymbol{x}^k), \boldsymbol{x}^\star - \boldsymbol{x}^k\rangle \leq 0$, by [23, Lem. 3.6], it holds that $\langle F(\boldsymbol{y}), \boldsymbol{y} - \boldsymbol{x}^\star\rangle = 0$ which, by Asm. 7 and Remark 2, implies that $\boldsymbol{y}$ is a solution to the VI at hand. Finally, [18, Lem. 1] implies that $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ converges to an NE of (1). ∎

V. Drivers inclination in traffic routing

Drivers usually choose routes based on their preferences and perception of traffic conditions to avoid congested paths. A traffic network can be described by a graph $\mathcal{T} := (\mathcal{V}, \mathcal{S})$, for which $\mathcal{L} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of all origin-destination (OD) pairs of nodes, $\ell := |\mathcal{L}|$, and $\mathcal{M}$ is the set of all paths, $m := |\mathcal{M}|$ [24]. The arc-path incidence matrix $B \in \mathbb{R}^{s\times m}$, $s := |\mathcal{S}|$, describes the edge structure of the paths: for all $(j,r) \in \mathcal{S}\times\mathcal{M}$, $b_{j,r} = 1$ if arc $j \in \mathcal{S}$ belongs to path $r \in \mathcal{M}$, $b_{j,r} = 0$ otherwise. We assume $\mathcal{T}$ being directed and that all OD pairs are connected by at least one path. We consider a population of drivers $\mathcal{I} := \{1,\ldots,N\}$ that aim at crossing the network in a noncooperative manner. Each agent $i \in \mathcal{I}$ is associated with an OD pair, for which it is allowed to select any path $r \in \mathcal{M}$ connecting such a pair. Each agent has a willingness $p_{i,r} \in [0,1]$ for pursuing path r, stacked into $p_i := \mathrm{col}((p_{i,r})_{r\in\mathcal{M}}) \in [0,1]^m$ with $\sum_{r\in\mathcal{M}} p_{i,r} = 1$. Considering some $j \in \mathcal{S}$, the common interest for passing through that edge corresponds to $e_j := \sum_{r\in\mathcal{M}} b_{j,r}(\sum_{i\in\mathcal{I}} p_{i,r}) \geq 0$. The (unit) travel time of going through the edge $j \in \mathcal{S}$ is a function $c_j : \mathbb{R}_{\geq 0}^{mN} \to \mathbb{R}_{\geq 0}$, evaluated as $c_j(e_j)$, and the cost experienced for choosing path $r \in \mathcal{M}$ is $\sum_{j\in\mathcal{S}} b_{j,r}c_j(e_j)$. By defining $\mathrm{col}((c_j(e_j))_{j\in\mathcal{S}}) =: c(p_i, \boldsymbol{p}_{-i}) : \mathbb{R}^{mN} \to \mathbb{R}_{\geq 0}^s$, $\boldsymbol{p}_{-i} := \mathrm{col}((p_k)_{k\in\mathcal{I}\setminus\{i\}})$, each driver aims at solving the optimization problem

$$\forall i \in \mathcal{I} : \min_{p_i\in\mathcal{P}_i} h_i(p_i) + p_i^\top B^\top c(p_i, \boldsymbol{p}_{-i}), \tag{12}$$

where $h_i : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ is a local cost to express path preferences, and $\mathcal{P}_i := \{p_i \in [0,1]^m | \sum_{r\in\mathcal{M}} p_{i,r} = 1,\ p_{i,r} = 0 \text{ for all } r \notin \mathcal{M}_i\}$, where $\mathcal{M}_i \subset \mathcal{M}$ contains the indices of those paths connecting the OD pair of the i-th driver. We consider a linear form for $c_j(p_i, \boldsymbol{p}_{-i}) := b_j(\boldsymbol{1}_N^\top \otimes I_m)\boldsymbol{p} = b_j\mathrm{col}((f_r)_{r\in\mathcal{M}})$, where b_j is the j-th row of B. With these considerations, the second term in the cost function (12) turns into $p_i^\top B^\top B p_i + p_i^\top B^\top B(\boldsymbol{1}_{N-1}^\top \otimes I_m)\boldsymbol{p}_{-i}$.

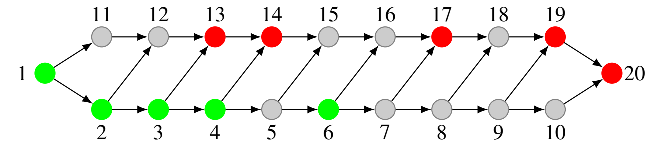

Fig. 1: Traffic network digraph $\mathcal{T}$ considered in §V-A. The green and red dots denote the origin and destination nodes, respectively.

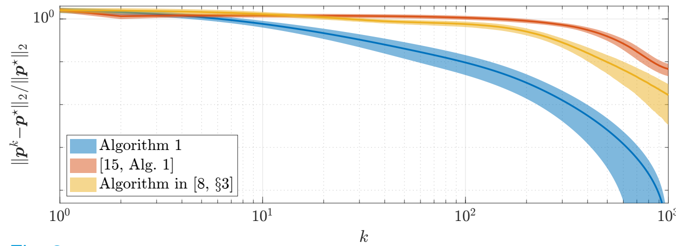

Fig. 2: Comparison test. The solid curves are the average evolution over 50 random instances; the dashed areas are the associated standard deviation.

A. Single equilibrium case

To test Algorithm 1, we set the local term $h_i(\cdot) = \|p_i\|_{Q_i}^2$ in (12), with $Q_i \in \mathbb{S}_{\succ 0}^m$. We randomly generate neighboring sets $\mathcal{N}_i$ so that the graph $\mathcal{G}$ meets Stand. Asm. 2. In Fig. 1 we show the digraph describing the topology of the traffic network $\mathcal{T}$ adopted here, which was also considered in [25]. This consists of 20 nodes and 28 edges, with $\mathcal{L} = \{(1,19), (1,20), (2,13), (2,17), (2,20), (3,14), (4,20), (6,19)\}$, and hence $\ell = 8$. By enumerating all possible paths connecting each OD pair we thus obtain $m = 49$, as well as the arc-path incidence matrix associated to the network, $B \in \mathbb{R}^{28\times 49}$.

For comparison, we contrast the average convergence performance of Algorithm 1 with those of the iterative schemes in [15] where the communication happens after the gradient step, and [12, §3], on 50 randomly generated, strongly monotone instances of the traffic network considered. In this case, we consider $N = 10$ agents, which are associated to an OD pair in $\mathcal{L}$ through a uniform distribution, and $A(\hat{\boldsymbol{x}}) = I_{490}$. Moreover, each Q_i is chosen as a diagonal matrix with entries uniformly sampled in $[1, 100]$. The parameters of the three algorithms are tuned separately to achieve the "best" converge performance. After few numerical experiments, we have then set $\kappa = 0.01$, $\alpha_i^k = 1/5k$, $i \in \mathcal{I}$ and $k \in \mathbb{N}$, for Algorithm 1, and $c = 0.5$, $\delta = 0$, and $\alpha_i^k = 8\times 10^{-3}$, $i \in \mathcal{I}$ and $k \in \mathbb{N}$, for [15, Alg. 1]. For the scheme reported in [12, §3], instead, $\alpha_i^k = 10^{-3}$ is taken constant too [12, §6], while the agents are woken up sequentially and choose a neighbor uniformly at random in $\mathcal{N}_i$. Once initialized the algorithms with the same point uniformly sampled in the polyhedron $\prod_{i\in\mathcal{I}} \mathcal{P}_i$, Fig. 2 shows how Algorithm 1 outperforms the other two.

B. Traffic problem with multiple equilibria

Besides testing Algorithm 2, the example discussed next aims at shedding further light on whether the agents' biases affect the NE computation in a monotone instance of the traffic problem described in §I featuring multiple equilibria.

We focus on the traffic topology in Fig. 3, consisting of 4 nodes and 5 edges, with only $\ell = 1$ OD pair, i.e., $\mathcal{L} = \{(1,2)\}$. The total number of paths is $m = 3$, while $B \in \mathbb{R}^{5\times 3}$. Calculating the eigenvalues of the matrix characterizing the pseudo-gradient mapping, which is affine in $\boldsymbol{x}$, reveals that

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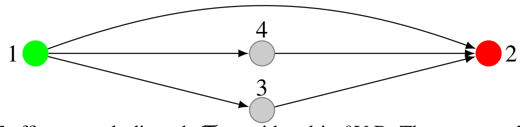

Fig. 3: Traffic network digraph $\mathcal{T}$ considered in §V-B. The green and red dots denote the origin and destination nodes, respectively.

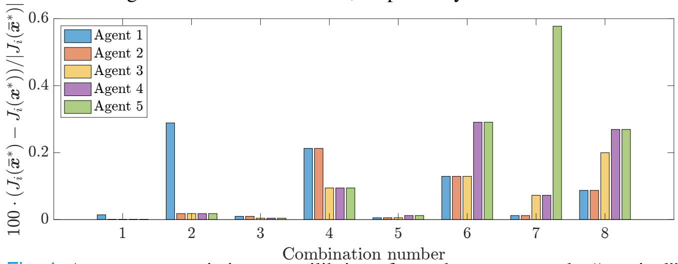

Fig. 4: Average cost variation at equilibrium, for each agent, w.r.t. the "nominal" one, i.e., $\bar{\boldsymbol{x}}^*$, obtained by setting the matrix function $A(\cdot) = I$.

the NEP considered is monotone. With $N = 5$ agents, we set $\mathcal{N}_i = \mathcal{I} \setminus \{i\}$ and $Q_i = 0$, for all i. Algorithm 2 is implemented here with $\alpha_i = \alpha = 5 \times 10^{-3}$, $\kappa = 0.01$ and $\delta = 10^{-3}$. We consider eight combinations: 1 :"SE-SP-SP-SP-SP," 2 :"SE-SN-SN-SN-SN," 3 :"SE-SE-SP-SP-SP," 4 :"SE-SE-SN-SN-SN," 5 :"SE-SE-SE-SP-SP,"; 6 :"SE-SE-SE-SN-SN," 7 :"SE-SE-SP-SP-SN," and 8 :"SE-SE-SP-SN-SN;" where, e.g., "SE-SP-SP-SP-SP" means that agent 1 is an SE, while agents 2 to 5 are SPs. In Fig. 4 we illustrate the effect the agents' attitude has on the computed NE. We compare the cost at equilibrium obtained in a specific configuration with the "nominal" one, $\bar{\boldsymbol{x}}^*$, computed by considering the same (random) initial condition with $A(\cdot) = I$. Since none of the bars take negative values, we firstly observe that adopting some sort of non-standard (i.e., different from the averaging scheme) opponents' strategy reconstruction mechanism in NEPs with multiple equilibria appears beneficial in terms of cost minimization, for all the agents. This might however be related with the fact that the considered traffic problem does not include shared constraints coupling the decision variables of the agents (as observed in [15]). We further note that when SE agents are mixed with other behaviors, they can drive the NE computation, improving their performance (first four columns in Fig. 4). However, this seems only true when SE agents are in the minority. As soon as they are the majority (fifth and sixth columns in Fig. 4), or are in the same number of other type of agents (seventh and eighth columns in Fig. 4), it seems that the opponents may take advantage, improving their cost up to almost the 60%.

VI. Conclusion

We proposed two iterative schemes for NEPs in a partial-decision information setup that include possibly stubborn behavior of the agents. The latter are modeled with a state-dependent communication dynamics, which in our iterative schemes is alternated with an optimization step. Despite the underlying biased attitude, we established results on the convergence rate of the communication dynamics, which allowed us to prove convergence to an NE of the proposed protocols. We verified our theoretical findings on a numerical instance of a traffic model, where the state-dependent nature of the problem is key for capturing the drivers' individual perceptions of traffic based on personal experiences.

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