Spiderwebs and sharp Lp bounds for the Hardy–Littlewood maximal operator on Gromov hyperbolic spaces

Journal of Functional Analysis 291 (2026) 111553

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Spiderwebs and sharp L^p bounds for the Hardy–Littlewood maximal operator on Gromov hyperbolic spaces [☆]

Nikolaos Chalmoukis [a], Stefano Meda [a], Federico Santagati [b,*]

[a] Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, via R. Cozzi 53, I-20125 Milano, Italy
[b] Dipartimento di Scienze Matematiche "Giuseppe Luigi Lagrange", Politecnico di Torino, corso Duca degli Abruzzi 24, 10129 Torino, Italy

A R T I C L E I N F O

Article history:
Received 9 September 2025
Accepted 1 May 2026
Available online 8 May 2026
Communicated by Camil Muscalu

This paper is dedicated to the memory of Saverio Giulini

MSC:
05C63
42B25
43A99

Keywords:
Gromov hyperbolic space
Centred Hardy–Littlewood maximal functions
Rough isometries
Spiderweb

A B S T R A C T

In this paper we prove that if $1 < a \leq b < a^2$ and X is a locally doubling δ-hyperbolic complete connected length metric measure space with (a, b)-pinched exponential growth at infinity, then the centred Hardy–Littlewood maximal operator $\mathscr{M}$ is bounded on $L^p(X)$ for all $p > \varrho$, and it is of weak type (ϱ, ϱ), where $\varrho := \log_a b$. A key step in the proof is a new structural theorem for Gromov hyperbolic spaces with (a, b)-pinched exponential growth at infinity, consisting in a discretisation of X by means of certain graphs, introduced in this paper and called spiderwebs, with "good connectivity properties". Our result applies to trees with bounded geometry, and Cartan–Hadamard manifolds of pinched negative curvature, providing new boundedness results in these settings. The index ϱ is optimal in the sense that if $p < \varrho$, then there exists X satisfying the assumptions above such that $\mathscr{M}$ is not of weak type (p, p). Furthermore, if $b > a^2$, then there are examples of spaces X satisfying the

☆ Chalmoukis and Santagati are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). Santagati was supported by The Australian Research Council, grant DP220100285.

* Corresponding author.
E-mail addresses: nikolaos.chalmoukis@unimib.it (N. Chalmoukis), stefano.meda@unimib.it (S. Meda), federico.santagati@polito.it (F. Santagati).

https://doi.org/10.1016/j.jfa.2026.111553

assumptions above such that $\mathscr{M}$ bounded on $L^p(X)$ if and only if $p = \infty$.

1. Introduction

The purpose of this paper is to prove sharp L^p bounds for the centred Hardy–Littlewood (HL) maximal operator $\mathscr{M}$ on a comparatively large class of metric measure spaces, including all Cartan–Hadamard manifolds of pinched negative curvature and trees with pinched exponential volume growth. Our main result, Theorem 1.1, is stated below in this introduction. A key ingredient in its proof is a discretisation of Gromov hyperbolic spaces with pinched exponential volume growth leading to certain graphs with "good connectivity properties", introduced in this paper and called *spiderwebs*. One of the advantages of our strategy is that it relies on flexible techniques from Geometric Analysis that can be applied to quite diverse settings and hopefully to various related problems.

Suppose that (X, d, μ) is a connected metric measure space, and μ is a Borel measure. Denote by $B_r(x)$ the open ball with centre x and radius r, i.e., $B_r(x) := \{y \in X : d(x, y) < r\}$, and assume that the measure of each ball is positive and finite. For each locally integrable function f on X, consider its *centred* HL maximal function $\mathscr{M}f$, defined by

$$\mathscr{M}f(x) := \sup_{r>0} \frac{1}{\mu\big(B_r(x)\big)} \int_{B_r(x)} |f| \,\mathrm{d}\mu.$$

Clearly $\mathscr{M}$ and the local version $\mathscr{M}_0$ thereof, defined by

$$\mathscr{M}_0 f(x) := \sup_{0<r\leq 1} \frac{1}{\mu\big(B_r(x)\big)} \int_{B_r(x)} |f| \,\mathrm{d}\mu, \tag{1.1}$$

are bounded on $L^\infty(X)$. It is known, and not hard to see (see Proposition 6.1), that if μ is locally doubling (see Subsection 2.1), then $\mathscr{M}_0$ is of weak type $(1, 1)$, hence bounded on $L^p(X)$ for all p in $(1, \infty]$. Thus, $\mathscr{M}$ is bounded on $L^p(X)$ [resp. is of weak type (p, p)] for some p in $(1, \infty)$ if and only if the "global" maximal operator $\mathscr{M}_\infty$, defined by

$$\mathscr{M}_\infty f(x) := \sup_{r>1} \frac{1}{\mu\big(B_r(x)\big)} \int_{B_r(x)} |f| \,\mathrm{d}\mu, \tag{1.2}$$

is bounded on $L^p(X)$ [resp. is of weak type (p, p)].

Denote by J_X the interval of all p's such that $\mathscr{M}_\infty$ is bounded on $L^p(X)$. Standard arguments show that $J_X = (1, \infty]$, and $\mathscr{M}_\infty$ is of weak type $(1, 1)$, on all doubling metric measure spaces (see [11, Theorem 2.2]).

The situation is less simple, but much more interesting, in the case where μ is locally doubling, but not doubling. The way J_X depends on the properties of the metric measure space (X, d, μ) is rather subtle, and, we believe, not fully understood. We shall make some comments on this later in this introduction. First, we describe some important contributions to the problem of determining J_X.

The paper [28], which has been a source of inspiration for our previous investigations on the subject [8,14,20,15,19], is a landmark in this field. J.-O. Strömberg [28, Theorem p. 115] proved that if X is a symmetric space of the noncompact type, then $J_X = (1, \infty]$ and $\mathscr{M}_\infty$ is of weak type $(1, 1)$. J.-Ph. Anker, E. Damek and C. Yacoub [1, Corollary 3.22] extended this result to Damek–Ricci spaces. Strömberg also addressed the problem of determining J_X on Riemannian surfaces X with pinched negative Gaussian curvature, and stated the following result [28, Remark 3, p. 126]: if A and B are positive numbers such that $A \leq B < 2A$, then there exists a perturbed hyperbolic metric on the upper half plane with Gaussian curvature K satisfying $-B^2 \leq K \leq -A^2$ such that $J_X = (B/A, \infty]$: furthermore $\mathscr{M}_\infty$ is unbounded on $L^p(X)$ if $p < B/A$. For a full proof of this result see [19, Theorem 7.1], where it is also shown that if $B > 2A$, then J_X is reduced to the point ∞.

Strömberg's example leads us naturally to conjecture that if X is a Cartan–Hadamard Riemannian manifold with pinched negative sectional curvature, i.e., $-B^2 \leq K \leq -A^2$ for some positive constants A and B, and $A \leq B < 2A$, then J_X contains $(B/A, \infty]$. Incidentally, this holds true (see Corollary 6.3), and follows from our main result. Note that compact perturbations of X can significantly affect the bounds on the sectional curvature, although it is reasonable to believe that the conclusion remains the same. For this and related reasons it is desirable to rely on more "robust" sets of assumptions than those involving curvature bounds.

Other results somewhat related to the geometric framework considered by Strömberg can be found in [16,17,19] and in the papers cited therein.

In particular, [16] and [17] focus on an interesting class of conic manifolds X of the form $X_0 \times (0, \infty)$, where X_0 is a length metric measure space. The distance d on X is defined much as the hyperbolic distance on the upper half plane, with the metric d_0 on X_0 playing the role of the Euclidean distance on the real line (see [16, formula (1)]) and the measure μ being the product of the measure μ_0 on X_0 and of the measure $\mathrm{d}y/y^{N+1}$ on $(0, \infty)$ for some nonnegative constant N. H.-Q. Li proves that J_X is either $(p_0, \infty]$, where p_0 depends on the parameters describing the volume of balls on X, or is reduced to the point ∞. The index p_0 is optimal in the class of the conic manifolds considered. Related results are contained in [17].

The results in [19] corroborate the idea that appropriate "perturbations" of a given Riemannian metric preserve the L^p boundedness properties of $\mathscr{M}$. Specifically, [19, Theorem 3.1] and [19, Theorem 5.5] deal with conformal perturbations, and with the case of strictly quasi-isometric length spaces with "controlled local geometry".

Finally, in [14] the authors focus on the class $\Upsilon_{a,b}$ of all trees T of (a, b)-*bounded geometry*, i.e., trees in which every vertex has at least $a+1$ and at most $b+1$ neighbours:

the integers a and b are assumed to satisfy the inequality $2 \leq a \leq b$. They prove that if $b \leq a^2$, then $\mathscr{M}$ is bounded on $L^p(\mathrm{T})$ if $p > \varrho$, where $\varrho := \log_a b$ and it is of restricted weak type (ϱ, ϱ). The proof hinges on the sharp form of the Kunze–Stein phenomenon (see [7, Theorem 1]). These trees may be considered as discrete counterparts of Riemannian manifolds with (a, b)-pinched exponential growth at infinity (see (1.3) below).

We observe that the threshold index in the Riemannian case (see Corollary 6.3) and that we found in this discrete setting agree. Indeed, on the one hand, the volume of balls in a tree T with (a, b)-bounded geometry satisfies the estimate

$$a^r \leq \mu_{\mathrm{T}}\big(B_r(x)\big) \leq 3\, b^r \qquad \forall x \in \mathrm{T} \quad \forall r \in \mathbb{N},$$

where μ_{T} denotes the counting measure on the vertices of T. On the other hand, by comparison results [25, Corollary 3.2 (ii)], if an n dimensional Cartan–Hadamard Riemannian manifold M satisfies $-B^2 \leq K \leq -A^2$, then

$$c\, \mathrm{e}^{(n-1)Ar} \leq \mu\big(B_r(x)\big) \leq C\, \mathrm{e}^{(n-1)Br} \qquad \forall x \in M \quad \forall r \geq 1,$$

where c and C are positive constants, depending only on n, and μ denotes the Riemannian measure on M. Thus, if we set $a := \mathrm{e}^{(n-1)A}$ and $b := \mathrm{e}^{(n-1)B}$, then the condition $B < 2A$ transforms to $b < a^2$, the volume bounds in the discrete and in the continuous case agree, and the threshold index B/A in the continuous case can be written as $(\log b)/(\log a)$, which agrees with ϱ.

An interesting observation is that the results in [19,14] are stable under *strict rough isometries* (see Definition 2.3 below, [14, Theorem 5.4] and [19, Theorem 5.5]). These are rough isometries in the sense of M. Kanai [12] that preserve the exponential rate of the volume growth of balls when the radius tends to infinity. This suggests that it may be worth looking for sets of assumptions that are “stable” under these maps. Rough isometries are also known as quasi-isometries (see, for instance, [10, Section 7.2]), and strict rough isometries are sometimes referred to as 1-quasi-isometries in the literature.

Suppose that δ, a and b are nonnegative numbers satisfying the condition $1 < a \leq b$. We denote by $\mathscr{X}^{\delta}_{a,b}$ the class of all connected noncompact complete δ-hyperbolic length spaces (X, d) (here d is the intrinsic metric associated to the length structure on X so that any pair of points in X is connected by a shortest path; see [4, Definition 8.4.1]), endowed with a locally doubling Borel measure μ with (a, b)-*pinched exponential growth at infinity*. By this we mean that for some a and b with $1 < a \leq b$ there exist positive constants c and C such that

$$c\, a^r \leq \mu\big(B_r(x)\big) \leq C\, b^r \qquad \forall x \in X \quad \forall r \in [1, \infty). \tag{1.3}$$

For notational convenience, set $\varrho := \log_a b$. Our main result is the following.

Theorem 1.1. *Suppose that $1 < a \leq b < a^2$ and that δ is a nonnegative number. If X belongs to the class $\mathscr{X}_{a,b}^{\delta}$, then the maximal operator $\mathscr{M}$ is bounded on $L^p(X)$ for all $p > \varrho$, and it is of weak type (ϱ, ϱ).*

The index ϱ is optimal in the sense that if $\varrho > 1$ and $p < \varrho$, then there exists a length space X in the class $\mathscr{X}_{a,b}^{\delta}$ such that $\mathscr{M}$ is not of weak type (p, p) for every $p < \varrho$. For instance, one may take Strömberg's counterexample mentioned above as X.

If $b > a^2$, then the conclusion of Theorem 1.1 fails, for there are examples of length spaces X in $\mathscr{X}_{a,b}^{\delta}$ such that $\mathscr{M}$ is bounded on $L^p(X)$ if and only if $p = \infty$: see, for instance, [19, Theorem 7.1 (ii)]. Another example is given by the natural metric tree ($\mathbb{R}$-tree) associated to the tree $\mathfrak{S}_{a,b}$, with $b > a^2$, considered in [14, Proposition 3.3 (ii)]. This metric tree (where each edge is isometric to the interval $[0, 1]$) belongs to $\mathscr{X}_{a,b}^{0}$, and $\mathscr{M}_\infty$ is bounded on $L^p(\mathfrak{S}_{a,b})$ if and only if $p = \infty$.

A noteworthy consequence of Theorem 1.1 is that if M is a Cartan–Hadamard manifold with sectional curvatures K_π satisfying the bound

$$-B^2 \leq K_\pi \leq -A^2$$

where $0 < A \leq B < 2A$, then the centred HL maximal operator $\mathscr{M}$ is bounded on $L^p(M)$ for all $p > B/A$, and it is of weak type $(B/A, B/A)$.

In particular, Theorem 1.1 improves Lohoué's results [18] in arbitrary dimensions and, when applied to Strömberg's counterexample, yields the endpoint result that $\mathscr{M}$ is of weak type (ϱ, ϱ), a fact which was previously unknown.

It is well known that Damek–Ricci spaces X are Gromov hyperbolic [13, Theorem 4.8] with pinched exponential growth, so that Theorem 1.1 implies the result of Anker et al. [1, Corollary 3.22] that $J_X = (1, \infty]$ and $\mathscr{M}$ is of weak type $(1, 1)$.

Theorem 1.1 also improves previous results on trees with (a, b)-bounded geometry satisfying $a < b < a^2$ [14, Theorem 3.2 (ii)] in two ways: it applies to a larger class of trees ((a, b)-bounded geometry implies (a, b)-pinched volume exponential growth, but not conversely), and it yields a better endpoint estimate (weak type (ϱ, ϱ) in place of restricted weak type (ϱ, ϱ)).

We observe also that higher rank noncompact symmetric spaces are not covered by Theorem 1.1, because they are not Gromov hyperbolic.

Our analysis hinges on the (new) notion of spiderweb, introduced in Definition 2.9. Loosely speaking, a spiderweb is a graph associated to a rooted tree that satisfies certain connectivity properties. The strategy of proof of Theorem 1.1 relies upon the following three facts:

(i) if X belongs to $\mathscr{X}_{a,b}^{\delta}$, then X is strictly roughly isometric to a spiderweb $\widehat{\Gamma}$, endowed with the graph distance $d_{\widehat{\Gamma}}$. Moreover, the measure of the metric balls in $\widehat{\Gamma}$ (with respect to the counting measure on the set of its vertices) satisfies the bound (1.3);

(ii) if $a \leq b < a^2$, then the maximal operator $\mathscr{M}$ is bounded on $L^p(\widehat{\Gamma})$ for $p > \varrho$, and it is of weak type (ϱ, ϱ);
(iii) the strict rough isometry in (i) above can be used to "transfer" to X the boundedness properties of $\mathscr{M}$ on $\widehat{\Gamma}$ described in (ii).

It is already known that δ-hyperbolic spaces are strictly roughly isometric to graphs [5, Section 5]. However, we emphasise that it is highly nontrivial to show that the approximating graph $\widehat{\Gamma}$ can be chosen to be a spiderweb: this richer structure is needed in the proof of (ii). Step (iii) is a variant of [14, Theorem 5.4] and of [19, Theorem 5.5], and reflects the fact that L^p boundedness properties of the maximal operator $\mathscr{M}$ depend essentially on the "large scale geometry" of X.

This paper is organised as follows. Section 2 is devoted to background material and preliminary results. Section 3 contains more information and results concerning spiderwebs associated to rooted trees. In Section 4 we extend [14, Theorem 3.2 (i)] concerning the L^p-boundedness of the maximal operator $\mathscr{M}$ on trees with bounded geometry to spiderwebs satisfying mild volume growth conditions. In Section 5 we prove that every connected noncompact complete δ-hyperbolic space X (with distance d) is strictly roughly isometric to a δ-hyperbolic spiderweb $\widehat{\Gamma}$, with graph distance $d_{\widehat{\Gamma}}$. Finally, in Section 6 we prove of our main result, and derive some consequences thereof.

We use the "variable constants convention", and denote by c and C constants whose value may vary from place to place and may depend on any factors quantified (implicitly or explicitly) before its occurrence, but not on factors quantified after.

2. Background and preliminary results

Suppose that (X, d, μ) is a metric measure space, where μ is a Borel measure on (X, d), and denote by $\mathscr{B}$ the family of all open balls in X. For each B in $\mathscr{B}$ we denote by r_B the radius of B. Given a ball $B_r(x)$ and a positive number k, sometimes we write $kB_r(x)$ for the ball $B_{kr}(x)$. For each s in $\mathbb{R}^+$, we denote by $\mathscr{B}_s$ the family of all balls B in $\mathscr{B}$ such that $r_B \leq s$.

2.1. The local doubling property

Assume that $0 < \mu(B) < \infty$ for every B in $\mathscr{B}$. We say that the metric measure space X possesses the *local doubling property* (LDP) if for every s in $\mathbb{R}^+$ there exists a constant L_s such that

$$\mu\big(2B_r(x)\big) \leq L_s\, \mu\big(B_r(x)\big) \qquad \forall x \in X \quad \forall r \leq s. \tag{2.1}$$

Remark 2.1. The LDP implies that for each $\tau \geq 1$ and for each s in $\mathbb{R}^+$ there exists a constant C such that

$$\mu\big(B'\big) \leq C\, \mu(B) \tag{2.2}$$

for each pair of balls B and B', with $B \subset B'$, B in $\mathscr{B}_s$, and $r_{B'} \leq \tau\, r_B$. We shall denote by $L_{\tau,s}$ the smallest constant for which (2.2) holds.

Indeed, consider the balls $B_r(x)$ and $B_{r'}(x')$, where $r \leq s$ and $r' \leq \tau\, r$, and assume that $B_r(x)$ is contained in $B_{r'}(x')$. Then the triangle inequality implies that $B_{r'}(x')$ is contained in the ball with centre x and radius $2r'$. Observe that

$$2r' \leq 2\,\tau\, r,$$

and denote by k the nonnegative integer such that $2^{k-1} \leq 2\tau < 2^k$. Then the ball $B_{r'}(x')$ is contained in $B_{2^k r}(x)$. The monotonicity of the measure and the local doubling condition imply that

$$\mu\big(B_{r'}(x')\big) \leq \mu\big(B_{2^k r}(x)\big) \leq L_{2^{k-1}s}\; \mu\big(B_{2^{k-1}r}(x)\big).$$

By iterating this argument we find that

$$\mu\big(B_{r'}(x')\big) \leq L_{2^{k-1}s} \cdots L_s\; \mu\big(B_r(x)\big).$$

Therefore

$$L_{\tau,s} \leq \big(L_{2\tau s}\big)^{2\tau}.$$

Remark 2.2. Observe that if X possesses the LDP and satisfies the growth condition (1.3), then the condition $0 < \mu(B) < \infty$ is automatically satisfied for every B in $\mathscr{B}$. Indeed, the monotonicity of μ and the right hand inequality in (1.3) imply that $\mu(B)$ is finite for every ball B.

Furthermore if $r_B > 1$, then $\mu(B) \geq c\, a^{r_B} > 0$ by the left hand inequality in (1.3). If, instead $r_B < 1$, and the centre of B is x, then

$$c\, a \leq \mu\big((1/r_B)\, B_{r_B}(x)\big) \leq L_{1/r_B, r_B}\, \mu\big(B_{r_B}(x)\big),$$

as required.

2.2. Rough isometries

A central notion in our investigation is the following.

Definition 2.3. Suppose that X and X' are two metric spaces, with distances d and d', respectively, and that λ and β are nonnegative numbers with $\lambda \geq 1$. A map $\varphi : X \to X'$ is a (λ, β)-rough isometry if

$$\frac{1}{\lambda}\, d(x,y) - \beta \leq d'\big(\varphi(x), \varphi(y)\big) \leq \lambda\, d(x,y) + \beta \qquad \forall x, y \in X$$

and sup $\{d'(\varphi(X), x') : x' \in X\} < \infty$. If $\lambda = 1$, then φ is called a *strict* β*-rough isometry* (or simply a strict rough isometry): in this case

$$d(x, y) - \beta \le d'(\varphi(x), \varphi(y)) \le d(x, y) + \beta \qquad \forall x, y \in X. \tag{2.3}$$

As mentioned in the introduction, rough isometries are also known as quasi-isometries (see, for instance, [10, Section 7.2]).

2.3. Gromov hyperbolicity

Suppose that (X, d) is a metric space. The *Gromov product* $(y, z)_x$ of two points y and z in X with respect to a third point x in X is defined by the formula:

$$(y, z)_x := \frac{1}{2}\left(d(x, y) + d(x, z) - d(y, z)\right).$$

Definition 2.4. Suppose that δ is a nonnegative number. The metric space (X, d) is δ-*hyperbolic* if and only if the following *four points condition* is fulfilled: for every x, y, z and w in X

$$(x, z)_w \ge \min\left((x, y)_w, (y, z)_w\right) - \delta. \tag{2.4}$$

The space (X, d) is called *Gromov hyperbolic* if it is δ-hyperbolic for some δ.

Note that if (2.4) is satisfied for all points x, y and z and one fixed base point w_0, then it is satisfied for all base points w with a constant 2δ (see, for instance, [6, pp. 2–3]). Thus, in order to check that a space is Gromov hyperbolic, it suffices to check the hyperbolicity condition (2.4) for one fixed base point.

In the case where X is a complete length space with intrinsic metric d, it is well known that (X, d) is δ-hyperbolic if and only if there exists δ' such that every geodesic triangle in X is δ'-*slim*, i.e., each of its sides is contained in the δ'-neighbourhood of the union of the other two: see, for instance, [3, Chapter III, Proposition 1.22].

For a length space X when we say that X is δ-hyperbolic we refer to the constant δ appearing in the Definition 2.4; if we want to refer to the constant δ' that controls the slimness of triangles in X, we say that X is a δ'-hyperbolic *length* space. In the cases where we are interested only in the slimness constant, we shall often call it δ instead of δ'.

Remark 2.5. We shall often use the following simple observation. Suppose that (X, d) is a δ-hyperbolic length space, and that oxy is a geodesic triangle in X, with edges $[ox]$, $[oy]$ and $[xy]$. Then there exists a point p in $[xy]$ such that

$$\min\left(d(p, [ox]), d(p, [oy])\right) < \delta.$$

Indeed, denote by E and F the set of all points in $[xy]$ at distance less than δ from $[ox]$ and from $[oy]$, respectively. Observe that E and F are nonempty, for E contains x and F contains y. Since X is a δ-hyperbolic length space, the δ-neighbourhood of $[ox] \cup [oy]$ contains $[xy]$, so that $E \cup F = [xy]$. Since $[xy]$, E and F are open (in the relative topology induced by d) and $[xy]$ is connected, $E \cap F$ cannot be empty. Any point in $E \cap F$ satisfies the inequality above.

Definition 2.6. Suppose that (X, d) is a length space and that ω and K are nonnegative numbers, with $\omega \geq 1$. We say that a path γ in X is an (ω, K)-*quasigeodesic* if for every pair of points p and q in γ

$$\ell_X([pq]) \leq \omega\, d(p, q) + K;$$

here $\ell_X([pq])$ denotes the length in (X, d) of the segment in γ with endpoints p and q. In particular, $(1, 0)$-quasigeodesics are just shortest paths.

Assume, in addition, that (X, d) is a δ-hyperbolic length space. We say that X is *geodesically stable* if for every ω in $[1, \infty)$ and each $K \geq 0$ there exists a constant $D_{\omega,K}$, depending on δ, ω, and K such that each (ω, K)-quasigeodesic γ belongs to the $D_{\omega,K}$-neighbourhood of any geodesic in X joining the endpoints of γ.

Next, we recall the well known Morse Lemma in hyperbolic geometry (see, for instance, [2, Proposition 3.1]).

Lemma 2.7. *Any δ-hyperbolic length space is geodesically stable.*

One of the nice features of Gromov hyperbolicity for length spaces is that it is preserved under rough isometric embeddings. Recall the following result (for its proof see, for instance, [3, Chapter III, Theorem 1.9]).

Theorem 2.8. *Suppose that X and X' are complete length metric spaces and that $\varphi : X' \to X$ is a (λ, β)-roughly isometric embedding. If X is a δ-hyperbolic length space, then X' is a δ'-hyperbolic length space, where δ' depends on δ, λ and β.*

We observe that the conclusion of Theorem 2.8 fails if we omit the assumption that X is a length space: see [9, Example 13, p. 89].

2.4. Graphs, trees and spiderwebs

Suppose that Γ is an undirected connected graph (without loops and multiedges). If x and y are vertices in Γ connected by an edge, then we say that x and y are *neighbours*, and write $x \sim y$. We shall always denote by μ_Γ the *counting measure* on the vertices of Γ.

In this paper we shall consider only graphs Γ such that for each vertex v in Γ the number $\nu(v)$ of its neighbours is finite: $\nu(v)$ is called the *valence* of v. If

$$\sup\{\nu(v) : v \in \Gamma\} < \infty, \tag{2.5}$$

then we say that Γ has *bounded valence.*

A path γ in Γ joining two vertices v and w is a finite sequence $[x_0 := v, x_1 \ldots, x_{N-1}, x_N =: w]$ of vertices such that $x_j \sim x_{j+1}$ for every j in $\{0, \ldots, N-1\}$. The *length* of γ is defined to be N. Since Γ is connected, given two vertices v and w, there exists at least one path joining them. The graph distance d_Γ between v and w is just the minimum of the lengths of all paths joining v and w.

A *tree* T is just a graph as above without circuits (sometimes called loops in the literature). A *rooted* tree is a tree T with a distinguished point o, called the root of the tree.

We now introduce the notion of spiderweb, which is central to our investigation. Consider a rooted tree T with root o. Given a vertex x in $\mathrm{T} \setminus \{o\}$, we write $p(x)$ for its predecessor, i.e., the unique neighbour y of x such that $d_\mathrm{T}(y, o) = d_\mathrm{T}(x, o) - 1$. For every x in T we set $p^0(x) := x$ and for every positive integer k we inductively define $p^k(x) = p(p^{k-1}(x))$. The (possibly empty) set of the neighbours of x such that $d_\mathrm{T}(y, o) = d_\mathrm{T}(x, o)+1$ is denoted by $s(x)$. Each vertex in $s(x)$ (if any) is called a *successor* of x. Similarly, for every x in T we set $s^1(x) := s(x)$ and, for every $r \geq 2$,

$$s^r(x) := \bigcup_{y \in s^{r-1}(x)} s(y).$$

Suppose that k is a nonnegative integer. We set

$$\Sigma_k := \big\{x \in \mathrm{T} : d_\mathrm{T}(x, o) = k\big\}. \tag{2.6}$$

Σ_k is called the *sphere* with centre o and radius k in the rooted tree T. We say that each point in Σ_k has *level* k. The *level* of a vertex x is denoted by $h(x)$. Note that $h(x)$ increases as we move away from o in T.

Definition 2.9. A *spiderweb* $\widehat{\Gamma}$ is a graph without loops (sometimes called self-loops in the literature) obtained from a rooted tree T by adding a (possibly empty) new set of edges, according to the following rules:

(i) two vertices from different tree levels are never joined in $\widehat{\Gamma}$ by a new edge;
(ii) if two vertices of the same level are connected by an edge, then their predecessors either coincide or they are neighbours.

A *quasi-spiderweb* Γ is a graph defined much as a spiderweb, with the only difference that condition (ii) above is replaced by the following weaker condition:

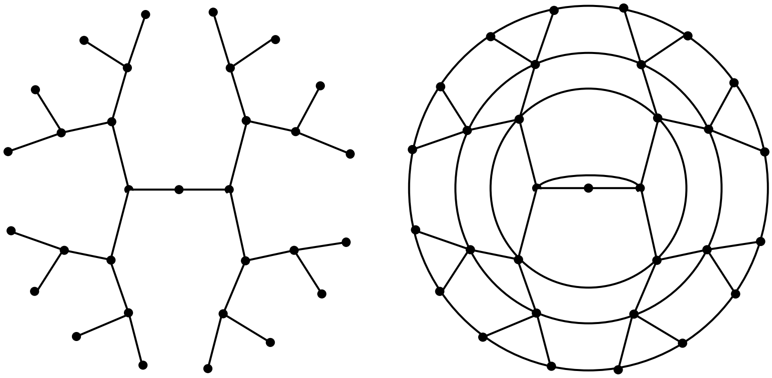

Fig. 1. Part of the rooted binary tree and of the associated dyadic spiderweb $\widehat{\Gamma}_2$.

(ii)′ there exists a positive integer m such that if any two vertices of Σ_n, with $n \geq m$ are connected by an edge, then for every integer k in $\{m, \dots, n\}$, their k^{th} predecessors either coincide or they are neighbours.

Note that a rooted tree is a very special spiderweb in which vertices belonging to the same level are never connected by an edge. We warn the reader that a spiderweb may very well be a nonplanar graph.

The prototype of spiderweb is the so-called dyadic spiderweb $\widehat{\Gamma}_2$, which we briefly discuss now. Let $\mathbb{F}_2$ the free monoid on two generators, i.e., the set of finite length words on two letters 0 and 1. Let an edge connect two words if and only if one is obtained by the other by adding one letter in the end (tree edges) or if two words have the same length and one is the immediate successor of the other in the lexicographic order or if one is the all 0's and the other is the all 1's word.

In Fig. 1 you can see part of the rooted binary tree and of the associated dyadic spiderweb $\widehat{\Gamma}_2$. There are two natural metrics on a spiderweb $\widehat{\Gamma}$: the tree metric d_{T} and the graph metric $d_{\widehat{\Gamma}}$. Obviously $d_{\mathrm{T}} \geq d_{\widehat{\Gamma}}$. Notice that d_{T} and $d_{\widehat{\Gamma}}$ may be nonequivalent metrics. For instance, consider the points $x_n := 0\underbrace{1\cdots 1}_{n\text{times}}$ and $y_n := 1\overbrace{0\cdots 0}^{n\text{times}}$ in $\widehat{\Gamma}_2$. Then

$$d_{\mathrm{T}}(x_n, y_n) = 2n + 2 \qquad \text{and} \qquad d_{\widehat{\Gamma}}(x_n, y_n) = 1.$$

2.5. Metric graphs embedded in length spaces

In what follows we shall encounter connected graphs Γ (without loops and multiedges) whose set of vertices is a discrete subset of a metric measure space (X, d, μ). In particular, we shall be concerned with the case where X is a complete length space with length function ℓ_X, d is the associated intrinsic metric [4, Definition 8.4.1], and μ is a locally doubling Borel measure on (X, d). For each pair of neighbours x and y in Γ, consider a geodesic $\gamma_{x,y}$ in the length space X connecting them.

We denote by $\widetilde{\Gamma}$ the *metric graph* defined as follows. The vertices of $\widetilde{\Gamma}$ agree with those of Γ; the edge in $\widetilde{\Gamma}$ connecting two neighbours x and y in Γ is the geodesic segment $\gamma_{x,y}$ chosen above. If z and w are points in $\gamma_{x,y}$, then we set

$$d_{\widetilde{\Gamma}}(z, w) := d(z, w). \tag{2.7}$$

In particular, $d_{\widetilde{\Gamma}}(x, y) = \ell_X(\gamma_{x,y})$. Now, if v and w are any two points in $\widetilde{\Gamma}$ (thus, v and w may be either vertices of Γ or points in the interior of geodesic segments), then we define $d_{\widetilde{\Gamma}}(v, w)$ to be the infimum of the lengths of all paths in $\widetilde{\Gamma}$ joining v and w. Clearly $d_{\widetilde{\Gamma}}$ is a distance on $\widetilde{\Gamma}$.

Definition 2.10. We say that the metric graph $\widetilde{\Gamma}$ associated to the graph Γ has *bounded geometry* if Γ has bounded valence and

$$0 < \inf \{\ell_X(\sigma) : \sigma \text{ edge in } \widetilde{\Gamma}\} \quad \text{and} \quad \sup \{\ell_X(\sigma) : \sigma \text{ edge in } \widetilde{\Gamma}\} < \infty.$$

Next we describe the *metric graph* Γ_0 associated to the connected graph Γ. Loosely speaking, Γ_0, as a set, agrees with $\widetilde{\Gamma}$, but each edge of Γ_0 (the image of a geodesic segment in X) is now declared to be isometric to the interval $[0, 1]$. More precisely, for each pair x and y of neighbours in Γ, consider the geodesic $\gamma_{x,y}$ in X joining x and y (which is an "edge" in $\widetilde{\Gamma}$), and its arc length parametrisation $s : [0, \ell_X(\gamma_{x,y})] \to \gamma_{x,y}$ such that $s(0) = x$ and $s\big(\ell_X(\gamma_{x,y})\big) = y$. Define

$$\iota(t) := s\big(t \cdot \ell_X(\gamma_{x,y})\big) \qquad \forall t \in [0, 1],$$

and set

$$d_{\Gamma_0}\big(\iota(t_1), \iota(t_2)\big) := |t_1 - t_2| \qquad \forall t_1, t_2 \in [0, 1]. \tag{2.8}$$

There is a natural notion of admissible paths in Γ_0 and a corresponding notion of length. Now d_{Γ_0} is just the metric associated to this length structure.

Remark 2.11. It is straightforward to check that if x and y are neighbours in Γ, then

$$1 = d_\Gamma(x, y) = d_{\Gamma_0}(x, y) = \frac{1}{\ell_X(\gamma_{x,y})}\, d_{\widetilde{\Gamma}}(x, y). \tag{2.9}$$

If $\widetilde{\Gamma}$ has bounded geometry, then there exist positive constants A_1 and A_2 such that

$$A_1 \leq \ell_X(\gamma_{x,y}) \leq A_2, \tag{2.10}$$

so that

$$A_1\, d_\Gamma(x,y) \le d_{\widetilde\Gamma}(x,y) \le A_2\, d_\Gamma(x,y) \qquad \forall x,y\in\Gamma\colon x\sim y.$$

Proposition 2.12. *Suppose that Γ, Γ_0 and $\widetilde\Gamma$ are as described above. If $\widetilde\Gamma$ has bounded geometry and $(\widetilde\Gamma, d_{\widetilde\Gamma})$ is a δ-hyperbolic length space for some nonnegative number δ, then the following hold:*

(i) $(\widetilde\Gamma, d_{\widetilde\Gamma})$ *and* (Γ_0, d_{Γ_0}) *are bilipschitz equivalent. More precisely*

$$A_1\, d_{\Gamma_0}(v,w) \le d_{\widetilde\Gamma}(v,w) \le A_2\, d_{\Gamma_0}(v,w) \qquad \forall v,w\in\widetilde\Gamma,$$

where A_1 and A_2 are as in (2.10);

(ii) (Γ_0, d_{Γ_0}) *is a δ'-hyperbolic length space for some nonnegative number δ';*

(iii) (Γ, d_Γ) *is a Gromov hyperbolic metric space (the four points condition holds).*

Proof. First we prove (i). Clearly, if x and y are neighbours in Γ, then

$$d_{\Gamma_0}(x,y) = 1 = \frac{1}{\ell_X(\gamma_{x,y})}\, d_{\widetilde\Gamma}(x,y).$$

If $\gamma_{x,y}$ is the edge in $\widetilde\Gamma$ joining x and y, denote by $s:[0,\ell_X(\gamma_{x,y})]\to\gamma_{x,y}$ is its arc length parametrisation (with $s(0)=x$ and $s\big(\ell_X(\gamma_{x,y})\big)=y$). Write L in place of $\ell_X(\gamma_{x,y})$ for short. Then for each pair of numbers t_1 and t_2 in $[0,L]$, we have that

$$\begin{aligned} d_{\Gamma_0}\big(s(t_1),s(t_2)\big) &= d_{\Gamma_0}\Big(s\Big(\frac{t_1}{L}\cdot L\Big), s\Big(\frac{t_2}{L}\cdot L\Big)\Big)\\ &= \frac{1}{L}\,|t_1-t_2|\\ &= \frac{1}{L}\, d_{\widetilde\Gamma}\big(s(t_1),s(t_2)\big); \end{aligned}$$

we have used (2.8) in the second to last equality above.

Since, by assumption, $\widetilde\Gamma$ has bounded geometry, (2.10) holds for every pair x and y of neighbours. As a consequence, we see that for each pair of points v and w belonging to an edge in Γ_0 the following estimate holds

$$A_1\, d_{\Gamma_0}(v,w) \le d_{\widetilde\Gamma}(v,w) \le A_2\, d_{\Gamma_0}(v,w).$$

It is straightforward to check that this estimate extends to all pairs of points v and w in $\widetilde\Gamma$, thereby concluding the proof of (i).

Observe that (ii) follows from (i) and [4, Theorem 8.4.16].

Finally, we prove (iii). Since (Γ_0, d_{Γ_0}) is a δ'-hyperbolic length space, in particular a δ'-hyperbolic space, the four point condition (2.4) holds (with δ' in place of δ). Since the metric graph d_Γ on Γ agrees with the restriction of d_{Γ_0} to Γ, the four points condition holds in (Γ, d_Γ). Thus, (Γ, d_Γ) is a δ'-hyperbolic metric space, as required. □

Caveat 2.13. *In what follows we shall consistently use the notation concerning graphs introduced in this section. In particular, given a geodesic metric space* (X, d), Γ *will always denote an abstract graph, whose vertices are points of* X*, endowed with the graph distance* d_Γ*. We associate to* Γ *the metric graphs* $(\widetilde{\Gamma}, d_{\widetilde{\Gamma}})$ *and* (Γ_0, d_{Γ_0}) *as follows:*

- (i) $\widetilde{\Gamma}$ *is obtained from* Γ *by connecting any pair* x *and* y *of neighbours in* Γ *by a geodesic* $\gamma_{x,y}$ *in* X *joining* x *and* y*, and declaring that* $d_{\widetilde{\Gamma}}(z,w) = d(z,w)$ *for every pair of points* z *and* w *in* $\gamma_{x,y}$*;*
- (ii) Γ_0 *agrees with* $\widetilde{\Gamma}$ *as a set, but* $d_{\Gamma_0}(z,w) = d(z,w)/\ell_X(\gamma_{x,y})$*, where* x*,* y*,* z *and* w *are as in* (i)*. In particular,* $d_{\Gamma_0}(x,y) = 1$*.*

The symbol $\widehat{\Gamma}$ *will always denote a spiderweb.*

Notation 2.14. Suppose that Γ and Γ_0 are as above, and denote by o a distinguished point in Γ. Assume that x belongs to $\Gamma_0 \setminus \Gamma$ and that γ is a geodesic in Γ_0 starting at o and containing x. Denote by $\lfloor x \rfloor$ the vertex on the segment $[ox] \subset \gamma$ closest to x. Observe that

$$d_{\Gamma_0}(\lfloor x \rfloor, o) \leq d_{\Gamma_0}(x, o) \leq d_{\Gamma_0}(\lfloor x \rfloor, o) + 1; \tag{2.11}$$

If x is in Γ, then we set $x := \lfloor x \rfloor$. The definition of $\lfloor x \rfloor$ depends on o and the choice of the geodesic γ; in order to avoid cumbersome formulae we shall suppress this dependence in our notation.

3. More on spiderwebs

Suppose that $\widehat{\Gamma}$ is a spiderweb with metric $d_{\widehat{\Gamma}}$. We say that a geodesic γ in $\widehat{\Gamma}$ is *ascending* if it is of the form $[x, p(x), \ldots, p^n(x)]$, where x is a vertex and $n \leq h(x)$. Similarly we say that a geodesic γ in $\widehat{\Gamma}$ is *descending* if it is of the form $[y_0, y_1, \ldots, y_n]$, where $y_j = p(y_{j+1})$ for every j in $\{0, \ldots, n-1\}$. Finally, we say that a geodesic γ connecting two points belonging to the same level Σ_k (see (2.6) for the notation) is *horizontal* if every vertex in γ belongs to Σ_k.

Our strategy to prove L^p bounds for $\mathscr{M}_\infty$ on spiderwebs will require estimates of the volume of balls in $\widehat{\Gamma}$ with respect to the graph distance $d_{\widehat{\Gamma}}$. As a preliminary step, we describe some of the geodesics in $\widehat{\Gamma}$. Interestingly, if the metric space $(\widehat{\Gamma}, d_{\widehat{\Gamma}})$ is Gromov hyperbolic, then it turns out that these geodesics have a form similar to that of the geodesics in the hyperbolic plane.

Definition 3.1. Suppose that $(\widehat{\Gamma}, d_{\widehat{\Gamma}})$ is a spiderweb. We say that a geodesic γ in $\widehat{\Gamma}$ is *standard* if it is the union of an ascending geodesic, a horizontal geodesic and a descending geodesic. Each of these three geodesic components may be reduced to a point.

Proposition 3.2. *Suppose that* $(\widehat{\Gamma}, d_{\widehat{\Gamma}})$ *is a spiderweb. The following hold:*

(i) *for each pair of vertices x and y in $\widehat{\Gamma}$ there exists a standard geodesic joining them;*
(ii) *if $(\widehat{\Gamma}, d_{\widehat{\Gamma}})$ is δ-hyperbolic for some nonnegative number δ, then the length of the horizontal part of any standard geodesic is at most $4\delta + 1$.*

Proof. First we prove (i). Suppose that γ_0, of the form $[x, x_1, \ldots, x_{\ell-1}, y]$, is a shortest path joining x and y. For convenience, set $x_0 := x$ and $x_\ell := y$. Each point in γ_0 belongs to one of the sets E_1 and E_2, defined by

$$E_1 := \{x_j \in \gamma_0 \setminus \{y\} : h(x_j) \neq h(x_{j+1})\},$$
$$E_2 := \{x_j \in \gamma_0 \setminus \{y\} : h(x_j) = h(x_{j+1})\} \cup \{y\}.$$

Observe that $\ell(\gamma_0) = |E_1| + |E_2| - 1$.

We shall prove that there exists a path γ of the form described in the statement of the proposition such that $\ell(\gamma) \leq \ell(\gamma_0)$.

Denote by n the smallest nonnegative integer k such that Σ_k (see (2.6) for the notation) has nonempty intersection with γ_0. Then $h(v) \geq n$ for every v in γ_0.

Since γ_0 starts at x and has nonempty intersection with Σ_n, it must "move towards o" at least $h(x) - n$ steps; here o denotes the root of the tree T associated to $\widehat{\Gamma}$. Similarly, since γ_0 ends at y, after intersecting Σ_n, it must "move away from o" at least $h(y) - n$ steps. Therefore

$$|E_1| \geq h(x) + h(y) - 2n. \tag{3.1}$$

Now, given a vertex v in γ_0, the point $p^{h(v)-n}(v)$ belongs to Σ_n, it is denoted by $\pi(v)$, and it is called the *projection* of v onto Σ_n.

Note the following two elementary facts:

(a) if x_j belongs to E_1, then $\pi(x_j) = \pi(x_{j+1})$, for $x_j = p(x_{j+1})$ if $h(x_j) > h(x_{j+1})$ and $p(x_j) = x_{j+1}$ if $h(x_j) < h(x_{j+1})$;
(b) if x_j and x_k are two consecutive elements in E_2 (hence $j < k$), then one of the following cases occurs:
 (b1) x_k agrees with x_{j+1} (i.e., x_j and x_k are consecutive points in γ_0) and therefore $h(x_j) = h(x_k)$;
 (b2) x_k is a successor of x_{j+1};
 (b3) x_k is a predecessor of x_{j+1}.
 Since $\widehat{\Gamma}$ is a spiderweb, $\pi(x_j)$ and $\pi(x_k)$ either agree or are neighbours, so that their distance in $\widehat{\Gamma}$ is at most 1.

Now, (a) implies that $\pi(\gamma_0)$ agrees with $\pi(E_2)$. Denote by $z_1, \ldots, z_p$ the points in E_2, and consider $\pi(z_1), \ldots, \pi(z_p)$, which is an enumeration of the points of $\pi(E_2)$. We single out an ordered subset $\{\xi_1, \ldots, \xi_q\}$ of $\{\pi(z_1), \ldots, \pi(z_p)\}$, as follows. Set $\xi_1 := \pi(z_1)$, and

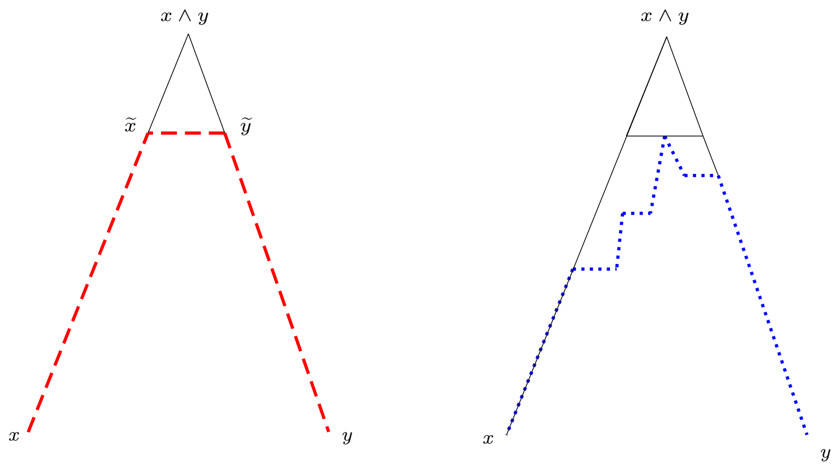

Fig. 2. Dashed lines represent the standard geodesic from x to y, while dotted lines represent an alternative path.

$\xi_2 := \pi(z_j)$, where j is the first integer greater than 1 such that $\pi(z_j)$ is distinct from ξ_1. Notice that ξ_1 and ξ_2 are neighbours, by (b) above. Then proceed iteratively.

Clearly $q \leq |\pi(E_2)|$. Furthermore $\widetilde{\gamma}_0 := [\xi_1, \ldots, \xi_q]$ is a path in $\widehat{\Gamma}$ contained in Σ_n connecting $\pi(x)$ and $\pi(y)$. Note that

$$\ell(\widetilde{\gamma}_0) \leq |\pi(E_2)| - 1 \leq |E_2| - 1. \tag{3.2}$$

Now denote by γ the path joining x and y consisting of the ascending geodesic $[x, p(x), \ldots, \pi(x)]$, the horizontal path $\widetilde{\gamma}_0$ and the descending geodesic $[\pi(y), \ldots, p(y), y]$. Clearly (3.1) and (3.2) imply that

$$\ell(\gamma) = h(x) + h(y) - 2n + \ell(\widetilde{\gamma}_0) \leq |E_1| + |E_2| - 1 = \ell(\gamma_0).$$

Thus, γ is a geodesic, as required. (See Fig. 2.)

Next we prove (ii). Suppose that γ is a standard geodesic joining x and y. The horizontal part $\widetilde{\gamma}$ of γ is a geodesic contained in Σ_n for some nonnegative integer n, with endpoints $\widetilde{x} := p^{h(x)-n}(x)$ and $\widetilde{y} := p^{h(y)-n}(y)$.

Consider the point $w := x \wedge y$ (the *confluent* of x and y with respect to the root o, i.e., the last point in common between the geodesics joining o to x and y). Denote by L the length of $\widetilde{\gamma}$ and consider the point p in $\widetilde{\gamma}$ such that $d_{\widehat{\Gamma}}(p, \widetilde{y}) = L/2$ in the case where L is even, and $d_{\widehat{\Gamma}}(p, \widetilde{y}) = \lfloor L/2 \rfloor + 1$ in the case where L is odd.

The spiderweb $\widehat{\Gamma}$ is, by assumption, δ-hyperbolic. Then the four points condition (2.4), applied to w, $\widetilde{x}$, $\widetilde{y}$ and p, yields

$$\begin{aligned} &d_{\widehat{\Gamma}}(p, \widetilde{x}) + d_{\widehat{\Gamma}}(p, \widetilde{y}) - d_{\widehat{\Gamma}}(\widetilde{x}, \widetilde{y}) \\ &\geq \min\left(d_{\widehat{\Gamma}}(p, w) + d_{\widehat{\Gamma}}(p, \widetilde{x}) - d_{\widehat{\Gamma}}(w, \widetilde{x}), d_{\widehat{\Gamma}}(p, w) + d_{\widehat{\Gamma}}(p, \widetilde{y}) - d_{\widehat{\Gamma}}(w, \widetilde{y})\right) - 2\delta. \end{aligned}$$

Notice that $d_{\widehat{\Gamma}}(p,w) = d_{\widehat{\Gamma}}(w,\widetilde{x}) = d_{\widehat{\Gamma}}(w,\widetilde{y})$, and that $d_{\widehat{\Gamma}}(p,\widetilde{x}) + d_{\widehat{\Gamma}}(p,\widetilde{y}) = L$. Thus, the inequality above simplifies to

$$\min\big(d_{\widehat{\Gamma}}(p,\widetilde{x}), d_{\widehat{\Gamma}}(p,\widetilde{y})\big) \leq 2\delta.$$

Now our choice of p implies that $d_{\widehat{\Gamma}}(p,\widetilde{x}) \leq 2\delta$, so that $d_{\widehat{\Gamma}}(p,\widetilde{y}) \leq 2\delta + 1$ and $L \leq 4\delta + 1$, as required.

This concludes the proof of the proposition. □

Suppose that $\widehat{\Gamma}$ is a spiderweb associated to the rooted tree T, that x is a vertex in $\widehat{\Gamma}$ and r is a nonnegative integer. We denote by $B_r^{\mathrm{T}}(x)$ and $B_r^{\widehat{\Gamma}}(x)$ the balls with centre x and radius r in $\widehat{\Gamma}$ with respect to the distances d_{T} and $d_{\widehat{\Gamma}}$, respectively.

The existence of standard geodesics joining any two points in a spiderweb, established in Proposition 3.2, has some noteworthy geometric consequences, one of which, concerning the volume of geodesic balls, is discussed in the following corollary.

Corollary 3.3. *Suppose that* T *is a rooted tree satisfying* (1.3)*, and that* $\widehat{\Gamma}$ *is a spiderweb associated to* T*. The following hold:*

(i) $d_{\widehat{\Gamma}} \leq d_{\mathrm{T}}$ *and* $\mu_{\widehat{\Gamma}}\big(B_r^{\widehat{\Gamma}}(x)\big) \geq \mu_{\widehat{\Gamma}}\big(B_r^{\mathrm{T}}(x)\big)$*;*

(ii) *if, in addition,* $\widehat{\Gamma}$ *has bounded valence, then there exist positive constants* c *and* C *such that*

$$c\,a^r \leq \mu_{\widehat{\Gamma}}\big(B_r^{\widehat{\Gamma}}(x)\big) \leq C\,b^r \qquad \forall r \in \mathbb{N}.$$

Proof. First we prove (i). Clearly every curve in T joining two vertices x and y is also an admissible curve joining x and y in $\widehat{\Gamma}$, whence $d_{\widehat{\Gamma}} \leq d_{\mathrm{T}}$. The second statement in (i) is a direct consequence of this.

Next we prove (ii). The left hand inequality is an obvious consequence of (i). It remains to prove the right hand inequality.

Consider a geodesic γ in $\widehat{\Gamma}$ of length r joining x and y. According to Proposition 3.2, we may assume that γ has one of the following forms:

(a) γ moves away from the root o of T at each step;

(b) γ moves h steps joining points at the same level as x, until it reaches a point z and then moves $r-h$ steps away from o;

(c) γ moves υ steps towards o, then goes horizontally h steps, and in the end it moves $r-\upsilon-h$ steps away from o.

The geodesics γ of the form (a) are precisely the geodesics in T starting from x and moving downwards at each step. Thus, the set of points in $\widehat{\Gamma}$ we can reach with these geodesics is $s^r(x)$ (see just above (2.6) for the definition of $s^r(x)$).

Similarly, the vertices that we can reach with geodesics of the form (b) are precisely those in $s^{r-h}(z)$. Clearly $s^{r-h}(z)$ is contained in $B^{\mathrm{T}}_{r-h}(z)$, which, by the right hand inequality in (1.3), has cardinality at most $C\, b^{r-h}$.

By Proposition 3.2, the estimate $h \leq 4\delta + 1$ holds, so that the number of vertices of $\widehat{\Gamma}$ reachable from x with a geodesic starting with a horizontal segment is at most

$$C \sum_{h=1}^{H} b^{r-h} \leq C\, b^r,$$

where $H := \lfloor 4\delta \rfloor + 2$ and C depends also on the maximum of the valence function on $\widehat{\Gamma}$.

A similar argument shows that the number of vertices of $\widehat{\Gamma}$ reachable from x with a geodesic of the form (c) is, at most,

$$C \sum_{v=1}^{r} \sum_{h=0}^{\min(r-v,H)} b^{r-v-h} \leq C\, b^r.$$

Altogether, we see that the cardinality of the sphere $S^{\widehat{\Gamma}}_r(x)$ is at most $C\, b^r$. Therefore

$$\mu_{\widehat{\Gamma}}\big(B^{\widehat{\Gamma}}_r(x)\big) = \sum_{j=0}^{r} \mu_{\widehat{\Gamma}}\big(S^{\widehat{\Gamma}}_j(x)\big) \leq C \sum_{j=0}^{r} b^j \leq C\, b^r,$$

as required. □

4. L^p bounds for $\mathscr{M}_\infty$ on spiderwebs

In this section we prove L^p bounds for the centred HL maximal operator on spiderwebs satisfying some mild additional assumptions. Our approach is a variant of the strategy that A. Naor and T. Tao follow to prove that $\mathscr{M}$ is of weak type $(1,1)$ on homogeneous trees (see [22], in particular Lemma 5.1 and the proof of Theorem 1.5). Such strategy was then generalised in [26, Theorem 4.1] and used in [14, Lemma 3.3]. For different proofs of the weak type $(1,1)$ estimate for $\mathscr{M}$, see [24] and [8].

We consider the metric measure space $\big(\widehat{\Gamma}, d_{\widehat{\Gamma}}, \mu_{\widehat{\Gamma}}\big)$. Notice that in this case $\mathscr{M}_0 f = f$ (see (1.1) for the definition of $\mathscr{M}_0$), so that $\mathscr{M}$ is bounded on $L^p(\widehat{\Gamma})$, or it is of weak type (p,p), if and only if so is the operator $\mathscr{M}_\infty$.

In this section the cardinality of the set E will be denoted by $|E|$. For a positive integer r, set

$$\mathbf{B}_r(x) := \big\{y \in \widehat{\Gamma} : d_{\widehat{\Gamma}}(y,x) \leq r\big\}.$$

Given a function f on $\widehat{\Gamma}$, denote by $A_r f(x)$ the mean of f over $\mathbf{B}_r(x)$, i.e.,

$$A_r f(x) := \frac{1}{\big|\mathbf{B}_r(x)\big|} \int_{\mathbf{B}_r(x)} f \,\mathrm{d}\mu_{\widehat{\Gamma}}.$$

Note the trivial bound

$$\mathscr{M}_\infty f \leq \sum_{r=1}^{\infty} A_r|f|.$$

The next theorem is the main result of this section. Recall that if a and b are positive numbers, then ϱ stands for $\log_a b$.

Theorem 4.1. *Suppose that $a \leq b < a^2$, and that $\widehat{\Gamma}$ is a spiderweb (associated to the tree* T*) satisfying the growth condition* (1.3)*. The following hold:*

(i) *there exists a positive constant C such that*

$$a^{-r}\left|\left\{(x,y) \in E \times F : d_{\widehat{\Gamma}}(x,y) \leq r\right\}\right| \leq C\,(\sqrt{b}/a)^r\sqrt{|E|\cdot|F|} \tag{4.1}$$

for every nonnegative integer r and every pair E, F of subsets of $\widehat{\Gamma}$;

(ii) *$\mathscr{M}_\infty$ is of weak type (ϱ,ϱ), and bounded on $L^p(\widehat{\Gamma})$ for every p in $(\varrho,\infty]$.*

Proof. First we prove (i). Set $U_r := \{(x,y) \in E \times F : d_{\widehat{\Gamma}}(x,y) \leq r\}$, and observe that $U_r = \sum_{p=0}^{r} G_p$, where $G_p := \{(x,y) \in E \times F : d_{\widehat{\Gamma}}(x,y) = p\}$. We shall prove that

$$|G_p| \leq C\,b^{p/2}\sqrt{|E|\cdot|F|} \tag{4.2}$$

for every p in $\{0,\ldots,r\}$, which implies that

$$|U_r| \leq C\,b^{r/2}\sqrt{|E|\cdot|F|}.$$

Since the left hand side of (4.1) is just $a^{-r}\,|U_r|$, part (i) follows directly from the previous estimate.

It remains to prove (4.2).

Suppose that x belongs to Σ_j. We estimate the number of the vertices in Σ_k at distance p from x, i.e., the cardinality of $S_p(x) \cap \Sigma_k$.

Denote by γ a geodesic of length p joining x and a point in Σ_k. Recall that by Proposition 3.2 we may assume that γ is a standard geodesic, i.e., it has one of the special forms described in (a), (b) and (c) in the proof of Corollary 3.3. In particular, denote by γ_0 the horizontal part (possibly reduced to a point) of γ, and set $\ell := d_{\widehat{\Gamma}}(x,\gamma_0)$. Then $0 \leq \ell \leq \min(j,p)$. Thus γ moves first ℓ steps towards o, then h steps horizontally, and finally $p-\ell-h$ steps away from o. Note also that $p-\ell-h$ is equal to the difference between k, the level of a point in Σ_k, and the level of γ_0, i.e., $j-\ell$. Thus, $p = \ell + h + k - (j-\ell)$, so that

$$\ell = \frac{1}{2}\,(p + j - k - h). \tag{4.3}$$

Since p, j and k are fixed parameters, (4.3) exhibits ℓ as a function of h. Recall that h is a nonnegative integer $\leq H$, where we have set $H := 4\delta + 1$ (see Proposition 3.2).

Now, given an integer h in $[0, H]$, how many points in Σ_k are reachable from x with a standard geodesic of length p and horizontal part of length h? We start from x and move $\ell = \big(p + k - j - h\big)/2$ steps towards o until we reach $p^\ell(x)$, which belongs to $\Sigma_{j-\ell}$. Next we move horizontally h steps, reaching a point z_γ. The number of points z_γ in $\Sigma_{j-\ell}$ that we can reach in this way is bounded from above by the volume of the ball with centre $p^\ell(x)$ and radius h, which has at most $C\, b^h$ points. Clearly

$$d_{\widehat{\Gamma}}\big(z_\gamma, \Sigma_k\big) = p - \ell - h = \frac{1}{2}\,\big(p + k - j - h\big).$$

Thus, the number of points in Σ_k reachable from z_γ is bounded by the measure of the ball centred at z_γ and of radius $\big(p + k - j - h\big)/2$, which has at most $C\, b^{(p+k-j-h)/2}$ points.

Altogether, the number of points in Σ_k reachable from x with a standard geodesic of length p and horizontal part of length h is bounded above by

$$C\, b^h\, b^{(p+k-j-h)/2} = C\, b^{(p+k-j+h)/2}$$

points. Hence

$$\big|S_p(x) \cap \Sigma_k\big| \leq C \sum_{h=0}^{H} b^{(p+k-j+h)/2} = C\, b^{(p+k-j)/2}. \tag{4.4}$$

A similar argument yields that if y belongs to Σ_k, then

$$\big|S_p(y) \cap \Sigma_j\big| \leq C\, b^{(p+j-k)/2}. \tag{4.5}$$

Furthermore, for every pair (j, k) of nonnegative integers define

$$E_j := E \cap \Sigma_j, \qquad F_k := F \cap \Sigma_k,$$

and

$$G_{j,k,p} := \big\{(x, y) \in E_j \times F_k : d_{\widehat{\Gamma}}(x, y) = p\big\}.$$

Observe that

$$|G_{j,k,p}| = \sum_{x\in E_j} \sum_{y\in F_k} \mathbf{1}_{S_p(x)}(y) = \sum_{y\in F_k} \sum_{x\in E_j} \mathbf{1}_{S_p(y)}(x).$$

Define $e_j := |E_j|/b^j$ and $d_k := |F_k|/b^k$. Now (4.4) and (4.5) imply that

$$|G_{j,k,p}| \leq C \min \left(|E_j|\, b^{(p-j+k)/2}, |F_k|\, b^{(p+j-k)/2}\right)$$
$$= C\, b^{(p+j+k)/2} \min\,(e_j, d_k).$$

Therefore

$$|G_p| = \sum_{j,k=0}^{\infty} |G_{j,k,p}| \leq C\, b^{p/2} \sum_{j,k=0}^{\infty} b^{(j+k)/2} \min\,\big(e_j, d_k\big).$$

Now we can apply almost *verbatim* the argument in the last part of the proof of [22, p. 759–760], and conclude that the last double series is dominated by $8\sqrt{|E|\cdot|F|}$. The estimate (4.2) follows directly from this.

Next we prove (ii). It suffices to consider nonnegative functions f in $L^{\varrho}(\widehat{\Gamma})$. We want to prove that there exists a constant $C > 0$ such that

$$\big|\{\mathscr{M}_\infty f > \lambda\}\big| \leq \frac{C}{\lambda^{\varrho}}\, \big\|f\big\|_{L^{\varrho}}^{\varrho} \qquad \forall \lambda > 0 \quad \forall f \in L^{\varrho}(\widehat{\Gamma}).$$

By replacing f by f/λ, we see that it suffices to prove the estimate above for $\lambda = 1$. Set $\Omega := \big\{x \in \widehat{\Gamma} : f(x) \leq 1/2\big\}$, and, for integers n and r, define

$$E_n := \big\{x \in \widehat{\Gamma} : 2^{n-1} < f(x) \leq 2^n\big\} \quad \text{and} \quad F_r := \big\{x \in \widehat{\Gamma} : f(x) > a^r/2\big\}.$$

We fix r, and decompose f accordingly. Denote by N the biggest integer such that $2^n \leq a^r$. Clearly

$$\widehat{\Gamma} = \Omega \cup \Big(\bigcup_{n=0}^{N} E_n\Big) \cup F_r,$$

so that

$$f \leq \frac{1}{2}\, \mathbf{1}_\Omega + \sum_{n=o}^{N} 2^n\, \mathbf{1}_{E_n} + f\, \mathbf{1}_{F_r}.$$

Consequently

$$A_r f \leq \frac{1}{2} + \sum_{n=o}^{N} 2^n\, A_r \mathbf{1}_{E_n} + A_r\big(f\, \mathbf{1}_{F_r}\big).$$

Define $g_r := \sum_{n=o}^{N} 2^n\, A_r \mathbf{1}_{E_n}$ and $h_r := A_r\big(f\, \mathbf{1}_{F_r}\big)$. Clearly

$$\big\{A_r f > 1\big\} \subseteq \big\{g_r + h_r > 1/2\big\},$$

and the latter set is contained in

$$\{g_r > 1/2\} \cup \{h_r > 0\}.$$

Denote by I_1 and I_2 the first and the second set in the union above. Clearly $|\{A_r f > 1\}| \leq |I_1| + |I_2|$. We estimate $|I_1|$ and $|I_2|$ separately.

First we estimate $|I_2|$. Observe that if $d_{\widehat{\Gamma}}(x, F_r) > r$, then $h_r(x) = 0$, so that I_2 is contained in $\bigcup_{y \in F_r} \mathbf{B}_r(y)$. Therefore

$$|I_2| \leq \sum_{y \in F_r} |\mathbf{B}_r(y)| \leq C\, b^r\, |F_r| = C\, a^{\varrho r}\, |F_r|.$$

Now we estimate $|I_1|$. Set $\beta := (2 - \varrho)/4$: notice that β is positive, for $\varrho < 2$ by assumption. It is convenient to set $\alpha := a^{-\beta r}\, (1 - 2^{-\beta})$ and

$$V_n := \{x : 2^n\, A_r(\mathbf{1}_{E_n})(x) \geq 2^{n\beta - 1}\, \alpha\}.$$

By arguing almost *verbatim* as in the proof of [23, p. 501], one can show that if x belongs to I_1, then x belongs to V_n for some integer n in $\{0, \ldots, N\}$. Thus,

$$|I_1| \leq \sum_{n=0}^{N} |V_n|. \tag{4.6}$$

Now, since $2^n\, A_r(\mathbf{1}_{E_n}) \geq 2^{n\beta-1}\, \alpha$ on V_n, we see that

$$\langle \mathbf{1}_{V_n}, 2^n\, A_r(\mathbf{1}_{E_n})\rangle \geq |V_n|\, 2^{n\beta-1}\, \alpha, \tag{4.7}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(\mu_{\widehat{\Gamma}})$. Also the lower estimate in (1.3) yields

$$\begin{aligned}\langle \mathbf{1}_{V_n}, 2^n\, A_r(\mathbf{1}_{E_n})\rangle &\leq \frac{2^n}{c\, a^r} \int_{V_n} \mathrm{d}\mu_{\widehat{\Gamma}}(y) \int_{\mathbf{B}_r(y)} \mathbf{1}_{E_n}(x)\, \mathrm{d}\mu_{\widehat{\Gamma}}(x) \\ &= \frac{2^n}{c\, a^r}\, \left|\{(x, y) \in E_n \times V_n\ : d_{\widehat{\Gamma}}(x, y) \leq r\}\right|\end{aligned}$$

This, combined with (i) and the lower estimate (4.7), implies that

$$|V_n| \leq C\, \frac{1}{2^{n\beta}\, \alpha}\, 2^n\, (\sqrt{b}/a)^r\, \sqrt{|V_n|\, |E_n|}.$$

Now recall that $b = a^{\varrho}$. Then the previous estimate may be rewritten as

$$|V_n| \leq C\, \frac{1}{2^{2n\beta}\, \alpha^2}\, 2^{2n}\, (b/a^2)^r\, |E_n| = C\, \left(\frac{2^n}{a^r}\right)^{1-\varrho/2} 2^{n\varrho}\, |E_n|,$$

which, together with (4.6), yields

$$|I_1| \le C \sum_{n=0}^{N} \left(\frac{2^n}{a^r}\right)^{1-\varrho/2} 2^{n\varrho}\, |E_n|.$$

Recall that $\{A_r f > 1\}$ is contained in $I_1 \cup I_2$, so that

$$\big|\{A_r f > 1\}\big| \le C \sum_{n=0}^{N} \left(\frac{2^n}{a^r}\right)^{1-\varrho/2} 2^{n\varrho}\, |E_n| + C\, a^{\varrho r}\, |F_r|. \tag{4.8}$$

Now, $\mathscr{M}_\infty f = \sup_{r\ge 1} A_r f$. Therefore if $\mathscr{M}_\infty f(x) > 1$, then there exists a nonnegative integer r such that $A_r f(x) > 1$. Consequently

$$\begin{aligned}\big|\{\mathscr{M}_\infty f > 1\}\big| &\le \sum_{r=0}^{\infty} \big|\{A_r f > 1\}\big| \\ &\le C \sum_{r=0}^{\infty} \sum_{n=0}^{N} \left(\frac{2^n}{a^r}\right)^{1-\varrho/2} 2^{n\varrho}\, |E_n| + C \sum_{r=0}^{\infty} a^{\varrho r}\, \big|F_r\big|.\end{aligned}$$

Now, trivially

$$\sum_{r=0}^{\infty} a^{\varrho r}\, \mathbf{1}_{F_r} \le \sum_{r=0}^{\lfloor \log_a(2f)\rfloor} a^{\varrho r} \le \frac{2^\varrho b}{b-1}\, f^\varrho,$$

whence

$$\sum_{r=0}^{\infty} a^{\varrho r}\, |F_r| = \int_{\widehat{\Gamma}} \sum_{r=0}^{\infty} a^{\varrho r}\, \mathbf{1}_{F_r}\, \mathrm{d}\mu_{\widehat{\Gamma}} \le \frac{2^\varrho b}{b-1}\, \big\|f\big\|^{\varrho}_{L^\varrho(\widehat{\Gamma})}\,.$$

Similarly, $\sum_{n=0}^{\infty} 2^{\varrho n}\, \mathbf{1}_{\widehat{E}_n} \le 2^{\varrho+1} f^\varrho$, where we have set $\widehat{E}_n := \{f > 2^{n-1}\}$. Clearly $E_n \subseteq \widehat{E}_n$. Using these facts and reversing the order of summation, we see that

$$\begin{aligned}\sum_{r=0}^{\infty}\sum_{n=0}^{N} \left(\frac{2^n}{a^r}\right)^{1-\varrho/2} 2^{\varrho n}\, |E_n| &\le \int_{\widehat{\Gamma}} \sum_{n=0}^{\infty} 2^{\varrho n}\, \mathbf{1}_{\widehat{E}_n} \sum_{r:a^r\ge 2^{n+1}} \left(\frac{2^n}{a^r}\right)^{1-\varrho/2} \mathrm{d}\mu_{\widehat{\Gamma}} \\ &\le C \int_{\widehat{\Gamma}} \sum_{n=0}^{\infty} 2^{n\varrho}\, \mathbf{1}_{\widehat{E}_n}\, \mathrm{d}\mu_{\widehat{\Gamma}} \\ &\le C \int_{\widehat{\Gamma}} f^\varrho\, \mathrm{d}\mu_{\widehat{\Gamma}}.\end{aligned}$$

By combining these estimates, we find that

$$|\{\mathscr{M}_\infty f > 1\}| \le C\, \big\|f\big\|^{\varrho}_{L^\varrho(\widehat{\Gamma})}\,,$$

as desired.

This concludes the proof of (ii), and of the theorem. □

Remark 4.2. Recall that a tree T is a very special spiderweb. Thus, if T satisfies (1.3) for some $b < a^2$, then Theorem 4.1 implies that the maximal operator $\mathscr{M}$ on T is of weak type (ϱ, ϱ). This improves [14, Theorem 3.2], in which it is shown that $\mathscr{M}$ is of *restricted* weak type (ϱ, ϱ) under stronger assumptions.

5. Spiderwebs "embedded" in δ-hyperbolic length spaces

Given a positive number η, we say that a set $\mathfrak{D}$ of points in a metric space (Y, d) is *η-separated* if $d(y, z) \geq \eta$ for every pair of distinct points y and z in $\mathfrak{D}$; the set $\mathfrak{D}$ is an *η-discretisation* of Y if it is a maximal (with respect to inclusion) η-separated set in Y. It is straightforward to show that η-discretisations exist for every η.

Notice that if $\mathfrak{D}$ a discretisation of Y, then

$$d(\mathfrak{D}, x) < \eta \qquad \forall x \in Y,$$

for otherwise there would exist a point x in Y such that $d(x, \mathfrak{D}) \geq \eta$, thereby violating the maximality of $\mathfrak{D}$.

We denote by $\Upsilon^{\delta}_{a,b}$ the class of all δ-hyperbolic spiderwebs $(\widehat{\Gamma}, d_{\widehat{\Gamma}})$ with the property that the counting measure $\mu_{\widehat{\Gamma}}$ on $\widehat{\Gamma}$ satisfies the growth condition (1.3).

The proof of the main result in this section, Theorem 5.1 below, hinges on a nontrivial variant of the construction described in [5, Section 5]. Recall the definition of the class $\mathscr{X}^{\delta}_{a,b}$ given just above Theorem 1.1.

Theorem 5.1. *Suppose that δ, a and b are positive numbers satisfying the condition $1 < a \leq b$. Assume that (X, d, μ) is in the class $\mathscr{X}^{\delta}_{a,b}$. Then there exist a positive number δ' and a spiderweb $\widehat{\Gamma}$ in the class $\Upsilon^{\delta'}_{a,b}$ that is strictly roughly isometric to (X, d).*

The proof consists of four big steps, each of which is further split up into smaller steps. Before we dive into the details of the proof we shall outline the general scheme. In order to construct the desired spiderweb $\widehat{\Gamma}$ we begin by defining the underlying tree T and an associated graph Γ, obtained from T by adding *horizontal* edges to the tree between points that are close enough in the metric space X. The graph Γ turns out to have bounded valence (Step I).

We observe that there is no *a priori* reason for which the graph Γ should be a spiderweb, because it might happen that two points x and y in Γ with the same level are neighbours, but their predecessors $p(x)$ and $p(y)$ are not. In Step IV we shall construct a spiderweb $\widehat{\Gamma}$, which has the same vertices as Γ, but where each vertex x may have more neighbours at its level than in Γ.

In order to do this, we need to establish a few preliminary facts, which occupy Steps II-III below. Recall the metric graphs Γ_0 and $\widetilde{\Gamma}$ associated to Γ (see Caveat 2.13).

Step II is devoted to the proof that Γ_0, $\widetilde{\Gamma}$ and Γ are δ'-hyperbolic. Step III, which contains the most technical part of the proof, is dedicated to the proof that Γ_0 is strictly roughly isometric to (X, d). The Morse lemma in hyperbolic geometry plays a crucial role in deriving the desired estimates here and in Step IV.

The last remaining link is proving that, although Γ might not be a spiderweb, because of hyperbolicity it is always a quasi-spiderweb and therefore can be "completed" to a true spiderweb $\widehat{\Gamma}$, which is still strictly roughly isometric to (X, d) (Step IV).

Proof. Step I: construction of the graph Γ**.** This step is split up into Step I_1, where we construct the tree T, and Step I_2, where we define the graph Γ and prove that it has bounded valence.

Step I_1*: construction of the tree* T. Fix a point o in X. We define a rooted tree T with root o embedded in X as follows. For each positive integer n consider the sphere $S_n(o) := \{x \in X : d(x, o) = n\}$, and a 1-discretisation Σ_n thereof (with respect to the distance d). In particular,

$$\bigcup_{x \in \Sigma_n} B_1(x) \supseteq S_n(o).$$

Notice that

$$B_1(o) \cup \Big(\bigcup_{n=1}^{\infty} \bigcup_{x \in \Sigma_n} B_2(x) \Big) = X. \tag{5.1}$$

Indeed, consider a point y in X. If $d(y, o) < 1$, then y is covered by $B_1(o)$, and if $d(y, o) = n$ for some positive integer n, then we already know that y is covered by $B_2(x)$ for some x in Σ_n.

Otherwise denote by n the positive integer such that $n < d(y, o) \leq n + 1$. Consider the point y_n where the geodesic joining o and y intersects the sphere $S_n(o)$. We have that

$$d(y_n, y) = d(o, y) - d(o, y_n) = d(o, y) - n \leq 1.$$

By construction of Σ_n, there exists a point x in Σ_n such that $d(x, y_n) < 1$, whence y belongs to $B_2(x)$.

Set $V := \{o\} \cup \Big(\bigcup_{n=1}^{\infty} \Sigma_n \Big)$. The points in V will be the vertices of the rooted tree T. It remains to indicate, for each x in $V \setminus \{o\}$, its predecessor $p(x)$. If x belongs to Σ_1, we set $p(x) = o$. If x is in Σ_n for some $n \geq 2$, then its predecessor $p(x)$ is one, no matter which, of the points in Σ_{n-1} at minimum distance from x. The construction above implies that

$$d\big(x, p(x)\big) \leq 2 \qquad \forall x \in \mathrm{T}. \tag{5.2}$$

Of course different choices of $p(x)$ will give rise to different trees. All of them share the properties we shall need in what follows. Note also that there may be vertices x in T such that $s(x)$ is empty, i.e., x has no successors.

Step I$_2$: definition of the graph Γ*.* The vertices of Γ agree with the vertices of T. Set

$$\theta := \max(15, 4D_{2,4} + 2\delta), \tag{5.3}$$

where δ is the hyperbolicity constant of (X, d) and $D_{2,4}$ is the constant $D_{\omega,K}$ of (X, d) (with $\omega = 2$ and $K = 4$) appearing in the definition of geodesically stable metric space (see just below Definition 2.6). We *declare* that two distinct vertices x and y in Γ are neighbours if and only if one of the following holds:

(i) x and y are neighbours in T;
(ii) they belong to the same level set Σ_n for some positive integer n and $d(x, y) \leq \theta$.

Observe that *the valence function* ν *on* Γ *is bounded*. Indeed, if $x \sim y$ in Γ, then either x and y are neighbours in T, whence $d(x, y) < 2$ by the construction of T (see Step I above), or they belong to the same level and $d(x, y) \leq \theta$. Altogether any neighbour of x in Γ belongs to $B_\theta(x)$. Recall that the points in Γ are 1-separated in X, so that

$$\mu\big(B_{\theta+1}(x)\big) \geq \sum_{y\sim x} \mu\big(B_{1/2}(y)\big) \geq \nu(x) \min_{y\sim x} \mu\big(B_{1/2}(y)\big).$$

Now, the LDP (see Remark 2.1) implies that $\nu(x) \leq L_{2\theta+2,1/2}$, as required.

The metric graphs Γ_0 and $\widetilde{\Gamma}$ are associated to Γ as in Caveat 2.13.

Step II: $(\widetilde{\Gamma}, d_{\widetilde{\Gamma}})$, (Γ_0, d_{Γ_0}) and (Γ, d_Γ) are δ'-hyperbolic for some positive δ'. This step is split up into Step II$_1$, where we prove that if two points in Γ are close enough with respect to d, then they are close enough also with respect to d_Γ, and Step II$_2$, where we show that $(\widetilde{\Gamma}, d_{\widetilde{\Gamma}})$ and (X, d) are roughly isometric. The δ'-hyperbolicity of $(\widetilde{\Gamma}, d_{\widetilde{\Gamma}})$ then follows from Step II$_2$ and Theorem 2.8. The δ'-hyperbolicity of (Γ_0, d_0) and (Γ, d_Γ) is a direct consequence of the δ'-hyperbolicity of $(\widetilde{\Gamma}, d_{\widetilde{\Gamma}})$ and of Proposition 2.12. Finally, the δ'-hyperbolicity of (Γ, d_Γ) follows from the fact that d_Γ is the restriction of d_{Γ_0} to Γ.

Step II$_1$: if x and y are in Γ and $d(x, y) \leq \theta/3$, then $d_\Gamma(x, y) \leq (\theta/3) + 1$.

The conclusion is trivial in the case where $x = y$. Thus, we may assume that $x \neq y$.

Observe that if $h(x) = h(y)$ and $d(x, y) \leq \theta/3$, then $x \sim y$ (by (ii) in Step I$_2$), so that $d_\Gamma(x, y) = 1$, and the conclusion follows.

If, instead, $h(x) \neq h(y)$, then, without loss of generality, we may assume that $h(x) > h(y)$. Set $\ell := h(x) - h(y)$. The triangle inequality implies that

$$d(x, y) \geq d(x, o) - d(o, y) = \ell.$$

Therefore $\ell \leq \theta/3$. Observe that

$$d\big(p^\ell(x), x\big) \leq \sum_{j=1}^{\ell} d\big(p^j(x), p^{j-1}(x)\big) \leq 2\ell;$$

the last inequality follows from (5.2). The points $p^\ell(x)$ and y belong to the same level, and the triangle inequality implies that

$$d\big(p^\ell(x), y\big) \leq d\big(p^\ell(x), x\big) + d(x, y) \leq 2\ell + \theta/3 \leq \theta.$$

Therefore $p^\ell(x)$ and y are neighbours (see the definition of Γ in Step I_2), and $[x, \dots, p^\ell(x), y]$ is a path in Γ of length at most $(\theta/3) + 1$, as required.

Step II_2: the identity map is a (λ, β)-rough isometry between $(\widetilde{\Gamma}, d_{\widetilde{\Gamma}})$ and (X, d), where

$$\lambda := \theta\,(\theta/3 + 1) \quad \text{and} \quad \beta := \theta\,(\theta/3 + 5).$$

Observe that $d(x, \widetilde{\Gamma}) \leq d(x, \Gamma)$ for every x in X. By (5.1),

$$\sup\{d(x, \Gamma) : x \in X\} < 2,$$

so that $\sup\{d(x, \widetilde{\Gamma}) : x \in X\} < 2$. Notice that

$$d(x, y) \leq d_{\widetilde{\Gamma}}(x, y) \qquad \forall x, y \in \widetilde{\Gamma}.$$

Proposition 2.12 (i) implies that

$$d_{\widetilde{\Gamma}}(x, y) \leq \theta\, d_{\Gamma_0}(x, y) \qquad \forall x, y \in \widetilde{\Gamma}. \tag{5.4}$$

Hence, it suffices to prove that

$$d_{\Gamma_0}(x, y) \leq (\theta/3 + 1)\, d(x, y) + \theta/3 + 5 \qquad \forall x, y \in \widetilde{\Gamma}. \tag{5.5}$$

Suppose that x and y are distinct points in $\widetilde{\Gamma}$, and denote by γ a geodesic in X joining them. If $d(x, y) \leq 1$ we set $a_0 := x$ and $a_1 := y$. If $d(x, y) > 1$, then we consider the points $\{a_0 := x, a_1, \dots, a_{N-1}, a_N := y\}$ in γ such that $d(a_j, a_{j+1}) = 1$ for every j in $\{0, ..., N-2\}$. Clearly $d(x, y) \geq N - 1$. Next, for every j in $\{0, ..., N\}$ choose a point z_j in Γ at minimum distance in X from a_j. In particular, $d(z_j, a_j) \leq 2$. The triangle inequality implies that

$$d(z_j, z_{j+1}) \leq d(z_j, a_j) + d(a_j, a_{j+1}) + d(a_{j+1}, z_{j+1}) \leq 5$$

for every j in $\{0, ..., N-1\}$. Since $5 \leq \theta/3$ (see (5.3)), Step II_1 yields

$$d_{\Gamma_0}(z_j, z_{j+1}) \leq \theta/3 + 1.$$

By combining the previous estimates, we see that

$$d_{\Gamma_0}(x,y) \le d_{\Gamma_0}(x,z_0) + \sum_{j=0}^{N-1} d_{\Gamma_0}(z_j,z_{j+1}) + d(y,z_N) \le 4 + N(\theta/3+1).$$

The right hand side above may be rewritten as

$$(N-1)(\theta/3+1) + \theta/3 + 5 \le d(x,y)(\theta/3+1) + \theta/3 + 5,$$

thereby proving (5.5).

Step III: (Γ_0, d_{Γ_0}) **and** (X,d) **are strictly roughly isometric.** We split up Step III into Steps III_1-III_4. Step III_1 contains some preliminary estimates. In Step III_2 we prove that $d_{\Gamma_0} \ge d - \eta$ for a suitable constant η. Step III_3 is devoted to the proof that geodesics in Γ_0 are quasigeodesics in X. Finally the upper estimate $d_{\Gamma_0} \le d + \eta$ is proved in Step III_4.

Observe that any geodesic in Γ_0 connecting o to a point z in Γ_0 coincides with the geodesic in the metric subtree of Γ_0 obtained from Γ_0 by removing the horizontal edges introduced in Step I_2.

Step III_1*: preliminary estimates. Suppose that* z *and* z' *are points in* Γ_0*, and that* z' *lies on a geodesic* $\gamma^0_{o,z}$ *in* Γ_0 *joining* o *and* z*. The following hold:*

(i) if z *belongs to* T*, then* $\gamma^0_{o,z}$ *is a* $(2,4)$*-quasigeodesic in* (X,d)*;*
(ii) if z *and* z' *belong to* T*, then*

$$d_{\mathrm{T}}(z,z') \ge d(z,z') - 4D_{2,4}, \tag{5.6}$$

where $D_{2,4}$ *is as in Step* I_2*, and* d_{T} *denotes the tree distance on* T*;*
(iii) $d_{\mathrm{T}}(\lfloor z\rfloor, \lfloor z'\rfloor) \le d_{\Gamma_0}(z,z') + 2$*; see Notation 2.14 for the definition of* $\lfloor z\rfloor$ *and* $\lfloor z'\rfloor$*;*
(iv) $d_{\Gamma_0}(z,z') \ge d(z,z') - 2\theta - 4D_{2,4} - 2$*.*

First we prove (i). It suffices to prove (i) in the case where $z \ne o$. Denote by

$$[a_0 := o, a_1, \dots, a_{N-1}, a_N := z]$$

the points in T that lie on $\gamma^0_{o,z}$, and by σ_j the geodesic segment in $\gamma^0_{o,z}$ joining a_j and a_{j+1}. Recall that σ_j is also a geodesic in (X,d).

Suppose that p and q are distinct points in $\gamma^0_{o,z}$. Without loss of generality we assume that $d(o,p) < d(o,q)$. Then there are nonnegative integers j_1 and j_2, with $j_1 \le j_2 \le N-1$, such that p and q belong to σ_{j_1} and to σ_{j_2}, respectively.

If $j_1 = j_2$, then $\ell_X\big(\gamma^0_{o,z}([p,q])\big) = d(p,q)$; here $\gamma^0_{o,z}([p,q])$ denotes the segment in σ_{j_1} with endpoints p and q.

If, instead, $j_1 < j_2$, then

$$\begin{aligned}\ell_X\big(\gamma^0_{o,z}([p,q])\big) &= d(p,a_{j_1+1}) + \sum_{j=j_1+1}^{j_2-1} d(a_j,a_{j+1}) + d(a_{j_2},q) \\ &\le 2\,(j_2 - j_1 - 1) + 4;\end{aligned} \tag{5.7}$$

we have used (5.2) in the last inequality, and we agree that the second summand on the first line above vanishes if $j_2 \leq j_1 + 1$. The triangle inequality implies that

$$d(p,q) \geq d(q,o) - d(p,o) \geq d(a_{j_2}, o) - d(a_{j_1+1}, o) = j_2 - j_1 - 1,$$

which, combined with (5.7), yields

$$\ell_X\big(\gamma^0_{o,z}([p,q])\big) \leq 2\, d(p,q) + 4,$$

as desired.

Next we prove (ii). Observe that if $z' = o$ and z belongs to Σ_n, then the construction of T (see Step I) and the definition of Σ_n imply that $d_{\mathrm{T}}(z,o) = n = d(z,o)$, so that (5.6) trivially holds.

If, instead, z' and o are distinct points, consider the geodesic γ in X joining o and z. Since, by (i), $\gamma^0_{o,z}$ is a $(2,4)$-quasigeodesic joining o and z, Lemma 2.7 ensures that there exists a positive constant $D_{2,4}$ such that $\gamma^0_{o,z}$ lies in a $D_{2,4}$-neighbourhood of γ. Therefore there exists a point x in γ such that $d(z',x) \leq D_{2,4}$, so that

$$\begin{aligned}\big|d(z,z') - d_{\mathrm{T}}(z,z')\big| &= \big|d(z,z') + d(z',o) - d(z,o)\big| \\ &= \big|d(z',o) - d(o,x) + d(z',z) - d(x,z)\big| \\ &\leq 2D_{2,4};\end{aligned}$$

the first equality follows from the fact that, by construction of the tree T, $d_{\mathrm{T}}(z,z') = d(o,z) - d(o,z')$, the second from the formula $d(o,z) = d(o,x) + d(z,x)$, which holds because z lies on γ, and the inequality from the triangle inequality applied to both the triangles oxz' and $z'xz$.

This proves (ii).

Observe that (iii) is trivial in the case where z and z' are vertices of Γ, for then z' lies on the geodesic in T joining o and z, whence $d_{\Gamma_0}(z,z') = d_{\mathrm{T}}(z,z')$.

Notice that $d_{\Gamma_0}(\lfloor z \rfloor, \lfloor z' \rfloor) = d_{\mathrm{T}}(\lfloor z \rfloor, \lfloor z' \rfloor)$, because $\lfloor z \rfloor$ and $\lfloor z' \rfloor$ are vertices of Γ lying on the geodesic in T joining $\lfloor z \rfloor$ and o. Furthermore $d_{\Gamma_0}(z, \lfloor z \rfloor) \leq 1$ and $d_{\Gamma_0}(\lfloor z' \rfloor, z') \leq 1$, because z lies on the edge in Γ_0 joining $\lfloor z \rfloor$ and one of its neighbours in Γ, and similarly for z'. Therefore

$$d_{\mathrm{T}}(\lfloor z \rfloor, \lfloor z' \rfloor) = d_{\Gamma_0}(\lfloor z \rfloor, \lfloor z' \rfloor) \leq d_{\Gamma_0}(\lfloor z \rfloor, z) + d_{\Gamma_0}(z,z') + d_{\Gamma_0}(z', \lfloor z' \rfloor),$$

as required.

Finally we prove (iv). By combining (iii) and (ii), we see that

$$d_{\Gamma_0}(z,z') \geq d_{\mathrm{T}}(\lfloor z \rfloor, \lfloor z' \rfloor) - 2 \geq d(\lfloor z \rfloor, \lfloor z' \rfloor) - 4D_{2,4} - 2. \tag{5.8}$$

Now, the point z lies on a geodesic in X joining $\lfloor z \rfloor$ and one of its neighbours. Such geodesics in X have length at most θ, so that $d(z, \lfloor z \rfloor) \leq \theta$. Similarly $d(\lfloor z' \rfloor, z') \leq \theta$. These estimates and the triangle inequality imply that

$$d(\lfloor z \rfloor, \lfloor z \rfloor') \geq d(z, z') - d(z, \lfloor z \rfloor) - d(\lfloor z' \rfloor, z') \geq d(z, z') - 2\theta.$$

Now, (iv) follows directly from this and (5.8).

Step III$_2$. The following lower estimate holds:

$$d_{\Gamma_0}(x, y) \geq d(x, y) - \eta \qquad \forall x, y \in \Gamma_0, \tag{5.9}$$

where

$$\eta := 2\theta\delta' + 2\delta' + 4\theta + 2 + \max\left(8\, D_{2,4} + 2, 6D_{\theta\lambda,\theta\beta}\right).$$

Consider the geodesic triangle oxy in Γ_0 with sides $\gamma^0_{o,x}$, $\gamma^0_{o,y}$ and $\gamma^0_{x,y}$. Since, by Step II$_2$, Γ_0 is a δ'-hyperbolic length space, Remark 2.5 implies the existence of a point p in (Γ_0, d_{Γ_0}), and points x' in $\gamma^0_{o,x}$ and y' in $\gamma^0_{o,y}$ such that

$$\max\left(d_{\Gamma_0}(p, x'), d_{\Gamma_0}(p, y')\right) \leq \delta'. \tag{5.10}$$

Since p is a point on the geodesic $\gamma^0_{x,y}$,

$$d_{\Gamma_0}(x, y) = d_{\Gamma_0}(x, p) + d_{\Gamma_0}(p, y). \tag{5.11}$$

Now the triangle inequality, applied to the triangle $xx'p$, and (5.10) imply that

$$d_{\Gamma_0}(x, p) \geq d_{\Gamma_0}(x, x') - d_{\Gamma_0}(x', p) \geq d_{\Gamma_0}(x, x') - \delta'. \tag{5.12}$$

By combining Step III$_1$ (iv) and (5.12) yields

$$d_{\Gamma_0}(x, p) \geq d(x, x') - \delta' - 4D_{2,4} - 2\theta - 2.$$

By arguing similarly, we see that

$$d_{\Gamma_0}(p, y) \geq d(y, y') - \delta' - 4D_{2,4} - 2\theta - 2.$$

Using (5.11) and the last two inequalities, we see that

$$\begin{aligned} d_{\Gamma_0}(x, y) &\geq d(x, x') + d(y, y') - 2\delta' - 8\, D_{2,4} - 4\theta - 4 \\ &\geq d(x, y) - d(x', y') - 2\delta' - 8\, D_{2,4} - 4\theta - 4 \\ &\geq d(x, y) - 2\theta\delta' - 2\delta' - 8\, D_{2,4} - 4\theta - 4; \end{aligned}$$

the penultimate inequality follows from the triangle inequality

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y),$$

and the last inequality from the fact that

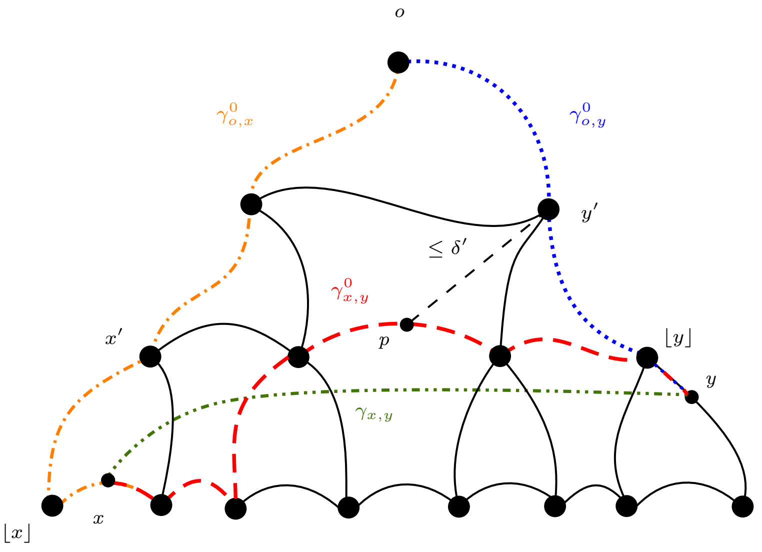

Fig. 3. The figure displays the points o, x, x', $\lfloor x \rfloor$, y, y', $\lfloor y \rfloor$, and p. The three geodesics in Γ_0 are represented using distinct line styles: a dash–dot curve for the geodesic from o to x, a dotted curve for the geodesic from o to y, and a long–dashed curve for the geodesic from x to y. The geodesic in X joining x to y is shown with a dash–dot–dot pattern.

$$d(x', y') \leq d_{\widetilde{\Gamma}}(x', y') \leq \theta \, d_{\Gamma_0}(x', y') \leq 2\theta\delta';$$

we have used (5.10) in the last inequality above. (See Fig. 3.)

The proof of (5.9) is complete.

Step III$_3$: geodesics in Γ_0 are $\big(\theta\lambda, \theta\beta\big)$-quasigeodesics in X, where λ and β are as in Step II$_2$.

Suppose that x and y are in Γ_0, and denote by $\gamma^0_{x,y}$ a geodesic in Γ_0 joining x and y. Observe that if $\gamma^0_{x,y}$ is contained in an edge, then for any pair of points u and u' in $\gamma^0_{x,y}$

$$\ell_X(\gamma^0_{x,y}([u, u'])) = d_{\widetilde{\Gamma}}(u, u') = d(u, u');$$

here $\gamma^0_{x,y}([u, u'])$ denotes the segment in $\gamma^0_{x,y}$ joining u to u'. Otherwise, denote by u_1 and u_1' the vertices in $\gamma^0_{x,y}$ at minimum distance in Γ_0 from u and u', respectively. Clearly

$$\ell_X\big(\gamma^0_{x,y}([u, u'])\big) = d_{\widetilde{\Gamma}}(u, u_1) + \ell_X\big(\gamma^0_{x,y}([u_1, u_1'])\big) + d_{\widetilde{\Gamma}}(u_1', u'). \tag{5.13}$$

Note that $\ell_X\big(\gamma^0_{x,y}([u_1, u_1'])\big)$ is equal to the sum of the lengths in X of the edges in $\gamma^0_{x,y}([u_1, u_1'])$. Since the length of each of these is dominated by θ,

$$\ell_X\big(\gamma^0_{x,y}([u_1, u_1'])\big) \leq \theta \, \#\big\{\text{edges in } \gamma^0_{x,y}([u_1, u_1'])\big\} = \theta \, d_{\Gamma_0}(u_1, u_1').$$

Furthermore $d_{\widetilde{\Gamma}}(u, u_1) \leq \theta \, d_{\Gamma_0}(u, u_1)$ and $d_{\widetilde{\Gamma}}(u_1', u') \leq \theta \, d_{\Gamma_0}(u_1', u')$ by the right hand inequality in (5.4). Now, these estimates and (5.13) imply that

$$\begin{aligned}\ell_X\left(\gamma^0_{x,y}([u,u'])\right) &\le \theta\left[d_{\Gamma_0}(u,u_1)+d_{\Gamma_0}(u_1,u_1')+d_{\Gamma_0}(u_1',u')\right]\\ &= \theta\, d_{\Gamma_0}(u,u').\end{aligned}$$

The construction of Γ implies that the length in X of any geodesic joining any two neighbours in Γ is at least 1. Therefore $d_{\Gamma_0}(u,u') \le d_{\widetilde{\Gamma}}(u,u')$, whence

$$\ell_X\left(\gamma^0_{x,y}([u,u'])\right) \le \theta\, d_{\widetilde{\Gamma}}(u,u').$$

Since, by Step II$_2$, the identity map is a (λ,β)-rough isometry between $(\widetilde{\Gamma}, d_{\widetilde{\Gamma}})$ and (X,d), we can conclude that

$$\ell_X\left(\gamma^0_{x,y}([u,u'])\right) \le \theta\lambda\, d(u,u') + \theta\beta,$$

i.e., $\gamma^0_{x,y}$ is a $(\theta\lambda, \theta\beta)$-quasigeodesic in X, as desired.

Step III$_4$: the following upper estimate holds:

$$d_{\Gamma_0}(x,y) \le d(x,y) + \eta \qquad \forall x,y \in \Gamma_0, \tag{5.14}$$

where η is as in Step III$_2$.

Preliminarily, observe that

$$d \le d_{\widetilde{\Gamma}} \le \theta\, d_{\Gamma_0} \tag{5.15}$$

on Γ_0. The left hand inequality is trivial, and the right hand inequality follows from Proposition 2.12 (i) (with θ in place of A_2).

Note that if $x = y$, then (5.14) holds trivially. If y agrees with o, then

$$d_{\Gamma_0}(x,o) \le d_{\Gamma_0}(x,\lfloor x\rfloor) + d_{\Gamma_0}(\lfloor x\rfloor, o) \le 1 + d_{\mathrm{T}}(\lfloor x\rfloor, o) = 1 + d(\lfloor x\rfloor, o).$$

Now,

$$d(\lfloor x\rfloor, o) \le d(\lfloor x\rfloor, x) + d(x,o) \le \theta + d(x,o).$$

Thus,

$$d_{\Gamma_0}(x,o) \le 1 + \theta + d(x,o),$$

and (5.14) holds, because $1+\theta < \eta$.

Thus, we can assume that x and y are distinct points both different from o. Denote by $\gamma_{x,y}$, $\gamma_{o,x}$ and $\gamma_{o,y}$ the edges of the geodesic triangle oxy in X, and by $\gamma^0_{x,y}$, $\gamma^0_{o,x}$ and $\gamma^0_{o,y}$ the edges of the corresponding geodesic triangle oxy in Γ_0.

Since, by Step II$_2$, the metric graph Γ_0 is a δ'-hyperbolic length space, Remark 2.5 guarantees the existence of points p, v and w in $\gamma^0_{x,y}$, $\gamma^0_{o,x}$ and $\gamma^0_{o,y}$, respectively, such that

$$d_{\Gamma_0}(p,v) < \delta' \quad \text{and} \quad d_{\Gamma_0}(p,w) < \delta'. \tag{5.16}$$

Step III$_3$ and Morse Lemma in hyperbolic geometry ensure that $\gamma^0_{o,x}$, $\gamma^0_{x,y}$ and $\gamma^0_{o,y}$ are contained in $D_{\theta\lambda,\theta\beta}$-neighbourhoods of $\gamma_{o,x}$, $\gamma_{o,y}$ and $\gamma_{x,y}$, respectively. In particular, there exist points v' in $\gamma_{o,x}$, w' in $\gamma_{o,y}$ and p' in $\gamma_{x,y}$ such that

$$\max\big(d(v,v'), d(w,w'), d(p,p')\big) < D_{\theta\lambda,\theta\beta}. \tag{5.17}$$

As an intermediate step, we derive a few consequences from (5.15), (5.16) and (5.17). The triangle inequality and (5.17) imply that

$$d(p',v') \le d(p',p) + d(p,v) + d(v,v') < d(p,v) + 2D_{\theta\lambda,\theta\beta}.$$

Moreover $d(p,v) \le \theta\, d_{\Gamma_0}(p,v) \le \theta\,\delta'$ by (5.15) and (5.16). By combining the last two estimates we see that

$$d(p',v') \le \theta\,\delta' + 2D_{\theta\lambda,\theta\beta}; \tag{5.18}$$

A similar argument yields

$$d(p',w') \le \theta\,\delta' + 2D_{\theta\lambda,\theta\beta}. \tag{5.19}$$

Now,

$$d_{\Gamma_0}(x,y) = d_{\Gamma_0}(x,p) + d_{\Gamma_0}(p,y),$$

because p is a point on the geodesic in Γ_0 joining x and y. We use the triangle inequality applied to the triangle xpv in Γ_0 and (5.16), and obtain that $d_{\Gamma_0}(x,p) \le d_{\Gamma_0}(x,v) + \delta'$. A similar argument applied to the triangle ypw in Γ_0 yields $d_{\Gamma_0}(p,y) \le d_{\Gamma_0}(w,y) + \delta'$. By combining these inequalities, we obtain that

$$d_{\Gamma_0}(x,y) \le d_{\Gamma_0}(x,v) + d_{\Gamma_0}(w,y) + 2\delta'. \tag{5.20}$$

Consider first x and v. Since v belongs to the geodesic $\gamma^0_{o,x}$,

$$d_{\Gamma_0}(x,v) = d_{\Gamma_0}(x,o) - d_{\Gamma_0}(v,o). \tag{5.21}$$

Now, $\lfloor v\rfloor$ lies on the geodesic in Γ_0 joining o and v, and $\lfloor x\rfloor$ lies on the geodesic in Γ_0 joining o and x. Furthermore, x belongs to a geodesic segment joining $\lfloor x\rfloor$ and one of its neighbours. Therefore

$$d_{\Gamma_0}(v,o) \ge d_{\Gamma_0}(\lfloor v\rfloor,o) \quad \text{and} \quad d_{\Gamma_0}(\lfloor x\rfloor,o) \le d_{\Gamma_0}(x,o) \le d_{\Gamma_0}(\lfloor x\rfloor,o) + 1.$$

These inequalities and (5.21) imply that

$$d_{\Gamma_0}(x,v) \leq d_{\Gamma_0}(\lfloor x \rfloor, o) - d_{\Gamma_0}(\lfloor v \rfloor, o) + 1 = d_{\Gamma_0}(\lfloor x \rfloor, \lfloor v \rfloor) + 1 :$$

in the equality above we have used the fact that $\lfloor v \rfloor$ lies on the geodesic joining o and $\lfloor x \rfloor$.

A similar inequality holds if we replace x and v with y and w. These observations and (5.20) imply that

$$d_{\Gamma_0}(x,y) \leq d_{\Gamma_0}(\lfloor x \rfloor, \lfloor v \rfloor) + d_{\Gamma_0}(\lfloor w \rfloor, \lfloor y \rfloor) + 2\delta' + 2.$$

Now, observe that

$$d_{\Gamma_0}(\lfloor x \rfloor, \lfloor v \rfloor) = d_{\Gamma_0}(\lfloor x \rfloor, o) - d_{\Gamma_0}(o, \lfloor v \rfloor) = d(\lfloor x \rfloor, o) - d(o, \lfloor v \rfloor);$$

the second equality follows from the construction of the tree T (see Step I). Similarly,

$$d_{\Gamma_0}(\lfloor y \rfloor, \lfloor w \rfloor) = d_{\Gamma_0}(\lfloor y \rfloor, o) - d_{\Gamma_0}(o, \lfloor w \rfloor) = d(\lfloor y \rfloor, o) - d(o, \lfloor w \rfloor).$$

Now, the triangle inequality implies that

$$d(\lfloor x \rfloor, o) \leq d(\lfloor x \rfloor, x) + d(x,o) \leq d_{\widetilde{\Gamma}}(\lfloor x \rfloor, x) + d(x,o) \leq \theta + d(x,o).$$

Similarly

$$d(\lfloor y \rfloor, o) \leq d(\lfloor y \rfloor, y) + d(y,o) \leq \theta + d(y,o).$$

For much the same reason we see that

$$d(o, \lfloor v \rfloor) \geq d(o,v) - \theta \quad \text{and} \quad d(o, \lfloor w \rfloor) \geq d(o,w) - \theta.$$

Now, the triangle inequality applied to the triangles ovv' and oww' in X, together with the estimate (5.17), imply that

$$d(o,v) \geq d(o,v') - d(v,v') \geq d(o,v') - D_{\theta\lambda,\theta\beta}$$

and

$$d(o,w) \geq d(o,w') - d(w,w') \geq d(o,w') - D_{\theta\lambda,\theta\beta}.$$

By combining the estimates above, we find that

$$d_{\Gamma_0}(x,y) \leq d(x,o) - d(v',o) + d(y,o) - d(w',o) + 4\theta + 2D_{\theta\lambda,\theta\beta} + 2\delta' + 2. \tag{5.22}$$

Observe that $d(x,o) - d(v',o) = d(x,v')$, because v' belongs to $\gamma_{o,x}$, and similarly $d(y,o) - d(w',o) = d(y,w')$. The triangle inequality applied to the triangles $p'xv'$ and $p'yw'$ in X and the fact that $d \leq d_{\widetilde{\Gamma}} \leq \theta \, d_{\Gamma_0}$ on Γ_0 give

$$\begin{aligned} d(x, v') &\le d(x, p') + d(p', p) + d(p, v) + d(v, v') \\ &\le d(x, p') + 2D_{\theta\lambda,\theta\beta} + \theta\, d_{\Gamma_0}(p, v) \\ &\le d(x, p') + 2D_{\theta\lambda,\theta\beta} + \theta\, \delta', \end{aligned}$$

and similarly,

$$d(y, w') \le d(y, p') + 2D_{\theta\lambda,\theta\beta} + \theta\, \delta'.$$

Now, (5.22) and the fact that $d(x, y) = d(x, p') + d(p', y)$ (which holds, because p' belongs to $\gamma_{x,y}$) imply that

$$\begin{aligned} d_{\Gamma_0}(x, y) &\le d(x, p') + d(p', y) + 2\theta\delta' + 4\theta + 6D_{\theta\lambda,\theta\beta} + 2\delta' + 2 \\ &= d(x, y) + 2\theta\delta' + 4\theta + 6D_{\theta\lambda,\theta\beta} + 2\delta' + 2, \end{aligned}$$

as required.

Step IV: construction of the spiderweb $\widehat{\Gamma}$ and conclusion of the proof. We split up Step IV into Step IV$_1$, where we prove that Γ is a quasi-spiderweb, and Step IV$_2$, where we show that Γ can be completed to a spiderweb $\widehat{\Gamma}$, which is still strictly roughly isometric to X.

Step IV$_1$: (Γ, d_Γ) is a quasi-spiderweb. Specifically, if x and y are neighbours in Γ, and $j > D_{2,4} + \delta + \theta$, then $p^j(x)$ and $p^j(y)$ either coincide or are neighbours in Γ.

Suppose that x and y are neighbours in Γ belonging to Σ_n, and consider the edges $\gamma_{x,y}$, $\gamma_{o,x}$ and $\gamma_{o,y}$ of the geodesic triangle oxy in (X, d). Consider also the geodesics $\gamma^0_{o,x}$ and $\gamma^0_{o,y}$ in (Γ_0, d_{Γ_0}), and for j in $\{1, \ldots, n-1\}$, the predecessors $p^j(x)$ and $p^j(y)$ of x and y, respectively.

By (5.2), $d\big(p^j(x), p^{j+1}(x)\big) \le 2$ for every j. We have already proved in Step III$_1$ (i) that $\gamma^0_{o,x}$ and $\gamma^0_{o,y}$ are $(2, 4)$-quasigeodesics in (X, d). By the Morse Lemma in hyperbolic geometry (see Lemma 2.7), there exists a constant $D_{2,4}$, depending on δ, such that $\gamma^0_{o,x}$ is contained in the $D_{2,4}$-neighbourhood of $\gamma_{o,x}$ and $\gamma^0_{o,y}$ is contained in the $D_{2,4}$-neighbourhood of $\gamma_{o,y}$. In particular, there exist points v in $\gamma_{o,x}$ and w in $\gamma_{o,y}$ such that

$$d\big(p^j(x), v\big) < D_{2,4} \quad \text{and} \quad d\big(w, p^j(y)\big) < D_{2,4}. \tag{5.23}$$

Since (X, d) is a δ-hyperbolic length space, the geodesic $\gamma_{o,y}$ is contained in a δ-neighbourhood of $\gamma_{o,x} \cup \gamma_{x,y}$. In particular, there exists a point z in $\gamma_{o,x} \cup \gamma_{x,y}$ such that $d(z, w) < \delta$.

We claim that if $j > D_{2,4} + \delta + \theta$, then z belongs to $\gamma_{o,x}$. It suffices to show that $d(w, \gamma_{x,y}) > \delta$. Indeed, if p is in $\gamma_{x,y}$, then

$$d(w, p) \ge d(w, y) - d(y, p) \ge d\big(p^j(y), y\big) - d\big(w, p^j(y)\big) - d(y, p);$$

the first inequality above follows from the triangle inequality applied to the triangle wpy and the second from the triangle inequality applied to the triangle $p^j(y)wy$.

Now, $d(y,p) \leq \theta$, because y and p belong to the geodesic $\gamma_{x,y}$ and this geodesic has length at most θ (for x and y are neighbours in Γ belonging to the same level Σ_n), $d\big(w, p^j(y)\big) < D_{2,4}$ by (5.23), and $d\big(p^j(y), y\big) \geq j$. Hence

$$d(w,p) \geq j - \theta - D_{2,4} > \delta,$$

as claimed.

Finally, we shall show that if $j > \theta + D_{2,4} + \delta$, then $d\big(p^j(x), p^j(y)\big) \leq \theta$, whence $p^j(x)$ and $p^j(y)$ are connected in Γ.

Observe that $p^j(x)$ and $p^j(y)$ belong to Σ_{n-j}, hence to $S_{n-j}(o)$. Furthermore, the triangle inequality in X, applied first to the triangle $op^j(y)w$, and then to the triangle ozw in X implies that

$$n - j - D_{2,4} \leq d(o,w) \leq n - j + D_{2,4},$$

and

$$d(w,o) - \delta \leq d(o,z) \leq d(w,o) + \delta,$$

so that

$$n - j - D_{2,4} - \delta \leq d(o,z) \leq n - j + D_{2,4} + \delta. \tag{5.24}$$

A similar argument gives that

$$n - j - D_{2,4} \leq d(v,o) \leq n - j + D_{2,4}. \tag{5.25}$$

Now, since v and z belong to $\gamma_{o,x}$,

$$d(v,z) = \big|d(v,o) - d(z,o)\big| \leq 2D_{2,4} + \delta;$$

the inequality above follows by combining (5.24) and (5.25). Finally, notice that, by the triangle inequality,

$$\begin{aligned} d\big(p^j(x), p^j(y)\big) &\leq d\big(p^j(x), v\big) + d(v,z) + d(z,w) + d\big(w, p^j(y)\big) \\ &\leq 4D_{2,4} + 2\delta \\ &\leq \theta; \end{aligned}$$

the last inequality follows from the definition of θ (see (5.3)). Therefore the points $p^j(x)$ and $p^j(y)$ either coincide or are connected in Γ, as required.

Step IV$_2$: the spiderweb $(\widehat{\Gamma}, d_{\widehat{\Gamma}})$ *is strictly roughly isometric to* (X,d). The graph $\widehat{\Gamma}$ has the same set of vertices as Γ. We endow $\widehat{\Gamma}$ the structure of a spiderweb by adding edges to the quasi-spiderweb Γ. We declare that two distinct vertices x and y in Γ are neighbours in $\widehat{\Gamma}$ if and only if one of the following holds:

(i) x and y are neighbours in Γ;
(ii) x and y are not neighbours in Γ, they belong to the same level, and there exists a positive integer j and vertices v and w, which are neighbours in Γ, such that $x = p^j(v)$ and $y = p^j(w)$.

Thus, we are adding edges to Γ in order to get $\widehat{\Gamma}$, but, loosely speaking, not too many, for Γ is already a quasi-spiderweb (see Step IV$_1$). Therefore there exists a positive integer m such that if v and w are neighbours in Γ and belong to the same level, Σ_n say, then for all j in $\{m, \ldots, n\}$ either the vertices $p^j(v)$ and $p^j(w)$ are neighbours in Γ or they coincide.

We endow $\widehat{\Gamma}$ with the graph distance $d_{\widehat{\Gamma}}$ and set $K := \lceil D_{2,4} + \delta + \theta + 1\rceil$. We claim that

$$d_{\widehat{\Gamma}}(x, y) \leq d_\Gamma(x, y) \leq d_{\widehat{\Gamma}}(x, y) + 2K. \qquad \forall x, y \in \Gamma. \tag{5.26}$$

The left hand inequality above is trivial. We prove the right hand inequality. Suppose that x and y are in Γ. By Proposition 3.2 (i), there exists a standard geodesic $\widehat{\gamma}$ in $\widehat{\Gamma}$ connecting x to y. We shall define a path γ in Γ such that

$$\ell_\Gamma(\gamma) \leq \ell_{\widehat{\Gamma}}(\widehat{\gamma}) + 2K,$$

which clearly implies (5.26).

Denote by $[x, \tilde{x}]$, $[\tilde{x}, \tilde{y}]$, and $[\tilde{y}, y]$ the ascending, the horizontal and the descending parts of $\widehat{\gamma}$, respectively. Observe that $\widetilde{x}$ and $\widetilde{y}$ belong to the same level, Σ_n say, so that the distance of both from o is equal to n in Γ and in $\widehat{\Gamma}$.

If $n \leq K$, then define γ to be the union of the ascending geodesic $[x, o]$ and the descending geodesic $[o, y]$.

If, instead, $n > K$, then define γ as the union of the ascending geodesic $[x, p^K(\widetilde{x})]$, the horizontal path γ_K, defined as the projection on $\Sigma_{h(\widetilde{x})-K}$ of $[\widetilde{x}, \widetilde{y}]$ and the descending geodesic $[p^K(\widetilde{y}), y]$.

Observe that, by definition of $\widehat{\Gamma}$ (see Step IV$_1$), if two neighbours u and v in $\widehat{\Gamma}$ belong to the same level, then $p^K(u)$ and $p^K(v)$ either coincide, or are neighbours in Γ. This implies that γ is a path in Γ connecting x to y. We conclude that

$$d_\Gamma(x, y) \leq \ell_\Gamma(\gamma) \leq \ell_{\widehat{\Gamma}}(\widehat{\gamma}) + 2K = d_{\widehat{\Gamma}}(x, y) + 2K,$$

as claimed.

By Step III, (Γ_0, d_{Γ_0}) and (X, d) are strictly roughly isometric. It is straightforward to check that (Γ_0, d_{Γ_0}) and (Γ, d_Γ) are strictly roughly isometric. Furthermore (5.26) implies that (Γ, d_Γ) and $(\widehat{\Gamma}, d_{\widehat{\Gamma}})$ are strictly roughly isometric. Since rough isometry is a transitive relation, we can conclude that $(\widehat{\Gamma}, d_{\widehat{\Gamma}})$ and (X, d) are strictly roughly isometric, as required. □

Remark 5.2. Observe that the lower estimate in Step III_1 (ii) holds for any pair of points z and z' in T. Indeed, denote by $z \wedge z'$ the confluent of z and z' in T with respect to the root o, i.e., the last point in common between the geodesics in T joining o to z and z'. Clearly

$$d_{\mathrm{T}}(z, z') = d_{\mathrm{T}}(z, z \wedge z') + d_{\mathrm{T}}(z', z \wedge z').$$

Now, the point $z \wedge z'$ belongs to both the tree geodesics joining o with z and o with z'. Therefore, by case (ii), applied twice, we see that

$$d_{\mathrm{T}}(z, z \wedge z') \geq d(z, z \wedge z') - 2\, D_{2,4} \quad \text{and} \quad d_{\mathrm{T}}(z', z \wedge z') \geq d(z', z \wedge z') - 2\, D_{2,4}.$$

By combining the formulae above, we see that

$$d_{\mathrm{T}}(z, z') \geq d(z, z') - 4\, D_{2,4},$$

as required.

6. The main result and its consequences

In this section we prove Theorem 1.1 and some of its consequences. We start with a well known result concerning $\mathscr{M}_0$ (see (1.1) for its definition). For the sake of completeness, we include a simple proof thereof.

Proposition 6.1. *Suppose that* (X, d, μ) *is a locally doubling metric measure space. Then the operator* $\mathscr{M}_0$ *is of weak type* $(1, 1)$ *and bounded on* $L^p(X)$ *for all* p *in* $(1, \infty]$.

Proof. The operator $\mathscr{M}_0$ is trivially bounded on $L^\infty(X)$. We prove the weak type $(1, 1)$ estimate: the boundedness of $\mathscr{M}_0$ on $L^p(X)$ for p in $(1, \infty)$ will follow from this by interpolation.

Consider a 1-discretisation $\mathfrak{D}$ of X. Then every point in X is covered by a ball $B_2(z)$ for some z in $\mathfrak{D}$. Set $\psi := \sum_{z \in \mathfrak{D}} \mathbf{1}_{B_2(z)}$. Observe that

$$1 \leq \psi \leq L_{12,1/2};$$

see, for instance, [21, Lemma 1 (i)]. The constant $L_{\varrho,s}$ is defined in Remark 2.1.

Suppose that f is in $L^1(X)$. For every z in $\mathfrak{D}$, set $f_z := f\, \mathbf{1}_{B_2(z)}/\psi$. Observe that $f = \sum_{z \in \mathfrak{D}} f_z$. By the sublinearity of $\mathscr{M}_0$ we have that

$$\mathscr{M}_0 f \leq \sum_{z \in \mathfrak{D}} \mathscr{M}_0 f_z \tag{6.1}$$

Observe that the support of $\mathscr{M}_0 f_z$ is contained in $\overline{B_4(z)}$. Thus, given a point x, the function $\mathscr{M}_0 f_z$ (possibly) does not vanish at x only if $d(x,z) < 4$, i.e., only if z belongs to $B_4(x)$. Now

$$\sharp\big(\mathfrak{D} \cap B_4(x)\big) \leq \frac{L_{12,1/2}}{L_{16}}$$

(see, for instance, [21, Lemma 1 (ii)]), where L_s is the constant appearing in (2.1)). Denote by N the constant on the right hand of the inequality above. Then for each point x there are at most N summands of the series in (6.1) (possibly) nonvanishing at x. Consequently, for each positive number λ the following containment holds:

$$\big\{\mathscr{M}_0 f > \lambda\big\} \subseteq \bigcup_{z\in\mathfrak{D}} \big\{\mathscr{M}_0 f_z > \lambda/N\big\}. \tag{6.2}$$

We shall prove that there exists a constant C, independent of z, such that

$$\big|\big\{\mathscr{M}_0 f_z > \sigma\big\}\big| \leq \frac{L_5^3}{\sigma}\, \big\|f_z\big\|_1 \qquad \forall \sigma > 0. \tag{6.3}$$

This will imply that

$$\big|\big\{\mathscr{M}_0 f > \lambda\big\}\big| \leq \frac{L_5^3 N}{\lambda} \sum_{z\in\mathfrak{D}} \big\|f_z\big\|_1 = \frac{L_5^3 N}{\lambda}\, \big\|f\big\|_1\,,$$

as required.

Thus, it remains to prove (6.3). Denote by E_σ the level set $\big\{\mathscr{M}_0 f_z > \sigma\big\}$. If x belongs to E_σ, then there exists a ball B_x, centred at x and of radius at most 1, such that

$$|B_x| \leq \frac{1}{\sigma} \int_{B_x} |f_z| \,\mathrm{d}\mu.$$

The collection of balls $\mathscr{F} := \{B_x : x \in E_\sigma\}$ covers E_σ. Now, we can argue *verbatim* as in the proof of [27, Lemma, p, 9] and select from $\mathscr{F}$ a (possibly finite) sequence $\{B_j\}$ of mutually disjoint balls such that $\{5B_j\}$ covers E_σ. Therefore

$$|E_\sigma| \leq \sum_j |5B_j| \leq L_{5,1} \sum_j |B_j| \leq \frac{L_{5,1}}{\sigma} \int_{B_j} |f_z| \,\mathrm{d}\mu \leq \frac{L_{5,1}}{\sigma}\, \big\|f_z\big\|_1\,,$$

as required. □

We prove our main result, Theorem 1.1, which we state in a more detailed form for the reader's convenience.

Theorem. *Suppose that $1 < a \le b < a^2$ and that δ is a nonnegative number. Assume that (X, d) is a δ-hyperbolic complete length space and μ is a locally doubling Borel measure on X such that (1.3) holds. Then the centred HL maximal operator $\mathscr{M}$ is bounded on $L^p(X)$ for all $p > \varrho$, and it is of weak type (ϱ, ϱ).*

Proof. By Proposition 6.1, $\mathscr{M}_0$ is of weak type $(1,1)$ and bounded on $L^p(X)$ for all p in $(1, \infty]$. Thus, it suffices to prove that $\mathscr{M}_\infty$ has the required mapping properties.

Since X belongs to the class $\mathscr{X}^\delta_{a,b}$, Theorem 5.1 implies that X is strictly roughly isometric to a spiderweb $\widehat{\Gamma}$ in the class $\Upsilon^{\delta'}_{a,b}$, endowed with its graph distance. More precisely, the proof of Theorem 5.1 shows that the set of vertices of $\widehat{\Gamma}$ is a discrete subset of X, and that the identity operator $\iota : \widehat{\Gamma} \to X$ is a strict β-rough isometric embedding of $(\widehat{\Gamma}, d_{\widehat{\Gamma}})$ into (X, d), i.e.,

$$d_{\widehat{\Gamma}}(x,y) - \beta \le d(x,y) \le d_{\widehat{\Gamma}}(x,y) + \beta \qquad \forall x, y \in \widehat{\Gamma}.$$

Now, by Theorem 4.1, the operator $\mathscr{M}_\infty$ is of weak type (ϱ, ϱ) and bounded on $L^p(\widehat{\Gamma})$ for all p in $(\varrho, \infty]$.

It remains to prove that the operator $\mathscr{M}_\infty$ on X inherits the abovementioned mapping properties of the operator $\mathscr{M}_\infty$ on $\widehat{\Gamma}$.

Recall that $\mathscr{C} := \{B_2(x) : x \in \widehat{\Gamma}\}$ is a covering of X (see (5.1)). For each point z in X denote by $\widehat{\Gamma}_z$ the set of all x in $\widehat{\Gamma}$ such that z belongs to $B_2(x)$, and set $\omega := \sup_{z \in X} \sharp(\widehat{\Gamma}_z)$. We call ω the *overlapping number* of the covering $\mathscr{C}$.

We *claim* that ω is finite. Indeed, observe that $d(z,x) < 2$ for every x in $\widehat{\Gamma}_z$. Since the vertices of $\widehat{\Gamma}$ consitute a 2-discretisation of X, the balls in $\{B_1(x) : x \in \widehat{\Gamma}\}$ are mutually disjoint and the balls in $\{B_1(x) : x \in \widehat{\Gamma}_z\}$ are contained in $B_3(z)$. This and the upper estimate in (1.3) imply that

$$\sharp(\widehat{\Gamma}_z) \cdot \inf \big\{\mu\big(B_1(x)\big) : x \in \widehat{\Gamma}_z\big\} \le \sup \big\{\mu\big(B_3(z)\big) : z \in X\big\} \le C\, b^3.$$

Now, the LDP and the lower estimate in (1.3) imply that

$$c\, a \le \mu\big(B_1(x)\big).$$

Altogether, we see that

$$\sharp(\widehat{\Gamma}_z) \le C\, \frac{b^3}{c\, a} \qquad \forall z \in X,$$

and the claim follows.

For each nonnegative measurable function f on X, consider the function πf on $\widehat{\Gamma}$, defined by the formula

$$(\pi f)(x) := \int_{B_2(x)} f \, \mathrm{d}\mu \qquad \forall x \in \widehat{\Gamma}.$$

For each z in X, denote by x_z a point in $\widehat{\Gamma}$ at minimum distance in X from z; in particular, $d(z, x_z) < 2$. There are at most ω such points, so that the choice of x_z is somewhat arbitrary. However, this will have no consequences in what follows. For each x in $\widehat{\Gamma}$, denote by Ω_x the set of all z in X such that $x_z = x$. Clearly Ω_x is contained in $B_2(x)$, and

$$X = \bigcup_{x \in \widehat{\Gamma}} \Omega_x.$$

Simple geometric considerations show that for each z in X and for every $R > 1$ the following containment holds

$$B_R(z) \subseteq \bigcup_{x \in \widehat{\Gamma} \cap B_{R+2}(z)} B_2(x) \subseteq B_{R+4}(z) \subseteq B_{R+6}(x_z),$$

so that

$$\mathbf{1}_{B_R(z)} \leq \sum_{x \in \widehat{\Gamma} \cap B_{R+2}(z)} \mathbf{1}_{B_2(x)} \leq \omega \, \mathbf{1}_{B_{R+6}(x_z)}.$$

Consequently

$$\begin{aligned} \int_{B_R(z)} f \, \mathrm{d}\mu &\leq \sum_{x \in \widehat{\Gamma} \cap B_{R+2}(z)} \int_X f \, \mathbf{1}_{B_2(x)} \, \mathrm{d}\mu \\ &= \sum_{x \in \widehat{\Gamma} \cap B_{R+2}(z)} \pi f(x) \\ &\leq \sum_{x \in \widehat{\Gamma} \cap B_{R+4}(x_z)} \pi f(x). \end{aligned}$$

Since the identity is a strict β-rough isometry between $(\widehat{\Gamma}, d_{\widehat{\Gamma}})$ and (X, d),

$$d_{\widehat{\Gamma}}(x, x_z) \leq d(x, x_z) + \beta,$$

so that $\widehat{\Gamma} \cap B_{R+4}(x_z) \subseteq B^{\widehat{\Gamma}}_{R+4+\beta}(x_z)$. Thus, we have proved that

$$\int_{B_R(z)} f \, \mathrm{d}\mu \leq \int_{B^{\widehat{\Gamma}}_{R+\beta+4}(x_z)} \pi f \, \mathrm{d}\mu_{\widehat{\Gamma}}, \tag{6.4}$$

where $\mu_{\widehat{\Gamma}}$ denotes the counting measure on the vertices of $\widehat{\Gamma}$. Now observe that

$$\begin{aligned}\mu_{\widehat{\Gamma}}\big(B^{\widehat{\Gamma}}_{R+\beta+4}(x_z)\big) &\le \sum_{x\in B^{\widehat{\Gamma}}_{R+\beta+4}(x_z)} \frac{\mu\big(B_1(x)\big)}{ca} \\ &\le \omega\,(c\,a)^{-1}\,\mu\big(B_{R+2\beta+5}(x_z)\big) \\ &\le \omega\,C_\beta\,(c\,a)^{-1}\,\mu\big(B_R(z)\big),\end{aligned} \tag{6.5}$$

where in the last line we have used the following easy observation:

$$B_{R+2\beta+5}(x_z) \subset B_{R+2\beta+7}(z) \subset \bigcup_{x\in\widehat{\Gamma}\cap B_R(z)} B_{2\beta+11}(x),$$

that implies

$$\begin{aligned}\mu(B_{R+\beta+5}(x_z)) &\le \sum_{x\in\widehat{\Gamma}\cap B_R(z)} \mu\big(B_{2\beta+11}(x)\big) \\ &\le C\,b^{2\beta}\,\mu_{\widehat{\Gamma}}\big(\{x\in\widehat{\Gamma}\cap B_R(z)\}\big) \\ &\le C_\beta\,\mu\big(B_R(x)\big),\end{aligned}$$

as desired. By combining (6.4) and (6.5), we find that to each z in X, we can associate a point x_z in $\widehat{\Gamma}$ such that $d(z,x_z)\le 2$ and

$$\mathscr{M}_\infty f(z) \le C_\beta\ \mathscr{M}_\infty(\pi f)(x_z).$$

For each $\alpha>0$ set

$$\widehat{\Gamma}(\alpha) := \big\{x\in\widehat{\Gamma}:\ \text{there exists } z\in E_{\mathscr{M}_\infty f}(\alpha) \text{ such that } x_z = x\big\}.$$

Observe that

$$E_{\mathscr{M}_\infty f}(\alpha) = \bigcup_{x\in\widehat{\Gamma}(\alpha)} \big(E_{\mathscr{M}_\infty f}(\alpha)\cap\Omega_x\big) \subseteq \bigcup_{x\in\widehat{\Gamma}(\alpha)} \big(E_{\mathscr{M}_\infty f}(\alpha)\cap B_2(x)\big),$$

so that

$$\mu\big(E_{\mathscr{M}_\infty f}(\alpha)\big) \le \sum_{x\in\widehat{\Gamma}(\alpha)} \mu\big(B_2(x)\big) \le C\,b^2\cdot\sharp\big(\widehat{\Gamma}(\alpha)\big). \tag{6.6}$$

Now, if x belongs to $\widehat{\Gamma}(\alpha)$, then $C_\beta\ \mathscr{M}_\infty(\pi f)(x)>\alpha$, i.e., x belongs to $E_{\mathscr{M}_\infty(\pi f)}(\alpha/C_\beta)$, whence

$$\sharp\big(\widehat{\Gamma}(\alpha)\big) \le \mu_{\widehat{\Gamma}}\big(E_{\mathscr{M}_\infty(\pi f)}(\alpha/C_\beta)\big).$$

Therefore the following inequality between the distribution functions of $\mathscr{M}_\infty f$ and $\mathscr{M}_\infty(\pi f)$ holds

$$\mu\big(E_{\mathscr{M}_\infty f}(\alpha)\big) \leq C\, b^2 \cdot \mu_{\widehat{\Gamma}}\big(E_{\mathscr{M}_\infty(\pi f)}(\alpha/C_\beta)\big) \qquad \forall \alpha > 0.$$

Now, observe that if f is in $L^\varrho(X)$, then πf is in $L^\varrho(\widehat{\Gamma})$. Indeed,

$$\begin{aligned} \|\pi f\|^\varrho_{L^\varrho(\widehat{\Gamma})} &= \sum_{x\in\widehat{\Gamma}} \Big(\int_{B_2(x)} f \,\mathrm{d}\mu\Big)^\varrho \\ &\leq \sum_{x\in\widehat{\Gamma}} \mu\big(B_2(x)\big)^{\varrho/\varrho'} \int_X f^\varrho\, \mathbf{1}_{B_2(x)} \,\mathrm{d}\mu \\ &\leq (Cb^2)^{\varrho/\varrho'}\, \omega\, \|f\|^\varrho_{L^\varrho(X)}\,; \end{aligned} \tag{6.7}$$

the first inequality follows from Hölder's inequality (with exponents ϱ and ϱ'), and the second from condition (1.3) and the fact that ω is the overlapping number of the cover $\mathscr{C}$.

Now, $\mathscr{M}_\infty$ is of weak type (ϱ,ϱ) on $\widehat{\Gamma}$ by assumption. This, together with the estimates (6.6) and (6.7), implies that

$$\mu\big(E_{\mathscr{M}_\infty f}(\alpha)\big) \leq C_\beta\, (Cb^2)^{1+\varrho/\varrho'}\, \frac{\omega}{\alpha^\varrho}\, \|f\|^\varrho_{L^\varrho(X)}\,,$$

i.e., $\mathscr{M}_\infty$ is of weak type (ϱ,ϱ) on X, as required.

This concludes the proof of the theorem. □

Remark 6.2. Observe that when $a = b$ in the assumptions of the above theorem, $\mathscr{M}$ is of weak type (1,1). This case includes all symmetric spaces of noncompact type of rank 1.

Corollary 6.3. *Suppose that A and B are positive numbers such that $A \leq B < 2A$, and M is a Cartan–Hadamard Riemannian manifold with pinched curvature sectional curvature, i.e., $-B^2 \leq K \leq -A^2$. Then $\mathscr{M}$ is of weak type $(B/A, B/A)$ and it is bounded on $L^p(M)$ for all $p > B/A$.*

Proof. It is well known that M is a δ-hyperbolic space. By comparison results [25, Corollary 3.2 (ii)],

$$\mathrm{e}^{(n-1)Ar} \leq \big|B_r(x)\big| \leq \mathrm{e}^{(n-1)Br} \qquad \forall x \in M \quad \forall r \geq 1.$$

Thus, if we set $a := \mathrm{e}^{(n-1)A}$ and $a := \mathrm{e}^{(n-1)B}$, then the condition $B < 2A$ transforms to $b < a^2$, so that M belongs to the class $\mathscr{X}^\delta_{a,b}$. Therefore, by Theorem 1.1, $\mathscr{M}$ is of weak type (ϱ,ϱ) and it is bounded on $L^p(M)$ for all p in $(\varrho,\infty]$. Observe that

$$\log_a b = \frac{\log b}{\log a} = \frac{B}{A}.$$

Hence $\mathscr{M}$ has the required mapping properties. □

Data availability

No data was used for the research described in the article.

References

[1] J.-Ph. Anker, E. Damek, C. Yacoub, Spherical analysis on harmonic AN groups, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 23 (1996) 643–679.
[2] M. Bonk, Quasi-geodesic segments and Gromov hyperbolic spaces, Geom. Dedic. 62 (1996) 281–298.
[3] M.R. Bridson, A. Haefliger, Metric Spaces of Non-positive Curvature, Springer, 1999.
[4] D. Burago, Y. Burago, S. Ivanov, A Course in Metric Geometry, Graduate Studies in Math., vol. 33, American Mathematical Society, 2001.
[5] D. Burago, S. Ivanov, Uniform approximation of metrics by graphs, Proc. Am. Math. Soc. 143 (2015) 1241–1256.
[6] M. Coornaert, T. Delzant, A. Papadopoulos, Géométrie et théorie des groupes. Les groupes hyperboliques de Gromov, Lecture Notes in Mathematics, vol. 1441, Springer-Verlag, 1990.
[7] M. Cowling, S. Meda, A.G. Setti, An overview of harmonic analysis on the group of isometries of a homogeneous tree, Expo. Math. 16 (1998) 385–423.
[8] M. Cowling, S. Meda, A.G. Setti, A weak type $(1,1)$ estimate for a maximal operator on a group of isometries of homogeneous trees, Collect. Math. 118 (2010) 223–232.
[9] P. de la Harpe, E. Ghys, Sur les groupes hyperboliques d'après Mikhael Gromov, Birkhäuser, 1990.
[10] M. Gromov, Hyperbolic groups, in: S.M. Gersten (Ed.), Essays in Group Theory, in: Mathematical Sciences Research Institute Publications, vol. 8, Springer, New York, 1987.
[11] J. Heinonen, Lectures on Analysis on Metric Spaces, London Math. Society Lecture Notes Series, vol. 162, Springer Verlag, 2001.
[12] M. Kanai, Rough isometries, and combinatorial approximation of non-compact Riemannian manifolds, J. Math. Soc. Jpn. 37 (1985) 391–413.
[13] G. Knieper, New results on noncompact harmonic manifolds, Comment. Math. Helv. 87 (2012) 669–703.
[14] M. Levi, S. Meda, F. Santagati, M. Vallarino, Hardy–Littlewood maximal operators on trees with bounded geometry, Trans. Am. Math. Soc. 378 (2025) 3951–3979, https://doi.org/10.1090/tran/9229.
[15] M. Levi, F. Santagati, Hardy–Littlewood fractional maximal operators on homogeneous trees, Math. Z. 308 (2024).
[16] H.-Q. Li, La fonction maximale de Hardy–Littlewood sur une classe d'espaces métriques measurables, C. R. Acad. Sci. Paris, Ser. I 338 (2004) 31–34.
[17] H.-Q. Li, Les fonctions maximales de Hardy–Littlewood pour des measures sur les variétés de type cuspidale, J. Math. Pures Appl. 88 (2007) 261–275.
[18] N. Lohoué, Fonction maximale sur les variétés de Cartan–Hadamard, C. R. Acad. Sci. Paris Sér. I 300 (1985) 213–216.
[19] S. Meda, S. Pigola, A.G. Setti, G. Veronelli, Hardy–Littlewood maximal operators on certain manifolds with bounded geometry, available at arXiv:2502.13109.
[20] S. Meda, F. Santagati, Triangular maximal operators on locally finite trees, Mathematika 70 (2024), https://doi.org/10.1112/mtk.12253.
[21] S. Meda, S. Volpi, Spaces of Goldberg type on certain measured metric spaces, Ann. Mat. Pura Appl. 196 (2017) 947–981.
[22] A. Naor, T. Tao, Random martingales and localization of maximal inequalities, J. Funct. Anal. 259 (2010) 731–779.
[23] S. Ombrosi, I.P. Rivera-Rios, Weighted L^p estimates on the infinite rooted k-ary tree, Math. Ann. 384 (2022) 491–510.
[24] R. Rochberg, M. Taibleson, Factorization of the Green's operator and weak-type estimates for a random walk on a tree, Publ. Math. 35 (1991) 187–207.

[25] T. Sakai, Riemannian Geometry, Translations of Mathematical Monographs, vol. 149, American Mathematical Society, 1992.
[26] J. Soria, P. Tradacete, Geometric properties of infinite graphs and the Hardy–Littlewood maximal operator, J. Anal. Math. 137 (2019) 913–937.
[27] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, N.J., 1970.
[28] J.-O. Strömberg, Weak type L^1 estimates for maximal functions on non-compact symmetric spaces, Ann. Math. 114 (1981) 115–126.