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Original
Low Complexity Fast Direct Solution for Multiscale Problems with Nested Equivalence Source Approximation / Zuo, Y., Li, M., Vipiana, F., Ding, D.. - (2025), pp. 1-3. (19th European Conference on Antennas and Propagation, EuCAP 2025 Stockholm (Swe) 30 March 2025 - 04 April 2025) [10.23919/eucap63536.2025.10999958].

Availability:
This version is available at: 11583/3011176 since: 2026-05-22T12:10:41Z

Publisher:
IEEE

Published
DOI:10.23919/eucap63536.2025.10999958

(Article begins on next page)

Low Complexity Fast Direct Solution for Multiscale Problems with Nested Equivalence Source Approximation

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***Abstract*—We propose a multi-level low complexity fast direct solution of electric field integral equations (EFIEs). Far coupling submatrices are compressed by a kernel-independent method with nested equivalence source approximation (NESA). In this work, equivalence sources are constructed at upper levels while skeletons are selected via the adaptive cross approximation (ACA) at the bottom level to further accelerate the computation. To factorize and inverse the system matrix, an elimination matrix is introduced to each group to reduce the far blocks dimension. Then, an LU factorization is performed to reduce the near blocks dimension. This process is repeated recursively from the bottom level to the top. The inverse of the system matrix is then represented as a multiplication of a series of matrices arising from the factorization process. Numerical results verify the accuracy and the linear complexity of our proposed method.**

***Index Terms*—fast direct solution, electric field integral equation (EFIE), nested equivalence source approximation (NESA).**

I. INTRODUCTION

In recent years, fast direct solutions for analyzing electromagnetic (EM) problems have gained considerable interest due to their significant advantages compared to iterative solutions over convergence and multiple right-hand-sides (RHSs) issues. For example, a RHS compression method commonly used for iterative solutions in the field of RCS computations is proposed [1]. A direct solver typically consists of a forward process to compress the far matrices and a backward process to obtain the final solution. A hierarchical matrix (H-matrix) [2]-[5] factorize the system matrix by representing off-diagonal blocks with low-rank approximations. However, this representation lacks of interactions between child and parent levels in a binary tree. The H^2-matrix is an improved structure of the H-matrix which represents the system matrix in a nested form [6], [7]. Transfer matrices are introduced between the child level and the parent level to represent the far matrices compression of parent levels by child levels. The nested property of an H^2-matrix is the main superiority over an H-matrix that reduces the complexity for both time and memory.

In this work, we propose a low complexity fast direct solution based on a purely algebraic matrix compression method. In the forward process, far coupling matrices are compressed by NESA, where the original unknowns are projected to corresponding equivalence sources [8], [9]. Because there are relatively less unknowns at the bottom level, we select dominant unknowns (i.e. skeletons) rather than equivalences to accelerate the computation, which can be viewed as a mixed equivalences and skeletons form. Elimination matrices are then introduced to reduce the redundant part of submatrices as well as to factorize the system matrix. In the backward process, the inversion is obtained by the multiplication a series of the matrices constructed during the factorization. The direct solution has a liner complexity at low frequencies.

II. THEORY AND FORMULATION

A. *Mixed Nested Equivalence Source Approximation*

Based on an octree, we select skeletons from the original unknowns at the bottom level and the matrix compression process of this step is illustrated in Fig. 1(a). The bottom level mixed NESA low-rank compression can be expressed as:

$$\begin{aligned} Z_{O,S} &= Z_{O,\sigma_O} Z^{\dagger}{}_{\tau_O,\sigma_O} Z_{\tau_O,\tau_S} Z^{\dagger}{}_{\sigma_S,\tau_S} Z_{\sigma_S,S} \\ &= U_O D_{O,S} V_S \end{aligned} \tag{1}$$

where τ, σ are the skeletons and test points, O denotes the *observation* group and S denotes the *source* group. U_O is the receiving matrix, V_S is the radiation matrix and D_O is the translation matrix.

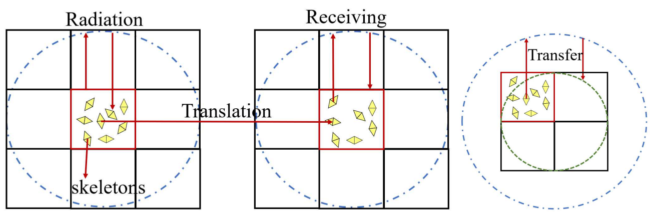


Fig. 1. (a) Mixed NESA low-rank representation at the bottom level. (b) Transfer process where the child skeletons are projected onto the parent equivalences.

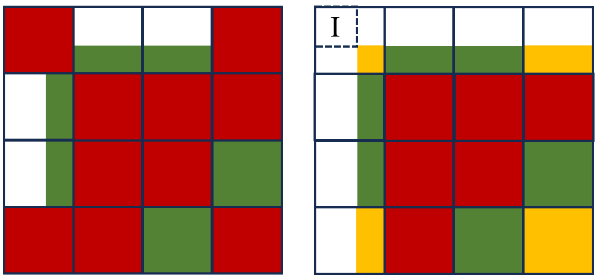
Similar to an H^2-matrix, NESA introduces transfer matrices to connect the child level with the parent level. This enables the reuse of radiation and receiving matrices at child levels to compress far coupling matrices at parent levels

At upper levels, the transfer process is performed where the child skeletons are projected onto the parent equivalences placed on a sphere surface to reconstruct the field radiated by the skeletons as shown in Fig. 1(b). The test points are also distributed around each group on a sphere surface.

B. Fast Direct Solution with Mixed NESA

In order to invers the system matrix in a reduced dimension, two kinds of elimination matrices are introduced for the near and far blocks respectively.

We first obtain the elimination matrix for the receiving matrix U as $E=\left[U^{or\perp},U^{or}\right]^H$, where H denotes the conjugate transpose, U^{or} is the orthogonalization of U and $U^{or\perp}$ is the complementary part of U^{or}. By multiplying, the complementary space of U zeros out the first several rows of the far groups. The radiation matrix V is processed in the same way according to the transpose relation in terms of EFIEs. Take the first group out of a total of four groups as an example, this process is shown in Fig. 2(a), where the red blocks are near matrices and the green ones are far matrices.



(a) (b)

Fig. 2. (a) Far blocks elimination, the red blocks are near matrices and the green ones are far matrices. (b) Near blocks elimination, the yellow blocks are the blocks added by fill-in blocks.

In the next step, a partial LU decomposition of the remaining part of the diagonal block is performed to reduce the near groups. As shown in Fig. 2(b), we need to revise the remaining part of the system matrix colored with yellow where Schur complements are added as the fill-in blocks, which are expressed by the following:

$$Z_r' = Z_r - Z_{r,d} U_d^{-1} L_d^{-1} Z_{d,r} \qquad (2)$$

where Z_r is the remaining part of the system matrix to be factorized, $Z_{r,d}$ and $Z_{d,r}$ are the near blocks associated with the diagonal block to be reduced and U_d, L_d are the LU factor of the diagonal block.

The fill-in blocks will destroy the matrices compressibility when added to far blocks, so we extract the column space of the fill-in blocks and add it to U^{or} to obtain the compression of the fill-in blocks. Here, the algorithm accuracy is controlled by the extraction tolerance. The elimination process continues to the rest of the groups from the bottom level to the top. After all groups are eliminated, we merge and permute the submatrices and apply an LU decomposition to the final remaining matrix to obtain the inversion of the system matrix.

III. Numerical Results

In this section, several results are presented to verify the correctness and the complexity of the proposed method.

A. Accuracy

We first compute the current of a PEC sphere with radius of 1.8 m at 300 MHz. The current norm error evaluated by the brute force LU factorization is shown in Fig. 3.

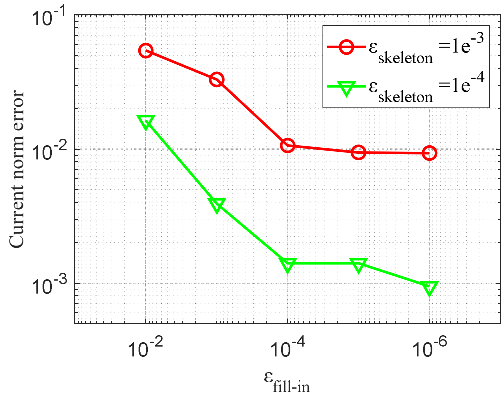


Fig. 3. The current norm error compared to the brute force LU factorization with different tolerance.

The threshold of ACA for skeleton selection is set to $1e^{-3}$ and $1e^{-4}$, and the extraction tolerance is set from $1e^{-2}$ to $1e^{-6}$. We observe that the current norm error can be controlled under $1e^{-4}$ when reducing the tolerance to $1e^{-6}$. Then the bistatic radar cross section (RCS) of the sphere compared with results from FEKO software is given in Fig. 4 where good agreement is observed.

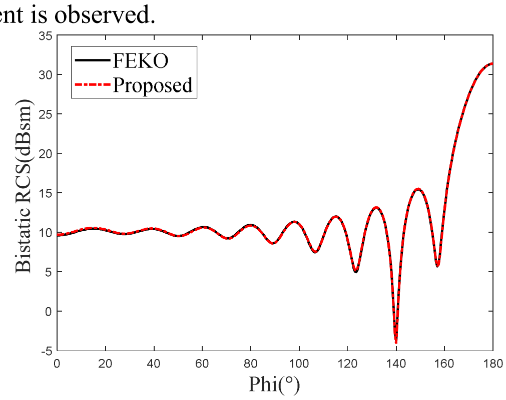


Fig. 4. Bistatic RCS of the PEC sphere.

Next, we plot the current distribution of the NASA almond model from the proposed method and the iterative methods in Fig. 5. The working frequency is set to 4 GHz and the incident direction is $\theta = 0°$, $\varphi = 0°$.

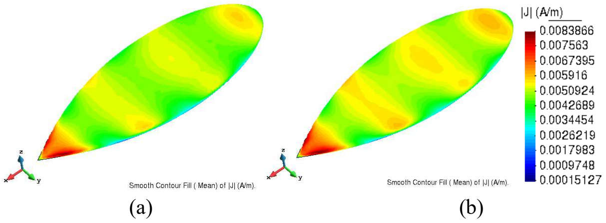


Fig. 5. Current distribution of the NASA almond model from. (a) Mixed NESA (b) The iterative solution.

The relative current norm error calculated by mixed NESA and the iterative method is 0.046.

B. Complexity

The computational complexity of the proposed direct solution is analyzed by both the factorization and total time with the aforementioned PEC sphere and given in Fig. 6. We reduce the mesh size to increase the number of unknowns.

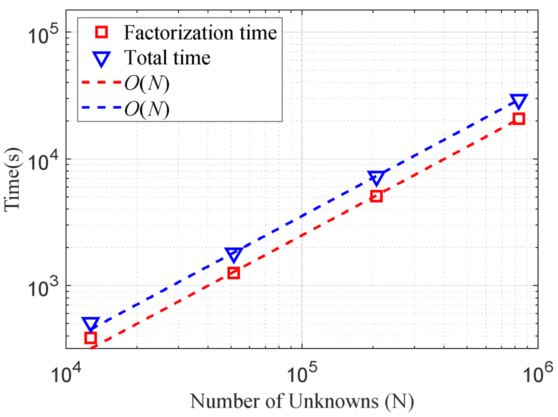


Fig. 6. Factorization time and total time cost with the increase of unknowns.

Clear linear complexities can be observed. In addition, we find that different from iterative methods, factorization time occupies the most part of the total time while the solution (i.e. the backward process) time is rather short. This indicates the advantage of direct solutions over iterative solutions in terms of multiple RHSs issues.

IV. Conclusion

This paper presents a kernel-independent fast direct solver based on mix NESA. The nested nature of NESA enables the direct solver to have a low complexity. RCS results and current distribution of several examples verify the correctness of our proposed method.

Acknowledgment

The research was supported also by the project “INSIGHT-Food – An innovative microwave sensing system for the evaluation and monitoring of food quality and safety”, joint research project within the Executive Program of Scientific and Technological Cooperation between Italy and China, funded by the National Natural Science Foundation of China (NSFC: 32261133623) and the Italian Ministry of Foreign Affairs and International Cooperation (MAECI, CUP E13C23000180005).

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