Abstract

This thesis investigates the existence and nonexistence of ground states for the focusing nonlinear Schrödinger equation (NLS) on noncompact metric graphs in the presence of external physical potentials. Metric graphs provide a natural framework for modeling wave propagation in branched structures such as quantum wires, optical networks, and Bose–Einstein condensates confined in ramified geometries. Here, the interaction between nonlinearity, graph geometry, and external potentials produces genuinely new effects, making the study of stationary states an interesting variational problem with clear physical motivation.

After an introduction to the topic, Chapter 2 revisits the classical variational problem associated with the NLS on graphs. In the absence of potentials, the existence of ground states, the minimizers of the NLS energy under a fixed mass constraint, depends delicately on the topology of the graph. A fundamental tool is the comparison with the soliton on the real line: if a function on the graph attains an energy strictly below the soliton energy, then a ground state exists; otherwise, minimizing sequences escape to infinity and no minimizer is achieved. This mechanism is governed by concentration–compactness and by geometric properties such as the presence of multiple unbounded directions.

Chapter 3 focuses on attractive potentials supported on compact subsets of the graph. The physical motivation is inspired by the case of a potential induced by the curvature of a graph. These potentials lower the energy of functions concentrating on their support and therefore favor the formation of bound states. We extend the classical existence criterion to this setting and establish two complementary existence results: ground states exist for sufficiently large mass and also for sufficiently small mass. In contrast, we show that for intermediate masses, nonexistence may occur when the compact core of the graph is sufficiently intricate and the potential is too weak to counterbalance the dispersive effect of the topology.

Chapter 4 examines magnetic potentials, introduced through the gauge-covariant derivative. While magnetic fields have no effect on trees, where they can be removed by a gauge transformation, they play a crucial role on graphs containing loops, in accordance with the Aharonov–Bohm effect. We show that the magnetic energy can be reformulated as an NLS with an effective repulsive potential supported on the cycles of the graph, with strength determined by the magnetic flux. This reduction allows us to extend the existence criterion to the magnetic setting and to analyze how magnetic fluxes influence the formation of ground states. As an application, we study the tadpole graph and prove that, depending on the magnitude of the effective magnetic potential, ground states may exist or fail to exist.

Overall, the thesis clarifies how external interactions, attractive or repulsive, influence the variational landscape of the NLS on graphs.