On a Calderón Preconditioned PMCHWT Integral Equation for Non-Perfectly Conductive Scatterers

(Article begins on next page)

On a Calderón Preconditioned PMCHWT Integral Equation for Non-Perfectly Conductive Scatterers

Valerio Morra[(1)(3)], Viviana Giunzioni[(1)], Adrien Merlini[(2)], and Francesco P. Andriulli[(1)]

[(1)] Department of Electronics and Telecommunications, Politecnico di Torino, 10129 Turin, Italy
[(2)] Microwave Department, IMT Atlantique, 29238 Brest, France
[(3)] Liceo Scientifico Newton, Chivasso, Italy

***Abstract*—The Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) equation is one of the most widespread full-wave integral formulations for modeling the electromagnetic scattering from penetrable obstacles. Unfortunately, it suffers from sources of ill-conditioning that include the low-frequency and the dense-discretization breakdowns, which manifest by an increase in condition number of the resulting discretized matrix both when the frequency goes to zero and when the mesh density increases. In addition, it also suffers from conductivity-related instabilities. These problems often limit the applicability of the equation in challenging scenarios. In fact, the low-frequency/high-conductivity regimes, including the eddy current regime, are very relevant for industrial applications. In this work, we propose a new Calderón-like preconditioning strategy for the PMCHWT equation combined with the use of quasi-Helmholtz projectors, which cures at once all the ill-conditioning issues described above. The strategy proposed here is compatible with several fast solvers and can be implemented on top of a standard PMCHWT code.**

I. Introduction

Full-wave electromagnetic modeling of the scattering from penetrable objects is a key enabling element of several applications [1]. Among diverse modeling strategies, integral equations discretized by Boundary Element Methods (BEMs) [1], [2] are particularly popular as they only require the discretization of interface boundaries instead of the full volume. Among integral equations for penetrable scatterers, the Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) equation [1], [3] is widely used due to its effectiveness. However, the PMCHWT is a first kind equation and its spectrum is unbounded [2], [4], which results upon discretization, in a condition number of the operator matrix that increases as the scatterer's mesh is refined. Moreover, the equation also suffers from low-frequency and conductivity related instabilities [5], [6].To circumvent these issues, ad-hoc solvers based on quasi-static approximations of Maxwell's equations are often employed. This, however, results in severe difficulties when handling multi-scale scenarios in which different regimes can potentially coexists in different parts of the structure.

Effective stabilization strategies have already been proposed for the PMCHWT equation for curing the dense-discretization breakdown [4] and preventing low-frequency cancellations [6]. Unfortunately, these strategies focus primarily on non-conductive dielectric scatterers and are not suitable for dealing with conducting regimes including the eddy current one.

The formulation proposed in [5], based on a proper rescaling of the quasi-Helmholtz components of the system obtained by an application of the quasi-Helmholtz projectors, already provides a frequency and conductivity stabilization of the formulation. In this work we present a left preconditioner for this frequency-stable equation that provides a new stabilization of the formulation with respect to the increasing mesh density. We fulfill this task by adopting a Calderón-like strategy. Indeed, it has already been shown that, when properly discretized, the PMCHWT is a valid preconditioner for itself [4]. Hence, we multiply the frequency and conductivity stabilized PMCHWT, dicretized with Rao-Wilton-Glisson (RWG) functions, by a dual counterpart discretized with the Buffa-Christiansen (BC) functions. This task is not trivial because, in contrast to the non-conductive case [6], the symmetry between the rescaling coefficients of the quasi-Helmholtz components is lost in presence of non-negligible losses. The resulting formulation is stable in frequency, conductivity, and refinement. In addition, current cancellations are avoided in the presence of an inductive magnetic frill excitation, allowing for the reconstruction of the scattered fields as shown in [5].

The paper is organized as follows: after introducing the required background and notation in Section II, we will introduce the proposed formulation and show its good conditioning properties in Section III. Numerical results in Section IV will attest of the validity of the preconditioning strategy proposed.

II. Background and Notation

Let $\Omega_1 \subset \mathbb{R}^3$ be a bounded conductive region with boundary $\Gamma = \partial\Omega_1$, characteristic length L, and outward pointing normal $\hat{\boldsymbol{n}}$, filled with a homogeneous conductive medium with conductivity σ, permeability $\mu_1 = \mu_r\mu_0$, and permittivity $\epsilon_1 = \epsilon_r'\epsilon_0 - j\sigma/\omega$, where ω is the angular frequency. The object is immersed in a vacuum region $\Omega_0 = \mathbb{R}^3\backslash\bar{\Omega}_1$, with permeability μ_0 and permittivity ϵ_0. The electric and magnetic field integral operators are defined as

$$(\mathcal{T}_k\,\boldsymbol{f})(\boldsymbol{r}) := -\mathrm{j}k(\mathcal{T}_{A,k}\,\boldsymbol{f})(\boldsymbol{r}) + \frac{1}{\mathrm{j}k}(\mathcal{T}_{\Phi,k}\,\boldsymbol{f})(\boldsymbol{r})\,, \tag{1}$$

$$(\mathcal{K}_k\,\boldsymbol{f})(\boldsymbol{r}) := \hat{\boldsymbol{n}} \times \text{p.v.}\int_\Gamma \nabla G_k(\boldsymbol{r},\boldsymbol{r}') \times \boldsymbol{f}(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'\,, \tag{2}$$

$$(\mathcal{T}_{A,k}\,\boldsymbol{f})(\boldsymbol{r}) := \hat{\boldsymbol{n}} \times \int_\Gamma G_k(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{f}(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'\,, \tag{3}$$

$$(\mathcal{T}_{\Phi,k}\,\boldsymbol{f})(\boldsymbol{r}) := \hat{\boldsymbol{n}} \times \nabla\int_\Gamma G_k(\boldsymbol{r},\boldsymbol{r}')\nabla'\cdot\boldsymbol{f}(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'\,, \tag{4}$$

where k is the wavenumber and $G_k(\boldsymbol{r},\boldsymbol{r}') := e^{-jkR}/(4\pi R)$ (with $R := ||\boldsymbol{r}-\boldsymbol{r}'||$) is the Green's function [2]; p.v. denotes the Cauchy principal value. An incident electromagnetic field

$(\boldsymbol{E}^i, \boldsymbol{H}^i)$ is impinging on the obstacle in Ω_1, with the electric and magnetic surface current densities $\boldsymbol{j}_s := \hat{\boldsymbol{n}}\times\boldsymbol{H}$ and $\boldsymbol{m}_s := -\hat{\boldsymbol{n}}\times\boldsymbol{E}$ that satisfy the PMCHWT equation [3],

$$\begin{pmatrix} \eta_0\mathcal{T}_{k_0}+\eta_1\mathcal{T}_{k_1} & -(\mathcal{K}_{k_0}+\mathcal{K}_{k_1}) \\ (\mathcal{K}_{k_0}+\mathcal{K}_{k_1}) & \frac{1}{\eta_0}\mathcal{T}_{k_0}+\frac{1}{\eta_1}\mathcal{T}_{k_1} \end{pmatrix} \begin{pmatrix} \boldsymbol{j}_s \\ \boldsymbol{m}_s \end{pmatrix} = \begin{pmatrix} -\hat{\boldsymbol{n}}\times\boldsymbol{E}^i \\ -\hat{\boldsymbol{n}}\times\boldsymbol{H}^i \end{pmatrix}, \tag{5}$$

where $k_{0,1} := \omega\sqrt{\epsilon_{0,1}\mu_{0,1}}$ and $\eta_{0,1} := \sqrt{\mu_{0,1}/\epsilon_{0,1}}$. After approximating the surface Γ with a triangular mesh with average edge length h we follow a Petrov-Galerkin discretization procedure with RWG functions $(\boldsymbol{f})$ obtaining the linear system of equations

$$\begin{pmatrix} \boldsymbol{T}_u & -\boldsymbol{K} \\ \boldsymbol{K} & \boldsymbol{T}_l \end{pmatrix} \begin{pmatrix} \boldsymbol{j} \\ \boldsymbol{m} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{h} \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{T}_u & -\boldsymbol{K} \\ \boldsymbol{K} & \boldsymbol{T}_l \end{pmatrix} =: \boldsymbol{Z} \tag{6}$$

where

$$\boldsymbol{K} := \boldsymbol{K}_{k_0} + \boldsymbol{K}_{k_1}, \tag{7}$$
$$\boldsymbol{T}_u := \eta_0\boldsymbol{T}_{k_0} + \eta_1\boldsymbol{T}_{k_1}, \tag{8}$$
$$\boldsymbol{T}_l := \frac{1}{\eta_0}\boldsymbol{T}_{k_0} + \frac{1}{\eta_1}\boldsymbol{T}_{k_1}. \tag{9}$$

Similarly, by expanding the unknown current densities with BC source basis functions $(\boldsymbol{g})$ and testing the equations with rotated BC functions we obtain the dual formulation

$$\begin{pmatrix} \mathbb{T}_u & -\mathbb{K} \\ \mathbb{K} & \mathbb{T}_l \end{pmatrix} =: \mathbb{Z}, \tag{10}$$

where the blocks are the dual of the ones defined above.

In this work, we will employ the definitions of low-frequency regimes proposed in [5]. Hence, after denoting by $\delta := \sqrt{2/(\omega\sigma\mu_1)}$ the skin depth parameter, we introduce the scalar coefficients [5]

$$\chi := k_0 L, \tag{11}$$
$$\gamma := \sqrt{\omega\epsilon_0/\sigma}, \tag{12}$$
$$\xi := \sqrt{2/\mu_r}L/\delta, \tag{13}$$

such that $\chi = \gamma\xi$.

To perform the conditioning analysis of the proposed equation in Section III, we introduce the quasi-Helmholtz basis $\boldsymbol{A} := (\boldsymbol{\Sigma} \quad \boldsymbol{H} \quad \boldsymbol{\Lambda})$, where $\boldsymbol{\Sigma}$, $\boldsymbol{H}$, and $\boldsymbol{\Lambda}$ are the transformation matrices respectively from the star, global loop, and local loop subspaces [7] to the RWG space. The primal and dual quasi-Helmholtz projectors are defined as

$$\boldsymbol{P}^{\boldsymbol{\Sigma}} := \boldsymbol{\Sigma}(\boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{\Sigma})^{+}\boldsymbol{\Sigma}^{\mathrm{T}}, \qquad \boldsymbol{P}^{\boldsymbol{\Lambda H}} := \boldsymbol{I} - \boldsymbol{P}^{\boldsymbol{\Sigma}}, \tag{14}$$
$$\mathbb{P}^{\boldsymbol{\Lambda}} := \boldsymbol{\Lambda}(\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda})^{+}\boldsymbol{\Lambda}^{\mathrm{T}}, \qquad \mathbb{P}^{\boldsymbol{\Sigma H}} := \boldsymbol{I} - \mathbb{P}^{\boldsymbol{\Lambda}}, \tag{15}$$

where the superscript $^{+}$ indicates the Moore-Penrose pseudoinvserse. The projector $\boldsymbol{P}^{\boldsymbol{\Lambda H}}$ projects the RWG functional space into into its solenoidal subspace, while the projector $\boldsymbol{P}^{\boldsymbol{\Sigma}}$ projects into the complementary, non-solenoidal, subspace. Similarly, $\mathbb{P}^{\boldsymbol{\Sigma H}}$ and $\mathbb{P}^{\boldsymbol{\Lambda}}$ are the solenoidal and non-solenoidal projectors for the dual BC space. Lastly, we introduce the mixed Gram matrices $\boldsymbol{G}$ and $\mathbb{G} = -\boldsymbol{G}^{\mathrm{T}}$, defined as $(\boldsymbol{G})_{mn} := (\hat{\boldsymbol{n}}\times\boldsymbol{f}_m, \boldsymbol{g}_n)_{L^2(\Gamma)}$ and $(\mathbb{G})_{mn} := (\hat{\boldsymbol{n}}\times\boldsymbol{g}_m, \boldsymbol{f}_n)_{L^2(\Gamma)}$.

III. The Proposed Formulation

In [5] the PMCHWT matrix was regularized in frequency and conductivity as

$$\tilde{\boldsymbol{Z}} := \begin{pmatrix} \frac{1}{\sqrt{\eta_0}}\mathbb{M}_1\boldsymbol{G}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \sqrt{\eta_0}\mathbb{M}_2\boldsymbol{G}^{-1} \end{pmatrix} \boldsymbol{Z} \begin{pmatrix} \frac{1}{\sqrt{\eta_0}}\boldsymbol{M}_3 & \boldsymbol{0} \\ \boldsymbol{0} & \sqrt{\eta_0}\boldsymbol{M}_4 \end{pmatrix}, \tag{16}$$

where

$$\mathbb{M}_1 := a_L\,\mathbb{P}^{\boldsymbol{\Sigma H}} + b_L\,\mathbb{P}^{\boldsymbol{\Lambda}}, \ \mathbb{M}_2 := c_L\,\mathbb{P}^{\boldsymbol{\Sigma H}} + d_L\,\mathbb{P}^{\boldsymbol{\Lambda}}, \tag{17}$$
$$\boldsymbol{M}_3 := a_R\,\boldsymbol{P}^{\boldsymbol{\Lambda H}} + b_R\,\boldsymbol{P}^{\boldsymbol{\Sigma}}, \ \boldsymbol{M}_4 := c_R\,\boldsymbol{P}^{\boldsymbol{\Lambda H}} + d_R\,\boldsymbol{P}^{\boldsymbol{\Sigma}}, \tag{18}$$

with scalar coefficients $\{a_L - d_R\}$ given in [5, Table III]. In this work, we propose a left regularization of the above system with respect to the mesh refinement, involving a matrix in the form

$$\tilde{\mathbb{Z}} := \begin{pmatrix} \frac{1}{\sqrt{\eta_0}}\overline{\mathbb{M}}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \sqrt{\eta_0}\overline{\mathbb{M}}_2 \end{pmatrix} \mathbb{Z} \begin{pmatrix} \frac{1}{\sqrt{\eta_0}}\boldsymbol{G}^{-1}\overline{\boldsymbol{M}}_3 & \boldsymbol{0} \\ \boldsymbol{0} & \sqrt{\eta_0}\boldsymbol{G}^{-1}\overline{\boldsymbol{M}}_4 \end{pmatrix}, \tag{19}$$

where

$$\overline{\mathbb{M}}_1 := \alpha_L\,\mathbb{P}^{\boldsymbol{\Sigma H}} + \beta_L\,\mathbb{P}^{\boldsymbol{\Lambda}}, \ \overline{\mathbb{M}}_2 := \gamma_L\,\mathbb{P}^{\boldsymbol{\Sigma H}} + \delta_L\,\mathbb{P}^{\boldsymbol{\Lambda}}, \tag{20}$$
$$\overline{\boldsymbol{M}}_3 := \alpha_R\boldsymbol{P}^{\boldsymbol{\Lambda H}} + \beta_R\,\boldsymbol{P}^{\boldsymbol{\Sigma}}, \ \overline{\boldsymbol{M}}_4 := \gamma_R\,\boldsymbol{P}^{\boldsymbol{\Lambda H}} + \delta_R\,\boldsymbol{P}^{\boldsymbol{\Sigma}}, \tag{21}$$

which represents a frequency and conductivity regularized PMCHWT matrix discretized with dual functions. Note that the proposed preconditioner is not simply the dual version of the frequency-regularized primal formulation, as in [6] for example, where matrix $\mathbb{Z}$ is regularized in the low-frequency limit by left multiplication with primal projectors and right multiplication with dual projectors. Indeed, it can be shown that, given the preconditioner $\tilde{\boldsymbol{Z}}$ proposed in [5] and given the need to cancel the inner projector matrices [6], this approach would introduce a static nullspace related to the genus of the geometry. Instead, resorting to a dual formulation in the form of (19) allows for the obtention of a well-posed and well-conditioned formulation.

In conclusion, notice that, for the effectiveness of the Calderón preconditioning strategy, a reordering of the columns of the system is required. Indeed, as shown in [4], the operator

$$\begin{pmatrix} \mathcal{K}_{k_0}+\mathcal{K}_{k_1} & -(\eta_0\mathcal{T}_{k_0}+\eta_1\mathcal{T}_{k_1}) \\ \frac{1}{\eta_0}\mathcal{T}_{k_0}+\frac{1}{\eta_1}\mathcal{T}_{k_1} & \mathcal{K}_{k_0}+\mathcal{K}_{k_1} \end{pmatrix}^2, \tag{22}$$

with magnetic field operators on the diagonal blocks and electric field operators on the off-diagonal blocks, is bounded and well-posed.

Finally, the new equation we propose, resulting from the joint use of a Calderón preconditioning technique and quasi-Helmholtz projectors to rescale the quasi-Helmholtz components of the system, reads

$$\boldsymbol{Z}_C \begin{pmatrix} \boldsymbol{y_m} \\ \boldsymbol{y_j} \end{pmatrix} := \boldsymbol{L} \begin{pmatrix} \boldsymbol{K} & -\boldsymbol{T}_u \\ \boldsymbol{T}_l & \boldsymbol{K} \end{pmatrix} \boldsymbol{R} \begin{pmatrix} \boldsymbol{y_m} \\ \boldsymbol{y_j} \end{pmatrix} = \boldsymbol{L} \begin{pmatrix} -\boldsymbol{e} \\ \boldsymbol{h} \end{pmatrix}, \tag{23}$$

where

$$\boldsymbol{L} := \begin{pmatrix} \frac{1}{\sqrt{\eta_0}}\overline{\mathbb{M}}_1 & \mathbf{0} \\ \mathbf{0} & \sqrt{\eta_0}\overline{\mathbb{M}}_2 \end{pmatrix} \begin{pmatrix} \mathbb{K} & -\mathbb{T}_u \\ \mathbb{T}_l & \mathbb{K} \end{pmatrix} \begin{pmatrix} \boldsymbol{G}^{-1}\overline{\boldsymbol{M}}_4\boldsymbol{G}\mathbb{M}_1\boldsymbol{G}^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{G}^{-1}\overline{\boldsymbol{M}}_3\boldsymbol{G}\mathbb{M}_2\boldsymbol{G}^{-1} \end{pmatrix}, \tag{24}$$

$$\boldsymbol{R} := \begin{pmatrix} \sqrt{\eta_0}\boldsymbol{M}_4 & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{\eta_0}}\boldsymbol{M}_3 \end{pmatrix}. \tag{25}$$

The solution of the system is retrieved as

$$\begin{pmatrix} \boldsymbol{m} \\ \boldsymbol{j} \end{pmatrix} = \boldsymbol{R} \begin{pmatrix} \boldsymbol{y_m} \\ \boldsymbol{y_j} \end{pmatrix}. \tag{26}$$

The choice of the coefficients of the dual system $\{\alpha_L - \delta_R\}$ should provide the stabilization of $\tilde{\mathbb{Z}}$ with respect to frequency and conductivity, as well as avoid any deterioration of the conditioning of the well-conditioned system resulting from the discretization of (22) [4]. To this purpose, the right coefficients of the new preconditioner for the dual system have been set to

$$\alpha_R = \frac{1}{d_L}, \quad \beta_R = \frac{1}{c_L}, \quad \gamma_R = \frac{1}{b_L}, \quad \delta_R = \frac{1}{a_L}. \tag{27}$$

The remaining left coefficients have been set in such a way as to obtain the well-conditioning of the dual formulation in frequency and conductivity.

In this work we focus on two different low-frequency regimes, referred to in [5] as quasi-static regime (QSR) and eddy-current-free eddy-current regime (ECFR). The chosen coefficients for these two regimes are summarized in Table I. As for the third regime described in [5], the technique proposed here creates matrices that suffer from poor conditioning properties. Further investigations will be needed to cover that scenario.

Using the same coefficients as in [5] for the primal formulation ensures that $\tilde{\boldsymbol{Z}}$ is frequency- and conductivity-stable in both regimes. We still have to show that, given the proposed coefficients $\{\alpha_L - \delta_R\}$ reported in Table I, $\tilde{\mathbb{Z}}$ shares the same propriety.

	QSR	ECFR		QSR	ECFR
a_L	1	$\chi^{1/2}\gamma^{-1}$	α_L	χ^{-2}	$\chi^{-1/2}$
b_L	χ^{-2}	$\chi^{-1/2}$	β_L	1	$\chi^{1/2}\gamma$
c_L	χ^{-1}	$\chi^{1/2}$	γ_L	χ^{-1}	$\chi^{-1/2}\gamma$
d_L	χ^{-1}	$\chi^{-1/2}\gamma$	δ_L	χ^{-1}	$\chi^{1/2}$
a_R	χ	$\chi^{-1/2}$	α_R	χ	$\chi^{1/2}\gamma^{-1}$
b_R	χ	$\chi^{1/2}\gamma$	β_R	χ	$\chi^{-1/2}$
c_R	1	$\chi^{-1/2}\gamma$	γ_R	χ^2	$\chi^{1/2}$
d_R	χ^2	$\chi^{1/2}$	δ_R	1	$\chi^{-1/2}\gamma$

TABLE I: Scalar multiplicative coefficients defining the proposed preconditioning strategy.

A. Analysis in the Quasi-Static Regime

The quasi-static regime is the regime of slowly varying field in which the eddy current approximation $\sigma \gg \omega\epsilon_0$ does not hold [5]. Hence, it is characterized by

$$\begin{cases} \chi \ll 1 \\ \gamma \gg 1 \end{cases}. \tag{28}$$

By setting the preconditioning coefficients as in Table I, the low-frequency scaling of the loop-star decomposition of $\tilde{\mathbb{Z}}$ reads

$$\begin{pmatrix} \boldsymbol{A}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A}^{\mathrm{T}} \end{pmatrix} \tilde{\mathbb{Z}} \begin{pmatrix} \boldsymbol{A} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A} \end{pmatrix} = \mathcal{O}\begin{pmatrix} 1 & 1 & 1 & 1 & \chi^2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \chi^2 & \chi^2 & 1 & 1 & \chi^2 & \chi^2 \\ \chi^2 & \chi^2 & 1 & 1 & \chi^2 & \chi^2 \\ \chi^2 & 1 & 1 & 1 & \chi^2 & \chi^2 \\ 1 & 1 & 1 & 1 & \chi^2 & 1 \end{pmatrix}. \tag{29}$$

By following a similar reasoning as in [8], it can be shown that $\tilde{\mathbb{Z}}$ does not suffer from a static nullspace in case of multiply connected geometries. Moreover the application of the Gershgorin circle theorem indicates favorable conditioning properties.

B. Analysis in the Eddy-Current-Free Eddy-Current Regime

We consider now the eddy current regime, characterized by $\gamma \ll 1$, under the condition of field penetration lengths longer than the characteristic size of the scatterer, that is,

$$\begin{cases} \chi \ll 1 \\ \gamma \ll 1 \\ \xi \ll 1 \end{cases}. \tag{30}$$

By setting the preconditioning coefficients as in Table I, the loop-star decomposition of $\tilde{\mathbb{Z}}$ follows as

$$\begin{pmatrix} \boldsymbol{A}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A}^{\mathrm{T}} \end{pmatrix} \tilde{\mathbb{Z}} \begin{pmatrix} \boldsymbol{A} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A} \end{pmatrix} = \mathcal{O}\begin{pmatrix} 1 & \chi\gamma^{-1} & \chi\gamma^{-1} & \chi\gamma^{-1} & \chi^2\gamma^{-2} & 1 \\ 1 & \chi\gamma^{-1} & \chi\gamma^{-1} & \chi\gamma^{-1} & 1 & 1 \\ \chi\gamma & \chi^2 & 1 & \gamma^2 & \chi\gamma & \chi\gamma \\ \chi\gamma^{-1} & \chi^2\gamma^{-2} & 1 & 1 & \chi\gamma^{-1} & \chi\gamma^{-1} \\ \chi\gamma^{-1} & 1 & 1 & 1 & \chi\gamma^{-1} & \chi\gamma^{-1} \\ 1 & \chi\gamma^{-1} & \chi\gamma^{-1} & \chi\gamma^{-1} & \chi^2\gamma^{-2} & 1 \end{pmatrix}. \tag{31}$$

For reasons equivalent to those above, given $\chi\gamma^{-1} = \xi \ll 1$, it can be deduced that also in this regime $\tilde{\mathbb{Z}}$ is well-conditioned in frequency and conductivity and well-posed in the static limit.

IV. Numerical Results

The proposed numerical results aim at assessing the good-conditioning properties of the new formulation with respect to both the frequency f and the refinement parameter h, when applied to simply and multiply connected geometries.

Figures 1 and 2 show the condition number of the formulation applied over a sphere with fixed conductivity, for decreasing values of frequency. The former shows the behavior of the conditioning of the formulation in the first regime, the latter in the second regime. As is clear from these figures, the PMCHWT is characterized by a condition number growing in the low-frequency regime as f^{-2}, while both the quasi-Helmholtz projectors stabilized formulation ($\tilde{\boldsymbol{Z}}$) and the equation proposed in this work ($\boldsymbol{Z}_C$) present

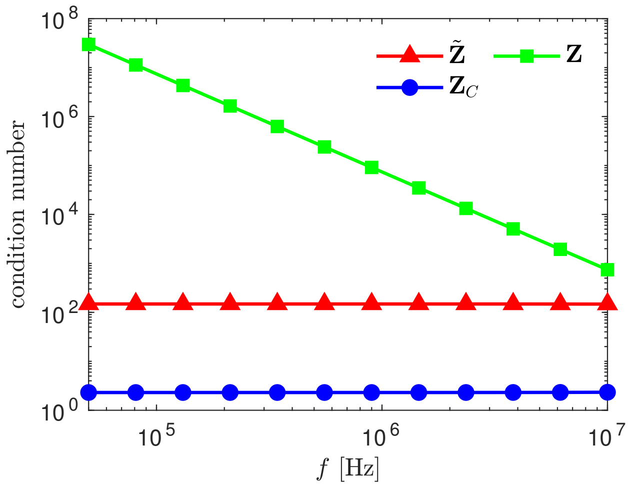

Fig. 1: Condition number of the formulations when applied to a sphere of radius $1\,\mathrm{m}$ of conductivity $10^{-8}\,\mathrm{S/m}$ discretized at $h = 0.71\,\mathrm{m}$ and excited at a frequency ranging from $5\times10^4\,\mathrm{Hz}$ to $10^7\,\mathrm{Hz}$.

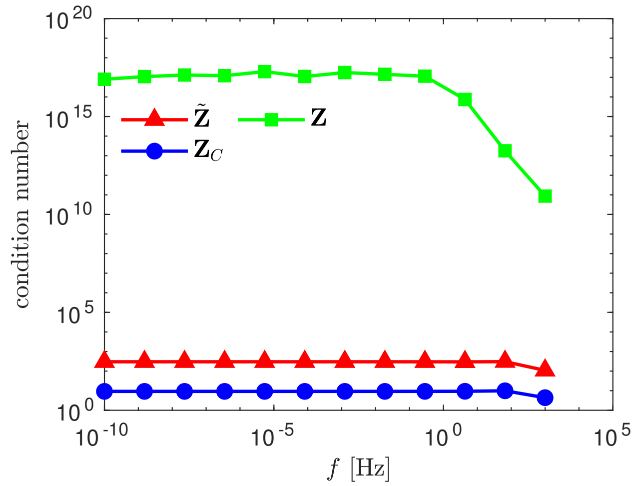

Fig. 2: Condition number of the formulations when applied to a sphere of radius $1\,\mathrm{m}$ of conductivity $10^2\,\mathrm{S/m}$ discretized at $h = 0.71\,\mathrm{m}$ and excited at a frequency ranging from $10^{-10}\,\mathrm{Hz}$ to $10^3\,\mathrm{Hz}$.

a constant condition number when the frequency decreases. Moreover notice that the Calderón preconditioning strategy applied ensures a condition number of $\boldsymbol{Z}_C$ lower than the one of $\tilde{\boldsymbol{Z}}$.

After having checked the frequency stability of the proposed equation, we can move on the numerical verification of the stability with respect to dense refinements. Figure 3 shows the condition number of the proposed formulation applied over a multiply-connected structure discretized at different values of h. As noted in [5], the condition number of $\tilde{\boldsymbol{Z}}$ grows as h^{-2}. On the contrary, we appreciate the effectiveness of the proposed Calderón preconditioning strategy in the fact that the condition number of $\boldsymbol{Z}_C$ remains constant for increasingly denser refinements.

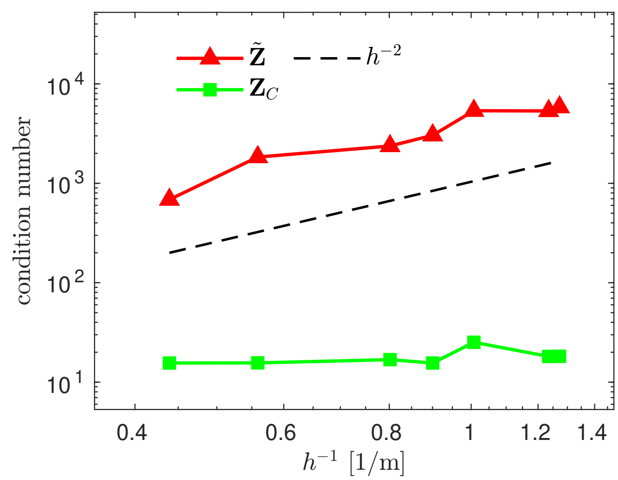

Fig. 3: Condition number of the formulations when applied to a square-ring structure of genus 1 with characteristic length of $6\sqrt{2}$ and conductivity $10^2\,\mathrm{S/m}$ excited at the frequency of $10^{-5}\,\mathrm{Hz}$ when varying the refinement parameter h.

ACKNOWLEDGEMENT

This work was supported by the European Innovation Council (EIC) through the European Union's Horizon Europe research Programme under Grant 101046748 (Project CEREBRO).

REFERENCES

[1] J.-M. Jin, *Theory and Computation of Electromagnetic Fields*. Piscataway, NJ: IEEE Press, 2nd ed ed., 2015.

[2] S. Sauter and C. Schwab, *Boundary Element Methods*, vol. 39 of *Springer Series in Computational Mathematics*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011.

[3] A. Poggio and E. Miller, "Integral equation solutions of three-dimensional scattering problems," in *Computer Techniques for Electromagnetics*, pp. 159–264, Elsevier, 1973.

[4] K. Cools, F. Andriulli, and E. Michielssen, "A Calderón multiplicative preconditioner for the PMCHWT integral equation," *IEEE Transactions on Antennas and Propagation*, vol. 59, pp. 4579–4587, Dec. 2011.

[5] V. Giunzioni, A. Scazzola, A. Merlini, and F. P. Andriulli, "Low-frequency stabilizations of the PMCHWT equation for dielectric and conductive media: On a full-wave alternative to eddy-current solvers," Aug. 2024.

[6] Y. Beghein, R. Mitharwal, K. Cools, and F. Andriulli, "On a low-frequency and refinement stable PMCHWT integral equation leveraging the quasi-Helmholtz projectors," *IEEE Transactions on Antennas and Propagation*, vol. 65, pp. 5365–5375, Oct. 2017.

[7] S. Adrian, A. Dely, D. Consoli, A. Merlini, and F. Andriulli, "Electromagnetic integral equations: Insights in conditioning and preconditioning," *IEEE Open Journal of Antennas and Propagation*, vol. 2, pp. 1143–1174, 2021.

[8] I. Bogaert, K. Cools, F. Andriulli, and H. Bagci, "Low-frequency scaling of the standard and mixed magnetic field and Müller integral equations," *IEEE Transactions on Antennas and Propagation*, vol. 62, pp. 822–831, Feb. 2014.