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Original
Coupled Harmonic Balance based approach for the non-linear dynamics of spur gear pairs / Saletti, Giacomo; Battiato, Giuseppe; Zucca, Stefano. - 1:(2025), pp. 161-169. (IMAC, A Conference and Exposition on Structural Dynamics 2025 Orlando, FL (USA) 10-13 February 2025) [10.13052/97887-438-0146-7_21].

Availability:
This version is available at: 11583/3010408 since: 2026-04-29T12:46:14Z

Publisher:
River Publishers

Published
DOI:10.13052/97887-438-0146-7_21

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19 November 2026

Chapter 21
Coupled Harmonic Balance based approach for the non-linear dynamics of spur gear pairs

Giacomo Saletti, Giuseppe Battiato, and Stefano Zucca

Abstract Noise, Vibration and Harshness analyses are of great interest for the latest developments of the gearboxes of electric vehicles. Gearboxes are now the main source of vibrations since electric powertrains are much quieter than internal combustion engines. Traditionally, the simulation of the non-linear gears dynamics is studied by first performing a series of preliminary static analyses to compute the static transmission error (STE). The STE (i.e. in the form of varying mesh stiffness) is then accepted as the system's excitation source to compute the dynamic transmission error (DTE). This paper presents a novel approach to analyze the non-linear dynamics of gears which does not require any preliminary static analyses. The method consists of a frequency domain approach based on the Harmonic Balance Method (HBM) and the Alternating Frequency-Time (AFT) scheme, allowing for much faster simulations when compared to the widely used direct time integration (DTI). The contact between the teeth is modeled as intermittent, penalty based with varying gap. The time varying gap between the teeth is initially approximated to a step function that guarantees the design contact ratio.

The methodology introduced is tested on a lumped parameter model of a spur gear pair already proposed and simulated in the literature. The results obtained with the novel approach are compared with the DTI simulation of the model as reference. The excellent match between the different approaches validates the reliability of developed methodology.

Keywords Gear dynamics · Non-linear dynamics · Harmonic Balance Method · Alternating Frequency-Time · Transmission error

Introduction

Gears serve as critical components in various mechanical systems, yet they are frequently subjected to issues such as vibrations. These vibrations are detrimental as they can induce damage, generate noise, and lead to malfunctions. The primary cause of vibrations in gears is the variation in contact stiffness, which fluctuates due to the changing number of teeth in contact during gear rotation [1].

The issue is well known and has been studied systematically since the mid-1800s with the early work of Walker who proposed a coefficient relating the static and dynamic loads operating on gears as reported by [2]. In the last century several lumped parameters models have been developed for the characterization of gear dynamics [3], addressing torsional vibrations [4], specifically teeth dynamics [1] and more in general geared rotor dynamics [5] where gears are often modelled as rigid disks with inertia on flexible shafts.

More recent studies that focused on torsional gear dynamics are still utilizing simple models for their analyses, Parker et al. [6] compared a single DOF model and a FE model in approximating the experimental dynamic response of a spur gear pair, Ambarisha and Parker [7] developed an analytical model able to capture the complex dynamics of a planetary gear when compared to its FE counterpart. In 2019 Dai et al. [8] proposed a hybrid analytical-computational approach that integrates the accurate FE static analyses with a compact lumped parameters model that can both accurately simulate the torsional vibrations of the gears and correctly capture the stresses on the teeth.

However, even dynamic analyses on lumped parameter models can require significant computational time if they are performed by time integration of the equation of motion of the system. The response of the gears is measured once steady state is reached, this requires performing the simulation of the initial transient until the response can be considered periodic.

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Ludovic Renson et al. (eds.), *Nonlinear Structures & Systems, Vol. 1*, Conference Proceedings
of the Society for Experimental Mechanics Series, https://doi.org/10.13052/97887-438-0146-7_21

Trying to overcome the limitations of time integration, frequency domain approaches were introduced in the field of gear dynamics. Harmonic Balance Method (HBM) allows for the determination of the steady state response without having to solve for initial transient.

Some definitions are necessary for the following discussion in this paper. Transmission errors (TE) are defined as the difference between the positions that perfectly rigid gears would occupy during ideal motion and the effective positions occupied by flexible gears. The static transmission error (STE) is typically defined as the TE obtained through a sequence of low-speed or quasi-static analyses on loaded gears [9]. The dynamic transmission error (DTE) is the counterpart of the STE when calculated at high rotational speeds. Instead of the STE, some prefer to introduce the concept of mesh stiffness, which is related to the ratio of the torque applied to the gears and the STE [6]. In practice, the STE and mesh stiffness provide similar information.

Commonly the study of gear dynamics in the frequency domain is performed by first computing the STE and subsequently using it as a dynamic excitation to obtain the DTE [6, 10, 11]. Initially solving the system to obtain the STE can be cumbersome, for large systems especially, moreover it represents an approximation because it assumes that the static (or quasi-static) solution of the system is not affected by the dynamics of the system. In the field of turbo-machinery, where HBM has been extensively applied, it has been proven that the uncoupled solution static-dynamic can lead to inaccurate results. Coupled HBM was developed to allow for accurate predictions of dynamic responses of non-linear models with friction damping. Zucca et al. [12] used coupled HBM with an alternating frequency time (AFT) on lumped parameter models with friction contact. They argue that considering the static and dynamic problem as uncoupled leads to inaccurate solution when comparing to DTI simulations as reference.

In this paper the use of coupled HBM is extended to gear dynamics developing a method capable of overcoming the differences between STE and DTE. The transmission error is obtained in a single analysis considering the coupling between the static solution and the dynamic one. The simulation algorithm is enabled by the use of the Alternating Frequency Time (AFT) approach [13]. During the Newton-Raphson iteration to solve the non-linear set of equations expressed in the frequency domain, the AFT approach allows to momentarily move to the time domain to compute the non-linear contact forces arising from the interaction between teeth pairs.

The methodology was implemented in a MATLAB script and the test system was solved making use of a continuation method to help convergence around turning points, however, it was not possible to reach convergence over the whole frequency ranges chosen.

The following sections will focus on:

- in section , Methodology, a comprehensive description of the proposed approach is presented, details on the numerical solution strategy are also highlighted;
- in section , Results and Discussion, comparative analyses are carried out on a simple lumped parameter model. Non-linear dynamic analyses are performed with both DTI (reference) and the developed methodology.
- In section , the final conclusions are drawn.

Methodology

The described approach is referred to as Coupled HBM and has been introduced in [12] for systems with dry friction contacts. In this paper the method is applied to systems with clearance type nonlinearities, specifically to gear dynamics. Several advantages of the methodology can be listed:

- Frequency domain approaches are faster than time domain approaches
- No preliminary static analyses are required
- No approximations/assumptions are required on the mesh stiffness
- Suitable for the integration with different contact models

The approach also presents some limitations:

- because of the clearance type nonlinearities, several harmonics might be required to obtain a converged solution
- The stiffness matrix can be singular for gear dynamics problems, leading to bad conditioning of the dynamic stiffness and hard to obtain Newton-Raphson convergence.

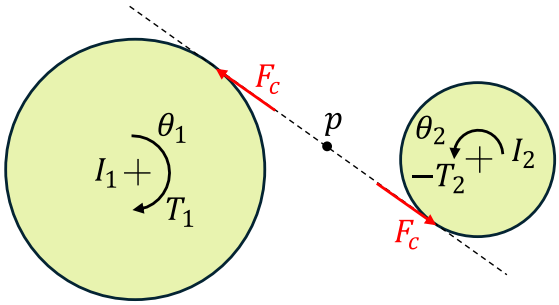


Fig. 1 Lumped parameters model

Single DOF model

The demonstration of the applicability of the method is performed through the introduction of a simple, lumped parameter model. The introduced one (see Figure 1) was already studied in [6], when trying to approximate some available experimental results. The model is composed of two gears with I_1 and I_2 moments of inertia, r_1 and r_2 base radii and T_1 and $T_2 = T_1 r_2/r_1$ the torques applied to the gear. Given that no torsional stiffness or damping constraint the rotational DOF of the gears, the equations of motion can be written as:

$$I_1\ddot{\theta}_1 + F_c r_1 = T_1 \tag{1}$$

$$I_2\ddot{\theta}_2 - F_c r_2 = -T_1 r_2/r_1 \tag{2}$$

From equations 1 and 2, the following set of equations can be easily obtained:

$$I_2 I_1\ddot{\theta}_1 r_1 + I_2 F_c r_1^2 = I_2 T_1 r_1 \tag{3}$$

$$I_1 I_2\ddot{\theta}_2 r_2 + I_1 F_c r_2^2 = -I_1 T_1 r_2^2/r_1 \tag{4}$$

Subtracting Eq. 3 from Eq. 4 is obtained

$$I_1 I_2\left(\ddot{\theta}_2 r_2 - \ddot{\theta}_1 r_1\right) - F_c\left(I_1 r_2^2 + I_2 r_1^2\right) = -T_1\left(\frac{I_1 r_2^2}{r_1} + I_2 r_1\right) \tag{5}$$

$$\frac{I_1 I_2}{I_1 r_2^2 + I_2 r_1^2}\left(\ddot{\theta}_2 r_2 - \ddot{\theta}_1 r_1\right) - F_c + T_1\frac{\frac{I_1 r_2^2}{r_1} + I_2 r_1}{I_1 r_2^2 + I_2 r_1^2} = 0 \tag{6}$$

Introducing the TE as the relative DOF between the gears $x = \theta_2 r_2 - \theta_1 r_1$ and introducing a damping term proportional to $\dot{x}$, the equation of motion can be expressed in the form:

$$\frac{I_1 I_2}{I_1 r_2^2 + I_2 r_1^2}\ddot{x} + C\dot{x} - F_c + T_1\frac{\frac{I_1 r_2^2}{r_1} + I_2 r_1}{I_1 r_2^2 + I_2 r_1^2} = 0 \tag{7}$$

$$M\ddot{x} + C\dot{x} - F_c + F_{ext} = 0 \tag{8}$$

The external excitation is collected in the F_{ext} term, it is directly dependent on the torque T_1 applied to the first gear. The external excitation is generally a periodic function known a priori.

F_c is the contact force being exchanged between the teeth of the gears. In the most general case, F_c is a non-linear function of time (t), the transmission error (x) and its derivative $(\dot{x})$.

Considering the gears rotating at a constant angular speed ω', once steady state is reached, every point of the gears will occupy the same position after a $T' = 2\pi/\omega'$ time period. However, considering the geometrical periodicity of the gears, already after a period of $T = 2\pi/z\omega'$ with z the number of teeth, the gears are in an equivalent position. Consequently, the contact forces are also periodic with the same period T and the effective pulsation of the system is $\omega = z\omega'$.

The displacement and forces in equation 8 can be expressed as Fourier series:

$$x(t) = \sum_{h=0}^{H} \tilde{x}_h e^{jh\omega t} \tag{9}$$

$$F_{ext}(t) = \sum_{h=0}^{H} \tilde{F}_{ext,h} e^{jh\omega t} \tag{10}$$

$$F_c(t, x, \dot{x}) = \sum_{h=0}^{H} \tilde{F}_{c,h} e^{jh\omega t} \tag{11}$$

The quantities $\tilde{x}_h$, $\tilde{F}_{ext,h}$ and $\tilde{F}_{c,h}$ are the complex Fourier coefficients associated with the h^{th} order, j is the imaginary unit. The series are truncated at an arbitrary order H.

Equations 9, 10 and 11 are substituted in equation 8 and the HBM is enforced. The equation for a generic order h is:

$$\left(-h^2\omega^2 M + jh\omega C\right)\tilde{x}_h - \tilde{F}_{c,h} + \tilde{F}_{ext,h} = 0 \tag{12}$$

The generic model presented in Figure 1 can be simulated using direct time integration (DTI) solving equation 8 or it can be solved in its frequency domain expression (equation 12) using non-linear solution methods (e.g. Newton-Raphson) with Alternating Frequency Time scheme (AFT) [13].

The AFT scheme starts from a given guess value of the Fourier coefficients of the generalized displacements. The displacements are then expressed in time domain over a single period via the Inverse Fast Fourier Transform (IFFT). The contact model allows to obtain the contact force evolution over time given the evolution of the generalized displacements. The sequence of generalized contact forces is eventually expressed back in the frequency domain via the Fast Fourier Transform (FFT), thus obtaining the Fourier coefficients of the contact forces $\tilde{F}_{c,h}$ for the different harmonics retained in the HBM.

If the residual of the equation of motion is still too large, a new guess of the Fourier coefficient of the generalized displacements must be provided. The iterations proceed until the residual of the equation of motion can be considered sufficiently small.

Contact model

A contact model is required to express F_c as a function of time (t), the transmission error (x) and its derivative $\dot{x}$. This model can then be used as is in the different formulations of the problem (time domain or frequency domain).

Considering the rigid rotation and mating of the gears, the contact ratio $c_r > 1$ defines the average number of teeth in contact during a period of the motion. The varying number of teeth in contact leads to changes in the equivalent stiffness of the contact which represent the source of dynamic excitation in the studied model. Moreover, the vibrations induced by this phenomenon can lead the teeth on opposing gear to close or open the contact in a different instant than the rigid case. The proposed contact model wants to mimic this behavior.

The basic idea is that, during one period of the rigid rotation of the gears a couple of teeth will be either in contact or not. The contact status (open or close) depends on the relative distance between the teeth. This is distance is referred to as *gap*.

The gap between two teeth can be either zero or a positive value. The relative position of the contact points $x = \theta_2(t)r_2 - \theta_1(t)r_1$ is positive if they tend to separate and negative if they tend to interpenetrate. A clearance type contact is defined in the form of

$$\begin{cases} F_c(t) = -k_c\left(x(t) + gap(t)\right) & \text{if } x + gap < 0 \\ F_c(t) = 0 & \text{if } x + gap \geq 0 \end{cases} \tag{13}$$

where k_c is the contact stiffness. The contact status is fully dependent on the total relative position of the contact at every time step. This contact model allows the system to find an equilibrium considering both the static and dynamic components of the response.

Considering a model with contact ratio $c_r = 1.75$, during one (rigid) rotation period of the gears, three couples of teeth will be in contact with each other at a certain point. Considering the representation in Figure 2, the central couple of teeth will be in contact for the entire rotation period, the *high* couple of teeth will start as separated and will enter contact after some time. On the contrary, the *low* couple of teeth will start in contact and will separate as the rotation continues.

To consider the interactions between the teeth described, the system presented in Figure 1 is completed in Figure 3 including three contact elements representing the couples of teeth.

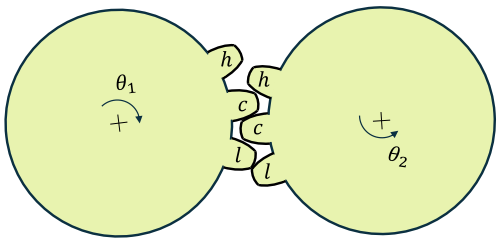


Fig. 2 Teeth in contact

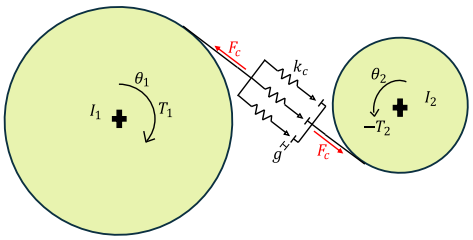


Fig. 3 Gears with contact elements

The gap operating in the different contact elements will vary over time. As a first approximation a step function is introduced (Figure 4), the gap for all the teeth is set to 0 when the teeth are in contact and to 1 when they are open. The gap jumps are set to enforce the design contact ratio. The approximation of the gap variation as a step function is not representative of the real behavior of the teeth contact, the results obtained with this model will not be quantitatively close to any experimental results possibly available. However, since the scope of this paper is to show the applicability of the coupled HBM to the simulation of gear dynamics having an accurate contact model is not a requirement. In future work, effort will be put into this aspect to improve the quality of the results and to investigate the applicability of the methodology to more representative models.

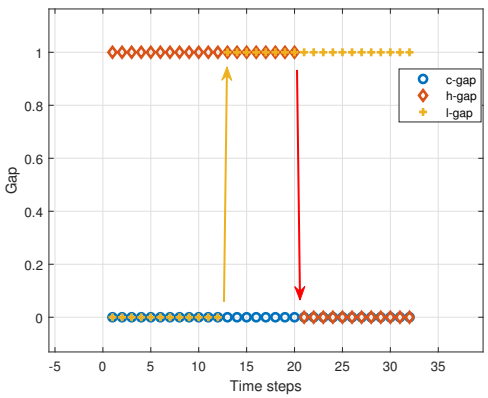


Fig. 4 Gap variation for the different contact elements

Results and discussion

The model considered for the following analyses considers a couple of identical gears. The equation 7 can then be simplified as follows:

$$\frac{I}{2r^2}\ddot{x} + C\dot{x} - F_c + T\frac{1}{r} = 0 \tag{14}$$

Table 1 System parameters.

$\mathbf{I}\ (Kg/m^2)$	$\mathbf{r}\ (m)$	$\mathbf{C}\ (Ns/m)$	$\mathbf{z}$	$\mathbf{c_r}$	φ	$\mathbf{k_c}\ (N/m)$
0.0158	0.0705	4038.2	50	1.75	20°	2.6e8

Where the parameters considered are collected in Table 1.

To validate the methodology proposed, the reference results are obtained by direct time integration of the system expressed in equation 14. The solutions are obtained implementing as a MATLAB algorithm the Newmark method with $\beta = 0.5$ and $\gamma = 0.25$ [14]. The simulation is carried out until steady state is reached.

Non-linear forced responses

Different analyses are performed for multiple levels of constant torques (T) applied, namely 100 Nm, 200 Nm and 300 Nm. The number of harmonics considered in the MHBM solution is $H = 14$. Since the system has only one degree of freedom (DOF), it is possible to simulate it using a large number of harmonics, although good results could be obtained by retaining fewer harmonics. It was also intention of the authors to properly capture the contact force behavior, considering that the unrealistic step gap function induces instant changes in the contact status leading to steep gradients of the contact stiffness during the rotation of the gears.

In Figure 5 the non-linear forced responses of the system are presented; the jumps phenomena are highlighted by black arrows. The HBM simulations are performed implementing a continuation method, however, it was hard to obtain the unstable branches of the response. The simulations for both DTI and HBM are carried out starting from the lowest frequency of the range (forward, "*forw*") and from the highest as well (backward, "*back*").

The different simulation methods are in excellent agreement with each other. All the peaks are predicted accurately both in terms of amplitude and frequency. The jump phenomena typical for this kind of systems are predicted accurately by the HBM simulations.

A slight difference seems to be present around the jump of the first super-harmonic (around 1250 Hz), in that region the continuation method struggled to obtain a converged solution due to the presence of an unstable branch of the response. The apparent difference is in fact due to the missed convergence of the numerical solution.

In Figure 6 a comparison of the non-linear forced responses is presented. It is worth noting that the different responses seem to be scaling linearly with the constant torque applied. This effect is not present in more accurate simulations or in the experimental results available in the literature [6, 15]. Typically, the increase of the constant torque level tends to "linearize" the system, resulting in less softened peaks. In fact, high levels of constant torque counters the complete separation of the teeth at excitation frequencies close to resonance.

The presented system does not completely capture this behavior as a consequence of the simplified contact modelling. More realistic contact modelling will be introduced in future work, to both improve convergence of the solution method and increase accuracy of the results obtained with respect to experimental results available in the literature.

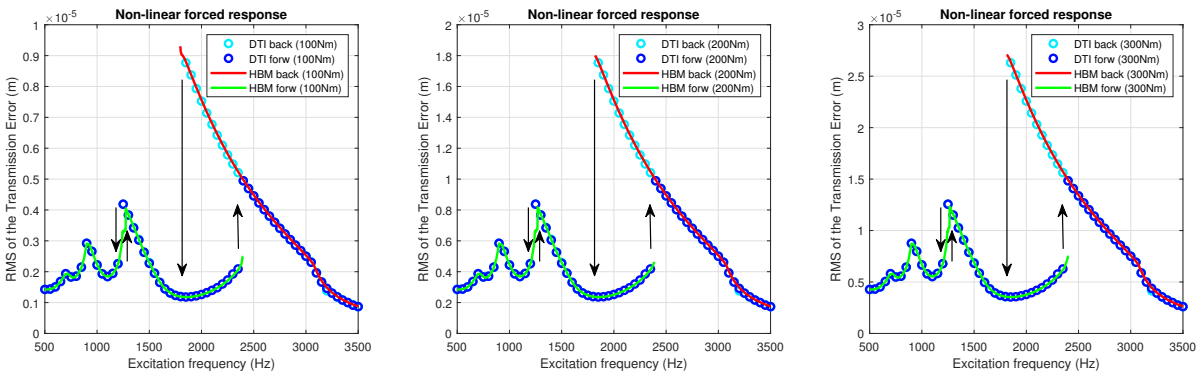


Fig. 5 Non-linear forced responses of the system simulated with DTI and HBM with different static torques applied

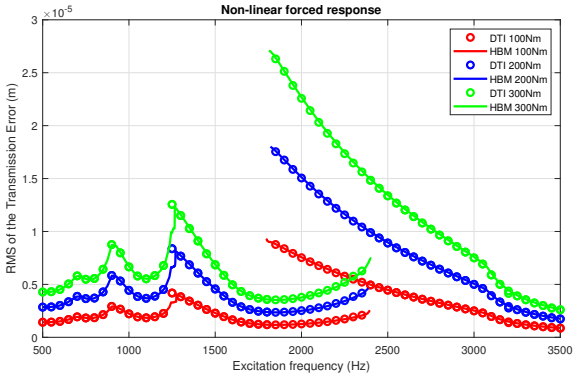


Fig. 6 Non-linear forced responses comparison for different torque levels

Contact force

In this sub-section, a specific focus on the contact quantities is presented. The accuracy of the proposed methodology is assessed by comparison of the contact force and equivalent contact stiffness in the HBM and DTI simulations. The following comparisons are performed for the case with $T = 100$ Nm constant torque applied, but equivalent results were obtained for the other cases.

In Figure 7 the plots associated with an excitation frequency of 3250 Hz are presented. Being relatively far from the resonances, the response at regime for a single period is close to a sinusoidal signal (on the left). The simulation carried out with the proposed methodology (HBM) and with the time integration (DTI) are in very good agreement. On the right side the computed contact force is presented, the steep changes are associated with the gap jumps shown in Figure 4. The instantaneous ratio between the contact force and the response is defined as the mesh stiffness, this quantity is plotted associated with the contact force. The mesh stiffness for this case is a step function that varies between k_c and $2k_c$, giving an indication on the number of contact elements (see Figure 3) operating at each time step. Both the mesh stiffness and the contact force are simulated remarkably well by the HBM solution method when compared to the DTI standard approach.

The same comparisons are presented for other significant excitation frequencies, in Figure 8 the focus is on 1250 Hz. Specifically, at the resonance frequency of the first super-harmonic, the displacement response presents two peaks. The mesh stiffness obtained for this excitation frequency shows three distinguished levels, for some time steps the stiffness is zero, which means that the two gears are completely separated and they are not exchanging any force. This effect is given by the dynamics of the system that is producing vibration levels high enough to open the contact.

In Figure 9 and Figure 10 the results at the main resonance's frequency are presented. At this frequency two stable results are possible, the response with the highest amplitude is presented in Figure 9, while the one with the lowest amplitude is presented in Figure 10. The response in Figure 9 is mainly mono-harmonic, the contact force reaches the highest values

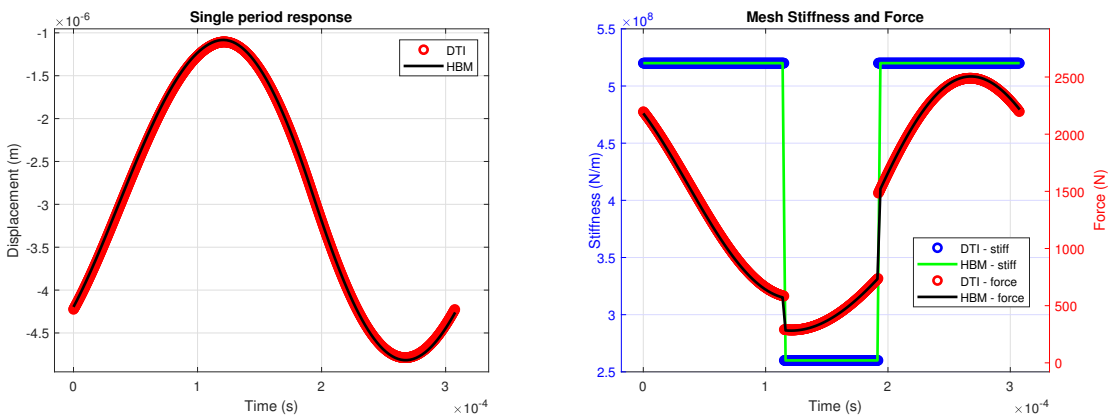


Fig. 7 Response, contact force and mesh stiffness at 3250 Hz

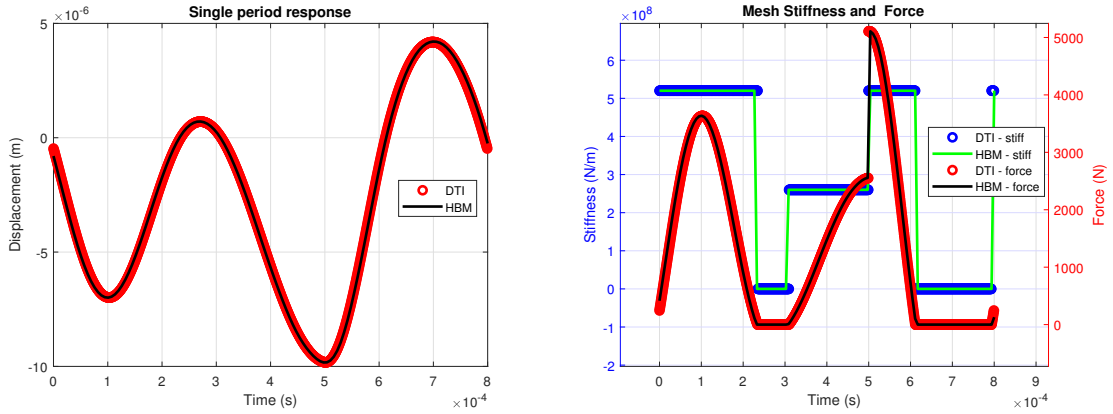


Fig. 8 Response, contact force and mesh stiffness at 1250 Hz

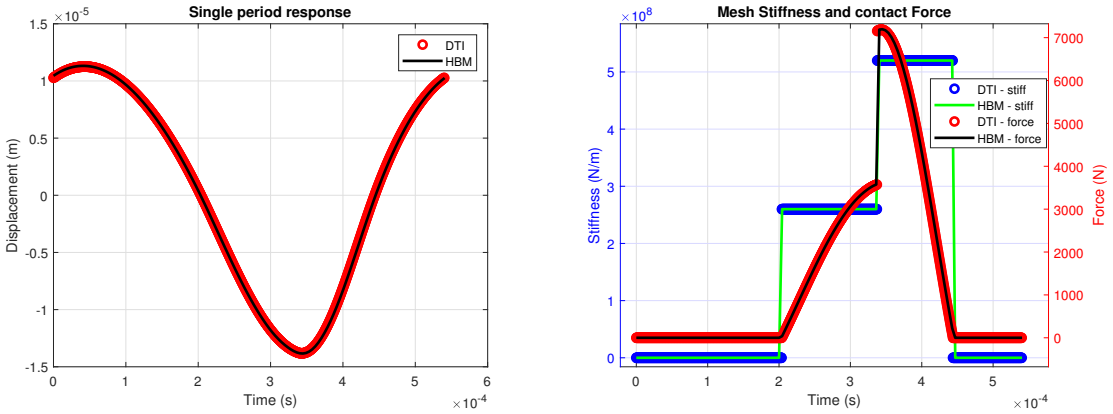


Fig. 9 Response, contact force and mesh stiffness at 1850 Hz, higher response level

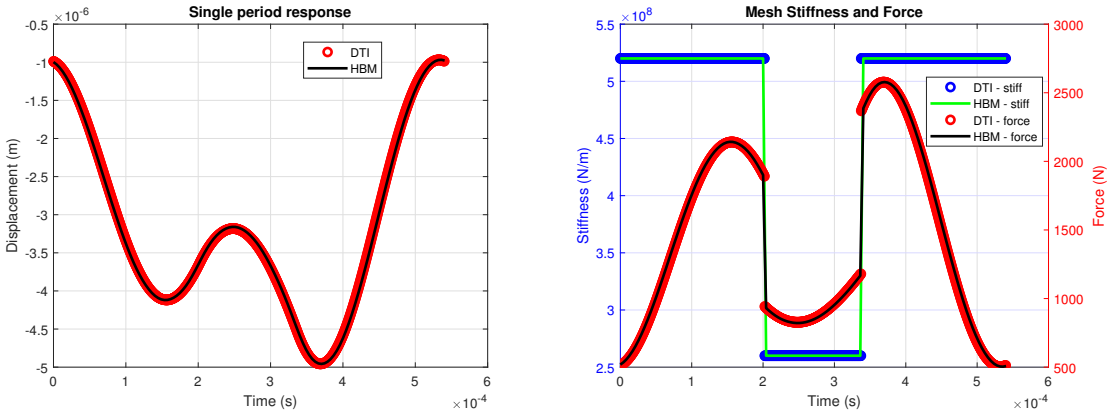


Fig. 10 Response, contact force and mesh stiffness at 1850 Hz, lower response level

of the frequency range and, for a significant portion of the period, separation occurs. While, in Figure 10, the response shows a non-neglectable contribution of the second harmonic. The mesh stiffness has assumed again the form it has far from resonance (as already seen in Figure 7), but in this case the contact force signal presents two peaks given that the excitation frequency of interest is closer to the first super-harmonic.

Conclusions

A new approach to the analyses of spur gear pairs is presented in this paper. The already well-established Harmonic Balance Method in its coupled static-dynamic formulation is extended to the equation of motion of a lumped parameters model of a spur gear pair. A solution scheme is proposed making use of the Alternating Frequency-Time method. The AFT allows to compute the non-linear contact forces moving temporarily to the time domain via a IFFT of the generalized displacements. Subsequently the Fourier coefficient of the contact forces are obtained with the FFT of the single period time signal of the contact forces.

A contact model is required to express the contact force as a function of the generalized displacements and its derivative. A basic penalty-based contact model is introduced defining a time varying gap between the teeth.

The formulation obtained allows to simulate the system in the frequency domain without the need for any preliminary static analysis.

In the results section, multiple non-linear responses are presented comparing the responses amplitudes obtained with the Newmark method and with the proposed methodology. The comparison holds for all the frequency range considered.

Contact force, mesh stiffness and one period responses are also compared at specific frequencies. HBM and DTI show overlapping results for all the cases.

Coupled HBM proved to be an effective alternative to DTI for the simulation of simple lumped parameters models. The results obtained are not close quantitatively to the experimental behavior of the modeled gears, however other studies (see [8]) have shown that lumped parameters model can simulate real components as long as an accurate contact model is introduced.

The proposed methodology allows for the incorporation of various contact models without affecting the overall framework. The authors plan to leverage this flexibility in future work by introducing more realistic contact models to enhance the accuracy of the simple lumped parameter model in relation to actual spur gear pairs.

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