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Existence of expansive solutions and Hamilton-Jacobi equations for the N-body problem

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To my parents and my brother

It's time to try defying gravity
-Elphaba Thropp, *Wicked*

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Introduction

This thesis brings together the recent results obtained in the works [66, 12, 11], providing a unified exposition of the research I carried out at the University of Turin during my PhD program in Pure and Applied Mathematics. These contributions concern the N-body problem in celestial mechanics, studied within the framework of calculus of variations.

The purpose of the current Introduction is to present a broad overview of the N-body problem and of the main results established in the aforementioned papers, which will be discussed in detail in the subsequent chapters. Particular emphasis will be placed on the variational techniques developed and applied in these works, as they provide the fundamental analytical tools used throughout the thesis.

Variational methods for the N-body problem

Despite being one of the oldest and most classical problems in dynamical systems and celestial mechanics, the N-body problem still presents a large set of open questions concerning the nature of its solutions and, particularly, the possible asymptotic behaviors of the masses as time tends to infinity.

Consider N point masses $m_1, ..., m_N$ whose positions in $\mathbb{R}^d$ are described by functions $r_i : I \subset \mathbb{R} \to \mathbb{R}^d$, with $d \geq 2$, evolving with respect to a time variable. The N-body problem consists in determining the motion of these bodies, which interact according to Newton's law of universal gravitation:

$$m_i \ddot{r}_i = - \sum_{j=1,...,N,\ j\neq i}^{N} m_i m_j \frac{r_i - r_j}{|r_i - r_j|^3} \quad \text{for all } i = 1, ..., N.$$

Thanks to the classical work of Newton and Kepler, the two-body problem is known to be completely integrable: its equations of motion possess sufficiently many independent first integrals to allow an explicit solution, yielding a full description of all possible trajectories, which can be ellipses, parabolas or hyperbolas. However, although its formulation is remarkably simple, we are still far from fully understanding the intricate dynamics of the solutions of the general N-body problem. One of the main obstacles is the lack of integrability when $N \geq 3$, which has been conjectured by Poincaré in 1890 [64].

Another difficulty in the analysis of the dynamics of the problem is the possible occurrence of singularities in finite time, where a motion is said to have a singularity at a time t^* if it cannot be extended beyond $t = t^*$. Such singularities typically correspond to collisions, where some mutual distances between bodies vanish.

Since the equations of motion of the N-body problem are invariant by translation, we fix the origin of our inertial frame at the center of mass of the system, so that we can work on the configuration space

$$\mathcal{X} = \left\{ x = (r_1, ..., r_N) \in \mathbb{R}^{dN},\ \sum_{i=1}^{N} m_i r_i = 0 \right\}.$$

Defining the Newtonian potential $U : \Omega \to \mathbb{R} \cup \{+\infty\}$ by

$$U(x) = \sum_{i<j} \frac{m_i m_j}{|r_i - r_j|},$$

we let Ω denote the set of collisionless configurations, that is,

$$\Omega = \{\, x = (r_1, ..., r_N) \in \mathcal{X} : r_i \neq r_j \text{ for all } i \neq j \,\},$$

so that $U(x) < +\infty$ precisely when no pair of bodies occupies the same position.

In this thesis, we focus on the application of variational methods to study the existence of solutions to the N-body problem. These techniques consist in finding solutions to the N-body problem as critical points of the Lagrangian action functional $\mathcal{A}_L : D \to \mathbb{R} \cup \{+\infty\}$,

$$\mathcal{A}_L(x) = \int_a^b L(x(t), \dot{x}(t))\, dt.$$

D is a suitable functional space and $L : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2} \|\dot{x}\|^2_{\mathcal{M}} + U(x),$$

where $\|\cdot\|_{\mathcal{M}}$ denotes the Euclidean mass scalar product in $\mathcal{X}$.

Let Δ denote the set of configurations with collisions. Clearly, if $x \in \Delta$, the Lagrangian action $\mathcal{A}_L(x)$ diverges, since $\lim_{x\to\Delta} U(x) = +\infty$.

By Hamilton's principle of least action, if a curve $\gamma : [a, b] \to \mathcal{X}$ is a minimizer of the Lagrangian action among all absolutely continuous curves with fixed initial and final configurations $\gamma(a)$ and $\gamma(b)$ respectively, then γ satisfies Newton's equations at every time $t \in [a, b]$ in which $\gamma(t)$ has no collisions, that is, whenever $\gamma(t) \in \Omega$. On the other hand, as Poincaré noticed in [65], there are curves both with isolated collisions and finite action. Consequently, classical minimizers of the action cannot exist, which calls for the development of techniques to handle or avoid collisions.

A key breakthrough in overcoming this difficulty was provided by Marchal [54], who introduced the main idea to prevent minimizers of the action from developing collisions. Marchal's Principle, later rigorously proved by Chenciner [21] and by Ferrario and Terracini [35], asserts that in $\mathbb{R}^d$, for $d \geq 2$, any path that minimizes the Lagrangian action among all trajectories connecting two given configurations in a fixed time T is collision-free on the open interval $(0, T)$. This result firmly established variational methods as powerful tools to study the existence of new classes of solutions to the N-body problem, a classical example being the celebrated figure-eight solution of the 3-body problem discovered by Chenciner and Montgomery [22].

Another difficulty in proving the existence of solutions to the N-body problem lies in establishing the existence of critical points of the Lagrangian action. The most direct approach would be to apply the classical direct method of the calculus of variations, which guarantees the existence of minimizers provided that the action is coercive. However, this strategy applies only in special situations. A classical counterexample that shows that the action is not coercive in general is obtained by considering a sequence of constant configurations $x^n = (x_1^n, \dots, x_N^n) \in \mathbb{R}^N$ such that $|x_i^n - x_j^n| \to +\infty$, as $n \to +\infty$, for every $i \neq j$. Then $\|x^n\|_{H^1} = \|x^n\|_{L^2} \to +\infty$, while the action $\mathcal{A}_L(x^n)$ does not diverge. Indeed, the kinetic term is identically zero, and the potential term tends to 0, so that $\mathcal{A}_L(x^n) \to 0$ as $n \to +\infty$.

On the other hand, the coercivity of the action holds for motions with fixed initial and final configurations, or when the action is restricted to the space of simple choreographies – motions in which the bodies move along a single closed curve, cyclically permuting their positions after a fixed time. For the case of periodic orbits, in the recent literature, the problem of coercivity has been overcome by imposing suitable symmetry constraints on the space of loops (see, for example, [3, 13, 75]).

In the following chapters, new techniques to prove the existence of action minimizers will be developed, leading to the construction of new classes of solutions to the N-body problem.

Main results

This thesis is organized as follows.

- Chapter 1 presents the variational techniques developed and employed in [66, 12, 11], with a particular focus on action-minimization principles.

- Chapter 2 concerns the existence of expansive solutions to the N-body problem – solutions for which all mutual distances diverge at infinity – and summarizes the results of [66].

- Chapter 3 is devoted to the Hamilton-Jacobi formulation of the N-body problem, following [12]. In this setting, we introduce an associated value function, study its viscosity properties, and analyze its regularity.

- Chapter 4 discusses the results of [11]. Here, the focus is on the stability analysis of the numerical symmetric solutions produced by the software `Symorb.jl`, a tool designed to compute new symmetric periodic orbits from algebraic symmetry constraints.

In the last paragraphs of the Introduction we collect the results that are presented in the subsequent chapters.

Expansive motions in the N-body problem

In 1922, in his fundational work [19], Chazy provided a classification of all possible final evolutions of the 3-body problem. His analysis concerns motions assumed to be free of future singularities and is based on the asymptotic behavior of the mutual distances between the bodies. While in the Keplerian case there are only three possible classes of motions – elliptic, parabolic or hyperbolic – this classification, which can be extended to the N-body problem as well, introduces new possible behaviors, such as oscillatory motions – motions that are unbounded but return infinitely often to a bounded region – or hyperbolic-elliptic – in which some distances diverge and others remain bounded.

The first part of this thesis is devoted to the study of expansive solutions to the N-body problem, which are defined as motions such that all the mutual distances between the bodies diverge, that is, when $|r_i(t) - r_j(t)| \to +\infty$ for all $i \neq j$, as $t \to +\infty$. Following classical results (see Marchal and Saari [55], and Pollard [67]), all expansive motions $\gamma : [t_0, +\infty) \to \mathcal{X}$ are described by the asymptotic expansion

$$\gamma(t) = at + O(t^{2/3}), \quad \text{as} \quad t \to +\infty,$$

where a is a configuration that can be either with or without collisions.

Chazy's classification can be applied, in particular, to expansive motions as follows, where we assume that the center of mass of the system is at rest (here, we write $f \approx g$ when $\frac{f}{g}$ is bounded between two positive constants):

(H) *hyperbolic*: if $a \in \Omega$ and $\|r_i(t) - r_j(t)\| \approx t$ for all $i < j$;

(P) *parabolic*: if $a = 0$ and $\|r_i(t) - r_j(t)\| \approx t^{2/3}$ for all $i < j$;

(HP) *hyperbolic-parabolic*: if $a \in \Delta \setminus \{0\}$.

The difficulty arising from the study of expansive solutions $\gamma : [1, +\infty) \to \Omega$ is that, in this type of orbits, the mutual distances between the bodies have at most a linear asymptotic growth. As a consequence, the Newtonian potential associated with an expansive motion is not integrable at infinity, thereby ruling out the possibility of minimizing the classical Lagrangian action $\mathcal{A}_L$. This lack of integrability at infinity makes it necessary to introduce new methods tailored to this setting.

In particular, we are interested in studying the existence of expansive solutions for all of the three subclasses, with fixed initial configuration of the bodies and fixed asymptotic expansion. One way to describe the asymptotic expansion of a motion is through its *limit shape*: we say that a motion $\gamma(t)$ has limit shape when there is a time dependent similarity $S(t)$ of the space $\mathbb{R}^d$ such that $S(t)\gamma(t)$ converges to some configuration $a = 0$.

Our aim is then that of proving, for each of the three subclasses, the existence of a motion $\gamma : [1, +\infty) \to \mathcal{X}$ with prescribed initial configuration and prescribed limit shape, according to the nature of the expansive regime under consideration. The possible limit shapes are summarized as follows:

- in the hyperbolic case, the limit shape is a configuration $a \in \Omega$, coinciding with the asymptotic velocity of the motion;
- in the parabolic case, the limit shape is a central configuration b, that is, a critical point of the Newtonian potential restricted to the inertial ellipsoid $\mathcal{E} = \{x \in \mathcal{X} : \|x\|^2 = 1\}$;
- in the hyperbolic-parabolic case, the limit shape again coincides with the asymptotic velocity, which is now a configuration with collisions $a \in \Delta$.

The first part of the thesis is devoted to the proof of such results, following the work in [66].

Existence of minimal expansive solutions to the N-body problem

The first existence result for expansive motions with prescribed initial configuration and limit shape is due to Maderna and Venturelli [50]. In 2009, they proved that for any initial configuration x^0 and any minimizing normalized central configuration b, there exists a collision-free parabolic solution asymptotic to b, which is globally minimizing on every compact time interval. Their approach consists in constructing minimizers of the action γ_n with initial configuration x^0, where compactness is ensured by uniform action bounds. By Ascoli's theorem and a diagonal extraction, a uniformly convergent subsequence is obtained, whose limit γ is collision-free by Marchal's Theorem. The limit curve solves the N-body equations and is shown to be parabolic by comparing its action with Keplerian arcs.

In 2020, Maderna and Venturelli established the existence of hyperbolic solutions with prescribed initial configuration and limit shape, which, in this case, may be any configuration $a \in \Omega$ [52]. By means of a PDE-based method, they studied global viscosity solutions of the Hamilton-Jacobi equation $H(x, \nabla u) = h > 0$.

Introducing h-calibrating curves – motions whose action realizes the exact value predicted by a given viscosity solution – they showed that their existence characterizes viscosity solutions and fixed points of the quotient Lax-Oleinik semigroup. The construction of global solutions is obtained through a compactification of the metric space $(\mathcal{X}, \phi_h)$, where ϕ_h is the Jacobi-Maupertuis metric, inspired by the one used by Gromov to compactify locally compact metric spaces in [40]. The calibrating curves are shown to be hyperbolic motions asymptotic to a given configuration $a \in \Omega$, yielding Jacobi-Maupertuis geodesic rays – motions such that each restriction to a compact interval are minimizing geodesics – with prescribed initial point and asymptotic direction.

In [66], the existence of expansive motions is proved for all three subclasses, including the first result on hyperbolic-parabolic motions with prescribed limit shape. The novelty of this work lies in the unified method employed: a single variational framework that applies to all three types of expansive motions, combining a direct application of the direct method in the calculus of variations with Marchal's Principle.

The main idea is that, rather than minimizing the classical Lagrangian action – which diverges along expansive motions – it is possible to work with a *renormalized* Lagrangian action functional $\mathcal{A}$, obtained by subtracting from the Lagrangian the non-integrable part responsible for the divergence at infinity. In this way, we can obtain a well-defined functional whose minimizers still correspond to genuine solutions of the N-body problem.

This Renormalized Action Principle consists in looking for expansive solutions $\gamma : [1, +\infty) \to \Omega$ of the form

$$\gamma(t) = r_0(t) + \varphi(t) + x^0 - r_0(1),$$

where the initial configuration $x^0 = \gamma(1) \in \mathcal{X}$ is fixed, and $r_0(t)$ is a reference trajectory – also called *guiding curve* – that determines the subclass of the expansive motion under consideration. The functions φ describe the deviation from the reference path and belong to a suitable variational space in which the renormalized action is finite and coercive. They are chosen to belong to the functional space $\mathcal{D}_0^{1,2}(1, +\infty)$, which is, essentially, a Sobolev space of functions with L^2-weak derivative. Working with this type of solutions, we can express the equations of motion of the N-body problem in terms of φ and then prove the existence of minimizers of a renormalized Lagrangian action, which is defined on the space $\mathcal{D}_0^{1,2}(1, +\infty)$. Indeed, if a function $\varphi \in \mathcal{D}_0^{1,2}(1, +\infty)$ is a minimizer of the renormalized Lagrangian action $\mathcal{A} : \mathcal{D}_0^{1,2}(1, +\infty) \to \mathbb{R}$,

$$\mathcal{A}(\varphi) = \int_1^{+\infty} \frac{1}{2}\|\dot\varphi(t)\|_{\mathcal{M}}^2 + U(r_0(t) + \varphi(t) + x^0 - r_0(1)) - U(r_0(t)) - \langle \mathcal{M}\ddot r_0(t), \varphi(t)\rangle \, dt,$$

then the associated expansive solution γ is a free-time minimizer – that is, it is a minimizer when restricted to any compact subinterval of $[1,+\infty)$ of the Lagrangian action $\mathcal{A}_L$ – and thus it is a solution of the N-body problem. In addition, since these solutions are free-time minimizers of the action, they correspond to geodesic rays of the Jacobi-Maupertuis metric.

This principle can be stated as follows.

Renormalized Action Principle (Polimeni and Terracini 2024 [66])**.** *Given x^0 and r_0 as above, if $\varphi^{min} \in \mathcal{D}_0^{1,2}(1,+\infty)$ is a minimizer of the renormalized Lagrangian action, then the corresponding expansive motion*

$$\gamma(t) = r_0(t) + \varphi^{min}(t) + x^0 - r_0(1)$$

is a free-time minimizer of the Lagrangian action and, in particular, is a solution of Newton's equations of the N-body problem for any $t \in (1,+\infty)$ (or for any $t \in [1,+\infty)$, if $x^0 \in \Omega$).

Using this new approach, we first obtain the same result of Maderna and Venturelli concerning hyperbolic motions.

Theorem (Maderna and Venturelli 2020 [52])**.** *Given $d \in \mathbb{N}$, $d \geq 2$, for the Newtonian N-body problem in $\mathbb{R}^d$ there is a hyperbolic motion $\gamma : [1,+\infty) \to \mathcal{X}$ of the form*

$$\gamma(t) = at - \log(t)\nabla U(a) + O(1) \quad \text{as } t \to +\infty,$$

for any initial configuration $x^0 = \gamma(1) \in \mathcal{X}$ and for any collisionless configuration $a \in \Omega$.

Next, we recover the result of Maderna and Venturelli for the parabolic case, also providing a sharper description of the remainder in the asymptotic expansion.

Theorem (Maderna and Venturelli 2009 [50], Polimeni and Terracini 2024 [66])**.** *Given $d \in \mathbb{N}$, $d \geq 2$, for the Newtonian N-body problem in $\mathbb{R}^d$ there is a parabolic solution $\gamma : [1,+\infty) \to \mathcal{X}$ of the form*

$$\gamma(t) = \beta b_m t^{2/3} + o(t^{1/3^+}) \quad \text{as } t \to +\infty,$$

for any initial configuration $x^0 = \gamma(1) \in \mathcal{X}$, for any minimal normalized central configuration b_m and for $\beta = \sqrt[3]{\frac{9}{2}U(b_m)}$.

The improvement lies in the sharper remainder estimate: here we have $o(t^{1/3^+}) = o(t^{1/3+\varepsilon})$ for all $\varepsilon > 0$, as $t \to +\infty$, whereas Maderna and Venturelli obtained $O(t^{2/3})$ as $t \to +\infty$.

While the proof in the hyperbolic and parabolic cases is relatively straightforward, the mixed hyperbolic-parabolic case requires a more delicate analysis, since the limit shape has collisions. To address this difficulty, our strategy consists in introducing a cluster decomposition of the bodies, determined by the collisions present in the limit configuration. Specifically, we define the following equivalence relation on the index set $\mathcal{N} = \{1, ..., N\}$:

$$i \sim j \quad \iff \quad a_i = a_j.$$

The partition classes are called *clusters*. Using this equivalence relation, we have that in each cluster, the bodies move, with respect to the center of mass of the cluster, with a parabolic expansion at infinity. On the other hand, the motion of the center of mass of each cluster follows a hyperbolic expansion at infinity.

The cluster decomposition is then used to decompose the renormalized Lagrangian action, accounting separately for the different types of expansion within the clusters and among the clusters, which significantly simplifies the proof of coercivity required for applying the direct method in the calculus of variations.

The main result in the hyperbolic-parabolic case is the following.

Theorem (Polimeni and Terracini 2024 [66])**.** *Given $d \in \mathbb{N}$, $d \geq 2$, for the Newtonian N-body problem in $\mathbb{R}^d$ there is a hyperbolic-parabolic motion $\gamma : [1, +\infty) \to \mathcal{X}$ of the form*

$$\gamma(t) = at + \beta b_m t^{2/3} + o(t^{1/3^+}) \quad \text{as } t \to +\infty,$$

for any initial configuration $x^0 = \gamma(1) \in \mathcal{X}$, for any collision configuration $a \in \Delta$, for any normalized minimal central configuration $b_m \in \mathcal{X}$ of the a-clustered potential and for any choice of the energy constant $h > 0$ (here, both b_m and β depends on a and the associated cluster decomposition).

Regularity of viscosity solutions to the Hamilton-Jacobi equations of the N-body problem

Hamilton-Jacobi equations play a central role in the study of dynamical systems, especially in celestial mechanics, and the Newtonian N-body problem offers a particularly rich setting for investigating their viscosity solutions. Over the past two decades, significant advances have been achieved in describing these solutions and in connecting them with the geometric and variational structure of the N-body system, as Maderna and Venturelli did in their analysis of hyperbolic solutions in [52]. In fact, a connection holds for all three different types of expansive motions. This idea was suggested in the final section of [66] and has been fully developed in [12].

The results in [12] extend previous ones of Cannarsa and Sinestrari, who established several fundamental results in [18] about viscosity solutions to Hamilton-Jacobi equations. Working in a more general setting, they proved that, given a continuous function $u_0 : X \to \mathbb{R}$ and fixing $t \in [0,T]$, the value function

$$u(t,x) = \min_{\xi \in \mathcal{C}(\xi(0),x)} \int_0^t L(\xi(s), \dot{\xi}(s))\ ds + u_0(\xi(0))$$

is a viscosity solution to the Hamilton-Jacobi Cauchy problem

$$\begin{cases} \partial_t u(t,x) + H(x, \nabla u(t,x)) = 0, & \text{in } [0,T] \times X, \\ u(0,x) = u_0(x), & \text{in } X. \end{cases}$$

We can thus understand that, in the framework of dynamical systems and the calculus of variations, Hamilton-Jacobi equations play a central role by encoding, in a single nonlinear PDE, the entire variational structure of the problem. Their viscosity solutions coincide with value functions of action-minimizing problems, thereby capturing all globally minimizing trajectories. Moreover, the regularity and singularity structure of these solutions reflect key dynamical features, such as the loss of uniqueness of minimizers.

In this thesis, we prove the same result, but adapted to the context of expansive solutions to the N-body problem: this means that in order to define a proper value function, we have to consider the fact that expansive motions are defined over the half-line $[1, +\infty)$, thus requiring a renormalization of the Lagrangian action.

The Hamiltonian of the N-body problem is defined on $\Omega \times \mathbb{R}^{dN}$ as

$$H(x,p) = \frac{1}{2}\|p\|_{\mathcal{M}^{-1}} - U(x).$$

Considering an expansive solution $\gamma(t) = r_0(t) + \varphi(t) + x^0 - r_0(1)$ as above, we can emphasize the dependence of the renormalized Lagrangian action on the initial configuration x^0 by defining the value function $v : \Omega \to \mathbb{R}$,

$$v(x^0) = \min_{\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)} \mathcal{A}(\varphi) - \langle a, x^0 \rangle_{\mathcal{M}}.$$

Here, $a \in \mathbb{R}^{dN}$ represents the configuration that controls the linear growth in the expansive motion. More specifically, $a \in \Omega$ if γ is hyperbolic, $a \in \Delta$ if γ is hyperbolic-parabolic and $a = 0$ if γ is parabolic.

The first main result in [12] is that the function v is a viscosity solution of the supercritical Hamilton-Jacobi equation

$$H(x, \nabla v(x)) = h,$$

where $h \geq 0$ is the constant energy of the motion. This extends the classical result of Cannarsa and Sinestrari to the present setting. The proof proceeds by introducing truncated value functions

$$\bar{v}(T,x) \;=\; \min_{\varphi\in\mathcal{D}_0^{1,2}(1,T)} \mathcal{A}(\varphi) \;-\; \langle \dot{r}_0(T),\, x\rangle, \qquad x\in\Omega,$$

which satisfy uniform coercivity estimates. From this, we show that, for each $x \in \Omega$, the functions $\bar{v}(T,x)$ converge to $v(x)$, as $T \to +\infty$, uniformly on compact subsets of Ω. Then, using the classical stability properties of viscosity solutions together with these uniform coercivity bounds, we pass to the limit and conclude that $v(x)$ indeed solves the desired Hamilton-Jacobi equation.

The second main result in [12] concerns the regularity of the value function v. We define the singular set Σ as the set of initial points for which the associated renormalized value function admits more than one minimizer. Moreover, we introduce the set of conjugate points Γ, understood heuristically as the set of points where the linearized problem degenerates. Mirroring the classical results of Cannarsa and Sinestrari in [18], and adapting them to our setting, we estimate the dimension of the singular set $\Sigma \cup \Gamma$.

These results can be summarized in the following.

Theorem (Berti, Polimeni and Terracini 2025 [12])**.** *Let $a \in \Omega$ (type (H)), or let b_m be a minimal central configuration of U (type (P)), or let $a \in \Delta$ and b_m be a normalized minimal central configuration of the a-clustered potential (type (HP)). Then, there exists a viscosity solution to the N-body Hamilton-Jacobi equation (3.1.2). The singular set of such a solution is a countably $\mathcal{H}^{d(N-1)-1}$-rectifiable subset of the configuration space $\mathcal{X}$. Moreover, we have*

$$\dim_{\mathcal{H}}(\Gamma \setminus \Sigma) \le d(N-1) - 2.$$

Stability of numerical solutions to the N-body problem

Despite being powerful tools for establishing the existence of new classes of solutions to the N-body problem, variational methods are often difficult to apply due to the lack of compactness in the open set of collisionless paths and the possible occurrence of collisions, which cannot always be ruled out by Marchal's Principle. Moreover, only a few explicit solutions of the N-body problem are known – such as the Keplerian motions and the figure-eight solution of Chenciner and Montgomery. For these reasons, numerical methods are frequently employed to approximate or identify further examples.

In 2004, in their celebrated paper [35], Ferrario and Terracini introduced a new variational method for proving the existence of periodic solutions by imposing algebraic constraints on the bodies. Their approach consists in prescribing symmetry conditions on the configurations through the action of a finite group G, and then seeking critical points of the action functional within the space of G-equivariant loops. Building on this theoretical framework, they developed an algorithm for computing numerical symmetric solutions, implemented in the software `Symorb` [28, 34]. This tool allows one to search efficiently for periodic orbits satisfying prescribed symmetry constraints, providing a practical counterpart to the variational existence theory. Based on a combination of Python, Fortran, and GAP, it allows to choose a finite group, consider the symmetry involved, and look for equivariant critical points of the action functional.

Recently, a new version of `Symorb` has been introduced in [5]. The new software `Symorb.jl` is introduced as a package written in Julia that unifies earlier implementations under a single framework. Blending traditional equivariant variational methods with modern computational tools, it enables large-scale exploration and classification of symmetric periodic solutions by minimizing the Lagrangian action among loops invariant under a prescribed finite symmetry group. The paper presents both the theoretical foundations of the method, based on the work of Ferrario and Terracini [35], and a description of the numerical implementation through explicit examples. In particular, `Symorb.jl` effectively works as a "factory" to discover and classify new periodic orbits, while exploring a wide range of symmetry types.

In [11], some benchmark examples of orbits produced by `Symorb.jl` are examined to analyze the stability of periodic solutions under variations of the action and under small perturbations of the initial conditions. Two stability indicators are used for this purpose: the discrete Morse index and the Floquet multipliers. The former counts the number of negative eigenvalues of the second variation of the action along a periodic solution, measuring how many independent directions decrease the action; the latter consists of the eigenvalues of the monodromy matrix of the linearized system along the periodic orbit, determining its stability over one period.

In this thesis, the results of [11] are presented: Section 1.5 develops the theoretical framework for the existence of symmetric periodic solutions, following [35], while Chapter 4 describes the numerical implementation of `Symorb.jl`.

Contents

Chapter 1

Variational approaches to the N-body problem

The lack of first integrals in the Newtonian N-body problem for $N \geq 3$, conjectured by Poincaré in [64], makes it difficult to detect solutions of this dynamical system. Since the 1990s, variational methods have proved to be powerful tools for establishing the existence of solutions to the classical N-body problem. These methods – which, roughly speaking, consist in finding solutions as critical points of a suitable functional – can be applied, for example, to prove the existence of fixed-end solutions, where the initial and final positions of the bodies are prescribed, and of simple choreography solutions, in which the bodies lie on the same curve and exchange their mutual positions after a fixed time.

The application of variational methods to the N-body problem typically follows the strategy outlined below.

1. Define a space of orbits that satisfy certain topological or symmetry constraints and match a prescribed boundary condition – such as a fixed initial configuration and/or a fixed asymptotic direction.

2. Introduce a functional, differentiable on this space, arising from the weak formulation of the equations of motion for the N-body problem. Typically, the Jacobi-length and Maupertuis functionals capture the Riemannian structure of the problem, whereas the energy and Lagrangian action functionals are more closely tied to its dynamical nature.

3. Formulate and prove a variational principle asserting that collisionless critical points of this functional correspond to solutions of the N-body problem.

4. Identify critical points of the functional by exploiting compactness properties of the space of paths. When the coercivity of the functional is ensured, the direct method in the calculus of variations can be applied to prove the existence of minimizers.

The motivation behind this thesis is the development and application of new variational techniques to establish the existence of new classes of solutions to the Newtonian N-body problem. In particular, we present the results obtained in [66], which prove the existence of expansive solutions, namely motions in which all mutual distances between the bodies diverge as time tends to infinity. Moreover, we exploit the variational formulation of the problem to study viscosity solutions of the associated Hamilton-Jacobi equations – again in the setting of expansive motions – as established in [12]. Finally, we show how variational techniques can also be used to generate explicit numerical solutions of the N-body problem, and we study the stability properties of such solutions, following the results in [11].

In this chapter, we introduce the variational approaches employed in proving these results and describe the corresponding variational framework. The application of these approaches can be found in the subsequent chapters.

1.1 The N-body problem: expansive motions and the Renormalized Action Principle

Consider the motion of N point masses $m_1, ..., m_N$ evolving under Newton's law of universal gravitation in $\mathbb{R}^d$. Their positions are collected in the configuration vector $x = (r_1, \ldots, r_N) \in \mathbb{R}^{dN}$, where each $r_i \in \mathbb{R}^d$. We denote by $|r_i - r_j|$ the Euclidean distance between the bodies of masses m_i and m_j. The dynamics of the system is governed by the Newtonian system of ordinary differential equations

$$\mathcal{M}\ddot{x} = \nabla U(x), \tag{1.1.1}$$

where $\mathcal{M} = \operatorname{diag}(m_1 I_d, ..., m_N I_d)$ denotes the diagonal matrix of the masses and I_d is the identity matrix in $\mathbb{R}^d$. Since these equations are invariant by translation, we can fix the origin of our inertial frame at the center of mass of the system, so that we can work on the configuration space

$$\mathcal{X} = \left\{ x = (r_1, ..., r_N) \in \mathbb{R}^{dN},\ \sum_{i=1}^{N} m_i r_i = 0 \right\}.$$

Denoting by $\Omega = \{x \in \mathcal{X} \mid r_i \neq r_j \ \forall\ i \neq j\} \subset \mathcal{X}$ the set of configurations without collisions, which is open and dense in $\mathcal{X}$, the Newtonian potential $U : \Omega \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$U(x) = \sum_{i<j} \frac{m_i m_j}{|r_i - r_j|}.$$

This function is homogeneous of degree -1, is of class $C^1(\Omega)$ and $\lim_{x \to \Delta} U(x) = +\infty$, where we denote by $\Delta = \mathcal{X} \setminus \Omega$ the complement of Ω, that is, the set of configurations with collisions.

Newton's equations define an analytic local flow on $\Omega \times \mathbb{R}^{dN}$, with a first integral given by the mechanical energy:

$$h = \frac{1}{2}\sum_{i=1}^{N} \frac{1}{m_i}|\dot{r}_i|^2 - U(x).$$

Studying the dynamics of this model is challenging due to the possible occurrence of singularities in finite time. A motion is said to have a singularity at time $t^* < +\infty$ if it cannot be extended beyond t^*. Von Zeipel showed in [81] that if there is a singularity at t^* and the positions of the N bodies remain bounded as t approaches t^*, then the singularity must be due to a collision, which means that a noncollision singularity is experienced only if the bodies become unbounded in finite time.

Theorem 1.1.1 (Von Zeipel 1908 [81])**.** *Let $x : (0, t^*) \to \mathbb{R}^{dN}$ be a maximal solution of the Newton's equations of the N-body problem with $t^* < +\infty$. If $\|x(t)\|$ is bounded in some neighborhood of t^*, then the limit $x_c = \lim_{t\to t^*} x(t)$ exists and the singularity is therefore due to collisions.*

It is worth to recall also Painlevé's Theorem about singularities of the classical solutions of N-body systems.

Theorem 1.1.2 (Painlevé 1897 [60])**.** *Let $\bar{x}$ be a classical solution for the N-body problem on the interval $[0, t^*)$. If $\bar{x}$ has a singularity at $t^* < +\infty$, then the potential associated to the problem diverges to $+\infty$ as t approaches t^*.*

Maximal solutions that end in finite time must either experience collisions at the last moment or have a pseudocollision. In this sense, we have the following definition.

Definition 1.1.3. Suppose that $x(t)$ has a singularity at t^*. This singularity is called a collision if there exists some $x^* \notin \Omega$ such that $x(t) \to x^*$ as $t \to t^*$. Otherwise, the singularity is called a pseudocollision.

The usual variational approach to prove the existence of solutions of the N-body problem consists in seeking orbits as critical points of the action functional associated to the system. Fixing two configurations $x, y \in \mathcal{X}$ and $T > 0$, we denote the subsets of absolutely continuous curves

$$\mathcal{C}(x, y, T) = \{\gamma : [a, b] \to \mathcal{X} \ : \ \gamma(a) = x,\ \gamma(b) = y,\ b - a = T\}$$

and

$$\mathcal{C}(x, y) = \bigcup_{T>0} \mathcal{C}(x, y, T),$$

and define the Lagrangian action of a curve $\gamma \in \mathcal{C}(x, y, T)$ as

$$\mathcal{A}_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t))\, dt. \tag{1.1.2}$$

Definition 1.1.4. We say that an absolutely continuous path $\gamma : [a, b] \to \mathcal{X}$ is a minimizer of the Lagrangian action if $\mathcal{A}_L(\gamma) \leq \mathcal{A}_L(\sigma)$ for every absolutely continuous path $\sigma : [a, b] \to \mathcal{X}$ having the same extremities.

The function $L : \mathbb{R}^{dN} \times \mathbb{R}^{dN} \to \mathbb{R}$,

$$L(\gamma, \dot{\gamma}) = \frac{1}{2}\|\dot{\gamma}\|^2_{\mathcal{M}} + U(\gamma), \tag{1.1.3}$$

is the Lagrangian of the N-body problem and $\|\cdot\|_{\mathcal{M}}$ denotes the norm induced by the mass scalar product

$$\langle x, y\rangle_{\mathcal{M}} = \sum_{i=1}^{N} m_i \langle r_i, s_i\rangle, \qquad \text{for any } x = (r_1, ..., r_N),\ y = (s_1, ..., s_N) \in \mathcal{X}.$$

With a small abuse of notation, $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^d$ and also in $\mathcal{X}$.

The Lagrangian action $\mathcal{A}_L$ defined in (1.1.2) is of class C^1 on the subset of collisionless motions in $\mathcal{C}(x, y, T)$. So, according to Hamilton's principle of least action, if a curve $\gamma : [a, b] \to \mathcal{X}$ is a minimizer of the Lagrangian action in $\mathcal{C}(x, y, T)$, then $\gamma(t)$ is a solution of (1.1.1) for all $t \in [a, b]$ in which $\gamma(t) \in \Omega$. However, being a critical point of the Lagrangian action is not enough to exclude the motion from having collisions, since, as Poincaré noticed in [65], there exist curves both with isolated collisions and finite action.

A major advance in this direction was made by Marchal [54], who introduced the idea of averaging over suitable sets of variations for the Keplerian potential and for discs or spheres, in order to prove that minimizers of the fixed-end (Bolza) problem are free of interior collisions. The results presented in this thesis rely fundamentally on Marchal's Principle (see Theorem 1.2.1), whose complete proofs were later given by Chenciner [21] and by Ferrario and Terracini [35].

Assuming motions without singularities in the future, every possible final evolution in the N-body problem has been classified in terms of the asymptotic behavior of the distance between the bodies, even though the existence of motions for any type of final evolution is not ensured. We can now describe the class of expansive motions, focusing on its properties and the classification given by Chazy in 1922 [19].

Definition 1.1.5. A motion $\gamma : [t_0, +\infty) \to \Omega$ is said to be expansive when all the mutual distances diverge, that is, when $|r_i(t) - r_j(t)| \to +\infty$, as $t \to +\infty$, for all

$i < j$.

Equivalently, a motion γ is expansive if $U(\gamma(t)) \to 0$, as $t \to +\infty$.

The conservation of energy implies that an expansive motion can only occur at nonnegative energy, since $\|\dot{\gamma}(t)\|_{\mathcal{M}}^2 \to h$ as $t \to +\infty$.

The following theorems are classical results about expansive motions and their asymptotic expansion, where we denote with

$$r(t) = \min_{i<j} |r_i(t) - r_j(t)| \quad \text{and} \quad R(t) = \max_{i<j} |r_i(t) - r_j(t)|$$

the minimum and maximum separation between the bodies at time t.

Notation 1. Given positive functions f and g, we write $f \approx g$ if their quotient is bounded between two positive constants.

Theorem 1.1.6 (Pollard 1967 [67])**.** *Let γ be a motion defined for all $t > t_0$. If r is bounded away from zero, then we have that $R = O(t)$ as $t \to +\infty$. In addition, $R(t)/t \to +\infty$ if and only if $r(t) \to 0$.*

Theorem 1.1.7 (Marchal-Saari 1976 [55])**.** *Let γ be a motion defined for all $t > t_0$. Then either $R(t)/t \to +\infty$ and $r(t) \to 0$, or there is a configuration $a \in \mathcal{X}$ such that $\gamma(t) = at + O(t^{2/3})$. In particular, for superhyperbolic motions (i.e. motions such that $\limsup_{t\to+\infty} R(t)/t = +\infty$) the quotient $R(t)/t$ diverges.*

Theorem 1.1.8 (Marchal-Saari 1976 [55])**.** *Suppose that $\gamma(t) = at + O(t^{2/3})$ for some $a \in \mathcal{X}$ and that the motion is expansive. Then, for each pair $i < j$ such that $a_i = a_j$, we have $|r_i(t) - r_j(t)| \approx t^{2/3}$.*

Theorem 1.1.6 infers that expansive motions can not be superhyperbolic, and hence can be described with the expansion

$$\gamma(t) = at + O(t^{2/3}), \text{ as } t \to +\infty,$$

for some limit configuration $a \in \mathcal{X}$.

In his well-known work from 1922 [19], Chazy gave a complete classification of expansive solutions of the 3-body problem – which can be extended to the N-body problem – based on the asymptotic order of growth of the mutual distances between the bodies. Assuming that the center of mass of the system is at rest, Chazy partitioned the class of expansive motions into the following subclasses:

(H) *Hyperbolic*: $a \in \Omega$ and $|r_i(t) - r_j(t)| \approx t$ for all $i < j$;

(P) *Parabolic*: $a = 0$ and $|r_i(t) - r_j(t)| \approx t^{2/3}$ for all $i < j$;

(HP) *Hyperbolic-parabolic*: $a \in \Delta$ but $a \neq 0$.

In Chapter 2 we will examine in detail the three subclasses of expansive motions, providing a more precise description of their asymptotic behavior, while in the present chapter, we focus on the variational framework associated with general expansive motions, without specifying the subclass to which they belong.

1.1.1 Outline of the proof

Referring to [66], we provide here a sketch of the proof for the Renormalized Action Principle, that will be employed in Chapter 2 to prove the existence of expansive solutions of the N-body problem.

To study the existence of expansive solutions, as well as the viscosity solutions of the associated Hamilton-Jacobi equations, we focus on motions having a specific structure. Specifically, we consider half-entire expansive motions

$$\gamma(t) = r_0(t) + \varphi(t) + x^0 - r_0(1), \tag{1.1.4}$$

where:

- r_0 is a "reference path", that is, the term whose definition determines the type of expansive motion represented by γ;
- φ is a function of lower order with respect to r_0 belonging to a proper functional space, say $\mathcal{D}$;
- x^0 is the (fixed) initial configuration of the bodies.

For all three subclasses, we will prove in Chapter 2 the existence of expansive solutions for any given initial configuration x^0 and for any limit shape, which is the configuration that determines the asymptotic behavior of the motion and is described as follows (see Section 2.1 for a complete description of the different limit shapes for expansive motions).

Definition 1.1.9. We say that a motion $\gamma(t)$ has limit shape when there is a time-dependent similarity $S(t)$ of the space $\mathbb{R}^d$ such that $S(t)\gamma(t)$ converges to some configuration which is different from the null one.

In our case, the action of $S(t)$ is diagonal, that is, $S(t)x = (S(t)r_1, ..., S(t)r_N)$, for $x \in \mathbb{R}^{dN}$.

Our goal is to find hyperbolic, parabolic and hyperbolic-parabolic trajectories as free-time minimizers, which have an *a priori* infinite Lagrangian action.

Definition 1.1.10. A curve $\gamma : I \to \mathcal{X}$ is a free-time minimizer for the Lagrangian action at energy h if for all intervals $[a, b], [a', b'] \subset I$ and for all curves $\sigma : [a', b'] \to \mathcal{X}$ such that $\gamma(a) = \sigma(a')$ and $\gamma(b) = \sigma(b')$, it holds

$$\int_a^b L(\gamma, \dot{\gamma})\, dt + h(b - a) \leq \int_{a'}^{b'} L(\sigma, \dot{\sigma})\, dt + h(b' - a').$$

To prove the existence of such solutions, the main strategy consists in expressing the equations of motion in terms of φ, so that we can work on the functional space $\mathcal{D}$ that ensures the Lagrangian action is well-defined and admits minimizers.

More specifically, we choose $\mathcal{D}$ as the Sobolev space (cfr. (1.2.2)):

$$\mathcal{D} = \{\varphi : \ \varphi(1) = 0 \text{ and} \int_1^\infty \|\dot{\varphi}\|^2_{\mathcal{M}} < +\infty\}.$$

This ensures $\|\varphi(t)\| = o(t^{1/2})$ as $t \to +\infty$ (see §1.2 for details), confirming that r_0 is the guiding term of the curve we are looking for, for large values of t, as, accordingly to (1.1.4), its minimal rate of growth in the three cases is $t^{2/3}$.

Given the equivalent equations of motion

$$\mathcal{M}\ddot{\varphi}(t) = \nabla U(r_0(t) + \varphi(t) + x^0 - r_0(1)) - \mathcal{M}\ddot{r}_0(t),$$

we can exploit the variational formulation of the problem, by which the solutions of the equations coincide with the minimizers of the associated Lagrangian action functional on $\mathcal{D}$. However, since, the guiding curves r_0 have, by definition, at most a linear expansion at infinity, the Lagrangian action cannot be minimized when considering all $t \in [1, +\infty)$, as it has an infinite value when $t \to +\infty$.

The approach adopted in [66] to overcome this issue is to consider a renormalized Lagrangian action instead of the usual one. Specifically, we add a term depending only on time to the Lagrangian, so that the new integrand of the action becomes integrable. Most importantly, this modification preserves the correspondence between the minimizers of the action and the solutions of the N-body problem, since the Euler-Lagrange equations satisfied by the minimizers of the renormalized action coincide with those of the original one.

The renormalized Lagrangian action is defined as follows.

Definition 1.1.11 (Renormalized Lagrangian action)**.** Given a reference path r_0 and an initial configuration x^0, we define the renormalized Lagrangian action $\mathcal{A}^{ren} : \mathcal{D} \to \mathbb{R}$ as

$$\mathcal{A}^{ren}(\varphi) = \int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|^2_{\mathcal{M}} + U(r_0(t) + \varphi(t) + x^0 - r_0(1)) - U(r_0(t)) - \langle \mathcal{M}\ddot{r}_0(t), \varphi(t)\rangle \, dt. \tag{1.1.5}$$

Unlike the standard Lagrangian action, the renormalized action is not positive definite. As a consequence, one of the main difficulties lies in establishing its coercivity. This property, combined with weak-lower semicontinuity, will allow us to apply the direct method of the calculus of variations and obtain the existence of minimizers. For completeness, we recall below the Weierstrass Theorem, which will be used to prove the existence of minimizers for the renormalized Lagrangian action.

Theorem 1.1.12 (Generalized Weierstrass Theorem)**.** *Let us consider a reflexive Banach space X, a non-empty, closed and convex subset $C \subset X$ and a functional $F : C \to \mathbb{R}$. If F is:*

1. *coercive, that is $F(u) \to +\infty$ if $\|u\| \to +\infty$, $u \in C$,*
2. *(sequentially) weakly lower semicontinuous, that is if $(u_n) \subset C$ weakly converges to u, then $F(u) \leq \liminf F(u_n)$,*

then there is a minimum point $\bar{u} \in C$ of F.

A central aspect of the proof is that, to be solutions of the N-body problem, the curves $\gamma(t)$ we are looking for must be free-time minimizers of the action functional. Being (fixed-time) minimizers is not sufficient, since we are working with curves defined on infinite time intervals.

The following principle summarizes the variational method used to prove the existence of expansive solutions. We give here only a sketch of the proof; further details, including the proof of the free-time minimization property, will be presented in the subsequent sections.

Renormalized Action Principle (Polimeni and Terracini 2024 [66])**.** *Given x^0, r_0 and $\mathcal{D}$ as above, if $\varphi^{min} \in \mathcal{D}$ is a minimizer of the renormalized Lagrangian action, then the corresponding expansive motion*

$$\gamma(t) = r_0(t) + \varphi^{min}(t) + x^0 - r_0(1)$$

is a free-time minimizer of the Lagrangian action and, in particular, is a solution of Newton's equations (1.1.1) for any $t \in (1, +\infty)$ (or for any $t \in [1, +\infty)$, if $x^0 \in \Omega$).

Proof. Assume that a curve $\varphi \in \mathcal{D}$ minimizes the renormalized Lagrangian action, and let $\gamma(t) = r_0(t) + \varphi(t) + x^0 - r_0(1)$ denote the corresponding expansive motion. By Hamilton's principle of least action, the minimality of φ implies that it satisfies the Euler-Lagrange equations associated with $\mathcal{A}^{\text{ren}}$, namely

$$\mathcal{M}\ddot{\varphi}(t) = \nabla U\Big(r_0(t) + \varphi(t) + x^0 - r_0(1)\Big) - \mathcal{M}\ddot{r}_0(t),$$

for every $t \in [1, +\infty)$ such that $\gamma(t) \in \Omega$. Furthermore, since Corollary 1.3.3 guarantees that $\gamma(t)$ is a free-time minimizer of the Lagrangian action, Marchal's Principle (Theorem 1.2.1) ensures that $\gamma(t)$ is collision-free for all $t \in (1, +\infty)$. Consequently, φ satisfies the above system, and equivalently, $\gamma(t)$ is a solution of equations (1.1.1) for all $t \in (1, +\infty)$. □

To simplify the notation, throughout this thesis, when there is no ambiguity in interpretation, we will write $\mathcal{A}$ instead of $\mathcal{A}^{ren}$.

1.2 Existence of expansive solutions: the variational setting

The problem arising in the presence of collisions in an orbit is that both the Lagrangian

$$L(x,v) = \frac{1}{2}\|v\|^2_{\mathcal{M}} + U(x),$$

and the Hamiltonian $H : \mathbb{R}^{dN} \times ((\mathbb{R}^d)^*)^N \to \mathbb{R}$,

$$H(x,p) = \frac{1}{2}\|p\|^2_{\mathcal{M}^{-1}} - U(x), \tag{1.2.1}$$

of an N-body system become infinite at collisions. This gives, in particular, an infinite Lagrangian action. Moreover, when proving the existence of an expansive solution, Hamilton's principle of least action is not sufficient to guarantee that a minimizer of the action is a solution of the N-body problem, since there exist curves which simultaneously present isolated collisions and finite action.

The key tool to overcome this issue is Marchal's Principle, which represents a major advance in the theory, as it enabled the use of variational techniques to study the existence of solutions to the N-body problem. Marchal introduced the central idea of the proof via averaged variations in [54], while more complete proofs were later given by Chenciner in [21] and by Ferrario and Terracini in [35].

Theorem 1.2.1 (Marchal [54], Chenciner [21], Ferrario and Terracini [35]). *Given $x, y \in \mathcal{X}$, if $\gamma \in \mathcal{C}(x,y)$ is defined on some interval $[a,b]$ and satisfies*

$$\mathcal{A}_L(\gamma) = \min\{\mathcal{A}(\sigma) \mid \sigma \in \mathcal{C}(x,y,b-a)\},$$

then $\gamma(t) \in \Omega$ for all $t \in (a,b)$.

Marchal's Theorem, combined with Hamilton's principle of least action, is employed in the proofs of Theorems 2.1.4, 2.1.3, and 2.1.7 to ensure that the action minimizers are genuine solutions of the N-body problem and are free of collisions. This strategy is in line with the approach previously used in the study of collisionless periodic solutions of the N-body problem (cfr. e.g. [35, 57]).

1.2.1 The space $\mathcal{D}^{1,2}_0(1,+\infty)$

In what follows, we present the variational framework that has been employed in [66] for the search of expansive solutions and, in particular, we describe the functional space in which the minimization of the renormalized Lagrangian action will be carried out.

Recalling that we are looking for solutions $\gamma(t) = r_0(t) + \varphi(t) + x^0 - r_0(1)$, we define the functional space

$$\begin{aligned} \mathcal{D} &= \mathcal{D}_0^{1,2}([1,+\infty), \mathcal{X}) \\ &= \{\varphi \in H^1_{loc}([1,+\infty), \mathcal{X}) \ : \ \varphi(1) = 0 \text{ and } \int_1^{+\infty} \|\dot{\varphi}(t)\|^2_{\mathcal{M}} \, dt < +\infty\}, \end{aligned} \tag{1.2.2}$$

which is endowed with the norm

$$\|\varphi\|_{\mathcal{D}} = \left(\int_1^{+\infty} \|\dot{\varphi}(t)\|^2_{\mathcal{M}} \, dt \right)^{1/2}. \tag{1.2.3}$$

Remark 1.2.2. Given a configuration $\varphi = (\varphi_1, ..., \varphi_n) \in \mathcal{D}_0^{1,2}([1,+\infty), \mathcal{X})$, we define the $\mathcal{D}_0^{1,2}$-norm of each component

$$\|\varphi_i\|_{\mathcal{D}} = \left(\int_1^{+\infty} |\dot{\varphi}_i(t)|^2 \, dt \right)^{1/2},$$

for $i = 1, ..., N$. We denote with $\mathcal{D}_0^{1,2}(1,+\infty)$ both the spaces $\mathcal{D}_0^{1,2}([1,+\infty), \mathcal{X})$ and $\mathcal{D}_0^{1,2}([1,+\infty), \mathbb{R}^d)$, since it is immediate to distinguish them.

Proposition 1.2.3 (Boscaggin, Dambrosio, Feltrin and Terracini 2021 [14])**.** *The space $\mathcal{D}_0^{1,2}(1,+\infty)$ is a Hilbert space containing the set $C_c^\infty(1,+\infty)$ as a dense subspace.*

Proof. Using the properties of the standard scalar product and the definition of $\mathcal{D}_0^{1,2}(1,+\infty)$, we prove that the application $(\cdot,\cdot) : \mathcal{D}_0^{1,2}(1,+\infty) \times \mathcal{D}_0^{1,2}(1,+\infty) \to \mathbb{R}$, given by

$$(\varphi, \psi) = \int_1^{+\infty} \langle \dot{\varphi}(t), \dot{\psi}(t) \rangle \, dt, \tag{1.2.4}$$

is an inner product:

1. $(\varphi + \psi, \eta) = \int_1^{+\infty} \langle \dot{\varphi}(t) + \dot{\psi}(t), \dot{\eta}(t) \rangle dt = \int_1^{+\infty} \langle \dot{\varphi}(t), \dot{\eta}(t) \rangle + \langle \dot{\psi}(t), \dot{\eta}(t) \rangle dt$
   $= (\varphi, \eta) + (\psi, \eta)$,
2. $(\alpha\varphi, \psi) = \int_1^{+\infty} \langle \alpha\dot{\varphi}(t), \dot{\psi}(t) \rangle dt = \alpha \int_1^{+\infty} \langle \dot{\varphi}(t), \dot{\psi}(t) \rangle dt = \alpha(\varphi, \psi)$,
3. $(\varphi, \psi) = \int_1^{+\infty} \langle \dot{\varphi}(t), \dot{\psi}(t) \rangle dt = \int_1^{+\infty} \langle \dot{\psi}(t), \dot{\varphi}(t) \rangle dt = (\psi, \varphi)$,
4. $(\varphi, \varphi) = \int_1^{+\infty} |\dot{\varphi}(t)|^2 \, dt = \|\dot{\varphi}\|^2_{L^2} \geq 0$,
5. $(\varphi, \psi) = 0 \leftrightarrow \int_1^{+\infty} |\dot{\varphi}(t)|^2 \, dt = 0 \Rightarrow \dot{\varphi}(t) = 0$ almost everywhere in $[1,+\infty) \Leftrightarrow \dot{\varphi}(t) \equiv 0$,

for all $\varphi, \psi, \eta \in \mathcal{D}_0^{1,2}(1,+\infty)$ and $\alpha \in \mathbb{R}$.

Noticing that the norm (1.2.3) is induced by the inner product (1.2.4), to prove that $\mathcal{D}_0^{1,2}$ is a Hilbert space (for the rest of the proof we can write without ambiguity $\mathcal{D}_0^{1,2}$ instead of $\mathcal{D}_0^{1,2}(1,+\infty)$) we just need to show that Cauchy sequences are convergent. To this end, let $(\varphi_n)_n \subseteq \mathcal{D}_0^{1,2}$ be a Cauchy sequence. Since, by definition of the norm, $(\dot{\varphi}_n)_n$ is a Cauchy sequence in L^2, there exists $v \in L^2$ such that $\dot{\varphi}_n \to v$ in L^2. Setting $\varphi(t) = \int_1^t v(s)ds$, we have $\varphi \in \mathcal{D}_0^{1,2}$ and $\varphi_n \to \varphi$ in $\mathcal{D}_0^{1,2}$, thus proving the completeness of $\mathcal{D}_0^{1,2}$.

To prove that $\mathcal{D}_0^{1,2}$ contains the set $C_c^\infty(1,+\infty)$ as a dense subspace, we need to show that, given a function $\varphi \in \mathcal{D}_0^{1,2}$, there exists a sequence $(\varphi_n)_n \subseteq C_c^\infty(1,+\infty)$ such that $\varphi_n \to \varphi$ in $\mathcal{D}_0^{1,2}$ or, equivalently, $\dot{\varphi}_n \to \dot{\varphi}$ in L^2. We consider the sequence $(v_n)_n \subseteq C_c^\infty(1,+\infty)$ such that $v_n \to \dot{\varphi}$ in L^2 and define $J_n = \operatorname{supp}(v_n)$. Setting $T_n = \sup J_n$ and taking $\gamma \in C^\infty([0,+\infty))$ such that $\gamma(t) = 1$ for $t \in [0,1]$ and $\gamma(t) = 0$ for $t \geq 2$, we define, for $t \geq 1$,

$$\gamma_n(t) = \gamma\left(\frac{t-1}{nT_n}\right)$$

and

$$\varphi_n(t) = \gamma_n(t) \int_1^t v_n(s)\, ds.$$

Clearly, $\varphi_n \in C_c^\infty(1,+\infty)$: from $v_n \in C^\infty(1,+\infty)\ \forall\, n$ and $\gamma_n \in C^\infty([0,+\infty))\ \forall\, n$, it follows that $\varphi_n \in C^\infty(1,+\infty)\ \forall\, n$; $\gamma_n(t) = 0$ for $t \geq 2nT_n + 1$, implying that $\operatorname{supp}(\varphi_n)$ is compact and $\operatorname{supp}(\varphi_n) \subseteq (1,+\infty)$.

We now need to show that $\dot{\varphi}_n \to \dot{\varphi}$ in L^2. We observe that

$$\dot{\varphi}_n(t) = \gamma_n(t)v_n(t) + \frac{1}{nT_n}\dot{\gamma}\left(\frac{t-1}{nT_n}\right)\int_1^t v_n(s)\, ds. \tag{1.2.5}$$

The first term on the right-hand side converges to $\dot{\varphi}$ in L^2, since

$$\begin{aligned}\|\gamma_n v_n - \dot{\varphi}\|_{L^2} &\leq \|\gamma_n(v_n - \dot{\varphi})\|_{L^2} + \|(\gamma_n - 1)\dot{\varphi}\|_{L^2} \\ &\leq \|\gamma\|_{L^\infty}\|v_n - \dot{\varphi}\|_{L^2} + \left(\int_1^{+\infty} |(\gamma_n(t) - 1)\dot{\varphi}(t)|^2\, dt\right)^{1/2},\end{aligned}$$

where $\|\gamma\|_{L^\infty}$ is bounded, $\|v_n - \dot{\varphi}\|_{L^2} \to 0$ for $n \to +\infty$ and the integral goes to zero by the dominated convergence Theorem. Indeed $\gamma_n(t) \to 1$, for $n \to +\infty$, and

$$\begin{aligned}|(\gamma_n(t) - 1)\dot{\varphi}(t)|^2 &\leq (|\gamma_n(t)\dot{\varphi}(t)| + |\dot{\varphi}(t)|)^2 \\ &\leq 2(|\gamma_n(t)\dot{\varphi}(t)|^2 + |\dot{\varphi}(t)|^2) \\ &\leq 2(|\dot{\varphi}(t)|^2 + |\dot{\varphi}(t)|^2) \\ &= 4|\dot{\varphi}(t)|^2 \in L^1,\end{aligned}$$

where we have used the inequality

$$(a+b)^p \le 2^{p-1}(a^p + b^p), \qquad \forall\, a, b \ge 0,\ \forall\, p \in (1, +\infty).$$

By the Cauchy-Schwarz inequality, it holds, for every $t \ge 1$,

$$\begin{aligned}\left|\int_1^t v_n(s)\, ds\right| &\le \left(\int_1^t |v_n(s)|^2\, ds\right)^{1/2} [\mathcal{M}(\mathrm{supp}(v_n))]^{1/2} \\ &= \sqrt{T_n}\|v_n\|_{L^2},\end{aligned}$$

(here, we denote by $\mathcal{M}(A)$ the measure of a set A) implying

$$\begin{aligned}\int_1^{+\infty} \left|\frac{1}{nT_n}\dot\gamma\left(\frac{t-1}{nT_n}\right)\int_1^t v_n(s)\, ds\right|^2 dt &\le \frac{\|v_n\|_{L^2}^2}{n^2 T_n}\int_1^{+\infty}\left|\dot\gamma\left(\frac{t-1}{nT_n}\right)\right|^2 dt \\ &= \frac{\|v_n\|_{L^2}^2}{n}\int_0^{+\infty} |\dot\gamma(s)|^2\, ds,\end{aligned}$$

which tends to zero as $n \to \infty$, since $\|v_n\|_{L^2} \to \|\dot\varphi\|_{L^2}$. From this, it follows that the second term on the right-hand side of (1.2.5) goes to zero in L^2. □

The following Hardy-type inequality, which states that the space $\mathcal{D}_0^{1,2}(1, +\infty)$ is continuously embedded in a weighted L^2-space with measure dt/t^2, is fundamental in our proofs.

Proposition 1.2.4 (*Hardy inequality*, Cfr. Boscaggin, Dambrosio, Feltrin and Terracini 2021 [14])**.** *For every $\varphi \in \mathcal{D}_0^{1,2}(1, +\infty)$, it holds*

$$\int_1^{+\infty} \frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^2}\, dt \le 4\int_1^{+\infty} \|\dot\varphi(t)\|_{\mathcal{M}}^2\, dt. \tag{1.2.6}$$

Moreover,

$$\sup_{t\in[1,+\infty)} \frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t-1} \le \int_1^{+\infty} \|\dot\varphi(t)\|_{\mathcal{M}}^2\, dt. \tag{1.2.7}$$

Proof. To prove inequality (1.2.6), assume first that $\varphi \in C_c^\infty(1, +\infty)$. Integrating by parts and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\int_1^{+\infty} \frac{\|\varphi(t)\|^2}{t}\, dt &= -\int_1^{+\infty} \frac{d}{dt}\left(\frac{1}{t}\right)\|\varphi(t)\|^2\, dt \\ &= 2\int_1^{+\infty} \frac{\langle \varphi(t), \dot\varphi(t)\rangle}{t}\, dt \\ &\le \int_1^{+\infty} \frac{2\|\varphi(t)\|\|\dot\varphi(t)\|}{t}\, dt \\ &\le 2\left(\int_1^{+\infty} \frac{\|\varphi(t)\|^2}{t^2}\, dt\right)^{1/2}\left(\int_1^{+\infty} \|\dot\varphi(t)\|^2\, dt\right)^{1/2}.\end{aligned}$$

The general case follows from Proposition 1.2.3.

Inequality (1.2.7) simply follows by Cauchy-Schwarz inequality:

$$\|\varphi(t)\| = \left\| \int_1^t \dot\varphi(s)\, ds \right\| \le \left(\int_1^t \|\dot\varphi(s)\|^2\, ds\right)^{1/2}\left(\int_1^t ds\right)^{1/2},$$

which holds for all $t \ge 1$ and implies the following pointwise estimate

$$\|\varphi(t)\| \le \|\dot\varphi\|_{L^2}\sqrt{t-1} \le \|\dot\varphi\|_{L^2}\sqrt{t}.$$

□

After renormalizing the action, we will prove its coercivity and weak-lower semicontinuity, so that the direct method of the calculus of variations can be applied to prove that there are minimizers of the renormalized action on the space $\mathcal{D}_0^{1,2}(1,+\infty)$.

1.2.2 The space $\mathcal{D}_0^{1,2}(1,T)$

When studying the Hamilton-Jacobi equations of the N-body problem, we need to consider finite-time minimizers in our arguments. To this end, it is useful to introduce an equivalent setting for the case where the end time is finite. Referring to the work in [12], we therefore extend, in this section, the discussion developed in Section 1.2.1.

For a given $T \in (1,+\infty]$, we define the functional space

$$\mathcal{D}_0^{1,2}(1,T) = \{\varphi \in H^1([1,T]) \;:\; \varphi(1) = 0,\; \int_1^T \|\dot\varphi(t)\|_{\mathcal{M}}\, dt < +\infty\},$$

and the norm

$$\|\varphi\|_{\mathcal{D}_T} = \left(\int_1^T \|\dot\varphi(t)\|_{\mathcal{M}}^2\, dt\right)^{1/2}, \quad \forall \varphi \in H^1([1,T]),\; \varphi(1) = 0,$$

which satisfies

$$\|\varphi(t)\|_{\mathcal{M}} \leq \|\varphi\|_{\mathcal{D}_T}\sqrt{t}, \quad \forall\varphi \in \mathcal{D}_0^{1,2}(1,T).$$

The conclusions of Proposition 1.2.4 admit the following extension, which shows that the space $\mathcal{D}_0^{1,2}(1,T)$ is compactly embedded into the weighted space $L^2(1,T)$ endowed with a suitable weight. To this purpose, for every $T \in (1,+\infty]$ and $\varepsilon \geq 0$, we define $L^2\Big(1,T;dt/t^{2+\varepsilon}\Big)$ as the space of functions φ such that

$$\int_1^T \frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^{2+\varepsilon}}\,dt < +\infty.$$

Proposition 1.2.5 (Berti, Polimeni and Terracini 2025 [12])**.** *Let $T \in (1,+\infty]$. Then, for all $\varepsilon \geq 0$ and for all $\varphi \in \mathcal{D}_0^{1,2}(1,T)$, the following Hardy-type inequality holds*

$$\int_1^T \frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^{2+\varepsilon}}\,dt \leq \frac{4}{(1+\varepsilon)^2}\int_1^T \|\dot{\varphi}(t)\|_{\mathcal{M}}^2\,dt, \tag{1.2.8}$$

that is, the space $\mathcal{D}_0^{1,2}(1,T)$ is continuously embedded in the space $L^2(1,T;dt/t^{2+\varepsilon})$. Besides, $\mathcal{D}_0^{1,2}(1,T)$ is compactly embedded in the space $L^2(1,T;dt/t^{2+\varepsilon})$ for all $\varepsilon >$ 0.

Proof. First, we prove that the embedding is continuous. For all fixed $T \in (1,+\infty)$ and $\varepsilon \geq 0$, this follows from

$$\begin{aligned}
\int_1^T \frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^{2+\varepsilon}}\,dt &= -\frac{1}{1+\varepsilon}\frac{\|\varphi(T)\|_{\mathcal{M}}^2}{T^{1+\varepsilon}} + \frac{2}{1+\varepsilon}\int_1^T \frac{\langle\varphi(t),\dot{\varphi}(t)\rangle_{\mathcal{M}}}{t^{1+\varepsilon}}\,dt \\
&\leq \frac{2}{1+\varepsilon}\int_1^T \frac{\langle\varphi(t),\dot{\varphi}(t)\rangle_{\mathcal{M}}}{t^{1+\varepsilon}}\,dt \\
&\leq \frac{2}{1+\varepsilon}\left(\int_1^T \frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^{2+2\varepsilon}}\,dt\right)^{1/2}\left(\int_1^T \|\dot{\varphi}(t)\|_{\mathcal{M}}^2\,dt\right)^{1/2} \\
&\leq \frac{2}{1+\varepsilon}\left(\int_1^T \frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^{2+\varepsilon}}\,dt\right)^{1/2}\left(\int_1^T \|\dot{\varphi}(t)\|_{\mathcal{M}}^2\,dt\right)^{1/2},
\end{aligned}$$

which yields the Hardy-type inequality (1.2.8). In the case $T = +\infty$, the claim follows from the density of $C_c^\infty(1,+\infty)$ in $\mathcal{D}_0^{1,2}(1,+\infty)$.

Now assume that $\varepsilon > 0$. If $T < +\infty$, compactness is a direct consequence of the Rellich-Kondrachov Theorem. When $T = +\infty$, let $(\varphi_n)_n$ be a bounded sequence in $\mathcal{D}_0^{1,2}(1,+\infty)$, that is, there exists a constant $C > 0$ such that $\|\varphi_n\|_{\mathcal{D}} \leq C$ for every $n \in \mathbb{N}$. Then one can extract a subsequence $(\varphi_{n_k})_k$ that converges pointwise on $[1,+\infty)$ to a function $\bar{\varphi}$. Since, by (1.2.7),

$$\frac{\|\varphi_{n_k}(t) - \bar{\varphi}(t)\|_{\mathcal{M}}^2}{t^{2+\varepsilon}} \leq \frac{\|\varphi_{n_k} - \bar{\varphi}\|_{\mathcal{D}}^2}{t^{1+\varepsilon}} \leq \frac{C}{t^{1+\varepsilon}} \in L^1(1,+\infty),$$

we can use the Dominated Convergence Theorem to say that

$$\lim_{k\to+\infty}\int_1^{+\infty}\frac{\|\varphi_{n_k}(t)-\bar{\varphi}(t)\|^2_{\mathcal{M}}}{t^{2+\varepsilon}}\,dt=0.$$

This concludes the proof. □

Remark 1.2.6. Clearly, for $T = +\infty$ and $\varepsilon = 0$, we obtain the same results of Proposition 1.2.4.

1.3 Existence of expansive free-time minimizers

In this section, we outline the proof of the existence of free-time minimizers for the Lagrangian action functional of the N-body problem, as established in [66].

N-body motions with energy h are geodesics of the Jacobi-Maupertuis' metric of level h

$$d\sigma^2 = (U+h)ds^2_{\mathcal{M}},$$

in the configuration space, with $ds^2_{\mathcal{M}}$ being the mass Euclidean metric in $\mathcal{X}$ (see Appendix A for further details on the jacobi-Maupertuis metric).

Definition 1.3.1. A curve $\gamma : [1,+\infty) \to E^N$ is a geodesic ray from $p \in E^N$ if $\gamma(1) = p$ and each restriction to a compact interval is a minimizing geodesic.

In [52], Maderna and Venturelli proved the following theorem.

Theorem 1.3.2 (Maderna and Venturelli 2020 [52])**.** *Let E be an Euclidean space. For any $h > 0$, $p \in E^N$ and $a \in \Omega$, there is geodesic ray of the Jacobi-Maupertuis' metric of level h with asymptotic direction a and starting at p.*

In facts, geodesic rays turn out to be unbounded free-time minimizers of the Lagrangian action at energy h in the sense of Definition 1.1.10 (see [46, 4]). In this section, we show that the existence results for expansive motions obtained via minimization of the renormalized action are consistent with Theorem 1.3.2. More precisely, we prove the following.

Corollary 1.3.3 (Polimeni and Terracini [66])**.** *Consider an expansive motion $\gamma : [1,+\infty) \to \mathcal{X}$ of the Newtonian N-body problem of the form*

$$\gamma(t) = r_0(t) + \varphi(t) + x_0 - r_0(1),$$

where $\varphi \in \mathcal{D}^{1,2}_0(1,+\infty)$ minimizes the renormalized action (1.1.5) and r_0 is the reference path in a hyperbolic, parabolic or hyperbolic-parabolic setting. Then γ is a free-time minimizer at its energy level and, therefore, it is a geodesic ray for the Jacobi-Maupertuis' metric.

Proof. Let $\gamma : [1, +\infty) \to \mathcal{X}$ be a curve of the form

$$\gamma(t) = r_0(t) + \varphi(t) + \tilde{x}^0,$$

where φ is a minimizer of the renormalized Lagrangian action on $\mathcal{D}_0^{1,2}(1, +\infty)$.

Arguing by contradiction, assume that there exist T and $\bar{T}$, with $\varepsilon > 0$, and a curve $\bar{\sigma} : [1, \bar{T}] \to \mathcal{X}$ satisfying $\gamma(T) = \bar{\sigma}(\bar{T})$, such that

$$\int_1^T L(\gamma, \dot{\gamma})\,\mathrm{d}t + hT > \int_1^{\bar{T}} L(\bar{\sigma}, \dot{\bar{\sigma}})\,\mathrm{d}t + h\bar{T} + \varepsilon. \tag{1.3.1}$$

By a density and continuity argument, we may construct a compactly supported function $\tilde{\varphi}$ satisfying $\tilde{\varphi}(t) = \varphi(t)$ on $[1, \hat{T}]$, where $\hat{T} \gg \max\{T, \bar{T}\}$, and such that $\tilde{\varphi}$ is sufficiently close to φ in the $\mathcal{D}_0^{1,2}$-norm to ensure

$$\mathcal{A}(\tilde{\varphi}) \leq \mathcal{A}(\varphi) + \varepsilon.$$

Using the minimizing property of φ, we deduce that

$$\mathcal{A}(\tilde{\varphi}) \leq \mathcal{A}(\psi) + \varepsilon, \qquad \forall \psi \in \mathcal{D}_0^{1,2}([1, +\infty)). \tag{1.3.2}$$

Finally, setting $\tilde{\gamma}(t) = r_0(t) + \tilde{\varphi}(t) + \tilde{x}^0$, we construct a curve $\tilde{\sigma} : [1, +\infty) \to \mathcal{X}$ such that

$$\tilde{\sigma}(t) = \begin{cases} \bar{\sigma}(t), & t \in [1, \bar{T}], \\ \tilde{\gamma}(t - \bar{T} + T), & t \in [\bar{T}, +\infty). \end{cases}$$

Since we assume that $\gamma(T) = \bar{\sigma}(\bar{T})$, it follows in particular that the path $\tilde{\sigma}$ is continuous. We then set

$$\bar{\varphi}(t) = \tilde{\sigma}(t) - r_0(t) - \tilde{x}_0,$$

so that $\bar{\varphi} \in \mathcal{D}_0^{1,2}(1, +\infty)$. By construction, we have

$$\bar{\varphi}(t) = r_0(t - \bar{T} + T) - r_0(t) = a(T - \bar{T}) + o(1), \qquad \forall\, t \gg \max\{T, \bar{T}\}, \tag{1.3.3}$$

since $\tilde{\varphi}$ has compact support.

We also observe that

$$\mathcal{A}(\tilde{\varphi}) = \int_1^{+\infty} L(\tilde{\gamma}, \dot{\tilde{\gamma}}) - L_0(t)\; dt,$$

which follows immediately from the fact that $L(\tilde{\gamma}, \dot{\tilde{\gamma}}) - L_0(t) \in L^1([1, +\infty))$. Moreover, using again the compact support of $\tilde{\varphi}$, we obtain

$$\int_1^{+\infty} -\langle \mathcal{M}\ddot{r}_0, \tilde{\varphi}\rangle\, dt = \int_1^{+\infty} \langle \mathcal{M}\dot{r}_0, \dot{\tilde{\varphi}}\rangle\, dt.$$

On the other hand, from (1.3.3), using $\dot{r}_0 \simeq t^{-1/3}$ at infinity, it follows

$$\int_1^{+\infty} -\langle \mathcal{M}\ddot{r}_0, \bar{\varphi}\rangle dt = \langle \mathcal{M}a, a\rangle(\bar{T} - T) + \int_1^{+\infty} \langle \mathcal{M}\dot{r}_0, \dot{\bar{\varphi}}\rangle dt$$
$$= 2h(\bar{T} - T) + \int_1^{+\infty} \langle \mathcal{M}\dot{r}_0, \dot{\bar{\varphi}}\rangle dt,$$

where $h = H(r_0, \dot{r}_0) = \|a\|^2_{\mathcal{M}}/2$ is the energy of r_0, which is positive in the hyperbolic and hyperbolic-parabolic cases and zero in the parabolic case. Consequently, we have

$$\mathcal{A}(\bar{\varphi}) = 2h(\bar{T} - T) + \int_1^{+\infty} L(\bar{\sigma}, \dot{\bar{\sigma}}) - L_0(t)\ dt.$$

Let us denote $L^h = L - h$ and $L_0^h(t) = L(r_0(t)) - h$. By (1.3.1), it holds

$$\begin{aligned}
&\int_1^{T} L^h(\gamma, \dot{\gamma})\ dt + \int_{\bar{T}}^{+\infty} L^h(\tilde{\sigma}, \dot{\tilde{\sigma}}) - L_0^h(t - \bar{T} + T)\ dt \\
&> \int_1^{\bar{T}} L^h(\bar{\sigma}, \dot{\bar{\sigma}})\ dt + \int_{\bar{T}}^{+\infty} L^h(\tilde{\sigma}, \dot{\tilde{\sigma}}) - L_0^h(t - \bar{T} + T)\ dt + \varepsilon + 2h(\bar{T} - T).
\end{aligned} \tag{1.3.4}$$

Working on left-hand side of equation (1.3.4), we obtain

$$\begin{aligned}
&\int_1^{T} L^h(\gamma, \dot{\gamma})\ dt + \int_{\bar{T}}^{+\infty} L^h(\tilde{\sigma}, \dot{\tilde{\sigma}}) - L_0^h(t - \bar{T} + T)\ dt \\
&= \int_1^{T} L^h(\gamma, \dot{\gamma})\ dt + \int_{\bar{T}}^{+\infty} L^h(\tilde{\gamma}(t - \bar{T} + T), \dot{\tilde{\gamma}}(t - \bar{T} + T)) - L_0^h(t - \bar{T} + T)\ dt \\
&= \int_1^{T} L^h(\gamma, \dot{\gamma}) - L_0^h(t)\ dt + \int_{T}^{+\infty} L^h(\tilde{\gamma}, \dot{\tilde{\gamma}}) - L_0^h(t)\ dt + \int_1^{T} L_0^h(t)\ dt \\
&= \int_1^{+\infty} L^h(\tilde{\gamma}, \dot{\tilde{\gamma}}) - L_0^h(t)\ dt + \int_1^{T} L_0^h(t)\ dt.
\end{aligned}$$

On the other hand, working on right-hand side of (1.3.4), we have

$$
\begin{aligned}
&\int_1^{\bar{T}} L^h(\bar{\sigma},\dot{\bar{\sigma}})\ dt + \int_{\bar{T}}^{+\infty} L^h(\tilde{\sigma},\dot{\tilde{\sigma}}) - L_0^h(t-\bar{T}+T)\ dt + 2h(\bar{T}-T) + \varepsilon \\
&= \int_1^{\bar{T}} L^h(\bar{\sigma},\dot{\bar{\sigma}}) - L_0^h(t)\ dt + \int_{\bar{T}}^{+\infty} L^h(\tilde{\sigma},\dot{\tilde{\sigma}}) - L_0^h(t-\bar{T}+T) + L_0^h(t) - L_0^h(t)\ dt \\
&\quad + \int_1^{\bar{T}} L_0^h(t)\ dt + 2h(\bar{T}-T) + \varepsilon \\
&= \int_1^{\bar{T}} L^h(\bar{\sigma},\dot{\bar{\sigma}}) - L_0^h(t)\ dt + \int_{\bar{T}}^{+\infty} L^h(\tilde{\sigma},\dot{\tilde{\sigma}}) - L_0^h(t)\ dt \\
&\quad + \int_1^{\bar{T}} L_0^h(t)\ dt + \int_{\bar{T}}^{+\infty} L_0^h(t) - L_0^h(t-\bar{T}+T)\ dt + 2h(\bar{T}-T) + \varepsilon \\
&= \int_1^{+\infty} L^h(\tilde{\sigma},\dot{\tilde{\sigma}}) - L_0^h(t)\ dt + \int_1^{\bar{T}} L_0^h(t)\ dt + \int_{\bar{T}}^{+\infty} L_0^h(t) - L_0^h(t-\bar{T}+T)\ dt \\
&\quad + 2h(\bar{T}-T) + \varepsilon.
\end{aligned}
$$

Thus, it follows

$$
\begin{aligned}
&\int_1^{+\infty} L^h(\tilde{\gamma},\dot{\tilde{\gamma}}) - L_0^h(t)\ dt \\
&> \int_1^{+\infty} L^h(\tilde{\sigma},\dot{\tilde{\sigma}}) - L_0^h(t)\ dt + \int_T^{\bar{T}} L_0^h(t)\ dt + \int_{\bar{T}}^{+\infty} L_0^h(t) - L_0^h(t-\bar{T}+T)\ dt \\
&\quad + 2h(\bar{T}-T) + \varepsilon.
\end{aligned}
$$

We recall the following property from functional analysis, leaving the proof to the reader.

Proposition 1.3.4. *Given a function $f \in L^1_{loc}(\mathcal{X})$ such that $f(t) \to 0$ as $t \to \pm\infty$ and such that $f(t) - f(t-\tau) \in L^1$ for some $\tau \in \mathbb{R}$, then*

$$
\int_{-\infty}^{+\infty} f(t) - f(t-\tau)\ dt = 0.
$$

Since

$$
\begin{aligned}
&\int_T^{\bar{T}} L_0^h(t)\ dt + \int_{\bar{T}}^{+\infty} L_0^h(t) - L_0^h(t-\bar{T}+T)\ dt \\
&= \int_{-\infty}^{+\infty} L_0^h(t)\mathcal{X}_{\{t>T\}} - L_0^h(t-\bar{T}+T)\mathcal{X}_{\{t>\bar{T}\}}\ dt,
\end{aligned}
$$

we can apply the Proposition 1.3.4 to the function $L_0^h(t)\mathcal{X}_{\{t>T\}}$. This eventually yields

$$\int_1^{+\infty} L^h(\tilde{\gamma},\dot{\tilde{\gamma}}) - L_0^h(t)\ dt > \int_1^{+\infty} L^h(\tilde{\sigma},\dot{\tilde{\sigma}}) - L_0^h(t)\ dt + 2h(\bar{T}-T) + \varepsilon,$$

and finally

$$\mathcal{A}(\tilde{\varphi}) > \mathcal{A}(\bar{\varphi}) + \varepsilon,$$

in clear contradiction with (1.3.2).

□

1.4 Value functions and Hamilton-Jacobi equations in finite horizon problems

As we mentioned in the Introduction, one of the main goals in [12] was to prove that a value function v, defined as a linear correction of a minimum of a renormalized Lagrangian action, is a viscosity solution of the Hamilton-Jacobi equations of the N-body problem

$$\frac{1}{2}\|\nabla v\|^2_{\mathcal{M}^{-1}} - U(x) = h,$$

for fixed values of the energy $h \geq 0$.

The definition of viscosity solutions we are referring to is the following.

Definition 1.4.1. For a given $x \in \mathcal{X}$, define the sets

$$D^-v(x) = \left\{ p \in \mathbb{R}^n : \liminf_{y\to x} \frac{v(y)-v(x)-\langle p, y-x\rangle_{\mathcal{M}}}{\|y-x\|_{\mathcal{M}}} \geq 0 \right\},$$

$$D^+v(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y\to x} \frac{v(y)-v(x)-\langle p, y-x\rangle_{\mathcal{M}}}{\|y-x\|_{\mathcal{M}}} \leq 0 \right\},$$

which are referred to as the Fréchet superdifferential and subdifferential of v at x, respectively. A function $v \in C(\mathcal{X})$ is called a *viscosity supersolution* of (3.1.2) if, for every $x \in \mathcal{X}$, it holds that

$$H(x,p) \geq h, \quad \forall p \in D^-v(x).$$

Similarly, v is a *viscosity subsolution* of (3.1.2) if, for every $x \in \mathcal{X}$, it holds that

$$H(x,p) \leq h, \quad \forall p \in D^+v(x).$$

A function v is a *viscosity solution* of (3.1.2) if it is both a viscosity supersolution and a viscosity subsolution.

Although our main interest lies in stationary solutions of the Hamilton-Jacobi equation, it is useful to emphasize some relevant features of the time-dependent problem in a finite-horizon setting. Within optimal control and Hamilton-Jacobi theory, the value function is of fundamental importance, since it represents the minimal cost needed to reach a prescribed state. A detailed understanding of its regularity properties is crucial both for theoretical considerations and for numerical implementations. In particular, lack of smoothness of the value function is often associated with the presence of multiple minimizers or with singular behavior of the underlying dynamics. To handle these phenomena, we make use of the classical results due to Cannarsa and Sinestrari, which we briefly review below.

In [18], Cannarsa and Sinestrari proved several basic results concerning the regularity of viscosity solutions to Hamilton-Jacobi equations. Let $t > 0$ be fixed, and denote by $\mathrm{AC}([0,t], X)$ the set of absolutely continuous curves $\xi : [0,t] \to X$. For a given $x \in X$, define the functional

$$\mathcal{J}_t(\xi) = \int_0^t L(\xi(s), \dot{\xi}(s))\, ds + u_0(\xi(0)), \quad \xi \in \mathcal{B}_{t;x}, \tag{1.4.1}$$

where $L : X \times X \to \mathbb{R}$ is a given Lagrangian and

$$\mathcal{B}_{t;x} = \{\xi \in \mathrm{AC}([0,t], X) : \xi(t) = x\}.$$

Definition 1.4.2. The function $u : [0,T] \times X \to \mathbb{R}$ defined as

$$u(t,x) = \min_{\xi \in \mathcal{B}_{t;x}} \mathcal{J}_t(\xi)$$

is called the value function associated with the problem of minimizing $\mathcal{J}_t(\xi)$ over all arcs $\xi \in \mathcal{B}_{t;x}$.

In [18, Theorem 6.4.3], it is shown that if u_0 is continuous function on X, then $u(t,\cdot)$ is locally semiconcave with linear modulus on X for every $t > 0$. Then, given $T > 0$, it is proved in [18, Theorem 6.4.5] that u is a viscosity solution of the Hamilton-Jacobi Cauchy problem

$$\begin{cases} \partial_t u(t,x) + H(x, \nabla u(t,x)) = 0, & \text{in } [0,T] \times X, \\ u(0,x) = u_0(x), & \text{in } X. \end{cases}$$

Remark 1.4.3. In [18], the final endpoint of the trajectory is prescribed. By contrast, here, following the work in [12], we adopt a reversed-time formulation, in which the initial configuration is fixed. This corresponds to analyzing of the backward Hamilton-Jacobi equation. Although the variational structure of the problem does not change, the reversal of time affects the interpretation of minimizers and the propagation of singularities.

Remark 1.4.4. A further distinction concerns the nature of the Lagrangian itself. In our framework, we deal with the classical N-body problem, for which the Lagrangian takes the form

$$L(x,v) = \sum_{i=1}^{N} \frac{1}{2} m_i |v_i|^2 + \sum_{i<j} \frac{m_i m_j}{|r_i - r_j|},$$

featuring a quadratic kinetic term coupled with a singular potential arising from pairwise gravitational forces. In contrast, [18] treats general smooth Lagrangians. Our analysis therefore takes place in a singular context, where collisions may arise. Nevertheless, by restricting our attention to the collision-free region $\Omega \subset \mathcal{X}$, the Lagrangian is smooth, and the arguments of [18] can be suitably adapted to this setting.

The goal of Cannarsa and Sinestrari was also that of analyzing the structure of the singular set of the value function, which is the set of points where the minimizing trajectory of the associated variational problem fails to be unique. In particular, they proved that, outside of the closure of this singular set, the value function exhibits the same regularity as the prescribed data.

We now introduce some definitions that will be used throughout the section. Given a point $z \in \mathbb{R}^n$, we denote by $\xi(\cdot, z)$ the characteristic curve of the Hamilton-Jacobi problem originating from the configuration z.

Definition 1.4.5. A point $(t,x) \in [0,T] \times \mathbb{R}^n$ is said to be *regular* if there exists a unique minimizer of (1.4.1). All other points are called *irregular*.

Definition 1.4.6. A point $(t,x) \in [0,T] \times \mathbb{R}^n$ is said to be *conjugate* if there exists $z \in \mathbb{R}^n$ such that $\xi(t,z) = x$, the arc $\xi(\cdot, z)$ is a minimizer of (1.4.1), and $\det(\xi_z(t,z)) = 0$.

We denote by Σ the set of irregular points and by Γ the set of conjugate points.

Cannarsa and Sinestrari carried out a careful analysis of the topological and measure-theoretic structure of the singular set Σ. Among their results, they proved that the closure $\overline{\Sigma}$ is itself countably $\mathcal{H}^n$-rectifiable, and therefore enjoys the same rectifiability properties as Σ. Moreover, they established the sharper estimate

$$\mathcal{H}^{n-1+\frac{2}{k-1}}(\Gamma \setminus \Sigma) = 0,$$

where $k \geq 3$ denotes the differentiability class of the data. As a consequence, the value function attains the same degree of smoothness as the problem data outside a closed rectifiable set of codimension one.

We now state their main result, adapted to our setting.

Theorem 1.4.7 (Cannarsa and Sinestrari 2004 [18]). *Let $n > 1$, and suppose that $L \in C^{R+1}([0,T] \times \mathbb{R}^n \times \mathbb{R}^n)$ is a positive Lagrangian, quadratic in the velocities,* *as in* (1.1.3)*, and that* $u_0 \in C^{R+1}(\mathbb{R}^n)$ *for some* $R \geq 1$*. Then the set of conjugate points* Γ *is countably* $\mathcal{H}^n$*-rectifiable, and*

$$\mathcal{H}^{n-1+\frac{2}{R}}(\Gamma \setminus \Sigma) = 0.$$

In particular, if $L \in C^\infty([0,T] \times \mathbb{R}^n \times \mathbb{R}^n)$ *and* $u_0 \in C^\infty(\mathbb{R}^n)$*, then the Hausdorff dimension satisfies* $\dim_{\mathcal{H}}(\Gamma \setminus \Sigma) \leq n-1$.

Their proofs rely on Sard's theorem, stated as in the following version ([31, Theorem 3.4.3]).

Theorem 1.4.8. *Let* $F : \mathbb{R}^N \to \mathbb{R}^M$ *be a* C^R *map for some* $R \geq 1$*. For each* $k \in \{0,1,\ldots,N-1\}$*, define*

$$A_k = \{x \in \mathbb{R}^N \mid \mathrm{rk}(DF(x)) \leq k\},$$

where $\mathrm{rk}(DF)$ *denotes the rank of the Jacobian matrix* DF*. Then*

$$\mathcal{H}^{k+\frac{N-k}{R}}(F(A_k)) = 0.$$

Our goal is to apply the same Sard-type theorem, carefully accounting for the differences between our problem and that considered by Cannarsa and Sinestrari.

1.4.1 Busemann functions, horofunctions, and the Gromov boundary in Hamilton-Jacobi theory

It is worth briefly discussing how the study of expansive motions in the N-body problem connects with the Gromov boundary in the theory of Hamilton-Jacobi equations, as well as with the associated Busemann functions.

As noticed in [52], hyperbolic solutions naturally fit within the framework of the Gromov boundary and other compactifications of the configuration space in the N-body problem. Consequently, they have attracted increasing attention in the recent literature (see, for instance, [16, 29, 33, 32, 43, 44, 45, 47, 66]). In particular, Maderna and Venturelli constructed hyperbolic solutions as horofunctions of the Hamilton-Jacobi equations, where the idea under their construction is that each point in a metric space (X, d) can be identified with the distance function to that point.

Consider the noncollision configuration space Ω, equipped with the Jacobi-Maupertuis metric j_h associated with the N-body problem, obtained by rescaling the Euclidean metric by a position-dependent factor. For a fixed energy level $h > 0$, j_h takes the form

$$j_{ij}(x) = 2\Big(h + U(x)\Big)\,\delta_{ij},$$

which is proportional to the usual mass-weighted Euclidean metric g_m. Free-time minimizers correspond exactly to geodesics of j_h; in particular, the expanding solutions studied in this thesis are geodesic rays, i.e. geodesics defined for all $t \geq 0$. Each such ray determines a point at infinity, and the collection of all these asymptotic endpoints forms the *ideal* (or *Gromov*) boundary of (Ω, j_h).

Let $\phi_h(x, y)$ denote the action potential – equivalently, the Jacobi-Maupertuis distance – between two configurations $x, y \in \Omega$ at energy $h > 0$. We now recall the following definition.

Definition 1.4.9. A function $f \in C(\mathcal{X})$ belongs to the ideal boundary at level h if there exists a sequence of configurations p_n with $\|p_n\| \to +\infty$ such that for every $x \in \mathcal{X}$,

$$f(x) = \lim_{n\to\infty} \Big(\phi_h(x, p_n) - \phi_h(0, p_n)\Big).$$

Any such function is called a *horofunction*.

Horofunctions arise naturally in the analysis of the geometry of metric spaces, describing the asymptotic behavior of distances. The key idea is that every point of a metric space (X, d) can be represented by its distance function. More precisely, we consider the map $X \to C(X)$ that associates to each point $x \in X$ the function $d_x(y) = d(y, x)$, which embeds X into the space $C(X)$ of continuous real-valued functions on X. This embedding is isometric, meaning that for any $x_0, x_1 \in X$, $\max_{y\in X} |d_{x_0}(y) - d_{x_1}(y)| = d(x_0, x_1)$.

Obviously, any sequence of functions d_{x_n} diverges if $x_n \to \infty$ – that is, if x_n escapes from any compact subset of X. However, if X is not compact, the boundary of the image of the induced embedding of X into the quotient space $C(X)/\mathbb{R}$ is in general non-trivial. Thus, it is possible to consider this boundary as an ideal boundary of X.

The set of all horofunction limits (modulo additive constants) is referred to as the *horofunction boundary*. In many settings, this boundary coincides with the Gromov boundary defined via geodesic rays. Informally, a horofunction acts like a generalized notion of distance from a point located "at infinity", with its level sets playing the role of horospheres.

Horofunctions in Hamilton-Jacobi theory

Within the context of the Jacobi-Maupertuis metric $j_h = 2(h + U(x))\, g_m$ on Ω, horofunctions naturally appear as limits of rescaled action functionals. Fix a noncollisional configuration $a \in \Omega$ that specifies a hyperbolic direction (for instance, a may represent the normalized velocity configuration of a hyperbolic solution). Let $(x_n)_n$ be a sequence in Ω diverging to infinity along a, in the sense that $x_n = \lambda_n\, a + o(\lambda_n)$ as $\lambda_n \to +\infty$. For each x_n, introduce the function $f_{x_n} : \Omega \to \mathbb{R}$ defined by

$$f_{x_n}(y) = \phi_h(y, x_n) - \phi_h(x_0, x_n),$$

which measures the extra action (or Jacobi-Maupertuis distance) needed to reach x_n from y compared to the reference point x_0; note that $f_{x_n}(x_0) = 0$ by construction.

If the sequence $(f_{x_n})_n$ converges uniformly on compact sets, the limit

$$f_a(y) = \lim_{n\to\infty} f_{x_n}(y)$$

defines a horofunction associated with the direction a, which we refer to as the *horofunction directed by a*. Standard results in the calculus of variations and viscosity theory imply that f_a is a viscosity solution of the stationary Hamilton-Jacobi equation

$$H\Big(x, \nabla f_a(x)\Big) = h.$$

The action-minimizing trajectories calibrated by f_a coincide with the expanding (hyperbolic) motions asymptotic to a. In terms of the Jacobi metric, these curves are precisely the geodesic rays of (Ω, j_h) that converge to the ideal point determined by the direction a.

Geometric and dynamical meaning

In the dynamical setting of hyperbolic motions, horofunctions – and in particular the functions f_a introduced above – admit several complementary interpretations:

- they yield canonical global viscosity solutions of the stationary Hamilton-Jacobi equation, which are often singled out by variational or selection principles;

- they describe the asymptotic behavior of action-minimizing trajectories or geodesic rays, effectively encoding the *direction at infinity* along which the motion proceeds as $t \to +\infty$;

- they coincide with the classical *Busemann functions* when one considers a specific geodesic ray. Indeed, if $\gamma : [0, +\infty) \to \Omega$ is a geodesic ray (for instance, the path traced by a hyperbolic solution), one defines

  $$f_\gamma(y) = \lim_{t\to+\infty} \Big(\phi_h\Big(y, \gamma(t)\Big) - \phi_h\Big(x_0, \gamma(t)\Big)\Big).$$

  The function f_γ is precisely the Busemann function of the ray γ, and it coincides – up to an additive constant – with the directed horofunction f_a, where a denotes the asymptotic configuration approached by $\gamma(t)$ as $t \to +\infty$.

Busemann functions are well defined because of the geodesic characteristic property of rays, which states that for any ray γ and for all $0 \le s \le t$, we have $\phi_h(\gamma(t), \gamma(s)) = t - s$ and hence $f_{\gamma(t)} \le f_{\gamma(s)}$. Indeed, if $\phi_h(\gamma(t), \gamma(s)) = t - s$, we have three cases:

- if x is between $\gamma(0)$ and $\gamma(s)$, then

$$\begin{aligned}&\phi_h(\gamma(t),x)-\phi_h(\gamma(s),x)+\phi_h(\gamma(s),\gamma(0))-\phi_h(\gamma(t),\gamma(0))\\&=\phi_h(\gamma(t),\gamma(s))-\phi_h(\gamma(t),\gamma(s))\\&=0;\end{aligned}$$

- if x is between $\gamma(s)$ and $\gamma(t)$, then

$$\begin{aligned}&\phi_h(\gamma(t),x)-\phi_h(\gamma(s),x)+\phi_h(\gamma(s),\gamma(0))-\phi_h(\gamma(t),\gamma(0))\\&\leq\phi_h(\gamma(t),\gamma(s))-\phi_h(\gamma(t),\gamma(s))\\&=0;\end{aligned}$$

- if x is not between $\gamma(0)$ and $\gamma(t)$, then

$$\begin{aligned}&\phi_h(\gamma(t),x)-\phi_h(\gamma(s),x)+\phi_h(\gamma(s),\gamma(0))-\phi_h(\gamma(t),\gamma(0))\\&=-2\phi_h(\gamma(t),\gamma(s))\\&=-2(t-s)\\&\leq 0.\end{aligned}$$

Each hyperbolic solution gives rise to a unique horofunction, defined up to an additive constant, and different asymptotic configurations a correspond to distinct points of the ideal boundary of (Ω, j_h).

Conversely, since the Gromov boundary of a geodesic space is obtained by identifying geodesic rays that remain at uniformly bounded distance from one another, any two rays having the same asymptotic configuration a necessarily define the same element of the Gromov boundary.

A crucial remark is that the uniqueness result established in [51] shows that, in the hyperbolic case, the value function v associated with the minimal renormalized action is in fact a Busemann function. It is also worth observing that the linear correction term appearing in (3.3.1) coincides with the Busemann function corresponding to the free particle.

So far, we have only discussed the hyperbolic setting. We now address the parabolic and mixed hyperbolic-parabolic cases, in which the notions of horofunction and ideal boundary require a more refined analysis. The parabolic case is particularly delicate: the Jacobi-Maupertuis metric is degenerate at infinity, and parabolic minimizing rays converge only to central configurations. This means that the ideal boundary must contain all minimal central configurations, each of which is represented by a parabolic solution of the form (2.1.1).

If the second term in the asymptotic expansion (2.1.1) were exactly of order 1/3, then the distance between any two distinct parabolic solutions would remain bounded as they tend to infinity. At present, however, this property cannot be rigorously proved.

The mixed hyperbolic-parabolic regime is even more subtle. The associated geodesic rays admit the two-term asymptotic expansion described in Theorem 2.1.7, and in a neighborhood of such expansive solutions the Jacobi-Maupertuis metric becomes locally equivalent to the Euclidean metric. Different choices of the minimal a-clustered central configuration b_m lead to geodesic rays that separate from one another; moreover, even rays corresponding to the same b_m may exhibit unbounded mutual distance. For these reasons, any meaningful definition of an ideal boundary in this mixed setting must be refined in order to capture and distinguish these various asymptotic behaviors.

1.5 Symmetry-constrained variational methods for the existence of periodic solutions

Since Poincaré famously conjectured in the 1890s [63] that periodic orbits are dense in the 3-body problem, which suggests that they play a central role in capturing the system's complexity, the search for periodic solutions in the N-body problem has attracted sustained interest within the mathematical community. To this end, both analytical and numerical approaches have been employed, together with increasingly sophisticated computational algorithms designed to detect and classify these orbits.

Starting in the 1990s, variational methods began to yield a wide range of collision-free periodic solutions (see, for example, [8, 20, 23, 39, 53, 75, 76]). As discussed earlier in this thesis, these techniques rely on minimizing the Lagrangian action functional under suitable constraints, producing periodic trajectories through the least-action principle.

A powerful strategy in the search for periodic orbits is the use of symmetry constraints, which can be imposed to simplify the identification of critical points of the Lagrangian action. The first successful application of this idea in the planar N-body problem is due to Bessi and Coti Zelati [13], who proved the existence of non-collision periodic solutions via variational methods. Subsequently, in the celebrated work [22], Chenciner and Montgomery rigorously established the existence of the figure-eight orbit for three equal masses – a solution first observed numerically by Moore [58].

In 2004, in their seminal paper [35], Ferrario and Terracini extended the use of variational methods to symmetric N-body problems, thereby enabling the treatment of a much broader class of symmetric configurations in which periodic solutions can be detected through simple algebraic conditions on the symmetry groups. Their method consists in encoding the symmetry constraints of the configurations of bodies through the action of a finite group G. In this way, the search for periodic solutions is reduced to identifying critical points of the action functional in the subspace of G-equivariant loops.

In their paper, alongside refined theoretical results for handling collisions and establishing a rigorous framework for G-equivariant loops, the authors also developed an algorithm for numerically generating symmetric solutions. The software `Symorb`, presented in [28, 34] on the basis of their theoretical results, implements these ideas. Written using a combination of Python, Fortran, and GAP, it allows the user to select a finite group, impose the corresponding symmetries, and search for equivariant critical points of the action functional.

Recently, a new version of `Symorb` has been developed [5, 27]. Among several refinements in its design and structure, the redesigned software `Symorb.jl` has been employed in [11] to compute periodic solutions of the N-body problem and to investigate their stability under small perturbations. In particular, the paper presents numerical algorithms for computing the Floquet multipliers and the Morse indices of such orbits, which serve as fundamental indicators of their linear and variational stability.

In this section, we review the theoretical foundations underlying `Symorb.jl`. In particular, we summarize the variational framework introduced by Ferrario and Terracini in [35]. In Chapter 4, we will provide further details on the numerical implementation of the algorithms used to compute the stability indicators and present several results illustrating their application to specific symmetric orbits.

1.5.1 Mathematical setting and problem statement

Consider $N \geq 2$ particles and their positions, denoted as $r_i \in \mathbb{R}^d$, where $d = 2{,}3$. As we did in the previous sections, for every particle we consider the equation of motion

$$m_i \ddot{r}_i(t) = \frac{\partial U}{\partial r_i}(r_1(t), \dots, r_n(t)) \tag{1.5.1}$$

and, since the center of mass of the system is invariant under translations, we fix it at the origin and work in the configuration space

$$\mathcal{X} = \left\{ x = (r_1, \dots, r_n) \in \mathbb{R}^{dN} \ : \ \sum_{i=1}^{n} m_i r_i = 0 \right\}.$$

Classical periodic solutions of (1.5.1) are trajectories $\gamma(t) = (r_1(t), \dots, r_n(t)) \in \Omega$ such that $\gamma(0) = \gamma(T)$ for a suitable $T > 0$. The smallest T that satisfies this condition is called the *period* of the orbit.

Given $T > 0$, we define the torus of length T as $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$, and consider the space of *T-periodic loops* in H^1 (possibly with collisions between the bodies)

$$\Lambda = H^1(\mathbb{T};\mathcal{X}) = \left\{\gamma, \dot\gamma \in L^2\left([0,T],\mathcal{X}\right) \ : \ \gamma(0)=\gamma(T)\right\}.$$

We then denote by

$$\hat\Lambda = H^1(\mathbb{T};\Omega) \subset \Lambda$$

the open subset of collisionless loops and consider the Lagrangian action functional $\mathcal{A}_L : \Lambda \to \mathbb{R}$,

$$\begin{aligned}\mathcal{A}_L(\gamma) &= \int_0^T L(\gamma(t),\dot\gamma(t))\,dt \\ &= \int_0^T \frac{1}{2}\sum_{i=1}^n m_i\|\dot r_i(t)\|^2 + \sum_{i<j}\frac{m_i m_j}{\|r_i(t)-r_j(t)\|}\,dt.\end{aligned} \tag{1.5.2}$$

Following the results in [35], the variational principle we rely on to obtain T-periodic solutions of the system states that

$$\gamma \in \hat\Lambda \text{ is a critical point of } \mathcal{A}_L \quad \Rightarrow \quad \gamma \text{ is a } T\text{-periodic solution of (1.5.1).}$$

1.5.2 Group actions

In the search for symmetric orbits, we are interested in certain finite groups that capture the geometric symmetries of the problem. In this section, referring mainly to [35], we introduce the concept of a *group action*, which formalizes the notion of symmetry of an object and allows us to encode all the relevant symmetry information within a group. In particular, the action on time and space of some particular classes of groups will model some reductions by symmetry of the N-body problem. A detailed discussion of G-equivariance and group actions can also be found in [5].

The groups relevant to our analysis are introduced in the following definitions.

Definition 1.5.1 (Dihedral groups)**.** A dihedral group is defined as the group of symmetries of a regular polygon, that is, the group of reflections and rotations which fix the polygon. Considering an n-gon (a polygon with n edges), we define D_{2n} as its dihedral group. The elements of D_{2n} are rotational symmetries, reflectional symmetries and their compositions. Note that rotations which fix a polygon with n edges are those of $2\pi/n$ and their multiples. The order of the dihedral group D_{2n} is exactly $2n$, since it contains n rotational symmetries and n reflectional symmetries.

The dihedral group D_{2n} can be generated by a unique rotation r and a unique reflection s, so that it is usually defined through the following representation:

$$\langle s, r \ : \ s^2 = r^n = (sr)^2 = 1\rangle.$$

Definition 1.5.2 (Symmetric groups)**.** If X is a set, we say that a permutation of X is any bijection from X to itself. The family of all permutations of X is a group under composition and we denote it by Σ_X.

If $n \in \mathbb{N}$ and $X = \{1,2,...,n\}$, we say the permutation group Σ_n is the symmetric group of degree n and its order is exactly $n!$.

Definition 1.5.3 (Cyclic groups)**.** Given $n_1, ..., n_k \in \{1, ..., n\}$, the permutation that maps n_1 to $n2$, n_2 to n_3, ..., n_{k-1} to n_k and finally n_k to n_1, and leaves the other elements unchanged, is called a k-cycle or a cyclic permutation of order k. The usual notation for this permutation is simply $(n_1, ..., n_k)$.

The arguments that follow rely on the next proposition from group theory. We denote by $O(d)$ the d-dimensional orthogonal group, and by $SO(d)$ the special orthogonal group.

Proposition 1.5.4. *Any finite subgroup H of O(2) is either cyclic or dihedral. In particular:*

- *if* $H \subseteq SO(2)$ *then* H *is cyclic and only contains rotations;*
- *if* $H = \{1, S\}$*, with* $S \in O(2) \setminus SO(2)$ *a reflection, then* H *is cyclic of order 2;*
- *if* H *contains both elements of* $SO(2)$ *and* $O(2) \setminus SO(2)$ *then it is dihedral.*

The first goal in the study of symmetric orbits is to impose a prescribed symmetry constraint on the space of H^1-periodic loops Λ, thereby effectively compactifying the configuration space $\mathcal{X}$. The action of a finite group G on Λ provides a natural and powerful way to encode such symmetries. We now give a precise definition of what is meant by a group action.

Definition 1.5.5. We say that a finite group G acts (on the left) on a set X if there exists a map $\phi : G \times X \to X$ such that

- $\phi(1, x) = x \quad \forall x \in X,$
- $\phi(g, \phi(h, x)) = \phi(gh, x) \quad \forall g, h \in G,\ x \in X.$

Since in this case there is no ambiguity, we will write gx instead of $\phi(g, x)$ to simplify the notations.

1.5.3 *G*-equivariance

We show here how requiring a symmetry constraint on a space of loops in the configuration space can be done by introducing a group action on such a space.

From a geometric perspective, groups can be viewed as sets of symmetries of an object or space that are closed under composition and taking inverses. For example, one may ask for the group of symmetries of a given mathematical object X. As Definition 1.5.1 indicates, if X is a square, its symmetry group is precisely the dihedral group D_8. Conversely, if a finite group G is fixed, one can study the objects on which G acts. This is one of the central problems of representation theory, which seeks to classify objects X up to isomorphism.

Since the configuration space $\mathcal{X}$ is a subset of $\mathbb{R}^{dN}$, with every component $r_i(t) \in \mathbb{R}^d$, it is reasonable to let the group G act in the same way on each component. We identify three finite dimensional objects underlying in Λ:

- the *space* $\mathbb{R}^d$ in which every component $r_i(t)$ of $\gamma(t)$ lies;
- the *time circle* $\mathbb{T} \subset \mathbb{R}^2$ which represents the period of a trajectory;
- the *index set* $\{1, ..., N\}$ on which the N-bodies are labeled.

We can define the action of G on Λ through three different representations of G as a group action, respectively on $\mathbb{R}^d$, $\mathbb{R}^2$ and $\{1, ..., n\}$. More precisely, G is represented as a subgroup of $O(d)$, of $O(2)$ and of the symmetric group Σ_n respectively via the homomorphisms

$$\rho : G \to O(d), \quad \tau : G \to O(2), \quad \sigma : G \to \Sigma_n.$$

Given $g \in G$, $\rho(g)$ describes how g acts on the d-dimensional space where every r_i lies, $\sigma(g)$ describes the possible interchanging of bodies along the loop, and $\tau(g)$ represents possible symmetries or recurrences of the orbit over a period T.

The action of G on the loop space Λ can be represented through the homomorphisms ρ, σ and τ as follows:

$$(gx)(t) = (\rho(g)x_{\sigma(g^{-1})1}(\tau(g^{-1})t), ..., \rho(g)x_{\sigma(g^{-1})N}(\tau(g^{-1})t))$$

which in short can be written as

$$(gx)(t) = (gx_{g^{-1}i}(g^{-1}t))_i$$

for any $g \in G$, $t \in \mathbb{T}$ and $x \in \Lambda$. We also define the set of loops in Λ fixed by G, or G-equivariant loops, as the set

$$\Lambda^G = \{x \in \Lambda : (gx)(t) = x(t), \forall t \in \mathbb{T}, g \in G\}$$

and its collisionless subset

$$\hat{\Lambda}^G = \{x \in \hat{\Lambda} : (gx)(t) = x(t), \forall t \in \mathbb{T}, g \in G\}.$$

In the following propositions, we show that this setting satisfies the variational framework, with the next result ensuring that Λ^G forms a loop space suitable for applying variational arguments to find periodic solutions as critical points of the action functional.

Proposition 1.5.6 (Ferrario and Terracini 2004 [35])**.** *Let G be a finite group, with orthogonal representations $\rho : G \to O(d), \quad \tau : G \to O(2), \quad \sigma : G \to \Sigma_n$ satisfying*

$$\forall g \in G : \{\sigma(g^{-1})i = j \Rightarrow m_i = m_j\}. \tag{1.5.3}$$

Then the following assertions hold true:

(i) *Λ^G is a closed linear subspace of Λ;*

(ii) *the Lagrangian action functional $\mathcal{A}_L$ is G-equivariant, i.e.,*

$$\mathcal{A}_L(gx) = \mathcal{A}_L(x), \forall g \in G, x \in \Lambda;$$

(iii) *the collision set Δ is G-equivariant, i.e.,*

$$x \in \Delta \Rightarrow gx \in \Delta, \forall g \in G.$$

Proof. (i) trivially follows from the definition of G on Λ and from the uniform convergence of a converging sequence in Λ^G.

(ii) follows from direct computations, taking into consideration that how the representations ρ, σ and τ act on the elements of G:

- $\rho(g)$ is always an isometry of $\mathbb{R}^d$;
- σ is a bijection and, in particular, verifies (1.5.3);
- $\tau(g)$ is either a reflection or a rotation.

Finally, assume that $x_i = x_j$ for some $i \neq j$ and take $g \in G$. Let $k, l \in \{1, ..., n\}$ such that

$$i = \sigma(g^{-1})k, \quad j = \sigma(g^{-1})l.$$

Then, clearly

$$\rho(g)x_{\sigma(g^{-1})k} = \rho(g)x_{\sigma(g^{-1})l},$$

so that $gx \in \Delta$ and (iii) is proved. □

As a consequence, we deduce this version of the Palais principle of symmetric criticality, where we refer to [61] for the proof.

Lemma 1.5.7 (Ferrario and Terracini 2004 [35])**.** *Let $\mathcal{A}_L\Big|_{\Lambda^G}$ be the restriction of the Lagrangian action functional $\mathcal{A}_L$ to the G equivariant loop space Λ^G. Then, a collisionless critical point $\bar{x} \in \hat{\Lambda}^G$ of $\mathcal{A}_L\Big|_{\Lambda^G}$ is also a (collisionless) critical point of $\mathcal{A}_L$ over the whole Λ.*

Denoting with $\mathcal{X}^G$ the set of points fixed by G, that is,

$$\mathcal{X}^G = \{x \in \mathcal{X} : gx = x, \ \forall g \in G\},$$

we also have the following proposition.

Proposition 1.5.8 (Ferrario and Terracini 2004 [35])**.** *Assume that $\mathcal{A}_L\Big|_{\Lambda^G}$ is not identically $+\infty$. Then $\mathcal{A}_L\Big|_{\Lambda^G}$ is coercive if and only if $\mathcal{X}^G = \{0\}$. As a consequence, if $\mathcal{X}^G = \{0\}$, there exists at least a minimizer of $\mathcal{A}_L$ in Λ^G.*

Proof. Assume $\mathcal{X}^G = \{0\}$. We prove that for any $x \in \Lambda^G$ it holds $[x] \in \mathcal{X}^G$, where

$$[x] = \frac{1}{T}\int_0^T x(t)\ dt.$$

Since $x \in \Lambda^G$, one has

$$[x] = \frac{1}{T}\int_0^T gx(g^{-1}t) = \frac{1}{T}\int_0^T gx(s)\ ds,$$

where $s = g^{-1}t = \tau(g^{-1})t$ and $\tau(g^{-1})$ could be either a rotation or a reflection, so that the T-periodicity of x gives the equalities above. Finally

$$\frac{1}{T}\int_0^T gx(s)\ ds = g\frac{1}{T}\int_0^T x(s)\ ds = g[x],$$

since, in this case, g is a matrix in $O(d)$ that acts on each component of x. This proves that $[x] \in \mathcal{X}^G$, and hence $[x] = 0$. Since the kinetic part of the action is an equivalent norm, the implication ($\Leftarrow$) follows.

To prove ($\Rightarrow$), let $x \in \Lambda^G$ be such that $\mathcal{A}_L(x) < +\infty$ and, by contradiction, assume that there exists $u \in \mathcal{X}^G \setminus \{0\}$. The sequence $x_k(t) = x(t) + ku(t)$, $k \in \mathbb{N}$, is such that $\|x_k\|_{H^1} \to +\infty$, but $\mathcal{A}_L(x_k) < +\infty$. This concludes the proof. □

The propositions above show that if $\bar{x} \in \hat{\Lambda}^G$ is a critical point of the restricted functional $\mathcal{A}_L|_{\Lambda^G}$, then it is a critical point of $\mathcal{A}_L$ over the whole Λ, and therefore a G-equivariant T-periodic solution of (1.5.1). With appropriate conditions on G, one can thus ensure *a priori* the existence of such collisionless critical points.

1.5.4 Fundamental domain and optimization

By the arguments developed above, the goal of the analysis thus becomes the identification of critical points of $\mathcal{A}_L|_{\Lambda^G}$. To this end, one can perform an additional reduction by restricting attention to a suitable subinterval $\mathbb{I} \subset \mathbb{T}$, referred to as the *fundamental domain*, and is enough to describe the action of G on the whole $\mathbb{T}$.

To properly define the fundamental domain, we need some preliminary definitions.

Definition 1.5.9 (Brake subgroups)**.** If H is a subgroup of $O(2)$ and $H = \{1, S\}$, with S a reflection matrix, then we say that H is a brake subgroup of $O(2)$. Furthermore, if $x(t) \in \Lambda$ and there exists $g \in G$ which acts as a reflection on T and such that $x(\tau(g)t) = x(t)$, for every $t \in T$, we say that $x(t)$ is a brake orbit.

Denote $\bar{G} = G/\ker\tau$. At this point, we can give a complete classification of a group action on Λ.

Definition 1.5.10. We classify the action of a finite group G on Λ as follows:

- if the quotient $\bar{G}$ acts trivially on the orientation of $\mathbb{T}$, then $\bar{G}$ is cyclic and the action of G on Λ is said to be of *cyclic type*. In this case, $\bar{G}$ is a subgroup of $SO(2)$ and only contains rotations;
- if $\bar{G}$ corresponds to a single reflection on $\mathbb{T}$, then the action of G on Λ is said to be of *brake type*. In particular, $\bar{G}$ is a cyclic group of order 2;
- if none of the above is satisfied, the action of G on Λ is said to be of *dihedral type* and $\bar{G}$ is a dihedral subgroup of $O(2)$.

We also recall the definition of the $\mathbb{T}$-isotropy subgroups, which will be needed when defining the fundamental domain.

Definition 1.5.11. The isotropy subgroups of the action of G on $\mathbb{T}$ through τ are called the $\mathbb{T}$-isotropy subgroups of G.

We finally have the following definition.

Definition 1.5.12. The fundamental domain $\mathbb{I} \subset \mathbb{T}$ of the action of G on $\mathbb{T}$ is defined as follows:

- if the action of G is of cyclic type, then $\mathbb{I}$ is the closed interval which connects the instant $t = 0$ with its image $\tau(g^{-1})0$, where g is the representative of a cyclic generator of $\bar{G}$;
- if the action of G is brake or dihedral, then $\mathbb{I}$ is a closed interval whose extrema are two distinct elements of $\mathbb{T}$ generating two distinct maximal $\mathbb{T}$-isotropy subgroups, such that no other instants related to maximal $\mathbb{T}$-isotropy subgroups are included in $\mathbb{I}$. Note that, if the action is of brake type, the unique maximal $\mathbb{T}$-isotropy subgroup is G itself.

Summed up, the main properties of the fundamental domain $\mathbb{I}$ are:

(A)

$$\mathbb{T} = \bigcup_{[g]\in\bar{G}} \tau(g^{-1})\mathbb{I}, \quad |\mathbb{I}| = \frac{|\mathbb{T}|}{|\bar{G}|},$$

(B) if $\bar{y} : \mathbb{I} \to \mathcal{X}$ is a critical point of the restricted action

$$\mathcal{A}_{\mathbb{I}}(y) = \int_{\mathbb{I}} L(y(t), \dot{y}(t))\ dt$$

over a suitable set of fixed-ends trajectories, then the symmetrized path given by the concatenation of $g\bar{y}$, $g \in \bar{G}$, is a solution of the G-equivariant N-body problem (see [5, Theorem 3.2]).

In particular, property (A) ensures that the action of $\tau(G)$ on the segment $\mathbb{I}$ is enough to reconstruct the entire T-periodic orbit. Property (B) makes it possible to restrict the search for critical points of the action functional to segments of T-periodic orbits that satisfy appropriate endpoint conditions. More precisely, the admissible paths y are required to lie in a class Y of fixed-endpoint-type paths, whose precise definition depends on the structure of $\tau(G)$ (see [5, Proposition 3.1]). Without loss of generality, we will take $\mathbb{I} = [0, \pi]$ and $T = l\pi$, $l \in \mathbb{N}$.

The optimization of the restricted action functional $\mathcal{A}_{\mathbb{I}}$ is the central numerical task addressed by the algorithm underlying `Symorb.jl`. Once a symmetry group is fixed – thus determining the corresponding fundamental domain $\mathbb{I}$ and the admissible set Y – the routine employs a combination of optimization and refinement methods to compute an approximation of a critical point $\bar{y}$ of $\mathcal{A}_{\mathbb{I}}$ over Y as a truncated Fourier series added to a linear interpolation between the boundary configurations:

$$y(t) = y_0 + \tfrac{t}{\pi}(y_1 - y_0) + \sum_{k=1}^{F} a_k \sin(kt), \quad t \in [0, \pi]. \tag{1.5.4}$$

Finally, by symmetrizing $\bar{y}$, an approximated solution $\bar{x}$ of (1.5.1) over the period T is obtained and can be expressed as a truncated Fourier series

$$\bar{x}(t) = A_0 + \sum_{k=1}^{\tilde{F}} A_k \cos\left(k\tfrac{2\pi}{T}t\right) + \sum_{k=1}^{\tilde{F}} B_k \sin\left(k\tfrac{2\pi}{T}t\right), \quad t \in [0, T]. \tag{1.5.5}$$

The coefficients A_0, A_k, B_k are computed analytically from the a_k as shown in B.

The output of the optimization is provided in two complementary forms: (i) the Fourier coefficients of $\bar{y}$ on $\mathbb{I}$, and (ii) a pointwise representation of $\bar{x}$ on $\mathbb{T}$.

After finding an approximated solution, it is possible to proceed with its stability analysis, which is the object of Chapter 4.

Chapter 2

Existence of minimal expansive solutions to the N-body problem

During the last century, renewed interest has risen in studying solutions of the N-body problem with prescribed asymptotic behavior, driven by the introduction of new methods of perturbative, variational, geometric and/or analytic functional nature (see [2, 19, 55, 71, 72]).

In particular, a lot of focus has been given to the variational properties of expansive motions, namely those in which the mutual distances between the bodies tend to infinity. Following the classification given by Chazy in [19], which depends on the order of growth of the mutual distances between the bodies, they can be classified into hyperbolic, parabolic and hyperbolic-parabolic motions. We quote here some recent results about the existence of hyperbolic solutions [29, 43, 47], parabolic [9, 10, 14, 48, 50, 73] and hyperbolic-parabolic ones [15].

Recent results by Maderna and Venturelli [50, 52] establish the existence of expansive solutions of parabolic and hyperbolic type, respectively, through variational approaches. Both results can be viewed as new applications of Marchal's Principle (Theorem 1.2.1), which has played a central role in the use of variational methods since its first proof in [54], as it allows to exclude the presence of collisions for minimizers of the Lagrangian action functional.

In [66], the Renormalized Action Principle (presented here in Chapter 1) has been applied to prove, in a unified manner, the existence of half-entire expansive solutions to the N-body problem in $\mathbb{R}^d$ of hyperbolic, parabolic and mixed hyperbolic-parabolic type, with prescribed initial position of the bodies and asymptotic growth. This chapter is devoted to presenting the proofs of these results.

2.1 Introduction and main results

The asymptotic behavior of an expansive motion is characterized by its limit shape configuration (see Definition 1.1.9). Depending on the type of expansive motion, the limit shape corresponds to a different kind of configuration. Consider a (half) expansive motion $\gamma : [1, +\infty) \to \mathcal{X}$:

- if γ is a (half) hyperbolic motion, then its limit shape coincides with its asymptotic velocity

$$a = \lim_{t \to +\infty} \frac{\gamma(t)}{t},$$

  which, by definition, is a configuration without collisions, that is $a \in \Omega$;

- if γ is a (half) parabolic motion, then its limit shape (if it exists) must be a central configuration, that is, a critical point of the potential U constrained to the inertial ellipsoid $\mathcal{E} = \{x \in \mathcal{X} : \|x\|_{\mathcal{M}} = 1\}$;

- if γ is a (half) hyperbolic-parabolic motion, then its limit shape coincides again with its asymptotic velocity $a = \lim_{t \to +\infty} \frac{\gamma(t)}{t}$, but, differently from the hyperbolic case, it is a configuration with collisions.

The main goal in [66] is to prove that for any given initial configuration, which can be either with or without collisions, and for any prescribed limit shape, there exists an expansive solution of the Newtonian N-body problem. Using the Renormalized Action Principle, introduced in Chapter 1, we establish the existence of such solutions for all three subclasses, taking into account that the limit shape must be chosen as described above.

The introduction of Marchal's Principle has been fundamental in the application of variational methods to prove the existence of solutions to the Newtonian N-body problem. In particular, in the last decades it has been used by Maderna and Venturelli to prove the existence of hyperbolic and parabolic solutions for the N-body problem.

In 2009, Maderna and Venturelli proved the existence of parabolic motions for any prescribed initial configuration and limit shape [50]. More specifically, they proved that for every initial configuration x^0 and for every minimizing normalized central configuration b, there exists a collision-free parabolic solution starting from x^0 and asymptotic to b. In addition, this solution is a minimizer of the Lagrangian action in every time interval. Their main theorem is the following.

Theorem 2.1.1 (Maderna and Venturelli 2009 [50]). *Given any initial configuration x^0 and any minimizing normalized central configuration b, there exists a parabolic solution $\gamma : [0, +\infty) \to \mathbb{R}^{dN}$ starting from x^0 at $t = 0$ and asymptotic to b for $t \to +\infty$. This solution is a minimizer of the Lagrangian action with fixed ends in every compact interval contained in $[0, +\infty)$ and it is collision-free for $t > 0$.*

The argument relies on the variational nature of the problem and consists in extracting a convergent subsequence from a family of minimizing trajectories, while the exclusion of collisions is ensured by Marchal's Principle. The parabolic solution γ is obtained as the limit of a sequence of minimizers $\gamma_n : [0, t_n] \to \mathbb{R}^{dN}$, each joining x^0 to a configuration homothetic to b in time t_n, with $t_n \to +\infty$. The first step to prove the Theorem 2.1.1 consists indeed in constructing the sequence γ_n and proving that it is uniformly convergent on every compact subset of $\mathbb{R}$.

We recall that a motion γ is said to be homothetic to a configuration b if it is of the form $\gamma(t) = \lambda(t)b$, for some function $\lambda : [a, b] \to (0, +\infty)$, meaning that all bodies move radially, expanding or contracting with respect to the center of mass. By classical results, if b is a normalized central configuration, the path

$$\gamma_0 : [0, +\infty) \to \mathcal{X}, \quad \gamma_0 = \alpha b t^{2/3}$$

is a solution of the N-body problem, for a proper constant $\alpha > 0$. γ_0 is called *homothetic-parabolic solution asymptotic to* b. The proof of Theorem 2.1.1 is based on the following.

Theorem 2.1.2 (Maderna and Venturelli 2009 [50])**.** *There exists a minimizing solution $\gamma : [0, +\infty) \to \mathcal{X}$ starting from x^0, a sequence of positive numbers $t_n \to +\infty$ and a sequence of minimizers $\gamma_n \in \Sigma(x^0, \gamma_0(t_n); t_n)$ such that γ_n converges uniformly to γ on every compact interval contained in $[0, +\infty)$. Moreover $\gamma(t)$ is collision-free for $t > 0$.*

Their next step consists of proving that the limit solution $\gamma(t)$ is parabolic and asymptotic to b.

In [66], the result concerning parabolic solutions introduces two main modifications. First, the method of resolution is based on the application of the Renormalized Action Principle, together with Marchal's Principle. Through this global variational approach, we again obtain the existence of solutions whose limit shape is given by a minimal normalized central configuration. In addition, by exploiting the properties of the functional space $\mathcal{D}_0^{1,2}(1, +\infty)$, we provide a more precise description of the remainder term in the asymptotic expansion of such solutions, reducing the order of growth to $o(t^{\frac{1}{3}+\varepsilon})$, for any $\varepsilon > 0$, instead of $O(t^{2/3})$.

Our result is stated in the following theorem.

Theorem 2.1.3 (Maderna and Venturelli 2009 [50], Polimeni and Terracini 2024 [66])**.** *Given $d \in \mathbb{N}$, $d \geq 2$, for the Newtonian N-body problem in $\mathbb{R}^d$ there is a parabolic solution $\gamma : [1, +\infty) \to \mathcal{X}$ of the form*

$$\gamma(t) = \beta b_m t^{2/3} + o(t^{1/3^+}) \quad \text{as } t \to +\infty, \tag{2.1.1}$$

for any initial configuration $x^0 = \gamma(1) \in \mathcal{X}$, for any minimal normalized central configuration b_m and for $\beta = \sqrt[3]{\frac{9}{2}U(b_m)}$.

In 2020, Maderna and Venturelli applied Marchal's Principle to prove the existence of hyperbolic motions for any prescribed limit shape, which in this case corresponds to a configuration without collisions [52]. This time, they adopted a different approach, based on the construction of global viscosity solutions of the Hamilton-Jacobi equation $H(x, \nabla u) = h$ and on the fact that these solutions are fixed points of the Lax-Oleinik semigroup.

In [66], we proved the same result from the viewpoint of the Renormalized Action Principle. This result is stated as follows.

Theorem 2.1.4 (Maderna and Venturelli 2020 [52])**.** *Given $d \in \mathbb{N}$, $d \geq 2$, for the Newtonian N-body problem in $\mathbb{R}^d$ there is a hyperbolic motion $\gamma : [1, +\infty) \to \mathcal{X}$ of the form*

$$\gamma(t) = at - \log(t)\nabla U(a) + O(1) \quad \text{as } t \to +\infty,$$

for any initial configuration $x^0 = \gamma(1) \in \mathcal{X}$ and for any collisionless configuration $a \in \Omega$.

While the existence of hyperbolic and parabolic solutions with prescribed limit shape was already known thanks to the work of Maderna and Venturelli, the existence of hyperbolic-parabolic solutions with a given asymptotic expansion was first established in [66]. The closest previous result was obtained by Burgos [15], who proved the existence of a partially hyperbolic motion with prescribed positive energy and prescribed initial configuration without collisions.

Theorem 2.1.5 (Burgos 2022 [15])**.** *Within any positive energy level h and starting at any given collisionless configuration x^0, provided that the underlying space has dimension at least two, there is a hyperbolic-parabolic motion.*

Burgos proved the theorem by showing the existence of a hyperbolic-parabolic motion γ that is an free time minimizer for the Lagrangian action $\mathcal{A}_{L+h}(\gamma)$ with energy level h. The proof follows from Theorem 2.1.4 and is based on a limiting procedure, taking a sequence of limit shapes of hyperbolic motions that approach the collision set.

To apply the Renormalized Action Principle to the class of hyperbolic-parabolic motions, we have to take into account that the limit shape is given by a configuration $a \in \Delta$. This calls for the introduction of an a-cluster partition of the bodies, where the clusters are the equivalent classes of the equivalence relation

$$i \sim j \iff a_i = a_j. \tag{2.1.2}$$

In order to state the main theorem, we need the following definition.

Definition 2.1.6. Let $a \in \Delta$ and consider the associated a-cluster partition determined by (2.1.2). Given a cluster K, the associated partial potential is defined

$$U_k(x) = \sum_{i<j,\ i,j \in K} \frac{m_i m_j}{|r_i - r_j|}.$$

We define the a-clustered potential U_a as the sum of all the cluster potentials of the partition.

Hyperbolic-parabolic motions can equivalently be defined as those expansive motions without collisions in the future having the form $\gamma(t) = at + o(t)$, as $t \to +\infty$, with collisional limit shape $a \in \Delta \setminus \{0\}$. Hyperbolic-parabolic motions can then be described as clusters of bodies with linear asymptotic growth, while the rate of growth of the mutual distances of the bodies inside each cluster has order $t^{2/3}$. Besides, the motion of each cluster's center of mass has a limit shape which is a minimal central configuration of the cluster's partial potential.

Theorem 2.1.7 (Polimeni and Terracini 2024 [66]). *Given $d \in \mathbb{N}$, $d \geq 2$, for the Newtonian N-body problem in $\mathbb{R}^d$ there is a hyperbolic-parabolic motion $\gamma : [1, +\infty) \to \mathcal{X}$ of the form*

$$\gamma(t) = at + \beta b_m t^{2/3} + o(t^{1/3^+}) \quad \text{as } t \to +\infty,$$

for any initial configuration $x^0 = \gamma(1) \in \mathcal{X}$, for any collision configuration $a \in \Delta$, for any normalized minimal central configuration $b_m \in \mathcal{X}$ of the a-clustered potential and for any choice of the energy constant $h > 0$ (see Section 2.4 for the exact definition of β and b_m).

We point out that the remainder term in the asymptotic expansion of such motions is the same as in the parabolic case.

To prove Theorem 2.1.7, we partition the set of bodies according to the natural cluster partition defined by (2.1.2), which was also introduced by Burgos and Maderna in [16]. It can be shown that, given a motion $\gamma(t) = (r_1(t), ..., r_N(t))$ and a limit shape $a = (a_1, ..., a_N) \in \Delta \setminus \{0\}$, we have $a_i = a_j$ if and only if $|r_i(t) - r_j(t)| = O(t^{2/3})$ as $t \to +\infty$ (see Figure 2.1 for an example of a cluster partition). Using this partition, the associated renormalized Lagrangian action can be decomposed into two terms: one describing the hyperbolic motion of the clusters' barycenters, and another corresponding to the parabolic motion of the bodies within the clusters. Following the same approach used in the proofs of Theorems 2.1.4 and 2.1.3, we apply the direct method of the calculus of variations to prove the existence of minimizers of the action, while Marchal's Principle is employed to ensure the absence of collisions among these minimizers.

Remark 2.1.8. In all three main theorems, the initial configuration may be chosen to lie in the collision set. If $x^0 \in \Delta$, the resulting motions γ provided by the theorems are continuous at $t = 1$ and extend to maximal solutions $\gamma(t)$ for all $t > 1$. For instance, when x^0 is the null configuration, the theorems guarantee the existence of ejection solutions emerging from total collision, with prescribed non-negative energy and an arbitrarily chosen limit shape.

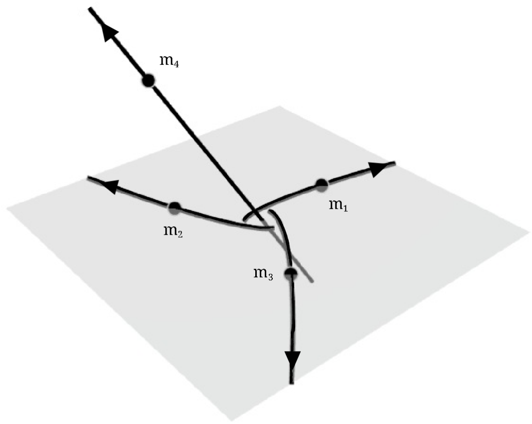

Figure 2.1: Example of a cluster decomposition for a four-body system exhibiting a hyperbolic-parabolic motion. We assume that $a_1 = a_2 = a_3 \neq a_4$. In this case, the index set $\mathcal{N} = \{1,2,3,4\}$ can be split into two clusters, namely $K_1 = \{1,2,3\}$ and $K_2 = \{4\}$. Inside the first cluster, the mutual distances satisfy $|r_i(t) - r_j(t)| \approx t^{2/3}$, for all $i < j$ with $i, j \in K_1$. This behavior shows that the point masses m_1, m_2, m_3 undergo a parabolic triangular expansion relative to the barycenter of their cluster, with a scaling of order $t^{2/3}$ as $t \to +\infty$, whereas the mass m_4 moves away from the center of mass of the remaining three bodies with a linear rate in time. The figure is taken from [66].

As a consequence of the variational setting of the problem, we also obtain the following corollary, where the free-time minimization property is proved in Corollary 1.3.3.

Corollary 2.1.9 (Polimeni and Terracini 2024 [66]). *The motions $\gamma(t)$ given by Theorems 2.1.4, 2.1.3 and 2.1.7 are continuous at $t = 1$ and collisionless for $t > 1$. Moreover, they are free-time action minimizers at their energy level.*

These results highlight the connection, via a Busemann function, between families of hyperbolic trajectories that are minimal in free time and solutions of the time-independent Hamilton-Jacobi equation of the N-body problem, as already stated by Maderna and Venturelli in [52]. In Chapter 3, we show that a linear correction of the value function is indeed a solution of the Hamilton-Jacobi equation.

2.2 Existence of minimal half hyperbolic motions

Homographic motions – trajectories whose configuration is always in the same similarity class – are the only explicitly known expansive motions. Such motions are characterized by the fact that $\gamma(t)$ is a central configuration for all t, which represents a strong limitation on the number of possible motions. Indeed, the Painlevé-Wintner conjecture, confirmed by Hampton and Moeckel [41] in the case of four bodies, and by Albouy and Kaloshin [1] for generic mass values in the planar 5-body problem, states that, up to similarity, there are always a finite number of central configurations. For $N = 3$, for example, the only central configurations are either equilateral or collinear.

On the other hand, Chazy proved in [19] that the set of initial points in the phase space that generate hyperbolic motions is an open set and that the limit shape depends continuously on the initial condition.

Theorem 2.2.1 (Chazy 1922 [19])**.** *Let $\gamma(t)$ be a motion with energy constant $h > 0$ and defined for all $t > t_0$.*

(i) The limit

$$\lim_{t\to+\infty} \frac{R(t)}{r(t)} = \lim_{t\to+\infty} \frac{\max_{i<j} |r_i(t) - r_j(t)|}{\min_{i<j} |r_i(t) - r_j(t)|} = L \in [1, +\infty]$$

always exists.

(ii) If $L < +\infty$, there are a configuration $a \in \Omega$ and some function P, which is analytic in a neighborhood of $(0,0)$, such that for every t large enough, we have

$$\gamma(t) = at - \log(t)\nabla U(a) + P(u, v),$$

where $u = 1/t$ and $v = \log(t)/t$.

This means that a motion close enough to some hyperbolic homographic motion is still hyperbolic: indeed, if an orbit is sufficiently close to a given hyperbolic motion, then, after some time, the distance between the bodies will be so large that the action of the gravitational forces will not be able to perturb their velocities too much. However, this result does not give additional information about the set of configurations that are realized as limit shapes.

Chazy's result was later generalized by Maderna and Venturelli, who provided a slightly more general version for homogeneous potentials of degree -1.

Lemma 2.2.2 (Maderna and Venturelli 2020 [52])**.** *Working on an Euclidean space E, which is endowed with an Euclidean norm $\|\cdot\|$, let $U : E^N \to \mathbb{R} \cup \{+\infty\}$ be a homogeneous potential of degree -1 of class C^2 on the open set $\Omega = \{x \in E^N \mid U(x) < +\infty\}$. Let $x : [0, +\infty) \to \Omega$ be a given solution of $\ddot{x} = \nabla U(x)$ satisfying $x(t) = at + o(t)$ as $t \to +\infty$ with $a \in \Omega$. Then we have the following.*

1. *The solution γ has asymptotic velocity a, meaning that*

$$\lim_{t\to+\infty} \dot{x}(t) = a.$$

2. *(Chazy's continuity of the limit shape). Given $\varepsilon > 0$, there are constants $t_1 > 0$ and $\delta > 0$ such that, for any maximal solution $y : [0, T) \to \Omega$ satisfying $\|y(0) - x(0)\| < \delta$ and $\|\dot{y}(0) - \dot{x}(0)\| < \delta$, we have*

   - *$T = +\infty$, $\|y(t) - at\| < t\varepsilon$ for all $t > t_1$;*
   - *there is $b \in \Omega$ with $\|b - a\| < \varepsilon$ for which $y(t) = bt + o(t)$.*

For the above reasons, proving that any collisionless configuration can occur as the limit shape of a hyperbolic motion represents a significant advance in the study of hyperbolic motions.

This section is devoted to the proof of Theorem 2.1.4 – following the work in [66] –, which states the existence of hyperbolic solutions to the N-body problem with prescribed initial configuration and limit shape.

We start by recalling the following definition, which is also due to Chazy [19].

Definition 2.2.3. Hyperbolic motions are those motions such that each body has a different limit velocity vector, that is, $\dot{r}_i(t) \to a_i \in \mathbb{R}^d$, as $t \to +\infty$, and $a_i \neq a_j$ whenever $i \neq j$.

To give an equivalent characterization of expansive motions, we notice that if $\gamma(t)$ is smooth in a normed vector space with asymptotic velocity a, then $\gamma(t) = at + o(t)$ as $t \to +\infty$, but the converse is in general not true. However, working on $\mathbb{R}^{dN}$, by Theorem 2.2.1, the converse is satisfied by solutions of the Newtonian N-body problem. We can thus characterize hyperbolic solutions as motions without singularities in the future and such that $\gamma(t) = at + o(t)$ as $t \to +\infty$ for some configuration $a \in \Omega$.

Fixing the initial configuration $x^0 \in \mathcal{X}$ and the limit shape $a \in \Omega$, we consider the differential system

$$\begin{cases} \mathcal{M}\ddot{\gamma} = \nabla U(\gamma) \\ \gamma(1) = x^0 \\ \lim_{t\to+\infty} \dot{\gamma}(t) = a. \end{cases} \tag{2.2.1}$$

To prove the existence of hyperbolic motions to Newton's equations (2.2.1), we look for solutions having the form $\gamma(t) = at + \varphi(t) + x^0 - a$, where $\varphi : [1, +\infty) \to \mathcal{X}$ belongs to the space $\mathcal{D}_0^{1,2}(1, +\infty)$. Writing the system in terms of φ, we obtain the equivalent system

$$\begin{cases} \mathcal{M}\ddot{\varphi}(t) = \nabla U(at + \varphi(t) + x^0 - a) \\ \varphi(1) = 0 \\ \lim_{t\to+\infty} \dot{\varphi}(t) = 0. \end{cases} \tag{2.2.2}$$

We notice that if we tried to prove the existence of hyperbolic solutions to (2.2.2) with the usual variational technique, which consists in proving the existence of minimizers of the Lagrangian action functional

$$\int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + U(at + \varphi(t) + x^0 - a)\ dt,$$

where

$$U(at + \varphi(t) + x^0 - a) = \sum_{i<j} \frac{m_i m_j}{|(a_i t + \varphi_i(t) + x_i^0 - a_i) - (a_j t + \varphi_j(t) + x_j^0 - a_j)|},$$

the major problem we encounter is that $U(at + \varphi(t) + x^0 - a)$ is not integrable at infinity. Indeed, when $\varphi \in C_0^\infty([1, +\infty))$, the term $U(at + \varphi(t) + x^0 - a)$ decays as $\frac{1}{t}$ for $t \to +\infty$.

However, since we can add arbitrary functions to the Lagrangian without changing the associated Euler-Lagrange equations, we can renormalize the action functional as in Definition 1.1.11 in order to have a finite integral. In this case, given the guiding curve $r_0(t) = at$, the renormalized Lagrangian action is

$$\mathcal{A}(\varphi) = \mathcal{A}^{ren}(\varphi) = \int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + U(at + \varphi(t) + x^0 - a) - U(at)\ dt.$$

2.2.1 Coercivity

In order to apply the direct method of the calculus of variations, we need to prove that the renormalized lagrangian action is coercive and weak-lower semicontinuous on the space $\mathcal{D}_0^{1,2}(1, +\infty)$. Here, we prove that the functional is coercive, that is to say, that

$$\mathcal{A}(\varphi) \to +\infty \text{ as } \|\varphi\|_{\mathcal{D}} \to +\infty.$$

From now on, we will use the notations $\varphi_{ij} = \varphi_i - \varphi_j$, $x_{ij}^0 = x_i^0 - x_j^0$ and $a_{ij} = a_i - a_j$.

We observe that the action can be equivalently written as

$$\mathcal{A}(\varphi) = \int_1^{+\infty} \frac{1}{2}\sum_{i=1}^{N} m_i|\dot{\varphi}_i(t)|^2 + U(at + \varphi(t) + x^0 - a) - U(at)\ dt,$$

where

$$\begin{aligned}
&U(at+\varphi(t)+x^0-a)-U(at)\\
&=\sum_{i<j}\left(\frac{m_im_j}{|(a_it+\varphi_i(t)+x_i^0-a_i)-(a_jt+\varphi_j(t)+x_j^0-a_j)|}-\frac{m_im_j}{|a_i-a_j|t}\right)\\
&=\sum_{i<j}\left(\frac{m_im_j}{|a_{ij}t+\varphi_{ij}(t)+x_{ij}^0-a_{ij}|}-\frac{m_im_j}{|a_{ij}|t}\right).
\end{aligned}$$

Since we are working in the space of configurations whose center of mass is null at every time, Leibniz's formula can be applied:

$$\sum_{i=1}^N m_i|\dot\varphi_i(t)|^2=\frac{1}{M}\sum_{i<j}m_im_j|\dot\varphi_{ij}(t)|^2, \tag{2.2.3}$$

where $M=\sum_{i=1}^N m_i$. Indeed, it holds

$$\begin{aligned}
\sum_{i<j}m_im_j|\dot\varphi_i(t)-\dot\varphi_j(t)|^2&=\frac12\sum_{i,j}m_im_j(|\dot\varphi_i(t)|^2+|\dot\varphi_j(t)|^2-2\langle\dot\varphi_i(t),\dot\varphi_j(t)\rangle)\\
&=\frac12\left(2M\sum_{i=1}^N m_i|\dot\varphi_i(t)|^2-2\langle\sum_{i=1}^N m_i\dot\varphi_i(t),\sum_{j=1}^N m_j\dot\varphi_j(t)\rangle\right)\\
&=M\sum_{i=1}^N m_i|\dot\varphi_i(t)|^2.
\end{aligned}$$

Using (2.2.3), we can write the Lagrangian action as

$$\mathcal{A}(\varphi)=\int_1^{+\infty}\sum_{i<j}m_im_j\left(\frac{|\dot\varphi_{ij}(t)|^2}{2M}+\frac{1}{|a_{ij}t+\varphi_{ij}(t)+x_{ij}^0-a_{ij}|}-\frac{1}{|a_{ij}|t}\right)dt.$$

Leibniz's formula (2.2.3) also states that $\|\dot\varphi\|_{L^2}\to+\infty$ if and only if there is $i<j$ such that $\|\dot\varphi_i-\dot\varphi_j\|_{L^2}\to+\infty$. Thus, setting

$$\mathcal{A}(\varphi)=\sum_{i<j}\mathcal{A}_{ij}(\varphi),$$

where

$$\mathcal{A}_{ij}(\varphi)=\int_1^{+\infty}m_im_j\left(\frac{|\dot\varphi_{ij}(t)|^2}{2M}+\frac{1}{|a_{ij}t+\varphi_{ij}(t)+x_{ij}^0-a_{ij}|}-\frac{1}{|a_{ij}|t}\right)dt,$$

if we prove the coercivity of each term $\mathcal{A}_{ij}$, the coercivity of the action trivially follows.

Using the inequality

$$|\varphi_i(t)| \leq \|\varphi_i\|_{\mathcal{D}}\sqrt{t}, \qquad \text{for every } i = 1, ..., N,\ t \geq 1 \text{ and } \varphi_i \in \mathcal{D}_0^{1,2}(1, +\infty), \tag{2.2.4}$$

which follows from (1.2.7), we have

$$U(at + \varphi(t) + x^0 - a) - U(at) \geq \sum_{i<j} \left(\frac{m_i m_j}{|a_{ij}t| + \|\varphi_{ij}\|_{\mathcal{D}}\sqrt{t} + |x^0_{ij} - a_{ij}|} - \frac{m_i m_j}{|a_{ij}|t} \right).$$

We can then look for an upper bound for the integral

$$\int_1^{+\infty} \left(\frac{1}{|a_{ij}|t} - \frac{1}{|a_{ij}|t + \|\varphi_{ij}\|_{\mathcal{D}}\sqrt{t} + |x^0_{ij} - a_{ij}|} \right) dt.$$

Using the change of variables $t = s^2$, we obtain

$$\frac{2}{|a_{ij}|} \int_1^{+\infty} \left(\frac{1}{s^2} - \frac{1}{s^2 + \frac{\|\varphi_{ij}\|_{\mathcal{D}}}{|a_{ij}|} s + \frac{|x^0_{ij} - a_{ij}|}{|a_{ij}|}} \right) s\ ds. \tag{2.2.5}$$

Since

$$\begin{aligned} s^2 + \frac{\|\varphi_{ij}\|_{\mathcal{D}}}{|a_{ij}|} s + \frac{|x^0_{ij} - a_{ij}|}{|a_{ij}|} &= \left(s + \frac{\|\varphi_{ij}\|_{\mathcal{D}}}{2|a_{ij}|} \right)^2 - \frac{\|\varphi_{ij}\|^2_{\mathcal{D}}}{4|a_{ij}|^2} + \frac{|x^0_{ij} - a_{ij}|}{|a_{ij}|} \\ &= \frac{\|\varphi_{ij}\|^2_{\mathcal{D}}}{4|a_{ij}|^2} \left[\left(\frac{2|a_{ij}|s}{\|\varphi_{ij}\|_{\mathcal{D}}} + 1 \right)^2 - 1 + \frac{4|x^0_{ij} - a_{ij}||a_{ij}|}{\|\varphi_{ij}\|^2_{\mathcal{D}}} \right], \end{aligned}$$

(2.2.5) is equal to

$$\frac{2}{|a_{ij}|} \frac{4|a_{ij}|^2}{\|\varphi_{ij}\|^2_{\mathcal{D}}} \int_1^{+\infty} \left[\frac{1}{\left(\frac{2|a_{ij}|s}{\|\varphi_{ij}\|_{\mathcal{D}}}\right)^2} - \frac{1}{\left(\frac{2|a_{ij}|s}{\|\varphi_{ij}\|_{\mathcal{D}}} + 1\right)^2 - 1 + \frac{4|x^0_{ij} - a_{ij}||a_{ij}|}{\|\varphi_{ij}\|^2_{\mathcal{D}}}} \right] s\ ds. \tag{2.2.6}$$

Changing variables again with $\tau = \frac{2|a_{ij}|s}{\|\varphi_{ij}\|_{\mathcal{D}}}$, we obtain that (2.2.6) is equal to

$$\frac{2}{|a_{ij}|} \int_{\frac{2|a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}}}^{+\infty} \left[\frac{1}{\tau^2} - \frac{1}{(\tau+1)^2 - 1 + \frac{4|x^0_{ij} - a_{ij}||a_{ij}|}{\|\varphi_{ij}\|^2_{\mathcal{D}}}} \right] \tau\ d\tau.$$

Since we are interested in large values of $\|\varphi_{ij}\|_{\mathcal{D}}$, we can suppose that there is some

$\lambda < 1$ such that $\frac{4|x_{ij}^0 - a_{ij}||a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}^2} \leq \lambda$. We then have

$$\begin{aligned} &\frac{2}{|a_{ij}|}\int_{\frac{2|a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}}}^{+\infty}\left[\frac{1}{\tau^2} - \frac{1}{(\tau+1)^2 - 1 + \frac{4|x_{ij}^0 - a_{ij}||a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}^2}}\right]\tau\ d\tau \\ &\leq \frac{2}{|a_{ij}|}\int_{\frac{2|a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}}}^{+\infty}\left[\frac{1}{\tau^2} - \frac{1}{(\tau+1)^2 - 1 + \lambda}\right]\tau\ d\tau. \end{aligned} \tag{2.2.7}$$

The integrand appearing in the last integral is a positive function. We note that it behaves asymptotically like $\frac{1}{\tau}$ as $\tau \to 0$ and like $\frac{1}{\tau^2}$ as $\tau \to +\infty$. In particular, the integral converges at infinity uniformly with respect to λ. By choosing $\|\varphi_{ij}\|_{\mathcal{D}}$ sufficiently large, we may equivalently analyze the integral

$$\int_{\varepsilon}^{+\infty}\left[\frac{1}{\tau^2} - \frac{1}{(\tau+1)^2 - 1 + \lambda}\right]\tau\ d\tau,$$

where $\varepsilon = \frac{2|a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}} < 1$. Since the integrand is asymptotic to $\frac{1}{\tau}$ as $\tau \to 0$, it is equivalent to consider the sum of integrals

$$\int_{\varepsilon}^{1}\frac{1}{\tau}\ d\tau + \int_{1}^{+\infty}\left[\frac{1}{\tau^2} - \frac{1}{(\tau+1)^2 - 1 + \lambda}\right]\tau\ d\tau,$$

where the second integral is constant (we will call it C_1) and does not depend on ε. We have

$$\int_{\varepsilon}^{1}\frac{1}{\tau}\ d\tau + \int_{1}^{+\infty}\left[\frac{1}{\tau^2} - \frac{1}{(\tau+1)^2 - 1 + \lambda}\right]\tau\ d\tau = -\log\varepsilon + C_1.$$

Then, as $\|\varphi_{ij}\|_{\mathcal{D}} \to +\infty$, we know that the integral on the right-hand side of (2.2.7) behaves like

$$\begin{aligned} \frac{2}{|a_{ij}|}\left(-\log\frac{2|a_{ij}|}{\|\varphi_{ij}\|_{\mathcal{D}}} + C_1\right) &= \frac{2}{|a_{ij}|}\left(\log\|\varphi_{ij}\|_{\mathcal{D}} + C_1 - \log 2|a_{ij}|\right) \\ &= \frac{2}{|a_{ij}|}(\log\|\varphi_{ij}\|_{\mathcal{D}} + C_2), \end{aligned}$$

where $C_2 = C_1 - \log 2|a_{ij}|$.

We have thus proved that

$$\int_{1}^{+\infty}\left(\frac{1}{|a_{ij}|t} - \frac{1}{|a_{ij}|t + \|\varphi_{ij}\|_{\mathcal{D}}\sqrt{t} + |x_{ij}^0 - a_{ij}|}\right)\ dt \leq \frac{2}{|a_{ij}|}(\log\|\varphi_{ij}\|_{\mathcal{D}} + C_2).$$

This means that given $R > 0$ large enough, when $\|\varphi_{ij}\|_{\mathcal{D}} \geq R$, it holds

$$\mathcal{A}_{ij}(\varphi) \geq m_i m_j \left[\frac{\|\varphi_{ij}\|_{\mathcal{D}}^2}{2M} - \frac{2}{|a_{ij}|} (\log \|\varphi_{ij}\|_{\mathcal{D}} + C_2) \right].$$

This concludes that $\mathcal{A}_{ij}(\varphi) \to +\infty$ as $\|\varphi_{ij}\|_{\mathcal{D}} \to +\infty$.

2.2.2 Weak-lower semicontinuity

To establish that the functional $\mathcal{A}$ is weakly lower semicontinuous, we first observe that the kinetic term $\frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2$ is a convex function. As a consequence, the weak-lower semicontinuity of the term

$$\int_1^{+\infty} \frac{1}{2} \|\dot{\varphi}(t)\|_{\mathcal{M}}^2 \, dt$$

follows directly. It is nevertheless important to remark that Fatou's Lemma cannot be applied to the term

$$\int_1^{+\infty} U(at + \varphi(t) + x^0 - a) - U(at) \, dt,$$

since the corresponding integrand is not nonnegative.

We claim that there exists at least one sequence of functions in $\mathcal{D}_0^{1,2}(1, +\infty)$ that converges uniformly on compact subsets of $[1, +\infty)$. To justify this, let $(\varphi^n)_n$ be a bounded sequence in $\mathcal{D}_0^{1,2}(1, +\infty)$. Then $\|\dot{\varphi}^n\|_{L^2([1,+\infty))} < +\infty$ and $\varphi^n(1) = 0$ for every n. From the inequality

$$\|\varphi(t)\|_{\mathcal{M}} \leq \|\dot{\varphi}\|_{L^2} \sqrt{t-1} \leq \|\dot{\varphi}\|_{L^2} \sqrt{t}, \qquad \text{for all } t \geq 1,$$

we deduce that

$$\|\varphi^n(t)\|_{\mathcal{M}} \leq \|\dot{\varphi}^n\|_{L^2} \sqrt{t}, \qquad \text{for all } t \geq 1 \text{ and all } n,$$

which implies that the L^∞-norm of φ^n on $[1, T]$ is uniformly bounded for every fixed $T \geq 1$. Moreover, since

$$\|\varphi^n(t_1) - \varphi^n(t_2)\|_{\mathcal{M}} \leq \|\dot{\varphi}^n\|_{L^2} \sqrt{t_1 - t_2},$$

for all $t_1, t_2 \in [1, +\infty)$ and all n, it follows that the sequence $(\varphi^n)_n$ is equicontinuous on each interval $[1, T]$, with T fixed. Therefore, by the Ascoli-Arzelà Theorem, for every $T \geq 1$ there exists a subsequence $(\varphi^{n_k})_k$ that converges uniformly on $[1, T]$, and hence converges pointwise on every compact interval. Furthermore, by a standard diagonal argument, one can extract a subsequence that converges pointwise on the whole interval $[1, +\infty)$.

Consider now a sequence $(\varphi^n)_n$ in $\mathcal{D}_0^{1,2}(1,+\infty)$ that converges weakly to some $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$. By the properties of weak convergence, the sequence $(\varphi^n)_n$ is bounded in $\mathcal{D}_0^{1,2}(1,+\infty)$, and from the previous discussion, there exists a subsequence $(\varphi^{n_k})_k$ that converges uniformly on compact subsets of $[1,+\infty)$, and hence pointwise on $[1,+\infty)$.

We can write the potential term as

$$\frac{1}{|a_{ij}t+\varphi_{ij}^n(t)+x_{ij}^0-a_{ij}|}-\frac{1}{|a_{ij}t|}=\int_0^1\frac{d}{ds}\left[\frac{1}{|a_{ij}t+s(\varphi_{ij}^n(t)+x_{ij}^0-a_{ij})|}\right]ds, \tag{2.2.8}$$

keeping in mind that this equality holds only when the denominator of the integrand is nonzero, which is guaranteed for sufficiently large t. In particular, for all $s\in(0,1)$ and $t\in[1,+\infty)$, we have

$$\begin{aligned}|a_{ij}t+s(\varphi_{ij}^n(t)+x_{ij}^0-a_{ij})| &\geq |a_{ij}|t-s\Big(\|\varphi_{ij}^n\|_{\mathcal{D}}\sqrt{t}+|x_{ij}^0-a_{ij}|\Big)\\ &> |a_{ij}|t-\Big(\|\varphi_{ij}^n\|_{\mathcal{D}}\sqrt{t}+|x_{ij}^0-a_{ij}|\Big).\end{aligned}$$

Since $|\varphi_{ij}^n(t)|\leq k\sqrt{t}$ for some $k\in\mathbb{R}^+$ large enough and for all $t\geq 1$, it follows that

$$|a_{ij}t+s(\varphi_{ij}^n(t)+x_{ij}^0-a_{ij})|>|a_{ij}|t-(k\sqrt{t}+|x_{ij}^0-a_{ij}|),$$

where the right-hand side is positive for $t\geq\bar{T}=\bar{T}(k)$, which can be explicitly computed by studying the function $g(t)=|a_{ij}|t-[k\sqrt{t}+|x_{ij}^0-a_{ij}|]$. For this reason, it is convenient to split the analysis of the potential term into the two intervals $[1,\bar{T}]$ and $[\bar{T},+\infty)$.

On $[1,\bar{T}]$, it is clear that $U(x^0-a+at+\varphi)\in L^1([1,\bar{T}])$. Moreover, since U is nonnegative, the pointwise convergence of $(\varphi^n)_n$ together with Fatou's Lemma gives

$$\int_1^{\bar{T}}\frac{1}{|a_{ij}t+\varphi_{ij}^n(t)+x_{ij}^0-a_{ij}|}\,dt\leq\liminf_{n\to+\infty}\int_1^{\bar{T}}\frac{1}{|a_{ij}t+\varphi_{ij}^n(t)+x_{ij}^0-a_{ij}|}\,dt.$$

Next, using the boundedness of the sequence $(\varphi^n)_n$, we aim to show that the term $U(\varphi^n(t)+x^0-a+at)-U(at)$ converges in $L^1([\bar{T},+\infty))$. By (2.2.8), we have

$$\begin{aligned}&\int_{\bar{T}}^{+\infty}\frac{1}{|a_{ij}t+\varphi_{ij}^n(t)+x_{ij}^0-a_{ij}|}-\frac{1}{|a_{ij}t|}\,dt\\ &\quad=\int_{\bar{T}}^{+\infty}\left(\int_0^1-\frac{[a_{ij}t+s(\varphi_{ij}^n(t)+x_{ij}^0-a_{ij})](\varphi_{ij}^n(t)+x_{ij}^0-a_{ij})}{|a_{ij}t+s(\varphi_{ij}^n(t)+x_{ij}^0-a_{ij})|^3}\,ds\right)dt.\end{aligned}$$

Our goal is to find an upper bound for the term

$$\int_{\bar{T}}^{+\infty} \left| \frac{1}{|a_{ij}t + \varphi_{ij}^n(t) + x_{ij}^0 - a_{ij}|} - \frac{1}{|a_{ij}t|} \right| dt.$$

To find the upper bound, we will need the inequality

$$\frac{|b+c|^2}{|b|^2 - |c|^2} \geq \frac{1}{3}, \qquad \text{for each } b, c \in \mathbb{R}^d \text{ such that } |b| \geq 2|c|, \tag{2.2.9}$$

which can easily be proved by elementary calculus. By (2.2.9) and using the fact that $|x_{ij}^0 - a_{ij}| + \|\varphi_{ij}^n\|_{\mathcal{D}}\sqrt{t} \leq k'\sqrt{t}$ for $k' \in \mathbb{R}^+$ large enough, we have

$$\begin{aligned}
\int_{\bar{T}}^{+\infty} \left| \int_0^1 - \frac{[a_{ij}t + s(\varphi_{ij}^n(t) + x_{ij}^0 - a_{ij})](\varphi_{ij}^n(t) + x_{ij}^0 - a_{ij})}{|a_{ij}t + s(\varphi_{ij}^n(t) + x_{ij}^0 - a_{ij})|^3} \, ds \right| dt \\
\leq \int_{\bar{T}}^{+\infty} \left(\int_0^1 \frac{|\varphi_{ij}^n(t) + x_{ij}^0 - a_{ij}|}{|a_{ij}t + s(\varphi_{ij}^n(t) + x_{ij}^0 - a_{ij})|^2} \, ds \right) dt \\
\leq \int_{\bar{T}}^{+\infty} \left(\int_0^1 3 \frac{\|\varphi_{ij}^n\|_{\mathcal{D}}\sqrt{t} + |x_{ij}^0 - a_{ij}|}{|a_{ij}t|^2 - s|\|\varphi_{ij}^n\|_{\mathcal{D}}\sqrt{t} + x_{ij}^0 - a_{ij}|^2} \, ds \right) dt \\
\leq \int_{\bar{T}}^{+\infty} \left(\int_0^1 \frac{3k'\sqrt{t}}{|a_{ij}|^2 t^2 - sk't} \, ds \right) dt.
\end{aligned}$$

By choosing $\bar{T}(k) \gg k'/|a_{ij}|^2$ so that $|a_{ij}|^2 t > sk'$ for all $s \in (0,1)$ and all $t \in [\bar{T}, +\infty)$ (which is possible for k sufficiently large), the last integral is finite. Consequently, we have shown that there exists $\hat{T}$ such that, for all $\bar{T} \geq \hat{T}$, the term

$$\int_{\bar{T}}^{+\infty} \left| \frac{1}{|a_{ij}t + \varphi_{ij}^n(t) + x_{ij}^0 - a_{ij}|} - \frac{1}{|a_{ij}t|} \right| dt$$

is dominated by an L^1-function. From this estimate, the L^1-convergence of the term

$$U(at + \varphi^n(t) + x^0 - a) - U(at)$$

follows. In particular, by the dominated convergence Theorem, we obtain

$$\lim_{n\to+\infty} \int_{\bar{T}}^{+\infty} U(at+\varphi^n(t)+x^0-a) - U(at) \, dt = \int_{\bar{T}}^{+\infty} U(at+\varphi^n(t)+x^0-a) - U(at) \, dt.$$

Thus, if we consider any sequence $(\varphi^n)_n$ in $\mathcal{D}_0^{1,2}(1,+\infty)$ converging weakly to some $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$, we have

$$\mathcal{A}(\varphi) \leq \liminf_{n\to+\infty} \int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}^n(t)\|_{\mathcal{M}}^2 + U(at + \varphi^n(t) + x^0 - a) - U(at) \, dt,$$

which proves the weak-lower semicontinuity of the renormalized Lagrangian action in the space $\mathcal{D}_0^{1,2}(1,+\infty)$.

Remark 2.2.4. Using the same arguments as above, we can prove the continuity of the renormalized action with respect to the strong topology for all φ that do not give rise to collisions.

2.2.3 Absence of collisions and hyperbolicity of the motion

Using the results of Sections 2.2.1 and 2.2.2, the direct method of the calculus of variations ensures the existence of at least one minimizer $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$ of the renormalized Lagrangian action. Consequently, the Renormalized Action Principle implies that φ satisfies

$$\mathcal{M}\ddot{\varphi}(t) = \nabla U(at + \varphi(t) + x^0 - a).$$

It remains to show that $\lim_{t\to+\infty} \dot{\varphi}(t) = 0$. We already know that $\dot{\varphi} \in L^2$ and that there exists $k \in \mathbb{R}^+$ such that $\|\varphi(t)\|_{\mathcal{M}} \leq k\sqrt{t}$. Using this estimate, we have

$$\sum_{i<j} m_i m_j \frac{1}{|a_{ij}t + \varphi_{ij}(t) + x_{ij}^0 - a_{ij}|} \leq \sum_{i<j} m_i m_j \frac{1}{|a_{ij}|t - k\sqrt{t} - |x_{ij}^0 - a_{ij}|},$$

and since $|a_{ij}|t - k\sqrt{t} - |x_{ij}^0 - a_{ij}| \to +\infty$ as $t \to +\infty$ for all $i,j = 1,\dots,N$, it follows that $\lim_{t\to+\infty} U(x(t)) = 0$. Moreover, from

$$\int_1^{+\infty} |\dot{\varphi}_{ij}(t)|^2 \, dt < +\infty,$$

we deduce

$$\liminf_{t\to+\infty} |\dot{\varphi}_{ij}(t)| = 0. \tag{2.2.10}$$

Remark 2.2.5. A solution $\gamma(t) = at + \varphi(t) + x^0 - a$ of $\mathcal{M}\ddot{\gamma} = \nabla U(\gamma)$ has positive energy. Indeed,

$$\frac{1}{2}\|\dot{\gamma}(t)\|_{\mathcal{M}}^2 - U(\gamma(t)) = \frac{1}{2}\sum_{i=1}^N m_i |\dot{\varphi}_i(t) + a_i|^2 - U(\gamma(t)) = h,$$

and since, by (2.2.10), there exists a sequence $t_k \to +\infty$ such that $\dot{\varphi}_i(t_k) \to 0$, we conclude that $h = \frac{1}{2}\|a\|_{\mathcal{M}}^2$.

By Remark 2.2.5, Chazy's Lemma (Lemma 2.2.2) applies, implying the existence of $\lim_{t\to+\infty} \dot{\gamma}(t)$ exists. Since, by (2.2.10), there is a sequence $(t_k)_k$ with $\dot{\gamma}(t_k) \to a$ as $t_k \to +\infty$, we can conclude that

$$\lim_{t\to+\infty} \dot{\gamma}(t) = a.$$

Furthermore, Chazy's Theorem (Theorem 2.2.1) guarantees that the minimizing motion γ admits the asymptotic expansion

$$\gamma(t) = at - \log(t)\nabla U(a) + o(1), \quad \text{as } t \to +\infty.$$

Therefore, we have shown that x is a solution of

$$\begin{cases} \mathcal{M}\ddot{\gamma} = \nabla U(\gamma), \\ \gamma(1) = x^0, \\ \lim_{t\to+\infty} \dot{\gamma}(t) = a, \end{cases}$$

i.e., γ is the hyperbolic motion of the N-body problem we sought.

2.3 Existence of minimal half parabolic motions

In this section, we discuss the proof of Theorem 2.1.3, which focuses on the class of parabolic motions, following the approach employed in [66]. These trajectories are characterized as motions of the form $\gamma(t) = at + O(t^{2/3})$ as $t \to +\infty$, with $a = 0$ and such that $|r_i(t) - r_j(t)| \approx t^{2/3}$ for every $i < j$. Equivalently, we have the following definition.

Definition 2.3.1. An expansive solution of the N-body problem is said to be parabolic if the velocity of each body converges to zero.

More precisely, our main result concerning parabolic motions asserts the existence of solutions to the N-body problem of the form

$$\gamma(t) = \beta b\, t^{2/3} + o(t^{1/3^+}), \qquad \text{as } t \to +\infty,$$

where $\beta \in \mathbb{R}$ is a suitable constant and b is a prescribed minimal central configuration. The remainder term is $o(t^{1/3^+})$ in the sense that it grows more slowly than $t^{1/3+\varepsilon}$ for every $\varepsilon > 0$.

We recall the following definitions.

Definition 2.3.2. We say that $b \in \mathcal{X}$ is a central configuration if it is a critical point of U restricted to the inertial ellipsoid

$$\mathcal{E} = \{x \in \mathcal{X} \ : \ \langle \mathcal{M}x, x\rangle = 1\}.$$

A central configuration $b_m \in \mathcal{E}$ is said to be minimal if

$$U(b_m) = \min_{b\in\mathcal{E}} U(b).$$

In what follows, we shall work with normalized central configurations, namely central configurations b satisfying $\langle \mathcal{M}b, b\rangle = 1$.

Remark 2.3.3. Since U diverges at collisions, any minimal central configuration b_m is collisionless, that is, $b_m \in \Omega$.

Remark 2.3.4. For a Kepler-type potential U, the definition of central configurations yields

$$\nabla U(b) = \lambda\, \mathcal{M}b,$$

where λ is a Lagrange multiplier. Moreover, it holds

$$\lambda = \lambda\langle \mathcal{M}b, b\rangle = \langle \nabla U(b), b\rangle = -U(b). \tag{2.3.1}$$

We also recall that, given a central configuration b, the curve

$$\gamma(t) = \beta b\, t^{2/3}$$

is a self-similar solution of Newton's equations, where β is a constant determined as follows:

$$\mathcal{M}\ddot{\gamma} = -\frac{2}{9}\mathcal{M}\beta b\, t^{-4/3} = \nabla U(x) = \nabla U(\beta b\, t^{2/3}) = \frac{1}{\beta^2} t^{-4/3}\nabla U(b) = \frac{1}{\beta^2} t^{-4/3}\lambda\, \mathcal{M}b.$$

By (2.3.1), it follows that

$$\beta^3 = \frac{9}{2}U(b).$$

Therefore, for $\beta = \sqrt[3]{\frac{9}{2}U(b)}$, the function $\gamma(t) = \beta b\, t^{2/3}$ is a homothetic solution of Newton's equations.

Now, set the guiding curve

$$r_0(t) = \beta b_m t^{2/3},$$

where $b_m \in \Omega$ is a normalized minimal central configuration and $\beta = \sqrt[3]{\frac{9}{2}U(b)}$. We want to prove the existence of solutions of the system

$$\begin{cases} \mathcal{M}\ddot{\gamma} = \nabla U(\gamma), \\ \gamma(1) = x^0, \\ \lim_{t\to+\infty} \dot{\gamma}(t) = 0, \end{cases}$$

where $x^0 \in \mathcal{X}$ is given. We seek solutions having the form

$$\gamma(t) = r_0(t) + \varphi(t) + \tilde{x}^0, \tag{2.3.2}$$

where $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$ and we denote $\tilde{x}^0 = x^0 - r_0(1)$. In this case, we have

$$\nabla U(\gamma(t)) = \mathcal{M}\ddot{\gamma}(t) = \mathcal{M}\ddot{r}_0(t) + \mathcal{M}\ddot{\varphi}(t) = \nabla U(r_0(t)) + \mathcal{M}\ddot{\varphi}(t),$$

which means that

$$\mathcal{M}\ddot{\varphi}(t) = \nabla U(r_0(t) + \varphi(t) + \tilde{x}^0) - \nabla U(r_0(t)).$$

We can thus write the renormalized Lagrangian action as

$$\mathcal{A}(\varphi) = \int_1^{+\infty} \frac{1}{2}\langle \mathcal{M}\dot{\varphi}(t), \dot{\varphi}(t)\rangle + U(r_0(t)+\varphi(t)+\tilde{x}^0) - U(r_0(t)) - \langle \nabla U(r_0(t)), \varphi(t)\rangle \, dt.$$

In addition to the coercivity and weak-lower semicontinuity of the Lagrangian action, we also need to verify that the action is well defined on the set of functions $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$ such that $r_0(t) + \varphi(t) + \tilde{x}^0(t) \neq 0$ for all $t \geq 1$, and that the action is continuous and of class C^1 on the set $\mathcal{D}_0^{1,2}(1,+\infty) \setminus \{\varphi \in \mathcal{D}_0^{1,2}(1,+\infty) : \exists\, t \text{ such that } r_0(t) + \varphi(t) + \tilde{x}^0(t) = 0\}$.

2.3.1 Coercivity

As we did in the hyperbolic case, we aim to use the direct method of the calculus of variations to prove the existence of minimizers of the action. To prove the coercivity of the renormalized Lagrangian action, we reconduct the problem to a Kepler problem. Denoting $U_{min} = \min_{b \in \mathcal{E}} U(b)$, it holds, for any orbit γ,

$$U(\gamma) \geq \frac{U_{min}}{\|\gamma\|},$$

where $\|\cdot\|$ represents the Euclidean norm on $\mathbb{R}^{dN}$. Indeed, because of the homogeneity of the potential,

$$U(x) = U\left(\|x\|\frac{x}{\|x\|}\right) = \frac{1}{\|x\|}U\left(\frac{x}{\|x\|}\right) \geq \frac{1}{\|x\|}U_{min}. \tag{2.3.3}$$

Besides, it holds

$$\nabla U(r_0) = \nabla U(\beta b_m t^{2/3}) = \frac{1}{\beta^2 t^{4/3}}\nabla U(b_m) = \frac{1}{\beta^2 t^{4/3}}\lambda \mathcal{M} b_m = -\frac{U_{min}}{\beta^2 t^{4/3}}\mathcal{M} b_m. \tag{2.3.4}$$

Using (2.3.3) and (2.3.4), we can then write

$$\begin{aligned}\mathcal{A}(\varphi) &\geq \int_1^{+\infty} \frac{1}{2}\langle \mathcal{M}\dot{\varphi}(t), \dot{\varphi}(t)\rangle + \frac{U_{min}}{\|r_0(t) + \varphi(t) + \tilde{x}^0\|} - \frac{U_{min}}{\|r_0(t)\|} \\ &\quad + \frac{1}{\beta^2 t^{4/3}}\langle U_{min}\mathcal{M}b_m, \varphi(t)\rangle \; dt \\ &= \int_1^{+\infty} \frac{1}{2}\langle \mathcal{M}\dot{\varphi}(t), \dot{\varphi}(t)\rangle + \frac{U_{min}}{\|r_0(t) + \varphi(t) + \tilde{x}^0\|} - \frac{U_{min}}{\|r_0(t)\|} \\ &\quad + \frac{\langle U_{min}\mathcal{M}r_0(t), \varphi(t)\rangle}{\|r_0(t)\|^3} \; dt.\end{aligned}$$

We have

$$\begin{aligned}\|r_0(t) + \varphi(t) + \tilde{x}^0\|^2 &= \|r_0(t)\|^2 + 2\langle \mathcal{M}r_0(t), \varphi(t)\rangle + 2\langle \mathcal{M}\varphi(t), \tilde{x}^0\rangle + 2\langle \mathcal{M}r_0(t), \tilde{x}^0\rangle \\ &\quad + \|\varphi(t)\|^2 + \|\tilde{x}^0\|^2 \\ &= u + v,\end{aligned}$$

where we define

$$\begin{aligned}&u := \|r_0(t)\|^2 \\ &v := 2\langle \mathcal{M}r_0(t), \varphi(t)\rangle + 2\langle \mathcal{M}\varphi(t), \tilde{x}^0\rangle + 2\langle \mathcal{M}r_0(t), \tilde{x}^0\rangle + \|\varphi(t)\|^2 + \|\tilde{x}^0\|^2.\end{aligned}$$

Remark 2.3.5. The following equalities hold:

$$\begin{aligned}U(b+s) - U(b) &= \int_0^1 \frac{d}{dt}U(b+st) \; dt = \int_0^1 \langle \nabla U(b+st), s\rangle \; dt, \\ U(b+s) - U(b) - \nabla U(b)s &= \int_0^1 \int_0^1 \langle \nabla^2 U(b + st_1 t_2)s, s\rangle t_2 \; dt_1 \; dt_2.\end{aligned}$$

Using Remark 2.3.5, we then have

$$\begin{aligned}\|r_0(t) + \varphi(t) + \tilde{x}^0\|^{-1} &= (u+v)^{-1/2} \\ &= u^{-1/2} - \frac{1}{2}u^{-3/2}v + \frac{3}{4}\int_0^1 \int_0^1 \langle (u + stv)^{-5/2}v, v\rangle s \; ds \; dt.\end{aligned}$$

Since the integral in the last expression is positive, it follows

$$
\begin{aligned}
&\|r_0(t)+\varphi(t)+\tilde{x}^0\|^{-1}\\
&=(u+v)^{-1/2}\\
&\geq u^{-1/2}-\frac{1}{2}u^{-3/2}v\\
&=\|r_0(t)\|^{-1}-\frac{1}{2\|r_0(t)\|^3}[2\langle\mathcal{M}r_0(t),\varphi(t)\rangle+2\langle\mathcal{M}\varphi(t),\tilde{x}^0\rangle+2\langle\mathcal{M}r_0(t),\tilde{x}^0\rangle\\
&\quad+\|\varphi(t)\|^2+\|\tilde{x}^0\|^2]\\
&=\|r_0(t)\|^{-1}-\frac{\langle\mathcal{M}r_0,\varphi(t)\rangle}{\|r_0(t)\|^3}-\frac{\langle\mathcal{M}\varphi(t),\tilde{x}^0\rangle}{\|r_0(t)\|^3}-\frac{\langle\mathcal{M}r_0(t),\tilde{x}^0\rangle}{\|r_0(t)\|^3}\\
&\quad-\frac{1}{2}\frac{\|\varphi(t)\|^2}{\|r_0(t)\|^3}-\frac{1}{2}\frac{\|\tilde{x}^0\|^2}{\|r_0(t)\|^3}.
\end{aligned}
\tag{2.3.5}
$$

At this point, we can use (2.3.5) to obtain

$$
\begin{aligned}
\mathcal{A}(\varphi)\geq&\int_1^{+\infty}\frac{1}{2}\langle\mathcal{M}\dot{\varphi}(t),\dot{\varphi}(t)\rangle+\frac{U_{min}}{\|r_0(t)+\varphi(t)+\tilde{x}^0\|}-\frac{U_{min}}{\|r_0(t)\|}\\
&+\frac{\langle U_{min}\mathcal{M}r_0(t),\varphi(t)\rangle}{\|r_0(t)\|^3}\,dt\\
\geq&\int_1^{+\infty}\frac{1}{2}\langle\mathcal{M}\dot{\varphi}(t),\dot{\varphi}(t)\rangle-\frac{U_{min}}{2}\frac{\|\varphi(t)\|^2}{\|r_0(t)\|^3}-\frac{\langle U_{min}\mathcal{M}\varphi(t),\tilde{x}^0\rangle}{\|r_0(t)\|^3}\,dt+C_3,
\end{aligned}
$$

where C_3 is a constant. By Hardy inequality (1.2.4) and the fact that, for $\beta=\sqrt[3]{\frac{9}{2}U(b_m)}$,

$$
\frac{U_{min}}{\|r_0(t)\|^3}=\frac{U_{min}}{\|\beta b_m t^{2/3}\|^3}=\frac{U_{min}}{\beta^3t^2\|b_m\|^3}=\frac{2}{9}\frac{1}{t^2},
\tag{2.3.6}
$$

we have

$$
\begin{aligned}
\mathcal{A}(\varphi)\geq&\int_1^{+\infty}\frac{1}{2}\left[\langle\mathcal{M}\dot{\varphi}(t),\dot{\varphi}(t)\rangle-\frac{8}{9}\langle\mathcal{M}\dot{\varphi}(t),\dot{\varphi}(t)\rangle\right]-\frac{U_{min}\langle\mathcal{M}\varphi(t),\tilde{x}^0\rangle}{\|r_0(t)\|^3}\,dt+C_3\\
=&\int_1^{+\infty}\frac{1}{18}\langle\mathcal{M}\dot{\varphi}(t),\dot{\varphi}(t)\rangle-\frac{U_{min}\langle\mathcal{M}\varphi(t),\tilde{x}^0\rangle}{\|r_0(t)\|^3}\,dt+C_3.
\end{aligned}
$$

Using again (2.3.6), we observe that

$$
\frac{U_{min}\langle\mathcal{M}\varphi(t),\tilde{x}^0\rangle}{\|r_0(t)\|^3}=\frac{2}{9}\frac{\langle\mathcal{M}\varphi(t),\tilde{x}^0\rangle}{t^2}.
$$

By Cauchy-Scwartz and Hardy inequalities, it follows

$$\begin{aligned}
\int_1^{+\infty} -\frac{U_{min}\langle \mathcal{M}\varphi(t), \tilde{x}^0\rangle}{\|r_0(t)\|^3}\, dt &\geq -\int_1^{+\infty} \frac{2}{9}\frac{|\langle \mathcal{M}\varphi(t), \tilde{x}^0\rangle|}{t^2}\, dt \\
&\geq -\int_1^{+\infty} \frac{2}{9}\frac{\|\varphi(t)\|_{\mathcal{M}}}{t}\frac{\|\tilde{x}^0\|_{\mathcal{M}}}{t}\, dt \\
&\geq -\frac{2}{9}\left(\int_1^{+\infty} \frac{\|\varphi(t)\|^2_{\mathcal{M}}}{t^2}\, dt\right)^{1/2}\left(\int_1^{+\infty} \frac{\|\tilde{x}^0\|^2_{\mathcal{M}}}{t^2}\, dt\right)^{1/2} dt \\
&\geq -\frac{4}{9}C_4\|\varphi\|_{\mathcal{D}},
\end{aligned}$$

where C_4 is constant. We can then conclude that

$$\mathcal{A}(\varphi) \geq \frac{1}{18}\|\varphi\|^2_{\mathcal{D}} - \frac{4}{9}C_4\|\varphi\|_{\mathcal{D}} + C_3,$$

which proves the coercivity of the action.

2.3.2 Weak-lower semicontinuity

Here, we focus on the proof of the weak-lower semicontinuity of the action. Consider a sequence of functions $(\varphi^n)_n \subset \mathcal{D}^{1,2}_0(1,+\infty)$ converging weakly in $\mathcal{D}^{1,2}_0(1,+\infty)$ to some φ, for $n \to +\infty$. It trivially follows that, for every n, $\|\varphi\|_{\mathcal{D}} < +\infty$ and $\|\varphi^n\|_{\mathcal{D}} < +\infty$.

Fixing a time $\bar{T} \in (1,+\infty)$, let us write the action as the sum of two terms:

$$\mathcal{A}(\varphi) = \mathcal{A}_{[1,\overline{T})}(\varphi) + \mathcal{A}_{[\overline{T},+\infty)}(\varphi),$$

where

$$\mathcal{A}_{[1,\overline{T})}(\varphi) = \int_1^{\overline{T}} \frac{\|\dot{\varphi}\|^2_{\mathcal{M}}}{2} + U(r_0+\varphi+\tilde{x}^0) - U(r_0) - \langle \nabla U(r_0), \varphi\rangle\, dt,$$

and

$$\mathcal{A}_{[\overline{T},+\infty)}(\varphi) = \int_{\overline{T}}^{+\infty} \frac{\|\dot{\varphi}\|^2_{\mathcal{M}}}{2} + U(r_0+\varphi+\tilde{x}^0) - U(r_0) - \langle \nabla U(r_0), \varphi\rangle\, dt.$$

Using Ascoli-Arzelà's Theorem, we can say that $\varphi^n \to \varphi$ uniformly on compact subsets of $[1,+\infty)$, which implies that $\langle \nabla U(r_0), \varphi^n\rangle \to \langle \nabla U(r_0), \varphi\rangle$ uniformly in $[1,\overline{T}]$, as $n \to +\infty$, for every $\overline{T} < +\infty$. By Fatou's Lemma, it easily follows that the term $\mathcal{A}_{[1,\overline{T})}(\varphi)$ is weak-lower semicontinuous.

Concerning the term $\mathcal{A}_{[\overline{T},+\infty)}(\varphi)$, we can write:

$$\begin{aligned}\mathcal{A}_{[\overline{T},+\infty)}(\varphi) = \int_{\overline{T}}^{+\infty} & \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \\ & + U(r_0(t) + \varphi(t) + \tilde{x}^0) - U(r_0(t)) - \langle \nabla U(r_0(t)), \varphi(t)\rangle \\ & - \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \; dt.\end{aligned}$$

Claim: The map $\varphi(t) \mapsto \left(\int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \; dt \right)^{1/2}$ is an equivalent norm to $\|\cdot\|_{\mathcal{D}}$. Indeed:

- by the homogeneity of the potential, it holds

$$\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \geq -\frac{2}{9}\frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^2}$$

  for each $t \in [1, +\infty)$ (see Remark 2.3.6). Then, by Hardy inequality, we have

$$\int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \; dt \geq \frac{1}{2}\left(1 - \frac{8}{9}\right)\|\varphi\|_{\mathcal{D}}^2 = \frac{1}{18}\|\varphi\|_{\mathcal{D}}^2;$$

- using the fact that, for some constant $C_5 > 0$,

$$\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \leq C_5 \frac{\|\varphi(t)\|_{\mathcal{M}}}{t^2}$$

  and Hardy inequality, we have

$$\int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \; dt \leq C_6\|\varphi\|_{\mathcal{D}}^2,$$

  for some constant $C_6 > 0$.

From the claim, it follows that also the term $\int_{\overline{T}}^{+\infty} \frac{1}{2}\|\dot{\varphi}\|_{\mathcal{M}}^2 + \frac{1}{2}\langle \nabla^2 U(r_0)\varphi, \varphi\rangle \; dt$ is weak-lower semicontinuous.

Using Taylor's series expansion, we can write

$$\begin{aligned}& U(r_0 + \varphi + \tilde{x}^0) - U(r_0) - \langle \nabla U(r_0), \varphi\rangle - \frac{1}{2}\langle \nabla^2 U(r_0), \varphi(t), \varphi\rangle \\ & = \int_0^1 \int_0^1 \int_0^1 \langle \nabla^3 U(r_0 + \tau_1\tau_2\tau_3(\varphi^n + \tilde{x}^0))(\varphi^n + \tilde{x}^0), \varphi^n + \tilde{x}^0, \varphi^n + \tilde{x}^0\rangle \tau_1\tau_2^2 \; d\tau_1 d\tau_2 d\tau_3.\end{aligned}$$

Obviously there is a $\tilde{t} > 1$ such that

$$\|r_0(t) + \tau_1\tau_2\tau_3(\varphi^n(t) + \tilde{x}^0)\|_{\mathcal{M}} > 0$$

for all $t \geq \tilde{t}$. Choosing $\overline{T} \geq \tilde{t}$, we have

$$\begin{aligned}
&\langle \nabla^3 U(r_0(t) + \tau_1\tau_2\tau_3(\varphi^n(t) + \tilde{x}^0))(\varphi^n(t) + \tilde{x}^0), \varphi^n(t) + \tilde{x}^0, \varphi^n(t) + \tilde{x}^0\rangle \\
&\leq C_7 \frac{\|\varphi^n(t) + \tilde{x}^0\|^3_{\mathcal{M}}}{t^{8/3}} \\
&\leq C_8 \frac{\|\varphi^n\|^3_{\mathcal{D}} t^{3/2}}{t^{8/3}} \\
&\leq \frac{C_9}{t^{7/6}},
\end{aligned}$$

for all $t \geq \overline{T}$ and for proper constants $C_7, C_8, C_9 > 0$. This means that the term $\langle \nabla^3 U(r_0(t) + \tau_1\tau_2\tau_3(\varphi^n(t) + \tilde{x}^0))(\varphi^n(t) + \tilde{x}^0), \varphi^n(t) + \tilde{x}^0, \varphi^n(t) + \tilde{x}^0\rangle \tau_1\tau_2^2$ is L^1-dominated and the weak-lower semicontinuity of $\mathcal{A}_{[\overline{T},+\infty)}$ follows from the dominated convergence Theorem.

2.3.3 Regularity of the renormalized action on non-collision sets

We claim that the action $\mathcal{A}$ is of class C^1 over the set $\mathcal{D}_0^{1,2}(1,+\infty) \setminus \{\varphi \in \mathcal{D}_0^{1,2}(1,+\infty) \ : \exists\, t \text{ such that } r_0(t) + \varphi(t) + \tilde{x}^0 = 0\}$. The term $\int_1^{+\infty} \frac{\langle \mathcal{M}\dot{\varphi}(t), \dot{\varphi}(t)\rangle}{2}\, dt = \frac{\|\varphi\|^2_{\mathcal{D}}}{2}$ is of course a smooth functional, so we focus on the potential term

$$\mathcal{A}^2(\varphi) := \int_1^{+\infty} K(t, \varphi(t))\, dt,$$

where we denote

$$K(t, \varphi(t)) := U(r_0(t) + \varphi(t) + \tilde{x}^0) - U(r_0(t)) - \langle \nabla U(r_0(t)), \varphi(t)\rangle.$$

It holds

$$\begin{aligned}
d\mathcal{A}^2(\varphi)[\psi] &= \int_1^{+\infty} \langle \nabla K(t, \varphi(t)), \psi(t)\rangle\, dt \\
&= \int_1^{+\infty} \langle \nabla U(r_0(t) + \varphi(t) + \tilde{x}^0) - \nabla U(r_0(t)), \psi(t)\rangle\, dt
\end{aligned}$$

for every $\psi \in \mathcal{D}_0^{1,2}(1,+\infty)$. Given a sequence $(\varphi^n)_n \subset \mathcal{D}_0^{1,2}(1,+\infty)$, we have to prove that if $\varphi^n \to \varphi$ in $\mathcal{D}_0^{1,2}(1,+\infty)$, then

$$\sup_{\|\psi\|_{\mathcal{D}} \leq 1} \left| \int_1^{+\infty} \langle \nabla K(t, \varphi^n(t)) - \nabla K(t, \varphi(t)), \psi(t)\rangle\, dt \right| \to 0.$$

Since

$$\nabla K(t,\varphi(t)) = \nabla U(r_0(t)+\varphi(t)+\tilde{x}^0) - \nabla U(r_0(t)) = \int_0^1 \nabla^2 K(t,s\varphi(t))\varphi(t)\ ds,$$

we can estimate

$$\|\nabla K(t,\varphi(t))\|_{\mathcal{M}} \leq \int_0^1 \|\nabla^2 K(t,s\varphi(t))\|_{\mathcal{M}} \|\varphi(t)\|_{\mathcal{M}}\ ds \leq C_{10}\frac{\|\varphi(t)\|_{\mathcal{M}}}{t^2}, \qquad (2.3.7)$$

where $C_{10} > 0$ is a proper constant. Then, by Cauchy-Schwartz inequality,

$$\begin{aligned}
&\sup_{\|\psi\|_{\mathcal{D}}\leq 1} \left| \int_1^{+\infty} \langle \nabla K(t,\varphi^n(t)) - \nabla K(t,\varphi(t)), \psi(t)\rangle\ dt \right| \\
&\leq \sup_{\|\psi\|_{\mathcal{D}}\leq 1} \int_1^{+\infty} t\|\nabla K(t,\varphi^n(t)) - \nabla K(t,\varphi(t))\|_{\mathcal{M}} \frac{\|\psi(t)\|_{\mathcal{M}}}{t}\ dt \\
&\leq \sup_{\|\psi\|_{\mathcal{D}}\leq 1} \left(\int_1^{+\infty} \frac{\|\psi(t)\|^2_{\mathcal{M}}}{t^2}\ dt \right)^{1/2} \left(\int_1^{+\infty} t^2 \|\nabla K(t,\varphi^n(t)) - \nabla K(t,\varphi(t))\|^2_{\mathcal{M}}\ dt \right)^{1/2} \\
&\leq 2\left(\int_1^{+\infty} t^2 \|\nabla K(t,\varphi^n(t)) - \nabla K(t,\varphi(t))\|^2_{\mathcal{M}}\ dt \right)^{1/2}.
\end{aligned}$$

Now, using (2.3.7)

$$\begin{aligned}
&\|\nabla K(t,\varphi^n(t)) - \nabla K(t,\varphi(t))\|^2_{\mathcal{M}} \\
&= \left| \int_0^1 \nabla^2 K(t,\varphi(t)+\sigma(\varphi^n(t)-\varphi))(\varphi^n(t)-\varphi(t))\ d\sigma \right|^2 \\
&\leq \left(\int_0^1 \|\nabla^2 K(t,\varphi(t)+\sigma(\varphi^n(t)-\varphi(t)))(\varphi^n(t)-\varphi(t))\|_{\mathcal{M}}\ d\sigma \right)^2 \\
&\leq \left(\int_0^1 \frac{\|\varphi^n(t)-\varphi(t)\|_{\mathcal{M}}}{t^2}\ d\sigma \right)^2 \\
&= \frac{\|\varphi^n(t)-\varphi(t)\|^2_{\mathcal{M}}}{t^4}.
\end{aligned}$$

From this last computation, it follows that

$$\begin{aligned}
&\left(\int_1^{+\infty} t^2\|\nabla K(t,\varphi^n(t)) - \nabla K(t,\varphi(t))\|_{\mathcal{M}}^2\,dt\right)^{1/2} \\
&\leq \left(\int_1^{+\infty} \frac{\|\varphi^n(t)-\varphi(t)\|_{\mathcal{M}}^2}{t^2}\,dt\right)^{1/2} \\
&\leq 2\left(\int_1^{+\infty} \|\dot\varphi^n(t)-\dot\varphi(t)\|_{\mathcal{M}}^2\,dt\right)^{1/2} \\
&= 2\|\varphi^n-\varphi\|_{\mathcal{D}}
\end{aligned}$$

and, since $\|\varphi^n-\varphi\|_{\mathcal{D}} \to 0$ as $n \to +\infty$, this proves our thesis.

2.3.4 Absence of collisions and parabolicity of the motion

We can now proceed using an argument analogous to that of Section 2.2: by the direct method of the calculus of variations, there exists a minimizer $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$ of the renormalized Lagrangian action. The Renormalized Action Principle then implies that φ satisfies

$$\mathcal{M}\ddot\varphi(t) = \nabla U\Big(\beta b_m t^{2/3} + \varphi(t) + x^0 - \beta b_m\Big) - \frac{2}{3}\frac{\beta b_m}{t^{1/3}}.$$

Next, consider

$$\gamma(t) = \varphi(t) + \beta b_m t^{2/3} + \tilde x^0,$$

so that

$$\dot\gamma(t) = \dot\varphi(t) + \frac{2}{3}\beta b_m t^{-1/3}.$$

To show that the motion γ is parabolic, it is sufficient to prove that

$$\lim_{t\to+\infty}\dot\gamma(t) = \lim_{t\to+\infty}\dot\varphi(t) = 0.$$

Since

$$\int_1^{+\infty} |\dot\varphi_{ij}(t)|^2\,dt < +\infty,$$

we have

$$\liminf_{t\to+\infty} |\dot\varphi_{ij}(t)| = 0.$$

By conservation of energy along the motion, it holds that

$$\frac{1}{2}\|\dot\gamma(t)\|_{\mathcal{M}}^2 - U(\gamma(t)) = \frac{1}{2}\sum_{i=1}^N m_i\left|\dot\varphi_i(t) + \frac{2}{3}\beta b_{m_i} t^{-1/3}\right|^2 - U(\gamma(t)) = h.$$

Since there exists a subsequence $(t_k)_k$ with $t_k \to +\infty$ such that $\lim_{t_k \to +\infty} \dot{\varphi}_i(t_k) = 0$, it follows that $h = 0$, and consequently

$$\frac{1}{2}\|\dot{\gamma}(t)\|^2_{\mathcal{M}} - U(\gamma(t)) = 0.$$

From this, we deduce that

$$\lim_{t \to +\infty} \dot{\gamma}(t) = 0.$$

2.3.5 Asymptotic estimates for half parabolic motions

With respect to the main result of Maderna and Venturelli in [50], in [66] a slightly better description of the parabolic motions' asymptotic expansion is provided. More specifically, inequality (2.2.4) is improved by saying that, for any minimizer $\varphi \in \mathcal{D}^{1,2}_0(1,+\infty)$ of the renormalized action in the parabolic case, it holds

$$\|\varphi(t)\|_{\mathcal{M}} \leq c t^{\frac{1}{3}+\varepsilon}, \quad \forall \varepsilon > 0, \tag{2.3.8}$$

for a proper constant $c \in \mathbb{R}$. This section is devoted to the proof of this estimate.

Consider a half parabolic motion $\gamma(t)$ having the form (2.3.2), where $\varphi \in \mathcal{D}^{1,2}_0(1,+\infty)$ is a solution of the equations of motion $\mathcal{M}\ddot{\varphi}(t) = \nabla U(r_0(t) + \varphi(t) + \tilde{x}^0) - \nabla U(r_0(t))$. It holds:

$$\begin{aligned}
\mathcal{M}\ddot{\varphi}(t) &= \frac{1}{\beta^2 t^{4/3}}\left[\nabla U\left(\frac{x(t)}{\beta t^{2/3}}\right) - \nabla U\left(\frac{r_0(t)}{\beta t^{2/3}}\right)\right] \\
&= \frac{1}{\beta^2 t^{4/3}}\left[\nabla U\left(b_m + \frac{\varphi(t)}{\beta t^{2/3}} + \frac{\tilde{x}^0}{\beta t^{2/3}}\right) - \nabla U(b_m)\right] \\
&= \frac{1}{\beta^3 t^2}\int_0^1 \nabla^2 U\left(b_m + \theta\frac{(\varphi(t)+\tilde{x}^0)}{\beta t^{2/3}}\right)(\varphi(t)+\tilde{x}^0)\, d\theta \\
&= \frac{1}{\beta^3 t^2}\left[\int_0^1 \nabla^2 U\left(b_m + \theta\frac{(\varphi(t)+\tilde{x}^0)}{\beta t^{2/3}}\right) d\theta\right](\varphi(t)+\tilde{x}^0),
\end{aligned}$$

where the integral term can be considered as a matrix.

Fixing a real constant $\delta \in (1,2)$ and a sufficiently big constant $k \in \mathbb{R}$, we define a test function $\psi_k : \mathbb{R} \to \mathcal{X}$ as

$$\psi_k(t) = \eta^2 \min\{k, \|\varphi(t)\|^{\delta-1}_{\mathcal{M}}\}\varphi(t)$$

where $\eta : \mathbb{R} \to \mathbb{R}$ is a C^∞-class cut-off function having the form

$$\eta(t) = \begin{cases} 0, & t \in [1,R] \\ 1, & t \in [2R,+\infty) \end{cases},$$

for R big enough, with $0 < \eta(t) < 1$, $\forall t \in (R,2R)$. We point out that k can be chosen such that $\eta \equiv 1$ when $\|\varphi(t)\|_{\mathcal{M}}^{\delta-1} > k$, so that it holds

$$\dot{\psi}_k(t) = \begin{cases} 2\eta\dot{\eta}\|\varphi(t)\|_{\mathcal{M}}^{\delta-1}\varphi(t) + \eta^2\delta\|\varphi(t)\|_{\mathcal{M}}^{\delta-2}\langle\varphi(t),\dot{\varphi}(t)\rangle_{\mathcal{M}}, & t \in I_k, \\ k\dot{\varphi}(t), & t \in \hat{I}_k, \end{cases}$$

given $I_k = \{t \in [1,+\infty) : \|\varphi(t)\|_{\mathcal{M}}^{\delta-1} \le k\}$ and $\hat{I}_k = [1,+\infty) \setminus I_k = \{t \in [1,+\infty) : \|\varphi(t)\|_{\mathcal{M}}^{\delta-1} > k\}$.

Multiplying the equations of motion by $\psi_k(t)$ and integrating on $[R,+\infty)$, we get

$$\begin{aligned}
&\int_R^{+\infty} -\langle\ddot{\varphi}(t),\psi_k(t)\rangle_{\mathcal{M}} + \left\langle\left[\int_0^1 \nabla^2 U\left(b_m + \theta\frac{(\varphi(t)+\tilde{x}^0)}{\beta t^{2/3}}\right)\,d\theta\right]\frac{(\varphi(t)+\tilde{x}^0)}{\beta^3 t^2},\psi_k\right\rangle\,dt \\
&= \int_R^{+\infty} \langle\dot{\varphi}(t),\dot{\psi}_k(t)\rangle_{\mathcal{M}} + \left\langle\left[\int_0^1 \nabla^2 U\left(b_m + \theta\frac{(\varphi(t)+\tilde{x}^0)}{\beta t^{2/3}}\right)\,d\theta\right]\frac{(\varphi(t)+\tilde{x}^0)}{\beta^3 t^2},\psi_k\right\rangle\,dt \\
&= 0.
\end{aligned}$$

Recalling that $\|\nabla^2 U(r_0+\theta(\varphi(t)+\tilde{x}^0))\|_{\mathcal{M}} \le \frac{C_{11}}{t^2}$ for a constant C_{11}, for all $t > 1$ and for all $\theta \in [0,1]$, we can use Hölder's and Hardy's inequalities to estimate

$$\begin{aligned}
&\int_R^{+\infty} \langle\dot{\varphi}(t),\dot{\psi}_k(t)\rangle_{\mathcal{M}} + \left\langle\left[\int_0^1 \nabla^2 U(r_0(t)+\theta(\varphi(t)+\tilde{x}^0))\,d\theta\right]\varphi(t),\psi_k(t)\right\rangle\,dt \\
&= -\int_R^{+\infty} \left\langle\left[\int_0^1 \nabla^2 U(r_0(t)+\theta(\varphi(t)+\tilde{x}^0))\,d\theta\right]\tilde{x}^0,\psi_k(t)\right\rangle\,dt \\
&\le C_{11}\int_R^{+\infty} \frac{\|\psi_k(t)\|_{\mathcal{M}}}{t^2}\,dt \\
&\le C_{11}\int_R^{+\infty} \frac{\|\varphi(t)\|_{\mathcal{M}}^{\delta}}{t^2}\,dt \\
&= C_{11}\int_R^{+\infty} \frac{1}{t^{2-\delta}}\frac{\|\varphi(t)\|_{\mathcal{M}}^{\delta}}{t^{\delta}}\,dt \\
&\le C_{11}\left(\int_R^{+\infty}\frac{1}{t^2}\,dt\right)^{(2-\delta)/2}\left(\int_R^{+\infty}\frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^2}\,dt\right)^{\delta/2} \\
&\le C_{12}\|\varphi\|_{\mathcal{D}}^{\delta},
\end{aligned}$$

where C_{12} is a proper constant.

Remark 2.3.6. We recall that, by the (-1)-homogeneity of the Keplerian potential $U(x) = \frac{1}{\|x\|}$,

$$U(r_0(t)) = U\left(\|r_0(t)\|_{\mathcal{M}}\frac{r_0(t)}{\|r_0(t)\|_{\mathcal{M}}}\right) = \frac{U(b_m)}{\|r_0(t)\|_{\mathcal{M}}}.$$

The Hessian matrix of $U(r_0(t))$ can then be written as

$$\nabla^2 U(r_0(t)) = -\frac{U(b_m)\mathcal{M}}{\|r_0(t)\|_{\mathcal{M}}^3} + 3\frac{U(b_m)}{\|r_0(t)\|_{\mathcal{M}}^5}\mathcal{M}r_0(t)\otimes\mathcal{M}r_0(t)$$
$$- 2\frac{\nabla_{b_m}U(b_m)\otimes\mathcal{M}r_0(t)}{\|r_0(t)\|_{\mathcal{M}}^4} + \frac{\nabla_{b_m}^2 U(b_m)}{\|r_0(t)\|_{\mathcal{M}}^3},$$

where $x\otimes x$ denotes the symmetric square matrix with components $(x\otimes x)_{ij} = r_i r_j$ for $i,j \in 1,\dots,N$, and $\nabla_{b_m}U(b_m)$ and $\nabla_{b_m}^2 U(b_m)$ represent the gradient and the Hessian matrix of U with respect to b_m, respectively. Since b_m is the minimum of the restricted potential, we have $\frac{\nabla_{b_m}U(b_m)\otimes\mathcal{M}r_0(t)}{\|r_0(t)\|_{\mathcal{M}}^4} = 0$. Besides, since $\mathcal{M}r_0(t)\otimes\mathcal{M}r_0(t)$ and $\nabla_{b_m}^2 U(b_m)$ are positive semidefinite quadratic forms, it holds

$$\langle \nabla^2 U(r_0(t))\varphi(t), \psi(t)\rangle \geq -\frac{U(b_m)\|\varphi(t)\|_{\mathcal{M}}\|\psi(t)\|_{\mathcal{M}}}{\|r_0(t)\|_{\mathcal{M}}^3}, \tag{2.3.9}$$

for $\varphi, \psi \in \mathcal{D}_0^{1,2}(1,+\infty)$.

Using a continuity argument and applying (2.3.9), we can also say that, for every $\mu > 0$, there is a $\bar{T} > 0$ such that for every $t > \bar{T}$,

$$\frac{1}{\beta^3 t^2}\nabla^2 U\left(b_m + \theta\frac{(\varphi(t)+\tilde{x}^0)}{\beta t^{2/3}}\right) \geq -\frac{2}{9}(1+\mu)\frac{\mathcal{M}}{t^2}$$

in the sense of quadratic forms. It follows

$$\int_R^{+\infty} \langle\dot\varphi(t), \dot\psi_k(t)\rangle_{\mathcal{M}} + \left\langle \frac{1}{\beta^3 t^2}\left[\int_0^1 \nabla^2 U\left(b_m + \theta\frac{(\varphi(t)+\tilde{x}^0)}{\beta t^{2/3}}\right)\, d\theta\right]\varphi(t), \psi_k(t)\right\rangle dt$$
$$\geq \int_R^{+\infty} \langle\dot\varphi(t), \dot\psi_k(t)\rangle_{\mathcal{M}} - \frac{2}{9}(1+\mu)\left\langle\frac{\varphi(t)}{t^2}, \psi_k(t)\right\rangle_{\mathcal{M}} dt.$$

To estimate the right-hand side of the last inequality, we study the integral separately over the two complementary sets I_k and $\hat{I}_k$. On I_k, it holds

$$\int_{I_k} \langle\dot\varphi(t), \dot\psi_k(t)\rangle_{\mathcal{M}} - \frac{2}{9}(1+\mu)\left\langle\frac{\varphi(t)}{t^2}, \psi_k(t)\right\rangle_{\mathcal{M}} dt$$
$$= \int_{I_k} 2\eta\dot\eta\|\varphi(t)\|_{\mathcal{M}}^{\delta-1}\langle\dot\varphi(t), \varphi(t)\rangle_{\mathcal{M}} + \eta^2\delta\|\varphi(t)\|_{\mathcal{M}}^{\delta-1}\|\dot\varphi(t)\|_{\mathcal{M}}$$
$$- \frac{2}{9}(1+\mu)\eta^2\frac{\|\varphi(t)\|_{\mathcal{M}}^{\delta+1}}{t^2}\, dt,$$

which implies

$$\begin{aligned}
&\int_{I_k} \eta^2\delta\|\varphi(t)\|_{\mathcal{M}}^{\delta-1}\|\dot\varphi(t)\|_{\mathcal{M}} - \frac{2}{9}(1+\mu)\eta^2\frac{\|\varphi(t)\|_{\mathcal{M}}^{\delta+1}}{t^2}\,dt\\
&\leq \int_{I_k} 2\eta\dot\eta\|\varphi(t)\|_{\mathcal{M}}^{\delta}\|\dot\varphi(t)\|_{\mathcal{M}}\,dt + C_{12}\|\varphi\|_{\mathcal{D}}^{\delta}.
\end{aligned}$$

Notice that the presence of the cut-off function ensures that the last integral is finite. In addition, it holds

$$\begin{aligned}
&\int_{I_k} \eta^2\delta\|\varphi(t)\|_{\mathcal{M}}^{\delta-1}\|\dot\varphi(t)\|_{\mathcal{M}} - \frac{2}{9}(1+\mu)\eta^2\frac{\|\varphi(t)\|_{\mathcal{M}}^{\delta+1}}{t^2}\,dt\\
&= \int_{I_k} \frac{4\delta}{(\delta+1)^2}\left(\eta\frac{d}{dt}\|\varphi(t)\|_{\mathcal{M}}^{\frac{\delta+1}{2}}\right)^2 - \frac{2}{9}(1+\mu)\eta^2\frac{\|\varphi(t)\|_{\mathcal{M}}^{\delta+1}}{t^2}\,dt.
\end{aligned}$$

On the other hand, working on the interval $\hat I_k$, it holds

$$\begin{aligned}
&\int_{\hat I_k} \langle\dot\varphi(t),\dot\psi_k(t)\rangle_{\mathcal{M}} - \frac{2}{9}(1+\mu)\left\langle\frac{\varphi(t)}{t^2},\psi_k(t)\right\rangle_{\mathcal{M}}\,dt\\
&= \int_{\hat I_k} k\|\dot\varphi(t)\|_{\mathcal{M}}^2 - \frac{2}{9}(1+\mu)k\frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^2}\,dt\\
&\geq \int_{\hat I_k} \frac{4\delta}{(\delta+1)^2}k\|\dot\varphi(t)\|_{\mathcal{M}}^2 - \frac{2}{9}(1+\mu)k\frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^2}\,dt,
\end{aligned}$$

where we used the fact that $\frac{4\delta}{(\delta+1)^2} < 1$ for every $\delta \in (1,2)$.

Now, we define a function $u_k : \mathbb{R} \to \mathbb{R}$ as

$$u_k(t) = \min\{\eta\|\varphi(t)\|_{\mathcal{M}}^{\frac{\delta-1}{2}}, k^{1/2}\}\|\varphi(t)\|_{\mathcal{M}}.$$

Putting everything together and using Hardy inequality, we get

$$\begin{aligned}
&\int_{I_k} \frac{4\delta}{(\delta+1)^2}\left(\eta\frac{d}{dt}\|\varphi(t)\|_{\mathcal{M}}^{\frac{\delta+1}{2}}\right)^2 - \frac{2}{9}(1+\mu)\eta^2\frac{\|\varphi(t)\|_{\mathcal{M}}^{\delta+1}}{t^2}\,dt\\
&+ \int_{\hat I_k} \frac{4\delta}{(\delta+1)^2}k\|\dot\varphi(t)\|_{\mathcal{M}}^2 - \frac{2}{9}(1+\mu)k\frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^2}\,dt\\
&= \int_1^{+\infty} \frac{4\delta}{(\delta+1)^2}\|\dot u_k(t)\|_{\mathcal{M}}^2 - \frac{2}{9}(1+\mu)\frac{\|u_k(t)\|_{\mathcal{M}}^2}{t^2}\,dt\\
&\geq \int_1^{+\infty} \left(\frac{4\delta}{(\delta+1)^2} - \frac{8}{9}(1+\mu)\right)\|\dot u_k(t)\|_{\mathcal{M}}^2\,dt.
\end{aligned}$$

In particular, we can choose μ such that $\frac{4\delta}{(\delta+1)^2} - \frac{8}{9}(1+\mu) > 0$, which proves that

$u_k \in \mathcal{D}_0^{1,2}(1,+\infty)$.

Since the estimates we obtained do not depend on k, we can take $k \to +\infty$ so that (2.2.4) leads us to the conclusion of our proof. We have thus shown that for any $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$ solution of the N-body problem in a parabolic setting and for any $\delta \in (1,2)$ there is a constant $c = c(\delta, \|\varphi\|_{\mathcal{D}})$ such that

$$\|\varphi(t)\|_{\mathcal{M}} \leq ct^{\frac{1}{\delta+1}}, \qquad \forall t \geq 1.$$

2.4 Existence of minimal half hyperbolic-parabolic motions

In order to prove the existence of hyperbolic-parabolic solutions with prescribed limit shape, which is the object of Theorem 2.1.7, we must take into account that the limit shape a exhibits collisions. This means that there exist $i \neq j$ such that $a_i = a_j$, which introduces additional difficulties in studying the coercivity of the action. In addition to determining an appropriate renormalization term for the action, the key idea in [66] to address this case is to introduce a *cluster partition* of the N bodies, which allows to decompose the renormalized Lagrangian action. In this way, the minimization of the action in $\mathcal{D}_0^{1,2}(1,+\infty)$ can be carried out in an analogous manner to the hyperbolic and parabolic cases.

Definition 2.4.1. Given a configuration $a \in \mathcal{X}$ and a motion $\gamma(t) = at + O(t^{2/3})$ as $t \to +\infty$, its corresponding natural partition (a-partition) of the index set $\mathcal{N} = \{1,...,N\}$ is the one for which $i,j \in \mathcal{N}$ belong to the same class if and only if the mutual distance $|r_i(t) - r_j(t)|$ grows as $O(t^{2/3})$. Equivalently, if $a = (a_1,...,a_N)$, then the natural partition is defined by the relation $i \sim j$ if and only if $a_i = a_j$. The partition classes will be called clusters.

We give now some definitions and basic notations related to a given partition $\mathcal{P}$ of the set of indexes $\mathcal{N} = \{1,...,N\}$.

Definition 2.4.2. Let $\mathcal{P}$ be a given partition of $\mathcal{N}$ and consider a configuration $x = (r_1,...,r_N) \in \mathcal{X}$. For each cluster $K \in \mathcal{P}$ we define the mass of the cluster as

$$M_K = \sum_{i \in K} m_i.$$

Definition 2.4.3. Let $\mathcal{P}$ be any given partition of $\mathcal{N}$. Then, for every given curve $x(t) = (r_1(t),...,r_N(t))$ in $\mathcal{X}$ and for each cluster $K \in \mathcal{P}$, we define the cluster potential

$$U_K(t) = \sum_{i,j \in K,\ i<j} \frac{m_i m_j}{|r_i(t) - r_j(t)|}.$$

which represents the restriction of the potential U to the cluster K.

Fixing the initial configuration x^0 and the limit shape $a \in \Delta$, we study the system

$$\begin{cases} \mathcal{M}\ddot{\gamma} = \nabla U(\gamma) \\ \gamma(1) = x^0 \\ \lim_{t\to+\infty} \dot{\gamma}(t) = a. \end{cases}$$

Since we are seeking solutions of the form $\gamma(t) = r_0(t) + \varphi(t) + x^0 - r_0(1)$, where $\varphi \in \mathcal{D}_0^{1,2}(1, +\infty)$, our problem equivalently reads

$$\begin{cases} \mathcal{M}\ddot{\varphi}(t) = \nabla U(r_0(t) + \varphi(t) + x^0 - r_0(1)) - \mathcal{M}\ddot{r}_0(t) \\ \varphi(1) = 0 \\ \lim_{t\to+\infty} \dot{\varphi}(t) = 0. \end{cases}$$

As in the previous cases, our goal is to exploit the Renormalized Action Principle in order to establish the existence of solutions to the system.

By grouping the indices according to the natural a-cluster decomposition, we obtain a partition of $\mathcal{N}$ of the form

$$K_1 := \{1, \dots, k_1\}, \quad K_2 := \{k_1 + 1, \dots, k_2\}, \quad K_3 := \{k_2 + 1, \dots, k_3\}, \ \dots$$

For each cluster K_i, we select a central configuration b^{K_i} that is minimal for the corresponding cluster, and we define the configuration

$$b = (b^{K_1}, b^{K_2}, \dots) \in \mathcal{X}.$$

Once b is fixed, we search for solutions of the form

$$\gamma(t) = at + \beta b\, t^{2/3} + \varphi(t) + \tilde{x}^0,$$

where $\tilde{x}^0 = x^0 - a - b$. Here β is a real vector whose number of components coincides with the number of clusters. More precisely,

$$\beta = (\beta_{K_1}, \beta_{K_2}, \dots),$$

with

$$\beta_{K_i} = \sqrt[3]{\frac{9}{2} U_{\min}^{K_i}}, \qquad i = 1,2, \dots,$$

where $U_{\min}^{K_i}$ denotes the minimum value of the potential U restricted to the i-th cluster. So, in this case, the guiding curve of the expansive motion we want to study, is defined as

$$r_0(t) = at + \beta b t^{2/3}.$$

With an abuse of notation, we denote $\beta b = (\beta_{K_1} b^{K_1}, \beta_{K_2} b^{K_2}, ...) \in \mathcal{X}$.

To apply the Renormalized Action Principle, we need to prove the existence of minimizers of the renormalized Lagrangian action

$$\mathcal{A}(\varphi) = \int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 + U(r_0(t) + \varphi(t) + \tilde{x}^0) - U(r_0(t)) - \langle \ddot{r}_0(t), \varphi(t)\rangle_{\mathcal{M}}\, dt.$$

In truth, the term $U(r_0)$ will be slightly reworded in order to avoid possible collisions and to facilitate the computations in the proof of coercivity and weak-lower semicontinuity of the functional (of course, this will not change the associated Euler-Lagrange equations).

To establish the existence of minimizers for the renormalized Lagrangian action, we employ the cluster partition of the bodies introduced above. The key idea is that the renormalized action can be expressed as the sum of two contributions: the first term accounts for the motion of the bodies within each cluster, while the second term captures the interactions between pairs of bodies belonging to different clusters. Following [16], we observe, for instance, that the Newtonian potential of $x = (r_1, \ldots, r_N)$ can be decomposed as

$$U(\gamma) = \sum_{K \in \mathcal{P}} \left(\sum_{i,j \in K,\ i<j} \frac{m_i m_j}{|r_i - r_j|} \right) + \frac{1}{2} \sum_{K_1, K_2 \in \mathcal{P},\ K_1 \neq K_2} \left(\sum_{i \in K_1,\ j \in K_2} \frac{m_i m_j}{|r_i - r_j|} \right).$$

For every $\varphi \in \mathcal{D}_0^{1,2}(1, +\infty)$, we can thus write the renormalized Lagrangian action as

$$\begin{aligned} \mathcal{A}(\varphi) &= \sum_{K \in \mathcal{P}} \mathcal{A}_K(\varphi) + \sum_{K_1, K_2 \in \mathcal{P},\ K_1 \neq K_2} \mathcal{A}_{K_1, K_2}(\varphi) \\ &= \sum_{K \in \mathcal{P}} \left(\sum_{i,j \in K,\ i<j} \mathcal{A}_K^{ij}(\varphi) \right) + \frac{1}{2} \sum_{K_1, K_2 \in \mathcal{P},\ K_1 \neq K_2} \left(\sum_{i \in K_1,\ j \in K_2} \mathcal{A}_{K_1, K_2}^{ij}(\varphi) \right), \end{aligned} \tag{2.4.1}$$

where

$$\begin{aligned} \mathcal{A}_K^{ij}(\varphi) = \int_1^{+\infty} & \frac{m_i m_j}{2M} |\dot{\varphi}_{ij}(t)|^2 + \frac{m_i m_j}{|\beta_K b_{ij}^K t^{2/3} + \varphi_{ij}(t) + \tilde{x}_{ij}^0|} - \frac{m_i m_j}{|\beta_K b_{ij}^K t^{2/3}|} \\ & + \frac{2}{9} \frac{\beta_K}{M} m_i m_j \frac{\langle b_{ij}^K, \varphi_{ij}(t)\rangle}{t^{4/3}}\, dt, \end{aligned} \tag{2.4.2}$$

$$\mathcal{A}_{K_1,K_2}^{ij}(\varphi) = \int_1^{+\infty} \frac{m_i m_j}{2M} |\dot{\varphi}_{ij}(t)|^2 + \frac{m_i m_j}{|a_{ij} t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \varphi_{ij}(t) + \tilde{x}_{ij}^0|} - \frac{m_i m_j}{|a_{ij} t|}\, dt. \tag{2.4.3}$$

Here, we use the notations:

$$b^{K_{1,2}} = (b^{K_1}, b^{K_2}),$$
$$\beta_{K_{1,2}} b^{K_{1,2}} = (\beta_{K_1} b^{K_1}, \beta_{K_2} b^{K_2}).$$

Remark 2.4.4. We point out that, in the decomposition above, we made a small change in the renormalization of the Lagrangian action functional. Indeed, if we used $-U(r_0(t))$ as the renormalization term, like we did for the hyperbolic and parabolic case, the cluster decomposition would require us to write this term as

$$-U(at + \beta b t^{2/3}) = -\sum_{K\in\mathcal{P}} \left(\sum_{i,j\in K,\ i<j} \frac{m_i m_j}{|\beta_K b_{ij}^K t^{2/3}|} \right) - \frac{1}{2} \sum_{K_1,K_2\in\mathcal{P},\ K_1\neq K_2} \left(\sum_{i\in K_1,\ j\in K_2} \frac{m_i m_j}{|a_{ij} t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3}|} \right).$$

However, we notice that, for small values of t, it may happen that $a_{ij}t+\beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3}$ vanishes for some indexes $i \in K_1$, $j \in K_2$, with $K_1, K_2 \in \mathcal{P}$, $K_1 \neq K_2$. To avoid this small issue, and also to simplify our computations even more, we use a slight variation of our usual renormalization term, which is given by

$$-\tilde{U}(r_0(t)) = -\sum_{K\in\mathcal{P}} \left(\sum_{i,j\in K,\ i<j} \frac{m_i m_j}{|\beta_K b_{ij}^K t^{2/3}|} \right) - \frac{1}{2} \sum_{K_1,K_2\in\mathcal{P},\ K_1\neq K_2} \left(\sum_{i\in K_1,\ j\in K_2} \frac{m_i m_j}{|a_{ij} t|} \right).$$

It is easy to prove that $\int_1^{+\infty} \tilde{U}(r_0(t)) - U(r_0(t))\ dt < +\infty$ and, since $-\tilde{U}(r_0(t))$ is still a fixed function of t, the minimization of the corresponding renormalized Lagrangian action will still lead us to a solution of Newton's equations.

We also point out that the term (2.4.2) refers to the (parabolic) motion of the bodies inside each cluster, while the term (2.4.3) refers to the interactions between pairs of bodies belonging to different clusters. In the following sections, we will work on the two terms separately, in order to apply the direct method of the calculus of variations and, consequently, the Renormalized Action Principle.

2.4.1 Coercivity

First, we prove the coercivity of the action when restricted to a general cluster K. In this case, since, by the natural a-cluster partition of the bodies, $a_i = a_j$ for all $i, j \in K$, the functional reads, for all $\varphi \in \mathcal{D}_0^{1,2}(1, +\infty)$,

$$\mathcal{A}_K(\varphi) = \sum_{i,j\in K,\ i<j} \int_1^{+\infty} \frac{1}{2M} m_i m_j |\dot{\varphi}_{ij}(t)|^2 + \frac{m_i m_j}{|\beta_K b_{ij}^K t^{2/3} + \varphi_{ij}(t) + \tilde{x}_{ij}^0|} - \frac{m_i m_j}{|\beta_K b_{ij}^K t^{2/3}|} + \frac{2}{9}\frac{\beta_K}{M} m_i m_j \frac{\langle b_{ij}^K, \varphi_{ij}(t)\rangle}{t^{4/3}}\ dt.$$

By the homogeneity of the potential, and denoting by U_K the restriction of the potential U to the cluster K, the inequality

$$U_K(\gamma) \geq \frac{U_K(b^K)}{\|x\|_{\mathcal{M}}} = \frac{U_{min}}{\|x\|_{\mathcal{M}}}$$

holds for any motion γ restricted to the cluster K. Then,

$$\mathcal{A}_K(\varphi) \geq \int_1^{+\infty} \sum_{i,j\in K,\ i<j} \left(\frac{1}{2M} m_i m_j |\dot{\varphi}_{ij}(t)|^2\right) + \frac{U_{min}}{\|\beta_K b^K t^{2/3} + \varphi(t) + \tilde{x}^0\|_{\mathcal{M}}} - \frac{U_{min}}{\|\beta_K b^K t^{2/3}\|_{\mathcal{M}}} + \frac{2}{9}\frac{\beta_K}{M}\langle \mathcal{M}_K b^K, \varphi(t)\rangle\ dt,$$

where we denote $\mathcal{M}_K = \sum_{i\in K} m_i$. By the inequality

$$\frac{1}{\|\varphi(t) + \beta_K b^K t^{2/3} + \tilde{x}^0\|_{\mathcal{M}}} \geq \frac{1}{\|\beta_K b^K t^{2/3}\|_{\mathcal{M}}} - \frac{1}{2\|\beta_K b^K\|_{\mathcal{M}}^3 t^2}(2t^{2/3}\beta_K\langle \mathcal{M}_K b^K, \varphi(t)\rangle + 2\langle \mathcal{M}_K \varphi(t), \tilde{x}^0\rangle + 2t^{2/3}\beta_K\langle \mathcal{M}_K b^K, \tilde{x}^0\rangle + \|\varphi(t)\|_{\mathcal{M}}^2 + \|\tilde{x}^0\|_{\mathcal{M}}^2),$$

which holds because of the convexity of the norm, we have

$$\begin{aligned}\mathcal{A}_K(\varphi) \geq &\int_1^{+\infty} \sum_{i,j\in K,\ i<j} \frac{1}{2M} m_i m_j |\dot{\varphi}_{ij}(t)|^2 + \frac{2}{9}\frac{\beta_K}{M}\langle \mathcal{M}_K b^K, \varphi(t)\rangle \\ &- \frac{U_{min}}{2\beta_K^3\|b^K\|_{\mathcal{M}}^3 t^2}(2t^{2/3}\beta_K\langle \mathcal{M}_K b^K, \varphi(t)\rangle + 2\langle \mathcal{M}_K \varphi(t), \tilde{x}^0\rangle \\ &+ 2t^{2/3}\beta_K\langle \mathcal{M}_K b^K, \tilde{x}^0\rangle + \|\varphi(t)\|_{\mathcal{M}}^2 + \|\tilde{x}^0\|_{\mathcal{M}}^2)\ dt \\ = &\int_1^{+\infty} \frac{1}{2}\|\dot{\varphi}(t)\|_{\mathcal{M}}^2 - \frac{U_{min}}{2\beta_K^3\|b^K\|_{\mathcal{M}}^3 t^2}(2\langle \mathcal{M}_K \varphi(t), \tilde{x}^0\rangle \\ &+ 2t^{2/3}\beta_K\langle \mathcal{M}_K b^K, \tilde{x}^0\rangle + \|\varphi(t)\|_{\mathcal{M}}^2 + \|\tilde{x}^0\|_{\mathcal{M}}^2)\ dt.\end{aligned}$$

We notice that the constant

$$C_{13} := \int_1^{+\infty} -\frac{U_{min}}{2\beta_K^3\|b^K\|_{\mathcal{M}}^3 t^2}(2t^{2/3}\beta_K\langle \mathcal{M}_K b^K, \tilde{x}^0\rangle + \|\tilde{x}^0\|_{\mathcal{M}}^2)\ dt < +\infty.$$

By Hardy and Cauchy-Schwartz inequalities we also have

$$\begin{aligned}
&-\int_1^{+\infty} \frac{U_{min}}{\beta_K^3 \|b^K\|_{\mathcal{M}}^3 t^2} \langle \mathcal{M}_K \varphi(t), \tilde{x}^0 \rangle \, dt \\
&= -\frac{2}{9} \int_1^{+\infty} \frac{1}{t^2} \langle \mathcal{M}_K \varphi(t), \tilde{x}^0 \rangle \, dt \\
&\geq -\frac{2}{9} \left(\int_1^{+\infty} \frac{\|\varphi(t)\|_{\mathcal{M}}^2}{t^2} \, dt \right)^{1/2} \left(\int_1^{+\infty} \frac{\|\tilde{x}^0\|_{\mathcal{M}}^2}{t^2} \, dt \right)^{1/2} \\
&\geq -C_{14} \|\varphi\|_{\mathcal{D}},
\end{aligned}$$

where $C_{14} := \frac{8}{9}\Big(\int_1^{+\infty} \frac{\|\tilde{x}^0\|_{\mathcal{M}}^2}{t^2} \, dt \Big)^{1/2}$ is constant. Again, by Hardy inequality, it holds

$$\mathcal{A}_K(\varphi) \geq \frac{1}{18} \|\varphi\|_{\mathcal{D}}^2 - C_{14} \|\varphi\|_{\mathcal{D}} + C_{13},$$

which implies that $\mathcal{A}_K$ is coercive.

Now, we focus on studying the interaction terms

$$\begin{aligned}
\mathcal{A}_{K_1,K_2}(\varphi) = \sum_{i \in K_1,\ j \in K_2} \int_1^{+\infty} &\frac{1}{2M} m_i m_j |\dot{\varphi}_{ij}(t)|^2 + \frac{m_i m_j}{|a_{ij} t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \varphi_{ij}(t) + \tilde{x}_{ij}^0|} \\
&- \frac{m_i m_j}{|a_{ij} t|} \, dt.
\end{aligned}$$

Remark 2.4.5. If two bodies of the configuration $b^{K_{1,2}}$ belong to different clusters and have collisions, that is, if there are $i \in K_1$ and $j \in K_2$ such that $b_i^{K_{1,2}} = b_j^{K_{1,2}}$, then the functional reads

$$\mathcal{A}_{K_1,K_2}(\varphi) = \sum_{i \in K_1,\ j \in K_2} \int_1^{+\infty} \frac{1}{2M} m_i m_j |\dot{\varphi}_{ij}(t)|^2 + \frac{m_i m_j}{|a_{ij} t + \varphi_{ij}(t) + \tilde{x}_{ij}^0|} - \frac{m_i m_j}{|a_{ij} t|} \, dt.$$

Since $a_i \neq a_j$ when $i \in K_1$, $j \in K_2$ and $K_1 \neq K_2$, we have already proved the coercivity of such action functional in this case.

Assuming $b^{K_{1,2}}$ without collisions, we proceed as follows. By the triangular inequality, it holds

$$\begin{aligned}
&\int_1^{+\infty} \frac{1}{|a_{ij} t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \varphi_{ij}(t) + \tilde{x}_{ij}^0|} - \frac{1}{|a_{ij} t|} \, dt \\
&\geq \int_1^{+\infty} \frac{1}{|a_{ij}| t + \beta_{K_{1,2}} |b_{ij}^{K_{1,2}}| t^{2/3} + \|\varphi_{ij}\|_{\mathcal{D}} t^{1/2} + |\tilde{x}_{ij}^0|} - \frac{1}{|a_{ij}| t} \, dt.
\end{aligned}$$

Using the change of variables $s = \|\varphi\|_{\mathcal{D}} u$, we obtain

$$\begin{aligned}
&\int_1^{+\infty} \frac{1}{|a_{ij}|t + \beta_{K_{1,2}}|b_{ij}^{K_{1,2}}|t^{2/3} + \|\varphi_{ij}\|_{\mathcal{D}} t^{1/2} + |\tilde{x}_{ij}^0|} - \frac{1}{|a_{ij}|t}\, dt \\
&= 2\int_1^{+\infty} \left(\frac{1}{|a_{ij}|s^2 + \beta_{K_{1,2}}|b_{ij}^{K_{1,2}}|s^{4/3} + \|\varphi_{ij}\|_{\mathcal{D}} s + |\tilde{x}_{ij}^0|} - \frac{1}{|a_{ij}|s^2} \right) s\, ds \\
&= \frac{2}{\|\varphi\|_{\mathcal{D}}|a_{ij}|} \int_1^{+\infty} \left(\frac{1}{s^2 + \frac{\beta_{K_{1,2}}|b_{ij}^{K_{1,2}}|}{|a_{ij}|} \frac{s^{4/3}}{\|\varphi\|_{\mathcal{D}}^{2/3}} + \frac{s}{|a_{ij}|\|\varphi\|_{\mathcal{D}}} + \frac{|\tilde{x}_{ij}^0|}{|a_{ij}|\|\varphi\|_{\mathcal{D}}}} - \frac{1}{\frac{s^2}{\|\varphi\|_{\mathcal{D}}^2}} \right) s\, ds \\
&= \frac{2}{|a_{ij}|} \int_{1/\|\varphi\|_{\mathcal{D}}}^{+\infty} \left(\frac{1}{u^2 + \frac{\beta_{K_{1,2}}|b_{ij}^{K_{1,2}}|}{|a_{ij}|} \frac{u^{4/3}}{\|\varphi\|_{\mathcal{D}}^{2/3}} + \frac{u}{|a_{ij}|} + \frac{|\tilde{x}_{ij}^0|}{|a_{ij}|\|\varphi\|_{\mathcal{D}}}} - \frac{1}{u^2} \right) u\, du.
\end{aligned}$$

We notice that, for $\|\varphi\|_{\mathcal{D}} \to +\infty$, it holds $\frac{\beta_{K_{1,2}}|b_{ij}^{K_{1,2}}|}{|a_{ij}|\|\varphi\|_{\mathcal{D}}^{2/3}} \leq 1$ and $\frac{|\tilde{x}_{ij}^0|}{|a_{ij}|\|\varphi\|_{\mathcal{D}}} \leq 1$. So, for $\|\varphi\|_{\mathcal{D}} \to +\infty$, it follows

$$\begin{aligned}
&\frac{2}{|a_{ij}|} \int_{1/\|\varphi\|_{\mathcal{D}}}^{+\infty} \left(\frac{1}{u^2 + \frac{\beta_{K_{1,2}}|b_{ij}^{K_{1,2}}|}{|a_{ij}|} \frac{u^{4/3}}{\|\varphi\|_{\mathcal{D}}^{2/3}} + \frac{u}{|a_{ij}|} + \frac{|\tilde{x}_{ij}^0|}{|a_{ij}|\|\varphi\|_{\mathcal{D}}}} - \frac{1}{u^2} \right) u\, du \\
&\geq \frac{2}{|a_{ij}|} \int_{1/\|\varphi\|_{\mathcal{D}}}^{+\infty} \left(\frac{1}{u^2 + u^{4/3} + \frac{u}{|a_{ij}|} + 1} - \frac{1}{u^2} \right) u\, du \\
&= \frac{2}{|a_{ij}|} \int_{1/\|\varphi\|_{\mathcal{D}}}^{+\infty} \frac{1}{u} \left(\frac{1}{1 + u^{-2/3} + \frac{u^{-1}}{|a_{ij}|} + u^{-1}} - 1 \right) du.
\end{aligned}$$

Since $1/\|\varphi\|_{\mathcal{D}} \leq 1$ for $\|\varphi\|_{\mathcal{D}}$ sufficiently large, we can study the integral separately on the intervals $[1/\|\varphi\|_{\mathcal{D}}, 1]$ and $[1, +\infty)$. On the second interval, the integral is constant (let us say that it is equal to a constant C_{15}). On the other interval, we have

$$\frac{2}{|a_{ij}|} \int_{1/\|\varphi\|_{\mathcal{D}}}^{1} \frac{1}{u} \left(\frac{1}{1 + u^{-2/3} + \frac{u^{-1}}{|a_{ij}|} + u^{-1}} - 1 \right) du \geq -\frac{2}{|a_{ij}|} \int_{1/\|\varphi\|_{\mathcal{D}}}^{1} \frac{du}{u}.$$

We have thus proved that

$$\int_1^{+\infty} \frac{1}{|a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \varphi_{ij}(t) + \tilde{x}_{ij}^0|} - \frac{1}{|a_{ij}t|}\, dt$$
$$\geq \frac{2}{|a_{ij}|} \log \frac{1}{\|\varphi\|_{\mathcal{D}}} + C_{15}$$
$$= -\frac{2}{|a_{ij}|} \log \|\varphi\|_{\mathcal{D}} + C_{15},$$

which concludes the proof of the coercivity of the Lagrangian action.

2.4.2 Weak-lower semicontinuity

As in the previous section, we employ the cluster decomposition (2.4.1) to establish the weak-lower semicontinuity of the Lagrangian action, analyzing separately the terms $\mathcal{A}_K$ and $\mathcal{A}_{K_1,K_2}$ for fixed, arbitrary clusters $K, K_1, K_2 \in \mathcal{P}$.

Regarding the term $\mathcal{A}_K$, we can refer to Section 2.3, since our choice of $\beta_K b^K$ leads to the same computations.

To prove the weak-lower semicontinuity of the interaction terms $\mathcal{A}_{K_1,K_2}$, consider a sequence $(\varphi^n)_n \subset \mathcal{D}_0^{1,2}(1,+\infty)$ that converges weakly in $\mathcal{D}_0^{1,2}(1,+\infty)$ to some φ as $n \to +\infty$. It then follows that there exists a constant $k \in \mathbb{R}$ such that $\|\varphi^n\|_{\mathcal{D}} \leq k$ and $\|\varphi\|_{\mathcal{D}} \leq k$ for every $n \in \mathbb{N}$.

The inequality

$$\begin{aligned} &\frac{1}{|a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \varphi_{ij}^n(t) + \tilde{x}_{ij}^0|} - \frac{1}{|a_{ij}t|} \\ &\quad = \int_0^1 \frac{d}{ds}\left[\frac{1}{|a_{ij}t + s(\beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \varphi_{ij}^n(t) + \tilde{x}_{ij}^0)|}\right] ds, \end{aligned} \tag{2.4.4}$$

holds true when the denominator of the integrand is not zero. For all $s \in (0,1)$ it holds

$$\begin{aligned} &|a_{ij}t + s(\beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \varphi_{ij}^n(t) + \tilde{x}_{ij}^0)| \\ &\quad \geq |a_{ij}|t - s(|\beta_{K_{1,2}} b_{ij}^{K_{1,2}}| t^{2/3} + \|\varphi_{ij}^n\|_{\mathcal{D}} t^{1/2} + |\tilde{x}_{ij}^0|) \\ &\quad > |a_{ij}|t - (|\beta_{K_{1,2}} b_{ij}^{K_{1,2}}| t^{2/3} + \|\varphi_{ij}^n\|_{\mathcal{D}} t^{1/2} + |\tilde{x}_{ij}^0|), \end{aligned}$$

and, since $\|\varphi_{ij}^n\|_{\mathcal{D}} \leq k$, it holds

$$|a_{ij}t + s(\beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \varphi_{ij}^n(t) + \tilde{x}_{ij}^0)| > |a_{ij}|t - (|\beta_{K_{1,2}} b_{ij}^{K_{1,2}}| t^{2/3} + k t^{1/2} + |\tilde{x}_{ij}^0|),$$

where the last term is larger than zero if $t \geq \bar{T} = \bar{T}(k)$, for a proper $\bar{T}$. Thus, we can study the weak-lower semicontinuity of the potential term separately on the two intervals $[1,\bar{T}]$ and $[\bar{T},+\infty)$.

On $[1,\bar{T}]$, the weak-lower semicontinuity easily follows from Fatou's Lemma. On $[\bar{T},+\infty)$, by (2.4.4) we have

$$\frac{1}{|a_{ij}t+\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}+\varphi_{ij}^{n}(t)+\tilde{x}_{ij}^{0}|}-\frac{1}{|a_{ij}t|}$$
$$=\int_0^1 -\frac{[a_{ij}t+s(\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}+\varphi_{ij}^{n}(t)+\tilde{x}_{ij}^{0})](\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}+\varphi_{ij}^{n}(t)+\tilde{x}_{ij}^{0})}{|a_{ij}t+s(\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}+\varphi_{ij}^{n}(t)+\tilde{x}_{ij}^{0})|^3}\,ds.$$

By (2.2.9), it follows

$$\begin{aligned}
&\int_{\bar{T}}^{+\infty}\left|\frac{1}{|a_{ij}t+\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}+\varphi_{ij}^{n}(t)+\tilde{x}_{ij}^{0}|}-\frac{1}{|a_{ij}t|}\right|dt\\
&\leq\int_{\bar{T}}^{+\infty}\left(\int_0^1\frac{|\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}+\varphi_{ij}^{n}(t)+\tilde{x}_{ij}^{0}|}{|a_{ij}t+s(\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}+\varphi_{ij}^{n}(t)+\tilde{x}_{ij}^{0})|^2}\,ds\right)dt\\
&\leq\int_{\bar{T}}^{+\infty}\left(\int_0^1\frac{3(|\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}|+kt^{1/2}+|\tilde{x}_{ij}^{0}|)}{|a_{ij}t|^2-s|\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}+kt^{1/2}+\tilde{x}_{ij}^{0}|^2}\,ds\right)dt\\
&\leq\int_{\bar{T}}^{+\infty}\left(\int_0^1\frac{3k't^{2/3}}{|a_{ij}|^2t^2-sk't^{4/3}}\,ds\right)dt,
\end{aligned}$$

where $k'\in\mathbb{R}$ is large enough so that $|kt^{1/2}|+|\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}+\tilde{x}_{ij}^{0}|\leq\sqrt{k'}t^{2/3}$. Since the denominator of the last integral is positive when

$$t>\left(\frac{|a_{ij}|^2}{k'}\right)^{2/3}=:\hat{T},$$

by choosing $\bar{T}(k)\gg\hat{T}$, the last integral is finite, which means that the term $\int_{\bar{T}}^{+\infty}\left|\frac{1}{|a_{ij}t+\beta_{K_{1,2}}b_{ij}^{K_{1,2}}t^{2/3}+\varphi_{ij}^{n}(t)+\tilde{x}_{ij}^{0}|}-\frac{1}{|a_{ij}t|}\right|dt$ is L^1-dominated. This implies the L^1-convergence of the potential term, which yields its weak-lower semicontinuity.

2.4.3 Regularity of the renormalized action on non-collision sets

It remains to prove that the action is of class C^1 on sets of motions that do not experience collisions. We have already established this result for the terms $\mathcal{A}_K$, so it suffices to focus on the terms $\mathcal{A}_{K_1,K_2}$. In particular, we aim to show that the differential of the potential term, namely

$$d\mathcal{A}_{K_1,K_2}(\varphi)[\psi] = \int_1^{+\infty} \Big\langle \nabla U\Big(at + \beta_{K_{1,2}} b^{K_{1,2}} t^{2/3} + \varphi(t) + \tilde{x}^0\Big), \psi(t)\Big\rangle \, dt,$$

is continuous for all $\varphi, \psi \in \mathcal{D}_0^{1,2}(1,+\infty)$, when restricted to the set of non-collisional configurations of the clusters K_1 and K_2.

First, we have

$$\begin{aligned} &\Big\| \nabla U(at + \beta_{K_{1,2}} b^{K_{1,2}} t^{2/3} + \varphi(t) + \tilde{x}^0) \Big\|_{\mathcal{M}} \\ &\le C_{16} \sum_{i \in K_1,\, j \in K_2} \frac{1}{\Big| a_{ij} t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \varphi_{ij}^n(t) + \tilde{x}_{ij}^0 \Big|^2}, \end{aligned}$$

for a suitable constant C_{16}, where the right-hand side behaves like $1/t^2$ as $t \to +\infty$. This ensures that the differential is well-defined, together with the Cauchy-Schwarz inequality.

Now, consider a sequence $(\varphi^n)_n \subset \mathcal{D}_0^{1,2}(1,+\infty)$ such that $\varphi^n \to \varphi$ in $\mathcal{D}_0^{1,2}(1,+\infty)$ for some φ. We wish to show that

$$\sup_{\|\psi\|_{\mathcal{D}} \le 1} \left| \int_1^{+\infty} \Big\langle \nabla U(t, \varphi^n(t)) - \nabla U(t, \varphi(t)), \psi(t) \Big\rangle \, dt \right| \longrightarrow 0, \qquad \text{as } n \to +\infty,$$

where we have set $U(t,\varphi(t)) := U\Big(at + \beta_{K_{1,2}} b^{K_{1,2}} t^{2/3} + \varphi(t) + \tilde{x}^0\Big)$ to simplify the notation. By the Cauchy-Schwarz and Hardy inequalities, we have

$$\begin{aligned} &\sup_{\|\psi\|_{\mathcal{D}} \le 1} \left| \int_1^{+\infty} \langle \nabla U(t, \varphi^n(t)) - \nabla U(t, \varphi(t)), \psi(t) \rangle \, dt \right| \\ &\le \sup_{\|\psi\|_{\mathcal{D}} \le 1} \int_1^{+\infty} t \|\nabla U(t, \varphi^n(t)) - \nabla U(t, \varphi(t))\|_{\mathcal{M}} \frac{\|\psi(t)\|_{\mathcal{M}}}{t} \, dt \\ &\le \sup_{\|\psi\|_{\mathcal{D}} \le 1} \left(\int_1^{+\infty} \frac{\|\psi(t)\|_{\mathcal{M}}^2}{t^2} \, dt \right)^{1/2} \left(\int_1^{+\infty} t^2 \|\nabla U(t, \varphi^n(t)) - \nabla U(t, \varphi(t))\|_{\mathcal{M}}^2 \, dt \right)^{1/2} \\ &\le 2 \left(\int_1^{+\infty} t^2 \|\nabla U(t, \varphi^n(t)) - \nabla U(t, \varphi(t))\|_{\mathcal{M}}^2 \, dt \right)^{1/2}. \end{aligned}$$

It holds

$$
\begin{aligned}
&\int_1^{+\infty} t^2 \|\nabla U(t,\varphi^n) - \nabla U(t,\varphi)\|^2_{\mathcal{M}}\, dt \\
&= \int_1^{+\infty} t^2 \left| \int_0^1 \nabla^2 U(at + \beta_{K_{1,2}} b^{K_{1,2}} t^{2/3} + \varphi + \tilde{x}^0 + s(\varphi^n - \varphi))(\varphi^n - \varphi)\, ds \right|^2 dt \\
&\leq \int_1^{+\infty} t^2 \Bigg(\int_0^1 C_{16} \\
&\cdot \sum_{i\in K_1,\ j\in K_2} \frac{1}{|a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \varphi_{ij} + \tilde{x}^0_{ij} + s(\varphi^n_{ij} - \varphi_{ij})|^3} \|\varphi^n - \varphi\|_{\mathcal{M}}\, ds \Bigg)^2 dt \\
&\leq \int_1^{+\infty} \Bigg(\int_0^1 C_{16} \\
&\cdot \sum_{i\in K_1,\ j\in K_2} \frac{1}{|a_{ij}t + \beta_{K_{1,2}} b_{ij}^{K_{1,2}} t^{2/3} + \varphi_{ij} + \tilde{x}^0_{ij} + s(\varphi^n_{ij} - \varphi_{ij})|^3} \|\varphi^n - \varphi\|_{\mathcal{D}} t^{3/2}\, ds \Bigg)^2 dt \\
&\leq C_{17} \|\varphi^n - \varphi\|_{\mathcal{D}}
\end{aligned}
$$

for a proper constant $C_{17} \in \mathbb{R}$, where the last term goes to zero as $n \to +\infty$. This concludes the proof.

2.4.4 Absence of collisions and hyperbolic-parabolicity of the motion

The existence of a minimizer $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$ for the renormalized Lagrangian action is obtained by the direct method of the calculus of variations. Moreover, the Renormalized Action Principle ensures that the curve $\gamma(t) = at + \beta bt^{2/3} + \varphi(t) + \tilde{x}^0$ solves Newton's equations (1.1.1) and satisfies the initial condition $\gamma(1) = x^0$.

In this case, the first derivative of the motion is given by

$$\dot{\gamma}(t) = a + \frac{2}{3}\beta bt^{-1/3} + \dot{\varphi}(t).$$

The conservation of the energy implies that the energy of the motion $h = \|a\|^2_{\mathcal{M}}/2 > 0$.

Remark 2.4.6. We observe that Chazy's Theorem applies only to the hyperbolic and hyperbolic-parabolic cases, since in the case of fully parabolic motions the energy constant of the internal motion vanishes. In such cases, the asymptotic shape of $\gamma(t)$ coincides with the configuration a. Moreover, $L = \lim_{t\to+\infty} \frac{\max_{i<j}|\gamma_{ij}(t)|}{\min_{i<j}|\gamma_{ij}(t)|} < +\infty$ if and only if γ is hyperbolic. If the energy is positive and $L = +\infty$, then the motion is either hyperbolic-parabolic or it fails to be expansive.

In our case, it is trivial to prove that $L = +\infty$. This implies that the motion is hyperbolic-parabolic.

Remark 2.4.7. We can observe that if the indexes i, j belong to the same cluster, we have $\dot{\gamma}_{ij}(t) \to 0$ when $t \to +\infty$, while if i, j belong to different clusters, we have $\dot{\gamma}_{ij}(t) \to a_{ij}$ when $t \to +\infty$.

2.4.5 Hyperbolic-parabolic motions' asymptotic expansion

As mentioned earlier, when applying an a-cluster partition to a hyperbolic-parabolic motion $\gamma(t) = at + \beta bt^{2/3} + \varphi(t) + \tilde{x}^0$, the centers of mass of the clusters follow a hyperbolic motion, while within each cluster the bodies undergo a parabolic expansion relative to the cluster's center of mass. This section is devoted to proving this fact.

We begin by proving that the centers of mass of each cluster exhibit a hyperbolic expansion. For a cluster K, let its center of mass be

$$c^K(t) = \frac{1}{M_K} \sum_{i \in K} m_i r_i(t),$$

so that its equations of motion are

$$\begin{aligned} M_K \ddot{c}^K(t) &= \sum_{i \in K} m_i \ddot{r}_i(t) \\ &= -\sum_{i \in K} \sum_{j \neq i} m_i m_j \frac{r_i(t) - r_j(t)}{|r_i(t) - r_j(t)|^3} \\ &= -\sum_{i \in K} \sum_{j \notin K} m_i m_j \frac{r_i(t) - r_j(t)}{|r_i(t) - r_j(t)|^3}. \end{aligned}$$

The right-hand side is an $O\left(\frac{1}{t^2}\right)$ term as $t \to +\infty$. Moreover,

$$-\sum_{i \in K} \sum_{j \notin K} m_i m_j \frac{r_i(t) - r_j(t)}{|r_i(t) - r_j(t)|^3} \simeq -\frac{1}{t^2} \sum_{i \in K} \sum_{j \notin K} m_i m_j \frac{a_i - a_j}{|a_i - a_j|^3} + O\left(\frac{1}{t^3}\right),$$

for $t \to +\infty$. We define

$$\tilde{\nabla} U(a^K) = -\sum_{i \in K} \sum_{j \notin K} m_i m_j \frac{a_i - a_j}{|a_i - a_j|^3},$$

which can be interpreted as the restriction of $\nabla U(a^K)$ to the cluster. Denoting by

a^K the restriction of the configuration a to K, we have

$$\lim_{t\to+\infty}\frac{M_K c^K(t)}{\log t}=\lim_{t\to+\infty}\frac{M_K \dot{c}^K(t)}{1/t}=-\lim_{t\to+\infty}\frac{M_K \ddot{c}^K(t)}{1/t^2}=-\tilde{\nabla}U(a^K).$$

Hence, the center of mass of K admits the hyperbolic asymptotic expansion

$$c^K(t)=a^K t-\tilde{\nabla}U(a^K)\log t+o(\log t),\quad \text{as } t\to+\infty.$$

Now, for an index $i\in K$, define the motion relative to the cluster's center of mass as

$$y_i(t)=r_i(t)-c_i^K(t),$$

and we aim to show that its asymptotic expansion is parabolic.

If the cluster contains only one body, then $y_i\equiv 0$, so we consider clusters with at least two bodies. The equation of motion for the i-th body reads

$$\begin{aligned} m_i\ddot{y}_i(t) &= m_i\ddot{r}_i(t)-m_i\ddot{c}_i^K(t)\\ &= -\sum_{j\in K} m_i m_j\frac{r_i(t)-r_j(t)}{|r_i(t)-r_j(t)|^3}-\sum_{j\notin K} m_i m_j\frac{r_i(t)-r_j(t)}{|r_i(t)-r_j(t)|^3}-m_i\ddot{c}_i^K(t). \end{aligned}$$

Since $-\sum_{j\notin K} m_i m_j\frac{r_i(t)-r_j(t)}{|r_i(t)-r_j(t)|^3}-m_i\ddot{c}_i^K(t)=O\big(1/t^2\big)$ as $t\to+\infty$, we obtain

$$m_i\ddot{y}_i(t)=-\sum_{j\in K} m_i m_j\frac{y_i(t)-y_j(t)}{|y_i(t)-y_j(t)|^3}+O\big(1/t^2\big).$$

Using the definition of $\gamma(t)$ and the asymptotic expansion of $c^K(t)$ above, we deduce

$$y_i(t)=\beta_K b_i^K t^{2/3}+\varphi_i(t)-\log t\sum_{j\notin K} m_i m_j\frac{a_i-a_j}{|a_i-a_j|^3}+o(\log t),\quad \text{as } t\to+\infty,$$

where $\beta_K=\sqrt[3]{\frac{9}{2}\min_K U}$. Defining

$$\psi_i(t):=\varphi_i(t)-\sum_{j\notin K} m_i m_j\frac{a_i-a_j}{|a_i-a_j|^3}+o(\log t),$$

it follows that $\psi_i\in\mathcal{D}^{1,2}(1,+\infty)$. Applying estimate (2.3.8), we finally obtain

$$y_i(t)=\beta_K b_i^K t^{2/3}+o(t^{1/3+}),\quad \text{as } t\to+\infty.$$

Chapter 3

Viscosity solutions to the Hamilton-Jacobi equations for the N-body problem

Hamilton-Jacobi equations are central in the study of dynamical systems and find many significant applications in celestial mechanics. The Newtonian N-body problem, in particular, represents a natural setting for the study of viscosity solutions of these equations. Over the last two decades, the characterization of such solutions, together with their geometric and variational properties – and their connections to the N-body problem – have been the focus of extensive research (cf., e.g., [49, 50, 52]). In the previous section, for instance, we recalled that Maderna and Venturelli proved the existence of hyperbolic solutions with prescribed initial configuration and limit shape by characterizing expansive solutions as fixed points of the Lax-Oleinik semigroup, which coincide with viscosity solutions of the Hamilton-Jacobi equation for the N-body problem.

The last section in [66] provides a deep connection between Hamilton-Jacobi equations and the expansive solutions of the N-body problem. More specifically, in the framework of expansive motions, it is shown that a value function which depends on the initial confiuration of the bodies can be associated to the minimizers of the renormalized Lagrangian action, and that this value function is a solution of the N-body problem's Hamilton-Jacobi equations in the viscosity sense. These results have later been developed in [12] and we present them in this chapter.

3.1 Introduction and main results

Consider an expansive motion of the N-body problem having the form

$$\gamma(t) = r_0(t) + \varphi(t) + x - r_0(1),$$

with fixed reference path $r_0(t)$ and initial configuration x. Emphasizing the dependence on x, we define the value function $v : \Omega \to \mathbb{R}$ by

$$\begin{aligned} v(x) = \min_{\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)} \Bigg\{ \int_1^{+\infty} \frac{1}{2} \|\dot\varphi(t)\|^2_{\mathcal{M}} \, \mathrm{d}t + U(r_0(t) + \varphi(t) + x - r_0(1)) \\ - U(r_0(t)) - \langle \ddot r_0(t), \varphi(t) \rangle_{\mathcal{M}} \, \mathrm{d}t \Bigg\} - \langle a, x \rangle_{\mathcal{M}}. \end{aligned} \tag{3.1.1}$$

In this chapter, we prove that v is a global viscosity solution of the stationary Hamilton-Jacobi equation

$$H(x, \nabla v(x)) = h \tag{3.1.2}$$

at a fixed energy level $h \geq 0$. Due to the lack of compactness or smoothness, or to the possible presence of singularities – especially in the case of singular potentials such as in the N-body problem – classical solutions of (3.1.2) may fail to exist (see, for example, [30, 49]). For this reason, it is more convenient to consider the weaker notion of viscosity solutions, which allows a meaningful interpretation even when v is not differentiable.

As a first remark, if we formally differentiate (3.1.1) with respect to a point of differentiability x, we obtain

$$\nabla v(x) = -\mathcal{M}\dot\gamma(1),$$

where $\gamma(t)$ is such that φ is a minimizer of the renormalized action associated with x. This yields

$$\mathcal{M}^{-1/2} \nabla v(x) = -\mathcal{M}^{1/2} \dot\gamma(1),$$

and, using the expression of the Hamiltonian (1.2.1), equation (3.1.2) follows immediately. In [12], the main idea was to adapt the method of [18], taking into account that the singular set is contained in a locally countable union of smooth hypersurfaces of codimension at least one.

The fact that the value function (3.1.1) solves (3.1.2) in the viscosity sense provides an additional justification for our choice of the renormalization of the action.

Moreover, it is worth noticing that the uniqueness result in [51] guarantees that, in the hyperbolic case, the value function v coincides with the Busemann function. In addition, the linear correction appearing in (3.1.1) itself corresponds to the Busemann function of a free particle.

The following is the main theorem of this chapter.

Theorem 3.1.1 (Berti, Polimeni and Terracini 2025 [12]). *Let $a \in \Omega$ (type (H)), or let b_m be a minimal central configuration of U (type (P)), or let $a \in \Delta$ and b_m be a normalized minimal central configuration of the a-clustered potential (type*

(HP)). Then, there exists a viscosity solution to the N-body Hamilton-Jacobi equation (3.1.2). The singular set of such a solution is a countably $\mathcal{H}^{d(N-1)-1}$-rectifiable subset of the configuration space $\mathcal{X}$. Moreover, we have

$$\dim_{\mathcal{H}}(\Gamma \setminus \Sigma) \leq d(N-1) - 2,$$

where Γ and Σ denote respectively the conjugate and irregular sets of points (Definitions 3.4.1 and 3.4.2).

Remark 3.1.2. Our viscosity solutions to the Hamilton-Jacobi equation extends continuously to collision configurations, although they are no longer Lipschitz continuous at those points. However, the results in [49] suggest that they remain 1/2-Hölder continuous at Δ.

3.2 Preliminary estimates

In this section, we describe some properties of minimizers of the (renormalized) Lagrangian action. As in [12], these properties will be used in the subsequent sections to prove that the renormalized value function is a viscosity solution of the Hamilton-Jacobi equation (3.1.2).

3.2.1 Uniform coercivity estimates

The goal of this subsection is to show that the renormalized Lagrangian action satisfies uniform coercivity estimates with respect to T and x, provided that the initial configuration lies in a compact set. This property will play a key role in the proof of the main theorems and will be repeatedly employed in the subsequent arguments.

Notation 2. Fix $T > 1$. For a function $\varphi \in \mathcal{D}_0^{1,2}(1,T)$, we will write

$$\mathcal{A}_{x,[1,T]}(\varphi) = \int_1^T \frac{1}{2}\|\dot{\varphi}(t)\|^2_{\mathcal{M}} + U(r_0(t)+\varphi(t)+x-r_0(1)) - U(r_0(t)) - \langle \mathcal{M}\ddot{r}_0(t), \varphi(t)\rangle \, dt.$$

To keep notations compact, in the case $T = +\infty$, with $\mathcal{A}_{x,[1,+\infty]}$ we mean the functional $\mathcal{A}_x$, defined on $\mathcal{D}_0^{1,2}(1,+\infty)$.

Remark 3.2.1. We have already established coercivity estimates for the renormalized Lagrangian action in Chapter 2. In this section, we prove analogous estimates, with the difference that we now work on intervals of the form $[1,T]$, with $T \in (1,+\infty]$. Moreover, the estimates obtained here are uniform with respect to both the initial configuration (ranging in a compact set) and the final time T.

Lemma 3.2.2 (Berti, Polimeni and Terracini 2025 [12]). *Given a compact set $\mathcal{K} \subset \mathcal{X}$, fix $x \in \mathcal{K}$ and $T \in (1, +\infty]$. Then, there exist positive constants $A, B, C \in \mathbb{R}$, which do not depend on x and T, such that for all φ belonging to a bounded subset of $\mathcal{D}_0^{1,2}(1,T)$, it holds*

$$\mathcal{A}_{x,[1,T]}(\varphi) \geq A\|\varphi\|^2_{\mathcal{D}_T} - B\|\varphi\|_{\mathcal{D}_T} + C.$$

Proof. For fixed $x \in \mathcal{X}$ and $T = +\infty$, coercivity estimates have already been proved in Chapter 2.

Now, we fix $x \in \mathcal{K}$, with $\mathcal{K}$ a compact subset of $\mathcal{X}$, and $T \in (1, +\infty]$. Let $\bar{k} \in \mathbb{R}$ be such that $\|x\|_{\mathcal{M}} \leq \bar{k}$ for all $x \in \mathcal{K}$. Our goal is to prove that the renormalized Lagrangian action satisfies coercivity estimates uniformly with respect to both x and T.

As usual, we investigate the three cases separately.

Hyperbolic case. We write the action as

$$\mathcal{A}_{x,[1,T]}(\varphi) = \sum_{i<j} m_i m_j \mathcal{A}_{x,[1,T]}(\varphi)_{ij},$$

where

$$\mathcal{A}_{x,[1,T]}(\varphi)_{ij} = \int_1^T \frac{1}{2M}|\dot{\varphi}_{ij}(t)|^2 + \frac{1}{|a_{ij}t + \varphi_{ij}(t) + x_{ij} - a_{ij}|} - \frac{1}{|a_{ij}t|}\, dt,$$

and $M = \sum_{i=1,\dots,N} m_i$. We study the uniform coercivity estimates for each term $\mathcal{A}_{x,[1,T]}(\varphi)_{ij}$.

We have

$$\begin{aligned} &\int_1^T \frac{1}{|a_{ij}t + \varphi_{ij}(t) + x_{ij} - a_{ij}|} - \frac{1}{|a_{ij}t|}\, dt \\ &\geq \int_1^T \frac{1}{|a_{ij}|t + |\varphi_{ij}(t)| + |x_{ij} - a_{ij}|} - \frac{1}{|a_{ij}|t}\, dt. \end{aligned}$$

By the convexity of $\frac{1}{t}$, we can use Taylor's expansion and then Hardy inequality to obtain

$$\begin{aligned}
&\int_1^T \frac{1}{|a_{ij}|t + |\varphi_{ij}(t)| + |x_{ij} - a_{ij}|} - \frac{1}{|a_{ij}|t}\,\mathrm{d}t \\
&\geq \int_1^T -\frac{|\varphi_{ij}(t)|}{|a_{ij}|t^2} - \frac{|x_{ij} - a_{ij}|}{|a_{ij}|t^2}\,\mathrm{d}t \\
&\geq -\frac{1}{|a_{ij}|}\left(\int_1^T \frac{|\varphi_{ij}(t)|^2}{t^2}\,dt\right)^{1/2}\left(\int_1^T \frac{1}{t^2}\,dt\right)^{1/2} - \int_1^T \frac{|x_{ij} - a_{ij}|}{|a_{ij}|t^2}\,dt \\
&\geq -\frac{2}{|a_{ij}|}\|\varphi_{ij}\|_{\mathcal{D}_T}\left(\int_1^{+\infty} \frac{1}{t^2}\,dt\right)^{1/2} - \int_1^{+\infty} \frac{|x_{ij} - a_{ij}|}{|a_{ij}|t^2}\,dt \\
&= -\frac{2}{|a_{ij}|}\|\varphi_{ij}\|_{\mathcal{D}_T} + \frac{|x_{ij} - a_{ij}|}{|a_{ij}|}.
\end{aligned}$$

This yields

$$\mathcal{A}_{x,[1,T]}(\varphi)_{ij} \geq \frac{1}{2M}\|\varphi_{ij}\|^2_{\mathcal{D}_T} - \frac{2}{|a_{ij}|}\|\varphi_{ij}\|_{\mathcal{D}_T} + \frac{|x_{ij} - a_{ij}|}{|a_{ij}|},$$

from which the uniform coercivity with respect to T and x trivially follows.

Parabolic case. In this case, it holds

$$\nabla U(r_0) = \nabla U(\beta b_m t^{2/3}) = -\frac{U_{min}}{\beta^2 t^{4/3}}\mathcal{M}b_m,$$

with $U_{min} = U(b_m)$.

Following the same arguments of Section 2.3.1, we get

$$\begin{aligned}
\mathcal{A}_{x,[1,T]}(\varphi) &= \frac{1}{2}\|\varphi\|^2_{\mathcal{D}_T} + \int_1^T U(\beta b_m t^{2/3} + \varphi(t) + \tilde{x}) - U(\beta b_m t^{2/3}) \\
&\quad - \langle \nabla U(\beta b_m t^{2/3}), \varphi(t)\rangle\, dt \\
&\geq \frac{1}{2}\|\varphi\|^2_{\mathcal{D}_T} + U_{min}\int_1^T \frac{1}{\beta t^{2/3}\|b_m\|_{\mathcal{M}} + \|\varphi(t)\|_{\mathcal{M}} + \|\tilde{x}\|_{\mathcal{M}}} - \frac{1}{\beta t^{2/3}\|b_m\|_{\mathcal{M}}} \\
&\quad + \frac{\langle \mathcal{M}b_m, \varphi(t)\rangle}{\beta^2 t^{4/3}}\, dt \\
&\geq \frac{1}{2}\|\varphi\|^2_{\mathcal{D}_T} - U_{min}\int_1^T \frac{\|\varphi(t)\|^2_{\mathcal{M}}}{2\beta^3\|b_m\|^3_{\mathcal{M}}t^2} + \frac{\langle \mathcal{M}\varphi(t), \tilde{x}\rangle}{\beta^3\|b_m\|^3_{\mathcal{M}}t^2}\, dt \\
&\quad - U_{min}\int_1^T \frac{\langle \mathcal{M}b_m, \tilde{x}\rangle\beta t^{2/3} + 1/2\|\tilde{x}\|_{\mathcal{M}}}{\beta^3\|b_m\|^3_{\mathcal{M}}t^2}\, dt,
\end{aligned}$$

where we set $\tilde{x} = x - \beta b_m$.

The following inequalities hold:

- $$-U_{min}\int_1^T \frac{\|\varphi(t)\|^2_{\mathcal{M}}}{2\beta^3\|b_m\|^3_{\mathcal{M}}t^2}\,dt \geq -\frac{8}{18}\|\varphi\|^2_{\mathcal{D}_T}.$$

- $$\begin{aligned}-U_{min}\int_1^T \frac{\langle \mathcal{M}\varphi(t),\tilde{x}\rangle}{\beta^3\|b_m\|^3_{\mathcal{M}}t^2}\,dt &\geq -\frac{2}{9}\int_1^T \frac{\|\varphi(t)\|_{\mathcal{M}}\|\tilde{x}\|_{\mathcal{M}}}{t^2}\,dt\\ &\geq -\frac{2}{9}\left(\int_1^T \frac{\|\varphi(t)\|^2_{\mathcal{M}}}{t^2}\,dt\right)^{1/2}\left(\int_1^T \frac{\|\tilde{x}\|^2_{\mathcal{M}}}{t^2}\,dt\right)^{1/2}\\ &\geq -\frac{4}{9}\|\varphi\|_{\mathcal{D}_T}\|\tilde{x}\|_{\mathcal{M}}.\end{aligned}$$

- $$\begin{aligned}&-U_{min}\int_1^T \frac{\langle \mathcal{M}b_m,\tilde{x}\rangle\beta t^{2/3}+1/2\|\tilde{x}\|_{\mathcal{M}}}{\beta^3\|b_m\|^3_{\mathcal{M}}t^2}\,dt\\ &\geq -\frac{2}{9}\int_1^T \frac{\|\tilde{x}\|_{\mathcal{M}}\beta t^{2/3}+1/2\|\tilde{x}\|_{\mathcal{M}}}{t^2}\,dt\\ &\geq -\frac{2}{9}\int_1^{+\infty} \frac{\|\tilde{x}\|_{\mathcal{M}}\beta t^{2/3}+1/2\|\tilde{x}\|_{\mathcal{M}}}{t^2}\,dt\\ &= \frac{\|\tilde{x}\|_{\mathcal{M}}}{9}(2\beta-1).\end{aligned}$$

It follows

$$\mathcal{A}_{x,[1,T]}(\varphi) \geq \frac{1}{18}\|\varphi\|^2_{\mathcal{D}_T} - \frac{4}{9}\|\varphi\|_{\mathcal{D}_T}\|\tilde{x}\|_{\mathcal{M}} + \frac{\|\tilde{x}\|_{\mathcal{M}}}{9}(2\beta-1),$$

and hence the proof for the parabolic case is completed.

Hyperbolic-Parabolic case. As in Section 2.4.1, we consider the a-cluster partition of the bodies determined by the equivalence relation (2.1.2). The renormalized Lagrangian action is then expressed as the sum of two terms as follows:

$$\begin{aligned}\mathcal{A}_{x,[1,T]}(\varphi) &= \sum_{K\in\mathcal{P}}\mathcal{A}_K(\varphi) + \sum_{K_1,K_2\in\mathcal{P},\ K_1\neq K_2}\mathcal{A}_{K_1,K_2}(\varphi)\\ &= \sum_{K\in\mathcal{P}}\left(\sum_{i,j\in K,\ i<j} m_im_j\mathcal{A}^{ij}_K(\varphi)\right)\\ &\quad+\frac{1}{2}\sum_{K_1,K_2\in\mathcal{P},\ K_1\neq K_2}\left(\sum_{i\in K_1,\ j\in K_2} m_im_j\mathcal{A}^{ij}_{K_1,K_2}(\varphi)\right),\end{aligned} \tag{3.2.1}$$

where

$$\mathcal{A}^{ij}_K(\varphi) = \int_1^T \frac{1}{2M}|\dot\varphi_{ij}(t)|^2 + \frac{1}{|\beta_K b^K_{ij} t^{2/3} + \varphi_{ij}(t) + \tilde x_{ij}|} - \frac{1}{|\beta_K b^K_{ij} t^{2/3}|} + \frac{2}{9}\frac{\beta_K}{M}\frac{\langle b^K_{ij}, \varphi_{ij}(t)\rangle}{t^{4/3}}\, dt,$$

$$\mathcal{A}^{ij}_{K_1,K_2}(\varphi) = \int_1^T \frac{1}{2M}|\dot\varphi_{ij}(t)|^2 + \frac{1}{|a_{ij}t + \beta_{K_{1,2}} b^{K_{1,2}}_{ij} t^{2/3} + \varphi_{ij}(t) + \tilde x_{ij}|} - \frac{1}{|a_{ij}t|}\, dt,$$

where $\beta_K > 0$, b^K is a minimal central configuration in the cluster K, $\beta_{K_{1,2}} b^{K_{1,2}} = (\beta_{K_1} b^{K_1}, \beta_{K_2} b^{K_2})$ with b^{K_i} minimal central configuration for the cluster K_i, $i = 1{,}2$, and $\beta_{K_i} \in \mathbb{R}$, $i = 1{,}2$ (see Section 2.4).

First, we examine the interaction term $\mathcal{A}^{ij}_{K_1,K_2}$, which exhibits behavior analogous to that of the renormalized Lagrangian action in the hyperbolic setting. The main distinction is that, in this case, the denominator of the potential also contains an additional term of order 2/3. By the triangular inequality, we have

$$\begin{aligned}\mathcal{A}^{ij}_{K_1,K_2}(\varphi) &= \int_1^T \frac{|\dot\varphi_{ij}(t)|^2}{2M} + \frac{1}{|a_{ij}t + \beta_{K_{1,2}} b^{K_{1,2}}_{ij} t^{2/3} + \varphi_{ij}(t) + \tilde x_{ij}|} - \frac{1}{|a_{ij}t|}\, dt \\ &\geq \frac{\|\varphi_{ij}\|^2_{\mathcal{D}_T}}{2M} + \int_1^T \frac{1}{|a_{ij}|t + |\beta_{K_{1,2}} b^{K_{1,2}}_{ij}| t^{2/3} + |\varphi_{ij}(t)| + |\tilde x_{ij}|} - \frac{1}{|a_{ij}|t}\, dt.\end{aligned}$$

By the convexity of the function $\frac{1}{t}$, it holds

$$\begin{aligned}&\int_1^T \frac{1}{|a_{ij}|t + |\beta_{K_{1,2}} b^{K_{1,2}}_{ij}| t^{2/3} + |\varphi_{ij}(t)| + |\tilde x_{ij}|}\, dt \\ &\geq -\frac{1}{|a_{ij}|^2}\int_1^T \frac{|\beta_{K_{1,2}} b^{K_{1,2}}_{ij}| t^{2/3} + |\varphi_{ij}(t)| + |\tilde x_{ij}|}{t^2}\, dt \\ &\geq -\frac{1}{|a_{ij}|^2}(3|\beta_{K_{1,2}} b^{K_{1,2}}_{ij}| + 4\|\varphi_{ij}\|_{\mathcal{D}_T} + |\tilde x_{ij}|).\end{aligned}$$

So, it holds

$$\mathcal{A}^{ij}_{K_1,K_2}(\varphi) > \frac{1}{2M}\|\varphi_{ij}\|^2_{\mathcal{D}_T} - \frac{4}{|a_{ij}|^2}\|\varphi_{ij}\|_{\mathcal{D}_T} + -\frac{1}{|a_{ij}|^2}(3|\beta_{K_{1,2}} b^{K_{1,2}}_{ij}| + |\tilde x_{ij}|).$$

To establish the uniform coercivity of the term $\mathcal{A}^{ij}_K(\varphi)$, we observe that its structure closely resembles that of the renormalized action in the parabolic case.

As a consequence, the same arguments apply and lead to the desired conclusion.

$$\mathcal{A}_K^{ij}(\varphi) \geq \frac{1}{18M_K}\|\varphi_{ij}\|_{\mathcal{D}_T}^2 - \frac{4}{9}\|\varphi_{ij}\|_{\mathcal{D}_T}\|\tilde{x}_{ij}\|_{\mathcal{M}} + \frac{\|\tilde{x}_{ij}\|_{\mathcal{M}}}{9}(2\beta_K - 1).$$

This concludes the proof. □

3.2.2 Distance of minimal solutions from collisions

As a direct consequence of the uniform coercivity estimates proved above, we deduce that any expansive motion $\gamma(t)$ with a collisionless initial configuration, obtained through the Renormalized Action Principle, stays uniformly separated from collisions for all $t \in [1, +\infty)$, locally uniformly with respect to x.

Lemma 3.2.3 (Berti, Polimeni and Terracini 2025 [12])**.** *For any $\bar{x} \in \Omega$, there is a neighborhood $\mathcal{U}(\bar{x})$ such that $\mathcal{U}(\bar{x}) \cap \Delta = \emptyset$ and there is a real constant $C > 0$ such that, for any $x \in \mathcal{U}(\bar{x})$, the motions $\gamma(t)$ given by Theorems 2.1.4, 2.1.3 and 2.1.7 with initial configuration x satisfy*

$$d(\gamma(t), \Delta) \geq C \quad \forall t \in [1, +\infty), \text{ uniformly with respect to } x. \tag{3.2.2}$$

Proof. By Marchal's principle, we already know that the minimizing curve

$$\gamma(t) = r_0(t) + \varphi^{\bar{x}}(t) + \bar{x} + r_0(1)$$

is free of collisions for all $t \in [1, +\infty)$.

From the uniform coercivity estimates in Lemma 3.2.2, it follows that for any $R > 0$ there exists a constant $C_R > 0$ such that $\|\varphi^x\|_{\mathcal{D}} \leq C_R$ for all $x \in B_R(0)$. Consequently, for any $R > 0$ there exists $C_R > 0$ such that

$$\begin{aligned}|\gamma_i(t) - \gamma_j(t)| &\geq |(a_i - a_j)t + (\beta b_i - \beta b_j)t^{2/3}| - |\varphi_i^x(t) - \varphi_j^x(t)| - |\tilde{x}_i - \tilde{x}_j| \\ &\geq C' t^{2/3} - C_R t^{1/2} - C'' \\ &\geq 1,\end{aligned}$$

for some $t \geq \tau = \tau(R) \geq 1$ and suitable constants $C', C'' > 0$.

The inequality $|\gamma_i(t) - \gamma_j(t)| \geq 1$ for all $i < j$ and $t \in [\tau, +\infty)$ implies the existence of a constant $C > 1$ such that $d(\gamma(t), \Delta) \geq C$. Indeed, the collision set Δ can be written as the union of hyperplanes

$$\Delta = \bigcup_{i<j} \Delta_{ij}, \qquad \Delta_{ij} := \{x \in \mathcal{X} : x_i = x_j\},$$

and it is immediate to see that

$$d(\gamma(t),\Delta) = \min_{i<j} d(\gamma(t),\Delta_{ij}).$$

We now prove that the estimate (3.2.2) also holds for all $t \in [1,\tau]$. Suppose, by contradiction, that there exists a sequence $(\bar{x}_n)_n \subset \mathcal{U}(\bar{x})$ such that $d(\gamma_n(t),\Delta) \to 0$ as $n \to +\infty$ for some $t \in [1,\tau]$, where for each $n \in \mathbb{N}$, $\gamma_n(t)$ is a minimizer of the value function with initial configuration $\bar{x}_n$. By the Ascoli-Arzelà theorem, we can extract a subsequence $(\gamma_{n_k}(t))_k$ converging uniformly on the compact interval $[1,\tau]$, and hence pointwise on $[1,\tau]$.

Let $\gamma_{n_k}(t) \to \bar{\gamma}(t) \in \Delta$ as $k \to +\infty$ for $t \in [1,\tau]$. In particular, $\bar{\gamma}(1) \in \Delta$. On the other hand, since $(\bar{x}_n)_n \subset \mathcal{U}(\bar{x})$, passing to the limit yields $\bar{\gamma}(1) \in \overline{\mathcal{U}(\bar{x})}$, which is a contradiction.

□

3.3 Viscosity solutions for the Hamilton-Jacobi equations

The main goal of this section is to describe the approach employed in [12] to prove the following.

Theorem 3.3.1 (Berti, Polimeni and Terracini 2025 [12])**.** *Fix $x \in \Omega$. The renormalized value function*

$$v(x) = \min_{\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)} \mathcal{A}_x(\varphi) - \langle a, x\rangle_{\mathcal{M}} \tag{3.3.1}$$

is a viscosity solution of the Hamilton-Jacobi equation

$$\frac{1}{2}\|\nabla v(x)\|^2_{\mathcal{M}^{-1}} - U(x) = \frac{\|a\|^2_{\mathcal{M}}}{2}. \tag{3.3.2}$$

In this and the following subsections, we establish some preliminary results that will be used in the proof of Theorem 3.3.1, which is presented in Section 3.3.3. Among these results, we highlight the following proposition, which concerns the uniform convergence on compact sets of $v(T,x)$ (defined via the functionals $\mathcal{A}_{x,[1,T]}$) to $v(x)$ as $T \to +\infty$.

Proposition 3.3.2 (Berti, Polimeni and Terracini 2025 [12])**.** *For $x \in \Omega$ and $T > 1$, let $v(x)$ be defined as in Theorem 3.3.1, and let $v(T,x)$ be defined as*

$$v(T,x) := \min_{\varphi \in \mathcal{D}_0^{1,2}(1,T)} \mathcal{A}_{x,[1,T]}(\varphi) - \langle \dot{r}_0(T), x\rangle_{\mathcal{M}}. \tag{3.3.3}$$

Then, $v(T,x)$ and $v(x)$ are continuous in Ω, and, if $(T_n)_n$ is a sequence such that $T_n \to +\infty$, then

$$\lim_{n\to+\infty} v(T_n,x) = v(x) \quad \textit{uniformly on compact subsets of } \ \Omega.$$

The proof of Proposition 3.3.2 follows directly as a corollary of Lemmas 3.3.4 and 3.3.5.

3.3.1 Finite horizon approximation

Define the functions $w : [1,+\infty) \times \Omega \to \mathbb{R}$ and $w_\infty : \Omega \to \mathbb{R}$ as the value functions

$$w(T,x) = \min_{\varphi\in\mathcal{D}_0^{1,2}(1,T)} \mathcal{A}_{x,[1,T]}(\varphi), \qquad w_\infty(x) = \min_{\varphi\in\mathcal{D}_0^{1,2}(1,+\infty)} \mathcal{A}_x(\varphi). \tag{3.3.4}$$

For $T > 1$, we refer to $w(T,\cdot)$ as the *finite-horizon value function associated with* T. In this section, our aim is to investigate the properties of these finite-horizon value functions and to prove their uniform convergence to w_∞ on compact subsets of Ω as $T \to +\infty$.

Lemma 3.3.3 (Structure Lemma, Berti, Polimeni and Terracini 2025 [12]). *For every $T \in (1,+\infty]$, we have*

$$\mathcal{A}_{x,[1,T]}(\varphi) = Q_T(\varphi,\varphi) + P_{x,T}(\varphi), \quad \textit{for} \ \ \varphi \in \mathcal{D}_0^{1,2}(1,T),$$

where

- *Q_T is a positive definite quadratic form on $\mathcal{D}_0^{1,2}(1,T) \times \mathcal{D}_0^{1,2}(1,T)$;*
- *$P_{x,T}$ is a functional on $\mathcal{D}_0^{1,2}(1,T)$ such that there exists $V : \Omega\times(1,+\infty)\times\mathcal{X} \to \mathbb{R}$ for which*
$$P_{x,T}(\varphi) = \int_1^T V(x,t,\varphi(t))\,dt.$$

Moreover, for every compact subset $\mathcal{K} \subset \Omega$ and $\hat{M} > 0$, there exists $\hat{T} > 1$, $C > 0$ and $\beta > 1$ such that

$$|V(x,t,\varphi(t))| \le Ct^{-\beta} \quad \textit{if} \ \ t \ge \hat{T}, \tag{3.3.5}$$

for every $x \in \mathcal{K}$ and φ such that $\|\varphi(t)\|_{\mathcal{M}} \le \hat{M}\sqrt{t}$.

Proof. **Hyperbolic case.** Since $\|\varphi\|_{\mathcal{D}_T} = \left(\int_1^T \|\dot\varphi(t)\|_{\mathcal{M}}^2 \,dt\right)^{1/2}$ is a norm, it is trivial that it is a definite positive quadratic form.

We can write down the second part of the action as

$$\mathcal{A}_{x,[1,T]}(\varphi) - \frac{\|\varphi\|^2_{\mathcal{D}_T}}{2} = \int_1^T \sum_{i<j} m_i m_j \left(\frac{1}{|a_{ij}t + \varphi_{ij}(t) + x_{ij} - a_{ij}|} - \frac{1}{|a_{ij}t|} \right) dt. \quad (3.3.6)$$

Since for all $s \in (0,1)$, $x \in \mathcal{K}$ and $a \neq 0$ it holds

$$\begin{aligned} |a_{ij}t + s(\varphi_{ij}(t) + x_{ij} - a_{ij})| &\geq |a_{ij}|t - s(\|\varphi_{ij}\|_{\mathcal{D}_T}\sqrt{t} + |x_{ij} - a_{ij}|) \\ &> |a_{ij}|t - (\|\varphi_{ij}\|_{\mathcal{D}_T}\sqrt{t} + |x_{ij} - a_{ij}|), \end{aligned}$$

there exists $T_1 = T_1(\|\varphi\|_{\mathcal{D}_T}, \mathcal{K})$ such that, for all $t \geq T_1$,

$$|a_{ij}t + s(\varphi_{ij}(t) + x_{ij} - a_{ij})| > |a_{ij}|t - (|x_{ij} - a_{ij}| + \|\varphi\|_{\mathcal{D}_T}\sqrt{t}) > 0. \quad (3.3.7)$$

Hence, for t large enough, by the Fundamental Theorem of Calculus, it holds

$$\frac{1}{|a_{ij}t + \varphi_{ij}(t) + x_{ij} - a_{ij}|} - \frac{1}{|a_{ij}t|} = \int_0^1 \frac{d}{ds}\left[\frac{1}{|a_{ij}t + s(\varphi_{ij}(t) + x_{ij} - a_{ij})|}\right] ds.$$

Then, we get

$$\begin{aligned} &\left| \frac{1}{|a_{ij}t + \varphi_{ij}(t) + x_{ij} - a_{ij}|} - \frac{1}{|a_{ij}t|} \right| \\ &= \left| \int_0^1 -\frac{[a_{ij}t + s(\varphi_{ij}(t) + x_{ij} - a_{ij})](\varphi_{ij}(t) + x_{ij} - a_{ij})}{|a_{ij}t + s(\varphi_{ij}(t) + x_{ij} - a_{ij})|^3} \, ds \right| \\ &\leq \int_0^1 \frac{|\varphi_{ij}(t) + x_{ij} - a_{ij}|}{|a_{ij}t + s(\varphi_{ij}(t) + x_{ij} - a_{ij}|^2} \, ds. \end{aligned}$$

By inequality (2.2.9), and taking into consideration that there is a constant $k' \in \mathbb{R}$ large enough such that $|x_{ij} - a_{ij}| + \hat{M}\sqrt{t} \leq k'\sqrt{t}$, we thus have

$$\begin{aligned} \int_0^1 \frac{|\varphi_{ij}(t) + x_{ij} - a_{ij}|}{|a_{ij}t + s(\varphi_{ij}(t) + x_{ij} - a_{ij})|^2} \, ds &\leq \int_0^1 3\frac{\hat{M}\sqrt{t} + |x_{ij} - a_{ij}|}{|a_{ij}t|^2 - s|M\sqrt{t} + x_{ij} - a_{ij}|^2} \, ds \\ &\leq \int_0^1 \frac{3k'\sqrt{t}}{|a_{ij}|^2t^2 - sk't} \, ds, \end{aligned}$$

where the last term is dominated, at infinity, by a term $\frac{k''}{t^\beta}$, with k'' not depending on T and x, and $\beta > 1$. This concludes the proof in the hyperbolic case, with $V(x,t,\varphi)$ given by the integrand function of the right-hand side of (3.3.6).

Parabolic case. Now, we investigate the parabolic setting. Adding and subtracting the term $\int_1^T \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \, dt$ to the action, we obtain

$$\begin{aligned}\mathcal{A}_{x,[1,T]}(\varphi) = \int_1^T & \frac{1}{2}\|\dot\varphi(t)\|^2_{\mathcal{M}} + \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \\ &+ U(r_0(t)+\varphi(t)+\tilde x) - U(r_0(t)) - \langle \nabla U(r_0(t)), \varphi(t)\rangle \\ &- \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \, dt.\end{aligned}$$

We prove the statement taking into account

$$Q_T(\varphi,\varphi) = \int_1^T \frac{1}{2}\|\dot\varphi(t)\|^2_{\mathcal{M}} + \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \, dt,$$

and

$$\begin{aligned}P_{x,T}(\varphi) = \int_1^T & U(r_0(t)+\varphi(t)+\tilde x) - U(r_0(t)) - \langle \nabla U(r_0(t)), \varphi(t)\rangle \\ &- \frac{1}{2}\langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \, dt.\end{aligned}$$

First, as noticed for the case $T = +\infty$ in Section 2.3.2, the map $\varphi \mapsto Q_T(\varphi,\varphi)$ defines a norm on $\mathcal{D}_0^{1,2}(1,T)$ that is equivalent to $\|\cdot\|_{\mathcal{D}_T}$. Indeed, as observed in Section 2.3.2, there exists a constant $C' > 0$ such that the following chain of inequalities holds pointwise.

$$-\frac{2}{9}\frac{\|\varphi(t)\|^2_{\mathcal{M}}}{t^2} \le \langle \nabla^2 U(r_0(t))\varphi(t), \varphi(t)\rangle \le C'\frac{\|\varphi(t)\|^2_{\mathcal{M}}}{t^2}.$$

Then, by Hardy inequality on $[1,T]$, we get

$$C''\|\varphi\|^2_{\mathcal{D}_T} \le Q_T(\varphi,\varphi) \le \frac{1}{18}\|\varphi\|^2_{\mathcal{D}_T},$$

for some positive $C'' > 0$.

Second, we prove (3.3.5) for large values of time as follows. If t is large enough so that (3.3.7) holds, we have

$$\begin{aligned}&U(r_0+\varphi+\tilde x) - U(r_0) - \langle \nabla U(r_0), \varphi(t)\rangle - \frac{1}{2}\langle \nabla^2 U(r_0), \varphi, \varphi\rangle \\ &= \int_0^1\int_0^1\int_0^1 \langle \nabla^3 U(r_0+\tau_1\tau_2\tau_3(\varphi+\tilde x))(\varphi+\tilde x), \varphi+\tilde x, \varphi+\tilde x\rangle \tau_1\tau_2^2 \, d\tau_1 \, d\tau_2 \, d\tau_3.\end{aligned}$$

Since $\|\varphi(t)\|_{\mathcal{M}} \le \hat M\sqrt{t}$, there exists $\tilde t > 1$ large enough such that, for all $t \ge \tilde t$,

$$\|r_0(t) + \tau_1\tau_2\tau_3(\varphi(t)+\tilde x)\|_{\mathcal{M}} > 0.$$

This implies that for all $t \geq \tilde{t}$ it holds (see Section 2.3.2):

$$\begin{aligned}&\left|\langle \nabla^3 U(r_0(t) + \tau_1\tau_2\tau_3(\varphi(t)+\tilde{x}))(\varphi(t)+\tilde{x}), \varphi(t)+\tilde{x}, \varphi(t)+\tilde{x}\rangle\right| \\ &\leq \hat{C}\frac{\|\varphi(t)+\tilde{x}\|^3_{\mathcal{M}}}{t^{8/3}} \\ &\leq \frac{\hat{C}'(\hat{M},\mathcal{K})}{t^{7/6}},\end{aligned}$$

where the constants $\hat{C}, \hat{C}'$ do not depend on T and $x \in \mathcal{K}$.

Hyperbolic-parabolic case. In the hyperbolic-parabolic setting, the renormalized Lagrangian action is decomposed as in (3.2.1), and the lemma is established using the same arguments as in the previous two cases.

□

Lemma 3.3.3 is crucial to prove the continuity of the value functions in (3.3.4).

Lemma 3.3.4 (Berti, Polimeni and Terracini 2025 [12])**.** *For every $T > 1$, $w(T,\cdot)$ is continuous in Ω, and w_∞ is continuous in Ω.*

Proof. We prove the statement for w_∞; the same result for the finite-horizon value functions $w(T,\cdot)$ follows a fortiori.

Upper semicontinuity. Consider a sequence $(x_n)_n$ converging to x. By the trivial fact that $\mathcal{A}_x$ is continuous with respect to x, it holds

$$\mathcal{A}_x(\varphi) = \limsup_{x_n\to x} \mathcal{A}_{x_n}(\varphi) \geq \limsup_{x_n\to x} w(x_n)$$

for all $\varphi \in \mathcal{D}^{1,2}_0(1,+\infty)$. This gives

$$w(x) \geq \limsup_{x_n\to x} w(x_n).$$

Lower semicontinuity. Let $(x_n)_n$ be a sequence converging to x. Consider any sequence $(\varphi^{x_n})_n \subset \mathcal{D}^{1,2}_0(1,+\infty)$ of minimizers of $\mathcal{A}_{x_n}$. By the uniform coercivity estimates in Lemma 3.2.2, the sequence $(\varphi^{x_n})_n$ is uniformly bounded. Moreover, up to a subsequence, it converges uniformly on compact subsets of $[1,+\infty)$ and weakly in $\mathcal{D}^{1,2}_0(1,+\infty)$ to a function $\bar{\varphi} \in \mathcal{D}^{1,2}_0(1,+\infty)$.

We decompose the action $\mathcal{A}_x$ as in Lemma 3.3.3. Since it is a quadratic form, we know that $\varphi \mapsto Q_\infty(\varphi,\varphi)$ is weakly lower semi-continuous on $\mathcal{D}^{1,2}_0(1,+\infty)$.

From previous arguments, there exists $\bar{T} > 1$ such that on the interval $[\bar{T},+\infty)$, the integrand in $P_{x,\infty}(\varphi)$ – here simply $P_x(\varphi)$ – is dominated by an L^1-function.

Splitting $P_{x_n}(\varphi^{x_n})$ into the two parts:

$$P_{x_n}(\varphi^{x_n}) = \int_1^{\bar{T}} V(x_n,t,\varphi^{x_n}(t))\,\mathrm{d}t + \int_{\bar{T}}^{+\infty} V(x_n,t,\varphi^{x_n}(t))\,\mathrm{d}t,$$

we observe the following. On $[1,\bar{T}]$, uniform convergence of φ^{x_n} to $\bar{\varphi}$ ensures convergence of the integral, while on $[\bar{T},+\infty)$, the dominated convergence theorem, together with (3.3.5), gives

$$P_x(\bar{\varphi}) = \lim_{n\to+\infty} P_{x_n}(\varphi^{x_n}).$$

Combining this with the weak-lower semicontinuity of Q, we obtain

$$\begin{aligned} w_\infty(x) &\le \mathcal{A}_x(\bar{\varphi}) = Q(\bar{\varphi},\bar{\varphi}) + P_x(\bar{\varphi}) \\ &\le \liminf_{n\to+\infty} Q(\varphi^{x_n},\varphi^{x_n}) + \liminf_{n\to+\infty} P_{x_n}(\varphi^{x_n}) \\ &\le \liminf_{n\to+\infty} \mathcal{A}_{x_n}(\varphi^{x_n}) = \liminf_{n\to+\infty} w_\infty(x_n). \end{aligned}$$

□

Now, we prove the convergence of $w(T,x)$ to $w_\infty(x)$.

Lemma 3.3.5 (Finite-horizon approximation, Berti, Polimeni and Terracini 2025 [12])**.** *Let $(T_n)_n \subset (1,+\infty)$ be a sequence of finite times such that $T_n \to +\infty$, as $n \to +\infty$. Then,*

$$\lim_{n\to+\infty} w(T_n,x) = w_\infty(x) \quad \textit{uniformly on compact subsets of } \Omega.$$

Proof. Consider a fixed compact subset $\mathcal{K}$ of $\mathcal{X}$. For every $x \in \mathcal{K}$, and $n \in \mathbb{N}$, by the coercivity of $\mathcal{A}_{x,[1,T_n]}$, there exists $\varphi^n \in \mathcal{D}_0^{1,2}(1,T_n)$ such that

$$w(T_n,x) = \mathcal{A}_{x,[1,T_n]}(\varphi^n).$$

Define an extension $\tilde{\varphi}^n \in \mathcal{D}_0^{1,2}(1,+\infty)$ of φ^n to $\mathcal{D}_0^{1,2}(1,+\infty)$ as follows:

$$\tilde{\varphi}^n(t) = \begin{cases} \varphi^n(t), & \text{in } [1,T_n] \\ \varphi^n(2T_n - t), & \text{in } (T_n, 2T_n - 1] \\ 0, & \text{in } (2T_n - 1, +\infty). \end{cases}$$

We can immediately notice that $\mathcal{A}_{x,[1,T_n]}(\tilde{\varphi}^n) = \mathcal{A}_{x,[1,T_n]}(\varphi^n)$. Since $\|\tilde{\varphi}^n\|_{\mathcal{D}} = 2\|\varphi^n\|_{\mathcal{D}_{T_n}}$, and by the uniform coercivity of the family of functionals $\mathcal{A}_{x,[1,T_n]}$, (see Section 3.2.1), we deduce that $(\tilde{\varphi}^n)_n$ is bounded in $\mathcal{D}_0^{1,2}(1,+\infty)$. This implies the existence of $\bar{\varphi} \in \mathcal{D}_0^{1,2}(1,+\infty)$ such that $\tilde{\varphi}^n \to \bar{\varphi}$, weakly on $\mathcal{D}_0^{1,2}(1,+\infty)$, uniformly on compact subsets of $(1,+\infty)$, and pointwise on $(1,+\infty)$.

Now, we want to prove that

$$\mathcal{A}_x(\bar{\varphi}) = \lim_{n\to+\infty} \mathcal{A}_{x,[1,T_n]}(\tilde{\varphi}^n) = \lim_{n\to+\infty} \mathcal{A}_{x,[1,T_n]}(\varphi^n),$$

uniformly on compact sets with respect to x.

Hyperbolic case. As in the analysis of coercivity for the hyperbolic case, we consider the renormalized Lagrangian action

$$\mathcal{A}_{x,[1,T_n]}(\tilde{\varphi}^n) = \sum_{i<j} m_i m_j \int_1^{T_n} \frac{1}{2M} \langle \dot{\tilde{\varphi}}^n_{ij}(t), \dot{\tilde{\varphi}}^n_{ij}(t)\rangle + U_{ij}(at+\tilde{\varphi}^n(t)+x-a) - U_{ij}(at)\, dt,$$

where

$$U_{ij}(at + \tilde{\varphi}^n(t) + x - a) = \frac{1}{|a_{ij}t + \tilde{\varphi}^n_{ij}(t) + x_{ij} - a_{ij}|}.$$

Since $\tilde{\varphi}^n$ is a critical point of the action, we have

$$d\mathcal{A}_{x,[1,T_n]}(\tilde{\varphi}^n)[\psi] = 0, \qquad \forall\, \psi \in \mathcal{D}^{1,2}_0(1,T_n),$$

that is,

$$\int_1^{T_n} \frac{1}{M} \langle \dot{\tilde{\varphi}}^n_{ij}(t), \dot{\psi}_{ij}(t)\rangle + \langle \nabla U_{ij}(at + \tilde{\varphi}^n(t) + x - a), \psi_{ij}(t)\rangle\, dt = 0$$

for all $\psi \in \mathcal{D}^{1,2}_0(1,T_n)$.

Letting $\bar{\varphi}^n = \bar{\varphi}|_{[1,T_n]}$, we may choose $\psi(t) = \tilde{\varphi}^n_{ij}(t) - \bar{\varphi}^n_{ij}(t)$, obtaining

$$\int_1^{T_n} \frac{1}{M} \langle \dot{\tilde{\varphi}}^n_{ij}(t), \dot{\tilde{\varphi}}^n_{ij}(t) - \dot{\bar{\varphi}}^n_{ij}(t)\rangle + \langle \nabla U_{ij}(at + \tilde{\varphi}^n(t) + x - a), \tilde{\varphi}^n_{ij}(t) - \bar{\varphi}^n_{ij}(t)\rangle\, dt = 0.$$

We first analyze the potential term. We have

$$\begin{aligned}\Big|\langle \nabla U_{ij}(at + \tilde{\varphi}^n(t) + x - a), \tilde{\varphi}^n_{ij}(t) - \bar{\varphi}^n_{ij}(t)\rangle\Big| &\le C \frac{|\tilde{\varphi}^n_{ij}(t) - \bar{\varphi}^n_{ij}(t)|}{|a_{ij}t + \tilde{\varphi}^n_{ij}(t) + x_{ij} - a_{ij}|^2} \\ &\le C' \frac{t^{1/2}}{|a_{ij}t + \tilde{\varphi}^n_{ij}(t) + x_{ij} - a_{ij}|^2},\end{aligned}$$

where the constants C, C' depend only on the bounded norms $\|\varphi_{ij}\|_{\mathcal{D}_T}$. As $t \to +\infty$, the right-hand side is dominated by $\frac{C'}{t^{3/2}} \in L^1(1,+\infty)$. Since $\tilde{\varphi}^n_{ij}(t) - \bar{\varphi}^n_{ij}(t) \to 0$ for $t \in (1,+\infty)$, the dominated convergence theorem yields

$$\int_1^{+\infty} \langle \nabla U_{ij}(at + \tilde{\varphi}^n(t) + x - a), \tilde{\varphi}^n_{ij}(t) - \bar{\varphi}^n_{ij}(t)\rangle\, dt \longrightarrow 0,$$

and in particular

$$\int_1^{+\infty} \langle \nabla U_{ij}(at + \tilde{\varphi}^n(t) + x - a), \tilde{\varphi}^n_{ij}(t)\rangle - \langle \nabla U_{ij}(at + \bar{\varphi}(t) + x - a), \bar{\varphi}^n_{ij}(t)\rangle\, dt \longrightarrow 0. \tag{3.3.8}$$

Moreover, by the weak convergence of $(\tilde{\varphi}^n)_n$ in $\mathcal{D}_0^{1,2}(1,+\infty)$,

$$\lim_{n\to+\infty}\int_1^{T_n}\langle\dot{\tilde{\varphi}}_{ij}^n(t),\dot{\bar{\varphi}}_{ij}^n(t)\rangle\,dt=\int_1^{+\infty}|\dot{\bar{\varphi}}_{ij}(t)|^2\,dt,$$

which implies

$$\lim_{n\to+\infty}\int_1^{T_n}|\dot{\tilde{\varphi}}_{ij}^n(t)|^2\,dt=\int_1^{+\infty}|\dot{\bar{\varphi}}_{ij}(t)|^2\,dt. \tag{3.3.9}$$

Combining (3.3.8) and (3.3.9), and recalling that

$$\begin{aligned}\mathcal{A}_{x,[1,T_n]}(\tilde{\varphi}^n)&=\sum_{i<j}m_im_j\int_1^{T_n}\frac{\langle\dot{\tilde{\varphi}}_{ij}^n(t),\dot{\tilde{\varphi}}_{ij}^n(t)\rangle}{2M}+U_{ij}(at+\tilde{\varphi}^n(t)+x-a)-U_{ij}(at)\,dt\\&=\sum_{i<j}m_im_j\int_0^1\int_1^{T_n}\frac{\langle\dot{\tilde{\varphi}}_{ij}^n(t),\dot{\tilde{\varphi}}_{ij}^n(t)\rangle}{2M}\\&\quad+\langle\nabla U_{ij}(at+s(\tilde{\varphi}^n(t)+x-a)),\tilde{\varphi}_{ij}^n(t)+x_{ij}-a_{ij}\rangle\,dt\,ds,\end{aligned}$$

we conclude that

$$\lim_{n\to+\infty}\mathcal{A}_{x,[1,T_n]}(\tilde{\varphi}^n)=\mathcal{A}_x(\bar{\varphi})\quad\text{pointwise.}$$

The convergence is in fact uniform on compact subsets with respect to x. Indeed,

$$|\mathcal{A}_x(\tilde{\varphi}^n)-\mathcal{A}_x(\bar{\varphi})|=$$
$$\left|\sum_{i<j}m_im_j\int_1^{+\infty}\int_0^1\langle\nabla U_{ij}(at+\bar{\varphi}(t)+x-a+\theta(\tilde{\varphi}^n(t)-\bar{\varphi}(t))),\tilde{\varphi}_{ij}^n(t)-\bar{\varphi}_{ij}(t)\rangle\,d\theta\,dt\right|.$$

Given $\varepsilon>0$, there exist $T_\varepsilon>1$ and N_ε such that for all $n\geq N_\varepsilon$,

$$\int_{T_\varepsilon}^{+\infty}\int_0^1\langle\nabla U_{ij}(at+\bar{\varphi}(t)+x-a+\theta(\tilde{\varphi}^n(t)-\bar{\varphi}(t))),\tilde{\varphi}_{ij}^n(t)-\bar{\varphi}_{ij}(t)\rangle\,d\theta\,dt<\frac{\varepsilon}{2},$$

since for large t there exists a constant C'', independent of x, such that

$$\left|\langle\nabla U_{ij}(at+\bar{\varphi}(t)+x-a+\theta(\tilde{\varphi}^n(t)-\bar{\varphi}(t))),\tilde{\varphi}_{ij}^n(t)-\bar{\varphi}_{ij}(t)\rangle\right|\leq\frac{C''}{t^{3/2}}.$$

The remaining integral over $[1,T_\varepsilon]$ is controlled using the continuity with respect to x, the uniform convergence $\tilde{\varphi}^n\to\bar{\varphi}$, and the fact that for n large enough

$$|at+\bar{\varphi}+x-a+\theta(\tilde{\varphi}^n-\bar{\varphi})|\geq\delta/2$$

for some $\delta>0$.

Parabolic case. In the parabolic case, as in Lemma 3.3.3, we consider

$$\begin{aligned}\mathcal{A}_{x,[1,T_n]}(\tilde{\varphi}^n) = \int_1^{T_n} \frac{1}{2}\|\dot{\tilde{\varphi}}^n(t)\|_{\mathcal{M}} + \frac{1}{2}\langle \nabla^2 U(\beta b_m t^{2/3})\tilde{\varphi}^n(t), \tilde{\varphi}^n(t)\rangle \\ + U(\beta b_m t^{2/3} + \tilde{\varphi}^n(t) + x - \beta b_m) - U(\beta b_m t^{2/3}) \\ - \langle \nabla U(\beta b_m t^{2/3}), \tilde{\varphi}^n(t)\rangle - \frac{1}{2}\langle \nabla^2 U(\beta b_m t^{2/3})\tilde{\varphi}^n(t), \tilde{\varphi}^n(t)\rangle \, dt \\ = Q_{T_n}(\tilde{\varphi}^n, \tilde{\varphi}^n) + P_{x,T_n}(\tilde{\varphi}^n).\end{aligned}$$

As before, the differential of the action vanishes at $\tilde{\varphi}^n$. Arguing as in the hyperbolic case and applying dominated convergence, we obtain

$$\lim_n P_{x,T_n}(\tilde{\varphi}^n) = P(\bar{\varphi}), \qquad \lim_n Q_{T_n}(\tilde{\varphi}^n, \tilde{\varphi}^n) = Q(\bar{\varphi}, \bar{\varphi}),$$

and therefore

$$\lim_{n\to+\infty} \mathcal{A}_{x,[1,T_n]}(\tilde{\varphi}^n) = \mathcal{A}_x(\bar{\varphi}).$$

Uniform convergence on compact sets follows as in the hyperbolic case.

Hyperbolic-parabolic case. Finally, in the hyperbolic-parabolic situation we use the decomposition of the renormalized Lagrangian action given in (3.2.1). By treating the resulting terms separately and repeating the arguments developed in the hyperbolic and parabolic cases, we reach the same conclusion.

□

3.3.2 Semiconcavity with linear modulus of the value function

By establishing that the value function v is semiconcave with linear modulus, we also obtain its local Lipschitz continuity on the set Ω.

Proposition 3.3.6 (Berti, Polimeni and Terracini 2025 [12])**.** *For any set $W \subset\subset \Omega$ there is a constant $C > 0$ such that for any $x, z \in \mathcal{X}$ with $x, x+z, x-z \in W$ it holds*

$$v(x+z) + v(x-z) - 2v(x) \le C\|z\|^2. \tag{3.3.10}$$

Proof. From the uniform coercivity estimates, we deduce that there exists a constant $\hat{C} > 0$ such that $\|\varphi^x\|_{\mathcal{D}} \le \hat{C}$ for all $x \in W$.

Fix a set $W \subset\subset \mathcal{X} \setminus \Delta$. By Lemma 3.2.3, there exists $\delta > 0$ such that

$$\inf_{t\ge 1} d(\gamma^x(t), \Delta) \ge \delta,$$

where $\gamma^x(t) = r_0(t) + \varphi^x(t) + x - r_0(1)$ and φ^x is a minimizer of the functional $\mathcal{A}_x$.

Since v is continuous on the set $\overline{W}$, it is bounded on $\overline{W}$. Therefore, it remains to verify (3.3.10) only for sufficiently small values of $\|z\|$.

For $\|z\|$ small enough, we obtain

$$\inf_{t\geq 1} d(\gamma^x(t) \pm sz, \Delta) \geq \frac{\delta}{2}, \quad s \in [0{,}1].$$

Thus

$$\begin{aligned}
&v(x+z) + v(x-z) - 2v(x)\\
&\leq \int_1^{+\infty} L^{ren}_{x+z}(\varphi^x(t)) + L^{ren}_{x-z}(\varphi^x(t)) - 2L^{ren}_{x}(\varphi^x(t))\ dt\\
&= \int_1^{+\infty} U(r_0(t) + \varphi^x(t) + x + z - r_0(1)) + U(r_0(t) + \varphi^x(t) + x - z - r_0(1))\\
&\quad - 2U(r_0(t) + \varphi^x(t) + x - r_0(1))\ dt\\
&= \int_1^{+\infty}\int_0^1 \langle \nabla U(r_0(t) + \varphi^x(t) + x - r_0(1) + sz), z\rangle\\
&\quad - \langle \nabla U(r_0(t) + \varphi^x(t) + x - r_0(1) - sz), z\rangle\ ds\ dt\\
&= \int_1^{+\infty}\int_0^1\int_0^1 -2\langle \nabla^2 U(r_0(t) + \varphi^x(t) + x - r_0(1) + (1-2\tau)sz)sz, z\rangle\ d\tau\ ds\ dt,
\end{aligned}$$

where we denote $L^{ren}_x(\varphi^x(t)) = \frac{1}{2}\|\varphi^x(t)\|^2_{\mathcal{M}} + U(r_0(t)+\varphi^x(t)+x-r_0(1)) - U(r_0(t)) - \langle \ddot r_0(t), \varphi^x(t)\rangle_{\mathcal{M}}$. The thesis follows from the fact that there is a constant $C' > 0$ with the property that there is $\varepsilon > 0$ such that for $\|z\| < \varepsilon$ and for any $x \in W$

$$\|\nabla^2 U(\gamma^x(t) + z)\| \leq \frac{C'}{t^2}, \quad \forall t \geq 1,$$

where we used the operator norm of a matrix.

□

In particular, the semiconcavity of the value function v implies its local Lipschitz continuity (see e.g. [18]).

3.3.3 Proof of Theorem 3.3.1

We prove Theorem 3.3.1 by employing the finite-horizon approximation provided in Proposition 3.3.2, together with the *stability principle* for viscosity solutions. The central idea is to demonstrate that each function $v(T, x)$ satisfies an appropriate differential equation, as stated in the following proposition.

We recall that, analogous to Definition 1.4.1, for a function $v : [1, +\infty) \times \Omega \to \mathbb{R}$, we define the Fréchet superdifferential and subdifferential of v at (T,x) as

$$D^-v(T,x) = \Big\{ (p_T,p_x) \in \mathbb{R}\times\mathcal{X} : \\ \liminf_{(\bar T,y)\to(T,x)} \frac{v(\bar T,y)-v(T,x)-p_T(\bar T-T)-\langle p_x,y-x\rangle_{\mathcal{M}}}{\sqrt{(\bar T-T)^2+\|y-x\|^2_{\mathcal{M}}}} \geq 0 \Big\},$$

and

$$D^+v(T,x) = \Big\{ (p_T,p_x) \in \mathbb{R}\times\mathcal{X} : \\ \limsup_{(\bar T,y)\to(T,x)} \frac{v(\bar T,y)-v(T,x)-p_T(\bar T-T)-\langle p_x,y-x\rangle_{\mathcal{M}}}{\sqrt{(\bar T-T)^2+\|y-x\|^2_{\mathcal{M}}}} \leq 0 \Big\}.$$

Given a function $F:[1,+\infty)\times\Omega\to\mathbb{R}$, we consider the equation

$$\frac{\partial v}{\partial T}(T,x)+H(x,\nabla v(T,x))+F(T,x)=0. \tag{3.3.11}$$

We say that a continuous function v is a viscosity supersolution of (3.3.11) if, for every (T,x) and $(p_T,p_x)\in D^-v(T,x)$, it holds

$$p_T+H(x,p_x)+F_*(T,x)\geq 0,$$

where F_* is the *lower semicontinuous envelope* of F; v is called viscosity subsolution of (3.3.11) if, for every (T,x) and $(p_T,p_x)\in D^+v(T,x)$, it holds

$$p_T+H(x,p_x)+F^*(T,x)\leq 0,$$

where F^* is the *upper semicontinuos envelope* of F.

If v is both a viscosity supersolution and a viscosity subsolution, then it is called a viscosity solution of (3.3.11).

Proposition 3.3.7 (Berti, Polimeni and Terracini 2025 [12]). *Let (T,x) be in $(1,+\infty)\times\Omega$. Then:*

- *there exists a minimizer $\varphi_{x,T}$ of $\mathcal{A}_{x,[1,T]}$ on $\mathcal{D}^{1,2}_0(1,T)$ such that $v(T,x)$ is a viscosity subsolution of*

$$\frac{\partial v}{\partial T}(T,x)+H(x,\nabla v(T,x))+\langle \ddot r_0(T),\varphi_{x,T}(T)+x\rangle_{\mathcal{M}}-H(r_0(T),\dot r_0(T))\leq 0,$$

- *for all $\varphi_{x,T}$ minimizers of $\mathcal{A}_{x,[1,T]}$ on $\mathcal{D}_0^{1,2}(1,T)$, $v(T,x)$ is a viscosity super-solution of*

$$\frac{\partial v}{\partial T}(T,x) + H(x, \nabla v(T,x)) + \langle \ddot{r}_0(T), \varphi_{x,T}(T) + x\rangle_{\mathcal{M}} - H(r_0(T), \dot{r}_0(T)) \geq 0.$$

In particular, the function $v(T,x)$ is a viscosity solution in $(1,+\infty)\times\Omega$ of

$$\frac{\partial v}{\partial T}(T,x) = -\frac{1}{2}\|\nabla v(T,x)\|^2_{\mathcal{M}^{-1}} + U(x) - \inf_{\varphi_{x,T}\in\mathcal{Z}} \langle \ddot{r}_0(T), \varphi_{x,T}(T) + x\rangle_{\mathcal{M}}$$
$$+ \frac{1}{2}\|\dot{r}_0(T)\|^2_{\mathcal{M}} - U(r_0(T)),$$

where $\mathcal{Z} = \{\varphi_{x,T} : \mathcal{A}_{x,[1,T]}(\varphi_{x,T}) = \min_{\varphi\in\mathcal{D}_0^{1,2}(1,T)} \mathcal{A}_{x,[1,T]}(\varphi)\}$.

Moreover, it holds

$$\lim_{T\to+\infty} \frac{\partial v}{\partial T}(T,x) = 0. \tag{3.3.12}$$

Taking Proposition 3.3.7, Theorem 3.3.1 follows at once, as shown in the proof below.

Proof of Theorem 3.3.1. From the uniform coercivity estimates, fixed x, we have that $|\varphi_{x,T}(T)| \leq C\sqrt{T}$, and hence

$$\left|\langle \ddot{r}_0(T), \varphi_{x,T}(T)\rangle_{\mathcal{M}}\right| \leq \|\ddot{r}_0(T)\|_{\mathcal{M}} C\sqrt{T} \to 0, \quad \text{as} \quad T\to+\infty.$$

From a direct inspection,

$$\frac{1}{2}\|\dot{r}_0(T)\|^2_{\mathcal{M}} - U(r_0(T)) \to \frac{\|a\|^2_{\mathcal{M}}}{2}, \quad \text{as} \quad T\to+\infty.$$

Therefore, combining (3.3.12) with the uniform convergence of $v(T,x)$ to $v(x)$ on compact subsets of Ω (Proposition 3.3.2), and invoking the stability principle for viscosity solutions (see, e.g., [25]), we deduce that the uniform limit $v(x)$ satisfies the limiting equation (3.3.2) in the viscosity sense.

Although the stability principle in [25] is stated for sequences of continuous equations, it nevertheless applies here, since the only potentially discontinuous term,

$$\inf_{\varphi_{x,T}\in\mathcal{Z}} \langle \ddot{r}_0(T), \varphi_{x,T}(T) + x\rangle_{\mathcal{M}},$$

vanishes as $T\to+\infty$, uniformly on compact sets with respect to x. □

In the last part of this section, we prove Proposition 3.3.7. To do this, we need several lemmas, which involve supplementary value functions $u(T,x)$ defined for any $x \in \Omega$ and $T > 1$. We define

$$\begin{aligned} u(T,x) &= \min_{\eta \in H^1([1,T]):\, \eta(1)=x} \int_1^T L(\eta(t),\dot\eta(t))\,dt - \langle \dot r_0(T), \eta(T)\rangle_{\mathcal{M}} \\ &= \int_1^T L(\gamma(t),\dot\gamma(t))\,dt - \langle \dot r_0(T), \gamma(T)\rangle_{\mathcal{M}}, \end{aligned} \tag{3.3.13}$$

where γ satisfies

$$\begin{cases} \ddot\gamma = \nabla U(\gamma) \ \text{ a.e. in } \ (1,T) \\ \gamma(1) = x \\ \dot\gamma(T) = \dot r_0(T). \end{cases} \tag{3.3.14}$$

Notice that, in general, Problem (3.3.14) is not uniquely solvable.

Furthermore, by setting $\hat\gamma(t) = \gamma(T+1-t)$, we get a solution of

$$\begin{cases} \ddot{\hat\gamma} = \nabla U(\hat\gamma) \ \text{ a.e. in } \ (1,T) \\ \hat\gamma(T) = x \\ \dot{\hat\gamma}(1) = -\dot r_0(T), \end{cases}$$

that satisfies

$$\begin{aligned} u(T,x) &= \int_1^T L(\hat\gamma(t),\dot{\hat\gamma}(t))\,dt - \langle \dot r_0(T), \hat\gamma(1)\rangle_{\mathcal{M}} \\ &= \min_{\eta \in H^1([1,T]):\, \eta(T)=x} \int_1^T L(\eta(t),\dot\eta(t))\,dt - \langle \dot r_0(T), \eta(1)\rangle_{\mathcal{M}}. \end{aligned} \tag{3.3.15}$$

Following the arguments of Cannarsa and Sinestrari, we aim to establish an inequality analogous to the dynamic programming principle (Theorem 1.2.2 in [18]). This leads to the following result.

Lemma 3.3.8 (Berti, Polimeni and Terracini 2025 [12])**.** *Let $T \in \mathbb{R}$, $T > 1$, and $x \in \Omega$, and consider a curve $\overline\gamma \in H^1([1,T])$ such that $\overline\gamma(T) = x$. Then, for all $T' \in [1,T]$,*

$$u(T,x) \le u(T',\overline\gamma(T')) + \int_{T'}^T L(\overline\gamma(t),\dot{\overline\gamma}(t))\ dt - \langle \dot r_0(T) - \dot r_0(T'), \tilde\gamma(1)\rangle_{\mathcal{M}}, \tag{3.3.16}$$

where $\tilde\gamma \in H^1([1,T'])$ realizes the minimum in $u(T',\overline\gamma(T'))$ and is such that $\tilde\gamma(T') = \overline\gamma(T')$.

Proof. Fix $T' \in [1,T]$ and let $\tilde\gamma \in H([1,T'])$ such that $\tilde\gamma(T') = \overline\gamma(T')$. Setting

$$\xi(t) = \begin{cases} \tilde\gamma(t), & t \in [1,T'] \\ \overline\gamma(t), & t \in [T',T] \end{cases}$$

we have $\xi \in H^1([1,T])$ and $\xi(T) = x$. Therefore,

$$\begin{aligned} u(T,x) &\leq \int_1^T L(\xi(t),\dot{\xi}(t))\,dt - \langle \dot{r}_0(T), \xi(1)\rangle_{\mathcal{M}} \\ &= \int_1^{T'} L(\tilde{\gamma}(t),\dot{\tilde{\gamma}}(t))\,dt + \int_{T'}^{T} L(\overline{\gamma}(t),\dot{\overline{\gamma}}(t))\,dt - \langle \dot{r}_0(T), \tilde{\gamma}(1)\rangle_{\mathcal{M}} \\ &= \int_1^{T'} L(\tilde{\gamma}(t),\dot{\tilde{\gamma}}(t))\,dt + \int_{T'}^{T} L(\overline{\gamma}(t),\dot{\overline{\gamma}}(t))\,dt - \langle \dot{r}_0(T'), \tilde{\gamma}(1)\rangle_{\mathcal{M}} \\ &\quad - \langle \dot{r}_0(T) - \dot{r}_0(T'), \tilde{\gamma}(1)\rangle_{\mathcal{M}}. \end{aligned}$$

Taking the infimum over all $\tilde{\gamma} \in H^1([1,T'])$, we get (3.3.16). □

We now exploit Lemma 3.3.8 to prove the next result.

Lemma 3.3.9 (Berti, Polimeni and Terracini 2025 [12]). *Fix (T,x) in $(1,+\infty)\times\Omega$. Then, we have the following:*

- *for every $(p_T, p_x) \in D^+u(T,x)$, there exists a curve γ realizing $u(T,x)$ in (3.3.15) such that, in the viscosity sense,*

$$p_T + H(x,p_x) + \langle \ddot{r}_0(T), \gamma(1)\rangle_{\mathcal{M}} \leq 0;$$

- *for every $(p_T, p_x) \in D^-u(T,x)$ and for every γ realizing $u(T,x)$ in (3.3.15) it holds, in the viscosity sense,*

$$p_T + H(x,p_x) + \langle \ddot{r}_0(T), \gamma(1)\rangle_{\mathcal{M}} \geq 0.$$

In particular, the function $u(T,x)$ defined in (3.3.13) satisfies, in the viscosity sense,

$$-\frac{\partial u}{\partial T}(T,x) = -H(x,\nabla u(T,x)) + \inf_{\gamma\in\mathcal{Y}} \langle \ddot{r}_0(T), \gamma(T)\rangle_{\mathcal{M}}, \qquad (3.3.17)$$

where $\mathcal{Y}$ is the set of curves $\gamma \in H^1([1,T])$, $\gamma(1) = x$ that realize $u(T,x)$.

Before providing the rigorous proof of Lemma 3.3.9, which establishes that $u(T,x)$ is a viscosity solution of (3.3.17), we first describe the heuristic argument underlying the result.

Formally differentiating the value function with respect to x at a point of differentiability gives

$$\nabla u(T,x) = -\mathcal{M}\dot{\gamma}(1).$$

Indeed, denoting $\psi(t) := \frac{\partial \gamma}{\partial x_i}(t)$ and with e_i the i-th element of the canonical basis, we have

$$\begin{aligned}
\frac{\partial u}{\partial x_i}(T,x) &= \int_1^T \frac{\partial L}{\partial \gamma}(\gamma(t),\dot\gamma(t))\psi(t) + \frac{\partial L}{\partial \dot\gamma}(\gamma(t),\dot\gamma(t))\dot\psi(t)\ dt - \langle \dot r_0(T), \psi(T)\rangle_{\mathcal{M}} \\
&= \frac{\partial L}{\partial \dot\gamma}(\gamma(T),\dot\gamma(T))\psi(T) - \frac{\partial L}{\partial \dot\gamma}(\gamma(1),\dot\gamma(1))\psi(1) \\
&\quad - \int_1^T \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\gamma}(\gamma(t),\dot\gamma(t))\right) - \frac{\partial L}{\partial \gamma}(\gamma(t),\dot\gamma(t))\right)\psi(t)\ dt - \langle \dot r_0(T), \psi(T)\rangle_{\mathcal{M}} \\
&= \langle \dot r_0(T), \psi(T)\rangle_{\mathcal{M}} - \langle \dot\gamma(1), \psi(1)\rangle_{\mathcal{M}} - \langle \dot r_0(T), \psi(T)\rangle_{\mathcal{M}} \\
&= -m_i\dot\gamma(1)\cdot e_i,
\end{aligned}$$

which follows from integration by parts. Besides, integrating by parts and using the conservation of the energy, we denote $\phi(t) := \frac{\partial\gamma}{\partial T}(t)$ and compute

$$\begin{aligned}
\frac{\partial u}{\partial T}(T,x) &= L(\gamma(T),\dot\gamma(T)) + \int_1^T \frac{\partial L}{\partial \gamma}(\gamma(t),\dot\gamma(t))\phi(t) + \frac{\partial L}{\partial \dot\gamma}(\gamma(t),\dot\gamma(t))\dot\phi(t)\ dt \\
&\quad - \langle \ddot r_0(T), \gamma(T)\rangle_{\mathcal{M}} - \langle \dot r_0(T), \dot\gamma(T)\rangle_{\mathcal{M}} - \langle \dot r_0(T), \phi(T)\rangle_{\mathcal{M}} \\
&= L(\gamma(T),\dot\gamma(T)) + \frac{\partial L}{\partial \dot\gamma}(\gamma(T),\dot\gamma(T))\phi(T) - \frac{\partial L}{\partial \dot\gamma}(\gamma(1),\dot\gamma(1))\phi(1) \\
&\quad - \int_1^T \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\gamma}(\gamma(t),\dot\gamma(t))\right) - \frac{\partial L}{\partial \gamma}(\gamma(t),\dot\gamma(t))\right)\phi(t)\ dt \\
&\quad - \langle \ddot r_0(T), \gamma(T)\rangle_{\mathcal{M}} - \langle \dot r_0(T), \dot\gamma(T)\rangle_{\mathcal{M}} - \langle \dot r_0(T), \phi(T)\rangle_{\mathcal{M}},
\end{aligned}$$

and therefore

$$\begin{aligned}
\frac{\partial u}{\partial T}(T,x) &= -\frac{1}{2}\|\dot\gamma(T)\|^2_{\mathcal{M}} + U(\gamma(T)) - \langle \ddot r_0(T), \gamma(T)\rangle_{\mathcal{M}} \\
&= -\frac{1}{2}\|\dot\gamma(1)\|^2_{\mathcal{M}} + U(\gamma(1)) - \langle \ddot r_0(T), \gamma(T)\rangle_{\mathcal{M}} \\
&= -H(x,\nabla u(T,x)) - \langle \ddot r_0(T), \gamma(T)\rangle_{\mathcal{M}}.
\end{aligned}$$

Now, we prove the same result in a rigorous way.

Proof of Lemma 3.3.9. In order to simplify the computations, we adopt the representation of $u(T,x)$ given in the last line of (3.3.15), so that the endpoint of the curve γ realizing $u(T,x)$ coincides with x.

Fix $T > 1$, $x \in \Omega$, and let $(p_T, p_x) \in D^+u(T,x)$. Then, for every $z \in \mathcal{X}$, we have

$$\limsup_{h\to 0^+} \frac{u(T-h, x-hz) - u(T,x) + h\Big(p_T + \langle z, p_x\rangle_{\mathcal{M}}\Big)}{h\sqrt{1+\|z\|^2_{\mathcal{M}}}} \le 0,$$

which is equivalent to

$$\limsup_{h\to 0^+} \frac{u(T-h, x-hz) - u(T,x)}{h} \leq -p_T - \langle z, p_x\rangle_{\mathcal{M}}.$$

Defining $\zeta(t) = x + (t-T)z$, with $z \in \mathcal{X}$, we may apply (3.3.16) to obtain

$$\begin{aligned} u(T,x) &\leq u(T-h, \zeta(T-h)) + \int_{T-h}^{T} L(\zeta(t), \dot\zeta(t))\, dt \\ &\quad - \langle \dot r_0(T) - \dot r_0(T-h), \gamma_h(1)\rangle_{\mathcal{M}} \\ &= u(T-h, x-hz) + \int_{T-h}^{T} L(x + (t-T)z, z)\, dt \\ &\quad - \langle \dot r_0(T) - \dot r_0(T-h), \gamma_h(1)\rangle_{\mathcal{M}}, \end{aligned}$$

where γ_h denotes a minimizer achieving $u(T-h, \zeta(T-h))$. It follows that

$$\begin{aligned} \limsup_{h\to 0^+} \frac{u(T-h, x-hz) - u(T,x)}{h} &\geq \lim_{h\to 0^+} \Big(-\frac{1}{h}\int_{T-h}^{T} L(x+(t-T)z, z)\, dt \\ &\qquad + \frac{1}{h}\langle \dot r_0(T) - \dot r_0(T-h), \gamma_h(1)\rangle_{\mathcal{M}} \Big) \\ &= -L(x,z) + \langle \ddot r_0(T), \gamma(1)\rangle_{\mathcal{M}}, \end{aligned}$$

where we used the convergence

$$\gamma_h(1) \to \gamma(1) \quad \text{as } h \to 0^+, \tag{3.3.18}$$

with γ a minimizer realizing $u(T,x)$. Observe that, in general, not every minimizer γ arises as such a limit of γ_h.

Combining the previous inequalities, we deduce that

$$p_T + \langle p_x, z\rangle_{\mathcal{M}} - L(x,z) + \langle \ddot r_0(T), \gamma(1)\rangle_{\mathcal{M}} \leq 0, \qquad \text{for all } z \in \mathcal{X},$$

which, upon choosing $z = p_x$, yields

$$p_T + H(x, p_x) + \langle \ddot r_0(T), \gamma(1)\rangle_{\mathcal{M}} \leq 0,$$

for every $\gamma \in \mathcal{Y}$ satisfying (3.3.18). Consequently,

$$p_T + H(x, p_x) + \inf_{\gamma\in\mathcal{Y}} \langle \ddot r_0(T), \gamma(1)\rangle_{\mathcal{M}} \leq 0,$$

that is, $u(T,x)$ is a viscosity subsolution of

$$\frac{\partial u}{\partial T} + H(x, \nabla u(T,x)) + \inf_{\gamma\in\mathcal{Y}} \langle \ddot r_0(T), \gamma(1)\rangle_{\mathcal{M}} = 0.$$

To establish the viscosity supersolution property, let γ be a minimizer for $u(T,x)$ with $x=\gamma(T)$, and set $w=\dot\gamma(T)$. Then

$$\begin{aligned}
&u(T-h,\gamma(T-h))-u(T,x)\\
&\le \int_1^{T-h} L(\gamma(t),\dot\gamma(t))\,dt-\langle \dot r_0(T-h),\gamma(1)\rangle_{\mathcal M}\\
&\quad-\int_1^{T} L(\gamma(t),\dot\gamma(t))\,dt+\langle \dot r_0(T),\gamma(1)\rangle_{\mathcal M}\\
&=-\int_{T-h}^{T} L(\gamma(t),\dot\gamma(t))\,dt+\langle \dot r_0(T)-\dot r_0(T-h),\gamma(1)\rangle_{\mathcal M}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{h\to0^+}\frac{u(T-h,\gamma(T-h))-u(T,x)}{h}\\
&\le \lim_{h\to0^+}\left(-\frac1h\int_{T-h}^{T} L(\gamma(t),\dot\gamma(t))\,dt+\frac1h\langle \dot r_0(T)-\dot r_0(T-h),\gamma(1)\rangle_{\mathcal M}\right)\\
&=-L(x,w)+\langle \ddot r_0(T),\gamma(1)\rangle_{\mathcal M}.
\end{aligned}$$

Given $(p_T,p_x)\in D^-u(T,x)$, the Lipschitz continuity of $u(T,x)$ (see Section 3.3.2) implies

$$\liminf_{h\to0^+}\frac{u(T-h,\gamma(T-h))-u(T,x)}{h}\ge -p_T-\langle p_x,w\rangle_{\mathcal M}.$$

Therefore,

$$p_T+\langle p_x,w\rangle_{\mathcal M}-L(x,w)+\langle \ddot r_0(T),\gamma(1)\rangle_{\mathcal M}\ge 0,$$

which leads to

$$p_T+H(x,p_x)+\langle \ddot r_0(T),\gamma(1)\rangle_{\mathcal M}\ge 0,$$

for every $\gamma\in\mathcal Y$. As a consequence, $u(T,x)$ is a viscosity supersolution of

$$\frac{\partial u}{\partial T}+H(x,\nabla u(T,x))+\inf_{\gamma\in\mathcal Y}\langle \ddot r_0(T),\gamma(1)\rangle_{\mathcal M}=0.$$

We have thus shown that, for any fixed $T>1$, the value function u satisfies

$$-\frac{\partial u}{\partial T}(T,x)=\frac12\|\nabla u(T,x)\|^2_{\mathcal M^{-1}}-U(x)+\inf_{\gamma\in\mathcal Y}\langle \ddot r_0(T),\gamma(T)\rangle_{\mathcal M}$$

in the viscosity sense. □

We finally make use of the following relation between $v(T,x)$ and $u(T,x)$.

Lemma 3.3.10 (Berti, Polimeni and Terracini 2025 [12])**.** *For every $T>1$ and*

$x \in \Omega$, *it holds*

$$v(T,x) = u(T,x) - \int_1^T \frac{1}{2}\|\dot{r}_0(t)\|^2_{\mathcal{M}} + U(r_0(t))\,dt + \langle \dot{r}_0(T), r_0(T) - r_0(1)\rangle_{\mathcal{M}}.$$

Proof. We recall that $v(T,x) = w(T,x) - \langle \dot{r}_0(T), x\rangle_{\mathcal{M}}$, as defined in (3.3.3), with $w(T,x)$ defined in (3.3.4).

The statement is then true if we prove that

$$u(T,x) = w(T,x) + \int_1^T \frac{1}{2}\|\dot{r}_0(t)\|^2_{\mathcal{M}} + U(r_0(t))\,dt - \langle \dot{r}_0(T), r_0(T) + x - r_0(1)\rangle_{\mathcal{M}}.$$

This follows from the next chain of identities:

$$\begin{aligned}
u(T,x) &= \int_1^T L(\bar{\gamma}(t), \dot{\bar{\gamma}}(t))\ dt - \langle \dot{r}_0(T), \bar{\gamma}(T)\rangle_{\mathcal{M}} \\
&= \int_1^T \frac{\|\dot{\bar{\varphi}}(t)\|^2_{\mathcal{M}}}{2} + \frac{\|\dot{r}_0(t)\|^2_{\mathcal{M}}}{2} + \langle \dot{\bar{\varphi}}(t), \dot{r}_0(t)\rangle_{\mathcal{M}} + U(r_0(t) + \bar{\varphi}(t) + x - r_0(1))\ dt \\
&\quad - \langle \dot{r}_0(T), r_0(T)\rangle_{\mathcal{M}} - \langle \dot{r}_0(T), \bar{\varphi}(T)\rangle_{\mathcal{M}} - \langle \dot{r}_0(T), x - r_0(1)\rangle_{\mathcal{M}} \\
&= \int_1^T \frac{\|\dot{\bar{\varphi}}(t)\|^2_{\mathcal{M}}}{2} + U(r_0(t) + \bar{\varphi}(t) + x - r_0(1)) - \langle \ddot{r}_0(t), \bar{\varphi}(t)\rangle_{\mathcal{M}} + \frac{\|\dot{r}_0(t)\|^2_{\mathcal{M}}}{2}\ dt \\
&\quad - \langle \dot{r}_0(T), r_0(T)\rangle_{\mathcal{M}} - \langle \dot{r}_0(T), x - r_0(1)\rangle_{\mathcal{M}},
\end{aligned}$$

which gives

$$\begin{aligned}
u(T,x) = &\int_1^T \frac{\|\dot{\bar{\varphi}}(t)\|^2_{\mathcal{M}}}{2} + U(r_0(t) + \bar{\varphi}(t) + x - r_0(1)) - U(r_0(t)) - \langle \ddot{r}_0(t), \bar{\varphi}(t)\rangle_{\mathcal{M}}\ dt \\
&+ \int_1^T \frac{\|\dot{r}_0(t)\|^2_{\mathcal{M}}}{2} + U(r_0(t))\ dt - \langle \dot{r}_0(T), r_0(T)\rangle_{\mathcal{M}} - \langle \dot{r}_0(T), x - r_0(1)\rangle_{\mathcal{M}},
\end{aligned}$$

and therefore

$$u(T,x) = w(T,x) + \int_1^T \frac{\|\dot{r}_0(t)\|^2_{\mathcal{M}}}{2} + U(r_0(t))\ dt - \langle \dot{r}_0(T), r_0(T)\rangle_{\mathcal{M}} - \langle \dot{r}_0(T), x - r_0(1)\rangle_{\mathcal{M}}.$$

Above, we have taken advantage of the fact that if $\bar{\gamma}$ realizes $u(T,x)$, then $\mathcal{D}^{1,2}_0(1,T) \ni \bar{\varphi} = \bar{\gamma} - r_0(t) - x + r_0(1)$ realizes $w(T,x)$. This follows from the fact that $\bar{\varphi}$ satisfies

$$\begin{cases} \ddot{\varphi} = \nabla U(r_0 + \varphi + x - r_0(1)) \ \text{ a.e. in } \ (1,T) \\ \varphi(1) = 0 \\ \dot{\varphi}(T) = 0, \end{cases}$$

that is, φ is a critical point of $\mathcal{A}_{x,[1,T]}$ on $\mathcal{D}^{1,2}_0(1,T)$. □

Proof of Proposition 3.3.7. This follows directly from (3.3.17), once we notice that

$\nabla v(T,x) = \nabla u(T,x)$ in the viscosity sense.

We claim that

$$\lim_{T\to+\infty}\left(-\frac{1}{2}\|\dot\gamma(T)\|^2_{\mathcal{M}} + U(\gamma(T)) - \langle \ddot r_0(T), \gamma(T)\rangle_{\mathcal{M}}\right) = -\frac{\|a\|^2_{\mathcal{M}}}{2}. \tag{3.3.19}$$

Assuming that (3.3.19) holds, and recalling that

$$\frac{\partial u}{\partial T}(T,x) = -\frac{1}{2}\|\dot\gamma(T)\|^2_{\mathcal{M}} + U(\gamma(T)) - \langle \ddot r_0(T), \gamma(T)\rangle_{\mathcal{M}}$$

in the viscosity sense, we infer that

$$\lim_{T\to+\infty}\frac{\partial u}{\partial T}(T,x) = -\frac{\|a\|^2_{\mathcal{M}}}{2},$$

again in the viscosity sense.

We now prove (3.3.19):

- We have
$$\lim_{T\to+\infty}\|\dot\gamma(T)\|^2_{\mathcal{M}} = \lim_{T\to+\infty}\|\dot r_0(T)\|^2_{\mathcal{M}} = \|a\|^2_{\mathcal{M}}.$$

- Moreover,
$$\lim_{T\to+\infty} U(\gamma(T)) = 0.$$
Indeed, for T sufficiently large and for x varying in a compact set, the uniform coercivity estimates of Section 3.2.1 guarantee the existence of a constant $c\in\mathbb{R}$, independent of both T and x, such that
$$\begin{aligned} U(\gamma(T)) &= \sum_{i<j}\frac{m_i m_j}{|r_{0,ij}(T) + \varphi_{ij}(T) + x_{ij} - r_{0,ij}(1)|} \\ &\le \sum_{i<j}\frac{m_i m_j}{|r_{0,ij}(T)| - |\varphi_{ij}(T)| - |x_{ij} - r_{0,ij}(1)|} \\ &\le \sum_{i<j}\frac{m_i m_j}{|r_{0,ij}(T)| - cT^{1/2} - |x_{ij} - r_{0,ij}(1)|}, \end{aligned}$$
and the last quantity converges to zero as $T\to+\infty$.

- Finally,
$$\lim_{T\to+\infty}\langle \ddot r_0(T), \gamma(T)\rangle_{\mathcal{M}} = 0.$$
This is shown by distinguishing the different cases. In the hyperbolic case,

we have $\ddot{r}_0(T) = 0$ for all $T \geq 1$. In the parabolic case,

$$\begin{aligned}
&\lim_{T\to+\infty} \langle \ddot{r}_0(T), \gamma(T)\rangle_{\mathcal{M}} \\
&= \lim_{T\to+\infty} -\frac{2}{9}\langle \beta b_m T^{-4/3}, \beta b_m T^{2/3} + \varphi(T) + x - \beta b_m\rangle_{\mathcal{M}} \\
&= \lim_{T\to+\infty} -\frac{2}{9}\frac{1}{T^2} = 0.
\end{aligned}$$

In the hyperbolic-parabolic case,

$$\begin{aligned}
&\lim_{T\to+\infty} \langle \ddot{r}_0(T), \gamma(T)\rangle_{\mathcal{M}} \\
&= \lim_{T\to+\infty} -\frac{2}{9}\langle \beta b_m T^{-4/3}, aT + \beta b_m T^{2/3} + \varphi(T) + x - a - \beta b_m\rangle_{\mathcal{M}} \\
&= 0,
\end{aligned}$$

since $\lim_{T\to+\infty}\langle \beta b_m T^{-4/3}, \varphi(T)\rangle_{\mathcal{M}} = 0$, owing to the fact that, for T large enough, there exists a constant $c' \in \mathbb{R}$, independent of T and x, such that $\|\varphi(T)\|_{\mathcal{M}} \leq c' T^{1/2}$.

□

Alternative direct proof of Theorem 3.3.1

Theorem 3.3.1 can also be established through a more direct computation, by verifying that the value function satisfies Definition 1.4.1. For completeness, we present this alternative argument.

Given $\varphi^x \in \mathcal{D}_0^{1,2}(1, +\infty)$ realizing $v(x)$, define $\gamma = r_0 + \varphi^x + x - r_0(1)$. Since energy conservation yields

$$H(\gamma(1), \dot{\gamma}(1)) = \frac{\|a\|_{\mathcal{M}}^2}{2},$$

in order to prove Theorem 3.3.1 it is sufficient to show, in the viscosity sense, that

$$\|\nabla v(x)\|_{\mathcal{M}} = \|\dot{\gamma}(1)\|_{\mathcal{M}},$$

where $\gamma(t)$ denotes an arbitrary expansive motion starting at x.

Let $x \in \Omega$ and $z \in \mathcal{X}$. For $h > 0$, let $x_h = x + hz$, and $\varphi^h \in \mathcal{D}_0^{1,2}(1, +\infty)$ be a minimizer of $\mathcal{A}_{x_h}$. It holds, $v(x_h) = \mathcal{A}_{x_h}(\varphi^h) - \langle a, x_h\rangle_{\mathcal{M}}$. Since $\varphi^h \in \mathcal{D}_0^{1,2}(1, +\infty)$, we have

$$v(x_h) - v(x) \geq \mathcal{A}_{x_h}(\varphi^h) - \mathcal{A}_x(\varphi^h) - \langle a, x_h - x\rangle_{\mathcal{M}}.$$

Hence, since $r_0(t) + \varphi^h(t) + x - r_0(1) + shz$ is collisionsless, due to the minimality of φ^h, we have

$$
\begin{aligned}
&v(x_h) - v(x)\\
&\geq \int_1^{+\infty} U(r_0(t)+\varphi^h(t)+x_h-r_0(1)) - U(r_0(t)+\varphi^h(t)+x-r_0(1))\,dt - h\langle a,z\rangle_{\mathcal{M}}\\
&= \int_1^{+\infty} h\int_0^1 \langle \nabla U(r_0(t)+\varphi^h(t)+x-r_0(1)+shz), z\rangle_{\mathcal{M}}\,ds\,dt - h\langle a,z\rangle_{\mathcal{M}},
\end{aligned}
$$

that is

$$
\frac{v(x_h)-v(x)}{h} \geq \int_1^{+\infty}\int_0^1 \langle \nabla U(r_0(t)+\varphi^h(t)+x-r_0(1)+shz), z\rangle\,ds\,dt - \langle a,z\rangle_{\mathcal{M}}.
$$

As noted in the proof of the continuity of v (see Proposition 3.3.2), there exists a minimizer $\bar\varphi$ of $\mathcal{A}_x$ such that $\varphi^h \to \bar\varphi$ pointwise on $[1,+\infty)$, uniformly on every interval $[1,T]$ for $T>1$, and weakly in $\mathcal{D}_0^{1,2}(1,+\infty)$. In particular, we also have $\varphi^h + shz \to \bar\varphi$ pointwise on $[1,+\infty)$ and uniformly on each compact interval $[1,T]$. It then suffices to note that, for every $\varepsilon>0$, there exists $T_\varepsilon>1$ such that

$$
\left|\langle \nabla U(r_0(t)+\varphi^h(t)+x-r_0(1)+shz), z\rangle\right| \leq \frac{\|z\|_{\mathcal{M}}}{(1-\varepsilon)\|r_0(t)\|^2_{\mathcal{M}}}, \quad \text{for} \quad t\geq T_\varepsilon,
$$

to apply dominated convergence Theorem and conclude that, as $h\to 0^+$,

$$
\begin{aligned}
\frac{v(x_h)-v(x)}{h} &\geq \int_1^{+\infty}\int_0^1 \langle \nabla U(r_0(t)+\bar\varphi(t)+x-r_0(1)), z\rangle\,ds\,dt - \langle a,z\rangle_{\mathcal{M}} + o(1)\\
&= \int_1^{+\infty} \langle \nabla U(r_0(t)+\bar\varphi(t)+x-r_0(1)), z\rangle\,dt - \langle a,z\rangle_{\mathcal{M}} + o(1)\\
&= \int_1^{+\infty} \langle \ddot\gamma(t), z\rangle_{\mathcal{M}}\,dt + \langle a,z\rangle_{\mathcal{M}} + o(1)\\
&= -\langle \dot\gamma(1), z\rangle_{\mathcal{M}} + o(1),
\end{aligned}
\tag{3.3.20}
$$

where $\gamma(t) = r_0(t)+\bar\varphi(t)+x-r_0(1)$. Therefore, if $p\in D^+v(x)$, then the following chain holds:

$$
\langle p,z\rangle_{\mathcal{M}} \geq \limsup_{h\to 0^+}\frac{v(x_h)-v(x)}{h} \geq -\langle \dot\gamma(1), z\rangle_{\mathcal{M}}, \quad \text{for every} \quad z\in\mathcal{X}.
$$

By choosing $z=-p$, we then obtain

$$
-\|p\|^2_{\mathcal{M}} \geq \langle \dot\gamma(1), p\rangle_{\mathcal{M}}, \tag{3.3.21}
$$

Instead, by choosing $z = \dot{\gamma}(1)$, we have

$$\langle \dot{\gamma}(1), p \rangle_{\mathcal{M}} \geq -\|\dot{\gamma}(1)\|_{\mathcal{M}}. \tag{3.3.22}$$

Putting together (3.3.21) and (3.3.22), we conclude that

$$\|p\|_{\mathcal{M}}^2 \leq \|\dot{\gamma}(1)\|_{\mathcal{M}}^2 = \|a\|_{\mathcal{M}}^2 + 2U(\gamma(1)) \tag{3.3.23}$$

by the conservation of the energy and Theorem 2.1.7.

On the other hand, if φ^x is a minimizer of $\mathcal{A}_x$, so that $v(x) = \mathcal{A}_x(\varphi^x) - \langle a, x \rangle_{\mathcal{M}}$, we have

$$v(x_h) - v(x) \leq \mathcal{A}_{x_h}(\varphi^x) - \mathcal{A}_x(\varphi^x) - h\langle a, z \rangle_{\mathcal{M}}.$$

With similar arguments as above, we then get

$$\begin{aligned}
&\frac{v(x_h) - v(x)}{h} \\
&\leq \int_1^{+\infty} U(r_0(t) + \varphi^x(t) + x_h - r_0(1)) - U(r_0(t) + \varphi^x(t) + x - r_0(1))\, dt - \langle a, z \rangle_{\mathcal{M}} \\
&= \int_1^{+\infty} \int_0^1 \langle \nabla U(r_0(t) + \varphi^x(t) + x + shz - r_0(1)), z \rangle\, dt - \langle a, z \rangle_{\mathcal{M}} \\
&\leq \int_1^{+\infty} \langle \nabla U(r_0(t) + \varphi^x(t) + x - r_0(1)), z \rangle\, dt - \langle a, z \rangle_{\mathcal{M}} + o(1), \quad \text{as} \quad h \to 0^+ \\
&= \int_1^{\infty} \langle \ddot{\gamma}^x(t), z \rangle\, dt - \langle a, z \rangle_{\mathcal{M}} + o(1),
\end{aligned}$$

that is,

$$\frac{v(x_h) - v(x)}{h} \leq -\langle \dot{\gamma}^x(1), z \rangle_{\mathcal{M}} + o(1), \quad \text{as } h \to 0^+. \tag{3.3.24}$$

Hence, if $p \in D^- v(x)$, then

$$\langle p, z \rangle_{\mathcal{M}} \leq \liminf_{h \to 0^+} \frac{v(x_h) - v(x)}{h} \leq -\langle \dot{\gamma}^x(1), z \rangle_{\mathcal{M}}, \quad \text{for every} \quad z \in \mathcal{X}. \tag{3.3.25}$$

As in the arguments to obtain (3.3.23), by putting together the inequalities from choosing $z = -p$ and $z = \dot{\gamma}^x(1)$ in (3.3.25), we get

$$\|p\|_{\mathcal{M}}^2 \geq \|\dot{\gamma}^x(1)\|_{\mathcal{M}}^2 = \|a\|_{\mathcal{M}}^2 + 2U(\gamma^x(1)),$$

by the conservation of the energy and Theorem 2.1.7.

To conclude, we proved that, in the viscosity sense,

$$\|\nabla v(x)\|_{\mathcal{M}}^2 = \|a\|_{\mathcal{M}}^2 + 2U(x),$$

that is

$$H(x, \nabla v(x)) = \frac{\|a\|^2_{\mathcal{M}}}{2}. \tag{3.3.26}$$

Remark 3.3.11. *En passant*, in the course of proving (3.3.26), we also established that whenever a point $x \in \Omega$ admits a unique minimizer φ^x of $\mathcal{A}_x$ in $\mathcal{D}^{1,2}_0(1, +\infty)$, the value function v is differentiable at x, and

$$\nabla v(x) = -\mathcal{M}\dot{\gamma}^x(1),$$

where $\gamma^x(t) = r_0(t) + \varphi^x(t) + x - r_0(1)$.

In fact, under this uniqueness assumption, relations (3.3.20) and (3.3.24) together yield

$$-\langle \dot{\gamma}^x(1), z\rangle_{\mathcal{M}} \leq \lim_{h\to 0^+} \frac{v(x+hz) - v(x)}{h} \leq -\langle \dot{\gamma}^x(1), z\rangle_{\mathcal{M}}, \quad \text{for every} \quad z \in \mathcal{X}.$$

3.4 Fine regularity results on the value function

In this section, devoted to regularity properties of the value function $v(x)$, we examine the size of two distinguished classes of points: the set of irregular points and the set of so-called conjugate points. In what follows, we refer to [12].

3.4.1 Irregular and conjugate points

Definition 3.4.1 (Irregular points)**.** We say that a configuration $x \in \Omega$ is regular if $\mathcal{A}_x$ admits a unique minimum φ^x on $\mathcal{D}^{1,2}_0(1, +\infty)$. All other points are called irregular. We denote by Σ the set of irregular points.

As noted in Remark 3.3.11, the set $\Omega \setminus \Sigma$ consists precisely of those points where v is of class C^1, so that the Hamilton-Jacobi equation (3.3.2) holds in the classical sense.

Following the approach of Cannarsa and Sinestrari [18], we now introduce the notion of *conjugate* points in a way adapted to our setting. The main difference is that our renormalized Lagrangian action is defined on curves whose domain is the half-line $[1, +\infty)$, a feature that must be incorporated into the definition.

We recall from [66] that over the set of non-collisional configurations, the differential $d\mathcal{A}_x(\varphi)$ is continuous to the dual space $(\mathcal{D}^{1,2}_0(1, +\infty))^*$, and is defined as follows:

$$d\mathcal{A}_x(\varphi)[\psi] = \int_1^{+\infty} \langle \dot{\varphi}(t), \dot{\psi}(t)\rangle_{\mathcal{M}} + \langle \nabla U(r_0(t) + \varphi(t) + x - r_0(1)), \psi(t)\rangle - \langle \ddot{r}_0(t), \psi(t)\rangle_{\mathcal{M}}\, dt. \tag{3.4.1}$$

By a composition between $d\mathcal{A}_x(\varphi)$ and the inverse of Riesz's isomorphism $\mathcal{R}_i^{-1}$: $(\mathcal{D}_0^{1,2}(1,+\infty))^* \to \mathcal{D}_0^{1,2}(1,+\infty)$, we can define a map $F : U \subset \Omega \times \mathcal{D}_0^{1,2}(1,+\infty) \to \mathcal{D}_0^{1,2}(1,+\infty)$ as:

$$\begin{aligned} F(x,\varphi) &:= \mathcal{R}_i^{-1} \circ d\mathcal{A}_x(\varphi) \\ &= \varphi + \mathcal{R}_i^{-1}[\nabla U(r_0 + \varphi + x - r_0(1)) - \mathcal{M}\ddot{r}_0]. \end{aligned}$$

Two remarks about F are in order. First, the zeros of F correspond to pairs $(\bar{x},\bar{\varphi})$ for which $\bar{\varphi}$ is a critical point of $\mathcal{A}_{\bar{x}}$. Second, if the differential $D_\varphi F(\bar{x},\bar{\varphi})$ is invertible, the Implicit Function Theorem guarantees the existence of a neighborhood $\bar{\mathcal{U}} \times \bar{\mathcal{V}}$ of $(\bar{x},\bar{\varphi}) \in \Omega \times \mathcal{D}_0^{1,2}(1,+\infty)$ such that, for every $x \in \bar{\mathcal{U}}$, there exists a unique critical point of $\mathcal{A}_x$ in $\bar{\mathcal{V}}$, denoted $\varphi^x \in \bar{\mathcal{V}}$, which depends on x with maximal regularity.

A sufficient condition for the invertibility of $D_\varphi F(x,\varphi)$ is the coercivity of $d^2\mathcal{A}_x(\varphi)$. Conversely, the set of conjugate points heuristically corresponds to points where the linearized problem becomes degenerate. This motivates the following definition.

Definition 3.4.2 (Conjugate points)**.** We define the set of conjugate points Γ as the subset of points $x \in \Omega$ such that:

- the set of minimizers of $\mathcal{A}_x$ is isolated on the set of critical points,
- $d^2\mathcal{A}_x(\varphi^x) \in \mathcal{L}(\mathcal{D}_0^{1,2}(1,+\infty);(\mathcal{D}_0^{1,2}(1,+\infty))^*)$ is not invertible, for every φ^x minimizer of $\mathcal{A}_x(\varphi^x)$,

where $d^2\mathcal{A}_x$ is the bilinear form corresponding to $D_\varphi F(x,\varphi^x)$, given by

$$d^2\mathcal{A}_x(\varphi^x)[\psi,\zeta] = \int_1^{+\infty} \langle\dot{\psi}(t),\dot{\zeta}(t)\rangle_{\mathcal{M}} + \langle\nabla^2 U(r_0(t)+\varphi^x(t)+x-r_0(1))\psi(t),\zeta(t)\rangle_{\mathcal{M}}\,dt. \tag{3.4.2}$$

We want to estimate the dimension of the set of conjugate points Γ.

3.4.2 Rectifiability properties of irregular, non-conjugate points.

In this and the following sections, we make use of the following right-tail T-actions:

$$\mathcal{A}_{x,[T,+\infty)}(\varphi) = \int_T^{+\infty} \frac{\|\dot{\varphi}\|^2_{\mathcal{M}}}{2} + U(r_0+\varphi+x-r_0(T)) - U(r_0) - \langle\ddot{r}_0,\varphi\rangle_{\mathcal{M}}\,dt, \tag{3.4.3}$$

where $\varphi \in \mathcal{D}_0^{1,2}(T,+\infty)$.

Proposition 3.4.3 (Berti, Polimeni and Terracini 2025 [12]). *For all $T > 1$, and for any $x \in \Omega$ and $\varphi \in \mathcal{D}_0^{1,2}(T, +\infty)$, the differential $d\mathcal{A}_{x,[T,+\infty)}(\varphi)$ is a compact perturbation of an invertible operator.*

Proof. We observe that $d\mathcal{A}_{x,[T,+\infty)}$ is as in (3.4.1), but with the integral evaluated in $[T, +\infty)$. Using Riesz isomorphism, we can write

$$d\mathcal{A}_{x,[T,+\infty)}(\varphi) = \mathbf{1} + \mathcal{R}_i^{-1}[\nabla U(r_0 + \varphi + x - r_0(1)) - \mathcal{M}\ddot{r}_0].$$

We will divide the proof accordingly to the class of expansive motion under consideration.

Hyperbolic case. Define $P_{x,T}^H \in \mathcal{L}(\mathcal{D}_0^{1,2}(T, +\infty); (\mathcal{D}_0^{1,2}(T, +\infty))^*)$ as

$$P_{x,T}^H(\varphi) \ : \ \psi \mapsto \int_T^{+\infty} \langle \nabla U(\gamma(t)), \psi(t) \rangle \, dt.$$

The statement follows once we observe that $P_{x,T}^H$ is compact.

Let $(\varphi_n)_n$ be a bounded sequence in $\mathcal{D}_0^{1,2}(T, +\infty)$. As observed repeatedly throughout this work, up to passing to a subsequence, $\varphi_n \to \bar{\varphi}$ for some $\bar{\varphi} \in \mathcal{D}_0^{1,2}(T, +\infty)$, with convergence being, in particular, pointwise on the entire half-line $[T, +\infty)$. Consequently, for every fixed $\psi \in \mathcal{D}_0^{1,2}(T, +\infty)$ with $\|\psi\|_{\mathcal{D}_0^{1,2}(T,+\infty)} = 1$, we have

$$\begin{aligned} &P_{x,T}^H(\varphi_n)[\psi] - P_{x,T}^H(\bar{\varphi})[\psi] \\ &= \int_T^{+\infty} \langle \nabla U(\gamma_n(t) - \nabla U(\bar{\gamma}(t)), \psi(t) \rangle \, dt \\ &= \int_T^{+\infty} \int_0^1 \langle \nabla^2 U(\gamma_n(t) + s(\varphi_n(t) - \bar{\varphi}(t)))(\varphi_n(t) - \bar{\varphi}(t)), \psi(t) \rangle \, ds \, dt. \end{aligned}$$

Since, for t large enough and for every $s \in [0,1]$,

$$\begin{aligned} \left| \langle \nabla^2 U(\gamma_n(t) + s(\varphi_n(t) - \bar{\varphi}(t)))(\varphi_n(t) - \bar{\varphi}(t)), \psi(t) \rangle \right| &\le \frac{C \, \|\varphi_n(t) - \bar{\varphi}(t)\| \, \|\psi(t)\|}{t^3} \\ &\le \frac{C'}{t^{5/2}} \in L^1(T, +\infty), \end{aligned}$$

by the Dominated Convergence Theorem and the pointwise convergence of φ_n to $\bar{\varphi}$, we deduce that

$$\|P_{x,T}^H(\varphi_n) - P_{x,T}^H(\bar{\varphi})\|_{(\mathcal{D}_0^{1,2}(T,+\infty))^*} = \sup_{\|\psi\|=1} \left| P_{x,T}^H(\varphi_n)[\psi] - P_{x,T}(\bar{\varphi})[\psi] \right| \to 0, \ n \to \infty,$$

that is, $P_{x,T}^H(\varphi_n)$ is convergent in $(\mathcal{D}_0^{1,2}(T, +\infty))^*$, which means that $P_{x,T}^H$ is compact.

Parabolic case. In this case, we need to start from $\mathcal{A}_{x,[T,+\infty)}(\varphi)$ written as follows

$$\begin{aligned}\mathcal{A}_{x,[T,+\infty)}(\varphi) = &\int_T^{+\infty} \frac{\|\dot\varphi(t)\|^2_{\mathcal{M}}}{2} + \langle \nabla^2 U(r_0(t))\varphi(t),\, \varphi(t)\rangle\, dt \\ &+ \int_T^{+\infty} U(\gamma(t)) - U(r_0(t)) - \langle \nabla U(r_0(t)), \varphi(t)\rangle \\ &\qquad - \langle \nabla^2 U(r_0(t))\varphi(t),\, \varphi(t)\rangle\, dt,\end{aligned}$$

where

$$\varphi \mapsto \int_T^{+\infty} \frac{\|\dot\varphi(t)\|^2_{\mathcal{M}}}{2} + \langle \nabla^2 U(r_0(t))\varphi(t),\, \varphi(t)\rangle\, dt$$

is a norm, coming from a positive quadratic form which is equivalent to $\|\cdot\|_{\mathcal{D}_0^{1,2}(T,+\infty)}$, as shown in [66, Section 4.2]. Hence, by Lax-Milgram Theorem, its differential is invertible. Setting

$$\mathcal{F}(\varphi) = \int_T^{+\infty} U(\gamma(t)) - U(r_0(t)) - \langle \nabla U(r_0(t)), \varphi(t)\rangle - \langle \nabla^2 U(r_0(t))\varphi(t),\, \varphi(t)\rangle\, dt,$$

it holds

$$\begin{aligned}&d\mathcal{F}(\varphi)[\psi] \\ &= \int_T^{+\infty} \langle \nabla U(\gamma(t)) - \nabla U(r_0(t)), \psi(t)\rangle - \langle \nabla^2 U(r_0(t))\varphi(t),\, \psi(t)\rangle\, dt \\ &= \int_T^{+\infty}\int_0^1 \langle \nabla^2 U(\gamma(t) + \sigma_1(\varphi(t)+\tilde x))(\varphi(t)+\tilde x), \psi(t)\rangle - \langle \nabla^2 U(r_0(t))\varphi(t),\, \psi(t)\rangle\, dt \\ &= \int_T^{+\infty}\int_0^1\int_0^1 \langle [\nabla^3 U(\gamma(t) + (\sigma_1+\sigma_2+\sigma_1\sigma_2)(\varphi(t)+\tilde x)) \\ &\qquad \cdot (\varphi(t) + \tilde x + \sigma_1(\varphi(t)+\tilde x)]\, \varphi(t),\, \psi(t)\rangle\, d\sigma_1\, d\sigma_2\, dt + \text{lower order terms}.\end{aligned}$$

Since, for t sufficiently large, we have

$$\begin{aligned}&\left|\left\langle\left[\nabla^3 U(\gamma(t) + (\sigma_1+\sigma_2+\sigma_1\sigma_2)(\varphi(t)+\tilde x))(\varphi(t) + \tilde x + \sigma_1(\varphi(t)+\tilde x)\right]\, \varphi(t),\, \psi(t)\right\rangle\right| \\ &\leq \frac{C\,\|\varphi(t)\|^2\,\|\psi(t)\|}{\|r_0(t)\|^4},\end{aligned}$$

with the same arguments used in the proof of the hyperbolic case for the operator $P^H_{x,T}$, we conclude that $d\mathcal{F} \in \mathcal{L}(\mathcal{D}_0^{1,2}(T,+\infty); (\mathcal{D}_0^{1,2}(T,+\infty))^*)$ is compact.

Hyperbolic-Parabolic case. The proof relies on the decomposition of the Lagrangian action induced by the cluster partition of the bodies, as presented in [66]. Specifically, the renormalized Lagrangian action is expressed as the sum of two terms: one accounts for the motion of the bodies within each cluster, while the other describes the interactions between pairs of bodies belonging to different clusters. Since the first term corresponds to a parabolic motion and the second to a hyperbolic motion, the computations proceed in a manner analogous to the two cases considered above. □

The following is the counterpart of [18, Theorem 6.4.9].

Theorem 3.4.4 (Berti, Polimeni and Terracini 2025 [12]). *If the set of functions $\{\varphi \in \mathcal{D}_0^{1,2}(1,+\infty) : \varphi$ is a minimizer of the renormalized value function $v(x)\}$ is not finite, then $x \in \Gamma$.*

Proof. Assume that $x \notin \Gamma$ and let $\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$. Then $d^2\mathcal{A}_x$ is invertible, meaning that for every $J \in (\mathcal{D}_0^{1,2}(1,+\infty))^*$ there exists a unique $\psi \in \mathcal{D}_0^{1,2}(1,+\infty)$ such that

$$J[\cdot] = d^2\mathcal{A}_x(\varphi)[\psi,\cdot] \quad \text{in } (\mathcal{D}_0^{1,2}(1,+\infty))^*.$$

It follows that the Implicit Function Theorem applies to the 0-level of

$$F(x,\varphi) := d\mathcal{A}_x(\varphi) \in (\mathcal{D}_0^{1,2}(1,+\infty))^*,$$

so that there exists a neighborhood $\mathcal{U}_x \times \mathcal{U}_\varphi$ of (x,φ) in $\Omega \times \mathcal{D}_0^{1,2}(1,+\infty)$ with the property that, for every $z \in \mathcal{U}_x$, there is a unique critical point $\bar\varphi = \bar\varphi^z = \bar\varphi(z) \in \mathcal{U}_\varphi$ of $\mathcal{A}_z$. Consequently, there are at most countably many minimizers, and they are isolated.

Suppose, by contradiction, that the number of such minimizers is infinite, and let $(\varphi_n)_n$ be a sequence of minimizers. Up to a subsequence, there exists $\bar\varphi \in \mathcal{D}_0^{1,2}(1,+\infty)$ such that $\varphi_n \to \bar\varphi$ weakly and pointwise. By Proposition 3.4.3, it follows that $\bar\varphi$ is a critical point, and in particular a minimizer. Therefore, we have

$$\begin{aligned}0 &= \|\varphi_n\|_{\mathcal{D}} - \|\bar\varphi\|_{\mathcal{D}} \\ &+ 2\int_1^{+\infty}\int_0^1 \Big\langle \nabla U\Big(r_0(t)+\varphi_n(t)+x-r_0(1)+s(\bar\varphi(t)-\varphi_n(t))\Big), \bar\varphi(t)-\varphi_n(t)\Big\rangle\, ds\, dt,\end{aligned}$$

and by the Dominated Convergence Theorem, it follows that $\|\varphi_n\|_{\mathcal{D}} \to \|\bar\varphi\|_{\mathcal{D}}$. This, in turn, implies $\|\varphi_n - \bar\varphi\|_{\mathcal{D}} \to 0$, contradicting the discreteness of $(\varphi_n)_n$ in $\mathcal{D}_0^{1,2}(1,+\infty)$.

□

Theorem 3.4.5 (Berti, Polimeni and Terracini 2025 [12]). *Let $x \in \Sigma \setminus \Gamma$ be given. Then there exists a neighborhood $\mathcal{U}_x$ of x and a finite number of $v_1,\dots,v_k : \mathcal{U}_x \to \mathbb{R}$ such that $\nabla v_i \neq \nabla v_j$ if $i \neq j$ and $u = \min\{v_i\}_i$ in $\mathcal{U}_x$.*

Proof. Let φ_i, for $i = 1,...,k$, denote the minimizers associated with $x \in \Sigma \setminus \Gamma$. We define

$$v_i(z) = \min_{\xi \in \mathcal{U}_{\varphi_i} \subset \mathcal{D}_0^{1,2}(1,+\infty)} \mathcal{A}_z(\xi), \qquad \text{for } z \in \mathcal{U}_x \text{ and for each } i = 1,\dots,k. \quad (3.4.4)$$

We note that every $z \in \mathcal{U}_x$ is a regular point for the restriction $\mathcal{A}_z|_{\mathcal{U}_{\varphi_i}}$, and that, possibly after shrinking the neighborhood $\mathcal{U}_x$, we have $z \notin \Gamma$, since Γ is closed. It follows that $v_i \in C^1(\mathcal{U}_x)$.

With the functions v_i, $i = 1, ..., k$, defined as in (3.4.4), the conclusion follows directly from Theorem 3.4.4 together with the Implicit Function Theorem applied to F. □

Recall that, since we are working in the space of configurations with fixed center of mass, we have $\dim \mathcal{X} = d(N-1)$. We know that $v_i(x) = v_j(x)$, and, since $\nabla v_i(x) = -\mathcal{M}\,\dot{\varphi}_i(1) \neq -\mathcal{M}\,\dot{\varphi}_j(1) = \nabla v_j(x)$, it follows that the set

$$\{y \in \mathcal{X} : v_i(y) = v_j(y)\}$$

is a $(d(N-1)-1)$-dimensional C^∞ hypersurface for any pair $i \neq j$. From this, we also obtain the following corollary.

Corollary 3.4.6 (Berti, Polimeni and Terracini 2025 [12])**.** *Let $x \in \Sigma \setminus \Gamma$ be given. Then, there exists $\mathcal{U}_x$, a neighborhood of x, such that $\mathcal{U}_x \cap \Sigma$ is contained in a finite union of $(d(N-1)-1)$-dimensional hypersurfaces of class C^∞.*

3.4.3 Bound on the dimension of the set of conjugate points

Definition of a local flow

Let $x_0 \in \Gamma$, and let φ^{x_0} be a corresponding minimizer of $\mathcal{A}_{x_0}$ in $\mathcal{D}_0^{1,2}(1, +\infty)$. Associated with φ^{x_0} is the curve γ^{x_0}, defined by

$$\gamma^{x_0}(t) = r_0(t) + \varphi^{x_0}(t) + x_0 - r_0(1).$$

Extend γ^{x_0} to the interval $(1-\varepsilon, 1]$ as the unique smooth solution of (1.1.1), and still denote this extension by γ^{x_0}, now defined on $(1-\varepsilon, +\infty)$.

The Hessian of the action is the bilinear form $d^2\mathcal{A}_{x_0}(\varphi)$ given in (3.4.2). Following $\varphi(t)$, and hence $\gamma(t)$, we consider the family of bilinear forms on $\mathcal{D}_0^{1,2}(T, +\infty) \times \mathcal{D}_0^{1,2}(T, +\infty)$:

$$\begin{aligned} &d^2\mathcal{A}_{\gamma^{x_0}(T),[T,+\infty)}(\tilde{\varphi})[\psi, \eta] \\ &= \int_T^{+\infty} \langle \dot{\psi}(t), \dot{\eta}(t)\rangle_{\mathcal{M}} + \langle \nabla^2 U(r_0(t) + \tilde{\varphi}(t) + \gamma^{x_0}(T) - r_0(T))\psi(t), \eta(t)\rangle\, dt \\ &= \int_T^{+\infty} \langle \dot{\psi}(t), \dot{\eta}(t)\rangle_{\mathcal{M}} + \langle \nabla^2 U(r_0(t) + \varphi^{x_0}(t) + x_0 - r_0(1))\psi(t), \eta(t)\rangle\, dt, \end{aligned}$$

where $\tilde{\varphi} = \varphi^{x_0}|_{[T,+\infty)} - \varphi^{x_0}(T)$. Notice that $d^2\mathcal{A}_{x_0}(\varphi)$ is not invertible, as, by hypothesis, $x \in \Gamma$.

Remark 3.4.7. We recall that $d^2\mathcal{A}_{x_0}(\varphi)$ is invertible if $\forall\, J \in (\mathcal{D}_0^{1,2}(1,+\infty))^*$ there is a unique $\psi \in \mathcal{D}_0^{1,2}(1,+\infty)$ such that $d^2\mathcal{A}_{x_0}(\varphi)[\psi,\cdot] = J[\cdot]$ in $(\mathcal{D}_0^{1,2}(1,+\infty))^*$ (or, using Riesz's isomorphism, $\mathcal{R}_i^{-1} \circ d^2\mathcal{A}_{x_0}(\varphi)[\psi,\cdot] = \mathcal{R}_i^{-1} \circ J[\cdot]$ in $\mathcal{D}_0^{1,2}(1,+\infty)$).

By Lax-Milgram's Theorem, $d^2\mathcal{A}_{\gamma^{x_0}(T),[T,+\infty)}(\tilde{\varphi})$ is invertible if it is coercive, that is, if there is a constant $\alpha > 0$ such that

$$d^2\mathcal{A}_{\gamma^{x_0}(T),[T,+\infty)}(\tilde{\varphi})[\psi,\psi] \geq \alpha \|\psi\|^2_{\mathcal{D}_0^{1,2}(T,+\infty)}, \tag{3.4.5}$$

for all $\psi \in \mathcal{D}_0^{1,2}(T,+\infty)$.

Now, we give the following lemma.

Lemma 3.4.8 (Berti, Polimeni and Terracini 2025 [12])**.** *Given $x_0 \in \Gamma$ there exist $T_0 > 1$ and $C > 0$ such that (3.4.5) holds for every $\psi \in \mathcal{D}_0^{1,2}(T_0,+\infty)$.*

Proof. We distinguish the hyperbolic case from the parabolic one to prove the claim.

Starting with the hyperbolic case, we notice that

$$\langle \nabla^2 U(at)\psi(t), \psi(t)\rangle \geq -C\frac{\|\psi(t)\|^2_{\mathcal{M}}}{\|\gamma^{x_0}(t)\|^3_{\mathcal{M}}} \geq -\frac{C}{\|a\|^3_{\mathcal{M}} T}\frac{\|\psi(t)\|^2_{\mathcal{M}}}{t^2},$$

for some constant $C > 0$ and for $t \geq T$. Then

$$\begin{aligned}\int_T^{+\infty} \|\dot{\psi}(t)\|^2_{\mathcal{M}} + \langle \nabla^2 U(\gamma^{x_0}(t))\psi(t), \psi(t)\rangle\, dt &\geq \int_T^{+\infty} \|\dot{\psi}(t)\|^2_{\mathcal{M}} - \frac{C'}{T}\frac{\|\psi(t)\|^2_{\mathcal{M}}}{t^2}\, dt \\ &\geq \left(1 - \frac{4C'}{T}\right)\int_T^{+\infty} \|\dot{\psi}(t)\|^2_{\mathcal{M}}\, dt.\end{aligned}$$

Choosing T big enough so that $1 - \frac{4C'}{T} > 0$, we obtain the coercivity of $L_{x_0,T}$.

To prove (3.4.5) in the parabolic case, we use the inequality

$$\langle \nabla^2 U(\gamma^{x_0}(t))\psi(t), \psi(t)\rangle \geq -U_{min}\frac{\|\psi(t)\|^2_{\mathcal{M}}}{\|\gamma^{x_0}(t)\|^3_{\mathcal{M}}},$$

which implies

$$\begin{aligned}\langle \nabla^2 U(\gamma^{x_0}(t))\psi(t), \psi(t)\rangle &\geq -U_{min}\frac{\|\psi(t)\|^2_{\mathcal{M}}}{\|r_0(t)\|^3_{\mathcal{M}}}\frac{1}{\left(1 + \frac{\|\varphi(t)\|_{\mathcal{M}}}{\|r_0(t)\|_{\mathcal{M}}} + \frac{\|x - r_0(1)\|_{\mathcal{M}}}{\|r_0(t)\|_{\mathcal{M}}}\right)^3} \\ &\geq -U_{min}\frac{\|\psi(t)\|^2_{\mathcal{M}}}{\|\beta b_m t^{2/3}\|^3_{\mathcal{M}}} \\ &= -\frac{2}{9}\frac{\|\psi(t)\|^2_{\mathcal{M}}}{t^2},\end{aligned}$$

being $r_0(t) = \beta b_m t^{2/3}$. Then,

$$\begin{aligned} d^2\mathcal{A}_{\gamma^{x_0}(T),[T,+\infty)}(\tilde{\varphi})[\psi,\psi] &\geq \|\psi\|^2_{\mathcal{D}^{1,2}_0(T,+\infty)} - \frac{2}{9}\int_T^{+\infty} \frac{\|\psi(t)\|^2_{\mathcal{M}}}{t^2}\,dt \\ &\geq \|\psi\|^2_{\mathcal{D}^{1,2}_0(T,+\infty)} - \frac{8}{9}\|\psi\|^2_{\mathcal{D}^{1,2}_0(T,+\infty)} \\ &= \frac{1}{9}\|\psi\|^2_{\mathcal{D}^{1,2}_0(T,+\infty)}. \end{aligned}$$

□

Let $T_0 > 1$ be chosen so that the bilinear form $d^2\mathcal{A}_{\gamma^{x_0}(T_0),[T_0,+\infty)}$ is coercive, as established in Lemma 3.4.8. Set $z_0 = \gamma^x(T_0)$. By the Implicit Function Theorem, there exist neighborhoods $\mathcal{U}_{z_0} \subset \mathbb{R}^{d(N-1)}$ and $\mathcal{U}_{\varphi^{z_0}} \subset \mathcal{D}^{1,2}_0(T_0,+\infty)$, together with a map $\varphi : \mathcal{U}_{z_0} \to \mathcal{U}_{\varphi^{z_0}}$ such that the function $\varphi(z) = \varphi^z$ is the unique critical point of the functional $\mathcal{A}_{z,(T_0,+\infty)}$ in $\mathcal{D}^{1,2}_0(T_0,+\infty)$, and in fact realizes its minimum. Moreover, the map $z \mapsto \varphi^z$ is smooth, and the function $u_{T_0}(z)$ defined by

$$u_{T_0}(z) = \mathcal{A}_{z,[T_0,\infty)}(\varphi^z)$$

is smooth as well, with gradient given by

$$\nabla u_{T_0}(z) = -\mathcal{M}\,\dot{\varphi}^z(T_0).$$

Possibly after restricting to a ball $B_R = B_R(z_0) \subset \mathcal{U}_{z_0}$, continuity ensures the existence of $\varepsilon > 0$ such that the function $\xi = \xi(z,t)$ is well defined, for every $z \in B_R$ and $t \in (1-\varepsilon, T_0]$, as the unique solution of

$$\begin{cases} \ddot{\xi} = \nabla U(\xi), \quad 1-\varepsilon < t \leq T_0 \\ \xi(z,T_0) = z \in B_R \\ \dot{\xi}(z,T_0) = -\mathcal{M}^{-1}\,\nabla u_{T_0}(z) + \dot{r}_0(T_0). \end{cases} \tag{3.4.6}$$

We briefly address the existence of ε. When $z = z_0$, we have $\xi(z_0,1) = x_0 \notin \Delta$. Therefore, there exists $\varepsilon > 0$ such that ξ can be extended backward in time up to $1-\varepsilon$. By continuity, the same argument applies in a neighborhood of x_0.

Now fix x such that there exist $z \in B_R$ and $s \in (1-\varepsilon, T_0)$ with $\xi(z,s) = x$. By translating the flow ξ by r_0, we define $\varphi(z,t) = \xi(z,t) - r_0(t)$, which satisfies

$$\begin{cases} \ddot{\varphi} = \nabla U(r_0+\varphi) - \ddot{r}_0, \quad 1-\varepsilon < t \leq T_0 \\ \dot{\varphi}(T_0) = -\mathcal{M}^{-1}\,\nabla u_{T_0}(z) \\ \varphi(s) = x - r_0(s). \end{cases} \tag{3.4.7}$$

Clearly, $\varphi(z,T_0) = \varphi^z(T_0) = 0$. Therefore, $\varphi(z,\cdot)$ defined in (3.4.7) can be extended as the unique solution of $(3.4.7)_1$ and $(3.4.7)_2$ satisfying $\varphi(z,T_0)=\varphi^z(T_0)$.

It follows that $\varphi(z,\cdot)$ is a critical point of

$$\begin{cases} \mathcal{J}_{x,s}(\varphi) = \int_s^{T_0} \frac{\|\dot\varphi(t)\|^2_{\mathcal{M}}}{2} + U(r_0(t)+\varphi(t)) - U(r_0(t)) - \langle \ddot r_0(t), \varphi(t)\rangle_{\mathcal{M}}\, dt \\ \qquad\qquad + u_{T_0}(\varphi(T_0)+r_0(T_0)), \\ \varphi \in H^1(s,T_0), \quad \varphi(s) = x - r_0(s), \quad \varphi(T_0)+r_0(T_0) \in B_R. \end{cases} \tag{3.4.8}$$

Remark 3.4.9. If $\mathcal{A}_x$ admits a unique minimizer φ^x, over $\mathcal{D}_0^{1,2}(1,+\infty)$, then $\mathcal{J}_{x,s}$ admits a unique minimizer, which coincides with $\varphi^x\big|_{(s,T_0)} - x + r_0(s)$.

Spectral theory along the flow

By the definitions of $\xi(z,T)$ and $\varphi(z,T)$ in (3.4.6)-(3.4.7), for each $z\in\mathcal{B}_0$ and every $T\in(1-\varepsilon,+\infty)$ we may consider the functional $\mathcal{A}_{\xi(z,T),[T,+\infty)}$, defined on $\mathcal{D}_0^{1,2}(T,+\infty)$ as in (3.4.3). The Hessian of this functional, evaluated at $\tilde\varphi(z,\cdot) = \varphi(z,\cdot)\big|_{[T,+\infty)} - \varphi(z,T)$, is the bilinear form on $\mathcal{D}_0^{1,2}(T,+\infty)\times\mathcal{D}_0^{1,2}(T,+\infty)$ given by

$$d^2\mathcal{A}_{\xi(z,T),[T,+\infty)}(\tilde\varphi(z,\cdot))[\psi,\eta] = \int_T^{+\infty} \langle\dot\psi(s),\dot\eta(s)\rangle^2_{\mathcal{M}} + \langle\nabla^2 U(\xi(z,T))\psi(s),\,\eta(s)\rangle\, ds.$$

For fixed z and T, we observe that $d^2\mathcal{A}_{\xi(z,T),T}(\tilde\varphi(z))[\psi,\psi]$ coincides with the numerator of the Rayleigh quotient

$$\mathcal{R}_a(\psi) = \frac{\int_T^{+\infty} \|\dot\psi(t)\|^2_{\mathcal{M}} + \langle\nabla^2 U(\xi(z,t))\psi(t),\psi(t)\rangle\, dt}{\int_T^{+\infty} \frac{\|\psi(t)\|^2_{\mathcal{M}}}{t^3}\, dt}, \qquad \psi\in\mathcal{D}_0^{1,2}(T,+\infty),$$

associated with the Sturm-Liouville problem

$$\begin{cases} -\ddot\psi(t) + \nabla^2 U(\xi(z,t))\psi(t) = -\dfrac{\lambda}{t^3}\psi(t) \quad \text{a.e. in } (T,+\infty), \\ \psi\in\mathcal{D}_0^{1,2}(T,+\infty). \end{cases} \tag{3.4.9}$$

Fix z and T. As already observed, since heuristically $\nabla^2U(\xi)\approx\nabla^2U(r_0)$ for large values of t, there exists $\hat T > T$ such that $d^2\mathcal{A}_{\xi(z,\hat T),\hat T}(\tilde\varphi(z,\cdot))$ is coercive on $\mathcal{D}_0^{1,2}(\hat T,+\infty)$. Furthermore, because $\xi(z,\cdot)$ stays away from collisions, there exists $C>0$ such that

$$\inf_{\eta:\,\|\eta\|=1} \langle\nabla^2U(\xi(z,t))\eta(t),\,\eta(t)\rangle \ge -\frac{C}{t^3}, \qquad \text{for every } t\in[T,\tilde T].$$

Consequently, by applying Lemma 3.4.8, there exist constants $\mu_0>0$ and $\tilde\alpha>0$

such that

$$
\begin{aligned}
&\int_T^{+\infty} \|\dot\psi(t)\|_{\mathcal{M}}^2 + \langle \nabla^2 U(\xi(z,t))\psi(t),\, \psi(t)\rangle + \mu_0 \frac{\|\psi(t)\|_{\mathcal{M}}^2}{t^3}\, dt \\
&\quad = \int_T^{\tilde T} \|\dot\psi(t)\|_{\mathcal{M}}^2 + \langle \nabla^2 U(\xi(z,t))\psi(t),\, \psi(t)\rangle + \mu_0 \frac{\|\psi(t)\|_{\mathcal{M}}^2}{t^3}\, dt \\
&\qquad + \int_{\tilde T}^{+\infty} \|\dot\psi(t)\|_{\mathcal{M}}^2 + \langle \nabla^2 U(\xi(z,t))\psi(t),\, \psi(t)\rangle + \mu_0 \frac{\|\psi(t)\|_{\mathcal{M}}^2}{t^3}\, dt \\
&\quad \geq \int_T^{\tilde T} \|\dot\psi(t)\|_{\mathcal{M}}^2 - C\frac{\|\psi(t)\|_{\mathcal{M}}^2}{t^3} + \mu_0 \frac{\|\psi(t)\|_{\mathcal{M}}^2}{t^3}\, dt + \alpha \int_{\tilde T}^{+\infty} \|\dot\psi(t)\|_{\mathcal{M}}^2\, dt \\
&\quad \geq \int_T^{\tilde T} \|\dot\psi(t)\|_{\mathcal{M}}^2\, dt + \alpha \int_{\tilde T}^{+\infty} \|\dot\psi(t)\|_{\mathcal{M}}^2\, dt \\
&\quad \geq \tilde\alpha \|\psi\|^2_{\mathcal{D}_0^{1,2}(T,+\infty)} \quad \text{for every} \quad \psi \in \mathcal{D}_0^{1,2}(T,+\infty),
\end{aligned}
$$

where α is given in Lemma 3.4.8.

Hence, for every $f \in L^2((T,+\infty); dt/t^3)$ there exists a unique ψ such that

$$
\begin{aligned}
&d^2\mathcal{A}_{\xi(z,T),[T,+\infty)}(\tilde\varphi(z,\cdot))[\psi,\eta] + \int_T^{+\infty} \frac{\langle \mu_0\psi(t), \eta(t)\rangle}{t^3}\, dt \\
&\quad = \int_T^{+\infty} \frac{\langle f(t), \eta(t)\rangle}{t^3}\, dt, \quad \text{for every} \quad \eta \in \mathcal{D}_0^{1,2}(T,+\infty).
\end{aligned}
\tag{3.4.10}
$$

As a consequence, we can define an operator

$$
\mathcal{F} : L^2\left((T,+\infty); \frac{dt}{t^3}\right) \longrightarrow L^2\left((T,+\infty); \frac{dt}{t^3}\right),
$$

with $\operatorname{im}(\mathcal{F}) = \mathcal{D}_0^{1,2}(T,+\infty)$, such that $\mathcal{F}(f) = \psi$, where ψ is the solution of (3.4.10). By Lemma 1.2.5, the operator $\mathcal{F}$ is compact (and positive). Standard arguments then imply that the problem (3.4.9) admits a sequence of eigenvalues

$$
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to +\infty,
$$

with

$$
\lambda_1(\xi(z,T),T) = \min_{\substack{\psi \in \mathcal{D}_0^{1,2}(T,+\infty) \\ \psi \not\equiv 0}} \frac{\int_T^{+\infty} \|\dot\psi(t)\|_{\mathcal{M}}^2 + \langle \nabla^2 U(\xi(z,t))\psi(t), \psi(t)\rangle\, dt}{\int_T^{+\infty} \frac{\|\psi(t)\|_{\mathcal{M}}^2}{t^3}\, dt}. \tag{3.4.11}
$$

We then introduce the function Λ defined by

$$
\Lambda(z,t) = \lambda_1(\xi(z,t),t).
$$

Remark 3.4.10. By the variational formulation of eigenvalues as minmax, we can write the k-th eigenvalue as

$$\Lambda_k(z,T) = \min_{M\subseteq\mathcal{D}_0^{1,2}(T,+\infty),\ \dim(M)=k} \max_{M\setminus\{0\}} \mathcal{R}_a(\psi),$$

where $\mathcal{R}_a(\psi)$ is the quotient in (3.4.11).

Remark 3.4.11. Let $z \in \mathcal{B}_0$. Then the map $t \mapsto \Lambda(z,t)$ is strictly decreasing. Indeed, fix $\delta > 0$ and consider an eigenfunction associated with $\Lambda(z,t+\delta)$. Extending this function by zero on the interval $(t,t+\delta)$, we obtain a function $\bar{\psi} \in \mathcal{D}_0^{1,2}(T,+\infty)$ such that $\mathcal{R}(\bar{\psi}) = \Lambda(z,t+\delta)$. By the variational characterization of $\Lambda(z,t)$ given in (3.4.11), it follows that

$$\Lambda(z,t) \leq \Lambda(z,t+\delta).$$

Assume now that $\Lambda(z,t+\delta) = \Lambda(z,t) = \lambda$ for some $\delta > 0$. In this case, an eigenfunction of (3.4.9) associated with λ in $\mathcal{D}_0^{1,2}(t+\delta,+\infty)$ can be extended by zero to produce an eigenfunction of (3.4.9) associated with λ in $\mathcal{D}_0^{1,2}(T,+\infty)$. This, however, is not possible, since eigenfunctions of (3.4.9) cannot vanish on an open subset of their domain, as a consequence of unique continuation, which here follows from the continuity of $\nabla^2 U(\xi(z,t))$. Therefore, for every $z \in \mathcal{B}_0$, there exists a unique time t such that $\Lambda(z,t) = 0$.

Remark 3.4.12. We notice that there exists $\mathcal{U}_x$, a neighborhood of x, such that

$$\Gamma \cap \mathcal{U}_x = \{y \in \mathbb{R}^n : y = \xi(z,t),\ \text{for some } (t,z) \in (1-\varepsilon,T_0)\times\mathcal{B}_0 \text{ s. t. } \Lambda(z,t) = 0\}.$$

For every $x \in \Gamma \cap \mathcal{U}_x$, if (z,t) is such that $\xi(z,t) = x$, then $\xi(\xi(z,t-t'),t') = x$.

Remark 3.4.13. For every $\mathcal{W}$ transversal section to ξ containing z_0, we can define

$$\xi_{\mathcal{W}}(z,t) = \xi(z,t), \quad \text{for every} \quad z \in \mathcal{W}, t \in (1-\varepsilon,T_0).$$

Consequently, we can define

$$\Lambda_{\mathcal{W}}(z,t) = \lambda_1(\xi_{\mathcal{W}}(z,t),t), \quad \text{for every} \quad z \in \mathcal{W}, t \in (1-\varepsilon,T_0).$$

Again, $t \mapsto \Lambda_{\mathcal{W}}(z,t)$ is strictly decreasing. Hence, for every $z \in \mathcal{W}$, there exists a unique t such that $\Lambda_{\mathcal{W}}(z,t) = 0$.

Case $\Lambda(z_0,1)$ simple

Since $\lambda_1(x_0,1) < \lambda_2(x_0,1)$, there exists $\delta > 0$ such that $\Lambda(z_0,t)$ is simple, i.e.,

$$\lambda_1(\xi(z_0,t),t) < \lambda_2(\xi(z_0,t),t), \quad t \in (1-\delta,1+\delta).$$

Let $t \mapsto \psi(t,T)$ denote the first eigenfunction in (3.4.9) corresponding to $\Lambda(T) := \Lambda(z_0,T)$, such that

$$\int_T^{+\infty} \frac{\|\dot\psi(t,T)\|_{\mathcal{M}}^2}{t^2}\, dt = 1.$$

Denoting $\psi_T(t,T) := \frac{\partial}{\partial T}\psi(t,T)$, we use (3.4.11) to compute $\Lambda(T)$:

$$\begin{aligned}
\frac{d}{dT}\Lambda(T) &= -\|\dot\psi(T,T)\|_{\mathcal{M}}^2 + \int_T^{+\infty} 2\langle \dot\psi_T(t,T), \dot\psi_T(t,T)\rangle_{\mathcal{M}} \\
&\quad + 2\langle V(t)\psi_T(t,T), \psi_T(t,T)\rangle_{\mathcal{M}}\, dt \\
&= -\|\dot\psi(T,T)\|_{\mathcal{M}}^2 - 2\langle \psi_T(T,T), \dot\psi_T(T,T)\rangle_{\mathcal{M}} \\
&\quad + 2\int_T^{+\infty} -\langle \ddot\psi(t,T), \psi_T(t,T)\rangle_{\mathcal{M}} + \langle V(t)\psi_T(t,T), \psi(t,T)\rangle_{\mathcal{M}}\, dt \\
&= -\|\dot\psi(T,T)\|_{\mathcal{M}}^2 - 2\langle \psi_T(T,T), \dot\psi_T(T,T)\rangle_{\mathcal{M}} \\
&\quad + 2\lambda_1(T)\int_T^{+\infty} \frac{\langle \psi(t,T), \psi_T(t,T)\rangle_{\mathcal{M}}}{t^2}\, dt.
\end{aligned}$$

From the boundary condition and the fact that we supposed $\int_T^{+\infty} \frac{\|\dot\psi(t)\|_{\mathcal{M}}^2}{t^2}\, dt \equiv 1\ \forall T$, it follows

$$\frac{d}{dT}\int_T^{+\infty} \frac{\|\dot\psi(t,T)\|_{\mathcal{M}}^2}{t^2}\, dt = 0,$$

which implies

$$\int_T^{+\infty} \frac{2\langle \psi(t,T), \psi_T(t,T)\rangle_{\mathcal{M}}}{t^2}\, dt = 0.$$

Besides, from the boundary condition, we have

$$\frac{\partial}{\partial T}\psi(T,T) = \dot\psi(T,T) + \psi_T(T,T) = 0.$$

We can thus conclude that

$$\begin{aligned}
\frac{d}{dT}\Lambda(T) &= -\|\dot\psi(T,T)\|_{\mathcal{M}}^2 - 2\langle \psi_T(T,T), \dot\psi(T,T)\rangle_{\mathcal{M}} \\
&= -\|\dot\psi(T,T)\|_{\mathcal{M}}^2 + 2\|\dot\psi(T,T)\|_{\mathcal{M}}^2 \\
&= \|\dot\psi(T,T)\|_{\mathcal{M}}^2 > 0.
\end{aligned}$$

This allows us to apply the Implicit Function Theorem to conclude that the function $t(z)$, defined for all $z \in \mathcal{U}_{z_0}$ by

$$\Lambda(z,t) = 0 \ \text{ if and only if } \ (z,t) = (z,t(z)),$$

is smooth. The same argument applies to $\Lambda_{\mathcal{W}}$ in place of Λ, showing that for every transversal section $\mathcal{W}$, the equation

$$\Lambda_{\mathcal{W}}(z,t) = 0 \quad \text{if and only if} \quad t = t_{\mathcal{W}}(z)$$

defines a smooth function on a neighborhood of z_0 in $\mathcal{W}$, namely $\mathcal{V}_{z_0} = \mathcal{U}_{z_0} \cap \mathcal{W}$.

As a consequence, $\Gamma' \cap \mathcal{U}_x$ is the image under $\xi_{\mathcal{W}}$ of the graph of a smooth function. Since $\Gamma' = \bigcup_{k\in\mathbb{N}} \Gamma_k$, where

$$\Gamma_k = \{x = \xi(z,t) : \; z \in \mathcal{W}, \; \lambda_1(\xi(z,t),t) \leq \lambda_2(\xi(z,t),t) + 1/k\}, \tag{3.4.12}$$

then $\Gamma' \cap \mathcal{U}_x$ is countably $\mathcal{H}^{d(N-1)-1}$-rectifiable. Indeed, if we define

$$\hat{\xi}(z) = \xi(z, t_{z_0}(z)), \quad \text{for} \quad z \in \mathcal{V}_{z_0},$$

we have

$$\Gamma' \subset \bigcup_{x_0\in\Gamma'} \hat{\xi}(\mathcal{V}_{z_0}) = \bigcup_{k\in\mathbb{N}} \bigcup_{x_0\in\Gamma_k} \hat{\xi}(\mathcal{V}_{z_0}) = \bigcup_{k\in\mathbb{N}} \hat{\xi}(\mathcal{V}_{z_k}).$$

Case Λ not simple

Let $x \in \Gamma$, for which a local flow (3.4.7) is defined. Assume that for some $z \in B_R$ and $s \in (1-\varepsilon, T_0]$, we have $\xi(z,s) \in \Gamma$ and

$$\lambda_1(z,s) = \cdots = \lambda_k(z,s) = 0, \quad k \geq 2.$$

Then, there exists linearly independent functions $w_1, \dots, w_k \in \mathcal{D}_0^{1,2}(s,+\infty)$ such that

$$\int_s^{+\infty} \|\dot{w}_i(t)\|^2_{\mathcal{M}} + \langle \nabla^2 U(\xi(z,t))\, w_i(t), \; w_i(t)\rangle_{\mathcal{M}} \, dt = 0, \quad \forall i = 1, \dots, k. \tag{3.4.13}$$

The functions w_i are independent solutions of

$$\begin{cases} \ddot{w} = \nabla^2 U(\xi(z,t))\, w, & t > 1, \\ w \in \mathcal{D}_0^{1,2}(s,+\infty). \end{cases} \tag{3.4.14}$$

Consider now the fundamental solution $\Phi = \Phi(z,t) \in \mathbb{R}^{d(N-1)}$, satisfying

$$\begin{cases} \ddot{\Phi} = \nabla^2 U(\xi(z,t))\, \Phi, & t > 1-\varepsilon, \\ \Phi(z,T_0) = I_d, \\ \Phi \in \mathcal{D}^{1,2}(1-\varepsilon,+\infty). \end{cases} \tag{3.4.15}$$

It is straightforward to verify that

$$w_i(t) = \Phi(z,t)\, w_i(T_0).$$

Consequently, the vectors $w_i(T_0)$ are linearly independent elements of $\operatorname{Ker}\Phi(z,s)$. Hence, we have

$$\dim(\operatorname{Ker}\Phi(z,s)) \geq k.$$

Vice versa, let $\theta_i \in \operatorname{Ker}\Phi(z,s)$. Hence, the function v_i defined by

$$v_i(t) := \Phi(z,t)\,\theta_i$$

is a solution of (3.4.14). Therefore, v_i satisfies (3.4.13), and hence

$$\dim(d^2\mathcal{A}_{\xi(z,s),[s,+\infty)}(\varphi(z,\cdot))) \geq k.$$

It follows

$$\dim(\operatorname{Ker} d^2\mathcal{A}_{\xi(z,s),[s,+\infty)}(\tilde{\varphi}(z,\cdot))) = \dim(\operatorname{Ker}\Phi(z,s)).$$

Therefore, if $x \in \mathcal{U}_{x_0}$ is such that $x = \xi(z,s)$, with $\lambda_1(\xi(z,s),s)$ not simple, then

$$\dim(\operatorname{Ker}\Phi(z,s)) \geq 2.$$

3.4.4 $\mathcal{H}^{d(N-1)-1}$-rectifiability of the set of conjugate points

Theorem 3.4.14 (Berti, Polimeni and Terracini 2025 [12]). *Γ is a countably $\mathcal{H}^{d(N-1)-1}$-rectifiable set.*

Proof. We can split Γ in two components:

$$\Gamma' = \{x \in \Gamma :\ \dim(\operatorname{Ker}(d^2\mathcal{A}_x(\varphi^x))) = 1\},$$

$$\Gamma'' = \{x \in \Gamma :\ \dim(\operatorname{Ker}(d^2\mathcal{A}_x(\varphi^x))) \geq 2\}.$$

For every $x \in \Gamma$, as in (3.4.6), we can define a local flow $\xi = \xi(z,t)$, for $z \in \mathcal{B}$ and $t > 1-\varepsilon$, whose image includes $\Gamma \cap \mathcal{U}_x$, with $\mathcal{U}_x$ a proper neighborhood of x.

We proved in Section 3.4.3 that for every transversal section $\mathcal{W}$ to $\dot{\xi}$ passing through $z = \gamma^x(T_0)$, we have

$$\Gamma' \cap \mathcal{U}_x = \{x = \xi(z,t),\ \text{for some } z \in \mathcal{W},\ \text{and } t \in (1-\varepsilon, T_0),\ \text{such that } \lambda_1(\xi(z,t),t) < \lambda_2(\xi(z,t),t)\},$$

and

$$\Gamma' = \bigcup_{k\in\mathbb{N}} \Gamma' \cap \mathcal{U}_{x_k},$$

for at most countably many points x_k. From (3.4.12), we conclude that Γ' is at most countably $(d(N-1)-1)$-rectifiable.

On the other hand, fixing $t=1$, and using the monotonicity of $t \mapsto \Lambda(z,t)$, we can describe $\Gamma'' \cap \mathcal{U}_x$ as

$$\Gamma'' \cap \mathcal{U}_x = \{x = \xi(z,1),\ \text{for some } z \in \mathcal{B},\ \text{and } \lambda_1(\xi(z,1),1) = \lambda_2(\xi(z,1),1)\}, \quad (3.4.16)$$

establishing a one-to-one correspondence between a subset of $\mathcal{B}$ and $\Gamma'' \cap \mathcal{U}_x$. This set of $z \in \mathcal{B}$ defined by (3.4.16) is not necessarily smooth, but we can still define the map

$$F : \mathcal{B} \to \mathbb{R}^{d(N-1)}, \quad F(z) = \xi(z,1).$$

Differentiating (3.4.6), we obtain (3.4.15), that is, $\xi_z(z,1) = \Phi(z,1)$. There exists a neighborhood $\mathcal{U}'_{x_0}$ such that

$$x \in \Gamma'' \cap \mathcal{U}'_{x_0} \text{if and only if } x = F(z) \text{ for some } z \in \mathcal{B} \text{ with } \operatorname{rk}\Phi(z,1) \leq d(N-1)-2.$$

Hence, by applying a Sard-type lemma as in [18, Lemma 6.6.1] to the map F, we obtain

$$\mathcal{H}^{d(N-1)-2+\varepsilon}(\Gamma'' \cap \mathcal{U}_x) \leq \mathcal{H}^{d(N-1)-2+\varepsilon}\Big(F\Big(\{z : \operatorname{rk} F_z(z) \leq d(N-1)-2\}\Big)\Big) = 0,$$

for all $\varepsilon > 0$. This in particular implies that $\Gamma'' \cap \mathcal{U}_x$ is at least $(d(N-1)-1)$-rectifiable. Again, since Γ'' is closed, there exists at most a countable collection of points x_k such that

$$\Gamma'' = \bigcup_{k \in \mathbb{N}} \Gamma'' \cap \mathcal{U}_{x_k},$$

where each $\mathcal{U}_{x_k}$ is a proper neighborhood of x_k. □

3.4.5 Hausdorff measure estimate for the set of regular conjugate points

Fix $x_0 \in \Gamma \setminus \Sigma$, and let $z_0 = \gamma^{x_0}(T_0)$. Then, for every $z \in B_R(z_0)$, define

$$\theta(z) \text{ generator of } \operatorname{Ker}\Phi(z,t(z)), \quad v(z) = \dot{\xi}(z,T_0).$$

Observe that $\theta(z)$ and $v(z)$ are not parallel. Since $\eta(\cdot) = \dot{\xi}(z,\cdot)$ satisfies

$$\begin{cases} \ddot{\eta} = \mathcal{M}^{-1}\nabla^2 U(\xi)\eta, \\ \eta(T_0) = v(z), \\ \eta \in \mathcal{D}^{1,2}(1,T_0), \end{cases}$$

it follows that $\eta(t) = \Phi(z,t)\, v(z)$. If $v(z) = k\,\theta(z)$ for some $k \in \mathbb{R}$, then $\eta(t(z)) = 0$, and hence $\dot{\xi}(z,t(z)) = 0$. This contradicts the conservation of mechanical energy at $t = t(z)$, since $h \geq 0$ and $U > 0$.

Now, consider the vector space of dimension $d(N-1)-2$, orthogonal to θ_0, v_0, and denote it by W_0. For each $w \in W_0$, we can define $\eta_w = \eta_w(s)$ by

$$\begin{cases} \dot{\eta}_w = \theta(\eta_w(s)) \\ \eta_w(0) = z_0 + w. \end{cases}$$

We can then define $\mathcal{W}$ as a $(d(N-1)-1)$-dimensional submanifold of $\mathcal{B}_0$, directly through the parameterization $U \ni (w,s) \mapsto \sigma(w,s) = \eta_w(s)$. It is straightforward to verify that $\partial_s\sigma(w,s) = \theta(\eta_w(s))$ and that $\partial_w\sigma(w,0) \in W_0$. This implies that $v(\eta_w(0)) \notin T_{\eta_w(0)}\mathcal{W}$. Consequently, by possibly restricting σ to a subset of U, we have $v(\eta_w(s)) \notin T_{\eta_w(s)}\mathcal{W}$ for every $(w,s) \in \mathcal{W}$.

Starting from the chart (U,σ), we compute

$$\begin{aligned} \partial_s\hat{\xi}(w,s,t(w,s)) &= \partial_s\Big(\xi(\sigma(w,s,t(w,s)))\Big) \\ &= \partial_z\xi(\sigma(w,s,t(w,s)))\,\partial_s\sigma(w,s,t(w,s)) \\ &= \partial_z\xi(z,t)\,\theta(z) + \partial_t\xi(z,t(z))\,\langle\nabla t(z),\theta(z)\rangle. \end{aligned}$$

Concerning the second term in the last expression, we adapt [18, Proposition 6.6.8] to the present framework by showing that, for every $z \in \mathcal{V}_{z_0} \subset \mathcal{B}_0$ such that $\xi(z,t(z)) \notin \Sigma$, we have

$$\langle\nabla t(z),\theta(z)\rangle = 0.$$

Indeed, following [18], if $\langle\nabla t(\bar{z}),\theta(\bar{z})\rangle \neq 0$, then by [18, Lemmas 6.6.6 and 6.6.11] there exists $\rho > 0$ such that the equation

$$\xi(z,t(\bar{z})) = \bar{x} + s\,w$$

admits no solution $z \in B_\rho(\bar{z})$ for $s < 0$ sufficiently small, where w is a generator of $\operatorname{Im}\Phi(\bar{z},t(\bar{z}))^\perp$ and $\bar{x} = \xi(\bar{z},t(\bar{z}))$.

Therefore, arguing as in the proof of [18, Proposition 6.6.8], but replacing $J_{\bar{t}}$ with $\mathcal{J}_{\xi(\bar{z},t(\bar{z})),t(\bar{z})}$ defined in (3.4.8), we can construct a sequence of arcs $(\varphi_n)_n$ such that each φ_n minimizes $\mathcal{J}_{x_n,t(\bar{z})}$, with

$$x_n = \xi(\bar{z},t(\bar{z})) \pm \frac{1}{n}\,w \longrightarrow \xi(\bar{z},t(\bar{z})),$$

and satisfies $\varphi_n(T_0) + r_0(T_0) \notin B_\rho(\bar{z})$.

Up to extracting a subsequence, letting $n \to +\infty$ yields another minimizer of $\mathcal{J}_{\xi(\bar{z},t(\bar{z})),t(\bar{z})}$ in (3.4.8), which is a contradiction.

This shows that $d\xi_{\mathcal{W}}(z,t(z))\,\theta(z)=0$, and consequently

$$\mathrm{rk}\Big(d\xi_{\mathcal{W}}(z,t(z))\Big)\le d(N-1)-2.$$

Applying the Sard-type Lemma for Banach manifolds [74], we infer that

$$\mathcal{H}^{d(N-1)-2+\varepsilon}(\Gamma'\cap\mathcal{U}_x)\le\mathcal{H}^{d(N-1)-2+\varepsilon}\Big(\{\xi_{\mathcal{W}}(z,t(z))\,:\,z\in\mathcal{W}\}\Big)=0,$$

for every $\varepsilon>0$.

Next, for $k\in\mathbb{N}$, define

$$\Gamma_k=\{x\in\Gamma'\,:\,\lambda_2(x)\ge\lambda_1(x)+1/k\},$$

which is a closed subset of Γ. Fixing $n\in\mathbb{N}$, we have

$$\Gamma_k\cap B(0,n)=\bigcup_{x\in\Gamma_k}\Big(\Gamma_k\cap\mathcal{U}_x\cap B(0,n)\Big)=\bigcup_{x_1,\dots,x_M\in\Gamma_k}\Big(\Gamma_k\cap\mathcal{U}_{x_i}\cap B(0,n)\Big),$$

where $B(0,n)$ denotes the ball of radius n centered at the origin. Hence,

$$\mathcal{H}^{d(N-1)-2+\varepsilon}(\Gamma_k\cap B(0,n))\le\sum_{i=1}^{M}\mathcal{H}^{d(N-1)-2+\varepsilon}(\Gamma_k\cap\mathcal{U}_{x_i}\cap B(0,n))=0.$$

It follows that

$$\mathcal{H}^{d(N-1)-2+\varepsilon}(\Gamma_k)\le\sum_{n=1}^{+\infty}\mathcal{H}^{d(N-1)-2+\varepsilon}(\Gamma_k\cap B(0,n))=0,$$

and therefore

$$\mathcal{H}^{d(N-1)-2+\varepsilon}(\Gamma')\le\sum_{k=1}^{+\infty}\mathcal{H}^{d(N-1)-2+\varepsilon}(\Gamma_k)=0.$$

Since we have already shown that $\mathcal{H}^{d(N-1)-2+\varepsilon}(\Gamma'')=0$ for every $\varepsilon>0$, we conclude that

$$\dim_{\mathcal{H}}(\Gamma\setminus\Sigma)\le d(N-1)-2.$$

Chapter 4

Stability of symmetric solutions of the N-body problem: a numerical analysis

Several years after the first study of Ferrario and Terracini [35], the new version `Symorb.jl` of the original software was introduced in [5, 27]. Distributed as a Julia package, it features a profound and modular redesign of the previous implementation. This new version integrates both quantitative and qualitative tools – such as stability analysis and topological index computations – and introduces more efficient optimization routines, which significantly improve computational performance. Finally, its new structure makes it possible to organize symmetric orbits into databases and to support advanced numerical methods.

Starting from a collection of numerical results produced with `Symorb.jl`, the aim of [11] is to establish a set of test cases by examining their stability properties under variations of the action and small perturbations of the initial conditions of the orbits. To this end, the authors consider two classical stability indicators: the Morse index and the Floquet multipliers.

The aim of this chapter is to present the numerical implementation developed in [11] for computing the Floquet multipliers and Morse indices of the periodic orbits produced by `Symorb.jl`. These stability indicators are subsequently applied to several representative examples of symmetric orbits.

4.1 Stability indicators

After obtaining an approximate solution of (1.5.1) using the `Symorb.jl` algorithm, it is possible to analyze its stability as a periodic orbit within the N-body problem. In this section, we present two complementary approaches for investigating stability.

The first approach involves Floquet multipliers, which capture how infinitesimal perturbations of the initial conditions evolve over a single period, thereby providing a criterion for linear stability. The second approach is based on the Morse index, which counts the number of independent directions along which the Lagrangian action decreases, offering insight into the variational stability of the orbit. For each of these quantities, we recall the essential definitions and describe the numerical methods adapted to our framework.

Further details on these stability indicators can be found in [59, 78, 79], while further applications of index theory in celestial mechanics are presented in [3, 6, 68].

The numerical algorithms developed to compute both indicators are then applied in Section 4.2 to ten representative test cases, covering planar and spatial orbits with varying numbers of bodies.

4.1.1 Floquet multipliers

Consider a nonlinear ordinary differential equation of the form

$$\dot{x}(t) = f(x(t)),$$

where $x : \mathbb{R} \to \mathbb{R}^d$, with $d \geq 1$, and $f : \mathbb{R}^d \to \mathbb{R}^d$. Assume that the system admits a T-periodic solution $\bar{x}(t)$ for some $T > 0$. The linear stability of such a periodic solution can be investigated by considering the associated linearized system

$$\dot{y}(t) = A_{\bar{x}}(t)\, y(t), \qquad A_{\bar{x}}(t) = Df(\bar{x}(t)), \tag{4.1.1}$$

where $A_{\bar{x}}(t)$ is a $d \times d$ matrix depending periodically on $\bar{x}(t)$.

According to the classical theory of linear systems with periodic coefficients, we introduce the *principal matrix solution* $X(t)$ as the unique solution of

$$\begin{cases} \dot{X}(t) = A_{\bar{x}}(t)\, X(t), \\ X(0) = I_d, \end{cases} \tag{4.1.2}$$

where $X(t) \in \mathbb{R}^{d\times d}$ for all t, and I_d denotes the identity matrix in dimension d. The *monodromy matrix* associated with (4.1.1) is then defined by

$$\mathcal{M}_{\bar{x}} = X(T).$$

The eigenvalues of the monodromy matrix, referred to as the *Floquet multipliers*, determine the linear stability of $\bar{x}$: if all the eigenvalues of $\mathcal{M}_{\bar{x}}$ lie inside or on the unit circle in the complex plane, then $\bar{x}$ is linearly stable; otherwise, it is linearly unstable (see, for example, [7, 17]).

In general, the explicit analytical computation of Floquet multipliers is a difficult task, and closed-form expressions are available only in very special situations. As a result, their evaluation is typically carried out by means of numerical methods.

Computation of Floquet multipliers for the N-body problem. Following the standard methodology adopted in the literature (see, for instance, [7, 17]), the N-body equations (1.5.1) can be reformulated as the first-order system

$$\begin{cases} \dot{x} = M^{-1}\, y, \\ \dot{y} = \nabla U(x), \end{cases}$$

where $M^{-1} \in \mathbb{R}^{d\times N}$ denotes the inverse of the diagonal mass matrix.

Within this setting, the linearized system (4.1.2) is represented by the block matrix

$$A_{\bar{x}}(t) = \begin{bmatrix} 0 & M^{-1} \\ \nabla^2 U(\bar{x}(t)) & 0 \end{bmatrix} \in \mathbb{R}^{(d\times N)^2},$$

where $0 = 0_{\mathbb{R}^{d\times N}}$ and $\nabla^2 U(\bar{x}(t))$ denotes the Hessian of the potential U evaluated along the periodic solution $\bar{x}(t)$.

For mechanical systems, the associated monodromy matrix $\mathcal{M}_{\bar{x}}$ is such that $\det(\mathcal{M}_{\bar{x}}) = 1$. Consequently, for any complex eigenvalue z with modulus $|z| = r$, there exists a corresponding eigenvalue w such that $|w| = 1/r$. It follows that a periodic orbit $\bar{x}$ is linearly stable if and only if all the eigenvalues of $\mathcal{M}_{\bar{x}}$ lie on the unit circle $\{\, |z| = 1 \,\} \subset \mathbb{C}$.

To compute the Floquet multipliers numerically, [11] implements the following algorithm, designed to evaluate the stability of a periodic orbit.

Algorithm 1 Floquet Multipliers Algorithm (Berti, Canneori, Ciccarelli, De Blasi, Introna, Polimeni 2025 [11])

1: **Input:** The periodic orbit, expressed in terms of its (truncated) Fourier coefficients.
2: **Output:** The eigenvalues of the monodromy matrix.
3: **Step 1.** Build the Hessian matrix $\nabla^2 U(\bar{x}(t))$
4: **Step 2.** Define the matrix differential equation (4.1.2)
5: **Step 3.** Compute the principal solution matrix $X(t)$
6: **Step 4.** Compute the complex eigenvalues of $\mathcal{M}_{\bar{x}} = X(T)$

4.1.2 Discrete Morse Index

The discrete Morse index counts the number of independent directions in which the action functional decreases, thereby distinguishing minima from saddle-type or variationally unstable configurations. A more detailed account of the Morse index and its properties can be found, for instance, in [59, 78].

In numerical studies of periodic orbits and choreographies, the Morse index serves as a valuable indicator for identifying bifurcations and qualitative transitions in stability [7, 37]. A discrete version of this index, motivated by Forman's combinatorial framework [36], transfers these ideas to finite-dimensional contexts where the action functional is defined on a discretized configuration space.

In [11], this discrete viewpoint is used to classify the critical points obtained through the numerical scheme and to interpret their stability properties from a variational perspective.

We recall the following general definition from [24].

Definition 4.1.1 (Computational discrete Morse index)**.** Let $f : H \to \mathbb{R}$ be a $\mathcal{C}^2$ function and p a non-degenerate critical point of f. The *index* of p is defined as the dimension of the maximal subspace of the tangent space at p where the Hessian is negative definite. Consequently, the *discrete Morse index* $\tilde{n}^-$ of p is the number of negative eigenvalues of the Hessian matrix $H_f(p)$.

In summary, the discrete Morse index offers a numerical criterion to classify critical points: a minimum is characterized by $\tilde{n}^-(p) = 0$, while a saddle point corresponds to $\tilde{n}^-(p) > 0$.

Computation of the Morse index for the N-body problem. Within the framework of the N-body problem, the computation of the discrete Morse index begins with a discretization of the action functional. This procedure converts the original infinite-dimensional variational problem into a finite-dimensional optimization problem, whose critical points approximate periodic solutions of the system. After discretization, the Morse index may be computed either on the *fundamental domain* $\mathbb{I}$ or, by exploiting the symmetries of the orbit, on the *full domain*. Clearly, the index evaluated on the fundamental domain cannot exceed the one computed along the entire orbit.

According to [28], the action functional (1.5.2) admits two principal discretization strategies. In the *point discretization* method, the integral defining the action is approximated by numerical quadrature on a uniform partition $0 = t_0 < t_1 < \cdots < t_{M+1} = T$. Time derivatives are replaced by finite differences, and the action is written in terms of discrete variables $y_i^k \approx x_i(t_k) \in \mathbb{R}^d$, for $i = 1, \ldots, n$ and $k = 0, \ldots, M+1$. This yields the discrete functional

$$f_1(y_i^k) = \mathcal{A}_h^{(1)},$$

defined on the block array $(y_i^k)_{i,k} \in \mathbb{R}^{d\times n\times(M+2)}$. Further details can be found in [26].

As an alternative, the *Fourier coefficients discretization* represents the trajectory through a truncated Fourier expansion – as in (1.5.4) and (1.5.5) – combined with a linear interpolation between the boundary configurations:

$$x(t) = x_0 + \frac{t}{\pi}(x_1 - x_0) + \sum_{k=1}^{F} A_k \sin(kt), \quad t \in [0, \pi],$$

where the sine terms enforce periodicity. Inserting this expression into (1.5.2) leads to

$$f_2(x_0, x_1, A_k) = \mathcal{A}_h^{(2)}.$$

In this case, the variables consist of the endpoints and the Fourier coefficients, for a total dimension of $(F + 2) \times d \times N$. A more detailed exposition is provided in [76, 77].

In both discretization approaches, the problem reduces to the minimization (or critical point analysis) of an objective function $f = f_i$ $(i = 1{,}2)$ with respect to the chosen discrete variables, whose critical points correspond to periodic motions of the system.

Once a discrete periodic orbit has been computed, the nature of the associated critical point can be further investigated by determining its discrete Morse index. This computation may be carried out in two equivalent frameworks, depending on whether the symmetries of the solution are taken into account: either on the *fundamental domain*, which captures a representative segment of the orbit, or on the *entire orbit*, where the analysis extends over the full period.

On the *fundamental domain* $\mathbb{I}$, as described in [5, 24], the Hessian of the discretized action functional can be evaluated using either the Fourier-based discretization, which relies on truncated Fourier series, or the pointwise discretization, where the time interval is subdivided into finitely many nodes.

On the other hand, in the *full-orbit* approach, the action functional is discretized over the complete period while imposing periodic boundary conditions. Since the trajectory is periodic, the initial and final configurations coincide, and only one of them is retained in the computation.

The procedure below describes the computation of the discrete Morse index over the full orbit.

Algorithm 2 Discrete Morse Index Algorithm (Berti, Canneori, Ciccarelli, De Blasi, Introna, Polimeni 2025 [11])

Input: Periodic solution given either
  (a) by its Fourier coefficients, or
  (b) by sampled positions $r_i(t_k)$ for each body $i = 1, \dots, N$ at times t_k, $k = 1, \dots, w$.

Output: Discrete Morse index $\tilde{n}^-(p)$.

Step 1. Build the Hessian matrix of the discretised action functional:
- Fourier case: compute the Hessian in coefficient space.
- Pointwise case: assemble the Hessian from the discretised action.

Step 2. Compute all eigenvalues λ_j of the Hessian.

Step 3. The discrete Morse index is the number of negative eigenvalues:

$$\tilde{n}^-(p) = \#\{\lambda_j < 0\}.$$

It is worth noting that, unlike previous studies that focused on specific families of solutions (such as the figure-eight choreography [22]), the approach adopted in the above algorithm is designed to be applicable to any numerically obtained periodic orbit. In this way, the abstract concepts introduced earlier are transformed into general computational tools, providing a direct link between the theoretical framework and the stability analysis in the N-body problem.

4.1.3 Variational and dynamical stability

Our numerical investigations (see Section 4.2) highlight, through the analysis of stability indicators, a classical distinction between *variational* stability and *dynamical* (or linear) stability. In the present setting, variational stability is quantified by the Morse index of the Lagrangian action, whereas dynamical stability is characterized by Floquet multipliers. These two notions generally describe different properties of a given orbit and do not necessarily coincide, as shown by several examples in this work and in earlier studies.

Variational stability concerns the behavior of the action functional under a restricted class of admissible perturbations, for example those satisfying imposed symmetries or corresponding to small deformations of the orbit. Within such a constrained framework, a zero Morse index indicates that the orbit is a local minimizer of the action.

By contrast, dynamical stability addresses the response of the orbit to arbitrary small perturbations governed by the linearized equations of motion, including perturbations that violate the variational constraints. Consequently, linear instability may arise even when the action attains a local minimum, since such kind of instability is not detected by the second variation of the action restricted to the admissible class.

An explanatory illustration of this discrepancy is provided by the Lagrange equilateral triangle solution (Figure 4.1). In the Newtonian 3-body problem with equal masses, this configuration minimizes the Lagrangian action among all periodic trajectories satisfying the same symmetry constraints, which is reflected by its vanishing Morse index. Nevertheless, the associated Floquet multipliers show linear instability, as some of them have modulus significantly larger than one. The source of this instability lies in perturbations excluded from the constrained variational problem but present in the full dynamical system, explaining how an action minimizer can still be linearly unstable.

The opposite situation is encountered for the figure-eight choreography (Figure 4.3). Here, the orbit does not minimize the action and possesses a positive Morse index, yet its Floquet multipliers are located close to the unit circle, indicating near linear stability. This phenomenon can be attributed to the strong symmetries of the solution, which remove many potentially unstable modes and render the linearized dynamics highly degenerate. The remaining perturbations increase the action but do not induce rapid dynamical divergence.

The numerical results reported in the following section therefore support the theoretical insight that minimization of the action is neither a necessary nor a sufficient condition for linear stability, even in the presence of high symmetry.

Comparison with Morse index computations. Only a limited number of works address the computation of Morse indices for periodic solutions of the 3-body problem, and most existing results rely on analytical or semi-analytical techniques rather than on a direct numerical evaluation of the second variation of the action. Nonetheless, a clear correspondence can be drawn between the discrete Morse indices computed in the present work and the Morse indices obtained for the figure-eight choreography by Fukuda, Fujiwara, and Ozaki [37, 38]. In their approach, the second variation of the action is studied via the spectral properties of the continuous Hessian operator, leading to the definition of three Morse indices depending on the admissible class of variations: the full periodic Morse index N, the choreographic Morse index N_c, and the figure-eight choreographic Morse index N_e. For the Newtonian figure-eight solution, they find $N = 2$ and $N_c = N_e = 0$, showing that the orbit is a saddle point of the unrestricted action functional, but becomes a local minimizer when variations are constrained by choreographic and figure-eight symmetries.

Our numerical results are fully consistent with this picture. When the Morse index is computed over the full period, we obtain a positive value equal to two, corresponding to unstable directions that break the choreographic symmetry. In contrast, restricting the computation to the fundamental domain associated with the figure-eight symmetry yields a vanishing discrete Morse index, indicating the absence of negative directions within the symmetric subspace. This agreement confirms that the discrete Hessian of the discretized action accurately reflects the symmetry-dependent variational properties of the figure-eight choreography and provides a faithful finite-dimensional approximation of the underlying continuous Morse index theory.

A closely related behavior is observed for other highly symmetric periodic solutions of the 3-body problem. In particular, the Morse index of the Lagrangian circular orbit has been analytically determined by Barutello, Jadanza, and Portaluri [7], who showed that this relative equilibrium is generically a saddle point of the Lagrangian action functional on the full loop space, while it becomes a minimizer only after imposing suitable constraints or restricting to appropriate invariant subspaces. Furthermore, they proved that changes in linear stability are accompanied by jumps in the Morse index of the orbit and its iterates. This mechanism closely resembles the situation for the figure-eight choreography, where unstable directions are present in the unrestricted variational setting but disappear once symmetry constraints are enforced.

4.2 Numerical results

In this section, the algorithms described in Section 4.1 are applied to a collection of ten distinct periodic orbits computed using `Symorb.jl`. Figures 4.1-4.10 illustrate a wide range of examples, including both planar and spatial configurations and involving different numbers of bodies. For each orbit, we report:

- the generators of the finite symmetry group G associated with the orbits. Recall that $\bar{G} = G/\ker\tau$ (see Section 1.5.4), and that the notation $R(\theta)$ denotes the rotation matrices of the form
$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix};$$
- the values of the action $\mathcal{A}$ and its gradient, to show the precision of the numerical method in locating a critical point of $\mathcal{A}$;
- the numerical Morse index, computed in both the fundamental domain $\mathbb{I}$ and over the whole period. Whenever this index is non-zero, we additionally report the maximal negative eigenvalue of the hessian matrix of $\mathcal{A}$;
- the maximal Floquet multiplier.

Within the variational framework, the value of the most negative eigenvalue of the Hessian gives a quantitative measure of how the negative spectrum of eigenvalues is far from zero, and hence of how reliable the computed Morse index is. In contrast, Floquet analysis focuses on the eigenvalue of the monodromy matrix with largest modulus: when this eigenvalue is sufficiently far from ± 1, we can be confident in saying that the numerically approximated orbit is linearly unstable.

The specific set of orbits analyzed in [11] has been selected to represent the diversity of periodic solutions that can be found in the N-body problem, focusing on non-collisional examples.

A notable subset of periodic motions is formed by the so-called *choreographies*, in which all bodies move along the same trajectory at evenly shifted times, as in Figures 4.1, 4.3, and 4.5. Moreover, several of the orbits shown in Figures 4.1-4.4 correspond to classical solutions previously examined in the literature (see, for instance, [7, 69, 70, 76, 80]). They are included here in order to validate the present computations against known benchmarks. Notably, Figure 4.2 depicts a *collinear relative equilibrium*, a classical configuration whose study dates back to early foundational works such as [56].

Among the examples considered, two cases illustrate the subtle relationship between variational and dynamical stability. The triangular Lagrange solution (Figure 4.1) is a global minimizer of the action functional, yet it is linearly unstable. By contrast, the celebrated figure-eight choreography (Figure 4.3) does not minimize the action over its full period, while nonetheless displaying strong linear stability – consistent with earlier analysis ([69, 76]).

Taken together, these examples show the intricate interplay between the variational and dynamical properties of periodic solutions to the N-body problem. The comparison between the Morse index, which captures the local variational structure of $\mathcal{A}$, and the Floquet multipliers, which encode linear dynamical stability, reveals a broad spectrum of different behaviors. This duality provides a useful tool for organizing and distinguishing families of periodic orbits according to their qualitative dynamical features.

Group generator(s)	$\ker\tau = \langle \mathrm{Id} \rangle, \quad \bar{G} = \langle r \rangle$ $\rho(r) = -\mathrm{Id}, \quad \sigma(r) = ()$	
Action value		9.8022
Gradient norm		7.72×10^{-11}
Morse (fund. domain)	**Index**	0
	Max negative eigenvalue	—
Morse (period)	**Index**	0
	Max negative eigenvalue	—
Floquet	**Norm of the max eigenvalue**	85.0213

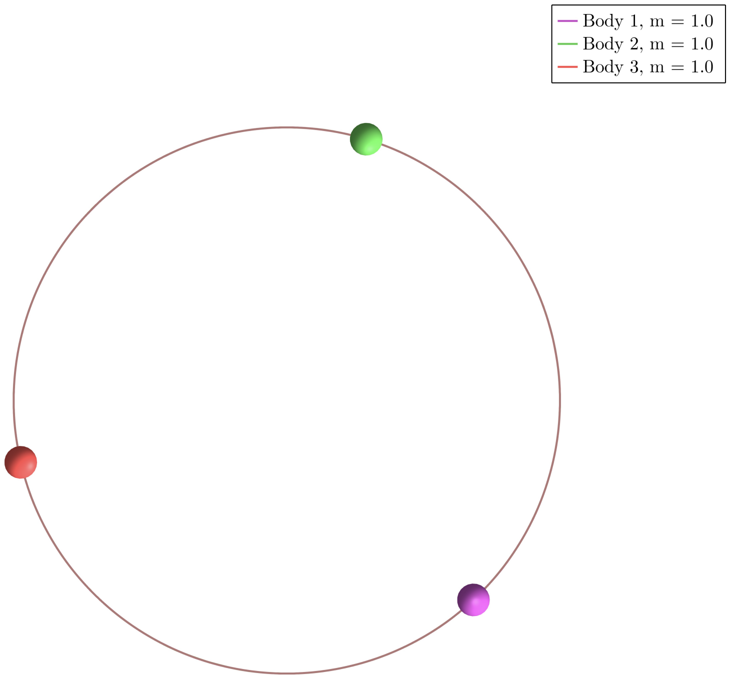

Figure 4.1: Lagrange equilateral triangle. This orbit is a minimizer of the action, with zero Morse index, but exhibits linear instability, as indicated by Floquet multipliers with modulus larger than one. The table and figure are taken from [11].

Group generator(s)	$\ker\tau = \langle \mathrm{Id} \rangle$, $\bar{G} = \langle r \rangle$ $\rho(r) = -\mathrm{Id}$, $\sigma(r) = ()$	
Action value		10.9365
Gradient norm		6.77×10^{-11}
Morse (fund. domain)	**Index**	1
	Max negative eigenvalue	-5.5278
Morse (period)	**Index**	3
	Max negative eigenvalue	-13.9510
Floquet	**Norm of the max eigenvalue**	5.8982×10^{4}

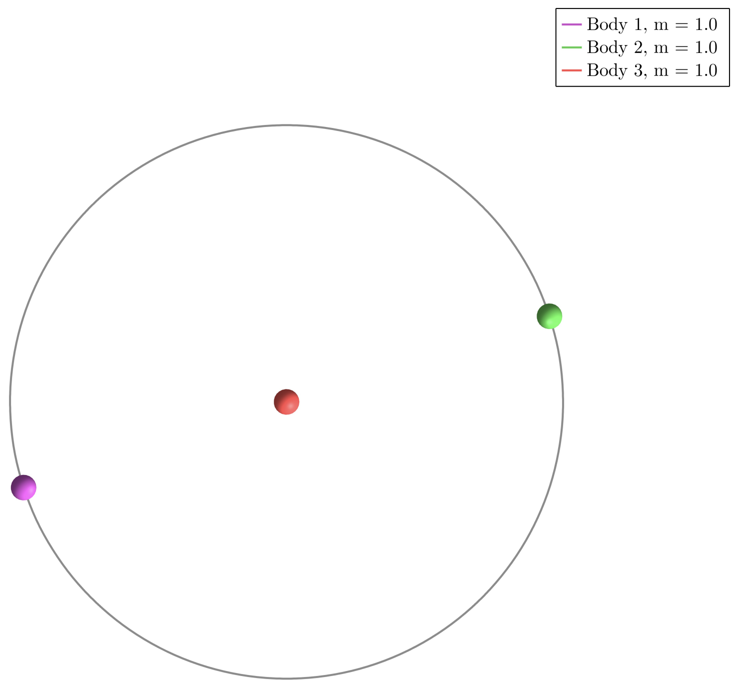

Figure 4.2: Collinear relative equilibrium. This orbit is a linearly unstable collinear periodic solution of the 3-body problem, as indicated by its Floquet multipliers. The table and figure are taken from [11].

Group generator(s)	$\ker\tau = \langle \mathrm{Id} \rangle$, $\quad \bar{G} = \langle r, s \rangle$ $\rho(r) = \mathrm{Id}$, $\quad \sigma(r) = (1,2,3)$ $\rho(s) = -\mathrm{Id}$, $\quad \sigma(s) = (2,3)$	
Action value		5.8584
Gradient norm		4.85×10^{-12}
Morse (fund. domain)	**Index**	0
	Max negative eigenvalue	—
Morse (period)	**Index**	2
	Max negative eigenvalue	-0.2257
Floquet	**Norm of the max eigenvalue**	1.0187

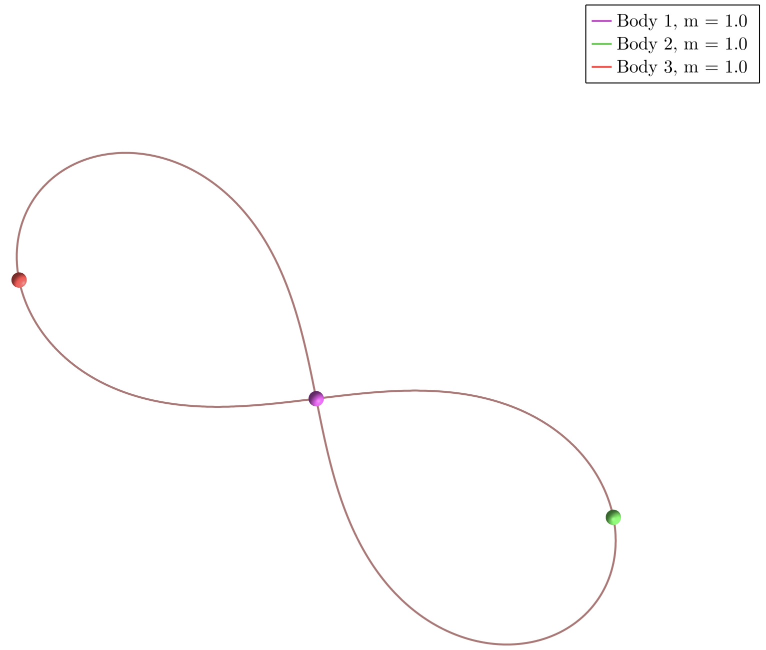

Figure 4.3: Figure-eight choreography. The orbit is not a minimizer of the action and has positive Morse index, but its Floquet multipliers lie close to the unit circle, indicating near-linear stability. The table and figure are taken from [11].

Group generator(s)	$\ker\tau = \langle \mathrm{Id} \rangle, \quad \bar{G} = \langle r \rangle$ $\rho(r) = -\mathrm{Id}, \quad \sigma(r) = ()$	
Action value		10.4421
Gradient norm		3.22×10^{-8}
Morse (fund. domain)	**Index**	0
	Max negative eigenvalue	—
Morse (period)	**Index**	2
	Max negative eigenvalue	-0.0617
Floquet	**Norm of the max eigenvalue**	1.0462

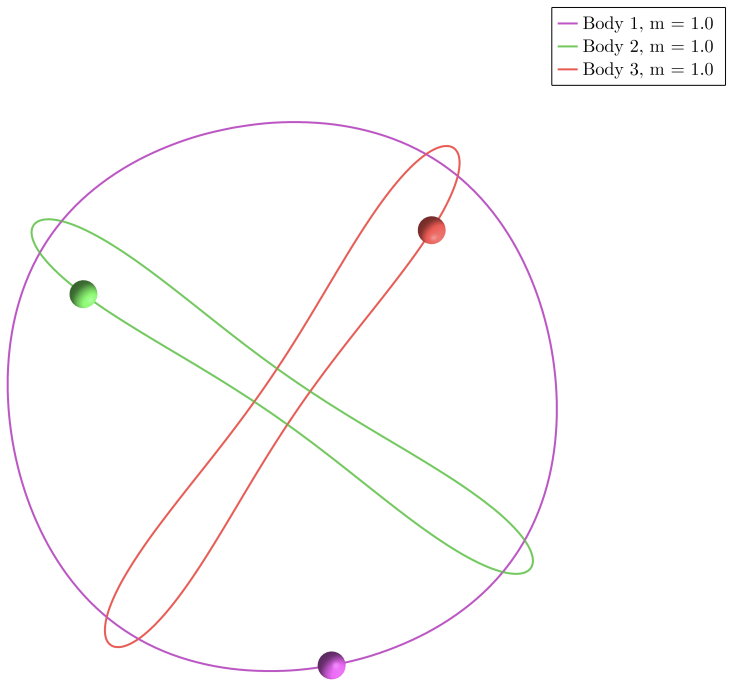

Figure 4.4: Solution of the 3-body problem, typically known as *Ducati* and first discovered in [58]. Its Floquet multipliers show that the orbit is linearly stable. The table and figure are taken from [11].

Group generator(s)	$\ker\tau = \langle \mathrm{Id} \rangle, \quad \bar{G} = \langle r \rangle$ $\rho(r) = \mathrm{Id}, \quad \sigma(r) = (1,2,3,4,5,6,7,8,9,10,11,12)$	
Action value		72.6212
Gradient norm		2.01×10^{-8}
Morse (fund. domain)	**Index**	0
	Max negative eigenvalue	—
Morse (period)	**Index**	4
	Max negative eigenvalue	-0.1103
Floquet	**Norm of the max eigenvalue**	4.1930×10^{12}

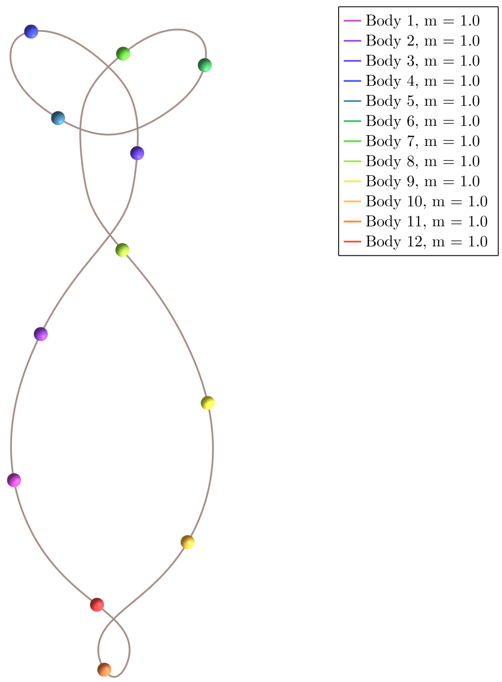

Figure 4.5: Linearly unstable choreography of the 12-body problem. The table and figure are taken from [11].

Group generator(s)	$\ker \tau = \langle \kappa \rangle, \quad \bar{G} = \langle r \rangle$ $\rho(\kappa) = R(2\pi/5)$, $\sigma(\kappa) = (1,2,3,4,5)(6,7,8,9,10)$ $\rho(r) = -\mathrm{Id}, \quad \sigma(r) = ()$	
Action value		114.3331
Gradient norm		4.11×10^{-7}
Morse (fund. domain)	**Index**	0
	Max negative eigenvalue	—
Morse (period)	**Index**	2
	Max negative eigenvalue	-0.0024
Floquet	**Norm of the max eigenvalue**	1173.9612

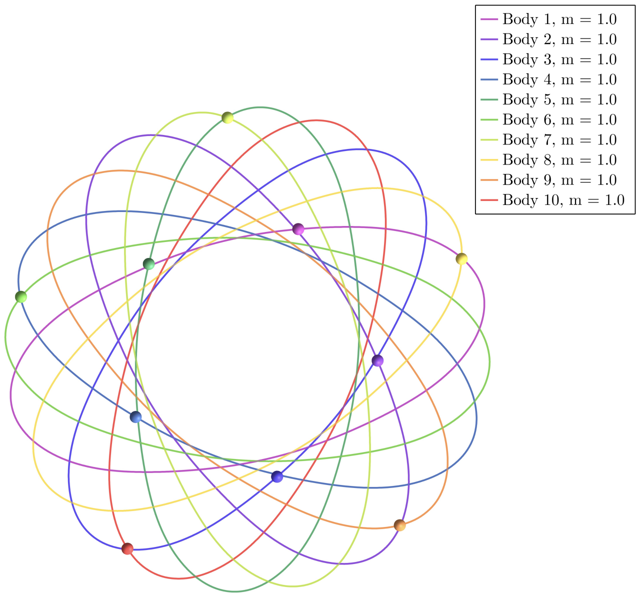

Figure 4.6: Linearly unstable periodic solution of the 10-body problem. The table and figure are taken from [11].

Group generator(s)	$\ker\tau = \langle \mathrm{Id}\rangle, \quad \bar{G} = \langle r\rangle$ $\rho(r) = \begin{pmatrix} R(2\pi/3) & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma(r) = ()$	
Action value		8.3409
Gradient norm		3.46×10^{-8}
Morse (fund. domain)	**Index**	1
	Max negative eigenvalue	-3.0423
Morse (period)	**Index**	19
	Max negative eigenvalue	-48.5411
Floquet	**Norm of the max eigenvalue**	6.3110×10^{8}

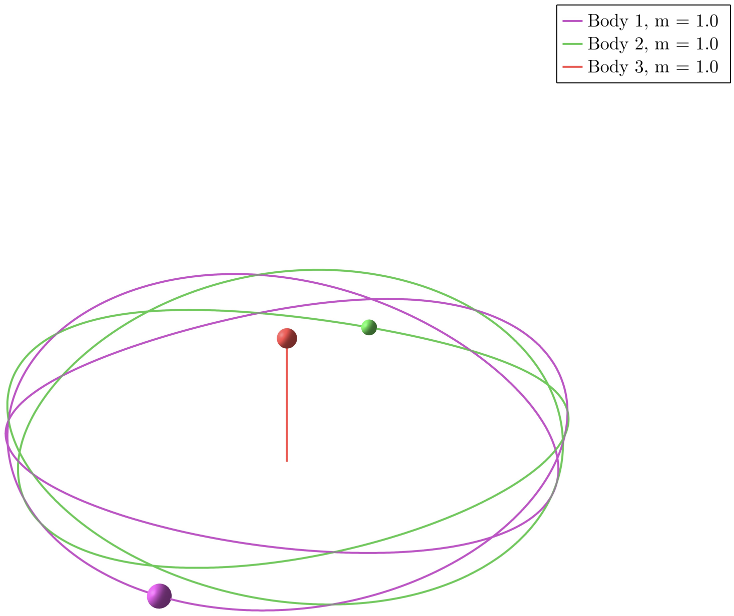

Figure 4.7: Linearly unstable periodic solution of the spacial isosceles 3-body problem. The table and figure are taken from [11].

Group generator(s)	$\ker\tau = \langle \mathrm{Id} \rangle, \quad \bar{G} = \langle r, s \rangle$ $\rho(r) = -\mathrm{Id}, \quad \sigma(r) = ()$ $\rho(s) = \begin{pmatrix} -Id_2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma(s) = ()$	
Action value		11.8999
Gradient norm		2.52×10^{-8}
Morse (fund. domain)	**Index**	0
	Max negative eigenvalue	—
Morse (period)	**Index**	9
	Max negative eigenvalue	-11.6320
Floquet	**Norm of the max eigenvalue**	25.1735

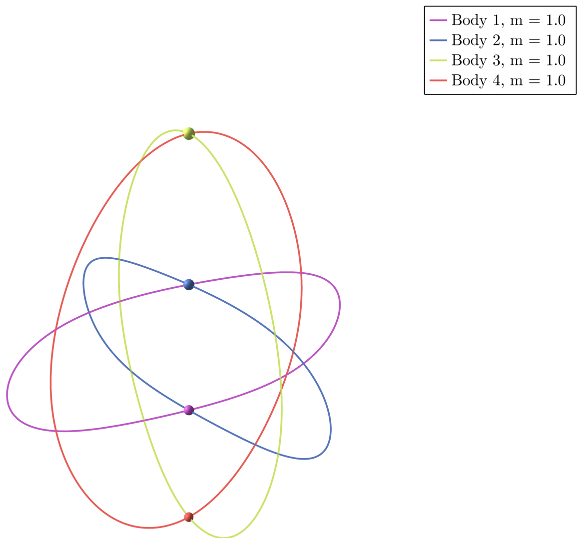

Figure 4.8: Linearly unstable periodic solution of the 4-body problem. The table and figure are taken from [11].

Group generator(s)	$\ker\tau = \langle \mathrm{Id} \rangle$, $\quad \bar{G} = \langle r, s \rangle$ $\rho(r) = -\mathrm{Id} \quad \sigma(r) = (1,2)(3,4)$ $\rho(s) = \begin{pmatrix} -\mathrm{Id}_2 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma(s) = ()$	
Action value		12.1428
Gradient norm		5.27×10^{-10}
Morse (fund. domain)	**Index**	0
	Max negative eigenvalue	—
Morse (period)	**Index**	11
	Max negative eigenvalue	-202.1842
Floquet	**Norm of the max eigenvalue**	2191.5747

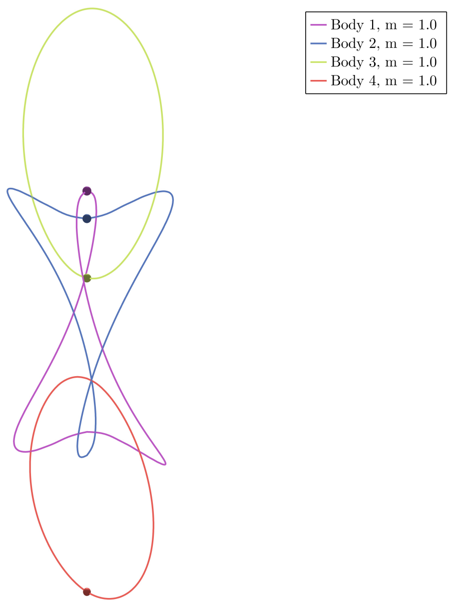

Figure 4.9: Linearly unstable periodic solution of the 4-body problem. The table and figure are taken from [11].

Group generator(s)	$\ker\tau = \langle \mathrm{Id} \rangle, \quad \bar{G} = \langle r \rangle$ $\rho(r) = R(\pi/6)$ $\sigma(r) = (1,2,3,4,5,6,7,8,9,10,11,12)$	
Action value		81.6410
Gradient norm		7.65×10^{-9}
Morse (fund. domain)	**Index**	0
	Max negative eigenvalue	—
Morse (period)	**Index**	4
	Max negative eigenvalue	-0.1778
Floquet	**Norm of the max eigenvalue**	8.5410×10^{12}

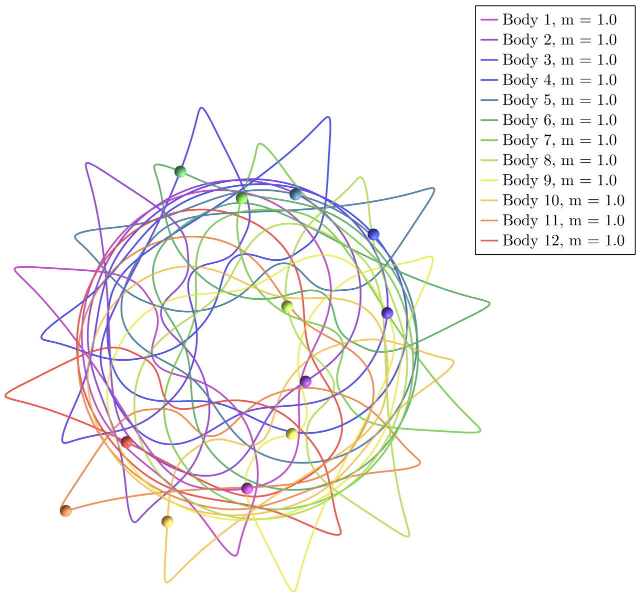

Figure 4.10: Linearly unstable periodic solution of the 12-body problem. The table and figure are taken from [11].

4.2.1 Validation and comparison with known results

Among the periodic solutions reported in [11], corresponding to Figures 4.1-4.10, the first four orbits (Figures 4.1-4.4) are previously known solutions of the 3-body problem. The remaining ones (Figures 4.5-4.10) appear, to the best of our knowledge, to be *new orbits*, and display favorable features in terms of the depth and exploratory capabilities of our algorithm. As a consequence, for the solutions shown in Figures 4.5-4.10, the stability indices introduced in this work cannot be directly compared with existing results in the literature. For the known solutions depicted in Figures 4.1 and 4.3, instead, the Morse index computations discussed in Section 4.1.3 already provide a meaningful validation, as they correctly recover the expected symmetry-dependent variational stability properties.

To further support our analysis, we compare the remaining numerical stability indicators with analytical and numerical results available in the literature.

Figures 4.1 and 4.2 represent classical solutions of the 3-body problem, namely *relative equilibria*. More precisely, Figure 4.1 corresponds to the *equilateral triangular* relative equilibrium, while Figure 4.2 depicts a *collinear* relative equilibrium. With regard to linear stability – which is closely related to the Floquet index introduced here – the following facts are well established. In the 3-body problem, collinear relative equilibria are *linearly unstable for all choices of the masses* (see, for instance, [70]). This fully accounts for the large value of the Floquet index observed in Figure 4.2.

The equilateral triangular configuration shown in Figure 4.1 is also known to be generically linearly unstable. This is a consequence of the *Gascheau-Routh criterion*, which asserts that linear stability holds if and only if

$$27\,(m_1m_2 + m_1m_3 + m_2m_3) < (m_1 + m_2 + m_3)^2.$$

This condition is clearly violated in the equal-mass case $m_1 = m_2 = m_3 = 1$. Furthermore, our results are consistent with those reported in [80, ”Lagrange3” in Table 2].

We now consider the solutions illustrated in Figures 4.3 and 4.4. The orbit shown in Figure 4.3 was proven to be linearly stable in [42], thereby settling a long-standing conjecture originally proposed by Simó, by means of a computer-assisted proof (see also [69, 80]). In our numerical experiments, the stability outcome depends on the tolerance used in the computations, which is consistent with the subtle and delicate nature of the stability analysis for this orbit.

The orbit displayed in Figure 4.4 is referred to as the *“Ducati”* orbit in [80, ”Ducati3” in Table 1], where its Floquet index is evaluated numerically. Our computed results are in good agreement with those previously reported.

These comparisons are summarized in Table 4.1.

Figure	**Validation method**	**Consistency**
Figure 1	Classical analytical results	✓
Figure 2	Classical analytical results	✓
Figure 3	Theoretical and numerical results	✓
Figure 4	Numerical results	✓
Figures 5-10	No previous results (unknown orbits)	

Table 4.1: Validation of the numerical results for the orbits considered in this chapter. Table taken from [11]

Appendix A

The Jacobi-Maupertuis' principle

In this appendix, we collect some classical results concerning the variational principle of Jacobi-Maupertuis for the Newtonian N-body problem. We recall its statement and describe its main properties, showing how it characterizes the dynamics of the N-body system as geodesics of a suitable metric associated with the energy level. The following are classical results and can be found in several books that treat classical and celestial mechanics, such as [62].

Consider the configuration space $M = \mathbb{R}^{dN}$, for $d \geq 2$, both as a Riemannian manifold and as its tangent space T_pM. Given a metric $G = (g_{ij})_{i,j}$ and a curve $\gamma\colon [a,b] \to M$, since $\dot\gamma(s) \in T_{\gamma(s)}M$, we can define the length of γ with respect to G as

$$\mathcal{L}(\gamma) = \int_a^b \|\dot\gamma(s)\|_{\gamma(s)}\, ds = \int_a^b \Big(\sum_{i,j} g_{ij}(\gamma(s))\gamma_i(s)\gamma_j(s)\Big)^{1/2} ds,$$

where $\|\cdot\|_{\gamma(s)}$ denotes the norm on $T_{\gamma(s)}M$.

Working in the setting of the N-body problem, consider Newton's equations

$$\mathcal{M}\ddot{x} = \nabla U(x),$$

where we take the matrix of the masses $\mathcal{M}$ to be the identity matrix for simplicity. The conservation of the energy states

$$h = \frac{1}{2}\|\dot{x}\|^2 - U(x),$$

where we consider $h > 0$ and the function U is the Newtonian potential.

The Jacobi metric (also called Jacobi-Maupertuis metric) is defined as

$$g_{ij}(x) = (U(x) + h)\delta_{ij},$$

where $x \in \mathbb{R}^{dN}$ and δ_{ij} is the Kronecker symbol, and the corresponding Jacobi length functional is

$$\mathcal{L}(\gamma) = \int_a^b \Big(\sum_{i,j} g_{ij}\dot{\gamma}_i\dot{\gamma}_j\Big)^{1/2} dt = \int_a^b \sqrt{U(\gamma(s)) + h}\|\dot{\gamma}(s)\| \ ds.$$

Remark A.0.1. $\mathcal{L}(\gamma)$ is invariant under time reparametrizations.

Maupertuis' variational principle states that the true path of a system described by N coordinates $p(t) = (p_1(t), ..., p_N(t))$ between two configurations x and y is a stationary point of the *abbreviated action functional*

$$\mathcal{S}_0(p) = \int q \ dp,$$

where the components of the vector $q = (q_1, ..., q_N)$ are the conjugate momenta of the generalized coordinates and are defined as

$$q_k = \frac{\partial L}{\partial \dot{p}_k}, \quad \text{for } k = 1, ..., N.$$

In particular, Maupertuis' principle implies that if γ is a geodesic curve for the Jacobi metric, then there exists a parametrization $x(t) = \gamma(\eta(t))$, with $t \in [0,1]$, that is a solution of the system

$$\begin{cases} \ddot{x} = \nabla U(x) \\ \frac{1}{2}\|\dot{x}\|^2 - U(x) = h. \end{cases} \tag{A.0.1}$$

The following are classical results concerning the energy functional of a motion. We leave the proofs to the reader.

Proposition A.0.2. *Minimizing $\mathcal{L}(\gamma)$ between the curves that join p_0 and p_1 is equivalent to minimize the energy functional*

$$\mathcal{E}(\gamma) = \int_0^1 \|\dot{\gamma}\|^2(h + U(\gamma)) \ dt.$$

Proposition A.0.3 (Necessary condition for extremality)**.** *Given p_0 and p_1, if there is a curve $\bar{\gamma}$ such that*

$$\min_{\gamma(0)=p_0, \ \gamma(1)=p_1} \mathcal{E}(\gamma) = \mathcal{E}(\bar{\gamma}),$$

then the curve $\bar{\gamma}$ satisfies the geodesic equation

$$-\frac{d}{dt}(2\dot{\bar{\gamma}}(h + U(\bar{\gamma})) + \|\dot{\bar{\gamma}}\|^2\nabla U(\bar{\gamma})) = 0 \tag{A.0.2}$$

and

$$\|\dot{\bar{\gamma}}\|^2(h+U(\bar{\gamma})) = \ constant \ = \mathcal{E}(\bar{\gamma}). \tag{A.0.3}$$

By the last proposition, we can deduce that any minimizer of the energy functional (or, equivalently, every minimizer of the length functional) satisfies the associated Euler-Lagrange equation almost everywhere.

The same holds for minimizers of the Lagrangian action $\mathcal{A}_L(\gamma) = \int_a^b L(\gamma,\dot{\gamma})\ dt$. Indeed, integrating by parts, we have, for any $v \in C^1([a,b])$ such that $v(a) = v(b) = 0$,

$$\begin{aligned} d\mathcal{A}_L(\gamma)[v] &= \int_a^b \frac{\partial L}{\partial \gamma}(\gamma,\dot{\gamma})v + \frac{\partial L}{\partial \dot{\gamma}}(\gamma,\dot{\gamma})\dot{v}\ dt \\ &= \int_a^b \left[\frac{\partial L}{\partial \gamma}(\gamma,\dot{\gamma}) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\gamma}}(\gamma,\dot{\gamma})\right)\right] v\ dt \\ &= 0 \end{aligned}$$

By the fundamental Lemma of the calculus of variations, it holds

$$\frac{\partial L}{\partial \gamma}(\gamma,\dot{\gamma}) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\gamma}}(\gamma,\dot{\gamma})\right) = 0$$

almost everywhere.

Proposition A.0.4. *Given a curve γ that satisfies* (A.0.2)*, there is a reparametrization $x(t) = \gamma(\eta(t))$ such that x solves equations* (A.0.1)*.*

Proof. We start with the second equation of (A.0.1). From (A.0.3), which follows from (A.0.2), we have that there is a real constant c such that

$$\left\|\frac{d}{d\eta}\gamma(\eta(t))\right\|^2 = \frac{c}{h+U(\gamma(\eta(t)))}.$$

Considering $x(t) = \gamma(\eta(t))$, we have

$$\begin{aligned} \frac{1}{2}\|\dot{x}(t)\|^2 - U(x) &= \frac{1}{2}\left\|\frac{d}{d\eta}\gamma(\eta(t))\right\|^2 \|\dot{\eta}(t)\|^2 - U(\gamma(\eta(t))) \\ &= \frac{1}{2}\frac{c}{h+U(\gamma(\eta(t)))}\|\dot{\eta}(t)\|^2 - U(\gamma(\eta(t))). \end{aligned}$$

Choosing

$$\dot{\eta}(t) = \sqrt{\frac{2}{c}}(h+U(\gamma(\eta(t)))), \tag{A.0.4}$$

we obtain $\frac{1}{2}\|\dot{x}(t)\|^2 - U(x) = h$, which is exactly what we wanted to prove.

Now, we want to prove that such reparametrization also satisfies Newton's equations (A.0.1). We have

$$\ddot{x}(t) = \frac{d^2}{d\eta^2}\gamma(\eta(t))\dot{\eta}(t)^2 + \frac{d}{d\eta}\gamma(\eta(t))\ddot{\eta}(t).$$

By computing the derivative in (A.0.2), we obtain

$$2\left(\frac{d^2}{d\eta^2}\gamma(\eta(t))\right)(h + U(\gamma(\eta(t)))) + \left\|\frac{d}{d\eta}\gamma(\eta(t))\right\|^2 \nabla U(\gamma(\eta(t))).$$

By this last expression and (A.0.4), it follows

$$\begin{aligned}
\ddot{x}(t) &= -\frac{\left\|\frac{d}{d\eta}\gamma(\eta(t))\right\|^2 \nabla U(\gamma(\eta(t)))}{2(h + U(\gamma(\eta(t))))} + \left(\frac{d}{d\eta}\gamma(\eta(t))\right)^2 \sqrt{\frac{2}{c}}\nabla U(\gamma(\eta(t)))\dot{\eta(t)}) \\
&= -\nabla U(\gamma(\eta(t))) + \left(\frac{d}{d\eta}\gamma(\eta(t))\right)^2 \frac{2}{c}\nabla U(\gamma(\eta(t)))(h + U(\gamma(\eta(t)))) \\
&= -\nabla U(\gamma(\eta(t))) + 2\nabla U(\gamma(\eta(t))) \\
&= \nabla U(\gamma(\eta(t))).
\end{aligned}$$

□

To be more specific, we can now give a relationship between the above argument and the property of being a free time minimizer of the action $\mathcal{A}_{L+h}$.

It is a well known argument that the variational property of being a free time minimizer at the energy level h is equivalent to the property of being a minimizing geodesic for the Jacobi-Maupertuis metric j_h.

In addition, if γ is a free time minimizer for $\mathcal{A}_{L+h}$, then γ satisfies the conservation of energy $\frac{1}{2}\|\dot{\gamma}\|^2 - U(\gamma) = h$. To see this, consider a curve $\gamma : [0, T] \to \mathbb{R}^{dN}$ such that $\gamma(0) = x$ and $\gamma(T) = y$. Our goal is to determine the critical points of the action functional $\mathcal{A}_{L+h}$ with respect to the time T.

Remark A.0.5. Unlike in Hamilton's principle of least action, in Maupertuis' principle the final time T is not fixed.

We can define a function $\eta : [0,1] \to \mathbb{R}^{dN}$ such that $\gamma(t) = \eta(t/T)$. Thus, it holds

$$\begin{aligned}
\mathcal{A}_{L+h}(\gamma) &= \int_0^T \frac{1}{2T^2}\|\dot{\eta}(t/T)\|^2 + U(\eta(t/T)) + h\ dt \\
&= \int_0^1 \frac{1}{2T}\|\dot{\eta}(\tau)\|^2 + T[U(\eta(\tau)) + h]\ d\tau.
\end{aligned}$$

We can define

$$A = \int_0^1 \frac{1}{2}\|\dot{\eta}(\tau)\|^2 \, d\tau \quad \text{and} \quad B = \int_0^1 U(\eta(\tau)) + h \, d\tau,$$

so that we can easily compute the minimum of the function

$$\frac{A}{T} + TB = 0,$$

which is reached at $T = \sqrt{\frac{A}{2B}}$. We can thus reduce our problem of finding a free-time minimizer for $\mathcal{A}_{L+h}$ to that of finding a minimizer of the length functional associated with the Jacobi-Maupertuis metric, which is equivalent to finding a minimizing geodesic for the Jacobi-Maupertuis metric.

Appendix B

Fourier coefficients on the full period from the fundamental domain

Let G be a finite group acting on the orbits as described in Section 1.5, and associate to each element $g \in G$ the matrix $M_g = \rho(g) \otimes p(\sigma(g))$, where $\otimes$ denotes the Kronecker product and p is the representation of permutations on the orthogonal group. Let $x(t)$ be an orbit defined on the fundamental domain $\mathbb{I} = [0, \pi]$ as in (1.5.4), and denote by $\tilde{x}(t)$ its extension to the full period T obtained through the action of G, as in (1.5.5). The corresponding Fourier coefficients A_0, A_k, and B_k are defined by

$$\begin{aligned} A_0 &= \frac{1}{T} \int_0^T \tilde{x}(t)\, dt, \\ A_k &= \frac{2}{T} \int_0^T \tilde{x}(t) \cos\left(\frac{2\pi}{T} kt\right)\, dt, \\ B_k &= \frac{2}{T} \int_0^T \tilde{x}(t) \sin\left(\frac{2\pi}{T} kt\right)\, dt. \end{aligned} \tag{B.0.1}$$

These integrals can be evaluated analytically, yielding explicit expressions for A_0, A_k, and B_k in terms of the coefficients a_k. The resulting formulae, however, depend crucially on the structure of the reduced group $\bar{G}$. As shown in [35], only three configurations are possible for $\bar{G}$:

- it consists solely of time rotations (*"cyclic action"*);
- it contains a single time reflection (*"brake action"*);
- it includes both time rotations and time reflections (*"dihedral action"*).

The cyclic case must therefore be treated separately from the brake and dihedral cases.

For later use, let $p, q \in \mathbb{N}$ and introduce the notation

$$c_q^p = \cos\left(\frac{p}{q}\pi\right), \qquad s_q^p = \sin\left(\frac{p}{q}\pi\right),$$

together with the integrals

$$I = \int_0^\pi x(t)\,dt = \frac{\pi}{2}(x_0 + x_1) + \sum_{\substack{h=1\dots F\\ h \text{ odd}}} \frac{2a_h}{h},$$

$$(I_{\cos})_q^p = \int_0^\pi x(t)\cos\left(\frac{pt}{q}\right)\,dt$$
$$= \begin{cases} \dfrac{q^2}{\pi p^2}(c_q^p - 1)(x_1 - x_0) + \dfrac{q}{p}s_q^p x_1 + \displaystyle\sum_{\substack{h=1\dots F\\ p\neq qh}} \left[c_q^p - (-1)^h\right] H_{q,h}^p a_h, & \text{if } p \neq q,\\[2ex] \dfrac{2}{\pi}(x_0 - x_1) + \displaystyle\sum_{\substack{h=1\dots F\\ h \text{ even}}} \dfrac{2h\,a_h}{h^2 - 1}, & \text{if } p = q,\end{cases}$$

$$(I_{\sin})_q^p = \int_0^\pi x(t)\sin\left(\frac{pt}{q}\right)\,dt$$
$$= \begin{cases} \dfrac{q^2}{\pi p^2}s_q^p(x_1 - x_0) + \dfrac{q}{p}(x_0 - c_q^p x_1) + \displaystyle\sum_{h=1\dots F} f_{q,h}^p a_h, & \text{if } p \neq q,\\[2ex] \dfrac{\pi}{2}a_1 + x_0 + x_1, & \text{if } p = q,\end{cases}$$

where

$$H_{q,h}^p = \frac{(-1)^h}{h\left[\left(\frac{p}{hq}\right)^2 - 1\right]}, \qquad f_{q,h}^p = \begin{cases} \dfrac{\pi}{2}, & \text{if } h = \dfrac{p}{q},\\ s_q^p H_{q,h}^p, & \text{otherwise.}\end{cases}$$

Cyclic action. Assume that $\bar{G}$ acts cyclically, so that $\bar{G} = \langle g \rangle$ and $\tau(g^{-1})t = t + \pi$. The extension of the orbit to the full period $T = m\pi$ is then given by

$$\tilde{x}(t) = (M_g)^l \cdot x(t - l\pi), \qquad t \in [l\pi, (l+1)\pi].$$

Substituting this expression into (B.0.1) yields

$$
\begin{aligned}
A_0 &= \frac{1}{m\pi}\sum_{l=0}^{m-1}\int_{l\pi}^{(l+1)\pi}\tilde{x}(t)\,dt = \frac{1}{m\pi}\sum_{l=0}^{m-1}(M_g)^l I,\\
A_k &= \frac{2}{m\pi}\sum_{l=0}^{m-1}\int_{l\pi}^{(l+1)\pi}\tilde{x}(t)\cos\left(\frac{2kt}{m}\right)\,dt\\
&= \frac{2}{m\pi}\sum_{l=0}^{m-1}(M_g)^l\Big[c_m^{2kl}(I_{\cos})_m^{2k} - s_m^{2kl}(I_{\sin})_m^{2k}\Big],\\
B_k &= \frac{2}{m\pi}\sum_{l=0}^{m-1}\int_{l\pi}^{(l+1)\pi}\tilde{x}(t)\sin\left(\frac{2kt}{m}\right)\,dt\\
&= \frac{2}{m\pi}\sum_{l=0}^{m-1}(M_g)^l\Big[s_m^{2kl}(I_{\cos})_m^{2k} + c_m^{2kl}(I_{\sin})_m^{2k}\Big].
\end{aligned}
$$

Dihedral or brake action. In the dihedral or brake case, $\bar{G} = \langle h_0, h_1\rangle$, where $\tau(h_0)t = -t$ and $\tau(h_1)t = 2\pi - t$. Defining $g = h_1 h_0$, its induced action on time is $\tau(g)t = t + 2\pi$. The extended orbit can therefore be written as

$$
\tilde{x}(t) = \begin{cases} (M_g)^l \cdot x(t - 2l\pi), & t \in [2l\pi, (2l+1)\pi],\\ (M_g)^l M_{h_1}\cdot x\big((2l+2)\pi - t\big), & t \in [(2l+1)\pi, (2l+2)\pi], \end{cases} \qquad l = 0,\dots,m-1.
$$

Inserting this expression into (B.0.1) leads to

$$
\begin{aligned}
A_0 &= \frac{1}{2m\pi}\sum_{l=0}^{m-1}\int_{2l\pi}^{2(l+1)\pi}\tilde{x}(t)\,dt\\
&= \frac{1}{2m\pi}\sum_{l=0}^{m-1}(M_g)^l\Big[Id + M_{h_1}\Big]I,\\
A_k &= \frac{1}{m\pi}\sum_{l=0}^{m-1}(M_g)^l\Bigg[c_m^{2kl}(I_{\cos})_m^k - s_m^{2kl}(I_{\sin})_m^k\\
&\qquad\qquad + M_{h_1}\Big(c_m^{2k(l+1)}(I_{\cos})_m^k + s_m^{2k(l+1)}(I_{\sin})_m^k\Big)\Bigg],\\
B_k &= \frac{1}{m\pi}\sum_{l=0}^{m-1}(M_g)^l\Bigg[s_m^{2kl}(I_{\cos})_m^k + c_m^{2kl}(I_{\sin})_m^k\\
&\qquad\qquad + M_{h_1}\Big(s_m^{2k(l+1)}(I_{\cos})_m^k - c_m^{2k(l+1)}(I_{\sin})_m^k\Big)\Bigg].
\end{aligned}
$$

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