Wiener-Hopf Analysis of Plane Wave Scattering by a Thick Slot Filled by Penetrable Media

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***Abstract*—The study of electromagnetic scattering problem related to a thick slot within a conducting screen is particularly significant in applications. The aim of the paper is two folds: to present a new effective semi-analytical solution for this problem in spectral domain and to present a novel mathematical methodology based on Wiener-Hopf techniques capable of dealing with finite Laplace transform physical unknowns.**

Keywords—thick metallic shield, loaded slot, screen, Wiener-Hopf method, scattering, diffraction, propagation, asymptotics

I. Introduction

The electromagnetic scattering problem constituted by thick slot filled by penetrable media within a conducting screen is of interest in applications such as antenna systems, periodic structures, screens and propagation, see Fig. 1.

Although the problem may be considered canonical; it receives constant attention for practical applications.

The problem can be analyzed by means of numerical, semi-analytical techniques and asymptotics, however an interest on full comprehension of physical phenomena and application at high frequency stimulates research finding in the field of spectral method with asymptotic evaluation of field components. For this reason, we focus the attention on semi-analytical methods with physical interpretation. For the sake of completeness we report that the initial studies using numerical methods may be referred to integral equation formulations, see [1-2]. Different attempts to get solutions have been reported for this problem and for very similar ones as the scattering by two sets of parallel plate waveguides with a hole and by parallel planes of finite length.

Among the works that implement semi-analytical methods we recognize the application of Wiener-Hopf (WH) technique in [3-8], of the mode-matching and coupled integral equations in [9-13], of Weber-Schafheitlin discontinuous integrals and Kobayashi-Nomura's method in [14-15], of combined modal expansion and ray tracing methods [16] and, of Kirchhoff approximation and asymptotics [17-18].

The proposed technique, in contrast to iterative methods such as Physical Optics and Ray Tracing, presents a complete representation of the problem and its structure through a comprehensive mathematical model in the spectral domain. This approach eliminates the need for multiple interaction steps between distinct diffraction canonical structures. Consequently, we derive the accurate spectra of field components, from which we extract physical and engineering phenomena, similar to the approach utilized in closed-form spectral analytical solutions.

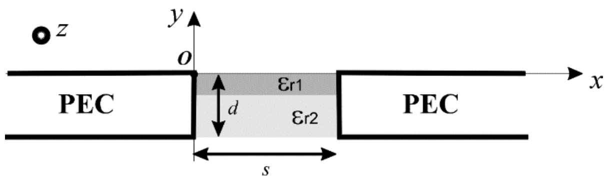

Figure 1. Thick slot filled by penetrable media within a conducting screen

The approach relies on the formulation and the solution of Wiener-Hopf equations in spectral domain, partitioning the problem's geometry into canonical sub-regions.

This method integrates the characteristic Green's function technique [19], the Mittag-Leffler theorem [20], and an innovative specialization of Fredholm factorization. The original version of Fredholm factorization was developed by these authors to address Wiener-Hopf problems not-amenable of closed-form solutions, systematically transforming the Wiener-Hopf factorization problem into a Fredholm integral equation of the second kind [21-23] that can be semi-analytically solved.

The method proposed in this paper is general and does not require any specialized (pre)factorization and solutions of systems of infinite equations for unknown coefficients as well known in the effective Jones' method for the solution of modified Wiener-Hopf equations [24].

Indeed, the proposed method provides explicit integral equation formulation for the solution of modified Wiener-Hopf equations such as in this problem. A similar methodology gas recently demonstrated its efficacy in problems with not entire spectral unknowns [25].

In this work we propose the theory and the demonstration that the Fredholm factorization is also the perfect instrument to consider entire function unknowns with exponential phase-factors. Due to the convergence properties of Fredholm integral equation modeling of the problems, simple quadrature schemes allow high precision results via reconstruction formula [26].

Moreover, both Wiener-Hopf formulation and Fredholm integral equation modeling can be interpreted systematically as network relations avoid redundancy.

The solution procedure is constituted by the following steps in order: 1) deduction of WH equations, 2) Fredholm factorization, 3) estimation of field components via asymptotics.

For completeness, we report that the theory was partially reported in [27], while only recently we have developed a solution method for modified Wiener-Hopf equations with entire unknows and exponential phase factors [28]. The precision of the first numerical validations seems to exceed the one of alternative techniques such as the one presented in [29] where pre-factorizations together with non -physical sampling (like in information theory) of entire unknows is used to get integral equation formulations to be solved numerically.

II. Formulation of the Problem

Fig. 1 illustrates the problem where the slot is of thickness d and wideness s. The figure reports that the slot is filled by an arbitrary penetrable stratification. In the following we use reference coordinate systems with different origins also for the definition of spectral transformations: Cartesian coordinates (x,y,z) and cylindrical coordinates (ρ,φ,z) with origin O, and shifted Cartesian coordinates (X=x-s,y,z) with origin O' (not reported in Fig. 1). Moreover, for asymptotic estimation of fields we use also coordinate system at the center of the port of the hole respectively for the top region and the bottom region, i.e. A'=(x,y,z)=(s/2,0,z) and A''=(x,y,z)=(s/2,-d,z).

Let us subdivide the geometry of the problem into three regions of simple canonical shapes. Using the (x,y,z) reference system with origin O, region 1 and 2 are homogenous regions and defined by respectively y>0 and y<-d, while region 3 is the slot region with 0<x<s, -d<y<0.

For the sake of simplicity, we consider E_z polarization, and we assume time harmonic fields with a time dependence specified $e^{+j\omega t}$ which is omitted. The problem is governed by the wave equation:

$$\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+k_i{}^2 E_z=0 \qquad (1)$$

where k_i is the propagation constant of the medium according to the three regions. For the complete mathematical formulation of the problem, we impose PEC boundary conditions on the two thick half planes, i.e. E_z=0 (PEC condition) for (x=0,s; -d<y<0;z), (x<0; y=0,-d;z), (x>s; y=0,-d;z). As radial distance from the four edges (x=0,s;y=0,-d;z) decrease to zero, E_z goes to zero (Meixner's edge condition). Radiation condition is enforced considering vanishing small losses in k_i and it is useful for the correct definition of WH unknows.

The source considered in this work is an *Ez*-polarized plane wave with incidence angle φ_o from region 1:

$$E_z^i(x,y)=E_o e^{jk\rho\cos(\varphi-\varphi_o)}=E_o e^{jk\left(x\cos\varphi_o+y\sin\varphi_o\right)} \qquad (2)$$

where *k* is the propagation constant (the region outside the slit can be considered free space).

The WH equations of the problem are written in terms of Laplace transforms along x direction of relevant field components at *y=0,-d*:

$$V_{1o}(\eta)=\int_0^s E_z(x,0)e^{j\eta x}dx,$$
$$V_{2o}(\eta)=\int_0^s E_z(x,-d)e^{j\eta x}dx, \qquad (3)$$

$$I_{1o}(\eta)=\int_0^s H_x(x,0)e^{j\eta x}dx,$$
$$I_{1+}(\eta)=e^{-j\eta s}\int_s^\infty H_x(x,-d)e^{j\eta x}dx,$$
$$I_{1\pi+}(\eta)=-\int_{-\infty}^0 H_x(x,0)e^{-j\eta x}dx, \qquad (4)$$
$$I_{2o}(\eta)=\int_o^s H_x(x,-d)e^{-j\eta x}dx,$$
$$I_{2+}(\eta)=e^{-j\eta s}\int_0^\infty H_x(x,-d)e^{j\eta x}dx,$$
$$I_{2\pi+}(\eta)=-e^{j\eta s}\int_{-\infty}^s H_x(x,-d)e^{-j\eta x}dx.$$

Using transverse equations formalism in spectral domain, from (1) we get network relations in spectral domain to model region 1 and region 2 which result to be after imposing PEC boundary conditions:

$$-I_{1\pi+}(-\eta)+I_{1o}(\eta)+e^{j\eta s}I_{1+}(\eta)=Y_c(\eta)[V_{1o}(\eta)] \qquad (4)$$

$$I_{2\pi+}(-\eta)-I_{2o}(\eta)-e^{j\eta s}I_{2+}(\eta)=Y_c(\eta)[V_{2o}(\eta)] \qquad (5)$$

with $Y_c(\eta)=\xi(\eta)/k\,Z_o,\xi(\eta)=\sqrt{k^2-\eta^2}$.

To model region 3 in the WH formulation we need to start by applying the finite Laplace transform (6) to the wave equations (1).

$$\tilde{E}_{zo}(\eta,y)=\int_0^s E_z(x,y)e^{j\eta x}dx \qquad (6)$$

For the sake of brevity in this theoretical formulation we consider a region 3 that is homogenous to the other two regions with propagation constant k. It yields

$$\left(\frac{d^2}{dy^2}+\tau^2\right)\tilde{E}_{zo}(\eta,y)=f_{o\eta}(y) \qquad (7)$$

with $f_{o\eta}(y)=-j\omega\mu\;H_y(s,y)e^{j\eta s}+j\omega\mu\;H_y(0,y)$.

By applying the Greens' function procedure, and observing the solution at y=0,-d, we get two-port network model

$$Y_{11}(\eta)V_{1o}(\eta)+Y_{12}(\eta)V_{2o}(\eta)=-e^{j\eta s}q_o(\eta)+p_o(\eta)-I_{1o}(\eta)$$
$$-Y_{21}(\eta)V_{1o}(\eta)-Y_{22}(\eta)V_{2o}(\eta)=-e^{j\eta s}s_o(\alpha)+r_o(\eta)-I_{2o}(\eta)$$
$$(8)$$

where

$$
\begin{aligned}
&Y_{11}(\eta)=Y_{22}(\eta)=-jY_c(\eta)\cot[\xi(\eta)d],\\
&Y_{12}(\eta)=Y_{21}(\eta)=Y_m(\eta)=j\frac{Y_c(\eta)}{\sin[\xi(\eta)d]}
\end{aligned}
\tag{9}
$$

and $q_o(\eta), p_o(\eta), s_o(\eta), r_o(\eta)$ are not explicit terms related to particular integrals due to $f_{o\eta}(y)$.

The application of Mittag-Leffler theorem to these terms allows a different representation in terms of samples of the spectra of WH unknows $V_{1o}(\eta), V_{2o}(\eta)$ at $\eta_n=\sqrt{k^2-(n\frac{\pi}{d})^2}$.

Notice that till now we have produced 4 WH equations reported at (4), (5), (8). Due to exponential phase factors reported in the equation, and in order to solve completely the system of equation with Fredholm factorization method we duplicate the equations by substituting η with $-\eta$ and using the auxiliary unknowns (10):

$$
\begin{aligned}
&V_{1\pi o}(\eta)=e^{j\eta s}V_{1o}(-\eta),\quad V_{2\pi o}(\eta)=e^{j\eta s}V_{2o}(-\eta),\\
&I_{1\pi o}(\eta)=e^{j\eta s}I_{1o}(-\eta),\quad I_{2\pi o}(\eta)=e^{j\eta s}I_{2o}(-\eta)
\end{aligned}
\tag{10}
$$

The motivation is that the application of Fredholm factorization requires regularization properties of the unknows and convergence of exponential factors [28].

The resulting system of equations allows to compute all the defined unknowns in terms of just the voltage entire unknowns.

In order to obtain explicit integral equation formulation, we need to represent the samples of spectral unknowns contained in $q_o(\eta), p_o(\eta), s_o(\eta), r_o(\eta)$ along the integration contour of the Fredholm integral equation. For this scope we apply a modified version of integral Cauchy representation formula that transforms sampled unknowns into augmented kernel terms while applying the Fredholm factorization method [25,28].

The final system of integral equations becomes explicit in terms of voltage entire unknows and of Fredholm type, thus it can be solved by simple quadrature and reconstruction formula.

III. SOLUTION OF THE PROBLEMS

Once solved the problem in terms of voltage spectra, via asymptotics we get diffraction coefficients and total fields of the problem for plane wave illumination.

Further details on the formulation, numerical validations and results will be shown during the presentation.

A first promising numerical result is reported in Fig. 2 where the slot is considered filled by free space and with dimensions kd=2, ks=7, the illumination is constituted by an Ez polarized plane wave with incidence from region 1 at $\varphi_o=2\pi/9$, and intensity Eo=1V/m. Fig. 2 reports the GO field, the GTD component and the total far field at kr=10.

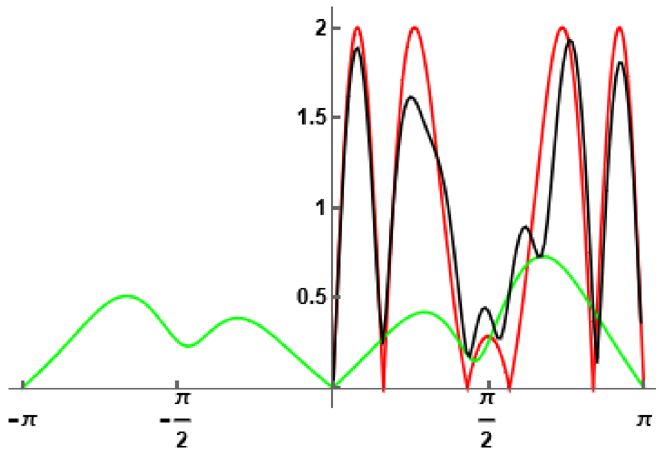

Figure 2. Thick slot filled by free space within a conducting screen: dimensions kd=2, ks=7. Ez polarized plane wave illumination from region 1with φo=2π/9, Eo=1V/m. The figure reports far filed computation at kr=10: GO component (red line), GTD component (green line), total far field (black line for top region 1 GO+GTD, green line for bottom region 2 GTD only).

IV. CONCLUSIONS

We present an effective method for the analysis of a thick slot within a conducting screen based on Wiener-Hopf technique. In particular, first numerical results show the effectiveness of the technique. Beyond the interest on the engineering application the paper presents also a method to solve modified Wiener-Hopf equations with entire unknowns and exponential phase factors.

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