

Dynamics of a charged Ziegler's double pendulum with a nonholonomic constraint under the joint action of a dead force and a Lorentz force

Original

Dynamics of a charged Ziegler's double pendulum with a nonholonomic constraint under the joint action of a dead force and a Lorentz force / Pastore, Andrea; Harrop, Joel C.; Bigoni, Davide; Grillo, Alfio. - In: APPLICATIONS IN ENGINEERING SCIENCE. - ISSN 2666-4968. - ELETTRONICO. - (2026).

Availability:

This version is available at: 11583/3008784 since: 2026-03-15T11:42:28Z

Publisher:

Elsevier

Published

DOI:

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

Elsevier preprint/submitted version

(Article begins on next page)

17 Dynamics of a charged Ziegler's double pendulum
18 with a nonholonomic constraint under the joint action
19 of a dead force and a Lorentz force

20 Andrea Pastore^a, Joel C. Harrop^b, Davide Bigoni^b, Alfio Grillo^{1a}

^a*Dipartimento di Scienze Matematiche (DISMA) 'G.L. Lagrange', Politecnico di Torino, Corso Duca degli Abruzzi
24, Torino, 10129, Italy*

^b*Instabilities Lab, University of Trento, Via Mesiano 77, Trento, 38123, Italy*

21 **Abstract**

A variant of viscoelastic Ziegler's double pendulum is investigated in which an electric charge is concentrated at one end of the device, at which a structural element imposing a nonholonomic constraint is attached. The obtained device is called 'charged Ziegler's double pendulum' and is subjected to the influence of a dead force and a Lorentz force. The latter results from an electric field and a magnetic induction field generated by an ideal solenoid. The mechanical system models a self-propelled microrobot and, in the absence of damping, is conservative up to the energy variation induced by the Lorentz force in the case of explicitly time dependent electric field. Within this framework, two situations are analyzed: Case I, in which the device is placed inside the solenoid and experiences a Lorentz force featuring both electric and magnetic part; Case II, in which the device is placed outside the solenoid and undergoes a Lorentz force due to the sole electric field. After determining the equilibrium for the system under study, a stability analysis is performed, and the conditions are obtained for flutter instability and Hopf bifurcation. In addition, the post-critical behavior of the structure is examined, focusing on the interaction between the Lorentz force and the other forces characterizing its dynamics. We investigate the possibility of revisiting, in the mechanical context provided by Ziegler's double pendulum, a phenomenology known in electromagnetism as Maxwell–Lodge effect.

22 *Keywords:* Ziegler's double pendulum, Nonholonomic constraints, Flutter instability, Hopf
23 bifurcation, Ideal solenoid, Lorentz force, Maxwell–Lodge effect, Microrobotics

24 *PACS:* 03.50.De, 05.45.-a, 45.05.+x, 45.20.Jj, 45.40.Ln, 87.85.gj, 87.85.St

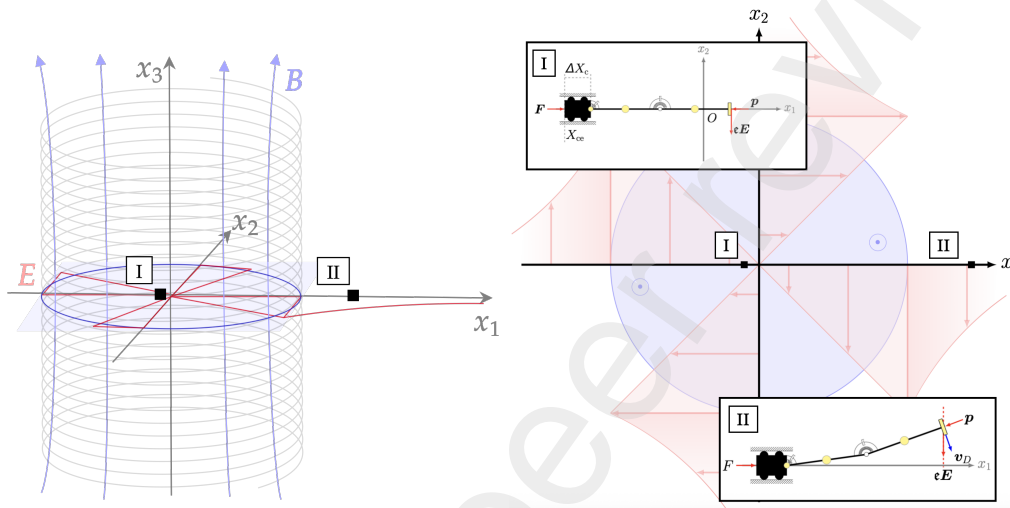
25 *2000 MSC:* 00A69, 37G15, 37J20, 37J60, 70E50, 70F25, 74F15

26 **Acknowledgements**

27 A.P and A.G. has been partially funded by the research project 'Innovative multiscale ap-
28 proaches, possibly based on Fractional Calculus, for the effective constitutive modeling of cell
29 mechanics, engineered tissues, and metamaterials in Biomedicine and related fields', PRIN
30 PNRR P2022KHFNB. D.B. and J.H. acknowledge financial support from the European Research
31 Council (ERC) under the European Union's Horizon Europe research and innovation programme
32 (Grant agreement No. ERC-ADG-2021-101052956-BEYOND). The methodologies developed
33 in this work fall within the aims of the GNFM (Gruppo Nazionale per la Fisica Matematica) of
34 the INDAM (Istituto Nazionale di Alta Matematica).

¹Corresponding author: alfio.grillo@polito.it

- 1 Graphical Abstract
- 2 Dynamics of a charged Ziegler's double pendulum
- 3 with a nonholonomic constraint under the joint action
- 4 of a dead force and a Lorentz force



5 **Highlights**

6 **Dynamics of a charged Ziegler's double pendulum**
7 **with a nonholonomic constraint under the joint action**
8 **of a dead force and a Lorentz force**

- 9 • The interplay between a follower force and the Lorentz force
- 10 • The dynamics of a charged microrobot is triggered by the Lorentz force generated by the
11 electromagnetic field inside an ideal solenoid and in the nearby outer space through a
12 time-dependent vector potential
- 13 • The appearance of the Lorentz force in the outer space seems to be a manifestation of the
14 Maxwell–Lodge effect

15 Dynamics of a charged Ziegler's double pendulum
16 with a nonholonomic constraint under the joint action
17 of a dead force and a Lorentz force

18 **Abstract**

A variant of viscoelastic Ziegler's double pendulum is investigated in which an electric charge is concentrated at one end of the device, at which a structural element imposing a nonholonomic constraint is attached. The obtained device is called 'charged Ziegler's double pendulum' and is subjected to the influence of a dead force and a Lorentz force. The latter results from an electric field and a magnetic induction field generated by an ideal solenoid. The mechanical system models a self-propelled microrobot and, in the absence of damping, is conservative up to the energy variation induced by the Lorentz force in the case of explicitly time dependent electric field. Within this framework, two situations are analyzed: Case I, in which the device is placed inside the solenoid and experiences a Lorentz force featuring both electric and magnetic part; Case II, in which the device is placed outside the solenoid and undergoes a Lorentz force due to the sole electric field. After determining the equilibrium for the system under study, a stability analysis is performed, and the conditions are obtained for flutter instability and Hopf bifurcation. In addition, the post-critical behavior of the structure is examined, focusing on the interaction between the Lorentz force and the other forces characterizing its dynamics. We investigate the possibility of revisiting, in the mechanical context provided by Ziegler's double pendulum, a phenomenology known in electromagnetism as Maxwell–Lodge effect.

19 *Keywords:* Ziegler's double pendulum, Nonholonomic constraints, Flutter instability, Hopf
20 bifurcation, Ideal solenoid, Lorentz force, Maxwell–Lodge effect, Microrobotics
21 *PACS:* 03.50.De, 05.45.-a, 45.05.+x, 45.20.Jj, 45.40.Ln, 87.85.gj, 87.85.St
22 *2000 MSC:* 00A69, 37G15, 37J20, 37J60, 70E50, 70F25, 74F15

23 **1. Introduction**

24 The double pendulum is a mechanical device that, because of its simplicity, is often employed
25 to exemplify more complicated structures, which appear in various mechanical contexts, ranging
26 from different branches of engineering [1, 2] to robotics [3, 4], and biomechanics [5].

27 One variant of the double pendulum is Ziegler's double pendulum [6], constructed by two
28 homogeneous and rigid bars, of equal length ℓ and equal mass m , connected one to the other via
29 a hinge. The masses of the bars are considered to be concentrated at their midpoints. One of the
30 two bars is hinged on a fixed frame, and the other is subjected to a force directed in parallel to
31 it, known as a follower force. Moreover, the hinges are regarded as viscoelastic, with angular
32 stiffness $k > 0$, and damping $c > 0$. These coefficients lump the viscoelastic properties of the
33 structure into the joints connecting the double pendulum to the fixed frame and the two bars to

34 each other [6, 7]. An important feature of Ziegler's double pendulum is that it can exhibit flutter
35 instability due to the non-conservative nature of the follower force [1, 6, 8, 9, 10].

36 Cazzolli et al. [3] introduced a modification of Ziegler's double pendulum that reproduces
37 the same bifurcation and instability landscape as the original structure, but *under a conservative*
38 *load*. In particular, one end of the first bar is pinned to a moving cart (Figure 1); at the tip
39 of the second bar, a nonholonomic constraint enforces orthogonality between the tip velocity
40 and the bar. This system is designed in a way that it can allow a translational motion of the
41 cart, when the latter is loaded with a dead force, possible through an oscillatory motion of the
42 arms of the pendulum, a dynamics resembling those of a certain family of microrobots or micro-
43 structures used in biomechanics [5, 11, 12, 13, 14]. One example of these microrobots is provided
44 by a '*magnetic flexible filament attached to a red blood cell*', as presented by Dreyfus et al.
45 [15], where the dead load plays the role of the propulsion thrust in aqueous environment at
46 low Reynolds numbers. This device prompted the present investigation of analogous systems in
47 microrobotics through a generalization of the loads considered in [3].

48 Microrobots for biomedical or biomechanical applications are often susceptible to electro-
49 magnetic interactions, because they feature magnetized [13, 14, 15] or electrically polarized
50 parts [16]. Therefore, also Lorentz-type forces contribute to determine the overall dynamics of
51 such devices in addition to self-propulsion [12] or other stimuli [5].

52 Despite its fundamental role in physics, certain foundational aspects of electromagnetism
53 have remained the subject of debate until recently. This applies, for instance, to the true role and
54 '*physical meaning*' [17] within classical electromagnetism of the vector potential \mathbf{A} , by which
55 the magnetic induction field \mathbf{B} is expressed as $\mathbf{B} = \text{curl}\mathbf{A}$. Indeed, whereas the electric and
56 the magnetic fields both have 'tangible' meaning, the vector potential works 'behind the scenes'
57 of the classical theory of electromagnetism. This issue has been thoroughly investigated in a
58 work by Rousseaux et al. [17], who conclude that there exist effects which cannot be understood
59 '*without the intervention of*' \mathbf{A} [17]. As an example for pursuing their goals, Rousseaux et
60 al. [17] took the paradigmatic problem of the ideal solenoid to unveil the Maxwell–Lodge effect
61 [18]. This phenomenon predicts that a current is generated in a wire wound around the solenoid in
62 response to the electric field induced outside it by the variation in time of the vector potential (not
63 of the magnetic induction field, which is zero outside the ideal solenoid). Although the purpose
64 of this work is not to address the intricacies of the debate surrounding the vector potential, we
65 embrace the viewpoint of Rousseaux et al. [17] and apply it to our mechanical device.

66 In light of the above considerations, the device investigated here can be regarded as a model
67 of an electrically charged microrobot, with dimensions comparable to those of eukaryotic cells.
68 It consists of a continuous viscoelastic rod attached to a small mass and is subjected to the
69 Lorentz force generated by a solenoid [5, 11, 12, 13, 14]. The rod is idealized as a viscoelastic
70 double pendulum, following the approach of Cazzolli et al. [3]. Compared to that approach, two
71 substantial differences are introduced: (i) the dead load is interpreted as a self-propulsion force;
72 (ii) the device carries a concentrated electric charge located in the structural element imposing
73 the nonholonomic constraint (similar problems were formulated by Maruskin et al. [19] and
74 Pastore et al. [20]).

75 Within this setting, the main objectives of our work are twofold:

- 76 1. We aim to assess the extent to which the Lorentz force modifies the results reported by
77 Cazzolli et al. [3] concerning stability, the onset of a Hopf bifurcation, and the overall
78 dynamical behavior of the device.
- 79 2. We revisit the Maxwell–Lodge effect [17] by placing it in the mechanical context of the

80 device under study. More specifically, we show that our theoretical framework predicts
 81 a Lorentz force on the charged structural element, even when it is located outside the
 82 solenoid. This result is achieved by placing the device first inside and then outside the
 83 solenoid. This may provide a potential—albeit presently theoretical—experimental setup to
 84 detect the Maxwell–Lodge effect, based on microrobotics and serving as an alternative to
 85 the experiment by Rousseaux et al. [17].

86 *Outline of the work.* The features of the device introduced by Cazzolli et al. [3] are outlined in
 87 Section 2, while in Section 3 the main electromagnetic aspects of our work are addressed (Sec-
 88 tion 3.1), including the way in which they affect the Lagrangian formalism of Ziegler’s double-
 89 pendulum (Section 3.2). Section 3.3 presents the governing dynamic equations, incorporating
 90 the Lorentz force, while Section 3.4 addresses the corresponding static equilibrium. Based on
 91 the determination of the static equilibrium, a stability analysis is carried out for neighboring con-
 92 figurations (Section 4). Specifically, after linearizing the dynamic equations around equilibrium
 93 and studying the nature of the roots of the associated characteristic polynomial (Section 4.1),
 94 the presence of a Hopf bifurcation is shown (Section 4.2). Finally, Section 5 presents numerical
 95 simulations of two virtual experimental tests in the context of microrobotics.

96 2. Basic problem description, Lagrangian function and the ‘skate’ constraint

97 The version of Ziegler’s double pendulum introduced by Cazzolli et al. [3] and used in the
 98 present investigation consists of two homogeneous and rigid bars, denoted by \mathcal{B}_1 and \mathcal{B}_2 , each
 99 of length ℓ and mass m , connected by a hinge that links the right end of \mathcal{B}_1 to left end of \mathcal{B}_2 .
 100 The left end of the bar \mathcal{B}_1 is hinged on a cart, which is free to move along a straight line on a
 101 fixed plane. A wheel is mounted at the right end of bar \mathcal{B}_2 , the bar acting as its axle, and rolls
 102 without slipping on a fixed plane. The non-slipping wheel is the realization of a nonholonomic
 103 constraint of the ‘skate’ type. Overall, the device may experience only planar motions, parallel
 104 to the fixed plane, Figure 1.

105 In a fixed Cartesian reference frame with origin O and axes aligned with the orthonormal
 106 vectors e_1 , e_2 , and e_3 , the motion of the device is confined to the plane spanned by e_1 and
 107 e_2 . Angles θ_1 and θ_2 measure the time-dependent inclinations of the bars \mathcal{B}_1 and \mathcal{B}_2 with the
 108 axis parallel to e_1 , respectively. Moreover, X_c denotes the instantaneous position, measured
 109 from O along e_1 , of the point of the cart to which \mathcal{B}_1 is hinged, Figure 1. The Lagrangian
 110 parameters θ_1 , θ_2 , and X_c , together with their velocities and accelerations, are collected in the
 111 arrays $q := (\theta_1, \theta_2, X_c)$, $\dot{q} := (\dot{\theta}_1, \dot{\theta}_2, \dot{X}_c)$, and $\ddot{q} := (\ddot{\theta}_1, \ddot{\theta}_2, \ddot{X}_c)$. Moreover, we introduce the
 112 following points of interest, whose spatial positions are expressed as functions of the Lagrangian
 113 coordinates:

$$x_A = \hat{x}_A(q) = (X_c, 0, 0), \quad (1a)$$

$$x_B = \hat{x}_B(q) = (X_c + \frac{1}{2}\ell \cos \theta_1, \frac{1}{2}\ell \sin \theta_1, 0), \quad (1b)$$

$$x_M = \hat{x}_M(q) = (X_c + \ell \cos \theta_1, \ell \sin \theta_1, 0), \quad (1c)$$

$$x_C = \hat{x}_C(q) = (X_c + \ell \cos \theta_1 + \frac{1}{2}\ell \cos \theta_2, \ell \sin \theta_1 + \frac{1}{2}\ell \sin \theta_2, 0), \quad (1d)$$

$$x_D = \hat{x}_D(q) = (X_c + \ell \cos \theta_1 + \ell \cos \theta_2, \ell \sin \theta_1 + \ell \sin \theta_2, 0), \quad (1e)$$

114 which represent, in order: the position of the cart (point A); the midpoint of \mathcal{B}_1 (point B); the
 115 hinge connecting \mathcal{B}_1 to \mathcal{B}_2 (point M); the midpoint of \mathcal{B}_2 (point C); and the position of the

116 massless wheel (point D). Accordingly, the instantaneous orientations of the bars \mathcal{B}_1 and \mathcal{B}_2 are
 117 given by the unit vectors

$$\mathbf{n}_1 \equiv \hat{\mathbf{n}}_1(q) = (\cos \theta_1)\mathbf{e}_1 + (\sin \theta_1)\mathbf{e}_2, \quad \mathbf{n}_2 \equiv \hat{\mathbf{n}}_2(q) = (\cos \theta_2)\mathbf{e}_1 + (\sin \theta_2)\mathbf{e}_2, \quad (2)$$

118 while the velocities of the points of interest are

$$\mathbf{v}_A = \hat{\mathbf{v}}_A(q, \dot{q}) = \dot{X}_c \mathbf{e}_1, \quad (3a)$$

$$\mathbf{v}_B = \hat{\mathbf{v}}_B(q, \dot{q}) = [\dot{X}_c - \frac{1}{2}\ell \dot{\theta}_1 \sin \theta_1] \mathbf{e}_1 + [\frac{1}{2}\ell \dot{\theta}_1 \cos \theta_1] \mathbf{e}_2, \quad (3b)$$

$$\mathbf{v}_M = \hat{\mathbf{v}}_M(q, \dot{q}) = [\dot{X}_c - \ell \dot{\theta}_1 \sin \theta_1] \mathbf{e}_1 + [\ell \dot{\theta}_1 \cos \theta_1] \mathbf{e}_2, \quad (3c)$$

$$\mathbf{v}_C = \hat{\mathbf{v}}_C(q, \dot{q}) = [\dot{X}_c - \ell \dot{\theta}_1 \sin \theta_1 - \frac{1}{2}\ell \dot{\theta}_2 \sin \theta_2] \mathbf{e}_1 + [\ell \dot{\theta}_1 \cos \theta_1 + \frac{1}{2}\ell \dot{\theta}_2 \cos \theta_2] \mathbf{e}_2, \quad (3d)$$

$$\mathbf{v}_D = \hat{\mathbf{v}}_D(q, \dot{q}) = [\dot{X}_c - \ell \dot{\theta}_1 \sin \theta_1 - \ell \dot{\theta}_2 \sin \theta_2] \mathbf{e}_1 + [\ell \dot{\theta}_1 \cos \theta_1 + \ell \dot{\theta}_2 \cos \theta_2] \mathbf{e}_2. \quad (3e)$$

119 The presence of the wheel and the postulate on its motion impose a *nonholonomic constraint*:
 120 the velocity \mathbf{v}_D of the end-point of \mathcal{B}_2 connected to the wheel is compelled to be at all times or-
 121 thogonal to \mathcal{B}_2 (see [3] for details). In the considered reference frame, this constraint is described
 122 by the relation $\mathbf{n}_2 \cdot \mathbf{v}_D = 0$, which, in terms of the Lagrangian parameters and up to the sign, reads

$$C \equiv \hat{C}(q, \dot{q}) = \dot{\theta}_1 \ell \sin(\theta_1 - \theta_2) - \dot{X}_c \cos \theta_2 = 0, \quad (4)$$

123 and is referred to as “*skate’ constraint*” [3].

124 To simplify the analysis of the dynamics of the considered device, we prescribe that the mass
 125 of each bar is concentrated at its midpoint, that the cart can be modeled as a material point of
 126 mass m_c , and that the wheel’s mass is negligible (*‘massless wheel’* [3]). Moreover, we assume
 127 that the cart and \mathcal{B}_1 as well as \mathcal{B}_1 and \mathcal{B}_2 are connected to each other by means of viscoelastic
 128 spring-like elements of angular stiffness $k > 0$ and damping coefficient $c > 0$. Finally, a constant
 129 force $\mathbf{F} = F \mathbf{e}_1$ of magnitude $F > 0$ (*‘dead load’* [3]) is applied to the cart, while a *‘follower*
 130 *force’* [3], generated by the nonholonomic constraint, acts on the skate parallel to the bar \mathcal{B}_2 , as
 131 shown in Figure 1.

132 Under the assumptions above, the system admits the Lagrangian function [3]

$$\begin{aligned} \mathcal{L}_o \equiv \hat{\mathcal{L}}_o(q, \dot{q}) = & \frac{1}{2} \left[\frac{5}{4} m \ell^2 \dot{\theta}_1^2 + \frac{1}{4} m \ell^2 \dot{\theta}_2^2 + (2m + m_c) \dot{X}_c^2 \right] \\ & + \frac{1}{2} m \ell^2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1) - \frac{3}{2} m \ell \dot{\theta}_1 \dot{X}_c \sin \theta_1 - \frac{1}{2} m \ell \dot{\theta}_2 \dot{X}_c \sin \theta_2 \\ & - \frac{1}{2} k \theta_1^2 - \frac{1}{2} k (\theta_2 - \theta_1)^2 + F X_c, \end{aligned} \quad (5)$$

133 which accounts for all interactions to which the system is subjected, with the exception of the
 134 damping and of the nonholonomic constraint (4). Following [3], damping is described through
 135 the Rayleigh dissipation function

$$\mathcal{R} = \hat{\mathcal{R}}(\dot{q}) = \frac{1}{2} c \dot{\theta}_1^2 + \frac{1}{2} c (\dot{\theta}_2 - \dot{\theta}_1)^2, \quad (6)$$

136 which yields the generalized dissipative forces

$$-\frac{\partial \hat{\mathcal{R}}}{\partial \dot{\theta}_1}(\dot{q}) = -c(2\dot{\theta}_1 - \dot{\theta}_2), \quad -\frac{\partial \hat{\mathcal{R}}}{\partial \dot{\theta}_2}(\dot{q}) = -c(\dot{\theta}_2 - \dot{\theta}_1), \quad -\frac{\partial \hat{\mathcal{R}}}{\partial \dot{X}_c}(\dot{q}) = 0. \quad (7)$$

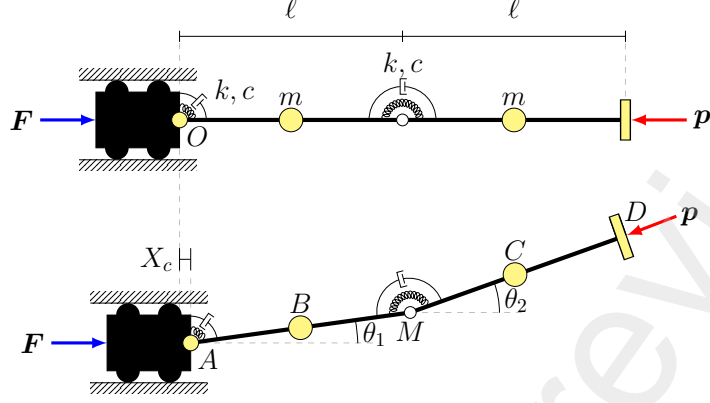


Figure 1: Sketch of the device introduced by Cazzoli et al. [3]. Upper part: the straight configuration ($\theta_1 = \theta_2 = 0$) with the cart located on the origin O , so that $X_c = 0$. Lower part: a deformed configuration ($\theta_1 \neq 0, \theta_2 \neq 0$), with the cart positioned at a point differing from the origin ($X_c \neq 0$). The hinges are characterized by an elastic stiffness $k > 0$ and a viscous damping coefficient $c > 0$. The reaction force \mathbf{p} associated with the nonholonomic constraint (the ‘wheel’ or the ‘skate’) is directed along the axis of the bar \mathcal{B}_2 .

137 The Lagrangian function \mathcal{L}_o defined by Equation (5) describes, up to the dissipative terms,
 138 the ‘original’ device (whence the subscript ‘o’) as it was conceived by Cazzoli et al. [3]. Thus,
 139 it features the external potential FX_c . Moreover, it does not depend explicitly on time.

140 To determine the equations of motion for the considered system, we apply the *Extended*
 141 *Hamilton Method*, following Lanczos [21],

$$\delta \int_{t_{\text{in}}}^{t_{\text{fin}}} \hat{\mathcal{L}}_o(q, \dot{q}) dt = \int_{t_{\text{in}}}^{t_{\text{fin}}} \left(\sum_{\alpha=1}^3 \frac{\partial \hat{\mathcal{R}}}{\partial \dot{q}_\alpha}(\dot{q}) \delta q_\alpha - \sum_{\alpha=1}^3 \mu \frac{\partial \hat{\mathcal{C}}}{\partial \dot{q}_\alpha}(q, \dot{q}) \delta q_\alpha \right) dt, \quad (8)$$

142 where the integral on the left-hand side is referred to as *action functional*, t_{in} and t_{fin} denote an
 143 initial and a final instant of time, and the function of time μ on the right-hand side is a Lagrange
 144 multiplier introduced to account for the nonholonomic constraint (4), see [21, 22, 23, 24] for
 145 details.

146 The variation of the action functional, performed as predicted by Hamilton’s Principle, and
 147 under the hypothesis $\delta q_\alpha(t_{\text{in}}) = \delta q_\alpha(t_{\text{fin}}) = 0$ for $\alpha = 1, 2, 3$, leads to

$$\delta \int_{t_{\text{in}}}^{t_{\text{fin}}} \hat{\mathcal{L}}_o(q, \dot{q}) dt = \int_{t_{\text{in}}}^{t_{\text{fin}}} \sum_{\alpha=1}^3 (\mathcal{E}_\alpha \hat{\mathcal{L}}_o) \delta q_\alpha dt, \quad (9)$$

148 with $\mathcal{E}_\alpha \hat{\mathcal{L}}_o$ being the Euler–Lagrange operator applied to $\hat{\mathcal{L}}_o$,

$$\mathcal{E}_\alpha \hat{\mathcal{L}}_o := \frac{\partial \hat{\mathcal{L}}_o}{\partial q_\alpha}(q, \dot{q}) - \frac{d}{dt} \left(\frac{\partial \hat{\mathcal{L}}_o}{\partial \dot{q}_\alpha}(q, \dot{q}) \right), \quad \alpha = 1, 2, 3. \quad (10)$$

149 Then, the local form of Equation (8) reads

$$\mathcal{E}_\alpha \hat{\mathcal{L}}_o - \frac{\partial \hat{\mathcal{R}}}{\partial \dot{q}_\alpha}(\dot{q}) + \mu \frac{\partial \hat{\mathcal{C}}}{\partial \dot{q}_\alpha}(q, \dot{q}) = 0, \quad \alpha = 1, 2, 3. \quad (11)$$

150 The third term on the left-hand side of Equation (11) identifies, for $\alpha = 1, 2, 3$, the Lagrangian
 151 components of the reaction force due to the constraint, which are given by

$$\mu \frac{\partial \hat{\mathcal{C}}}{\partial \theta_1}(q, \dot{q}) = \mu \ell \sin(\theta_1 - \theta_2) \equiv \{\ell \hat{\mathbf{n}}_1(q) \times \hat{\mathbf{p}}(q)\} \cdot \mathbf{e}_3, \quad (12a)$$

$$\mu \frac{\partial \hat{\mathcal{C}}}{\partial \theta_2}(q, \dot{q}) \equiv \{\ell \hat{\mathbf{n}}_2(q) \times \hat{\mathbf{p}}(q)\} \cdot \mathbf{e}_3 = 0, \quad (12b)$$

$$\mu \frac{\partial \hat{\mathcal{C}}}{\partial \dot{X}_c}(q, \dot{q}) = -\mu \cos \theta_2 \equiv \hat{\mathbf{p}}(q) \cdot \mathbf{e}_1, \quad (12c)$$

152 where ‘ \times ’ denotes the cross product and $\mathbf{p} = \hat{\mathbf{p}}(q) = -\mu \mathbf{n}_2$ is referred to as a *follower force* [3].
 153 We notice that, in a generic configuration of the device, Equation (12a) is the (scalar) torque that
 154 the follower force exerts on \mathcal{B}_1 around the instantaneous position x_A of the point A. Similarly,
 155 Equation (12b) is the *null* (scalar) torque that \mathbf{p} exerts on \mathcal{B}_2 , since \mathbf{p} is parallel to \mathcal{B}_2 by design.
 156 Moreover, Equation (12c) provides the \mathbf{e}_1 -component of \mathbf{p} acting on the cart.

157 3. Charged Ziegler’s double-pendulum interacting with Lorentz force

158 With reference to the mechanical system described in the previous section and, in particular,
 159 to the Lagrangian function reported in Equation (5), we introduce here two *major modifications*:

- 160 (1) The massless wheel, concentrated at the point D (see Figure 1), carries *electric charge*
 161 $e \neq 0$ (it will be assumed $e > 0$ in the following);
- 162 (2) A solenoid \mathcal{S} as described in detail in Section 3.1 is considered and two separate cases are
 163 analyzed: the device is placed in the interior of the solenoid (*Case I*); the device is placed
 164 in the exterior of the solenoid (*Case II*).

165 We refer the reader to Figure 2 for a graphical representation of the solenoid and of the device.

166 3.1. The paradigmatic solenoid problem and the Maxwell–Lodge effect

167 The solenoid \mathcal{S} has virtually infinite height, axis parallel to \mathbf{e}_3 , and circular cross section \mathcal{C}
 168 of radius R_s much larger than 2ℓ . An electric current flowing in the wire constituting the solenoid
 169 generates a magnetic field \mathbf{H} , which, given the magnetic permeability of the vacuum μ_0 , yields
 170 the magnetic induction field $\mathbf{B} = \mu_0 \mathbf{H}$.

171 Assuming appropriate regularity, the magnetic induction field \mathbf{B} and the electric field \mathbf{E} must
 172 fulfill the equations

$$\operatorname{div} \mathbf{B} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{E} = -\partial_t \mathbf{B}, \quad (13)$$

173 which constitute two of the four Maxwell equations written in local form. These conditions make
 174 it possible to express \mathbf{B} and \mathbf{E} in terms of a vector potential \mathbf{A} and of a scalar potential Φ —both
 175 at least of class C^2 —through the relations $\mathbf{B} = \operatorname{curl} \mathbf{A}$ and $\mathbf{E} = -\operatorname{grad} \Phi - \partial_t \mathbf{A}$ [25]. Because
 176 of the differential structure of these relations, the potentials \mathbf{A} and Φ are not unique. Indeed, for
 177 any scalar field χ at least of class C^2 , \mathbf{B} and \mathbf{E} remain invariant under transformations of the type

$$\mathbf{A} \mapsto \tilde{\mathbf{A}} = \mathbf{A} + \operatorname{grad} \chi, \quad \Phi \mapsto \tilde{\Phi} = \Phi - \partial_t \chi, \quad (14)$$

178 known as *gauge transformations* [25]. The field χ is the *generator* of the gauge transformations
 179 and allows for obtaining new forms of the vector potential and of the scalar potential, therefore
 180 also referred to as *gauge potentials*.

181 Although the ideal solenoid considered in this section constitutes a paradigmatic and widely
 182 studied physical problem, we review it to contextualize it within the mechanical framework of
 183 the device presented in Section 2.

184 The solenoid problem is grounded on the fundamental hypothesis that the magnetic induction
 185 field \mathbf{B} is non-null and homogeneous in the interior of \mathcal{S} and identically null outside it, whence

$$\mathbf{B}(x, t) = B_h(t) \mathbf{e}_3, \quad \text{for } x \in \text{Int}(\mathcal{S}) \quad \text{and} \quad \mathbf{B}(x, t) = \mathbf{0}, \quad \text{for } x \in \mathbb{R}^3 \setminus \mathcal{S}, \quad (15)$$

186 where x is a spatial point of Cartesian coordinates (x_1, x_2, x_3) , enumerated consistently with \mathbf{e}_1 ,
 187 \mathbf{e}_2 , and \mathbf{e}_3 , while $B_h(t)$ is a prescribed function of time t [25].

188 We notice that the vanishing of \mathbf{B} in the exterior of an ideal solenoid is exact '*in the stationary*
 189 *regime*' [17], although it can be maintained also under the hypothesis that the current flowing in
 190 the solenoid '*varies slowly in time*' [17]. The latter condition is fundamental in our setting and
 191 will be kept throughout the remainder of our work.

192 Within the so-called *temporal gauge* [26], which prescribes a vanishing scalar potential, the
 193 distribution of \mathbf{B} given in Equation (15) and the relation $\mathbf{B} = \text{curl} \mathbf{A}$ imply [25]

$$\mathbf{A}(x, t) = -\frac{1}{2} B_h(t) x_2 \mathbf{e}_1 + \frac{1}{2} B_h(t) x_1 \mathbf{e}_2, \quad \Phi(x, t) = 0, \quad \text{in } \text{Int}(\mathcal{S}), \quad (16a)$$

$$\mathbf{A}(x, t) = -\frac{1}{2} B_h(t) R_s^2 \frac{x_2}{x_1^2 + x_2^2} \mathbf{e}_1 + \frac{1}{2} B_h(t) R_s^2 \frac{x_1}{x_1^2 + x_2^2} \mathbf{e}_2, \quad \Phi(x, t) = 0, \quad \text{in } \mathbb{R}^3 \setminus \mathcal{S}, \quad (16b)$$

194 with $\text{Int}(\mathcal{S})$ being the set of internal points of \mathcal{S} . Therefore, \mathbf{A} is continuous on $\partial\mathcal{S}$. Moreover,
 195 it is divergence free both in the interior and in the exterior of the solenoid, since the equality
 196 $\text{div} \mathbf{A} = 0$ is identically satisfied both in $\text{Int}(\mathcal{S})$ and in $\mathbb{R}^3 \setminus \mathcal{S}$, and it is irrotational outside the
 197 solenoid, since $\text{curl} \mathbf{A} = \mathbf{0}$ in $\mathbb{R}^3 \setminus \mathcal{S}$.

198 It is worth recalling that the Euclidean norm of $\mathbf{A}(x, t)$ reads

$$\|\mathbf{A}(x, t)\| = \frac{1}{2} |B_h(t)| \sqrt{x_1^2 + x_2^2} = \frac{1}{2} |B_h(t)| \|\mathbf{r}(x)\|, \quad \text{in } \text{Int}(\mathcal{S}), \quad (17a)$$

$$\|\mathbf{A}(x, t)\| = \frac{1}{2} |B_h(t)| R_s^2 \frac{1}{\sqrt{x_1^2 + x_2^2}} = \frac{1}{2} |B_h(t)| \frac{R_s^2}{\|\mathbf{r}(x)\|}, \quad \text{in } \mathbb{R}^3 \setminus \mathcal{S}, \quad (17b)$$

199 with $\mathbf{r}(x) = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$, thereby increasing linearly with $\varrho = \|\mathbf{r}(x)\|$ until $\varrho = R_s$, and then
 200 decreasing asymptotically towards zero as ϱ^{-1} for $\varrho > R_s$ [17, 25].

201 According to Equations (15), (16a), and (16b), for any open surface Σ with piecewise regular
 202 contour $\partial\Sigma$ lying on a plane that intersects transversely the solenoid, and either containing the
 203 cross section \mathcal{C} of the solenoid or being equal to \mathcal{C} , the magnetic flux is

$$\Phi_M(\Sigma; t) = \int_{\Sigma} \mathbf{B}(x, t) \cdot \mathbf{e}_3 \, da(x) = \int_{\partial\Sigma} \mathbf{A}(x, t) \cdot \boldsymbol{\tau}(x) \, ds(x) = B_h(t) \pi R_s^2, \quad (18)$$

204 where $\boldsymbol{\tau}(x)$ is the unit vector tangent to $\partial\Sigma$ at x . When Σ contains \mathcal{C} , the magnetic induction
 205 field \mathbf{B} is null in $\Sigma \setminus \mathcal{C}$ and the first integral in Equation (18) reduces to a surface integral over
 206 \mathcal{C} , whence $\Phi_M(\Sigma; t) = B_h(t) \pi R_s^2$, with $\pi R_s^2 = \text{Area}(\mathcal{C})$. Since $\Phi_M(\Sigma; t)$ must be independent of

207 $\Sigma \supseteq \mathcal{C}$, the magnetic flux can be obtained by directly computing the contour integral in Equation
 208 (18), with \mathbf{A} given by Equation (16b) [25].

209 By employing Equations (16a) and (16b) for the calculation of the electric field within the
 210 temporal gauge ($\Phi = 0$), which implies $\mathbf{E} = -\text{grad}\Phi - \partial_t \mathbf{A} = -\partial_t \mathbf{A}$, we obtain

$$\mathbf{E}(x, t) = \frac{1}{2} \dot{B}_h(t) x_2 \mathbf{e}_1 - \frac{1}{2} \dot{B}_h(t) x_1 \mathbf{e}_2, \quad \text{in Int}(\mathcal{S}), \quad (19a)$$

$$\mathbf{E}(x, t) = \frac{1}{2} \dot{B}_h(t) R_s^2 \frac{x_2}{x_1^2 + x_2^2} \mathbf{e}_1 - \frac{1}{2} \dot{B}_h(t) R_s^2 \frac{x_1}{x_1^2 + x_2^2} \mathbf{e}_2, \quad \text{in } \mathbb{R}^3 \setminus \mathcal{S}. \quad (19b)$$

211 A consequence of Equations (19a) and (19b) is the Lorentz force exerted on the charged skate
 212 placed at the end-point D of the bar \mathcal{B}_2 of the device under study. In this respect, as mentioned
 213 at the beginning of this section, we consider the following two difference cases (Figure 2):

214 *Case I: Device inside the solenoid.* If the cross section of the solenoid is big enough to host the
 215 device, and if we assume that the motion of the device remains confined to this cross section, so
 216 that the skate does not transit through the lateral boundary of the solenoid, then the Lorentz force
 217 acting on the skate in $\text{Int}(\mathcal{S})$ reads

$$\begin{aligned} \mathbf{F}_L(x_D(t), t) &= e [\mathbf{E}(x_D(t), t) + \mathbf{v}_D(t) \times \mathbf{B}(x_D(t), t)] \\ &= \left(\frac{e \dot{B}_h(t)}{2} x_{D2}(t) + e B_h(t) \dot{x}_{D2}(t) \right) \mathbf{e}_1 - \left(\frac{e \dot{B}_h(t)}{2} x_{D1}(t) + e B_h(t) \dot{x}_{D1}(t) \right) \mathbf{e}_2, \end{aligned} \quad (20)$$

218 meaning that it consists both of an electric and of a magnetic contribution. Note that the fields
 219 \mathbf{E} and \mathbf{B} are defined at all spatial points, and, consequently, they couple dynamically with the
 220 system through their evaluation at time t and at the spatial point $x = x_D(t)$ occupied by the
 221 material point D at the same time.

222 For this reason, the Lorentz force is evaluated at $x = x_D(t)$, with velocity $\mathbf{v}_D(t)$ at time t . The
 223 same considerations apply to the vector potential \mathbf{A} .

224 *Case II: Device outside the solenoid.* The nontrivial electric field present also in the exterior of
 225 the solenoid, as given by Equation (19b), generates a Lorentz force acting on the charged skate,
 226 even in the case in which the skate is located outside the solenoid. In this setting, the Lorentz
 227 force in $\mathbb{R}^3 \setminus \mathcal{S}$ reads

$$\begin{aligned} \mathbf{F}_L(x_D(t), t) &= e \mathbf{E}(x_D(t), t) \\ &= \frac{e \dot{B}_h(t) R_s^2}{2} \frac{x_{D2}(t)}{[x_{D1}(t)]^2 + [x_{D2}(t)]^2} \mathbf{e}_1 - \frac{e \dot{B}_h(t) R_s^2}{2} \frac{x_{D1}(t)}{[x_{D1}(t)]^2 + [x_{D2}(t)]^2} \mathbf{e}_2. \end{aligned} \quad (21)$$

228 **Remark 1** (The Maxwell–Lodge effect revisited).

229 Equations (16b) and (19b) evidence that the vector potential \mathbf{A} and the electric field $\mathbf{E} = -\partial_t \mathbf{A}$
 230 are non-null also outside the solenoid, where the magnetic induction field \mathbf{B} is null. This result
 231 has many important physical consequences, one of which is known as Maxwell–Lodge effect
 232 [18]: by positioning a coil of given electric conductivity around the solenoid, but not in contact
 233 with it, an electric current flowing in this coil is observed in response to the electromotive force
 234 generated by the negative of the time derivative of the magnetic flux $B_h(t) \pi R_s^2$, due to \mathbf{B} inside
 235 the solenoid [17]. While this result is exactly the Faraday–Neumann–Lenz law, with the elec-
 236 tromotive force being $\mathcal{E}(t) := \int_{\partial \Sigma} \mathbf{E}(x, t) \cdot \boldsymbol{\tau}(x) \, ds(x) = -\dot{B}_h(t) \pi R_s^2$, and $\partial \Sigma$ representing the coil

237 —assumed filiform—, the fact that the current in the coil occurs in the region of space in which
 238 the magnetic flux is null (but $\mathbf{E} = -\partial_t \mathbf{A}$ is non-null) constitutes the essence of the Maxwell–
 239 Lodge effect. This has generated considerable debate aiming at clarifying the role and intrinsic
 240 meaning of the vector potential \mathbf{A} in classical physics. Although reviewing the ‘paradoxes and
 241 controversies around the Maxwell–Lodge effect’ [17] is out of the scopes of our work (the in-
 242 terested reader is referred to [17]), we argue here that the Maxwell–Lodge effect manifests itself
 243 also in the dynamics of the charged Ziegler double pendulum in Case II, when the pendulum is
 244 placed outside the solenoid. Indeed, because of the electric charge e concentrated at the point
 245 D of the device, this point is subject to the Lorentz force $\mathbf{F}_L(x_D(t), t)$ as prescribed by Equation
 246 (21) when the device is in $\mathbb{R}^3 \setminus \mathcal{S}$. In turn, $\mathbf{F}_L(x_D(t), t)$ influences the dynamics of the device
 247 and interacts with the dead load F applied to the cart, at least until the device exceeds a certain
 248 distance from the solenoid’s boundary, beyond which the amplitude of $\mathbf{F}_L(x_D(t), t)$, decreasing
 249 radially as $\{[x_{D1}(t)]^2 + [x_{D2}(t)]^2\}^{-1/2}$, becomes negligible compared to F .

250 On the basis of well documented results on the gauge transformations presented in Equation
 251 (14) [25], one can choose the generators

$$\chi(x, t) = \frac{1}{2} B_h(t) x_1 x_2, \quad \text{in Int}(\mathcal{S}), \quad (22a)$$

$$\chi(x, t) = \frac{1}{2} B_h(t) R_s^2 \arctan(x_1/x_2), \quad \text{in } \mathbb{R}^3 \setminus \mathcal{S} \text{ and for } x_2 \neq 0, \quad (22b)$$

252 thereby moving out from the temporal gauge and switching to the transformed gauge potentials

$$\tilde{\mathbf{A}}(x, t) = B_h(t) x_1 \mathbf{e}_2, \quad \tilde{\Phi}(x, t) = -\frac{1}{2} \dot{B}_h(t) x_1 x_2, \quad \text{in Int}(\mathcal{S}), \quad (23a)$$

$$\tilde{\mathbf{A}}(x, t) = \mathbf{0}, \quad \tilde{\Phi}(x, t) = -\frac{1}{2} \dot{B}_h(t) R_s^2 \arctan(x_1/x_2), \quad \text{in } \mathbb{R}^3 \setminus \mathcal{S} \text{ and for } x_2 \neq 0 \quad (23b)$$

253 (note that χ has null Laplacian in the region in which it is defined). The new gauge potentials,
 254 however, yield exactly the same Lorentz forces as those reported in Equations (20) and (21),
 255 with the only (irrelevant) technical difference being that \mathbf{B} and \mathbf{E} are now to be computed as
 256 $\mathbf{B} = \text{curl} \tilde{\mathbf{A}}$ and $\mathbf{E} = -\text{grad} \tilde{\Phi} - \partial_t \tilde{\mathbf{A}}$. In particular, outside the solenoid, $\mathbf{E} = -\text{grad} \tilde{\Phi}$ holds
 257 at all points $x \in \mathbb{R}^3 \setminus \mathcal{S}$ such that $x_2 \neq 0$. Therefore, for any open surface $\Sigma \supset \mathcal{C}$ having
 258 piecewise regular contour $\partial \Sigma$, \mathbf{E} is undefined at both points in which $\partial \Sigma$ intersects the x_1 -axis.
 259 However, since these two points constitute a set of null measure, the electromotive force $\mathcal{E}(t) =$
 260 $-\int_{\partial \Sigma} \text{grad} \tilde{\Phi}(x, t) \cdot \boldsymbol{\tau}(x) ds(x)$ remains unchanged and delivers $-\dot{B}_h(t) \pi R_s^2$, as is the case of the
 261 temporal gauge.

262 3.2. The Lagrangian function of the charged Ziegler double pendulum

263 The Lagrangian function of the device subjected to the additional magnetic interaction is
 264 obtained from the Lagrangian function \mathcal{L}_o in Equation (5) by adding the generalized, velocity-
 265 dependent potential $e\mathbf{A}(x_D(t), t) \cdot \mathbf{v}_D(t)$, in which \mathbf{A} is evaluated at time t and at the spatial point
 266 $x_D(t)$ that is instantaneously occupied by the moving charge at time t [20, 25]. In terms of the
 267 system’s Lagrangian parameters and generalized velocities, this yields (see also [19] for the case
 268 of the ‘knife edge with a dipole moment in a magnetic field’)

$$\mathcal{L} \equiv \hat{\mathcal{L}}(q, \dot{q}, t) = \hat{\mathcal{L}}_o(q, \dot{q}) + e[\mathcal{A}_1(q, t)\dot{\theta}_1 + \mathcal{A}_2(q, t)\dot{\theta}_2 + \mathcal{A}_3(q, t)\dot{X}_c], \quad (24)$$

269 where, for $\beta = 1, 2, 3$, the functions $\mathcal{A}_\beta(q, t) = \frac{1}{2}B_h(t)\mathcal{U}_\beta(q)\ell^2$ express explicitly how the device
 270 interacts with the vector potential and are defined by the auxiliary quantities

$$\mathcal{U}_1(q) := \begin{cases} 1 + \cos(\theta_2 - \theta_1) + (X_c/\ell) \cos \theta_1 & \text{Case I} \\ \frac{R_s^2}{\ell^2} \frac{1 + \cos(\theta_2 - \theta_1) + (X_c/\ell) \cos \theta_1}{[\cos \theta_1 + \cos \theta_2 + (X_c/\ell)]^2 + [\sin \theta_1 + \sin \theta_2]^2} & \text{Case II} \end{cases} \quad (25a)$$

$$\mathcal{U}_2(q) := \begin{cases} 1 + \cos(\theta_2 - \theta_1) + (X_c/\ell) \cos \theta_2 & \text{Case I} \\ \frac{R_s^2}{\ell^2} \frac{1 + \cos(\theta_2 - \theta_1) + (X_c/\ell) \cos \theta_2}{[\cos \theta_1 + \cos \theta_2 + (X_c/\ell)]^2 + [\sin \theta_1 + \sin \theta_2]^2} & \text{Case II} \end{cases} \quad (25b)$$

$$\mathcal{U}_3(q) := \begin{cases} -\frac{\sin \theta_1 + \sin \theta_2}{\ell} & \text{Case I} \\ -\frac{R_s^2}{\ell^3} \frac{\sin \theta_1 + \sin \theta_2}{[\cos \theta_1 + \cos \theta_2 + (X_c/\ell)]^2 + [\sin \theta_1 + \sin \theta_2]^2} & \text{Case II.} \end{cases} \quad (25c)$$

271 More in detail, it can be shown that, both in Case I and in Case II, the following identifications
 272 apply:

$$\mathcal{A}_1(q(t), t) = [\ell \mathbf{n}_1(t) \times \mathbf{A}(x_D(t), t)] \cdot \mathbf{e}_3, \quad (26a)$$

$$\mathcal{A}_2(q(t), t) = [\ell \mathbf{n}_2(t) \times \mathbf{A}(x_D(t), t)] \cdot \mathbf{e}_3, \quad (26b)$$

$$\mathcal{A}_3(q(t), t) = \mathbf{A}(x_D(t), t) \cdot \mathbf{e}_1. \quad (26c)$$

273 We remark that the vector potential \mathbf{A} , although having only two nonzero components, generates
 274 the three nonzero components \mathcal{A}_1 , \mathcal{A}_2 , and \mathcal{A}_3 , with the first two being conjugate to the angular
 275 velocities of the bars $\dot{\theta}_1$ of $\dot{\theta}_2$, and the last one being conjugate to the linear velocity \dot{X}_c of the cart
 276 along the \mathbf{e}_1 -axis. In addition, functions \mathcal{A}_1 , \mathcal{A}_2 , and \mathcal{A}_3 , explained in Equations (26a)–(26c),
 277 describe how the vector potential couples at each instant of time t with the configuration $q(t)$
 278 assumed by the system as a whole at time t .

279 We emphasize that the charge of the skate is assumed to be sufficiently weak so that the
 280 magnetic and the electric fields that it generates can be neglected.

281 A major difference with respect to the work by Cazzoli et al. [3] is that, in addition to the
 282 dead load applied to the cart, the system is also subjected to the *Lorentz force* $\mathbf{F}_L(x_D(t), t)$, which
 283 assumes the expression given in Equation (20) or in Equation (21), depending on whether the
 284 device is inside or outside the solenoid, respectively.

285 In Case I, and when \mathbf{A} does not depend explicitly on time, the Lorentz force reduces to
 286 $\mathbf{F}_L(x_D(t), t) = e \mathbf{v}_D(t) \times \mathbf{B}(x_D(t))$ and, because of the constraint (4), it remains parallel to the
 287 bar \mathcal{B}_2 , and thus with the follower force $\mathbf{p} = -\mu \mathbf{n}_2$, at all times. In addition, the Lorentz force
 288 vanishes whenever $\mathbf{v}_D(t)$ is null.

289 In Case II, the Lorentz force features solely the purely electric contribution, which is due
 290 to the explicit time dependence of \mathbf{A} , that is, $\mathbf{F}_L(x_D(t), t) = e \mathbf{E}(x_D(t), t) = -e \partial_t \mathbf{A}(x_D(t), t)$.
 291 Therefore, if \mathbf{A} is assumed to not be explicitly dependent on time, then the Lorentz force vanishes
 292 identically.

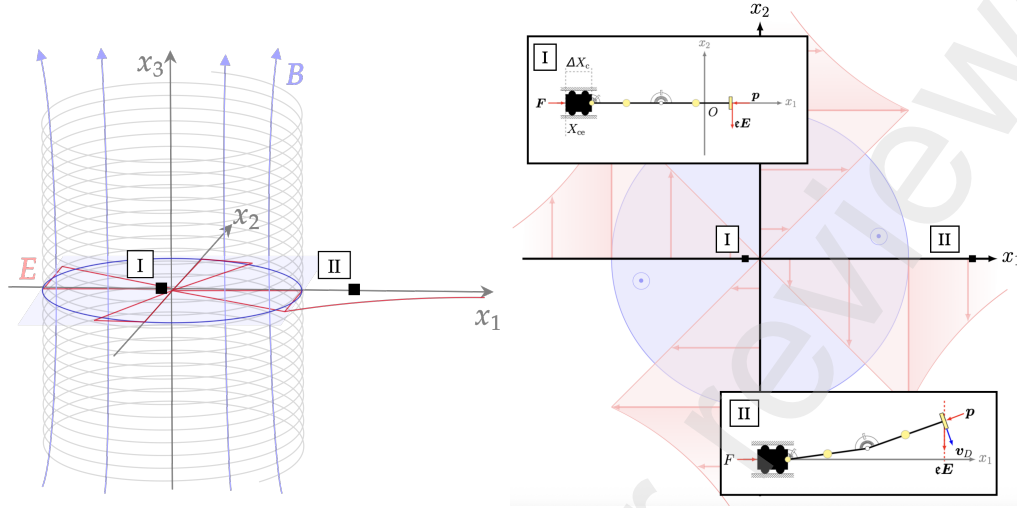


Figure 2: Sketch of the solenoid, illustrating the electric field \mathbf{E} (red) and the magnetic induction field \mathbf{B} (blue). The system is shown both in a three-dimensional perspective and in a vertical cross-section, with the device positions for Case I and Case II indicated in both views. In the position referring to Case I, $\Delta X_c := X_{c0} - X_{ce}$, with X_{c0} being the initial position of the cart. The images correspond to assuming $B_h(t) > 0$ and $\dot{B}_h(t) > 0$. In the zoomed-in image referring to Case II, $e\mathbf{E}$ is directed tangentially to the circumference centered in O and passing through D , but it can be regarded as approximately parallel to the x_2 -axis because the device is placed at a relatively long distance from O .

293 3.3. Dynamic equations

294 If the dynamic equations (11), featuring the Lagrangian function $\hat{\mathcal{L}}_o$, are written explicitly,
 295 they lead to the set of equations presented by Cazzoli et al. [3]. In our framework, however, since
 296 electromagnetic interactions are considered, the dynamic equations follow from the Lagrangian
 297 function $\hat{\mathcal{L}}$ expressed in Equation (24). They read

$$\mathcal{E}_\alpha \hat{\mathcal{L}} - \frac{\partial \hat{\mathcal{R}}}{\partial \dot{q}_\alpha}(\dot{q}) + \mu \frac{\partial \hat{\mathcal{C}}}{\partial \dot{q}_\alpha}(q, \dot{q}) = 0, \quad \alpha = 1, 2, 3, \quad (27)$$

298 where, as in Equation (11), the contributions $-\partial_{\dot{q}_\alpha} \hat{\mathcal{R}}(\dot{q})$ and $\mu \partial_{\dot{q}_\alpha} \hat{\mathcal{C}}(q, \dot{q})$ identify the dissipative
 299 generalized forces associated with damping and the reaction forces due to the nonholonomic
 300 “skate’ constraint’ [3], respectively.

301 Before deriving the new Euler–Lagrange equations, we notice that the Euler–Lagrange oper-
 302 ator applied to $\hat{\mathcal{L}}$ produces

$$\mathcal{E}_\alpha \hat{\mathcal{L}} = \mathcal{E}_\alpha \hat{\mathcal{L}}_o + e \sum_{\beta=1}^3 \mathcal{F}_{\alpha\beta} \dot{q}_\beta + e \mathcal{E}_\alpha, \quad \alpha = 1, 2, 3, \quad (28)$$

303 where the quantities $\mathcal{F}_{\alpha\beta}$ and \mathcal{E}_α are defined as

$$\mathcal{F}_{\alpha\beta} := \frac{\partial \mathcal{A}_\beta}{\partial q_\alpha} - \frac{\partial \mathcal{A}_\alpha}{\partial q_\beta}, \quad \mathcal{E}_\alpha := -\frac{\partial \mathcal{A}_\alpha}{\partial t} \quad \alpha, \beta = 1, 2, 3, \quad (29)$$

304 and represent, in terms of the Lagrangian parameters selected, the components of the *generalized*
 305 *Faraday tensor* and of the *generalized electric field*, respectively. From the discussion in Section
 306 3.1, in which the electric and the magnetic contributions of the Lorentz force have been analyzed

307 in Cartesian coordinates, it follows that the magnetic contribution is active only in the interior
 308 of the solenoid. Accordingly, the equations of motion either include or exclude the magnetic
 309 contribution of the Lorentz force, depending on whether the device is located inside or outside
 310 the solenoid (Case I and Case II, respectively). Since this result must be consistently described
 311 when the device is analyzed in terms of the components of the generalized Faraday tensor and of
 312 the generalized electric field, defined by Equation (29), the quantities $\mathcal{F}_{\alpha\beta}$ must vanish identically
 313 in Case II, when the device moves outside the solenoid.

314 Specifically, the generalized Faraday tensor has components

$$\begin{aligned} \mathcal{F}_{12} &= -\ell^2 B_h(t) \sin(\theta_1 - \theta_2), & \mathcal{F}_{13} &= -\ell B_h(t) \cos \theta_1, & \mathcal{F}_{23} &= -\ell B_h(t) \cos \theta_2, & \text{Case I} & (30a) \\ \mathcal{F}_{12} &= 0, & \mathcal{F}_{13} &= 0, & \mathcal{F}_{23} &= 0, & \text{Case II} & (30b) \end{aligned}$$

315 with $\mathcal{F}_{\alpha\beta} = -\mathcal{F}_{\beta\alpha}$ for $\alpha, \beta = 1, 2, 3$, while the generalized electric field has components

$$\mathcal{E}_\alpha \equiv -\partial_t \mathcal{A}_\alpha = -\frac{1}{2} \dot{B}_h(t) \mathcal{U}_\alpha(q) \ell^2, \quad \alpha = 1, 2, 3, \quad (31)$$

316 where functions $\mathcal{U}_\alpha(q)$ are given in Equations (25a)–(25c), depending on whether Case I or Case
 317 II is considered.

318 Finally, upon introducing the components of the generalized Lorentz force

$$\mathfrak{F}_{L\alpha} := e \sum_{\beta=1}^3 \mathcal{F}_{\alpha\beta} \dot{q}_\beta + e \mathcal{E}_\alpha, \quad \alpha = 1, 2, 3, \quad (32)$$

319 the dynamic equations (27) read

$$\mathcal{E}_\alpha \hat{\mathcal{L}}_\alpha - \frac{\partial \hat{\mathcal{R}}}{\partial \dot{q}_\alpha}(\dot{q}) + \mu \frac{\partial \hat{\mathcal{C}}}{\partial \dot{q}_\alpha}(q, \dot{q}) + \mathfrak{F}_{L\alpha} = 0, \quad \alpha = 1, 2, 3, \quad (33)$$

320 and, in explicit form, they become

$$\begin{aligned} \frac{5}{4} m \ell^2 \ddot{\theta}_1 + \frac{1}{2} m \ell^2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 - \frac{3}{2} m \ell (\sin \theta_1) \ddot{X}_c + \frac{1}{2} m \ell^2 \sin(\theta_1 - \theta_2) \dot{\theta}_2^2 \\ = -k(2\theta_1 - \theta_2) - c(2\dot{\theta}_1 - \dot{\theta}_2) + \mu \ell \sin(\theta_1 - \theta_2) + \mathfrak{F}_{L1}, \end{aligned} \quad (34a)$$

$$\begin{aligned} \frac{1}{2} m \ell^2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 + \frac{1}{4} m \ell^2 \ddot{\theta}_2 - \frac{1}{2} m \ell (\sin \theta_2) \ddot{X}_c - \frac{1}{2} m \ell^2 \sin(\theta_1 - \theta_2) \dot{\theta}_1^2 \\ = -k(\theta_2 - \theta_1) - c(\dot{\theta}_2 - \dot{\theta}_1) + \mathfrak{F}_{L2}, \end{aligned} \quad (34b)$$

$$\begin{aligned} -\frac{3}{2} m \ell (\sin \theta_1) \ddot{\theta}_1 - \frac{1}{2} m \ell (\sin \theta_2) \ddot{\theta}_2 + M \ddot{X}_c - \frac{3}{2} m \ell (\cos \theta_1) \dot{\theta}_1^2 - \frac{1}{2} m \ell (\cos \theta_2) \dot{\theta}_2^2 \\ = -\mu \cos \theta_2 + F + \mathfrak{F}_{L3}. \end{aligned} \quad (34c)$$

321 where, together with the total mass $M := 2m + m_c$, the explicit expressions for the generalized
 322 components of the Lorentz force are introduced:

$$\mathfrak{F}_{L1} = \begin{cases} -\frac{1}{2} e \dot{B}_h(t) \mathcal{U}_1(q) \ell^2 - e B_h(t) [\sin(\theta_1 - \theta_2) \dot{\theta}_2 + (\cos \theta_1) (\dot{X}_c / \ell)] \ell^2 & \text{Case I} \\ -\frac{1}{2} e \dot{B}_h(t) \mathcal{U}_1(q) \ell^2 & \text{Case II,} \end{cases} \quad (35a)$$

$$\mathfrak{F}_{L2} = \begin{cases} -\frac{1}{2} e \dot{B}_h(t) \mathcal{U}_2(q) \ell^2 + e B_h(t) [\sin(\theta_1 - \theta_2) \dot{\theta}_1 - (\cos \theta_2) (\dot{X}_c / \ell)] \ell^2 & \text{Case I} \\ -\frac{1}{2} e \dot{B}_h(t) \mathcal{U}_2(q) \ell^2 & \text{Case II,} \end{cases} \quad (35b)$$

$$\mathfrak{F}_{L3} = \begin{cases} -\frac{1}{2} e \dot{B}_h(t) \mathcal{U}_3(q) \ell^2 + e B_h(t) [(\cos \theta_1) \dot{\theta}_1 + (\cos \theta_2) \dot{\theta}_2] \ell & \text{Case I} \\ -\frac{1}{2} e \dot{B}_h(t) \mathcal{U}_3(q) \ell^2 & \text{Case II.} \end{cases} \quad (35c)$$

323 We emphasize that the magnetic part of the generalized force component in Equation (35b) is
 324 zero also in Case I since the terms in square brackets return the constraint (4). This is because
 325 the magnetic part of the Lorentz force $e \mathbf{v}_D \times \mathbf{B}$ is parallel to the bar of unit vector \mathbf{n}_2 and also
 326 with the follower force, as explained at the end of Section 2.

327 Equations (34a)–(34c) are to be solved together with the constraint (4), which we differentiate
 328 with respect to time and turn into the second-order differential equation

$$-\ell \sin(\theta_1 - \theta_2)\ddot{\theta}_1 + (\cos \theta_2)\ddot{X}_c = \dot{\theta}_2 \dot{X}_c \sin \theta_2 + \ell \dot{\theta}_1 \cos(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2). \quad (36)$$

329 Equation (36), obtained by changing sign to the time derivative of Equation (4), is necessary to
 330 apply Schur's complement technique to the system (34a)–(34c) and (36) (see [20, 27] for details;
 331 the sign change is motivated by convenience in the construction of the Schur matrix).

332 Alongside the elastic and viscous forces featuring in the right-hand sides of Equations (34a)–
 333 (34c), the additional generalized Lorentz forces \mathfrak{F}_{L1} , \mathfrak{F}_{L2} and \mathfrak{F}_{L3} are present, and, coherently
 334 with Equations (12a)–(12c), the following identifications apply

$$\mathfrak{F}_{L1} = [\ell \mathbf{n}_1(t) \times \mathbf{F}_L(x_D(t), t)] \cdot \mathbf{e}_3, \quad (37a)$$

$$\mathfrak{F}_{L2} = [\ell \mathbf{n}_2(t) \times \mathbf{F}_L(x_D(t), t)] \cdot \mathbf{e}_3, \quad (37b)$$

$$\mathfrak{F}_{L3} = \mathbf{F}_L(x_D(q), t) \cdot \mathbf{e}_1. \quad (37c)$$

335 Equations (37a) and (37b) are the torques generated by the Lorentz force acting on the bars \mathcal{B}_1
 336 and \mathcal{B}_2 , respectively, while Equation (37c) is the component of the Lorentz force acting on the
 337 cart.

338 3.4. Determination of static equilibrium

339 To study the static equilibrium of the system, we consider an initial instant of time t_0 at which
 340 the system occupies the configuration $q(t_0) = q_0$, with null velocity $\dot{q}(t_0)$, and, for $t > t_0$, we look
 341 for solutions to Equations (34a)–(34c) of the type $q(t) = q_e := (\theta_{1e}, \theta_{2e}, X_{ce}) \equiv q_0$, satisfying
 342 $\dot{q}(t) = (0, 0, 0)$ and

$$-\mu_e \ell \sin(\theta_{1e} - \theta_{2e}) = -\frac{1}{2} e \dot{B}_h(t) \mathcal{U}_1(q_e) \ell^2 - k(2\theta_{1e} - \theta_{2e}), \quad (38a)$$

$$0 = -\frac{1}{2} e \dot{B}_h(t) \mathcal{U}_2(q_e) \ell^2 - k(\theta_{2e} - \theta_{1e}), \quad (38b)$$

$$\mu_e \cos \theta_{2e} = -\frac{1}{2} e \dot{B}_h(t) \mathcal{U}_3(q_e) \ell^2 + F, \quad (38c)$$

343 where μ_e is the (for the moment unknown) Lagrange multiplier at equilibrium [24, p. 25].

344 Note that the nonholonomic constraint (4) is trivially satisfied when the generalized velocities
 345 are all set equal to zero. For this reason, the number of equations characterizing equilibrium
 346 reduces to the three force balances (38a)–(38c), which, however, still feature the four unknowns
 347 θ_{1e} , θ_{2e} , X_{ce} , and μ_e . We also notice that the torques due to the magnetic part of the Lorentz force
 348 are not present in Equations (38a)–(38c) because the velocities are set equal to zero.

349 Assuming $\theta_{1e} = \theta_{1e} = \theta_e$, Equations (38a)–(38c) become

$$0 = -\frac{1}{2} e \dot{B}_h(t) \mathcal{U}_1(q_e) \ell^2 - k\theta_e, \quad (39a)$$

$$0 = -\frac{1}{2} e \dot{B}_h(t) \mathcal{U}_2(q_e) \ell^2, \quad (39b)$$

$$\mu_e \cos \theta_e = -\frac{1}{2} \epsilon \dot{B}_h(t) \mathcal{U}_3(q_e) \ell^2 + F, \quad (39c)$$

with $\mathcal{U}_1(q_e) = \mathcal{U}_2(q_e)$. Thus, equilibrium in the general case in which $\dot{B}_h(t)$ is different from zero requires from Equation (39b) $\mathcal{U}_2(q_e) = 0$, which may hold true only in Case I. Hence, we find

$$\mathcal{U}_2(q_e) = 0 \quad \Rightarrow \quad 2 + \frac{X_{ce}}{\ell} \cos \theta_e = 0 \quad \Rightarrow \quad X_{ce} = -\frac{2\ell}{\cos \theta_e}. \quad (40)$$

Moreover, since $\mathcal{U}_1(q_e)$ vanishes too, Equation (39a) yields the more stringent condition

$$\theta_e = 0. \quad (41)$$

This implies that $\mathcal{U}_3(q_e)$ also vanishes identically, thereby yielding the force balance

$$\mu_e = F. \quad (42)$$

By gathering the results (40)–(42), we find

$$q_e = (\theta_{1e}, \theta_{2e}, X_{ce}) = (0, 0, -2\ell), \quad \mu_e = F. \quad (43)$$

The condition $\mathcal{U}_1(q_e) = \mathcal{U}_2(q_e) = 0$ for nonzero $\dot{B}_h(t)$ amounts to placing the device in a configuration in which the electric part ϵE of the Lorentz force acting on the massless wheel is null (see Figure 3a). Indeed, given the expression $\epsilon E(\hat{x}_D(q_e), t) = -\frac{1}{2} \epsilon \dot{B}_h(t) (X_{ce} + 2\ell) e_2$, the requirement of the vanishing of this force means that the skate has to occupy the origin of the considered reference frame, where $\hat{x}_D(q_e) = 0$ and, thus, $X_{ce} = -2\ell$.

We note that no equilibrium configuration exists in Case II, since, when the device is outside the solenoid, Equation (39b) cannot be satisfied unless the trivial condition of time independent B_h is considered. On the other hand, if $\dot{B}_h(t) = 0$ for all $t \geq t_0$, every triple $(\theta_1(t), \theta_2(t), X_c(t)) = (0, 0, X_{ce})$, with arbitrary $X_{ce} > R_s$, satisfies the equilibrium conditions.

See Figure 3 for a graphical depiction of the device in the four configurations of interest inside the solenoid (Case I).

Remark 2 (One static equilibrium configuration extracted from the manifold in [3]).

The configuration q_e determined in Equation (43) is one particular configuration of the manifold $\{(0, 0, X_{ce}) : X_{ce} \in \mathbb{R}\}$ of equilibrium configurations obtained in [3] in the absence of magnetic interactions. This constitutes an important difference from the work by Cazzolli et al. [3]. In our setting, indeed, X_c cannot be arbitrary, since the equilibrium configuration is obtained by imposing the additional condition of null electric part of the Lorentz force acting on the skate. However, we retrieve the same equilibrium manifold as Cazzolli et al. [3] if we assume that B_h is independent of time, which implies $\dot{B}_h(t) = 0$ at all times. Indeed, under the hypothesis of time-independent B_h , Equation (39a) returns $\theta_e = 0$, Equation (39b) is trivially satisfied (thus, there is no counterpart of Equation (40) to determine X_{ce} , which remains unprescribed), and Equation (39c) reduces to $\mu_e = F$.

4. Stability analysis: flutter instability and bifurcations in Case I

In this section, we carry out a stability analysis of the system described by Equations (34a)–(34c) and (36). To this end, since we need to perturb the system from a static equilibrium configuration, we restrict our study to Case I only. For the methodologies used hereafter, we refer to [3, 6, 23, 28].

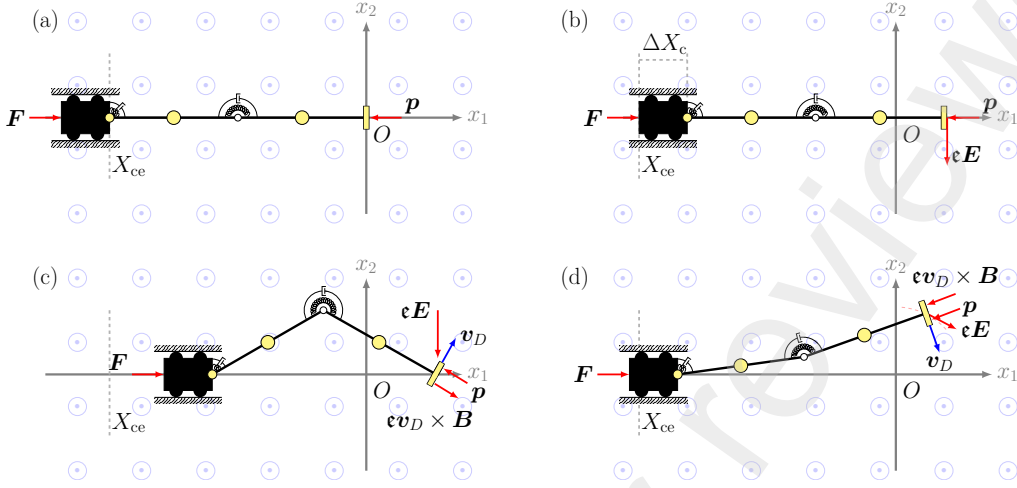


Figure 3: Body diagram showing the forces acting on the device in the case in which the radius $R_s > 2\ell$ of the solenoid is sufficiently large to fully host the device and its motion (Case I). Four scenarios are depicted: (a) the device is in the static equilibrium configuration determined in Equation (43); (b) although the device is in a configuration with initial null velocities, such configuration is not of equilibrium because the Lorentz force, composed only by its electric contribution, produces a non-zero torque, thereby triggering the motion; (c and d) in each case the device is in a deformed configuration with nonzero velocities and, thus, the magnetic part of the Lorentz force acts on the device in addition to the electric part. In particular, we emphasize that, while $\epsilon v_D \times B$ is parallel to n_2 by construction, ϵE has to be tangential to the circumference instantaneously centered in the origin and passing through $x_D(t)$ (indeed, $E(x_D(t), t)$ is proportional to $e_3 \times r(x_D(t))$, with $r(x_D(t)) = x_{D1}(t)e_1 + x_{D2}(t)e_2$).

383 Before proceeding with the stability analysis, we notice that Equations (34a)–(34c) feature
 384 together six physical parameters to which we add the magnitude F of the dead load applied to
 385 the cart and the e_3 -component $B_h(t)$ of the magnetic induction field:

$$\mathcal{P} := \{m_c; m, c, k, \ell; \epsilon; F, B_M\}, \quad (44)$$

386 with $B_M := \max_t |B_h(t)|$.

387 Dimensionless quantities are introduced via the mappings

$$X_c \mapsto \frac{X_c}{\ell}, \quad t \mapsto \frac{t}{\sqrt{m\ell^2/k}}, \quad \mu \mapsto \frac{\mu\ell}{k}, \quad F \mapsto \frac{F\ell}{k}, \quad B_M \mapsto \frac{\epsilon B_M \ell}{\sqrt{mk}}, \quad (45a)$$

$$M \mapsto \frac{M}{m}, \quad c \mapsto \frac{c}{\ell \sqrt{mk}}. \quad (45b)$$

388 For simplicity, we do not introduce additional notation to indicate the dimensionless quantities
 389 in Equations (45a) and (45b). Rather, from here on, we simply rename them as X_c , t , μ , F , B_M ,
 390 M , and c , and we differentiate with respect to the dimensionless time variable. We emphasize
 391 that dimensionless variables are used throughout Section 4, and physical dimensions are restored
 392 in Section 5 to contextualize the numerical results.

393 4.1. Linearization of the equations governing the dynamics of the structure

394 To perform the stability analysis in a neighborhood of the pair $(q_e; \mu_e) = (0, 0, -2; F)$, we
 395 consider small *perturbations* around q_e and μ_e . This is achieved by introducing the homotopies

396 \tilde{q} and $\tilde{\mu}$, which define the perturbed configuration $\tilde{q}(t, \varepsilon)$ and the perturbed Lagrange multiplier
 397 $\tilde{\mu}(t, \varepsilon)$, with ε being a smallness parameter, and by requiring them to comply with the conditions
 398 $\tilde{q}(t, 0) = q_e$ and $\tilde{\mu}(t, 0) = \mu_e$. We assume \tilde{q} and $\tilde{\mu}$ to be at least C^2 functions of their arguments.

399 The first-order expansions of \tilde{q} and $\tilde{\mu}$ around $\varepsilon = 0$ and for all t can be written as:

$$\tilde{q}(t, \varepsilon) = q_e + \eta(t)\varepsilon + o(\varepsilon), \quad \varepsilon \rightarrow 0, \quad (46a)$$

$$\tilde{\mu}(t, \varepsilon) = \mu_e + \varrho(t)\varepsilon + o(\varepsilon), \quad \varepsilon \rightarrow 0, \quad (46b)$$

400 with $\eta(t) := \partial_\varepsilon \tilde{q}(t, 0)$ and $\varrho(t) := \partial_\varepsilon \tilde{\mu}(t, 0)$. In the following, we set $\eta := (\vartheta_1, \vartheta_2, x_c)$.

401 By substituting Equations (46a) and (46b) into the dimensionless form of Equations (34a)–
 402 (34c) and (36), and dropping all terms of order higher than the first in ε , we obtain the dynamic
 403 equations linearized around q_e and μ_e :

$$\frac{5}{4}\ddot{\vartheta}_1 + \frac{1}{2}\ddot{\vartheta}_2 + c(2\dot{\vartheta}_1 - \dot{\vartheta}_2) + (2\vartheta_1 - \vartheta_2) - F(\vartheta_1 - \vartheta_2) + B_h(t)\dot{x}_c + \frac{1}{2}\dot{B}_h(t)x_c = 0, \quad (47a)$$

$$\frac{1}{2}\ddot{\vartheta}_1 + \frac{1}{4}\ddot{\vartheta}_2 + c(\dot{\vartheta}_2 - \dot{\vartheta}_1) + (\vartheta_2 - \vartheta_1) + B_h(t)\dot{x}_c + \frac{1}{2}\dot{B}_h(t)x_c = 0, \quad (47b)$$

$$M\ddot{x}_c + \varrho - B_h(t)(\dot{\vartheta}_1 + \dot{\vartheta}_2) - \frac{1}{2}\dot{B}_h(t)(\vartheta_1 + \vartheta_2) = 0, \quad (47c)$$

$$\ddot{x}_c = 0, \quad (47d)$$

404 where $B_h(t)$ and $\dot{B}_h(t)$ are dimensionless functions of dimensionless time. Accordingly, the di-
 405 mensionless characteristic amplitude B_M of the magnetic induction field, as shown in Equation
 406 (45a)₅, is understood as $\max_t |B_h(t)| = B_M$.

407 **Remark 3** (Lorentz force couples the translational and the rotational degrees of freedom).

408 *In the linearized dynamic equations (47a)–(47c), $B_h(t)$ and $\dot{B}_h(t)$ make it impossible to decouple*
 409 *x_c from ϑ_1 and ϑ_2 . This holds true even in the case in which the condition $\dot{x}_c = 0$ (representing*
 410 *the linearization of the constraint (4)) is substituted in (47a)–(47c), because the terms involving*
 411 *$\dot{B}_h(t)$ do not cancel, as they represent the electric part of the Lorentz force. This is another*
 412 *difference with the case studied by Cazzolli et al. [3] in which the absence of the Lorentz force*
 413 *allows for decoupling the dynamics of the cart from the dynamics of the double pendulum.*

414 Following Neimark and Fufaev [23], upon introducing the 3×1 array $\eta = \{\vartheta_1 \ \vartheta_2 \ x_c\}^T$ and
 415 the 1×1 array $\rho = \{\varrho\}$, Equations (47a)–(47d) can be recast in matrix formalism as

$$M_e \ddot{\eta} + [C_e + B_h L_e] \dot{\eta} + [K_e + F G_e + \frac{1}{2} \dot{B}_h L_e] \eta - A_e^T \rho = 0, \quad (48a)$$

$$-A_e \dot{\eta} = 0, \quad (48b)$$

416 where M_e , K_e , and C_e represent the mass, stiffness, and damping matrices, respectively, G_e is
 417 an auxiliary (singular) matrix introduced to account for the force F , A_e is the constraint matrix,
 418 while L_e is the matrix accounting for the contributions due to the Lorentz force. These matrices
 419 read

$$M_e := \begin{bmatrix} \frac{5}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & M \end{bmatrix}, \quad K_e := \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_e := \begin{bmatrix} 2c & -c & 0 \\ -c & c & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (49a)$$

$$G_e := \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_e := [0 \ 0 \ -1], \quad L_e := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}. \quad (49b)$$

420 **Remark 4** (Faraday tensor in matrix formalism and its coupling with C_e , K_e , and FG_e).

421 *In the linearized setting, the dead load F , although being applied to the cart, influences the*

422 *double pendulum through the introduction of the ‘apparent’ stiffness matrix FG_e , referred to as*

423 *‘geometric stiffness matrix’ in the literature [10] and adding to the elastic stiffness matrix K_e .*

424 *Similarly, B_h and \dot{B}_h yield the ‘apparent’ damping matrix $B_h L_e$ and the additional ‘apparent’*

425 *stiffness matrix $\frac{1}{2}\dot{B}_h L_e$, both due to the linearization of the Lorentz force. It is worth noticing that*

426 *$B_h L_e$ is the matrix associated with the generalized Faraday tensor, evaluated at the equilibrium*

427 *configuration q_e in dimensionless form and in the case in which the device is inside the solenoid*

428 *(cfr. Equation (30a)). Analogously, $\frac{1}{2}\dot{B}_h L_e$ is the matrix associated with the electric contribution*

429 *$\frac{1}{2}\dot{B}_h L_e \eta$ of the Lorentz force. We emphasize that the terms $B_h L_e$ and $\frac{1}{2}\dot{B}_h L_e$ constitute another*

430 *relevant difference from the work by Cazzolli et al. [3] and add to the damping matrix C_e and to*

431 *the stiffness matrix K_e , respectively.*

432 To solve Equation (48a) and (48b), we follow standard methods [23], and we enforce the

433 *ansatz*

$$\eta(t) = H e^{lt}, \quad \rho(t) = R e^{lt}, \quad \lambda \in \mathbb{C}, \quad (50)$$

434 where H and R are two column vectors of constant amplitudes, having sizes 3×1 and 1×1 ,

435 respectively, and λ is a complex number. Substituting Equation (50) into (48a) and (48b) yields

$$\left[\begin{array}{c|c} \lambda^2 M_e + \lambda[C_e + B_h L_e] + [K_e + FG_e + \frac{1}{2}\dot{B}_h L_e] & -A_e^T \\ \hline & 0 \end{array} \right] \begin{Bmatrix} H \\ R \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad (51)$$

436 where 0 denotes here the 3×1 null array.

437 To find nontrivial solutions to Equation (51), we impose that the determinant of the block-

438 wise matrix on the left-hand side of Equation (51) vanish. Before proceeding, we remark that the

439 term $\lambda^2 A_e$ in the block-wise matrix of Equation (51) would lead to a sixth-grade characteristic

440 polynomial in λ featuring a factor λ^2 . This corresponds to a vanishing root of algebraic multi-

441 plicity two, which, as explained by Neimark and Fufaev [23, p. 265–267], does not affect the

442 overall stability properties of the system. Hence, we introduce the alternative system

$$\left[\begin{array}{c|c} \lambda^2 M_e + \lambda[C_e + B_h L_e] + [K_e + FG_e + \frac{1}{2}\dot{B}_h L_e] & -A_e^T \\ \hline & 0 \end{array} \right] \begin{Bmatrix} H \\ R \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (52)$$

443 Computing the determinant of the block-wise matrix on the left-hand side of Equation (52) yields

444 a polynomial equation of the fourth-grade in λ , which can be written as

$$Q(\lambda) := a_0 \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0, \quad (53)$$

445 with the coefficients a_0, a_1, a_2, a_3, a_4 being all real and given by

$$a_0 = 1, \quad a_1 = 44c, \quad a_2 = 44 + 16c^2 - 12F, \quad a_3 = 32c, \quad a_4 = 16. \quad (54)$$

446 An important result unfolded by the coefficients a_0, \dots, a_4 is that, in spite of the Lorentz force

447 contributions to Equation (52), the characteristic polynomial (53) depends neither on $B_h(t)$ nor

448 on $\dot{B}_h(t)$. To see why this result holds true from the algebraic point of view, we report in Figure 4

449 a sketch of the structure of the block-wise matrix in Equation (52) and of the calculation of its

$$\begin{array}{l}
\left[\begin{array}{ccc|c}
Z & Z & L & 0 \\
Z & Z & L & 0 \\
L & L & Z & 1 \\
\hline
0 & 0 & 1 & 0
\end{array} \right] \quad \begin{array}{l}
Z: \text{Ziegler's contribution} \\
L: \text{Lorentz's contribution}
\end{array} \\
\hline
\left[\begin{array}{ccc|c}
Z & Z & L & 0 \\
Z & Z & L & 0 \\
L & L & Z & 1 \\
0 & 0 & 1 & 0
\end{array} \right] = -1 \left[\begin{array}{ccc|c}
Z & Z & 0 \\
Z & Z & 0 \\
L & L & 1
\end{array} \right] = -1 \left[\begin{array}{cc|c}
Z & Z & \\
Z & Z & \\
\hline
Z & Z & 1
\end{array} \right] = \dots
\end{array}$$

Figure 4: Sketch of the proof that the determinant of the block-wise matrix on the left-hand side of Equation (52) depends neither on $B_h(t)$ nor on $\dot{B}_h(t)$.

450 determinant. Thus, the linear stability analysis falls back to that of Ziegler's double pendulum
451 [3, 6, 10].

452 Although a_0 is equal to unity, a_1 and a_3 depend only on c , and a_2 depends on both c and
453 F , it is convenient to regard all coefficients as *formally* depending on c and F . In particular,
454 as the purpose of this study is to analyze the stability of the system under the dead load F , the
455 coefficients a_0, a_1, \dots, a_4 are regarded as functions of F , parameterized by c .

456 The polynomial equation (53) has in general four complex roots. Since it has real coefficients
457 and is of even grade, its four roots, hereafter denoted as $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, are paired in the sense that,
458 if λ_1 and λ_2 are complex numbers, then $\lambda_4 \equiv \bar{\lambda}_1$ and $\lambda_3 \equiv \bar{\lambda}_2$. We also notice that the roots
459 $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are functions of F depending parametrically on c .

460 4.1.1. The case of the undamped double pendulum: symmetric diagram of the roots

461 If c is set equal to zero, Equation (53) simplifies to a biquadratic equation in λ , since the
462 coefficients a_1 and a_3 disappear. Hence, by setting $\Delta(F) := [a_2(F)]^2 - 4a_4(F)$, the roots of
463 Equation (53) can be solved for λ^2 as

$$\lambda_1^2(F) = -\frac{a_2(F)}{2} - \frac{\sqrt{\Delta(F)}}{2}, \quad \lambda_2^2(F) = -\frac{a_2(F)}{2} + \frac{\sqrt{\Delta(F)}}{2}, \quad \text{if } \Delta(F) \geq 0, \quad (55a)$$

$$\lambda_1^2(F) = -\frac{a_2(F)}{2} - i\frac{\sqrt{|\Delta(F)|}}{2}, \quad \lambda_2^2(F) = -\frac{a_2(F)}{2} + i\frac{\sqrt{|\Delta(F)|}}{2}, \quad \text{if } \Delta(F) < 0. \quad (55b)$$

464 Note that the form of $\lambda_1^2(F)$ and $\lambda_2^2(F)$ in Equations (55a) and (55b) depends on the asymmetry
465 of the matrix $K_e + FG_e$ when $F \neq 0$. Indeed, when this matrix is symmetric, the roots λ are
466 purely imaginary [10].

467 For the following calculations, it is convenient to introduce the auxiliary quantities

$$H_1(F) := \frac{a_2(F)}{2} + \frac{\sqrt{\Delta(F)}}{2}, \quad H_2(F) := \frac{a_2(F)}{2} - \frac{\sqrt{\Delta(F)}}{2}, \quad \text{if } \Delta(F) \geq 0, \quad (56a)$$

$$K_1(F) := \frac{a_2(F)}{2} + i\frac{\sqrt{|\Delta(F)|}}{2}, \quad K_2(F) := \frac{a_2(F)}{2} - i\frac{\sqrt{|\Delta(F)|}}{2}, \quad \text{if } \Delta(F) < 0. \quad (56b)$$

468 For notational convenience, we set $K(F) := K_1(F) \equiv \bar{K}_2(F)$ for $\Delta(F) < 0$.

469 *Case #1:* $\Delta = 0$. We have $H_1(F) \equiv H_2(F) \equiv H(F) := a_2(F)/2$, and, in terms of the values
 470 assumed by a_2 and a_4 , the discriminant $\Delta := a_2^2 - 4a_4$ vanishes at the two *critical* values of F :

$$F_1 = 3, \quad F_2 = \frac{13}{3}. \quad (57)$$

471 For these values, the squared eigenvalues are

$$\lambda_1^2(F_1) = \lambda_2^2(F_1) = -H(F_1) = -\frac{a_2(F_1)}{2} = -4 = i^2 4, \quad (58)$$

$$\lambda_1^2(F_2) = \lambda_2^2(F_2) = -H(F_2) = -\frac{a_2(F_2)}{2} = +4, \quad (59)$$

472 leading to

$$\lambda_{1,1}(F_1) = \lambda_{2,1}(F_1) = -i2, \quad \lambda_{1,2}(F_1) = \lambda_{2,2}(F_1) = +i2, \quad (60a)$$

$$\lambda_{1,2}(F_2) = \lambda_{2,1}(F_2) = -2, \quad \lambda_{1,1}(F_2) = \lambda_{2,2}(F_2) = +2. \quad (60b)$$

473 *Case #2:* $\Delta > 0$. The discriminant Δ is positive for the values of F :

$$F \in [0, F_1[\cup]F_2, +\infty[. \quad (61)$$

474 Two situations occur:

475 2.1 For $F \in [0, F_1[$, the coefficient $a_2(F)$ is strictly positive. The following two different *real*
 476 and *negative* values of λ^2 exist

$$0 > \lambda_1^2(F) = i^2 H_1(F), \quad H_1(F) > 0, \quad (62a)$$

$$0 > \lambda_2^2(F) = i^2 H_2(F), \quad H_2(F) > 0. \quad (62b)$$

477 Since $\lambda_1^2(F)$ and $\lambda_2^2(F)$ are negative, each of them produces a pair of purely imaginary
 478 conjugate roots,

$$\lambda_{1,1}(F) = -i\sqrt{H_1(F)}, \quad \lambda_{1,2}(F) = +i\sqrt{H_1(F)}, \quad (63a)$$

$$\lambda_{2,1}(F) = -i\sqrt{H_2(F)}, \quad \lambda_{2,2}(F) = +i\sqrt{H_2(F)}. \quad (63b)$$

479 Accordingly, the system linearized about the static equilibrium is *marginally stable* [29].
 480 So, if perturbed from equilibrium, the system oscillates around its equilibrium configura-
 481 tion. We emphasize that the necessary and sufficient condition for stability, which requires
 482 the real part of all the roots to be strictly negative [29], is not met in this first case.

483 2.2 For $F \in]F_2, +\infty[$, the coefficient $a_2(F)$ is strictly negative and the following two *real* and
 484 *positive* values for λ^2 exist

$$0 < \lambda_1^2(F) = -H_1(F), \quad H_1(F) < 0, \quad (64a)$$

$$0 < \lambda_2^2(F) = -H_2(F), \quad H_2(F) < 0, \quad (64b)$$

485 each of which produces one pair of real roots,

$$\lambda_{1,1}(F) = +\sqrt{-H_1(F)}, \quad \lambda_{1,2}(F) = -\sqrt{-H_1(F)}, \quad (65a)$$

$$\lambda_{2,1}(F) = -\sqrt{-H_2(F)}, \quad \lambda_{2,2}(F) = +\sqrt{-H_2(F)}. \quad (65b)$$

486 Since $\lambda_{1,1}(F)$ and $\lambda_{2,2}(F)$ are real and strictly positive for $F \in]F_2, +\infty[$, the phenomenon
 487 of ‘blow up’, or divergence, is expected to occur in this range of values of F , where the
 488 solution is *not* oscillatory.

489 *Case #3:* $\Delta < 0$. The discriminant Δ is negative for the following values of F :

$$F \in]F_1, F_2[. \quad (66)$$

490 Two complex conjugate values for λ^2 exist,

$$\lambda_1^2(F) = -K_1(F) = -K(F), \quad (67a)$$

$$\lambda_2^2(F) = -K_2(F) = -\bar{K}(F), \quad (67b)$$

491 whose roots are given by

$$\lambda_{1,1}(F) = -i\sqrt{\rho(F)} \exp\left(+i\frac{\phi(F)}{2}\right), \quad \lambda_{1,2}(F) = +i\sqrt{\rho(F)} \exp\left(+i\frac{\phi(F)}{2}\right), \quad (68a)$$

$$\lambda_{2,1}(F) = -i\sqrt{\rho(F)} \exp\left(-i\frac{\phi(F)}{2}\right), \quad \lambda_{2,2}(F) = +i\sqrt{\rho(F)} \exp\left(-i\frac{\phi(F)}{2}\right). \quad (68b)$$

492 where

$$\rho(F) := \sqrt{\operatorname{Re}[K(F)]^2 + \operatorname{Im}[K(F)]^2}, \quad \phi(F) := \arctan\left(\frac{\operatorname{Im}[K(F)]}{\operatorname{Re}[K(F)]}\right). \quad (69)$$

493 Note that the following conjugacy relations hold: $\lambda_{1,1}(F) = \bar{\lambda}_{2,2}(F)$ and $\lambda_{1,2}(F) = \bar{\lambda}_{2,1}(F)$.

494 For future reference, we also provide for these roots the equivalent expressions

$$\lambda_{1,1}(F) = \sqrt{\rho(F)} \left(+\sin\frac{\phi(F)}{2} - i\cos\frac{\phi(F)}{2} \right), \quad \lambda_{1,2}(F) = \sqrt{\rho(F)} \left(-\sin\frac{\phi(F)}{2} + i\cos\frac{\phi(F)}{2} \right), \quad (70a)$$

$$\lambda_{2,1}(F) = \sqrt{\rho(F)} \left(-\sin\frac{\phi(F)}{2} - i\cos\frac{\phi(F)}{2} \right), \quad \lambda_{2,2}(F) = \sqrt{\rho(F)} \left(+\sin\frac{\phi(F)}{2} + i\cos\frac{\phi(F)}{2} \right). \quad (70b)$$

495 Referring to Figure 5a, since the real parts $\operatorname{Re}[\lambda_{1,2}(F)] = \operatorname{Re}[\lambda_{2,1}(F)] < 0$ are negative for
 496 all the considered values of $F \in]F_1, F_2[$, the roots $\lambda_{1,2}(F)$ and $\lambda_{2,1}(F)$ yield oscillating solutions
 497 with amplitude decreasing to zero as time increases. However, since the real parts $\operatorname{Re}[\lambda_{1,1}(F)] =$
 498 $\operatorname{Re}[\lambda_{2,2}(F)]$ are positive, the roots $\lambda_{1,1}(F)$ and $\lambda_{2,2}(F)$ yield oscillating solutions with amplitudes
 499 increasing in time, thereby being responsible for the globally unstable behavior of the system.
 500 This phenomenon is known as *flutter instability* [10].

501 With these cases in mind, the critical values F_1 and F_2 henceforth will be renamed $F_{\text{flu}}^{\text{nd}}$ and
 502 $F_{\text{div}}^{\text{nd}}$ to indicate that they specify, in the absence of damping (superscript ‘nd’), the values of F
 503 delimitating the region of flutter. In particular, for $F > F_{\text{flu}}^{\text{nd}}$ at least one root has positive real
 504 part, and, thus, the system passes from the marginally stable region to the first unstable region in
 505 which flutter occurs. The second critical value $F_{\text{div}}^{\text{nd}} > F_{\text{flu}}^{\text{nd}}$ marks the end of the flutter regime,
 506 and, for $F > F_{\text{div}}^{\text{nd}}$, the region of divergence instability, without oscillatory behavior. It is worth
 507 emphasizing that, at $F_{\text{flu}}^{\text{nd}}$ and $F_{\text{div}}^{\text{nd}}$, the real parts of the roots split into different branches, whereas
 508 the imaginary parts coalesce. This is illustrated in Figure 5a, where the points B_0 , B_1 , and B_2
 509 highlight the three branching points of the real parts when $c = 0$.

510 From now on, we adopt the following numbering of the roots, which ensures that both the real
 511 and imaginary parts of all roots are continuous functions of F , while preserving pairwise conju-
 512 gacy (i.e., $\lambda_1(F)$ and $\lambda_4(F)$ are complex conjugate, and $\lambda_2(F)$ and $\lambda_3(F)$ are complex conjugate
 513 for $F \in [0, F_{\text{div}}^{\text{nd}}]$):

$$\lambda_1(F) = \begin{cases} \lambda_{1,2}(F) & \text{if } F \in [0, F_{\text{flu}}^{\text{nd}}[, \\ \lambda_{1,2}(F) & \text{if } F \in [F_{\text{flu}}^{\text{nd}}, F_{\text{div}}^{\text{nd}}], \\ \lambda_{2,1}(F) & \text{if } F \in]F_{\text{div}}^{\text{nd}}, +\infty[. \end{cases} \quad \lambda_2(F) = \begin{cases} \lambda_{2,2}(F) & \text{if } F \in [0, F_{\text{flu}}^{\text{nd}}[, \\ \lambda_{2,2}(F) & \text{if } F \in [F_{\text{flu}}^{\text{nd}}, F_{\text{div}}^{\text{nd}}], \\ \lambda_{2,2}(F) & \text{if } F \in]F_{\text{div}}^{\text{nd}}, +\infty[. \end{cases} \quad (71a)$$

$$\lambda_3(F) = \begin{cases} \lambda_{2,1}(F) & \text{if } F \in [0, F_{\text{flu}}^{\text{nd}}[, \\ \lambda_{1,1}(F) & \text{if } F \in [F_{\text{flu}}^{\text{nd}}, F_{\text{div}}^{\text{nd}}], \\ \lambda_{1,1}(F) & \text{if } F \in]F_{\text{div}}^{\text{nd}}, +\infty[. \end{cases} \quad \lambda_4(F) = \begin{cases} \lambda_{1,1}(F) & \text{if } F \in [0, F_{\text{flu}}^{\text{nd}}[, \\ \lambda_{2,1}(F) & \text{if } F \in [F_{\text{flu}}^{\text{nd}}, F_{\text{div}}^{\text{nd}}], \\ \lambda_{1,2}(F) & \text{if } F \in]F_{\text{div}}^{\text{nd}}, +\infty[. \end{cases} \quad (71b)$$

514 *4.1.2. The case of the damped double pendulum: asymmetric diagram of the roots*

515 In this section the critical loads of flutter and divergence are obtained for $c > 0$. Two different
516 techniques are used, one to determine the critical load for flutter and another to derive the critical
517 load for divergence, which are now explicitly regarded as depending on c , as indicated by the
518 notation $F_{\text{flu}}(c)$ and $F_{\text{div}}(c)$. The first method is the Routh–Hurwitz stability criterion [6, 29] and
519 provides an explicit expression for $F_{\text{flu}}(c)$. The second method relies on the direct inspection
520 of the characteristic polynomial provided in Equation (53), whereby we compute numerically
521 $F_{\text{div}}(c)$ for given values of $c > 0$.

522 *Critical load for flutter.* The Routh–Hurwitz criterion is a direct test done on the coefficients
523 of a polynomial to determine if all of its roots lie on the left half of the complex plane, i.e., if
524 they all have negative real parts. This criterion is applied to the characteristic equation (53), to
525 determine necessary and sufficient conditions on F that guarantee that the roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ all
526 have negative real parts. This yields the threshold value, $F_{\text{flu}}(c)$, approached from the left, i.e.,
527 in the limit $F \rightarrow [F_{\text{flu}}(c)]^-$, with the force range $[0, F_{\text{flu}}(c)[$ corresponding to the region, in the
528 undamped case, in which all the roots have zero real part, see Figure 5a for $F \in [0, F_{\text{flu}}^{\text{nd}}[$.

529 The Hurwitz matrix associated with the characteristic polynomial in (53) is

$$H_4 = \begin{bmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{bmatrix}. \quad (72)$$

530 Considering now all the principle minors which are generated by H_4 , the condition to guarantee
531 stability of the system is that *all* of these principle minors must be strictly positive, leading to the
532 following stability conditions:

$$0 < a_1, \quad (73a)$$

$$0 < a_1 a_2 - a_0 a_3, \quad (73b)$$

$$0 < a_3(a_1 a_2 - a_0 a_3) - a_1^2 a_4, \quad (73c)$$

$$0 < (a_3(a_1 a_2 - a_0 a_3) - a_1^2 a_4) a_4. \quad (73d)$$

533 Since $a_1 = 44c$ (see Equation (54)₂), the inequality (73a) is identically satisfied for positive
534 damping coefficients. Moreover, the condition (73d) coincides with Equation (73c) multiplied
535 by a_4 , which is positive (see Equation (54)₄, in which $a_4 = 16$). Hence, the inequality (73d) is
536 satisfied automatically if the condition (73c) is fulfilled. Therefore, only inequalities (73b) and
537 (73c) have to be considered. Their solution leads to the threshold value for F , $F_{\text{flu}}(c)$, below
538 which all the roots have negative real part. Such condition reads [10]

$$F < F_{\text{flu}}(c), \quad F_{\text{flu}}(c) = \frac{39}{22} + \frac{4c^2}{3}. \quad (74)$$

539 Therefore, the system is stable for $F < F_{\text{flu}}(c)$, and it experiences flutter or divergence instability
540 otherwise. More in detail, $F_{\text{flu}}(c)$ defines, for varying damping coefficient $c > 0$, the onset of
541 *flutter* when damping is present.

542 *Critical load for divergence.* The critical load for divergence is obtained by directly examining
 543 the discriminant of the characteristic polynomial (53). The general formula for the discriminant
 544 of a polynomial $Q(\lambda)$ of grade n is given in [30] as

$$D[Q] = (-1)^{n(n-1)/2} a_0^{-1} \text{Res}[Q, Q'], \quad Q(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n, \quad (75)$$

545 where $Q'(\lambda)$ is the derivative of $Q(\lambda)$ with respect to λ , and $\text{Res}[Q, Q']$ is the ‘resultant’ of the
 546 polynomials $Q(\lambda)$ and $Q'(\lambda)$. Since the grade n is equal to 4, $\text{Res}[Q, Q']$ can be computed as the
 547 determinant of the Sylvester matrix $S_n \equiv S_4$ formed by the polynomials $Q(\lambda)$ and $Q'(\lambda)$ [31, 32],

$$\text{Res}[Q, Q'] := \det(S_4) = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 & 0 \\ 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 \\ 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 \\ 0 & 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 \end{vmatrix}. \quad (76)$$

548 Note that, in our case, $a_0 = 1$. We emphasize that, for $c \neq 0$, the general formula $D[Q]$
 549 must be used because, although $n = 4$, the polynomial $Q(\lambda)$ no longer reduces to a quadratic
 550 polynomial in λ^2 , as it does for $c = 0$. Thus, Equation (75)₁ becomes

$$\begin{aligned} D[Q] &= 256a_4^3 - 192a_1a_3a_4^2 - 128a_2^2a_4^2 + 144a_2a_3^2a_4 - 27a_3^4 + 144a_1^2a_2a_4^2 \\ &\quad - 6a_1^2a_3^2a_4 - 80a_1a_2^2a_3a_4 + 18a_1a_2a_3^3 + 16a_2^4a_4 - 4a_3^3a_3^2 \\ &\quad - 27a_1^4a_4^2 + 18a_1^3a_2a_3a_4 - 4a_1^3a_3^3 - 4a_1^2a_2^3a_4 + a_1^2a_2^2a_3^2. \end{aligned} \quad (77)$$

551 Since a_1, a_2, a_3 , and a_4 can be formally regarded as functions of F —although, in fact, only
 552 a_2 depends on F (see Equation (54))—we may write $D[Q] \equiv P(F)$, where $P(F)$ is a quartic poly-
 553 nomial in F whose coefficients depend parametrically on $c > 0$. The expression for $P(F)$ is too
 554 lengthy to be reported here. However, the equation $P(F) = 0$ is solved numerically to determine
 555 the values of F that, for fixed $c > 0$, cause the discriminant (77) to vanish. In correspondence
 556 of these values of F , the nature of the roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of $Q(\lambda)$ changes. Specifically, two of
 557 the four solutions to $P(F) = 0$ are complex conjugate, and can be disregarded, because the force
 558 F should be represented by real numbers only. The two remaining solutions are real numbers
 559 corresponding to the branching points of the real parts of the roots of $Q(\lambda)$ in Equation (53),
 560 represented by points B_1 and B_2 in Figure 5b. In Figure 5b, corresponding to the case $c = 0.1$,
 561 the larger of the two real solutions identifies the branching point B_2 . For values of F above this
 562 threshold, all the roots of $Q(\lambda)$ have zero imaginary part. In particular, two of them have a nega-
 563 tive real part, and the other two have a positive real part, which leads to blow-up, or divergence
 564 (without oscillations), of the solutions ϑ_1 and ϑ_2 of the double pendulum.

565 By using the Routh–Hurwitz criterion, Equation (74), and determining the zeros of $P(F)$ for
 566 $c = 0.1$, the threshold values for F are found

$$F_{\text{flu}}(c = 0.1) = 1.7861, \quad F_{\text{div}}(c = 0.1) = 5.1623, \quad (78)$$

567 corresponding to the onset of the flutter and divergence instabilities, respectively, in the presence
 568 of damping. The λ - F diagram for $c = 0.1$ is presented in Figure 5b.

569 **Remark 5** (Ziegler's paradox [8, 9, 10]). *If one assumes $c = 0$, the results presented in Section*
570 *4.1.1 must be recovered. Indeed, one obtains $D[Q] \equiv P(F) \equiv 16 a_4(F)[\Delta(F)]^2$ and, thus, the*
571 *vanishing of $P(F)$ yields the two real and positive values $F = F_{\text{flu}}^{\text{nd}} = 3$ and $F = F_{\text{div}}^{\text{nd}} = 13/3$, but*
572 *both with multiplicity two. In this case, in which c is set equal to zero, there are no dissipative*
573 *mechanisms in the double pendulum subsystem that can produce limit cycles in the fluttering*
574 *regime. However, Ziegler's paradox [8, 9, 10] consists precisely in the fact that $F_{\text{flu}}^{\text{nd}} = 3$ exceeds*
575 *the infimum of $F_{\text{flu}}(c)$ for $c \in]0, +\infty[$, as computed via the Routh–Hurwitz criterion in Equation*
576 (74),

$$F_{\text{flu}}^{\text{nd}} = 3 > F_{\text{flu}}^{(0)} := \lim_{c \rightarrow 0^+} F_{\text{flu}}(c) = 39/22 \simeq 1.7727. \quad (79)$$

577 *Thus, for $F \in]F_{\text{flu}}^{(0)}, F_{\text{flu}}^{\text{nd}}]$, the roots of $Q(\lambda)$ may take on non-negative values. Hence, one may find*
578 *damping coefficients $c > 0$ such that for F compliant with $F_{\text{flu}}^{(0)} < F_{\text{flu}}(c) \leq F \leq F_{\text{flu}}^{\text{nd}}$, the roots of*
579 *$Q(\lambda)$ can have non-negative real part and non-zero imaginary part, so that flutter occurs.*

580 **Remark 6** (Symmetry breaking of the λ - F diagram due to the presence of damping). *It is of*
581 *interest to examine how the branching and merging points of the real and imaginary parts of the*
582 *roots of $Q(\lambda)$ change from Figure 5a to Figure 5b when switching from null damping to nonzero*
583 *damping (in this case, $c = 0.1$). Specifically, the value $F = F_{\text{flu}}^{\text{nd}} = 3$ corresponds to the branching*
584 *point B_0 in the λ - F diagram in the case without damping, Figure 5a, whereas branching does*
585 *not occur at $F = F_{\text{flu}}(c = 0.1)$, when damping is present, as shown in Figure 5b (point H in*
586 *the magnified region). The beginning of the flutter instability depends only on whether the load*
587 *F exceeds the critical value $F_{\text{flu}}(c = 0.1)$, not on the presence of a branching point in the λ -*
588 *F diagram. Moreover, in the damped case the real parts of the roots $\lambda_1(F)$ (solid line marked*
589 *with diamond) and $\lambda_4(F)$ (solid line marked with circles) overlap from $F = 0$ until the value of*
590 *F corresponding to B_1 . The same applies also to $\lambda_2(F)$ (solid line marked with triangles) and*
591 *$\lambda_3(F)$ (solid line marked with squares) from $F = 0$ until the value of F corresponding to B_2 .*
592 *Another consequence of damping is that the two real solutions of $P(F) = 0$, corresponding to the*
593 *abscissae of the points B_1 and B_2 in Figure 5b, move farther apart as c increases. Finally, for*
594 *$c \neq 0$, the roots $\lambda_1(F)$, $\lambda_2(F)$, $\lambda_3(F)$, and $\lambda_4(F)$ are defined analogously to those in Equations*
595 *(71a) and (71b), with $F_{\text{flu}}^{\text{nd}}$ and $F_{\text{div}}^{\text{nd}}$ replaced by $F_{\text{flu}}(c)$ and $F_{\text{div}}(c)$, respectively.*

596 With reference to Figures 5a and 5b, we now discuss the type of bifurcation occurring in
597 the cases analyzed in Sections 4.1.1 and 4.1.2. In the case without damping, Figure 5a (Section
598 4.1.1), the double pendulum subsystem becomes a Hamiltonian system and the occurrence of a
599 Hamiltonian Hopf bifurcation is observed [33, 34]. Such a bifurcation occurs when the roots of
600 the characteristic polynomial $Q(\lambda)$ behave as follows (see Figure 6 for a graphical representation
601 of points a., b., and c., below):

- 602 a. For $F \in [0, F_{\text{flu}}^{\text{nd}}]$, the four roots of $Q(\lambda)$ are purely imaginary and occur in two complex-
603 conjugate pairs, all distinct from one another. Two of them have strictly positive imaginary
604 part, while the other two have strictly negative imaginary part (see the four distinct curves
605 on the left-hand side of Figure 5a).
- 606 b. As the control parameter increases and reaches the critical value $F \equiv F_{\text{flu}}^{\text{nd}}$, the two roots
607 with strictly positive imaginary part coalesce into a single purely imaginary root with pos-
608 itive imaginary part and zero real part. The two roots with strictly negative imaginary part
609 undergo an analogous coalescence, merging into a purely imaginary root with negative
610 imaginary part. The two resulting roots still form a complex-conjugate pair.

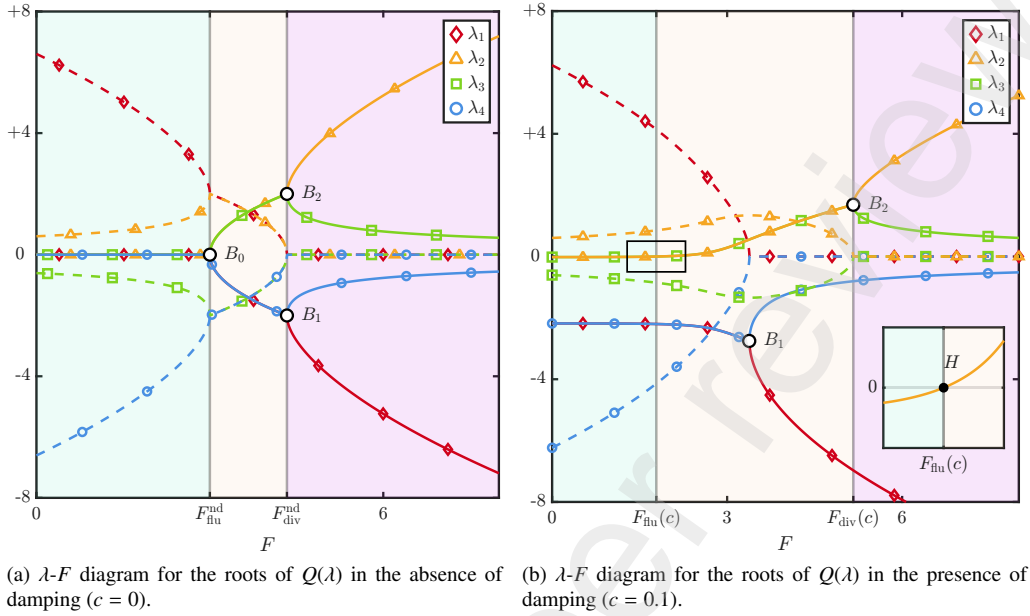


Figure 5: Diagrams of the roots of $Q(\lambda)$ in Equation (53) for $c = 0$ and $c = 0.1$.

611 c. For $F \in]F_{\text{flu}}^{\text{nd}}, F_{\text{div}}^{\text{nd}}[$ (the flutter region), the two coalesced pairs split and acquire nonzero
 612 real parts: one with positive real part and the other with negative real part. The corre-
 613 sponding roots are symmetric with respect to the real axis in the λ - F diagram, but they are
 614 no longer complex conjugates of one another.

615 When addressing the case with damping (Figure 5b), a Hopf bifurcation occurs, which will
 616 be discussed in detail in the next section.

617 4.2. Investigation of Hopf bifurcation: from Schur's complement technique to dynamical systems

618 The Schur's complement technique [20, 27] is used in this section to decouple the computa-
 619 tion of the Lagrange multiplier μ from the determination of θ_1 , θ_2 , and X_c . To this end, Equations

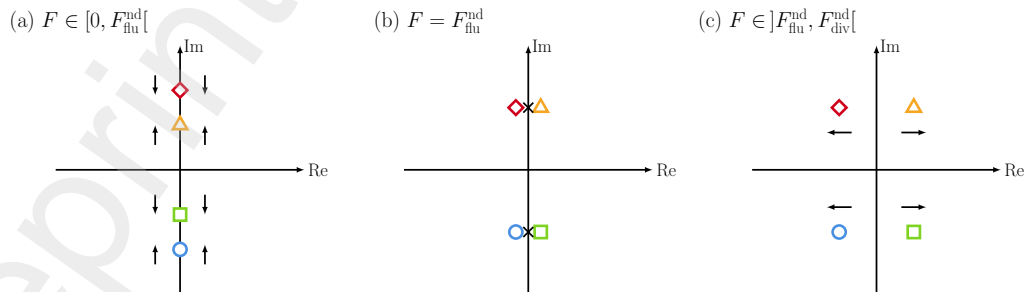


Figure 6: Graphical representation of the behavior of the roots in the case of the Hamiltonian Hopf bifurcation.

620 (34a)–(34c) and (36) are rewritten in matrix form, isolating the array $\ddot{\mathbf{q}}$ collecting the second-
621 order derivatives $\ddot{\mathbf{q}} \equiv (\ddot{\theta}_1, \ddot{\theta}_2, \ddot{X}_c)$ together with the Lagrange multiplier μ . We then reformulate
622 the problem in the *state-variable formalism* and apply dynamical-systems techniques to investi-
623 gate the stability of the device.

624 In matrix form, Equations (34a)–(34c) and (36) read

$$\begin{bmatrix} \hat{M}(\mathbf{q}) & -[\hat{A}(\mathbf{q})]^T \\ -\hat{A}(\mathbf{q}) & \mathbf{O} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \mu \end{Bmatrix} = \begin{Bmatrix} \hat{\mathbf{b}}_q(\mathbf{q}, \dot{\mathbf{q}}, t) \\ \hat{\mathbf{b}}_c(\mathbf{q}, \dot{\mathbf{q}}) \end{Bmatrix}, \quad (80)$$

625 where $\hat{M}(\mathbf{q})$ denotes the 3×3 mass matrix; $\hat{A}(\mathbf{q})$ is the 1×3 constraint matrix; \mathbf{O} is the 1×1
626 null matrix; $\hat{\mathbf{b}}_q(\mathbf{q}, \dot{\mathbf{q}}, t)$ is a 3×1 array collecting all the terms appearing in Equations (34a)–(34c)
627 that involve neither the generalized accelerations $\ddot{\theta}_1$, $\ddot{\theta}_2$, and \ddot{X}_c , nor the Lagrange multiplier μ ;
628 $\hat{\mathbf{b}}_c(\mathbf{q}, \dot{\mathbf{q}})$ is the 1×1 array representing the right-hand side of Equation (36). Note that $\hat{\mathbf{b}}_q(\mathbf{q}, \dot{\mathbf{q}}, t)$
629 depends explicitly on time through $B_h(t)$ and $\dot{B}_h(t)$. The explicit expressions for $\hat{M}(\mathbf{q})$, $\hat{A}(\mathbf{q})$,
630 $\hat{\mathbf{b}}_q(\mathbf{q}, \dot{\mathbf{q}})$, and $\hat{\mathbf{b}}_c(\mathbf{q}, \dot{\mathbf{q}})$ are

$$\mathbf{M} \equiv \hat{M}(\mathbf{q}) := \begin{bmatrix} \frac{5}{4} & \frac{1}{2} \cos(\theta_1 - \theta_2) & -\frac{3}{2} \sin \theta_1 \\ \frac{1}{2} \cos(\theta_1 - \theta_2) & \frac{1}{4} & -\frac{1}{2} \sin \theta_2 \\ -\frac{3}{2} \sin \theta_1 & -\frac{1}{2} \sin \theta_2 & M \end{bmatrix}, \quad (81a)$$

$$\mathbf{b}_q \equiv \hat{\mathbf{b}}_q(\mathbf{q}, \dot{\mathbf{q}}) := \begin{Bmatrix} -\frac{1}{2} \sin(\theta_1 - \theta_2) \dot{\theta}_2^2 - c(2\dot{\theta}_1 - \dot{\theta}_2) - (2\theta_1 - \theta_2) \\ \frac{1}{2} \sin(\theta_1 - \theta_2) \dot{\theta}_1^2 - c(\dot{\theta}_2 - \dot{\theta}_1) - (\theta_2 - \theta_1) \\ \frac{3}{2} (\cos \theta_1) \dot{\theta}_1^2 + \frac{1}{2} (\cos \theta_2) \dot{\theta}_2^2 + F \end{Bmatrix} + \begin{Bmatrix} \mathfrak{F}_{L1} \\ \mathfrak{F}_{L2} \\ \mathfrak{F}_{L3} \end{Bmatrix}, \quad (81b)$$

$$\mathbf{b}_c \equiv \hat{\mathbf{b}}_c(\mathbf{q}, \dot{\mathbf{q}}) := \{(\sin \theta_2) \dot{X}_c \dot{\theta}_2 + \cos(\theta_1 - \theta_2) (\dot{\theta}_1 - \dot{\theta}_2) \dot{\theta}_1\}, \quad (81c)$$

$$\mathbf{A} \equiv \hat{A}(\mathbf{q}) := [\sin(\theta_1 - \theta_2) \quad 0 \quad -\cos \theta_2]. \quad (81d)$$

631 Since $\hat{M}(\mathbf{q})$ is an invertible matrix for all values of \mathbf{q} , we can apply the Schur complement
632 technique [20, 27, 35] to rewrite system (80) in normal form and, upon introducing the Schur
633 complement $\mathbf{S} := \mathbf{A} \mathbf{M}^{-1} \mathbf{A}^T$ of \mathbf{M} , we obtain

$$\ddot{\mathbf{q}} = [\mathbf{M}^{-1} - (\mathbf{M}^{-1} \mathbf{A}^T) \mathbf{S}^{-1} (\mathbf{A} \mathbf{M}^{-1})] \mathbf{b}_q - (\mathbf{M}^{-1} \mathbf{A}^T) \mathbf{S}^{-1} \mathbf{b}_c, \quad (82a)$$

$$\mu = -\mathbf{S}^{-1} [(\mathbf{A} \mathbf{M}^{-1}) \mathbf{b}_q + \mathbf{b}_c]. \quad (82b)$$

634 Equation (82a) unfolds three scalar equations for the three unknowns collected in \mathbf{q} , and, once
635 these are computed, Equation (82b) determines the Lagrange multiplier μ .

636 A further step beyond Equation (82a) consists in introducing the *state-variable formalism*,
637 which allows us to recast (82a) as a system of six first-order scalar ordinary differential equations.
638 In fact, by setting $Y_1 := \theta_1$, $Y_2 := \theta_2$, $Y_3 := X_c$, $Y_4 := \dot{\theta}_1$, $Y_5 := \dot{\theta}_2$, $Y_6 := \dot{X}_c$, and introducing the
639 6×1 arrays $\mathbf{Y} = \{Y_1 \cdots Y_6\}^T$ and $\dot{\mathbf{Y}} = \{\dot{Y}_1 \cdots \dot{Y}_6\}^T$, Equation (82a) is equivalent to

$$\dot{\mathbf{Y}} = \mathbf{f}(\mathbf{Y}, t; F), \quad (83)$$

640 where, for $k = 1, 2, 3$, $f_k(\mathbf{Y}, t; F) = Y_{k+3}$, while $f_4(\mathbf{Y}, t; F)$, $f_5(\mathbf{Y}, t; F)$, and $f_6(\mathbf{Y}, t; F)$ are the three
641 components of the right-hand side of Equation (82a), written in the state variable formalism.

642 In Equation (83), the vector $\mathbf{f}(\mathbf{Y}, t; F)$ inherits its explicit time dependence from $\hat{\mathbf{b}}_q(\mathbf{q}, \dot{\mathbf{q}}, t)$.
643 However, this dependence appears only in the components $f_4(\mathbf{Y}, t; F)$, $f_5(\mathbf{Y}, t; F)$, and $f_6(\mathbf{Y}, t; F)$.

644 We also emphasize that we have expressed the functions as parametrically dependent on the dead
 645 load F , which plays the role of the bifurcation parameter. Although the characteristic magnetic
 646 induction B_M may, in principle, influence the stability of the device [5], it is treated here as a
 647 fixed parameter for simplicity, and F is taken as the sole bifurcation parameter of the study.

648 We now investigate the stability of the equilibrium configuration q_e , given in Equation (43),
 649 following standard dynamical-systems procedures. In particular, we refer to the dynamical sys-
 650 tem described by Equation (83). Our aim is to assess whether or not the system undergoes a
 651 Hopf bifurcation at $F = F_{\text{flu}}(c)$. To this end, we analyze the eigenvalues of the Jacobian matrix
 652 of the map f in Equation (83), evaluated at the equilibrium point $Y = Y_e = \{0 \ 0 \ -2 \ 0 \ 0 \ 0\}^T$. This
 653 corresponds to expressing q_e in the state-variable formalism of Section 4.2, with $X_{ce} = -2$.

654 We start considering the first-order Taylor expansion of f at Y_e and for a generic value of F .
 655 Hence, for $j = 1, \dots, 6$, and in the limit $\|Y - Y_e\|_2 \rightarrow 0$, we write

$$[f(Y, t; F)]_j = \underbrace{[f(Y_e, t; F)]_j}_{=0} + \sum_{k=1}^6 \underbrace{\left[\frac{\partial f_j}{\partial Y_k}(Y_e, t; F) \right]}_{=: [J_e]_{jk}} [Y_k - (Y_e)_k] + o(\|Y - Y_e\|_2), \quad (84)$$

656 where $\|p\|_2 := \sqrt{p^T p}$ denotes the 2-norm of a generic (column) array p . The 6×6 Jacobian
 657 matrix $J_e \equiv \hat{J}(Y_e, t; F)$ features six eigenvalues given by the roots of the characteristic equation

$$P(\omega; F) = \det(J_e - \omega I_6) = \frac{1}{M} Q(\omega; F) \omega^2 = 0, \quad (85)$$

658 where $Q(\omega; F)$ is a rewriting of the fourth-grade characteristic polynomial in Equation (53),
 659 associated with the double pendulum, with ω replacing λ and the dependence on F made explicit.
 660 The roots of $P(\omega; F)$ define the eigenvalues $\omega_1, \dots, \omega_6$ as functions of F , so that we may write
 661 $\omega_j = \hat{\omega}_j(F)$, for $j = 1, \dots, 6$.

662 As noticed in Section 4.1 and sketched in Figure 4, the characteristic polynomial $Q(\omega; F)$
 663 remains unchanged regardless of whether or not the Lorentz force is considered, and the same
 664 applies to $P(\omega; F)$. Hence, $P(\omega; F)$ and its roots can be used to analyze the occurrence of a
 665 Hopf bifurcation as if the Lorentz force were absent, in which case the dynamical system would
 666 behave as autonomous.

667 Although the characteristic equation (85) describes the device as a whole—including the
 668 skate, which contributes the factor ω^2 arising from the constraint—the decoupling, in terms of
 669 eigenvalues, between the double pendulum and the cart in the linearized analysis of Section 4
 670 is manifested in the factorization shown in Equation (85). Indeed, the characteristic polynomial
 671 features two distinct factors: one of grade four for the double pendulum, $Q(\omega; F)$; another of
 672 grade two, ω^2 , related to the nonholonomic constraint. The presence of the latter can be related
 673 to the linearized constraint and, consequently, to the acceleration of the cart \ddot{x}_c in Equation (47d).

674 When Equation (85) is evaluated at $F = F_{\text{flu}}(c)$, the polynomial $Q(\omega; F)$ becomes

$$Q(\omega; F_{\text{flu}}(c)) = (22 + 44c\omega + \omega^2)(8/11 + \omega^2) = 0, \quad (86)$$

675 and the eigenvalues of $J_e \equiv \hat{J}(Y_e, t; F_{\text{flu}}(c))$ are the roots of the characteristic equation

$$P(\omega; F_{\text{flu}}(c)) = \frac{1}{M} (22 + 44c\omega + \omega^2)(8/11 + \omega^2)\omega^2 = 0, \quad (87)$$

676 which read

$$\underbrace{\omega_{1,2} = -22c \pm \sqrt{-22 + (22c)^2}}_{\text{double pendulum}}, \quad \underbrace{\omega_{3,4} = \pm i \sqrt{8/11}}_{\text{constraint}}, \quad \underbrace{\omega_{5,6} = 0}_{\text{constraint}} \quad (88)$$

677 Moreover, the eigenvalues $\omega_{1,2} \equiv \hat{\omega}_{1,2}(F_{\text{flu}}(c))$ depend parametrically on c both through $F_{\text{flu}}(c)$
 678 and through the coefficients of $Q(\omega; F)$, as shown in Equation (86), although this is not explicitly
 679 expressed by the notation used.

680 The first four roots obtained from Equation (87), $\omega_1, \omega_2, \omega_3$ and ω_4 , are also roots of Equation
 681 (86), and refer to the double pendulum, which experiences flutter instability. In particular, the
 682 eigenvalues ω_1 and ω_2 are complex conjugate for $c \in]0, 1/\sqrt{22}[$, while they are purely real
 683 for $c > 1/\sqrt{22}$, and are coincident for $c = 1/\sqrt{22}$. In all these cases, $\text{Re}[\omega_1]$ and $\text{Re}[\omega_2]$ are
 684 negative for all $c > 0$. Hence, $\omega_1, \omega_2, \omega_3$, and ω_4 characterize the onset of flutter through a Hopf
 685 bifurcation [3]. In particular, a Hopf bifurcation at the equilibrium Y_e , occurring at $F \equiv F_{\text{flu}}(c)$,
 686 arises if the following three conditions are satisfied [36]:

- 687 1. At the onset of flutter, Y_e is an equilibrium state, $f(Y_e, t; F_{\text{flu}}(c)) = 0$;
- 688 2. At $F = F_{\text{flu}}(c)$, exactly one pair of purely imaginary complex-conjugate eigenvalues exists,
 689 each of algebraic multiplicity one, while all remaining eigenvalues have nonzero real part
 690 (in our case, negative real part).
- 691 3. The eigenvalues $\omega_3(F)$ and $\omega_4(F) \equiv \bar{\omega}_3(F)$ that, at $F = F_{\text{flu}}(c)$, form to the aforementioned
 692 pair of conjugate imaginary eigenvalues must depend on F in such a way that their real
 693 part be differentiable at $F_{\text{flu}}(c)$ and compliant with the condition

$$\frac{d \text{Re}[\omega_{3,4}]}{dF}(F_{\text{flu}}(c)) \neq 0. \quad (89)$$

694 The fulfillment of these conditions is highlighted in Figure 5b.

695 **Remark 7** (Lorentz force does not affect the onset of the Hopf bifurcation for $F = F_{\text{flu}}(c)$).
 696 *It has been shown that, despite the inclusion of the Lorentz force in the model studied by Cazzolli*
 697 *et al. [3], the system still undergoes a Hopf bifurcation under the same conditions, namely when*
 698 *the dead load F reaches the critical flutter load $F_{\text{flu}}(c)$.*

699 We conclude by noting that a Hopf bifurcation occurs even though the eigenvalues ω_5 and
 700 ω_6 vanish and, strictly speaking, do not satisfy the second Hopf condition. Moreover, they make
 701 the Jacobian matrix J_e singular and preclude the possibility of computing the first Lyapunov
 702 coefficient for the problem as is currently formulated. However, as recalled in Section 4, ω_5
 703 and ω_6 can be disregarded from the study of stability and, thus, of the bifurcations [23]. This
 704 argument is grounded in the same reasoning used to derive Equation (52) from Equation (51) in
 705 the stability analysis.

706 5. Post-critical behavior: the Lorentz force influences the flutter instability

707 In this section, we aim to assess how the Lorentz force can influence the post-critical behavior
 708 of the device both in Case I and in Case II.

709 As anticipated in the preamble of Section 4, we restore the physical dimensions of the prob-
 710 lem to provide context for the following numerical test.

711 5.1. The scenario of a soft bodied microrobot for biological applications

712 Among the various designs that are often employed in microrobotics [37], we model our
 713 microrobot as a continuous rod (discretized by the double pendulum) attached to a small mass
 714 (the cart), having both viscous and elastic material properties (the viscoelastic hinges), and a

715 concentrated electric charge at one end (the point charge e in D). Specifically, we assume the rod
 716 to be made of an idealized rubber-like material with density $\rho = 10^3 \text{ kg/m}^3$ and Young modulus
 717 $E_Y = 10^6 \text{ Pa}$, [14].

718 The device's geometry is characterized by slender bars, with length $\ell = 10 \mu\text{m}$, width
 719 $w = 1 \mu\text{m}$, and thickness $\tau = 0.1 \mu\text{m}$, so that their volumes and moments of inertia read $V = \ell w \tau$
 720 and $J = (1/12)\tau^3 w$, respectively (the aspect ratio of the bars and the notation are taken from [5]).
 721 Consequently, the mass of one bar reads $m = \rho V$, while the stiffness coefficient k is computed as
 722 $k = E_Y J / \ell$ [5]. Once k is obtained, the damping coefficient c (modeling the overall viscous be-
 723 havior of the robot) is selected coherently with the work by Cazzolli et al. [3]. Hence, c is chosen
 724 in such a way that the dimensionless damping coefficient of Equation (45b)₂ is $c / (\ell \sqrt{mk}) = 0.1$
 725 (cfr. Section 4).

726 The value of the dead load F is in line with the typical thrust and propulsion forces reported
 727 for microrobots in biological applications [11]. Specifically, we choose the value $F = 1.6 \text{ pN}$,
 728 for which flutter can occur, since $F \simeq 1.1 F_{\text{fl}}(c)$.

729 The solenoid is assumed to be sufficiently long, with height at least four times its diameter
 730 [17], and radius $R_s = 5 \text{ cm}$, and capable of generating a magnetic induction field of maximum
 731 amplitude $B_M = 1 \text{ T}$. We emphasize that this value of B_M is higher than those typically used
 732 to study, for instance, the motility of micro- or nano-robots in biological environments (usually
 733 around the order of magnitude of 10^{-3} T , [11]). Although the value $B_M = 1 \text{ T}$ may be considered
 734 extreme, we adopt it to make the Lorentz force comparable to the propulsion forces (dead load)
 735 considered in [11] and to the follower force. Solenoids with relatively small diameters can nev-
 736 ertheless reach MRI-level magnetic induction fields [38, 39] and can be used for micro-motility
 737 applications [40].

738 The magnetic induction field is assumed to be sinusoidal in time by setting

$$B_h(t) = B_M \sin(\Omega t), \quad \text{so that} \quad \dot{B}_h(t) = B_M \Omega \cos(\Omega t), \quad (90)$$

739 with Ω being its frequency of oscillation. In the following sections, we examine the cases $\Omega =$
 740 5 kHz and $\Omega = 10 \text{ kHz}$, consistent with the frequency range adopted by Rousseaux et al. [17] to
 741 justify the low-frequency approximation for the vector potential \mathbf{A} . We restrict the analysis to a
 742 parametric study of ω , while all the other parameters are kept fixed.

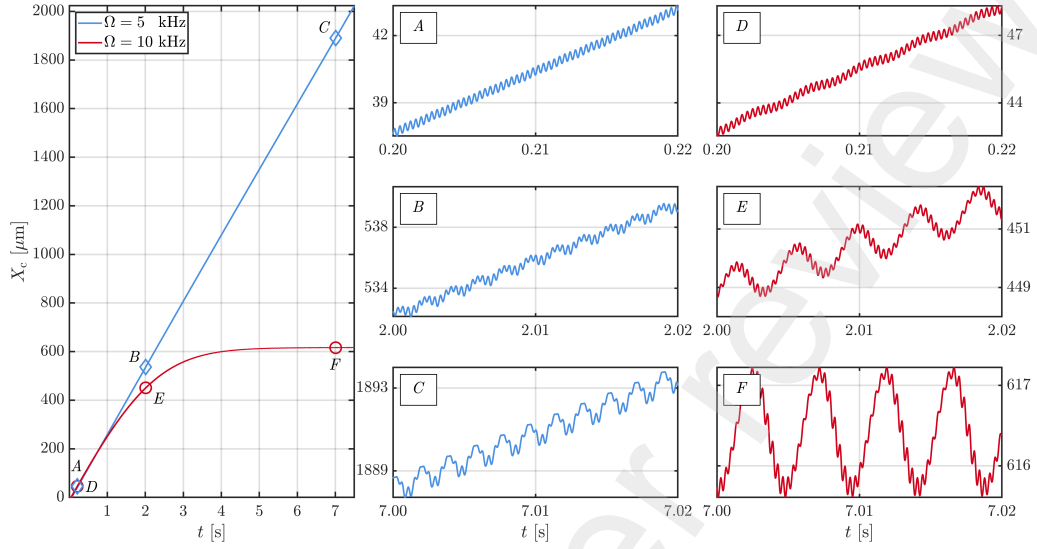
743 The point charge concentrated at the skate is in agreement with the effective net charge of
 744 micro-metric entities in aqueous media, as documented in electrokinetic studies [12]. We select
 745 $e = 0.03 \text{ pC}$ rather than the more natural value $e = 0.01 \text{ pC}$ to amplify the influence of the Lorentz
 746 force, without allowing the magnetic field amplitude to exceed 1 T .

747 **Remark 8** (Implementation details).

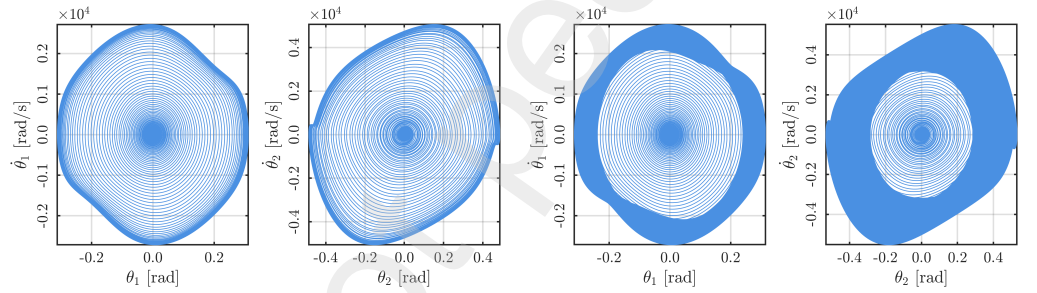
748 *The simulations are obtained by numerically solving the Cauchy problem defined by Equa-*
 749 *tion (83), together with the array of initial conditions $\mathbf{Y}_0 = \{0 \ 0 \ X_{c0} \ 0 \ 0 \ 0\}^T$. In Case I, X_{c0}*
 750 *is chosen as a small deviation from X_{ce} in Equation (43), while in Case II it is $X_{c0} > R_s$. Null*
 751 *initial velocities are assumed. Numerical simulations are performed in MATLAB [41] using the*
 752 *adaptive Runge–Kutta (4,5) scheme implemented in the ode45 function for time integration [42].*

753 *5.2. Case I: the Lorentz force triggers flutter and can suppress the horizontal motion*

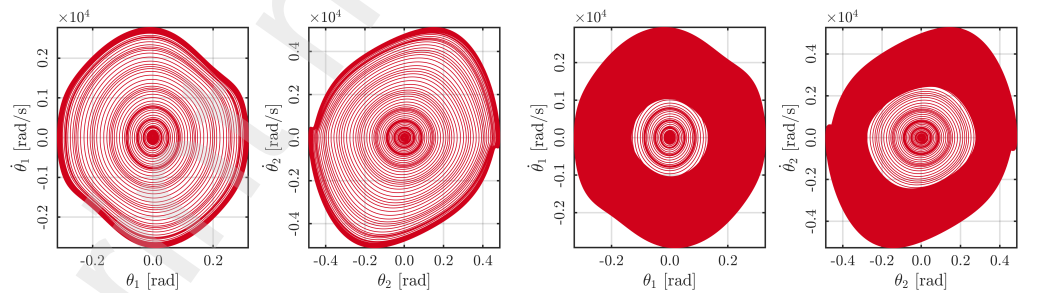
754 We consider the case in which the device is fully contained within the solenoid and the cart
 755 is initially positioned at $X_{c0} = -1.5\ell$. For this configuration, corresponding to Figure 3b, with
 756 $|X_{c0} - X_{ce}| = 0.5\ell$, flutter occurs as induced by the nonzero torque exerted by the electric part



(a) Time trend of X_c for $\Omega = 5$ kHz (blue line with diamonds) and for $\Omega = 10$ kHz (red line with circles).



(b) Phase portraits of θ_1 and θ_2 , for $\Omega = 5$ kHz, over the interval $[0 \text{ s}, 0.1 \text{ s}]$, ending with the zoomed-in region A. (c) Phase portraits of θ_1 and θ_2 , for $\Omega = 5$ kHz, over the interval $[0 \text{ s}, 7 \text{ s}]$, ending with the zoomed-in region C.



(d) Phase portraits of θ_1 and θ_2 , for $\Omega = 10$ kHz, over the interval $[0 \text{ s}, 0.1 \text{ s}]$, ending with the zoomed-in region D. (e) Phase portraits of θ_1 and θ_2 , for $\Omega = 10$ kHz, over the interval $[0 \text{ s}, 7 \text{ s}]$, ending with the zoomed-in region F.

Figure 7: Time trend of X_c and phase portraits of θ_1 and θ_2 concerning the parametric study conducted for Case I and representing the dynamic motion induced by flutter instability on a movable double pendulum with electrically charged nonholonomic constraint and embedded in a magnetic field.

Parameter	Symbol	Value	Unit	Ref.
Length of each bar	ℓ	10	μm	[11, 13]
Mass of each bar	m	0.001	ng	[14]
Mass of the cart	m_c	0.01	ng	–
Spring stiffness	k	10	$\text{pC } \mu\text{m}$	[5]
Damping coefficient	c	0.0001	$\text{pN } \mu\text{m s}^{-1}$	[3]
Dead load	F	1.6	pN	[11]
Electric charge	e	0.03	pC	[12]
Solenoid radius	R_s	5	cm	[38, 39]
Magnetic amplitude	B_M	1	T	[40, 39]
Magnetic frequency	Ω	{5, 10}	kHz	[17]

Table 1: Numerical values used for the simulations.

of the Lorentz force. This phenomenon does not arise in the problem studied by Cazzolli et al. [3], where flutter is triggered only by perturbations of the angles θ_1 or θ_2 , or of their associated angular velocities.

We begin with the case $\Omega = 5$ kHz, represented in Figure 7 by the solid blue line marked with diamonds. Figure 7a shows that for $\Omega = 5$ kHz the motion of the cart, described by X_c , exhibits oscillations around a *main, roughly linear, trend*. In spite of the different setting considered in our work, this result is similar to what was found in the paper by Cazzolli et al. [3], in which the cart advances on a large scale with a mean velocity, but it experiences oscillations around such mean velocity on a smaller scale (see the top left panel of Figure 12 of [3]). Regarding the large scale motion, the Lorentz force seems not to affect significantly the dynamics of the cart: the cart moves away from the solenoid center with a comparatively large and approximately constant mean velocity of about $260 \mu\text{m s}^{-1}$, thereby covering about 13 body lengths (2ℓ) per second. However, while the cart's small scale oscillations in our work and in [3] are comparable to each other in the first instants of our simulation (see the zoomed-in region labeled with 'A' in Figure 7a), they are modified by the Lorentz force over time, as the device advances into regions in which the Lorentz force is stronger (see the zoomed-in regions B and C in Figure 7a).

Figures 7b and 7c show the phase portraits of θ_1 and θ_2 for the case $\Omega = 5$ kHz and over the time windows $[0 \text{ s}, 0.1 \text{ s}]$ and $[0 \text{ s}, 7 \text{ s}]$, respectively. For brevity, the phase portraits corresponding to the intermediate time interval $[0 \text{ s}, 2 \text{ s}]$ are not shown.

The increasing amplitude of the oscillations of X_c should be compared with the phase plots of θ_1 and θ_2 in Figures 7b and 7c, and, in particular, with the progressive clustering of the orbits, as time goes by, from Figure 7b (corresponding to the region A) to Figure 7c (corresponding to the region C). Our conclusion is that the Hopf bifurcation, rather than resulting into a stable limit cycle, is characterized by an almost chaotic behavior.

A different situation occurs for $\Omega = 10$ kHz. At this frequency, and with reference to the solid red line marked with circles in Figure 7a, the device no longer experiences a motion similar to that reported by Cazzolli et al. [3]. Indeed, although the cart initially advances as in the case $\Omega = 5$ kHz (and thus as predicted in [3]), exhibiting micro-oscillations in the evolution of X_c (see the zoomed-in regions D, E, and F of Figure 7a), the increasing Lorentz force causes a qualitative change in the motion. The zoomed-in regions D, E, and F in Figure 7a show that X_c oscillates around a *non-linear trend* and that, in this case, the oscillations gain a much more

788 marked two-scale character with increasing time (from D to F). For $t \geq 7$ s, the cart swings
789 in a bounded region of space, without appreciable further advancement, thereby suggesting an
790 average-in-time balance among the dead load, the follower force and the Lorentz force. Also in
791 this case, the oscillations of the cart are to be compared with the progressive thickening of the
792 clusters of orbits reported in Figures 7d and 7e, which show the phase portraits of θ_1 and θ_2 for
793 the case $\Omega = 10$ kHz and over the time windows $[0 \text{ s}, 0.1 \text{ s}]$ and $[0 \text{ s}, 7 \text{ s}]$, respectively. Again, the
794 agglomeration of orbits seems to indicate the tendency of the system towards chaos rather than
795 towards stable limit cycles, with a quasi-periodic behavior.

796 The increasing impact of the Lorentz force on the flutter instability is also reflected in the
797 phase diagrams of θ_1 and θ_2 . Figures 7c and 7e show that the phase portraits of θ_1 and θ_2 do not
798 converge to single closed orbits corresponding to stable limit cycles, in contrast to the behavior
799 reported by Cazzoli et al. [3]. As time progresses, the simulations predict the formation of
800 progressively larger regions, or ‘bands’, of trajectories in the corresponding phase spaces. This
801 behavior is likely due to the modulations induced by the Lorentz force (Figures 7c and 7e), and
802 is more pronounced in the case with 10 kHz, since the oscillations of the cart around the main
803 trend are stronger than for $\Omega = 5$ kHz.

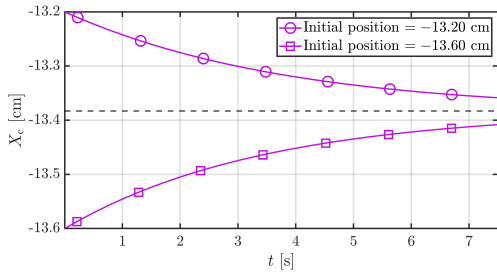
804 Lastly, we find it worth mentioning that, both for $\Omega = 5$ kHz (region C, corresponding to
805 $t = 7$ s) and for $\Omega = 10$ kHz (regions D, E, and F) the Lorentz force seems to induce three-scale
806 dynamics: a main trend, which is linear for $\Omega = 5$ kHz and nonlinear for $\Omega = 10$ kHz, ‘slow’
807 oscillations, and ‘fast’ oscillations. The regions C and E of Figure 7a are those in which this
808 phenomenon is best displayed.

809 5.3. Case II: the Maxwell–Lodge effect in the dynamics of the microrobot

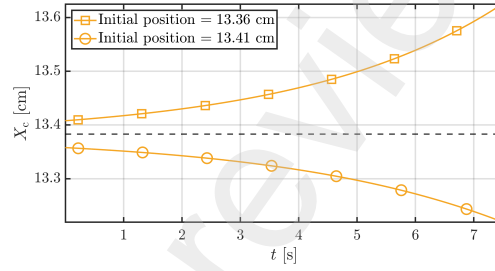
810 We now consider Case II, in which the device is located outside the solenoid. To characterize
811 its overall motion, we examine the time evolution of the cart position. Owing to the geometric
812 symmetry of the configuration, we analyze two initial placements: one to the left of the solenoid
813 and one to the right, both equidistant from the origin.

814 Figure 8a illustrates the case in which the cart is positioned to the left of the solenoid. Each of
815 the branches in Figure 8a represents the motion of the cart for different initial conditions X_{c0} , with
816 the upper branch (marked with circles) associated with $X_{c0} = -13.20$ cm and the lower branch
817 (marked with squares) corresponding to $X_{c0} = -13.60$ cm. We recall that outside the solenoid
818 any initial position X_{c0} of the cart is of non-equilibrium and therefore Figure 8a shows how the
819 cart evolves, either moving away from the solenoid’s boundary (curve marked with circles) or
820 moving towards it (curve marked with squares). Moreover, even if the device is initially in a
821 straight configuration, with $\theta_1(0) = \theta_2(0) = 0$ and $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$, the Lorentz force acting
822 on the skate generates an unbalanced torque that induces angular motion of the bars, ultimately
823 leading to flutter, as discussed in Section 4.1. The observed dynamic behavior is argued to be
824 a consequence of the Maxwell–Lodge effect, according to which, even in the absence of the
825 magnetic induction field (see Remark 1 and Equations (16b) and (19b)), the Lorentz force acting
826 on the skate remains nonzero.

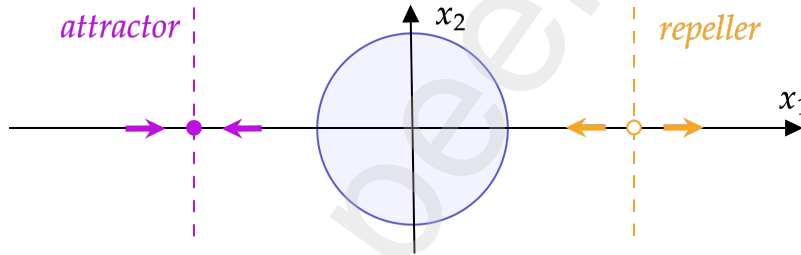
827 In addition, both mean positions of the cart—one with initial position at -13.20 cm (circles)
828 and the other at -13.60 cm (squares)—tend asymptotically toward a position at approximately
829 -13.38 cm, about which they oscillate. In view of this behavior, and with a slight abuse of termi-
830 nology, we refer to the point $(-13.38 \text{ cm}, 0)$ as an ‘attractor’. Complementary to the trajectories
831 of the cart X_c converging to the attractor, Figures 8d and 8e also present the phase portraits of
832 the angles θ_1 and θ_2 when the cart is at the attractor, together with a zoomed-in view of the cart



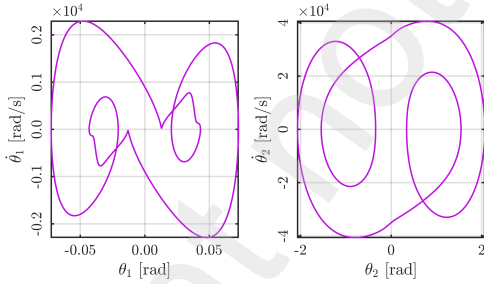
(a) Time trend of X_c , for $\Omega = 10$ kHz, when the cart is placed outside of the solenoid on its left, at a position of -13.20 cm (circles) and -13.60 cm (squares).



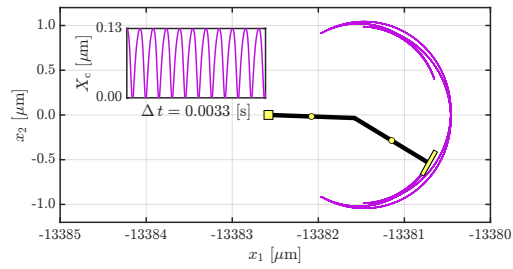
(b) Time trend of X_c , for $\Omega = 10$ kHz, when the cart is placed outside of the solenoid on its right, at a position of 13.36 cm (circles) and 13.41 cm (squares).



(c) Graphical depiction of the 'attractor' point and 'repeller' point. The picture is not in scale.



(d) Phase plots of θ_1 and θ_2 , at when the device is placed at the point attractor $(-13.38$ cm, 0).



(e) Trace of the position of D alongside the cart oscillations (zoomed in section) at the point attractor $(-13.38$ cm, 0).

Figure 8: Time trend of X_c (a) and (b), picture of the 'attractor' and 'repeller' points (c), phase portraits of θ_1 and θ_2 (d), and trajectory of the point D (e) for the parametric study of Case II and representing the dynamic motion induced by flutter instability on a movable double pendulum with electrically charged nonholonomic constraint, embedded in a magnetic field.

833 displacement and the trajectory of point D. The dynamical regime in which the device now oper-
834 ates, as observed from the phase portraits, appears to lead to stable limit cycles. This is further
835 illustrated in the zoomed-in view of X_c , where no micro-oscillations are observed to disturb the
836 motion. We believe that, in this case, the Lorentz force acting at the tip of the pendulum is the
837 dominant contribution.

838 Analogous considerations apply to Figure 8b, although the opposite behavior is observed.
839 For symmetry, we place the cart at two initial positions near the point (13.38 cm, 0), on the right-
840 hand side of the solenoid: $X_{c0} = 13.36$ cm (corresponding to the curve marked with squares) and
841 $X_{c0} = 13.41$ cm (corresponding to the curve marked with circles). The mean trajectories of the
842 cart are now repelled from this point: the one with initial position $X_{c0} = 13.36$ cm moves back
843 toward the solenoid, whereas the one with initial position $X_{c0} = 13.41$ cm moves away from it.
844 Accordingly, we refer to the point (+13.38 cm, 0) as a ‘repeller’.

845 Consistent with the discussion above, we emphasize that the device may also undergo back-
846 ward motion in order to approach the ‘attractor’ or move away from the ‘repeller’, as shown in
847 Figure 8. Such behavior is not achievable in the context of the problem studied by Cazzolli et
848 al. [3].

849 6. Discussion and conclusions

850 A mechanical structural system has been proposed and analyzed that is capable of generating
851 propulsion from an applied conservative force and a magnetic field by exploiting flutter insta-
852 bility and Hopf bifurcation. The system provides a simple model of a microrobot suitable for
853 operating in magnetically controlled environments, without relying on conventional propulsion
854 mechanisms. Note that, when viscosity is absent, the sole responsible for the variation of energy
855 is the electric part of the Lorentz force. If the latter is switched off, the system becomes indeed
856 conservative.

857 Our design has coupled a Ziegler double pendulum with a movable cart subject to a dead
858 load and an electrically charged nonholonomic constraint. Electromagnetic interactions provide
859 a Lorentz force applied on the constraint, when placed inside or outside an ideal solenoid. Our
860 setup differs from previously analyzed devices based on electric dipoles or magnetic torques [5].

861 A primary consequence of the theoretical framework underlying our work is the recognition
862 that the Maxwell–Lodge effect [17, 18] may also be observed in systems such as the microrobot
863 considered here. Owing to the significance of this effect in physics, from both theoretical and
864 experimental perspectives—Rousseaux et al. [17] describe it as the ‘*classical equivalent*’ of the
865 ‘*Aharonov–Bohm effect [...] within quantum physics*’—we found it appropriate to revisit it
866 within our framework (see Remark 1) and to assess its role in the dynamics of the device.

867 We have proposed two ‘virtual’ experimental configurations (Case I and Case II), formulated
868 as dedicated Cauchy problems, which may serve as benchmarks for future investigations of the
869 post-critical dynamics of systems similar to ours.

870 Although the device exhibits behaviors not reported in previous studies, and although the
871 parameters adopted in our simulations are drawn from the literature (see Table 1), the intensity
872 B_M of the magnetic induction field may be relatively high. In particular, we selected $B_M = 1$ T—a
873 value achievable in MRI-based and electron-beam experiments [39]—in order to ensure that the
874 Lorentz force is comparable in magnitude to the dead load F typically considered in Ziegler’s
875 double pendulum models in microrobotics, as well as to the follower force.

876 A systematic analysis for smaller values of B_M would be desirable to further delineate the
877 range of validity of the present results. Notwithstanding this limitation, the findings summarized

878 in Figures 7a–7e and 8a–8b provide a basis for further discussion on the interplay between flutter
879 instability and Hopf bifurcation in devices controlled through additional forces such as the
880 Lorentz force (see Remark 7).

881 As shown in Figures 8a and 8b, the Lorentz force outside the solenoid—arising solely from
882 the induced electric field—can either attract or repel the microrobot (through its action on the
883 cart), depending on whether the cart is initially positioned below or above a critical distance
884 from the left or right boundary of the solenoid.

885 The present study contributes to the analysis of the stability of microrobots and related structures
886 inspired by Ziegler’s double pendulum, particularly when interactions beyond purely mechanical
887 effects are taken into account. The results may provide a reference framework for future
888 investigations based on alternative experimental configurations and may stimulate the design of
889 new experimental studies in this area.

890 **Authors’ contributions**

891 All authors contributed equally to the manuscript.

892 **Declaration of competing interests statement**

893 The authors have no competing interests to declare.

894 **Use of generative Artificial Intelligence (AI)**

895 The authors have *not* used any generative AI tool for writing this work.

896 **References**

- 897 [1] O. N. Kirillov, Structural optimization of the ziegler’s pendulum: singularities and exact
898 optimal solutions, 2010.
- 899 [2] D. Bigoni, G. Noselli, Experimental evidence of flutter and divergence instabilities induced
900 by dry friction, *Journal of the Mechanics and Physics of Solids* 59 (2011) 2208–2226.
- 901 [3] A. Cazzoli, F. Dal Corso, D. Bigoni, Non-holonomic constraints inducing flutter instability
902 in structures under conservative loadings, *Journal of the Mechanics and Physics of Solids*
903 138 (2020) 103919.
- 904 [4] F. D’Annibale, G. Rosi, A. Luongo, Controlling the limit-cycle of the ziegler column via a
905 tuned piezoelectric damper, *Mathematical Problems in Engineering* 2015 (2015) 1–9.
- 906 [5] F. Alouges, A. DeSimone, L. Giraldo, M. Zoppello, Can magnetic multilayers propel artificial
907 microswimmers mimicking sperm cells?, *Soft Robotics* 2 (2015) 117–128.
- 908 [6] H. Ziegler, *Principles of Structural Stability*, Lehr- und Handbücher der Ingenieurwissenschaften,
909 Birkhäuser Basel, 1977.
- 910 [7] X. Wang, C. Fan, S. Yang, Z. Yang, Y. Yang, M. Sinapius, Flutter instability characteristics
911 and mechanisms of ziegler double pendulum with arbitrary masses, stiffness and damping,
912 *Nonlinear Dynamics* 112 (2024) 20771–20792.

- 913 [8] O. Kirillov, F. Verhulst, Paradoxes of dissipation-induced destabilization or who opened
914 whitney's umbrella?, *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift*
915 *für Angewandte Mathematik und Mechanik* 90 (2010) 462–488.
- 916 [9] O. Kirillov, *Nonconservative stability problems of modern physics*, volume 14 of *De*
917 *Gruyter Studies in Mathematical Physics*, 2013.
- 918 [10] D. Bigoni, *Flutter from Friction in Solids and Structures*, 2018.
- 919 [11] L. Zhang, K. E. Peyer, B. J. Nelson, Artificial bacterial flagella: Fabrication and magnetic
920 control, *Nano Letters* 9 (2009) 3663–3667.
- 921 [12] A. T. Brown, W. C. K. Poon, Ionic effects in self-propelled pt-coated janus swimmers,
922 *Physical Chemistry Chemical Physics* 16 (2014) 2446–2458.
- 923 [13] H. Shen, S. Cai, Z. Wang, Z. Ge, W. Yang, Magnetically driven microrobots: Recent
924 progress and future development, *Materials & Design* 227 (2023) 111735.
- 925 [14] Y. Kim, G. A. Parada, S. Liu, X. Zhao, Ferromagnetic soft continuum robots, *Nature* 573
926 (2019) 82–87.
- 927 [15] R. Dreyfus, J. Baudry, M. L. Roper, M. Fermigier, H. A. Stone, J. Bibette, Microscopic
928 artificial swimmers, *Nature* 437 (2005) 862–865.
- 929 [16] F. Katzmeier, F. C. Simmel, Microrobots powered by concentration polarization elec-
930 trophoresis (cpep), *Nature Communications* 14 (2023).
- 931 [17] G. Rousseaux, R. Kofman, O. Minazzoli, The Maxwell-Lodge effect: significance of elec-
932 tromagnetic potentials in the classical theory, *The European Physical Journal D* 49 (2008)
933 249–256.
- 934 [18] O. Lodge, On an electrostatic field produced by varying magnetic induction, *Proceedings*
935 *of the Physical Society of London* 10 (1889) 116–128.
- 936 [19] J. Maruskin, A. Bloch, J. Marsden, D. Zenkov, A fiber bundle approach to the transposi-
937 tional relations in nonholonomic mechanics, *J. Nonlinear Sci.* 22 (2012) 431–461.
- 938 [20] A. Pastore, A. Giammarini, A. Grillo, Reconciling Kozlov's vakonomic method with the
939 traditional non-holonomic method: solution of two benchmark problems, *Acta mechanica*
940 (2024).
- 941 [21] C. Lanczos, *The Variational Principles of Mechanics*, Dover Publications, Inc. New York,
942 1970.
- 943 [22] L. A. Pars, *A Treatise on Analytical Dynamics*, Heinemann, London, 1965.
- 944 [23] J. Neimark, N. Fufaev, *Dynamics of Nonholonomic Systems*, American Mathematical So-
945 ciety, Rhode Island, 1972.
- 946 [24] F. Gantmacher, *Lectures in Analytical Mechanics*, MIR Publishers, Moscow, 1975.
- 947 [25] B. Felsager, *Geometry, particles, and fields*, Springer, Heidelberg, 1998.

- 948 [26] K. Haller, Quantum electrodynamics in the temporal gauge, *Phys. Rev. D* 36 (1987) 1830–
949 1838.
- 950 [27] G. Zampieri, Nonholonomic versus vakonomic dynamics, *J. Differ. Equ.* 163 (2000) 335–
951 347.
- 952 [28] A. Agúndez, D. García-Vallejo, E. Freire, Linear stability analysis of nonholonomic multi-
953 body systems, *International journal of mechanical sciences* 198 (2021) 106392.
- 954 [29] B. C. Kuo, *Automatic Control Systems*, 3rd ed., Prentice Hall, 1975.
- 955 [30] B. Sturmfels, *Solving Systems of Polynomial Equations*, Conference Board of the Mathe-
956 matical Sciences Regional Confe, Conference Board of the Mathematical Sciences, 2002.
- 957 [31] J. Sylvester, Xxiii. a method of determining by mere inspection the derivatives from two
958 equations of any degree, *The London, Edinburgh, and Dublin Philosophical Magazine and*
959 *Journal of Science* 16 (1840) 132–135.
- 960 [32] H. Woody, *Polynomial resultants*, 2016.
- 961 [33] J.-C. van der Meer, *The Hamiltonian Hopf Bifurcation*, volume 1160 of *Lecture Notes in*
962 *Mathematics*, Springer-Verlag, Berlin, Heidelberg, 1985.
- 963 [34] R. Mazrooei-Sebdani, E. Hakimi, Nondegenerate Hamiltonian Hopf bifurcations in $\omega : 3 :$
964 6 resonance ($\omega = 1$ or 2), *Regular and Chaotic Dynamics* 25 (2020) 522–536.
- 965 [35] A. Grillo, A. Pastore, S. Di Stefano, An Approach to Growth Mechanics based on the
966 Analytical Mechanics of Nonholonomic Systems, *J. Elast.* 157 (2025).
- 967 [36] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, volume 112 of *Applied Mathe-*
968 *matical Sciences*, 4 ed., Springer Cham, 2023.
- 969 [37] R. Xu, Q. Xu, A survey of recent developments in magnetic microrobots for micro-/nano-
970 manipulation, *Micromachines* 15 (2024) 468.
- 971 [38] D. B. Montgomery, *Solenoid Magnet Design: The Magnetic and Mechanical Aspects of*
972 *Resistive and Superconducting Systems*, Wiley-Interscience, New York, 1969.
- 973 [39] T. Xu, C. J. R. Duncan, P. Denham, B. H. Schaap, A. Kulkarni, D. Garcia, S. D. Ander-
974 son, P. Musumeci, R. J. England, Focusing of relativistic electron beams with permanent
975 magnetic solenoid, *Phys. Rev. Accel. Beams* 28 (2025) 082401.
- 976 [40] S. Martel, et al., Mri-based medical nanorobotics: the modulation of magnetic gradients
977 for propulsion and navigation of ferromagnetic core nanocapsules, *Applied Physics Letters*
978 90 (2007) 114105.
- 979 [41] The MathWorks, Inc., *MATLAB*, MathWorks, Natick, Massachusetts, 2024. Version
980 R2024b.
- 981 [42] L. Shampine, M. Reichelt, The MATLAB ODE Suite, *SIAM Journal on Scientific Com-*
982 *puting* 18 (1997).