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Measuring the distance between single random inputs and OWA operators

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ABSTRACT

Considered a random sample of fixed cardinality extracted from a population with unknown distribution, this paper deals with the ability of an Ordered Weighted Averaging (OWA) operator to approximate the distance between the values it assumes and the single observation of the sample. To this purpose, a measure of distance between random quantities recently introduced in the literature is considered, which takes into account both their mutual dependence and the shape of their distributions. Conditions are identified on the weights of the OWA operator that minimize this distance, or for which the different distances are ordered as the weights of two different operators vary. The paper also considers the case of operators defined as mixture of order statistics and, subsequently, the case of input values from populations with different distributions, showing conditions on these distributions that highlight the importance of the inputs in the value assumed by this distance.

1. Introduction and preliminaries

Aggregation functions, and Ordered Weighted Averaging (OWA) operators in particular, are widely used for fusing the information from different sources in different applications such as decision making, computer science, economics, finance, mathematics and statistics. In the literature, various techniques have been proposed for determining the weights to be assigned to the order statistics that make up these operators, depending on the application context in which they are used, see [1,2] or [3] for some examples in this regard. For instance, for different applications, symmetric weights that prioritize central order statistics are preferred (see [4]). This choice is particularly effective when the operator is used to synthesize the characteristics of observations from a population of unknown distribution.

In this context, it is particularly useful to determine how well the operator, or rather the choice of the weights that define it, can approximate the observed input values and their distribution. This paper addresses this issue, using a measure of distance between random variables recently introduced in the literature, described in detail below, which measures the average difference while also taking into account the dependence between them. In Section 2, this measure is used to formally show that, in the case of symmetric weights, the distance between the individual random input and the operator decreases as the weights increase centrally, as suggested by intuition. Conditions are then shown that lead to ordering between distances when operators with different weight assignments are considered, and examples are shown in which such ordering is not guaranteed. A similar analysis is carried out in Section 3, where, however, operators of a different nature are considered, still based on sample order statistics, but defined as mixtures of them. Finally,

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the last section considers the case where the inputs do not come from the same population, but each input has its own distribution, and the sample median is used as an aggregation function to synthesize the sample. For this case, conditions are identified on the individual input's distribution such that the measure of distance mentioned above decreases in value, in order to understand which components of the sample have the greatest effect on the sample median and how.

We now recall here some notions and properties that will be used in the following sections. The measure of distance we are going to consider to evaluate how the OWA operator is able to approximate the single observation has been recently introduced and studied in detail in the recent paper [5]. This index is essentially a generalization of the well-known Gini Mean Difference index, defined as the expected value of the absolute difference between two independent and identically distributed copies of the same variable X, which is used as a measure of variability of X or as a measure of income inequality in a population in economics and social sciences (see, for instance, Arnold and Sarabia [6]). The generalization considered in [5], simply named *bivariate Gini Mean Difference* (shortly $GMD(X,Y)$) refers instead to a vector (X,Y), where X and Y can have different distributions and can be dependent, and it is defined as the expected value of the absolute difference between X and Y, i.e. $GMD(X,Y)=E|X-Y|$ (if such an expectation exists). As shown in the mentioned paper, this measure falls within the category of "dependence-dispersion" indices, decreasing in value as dependence increases and increasing in value as the variability of X and Y increases. We refer the reader for more information about properties and examples of application of the bivariate Gini Mean Difference measure in various fields of research to [5].

Concerning the calculation of the bivariate Gini Mean Difference $GMD(X,Y)$, several alternative formulas are provided in [5]. Among them, one which will be often used in the paper is based on the copula describing the dependence between X and Y, thus the concept of copula of a random vector (for the bivariate case) is recalled here. Given the vector (X,Y) with marginal distributions F_X and F_Y and joint distribution function F, one can decompose F as

$$F(x,y)=C(F_X(x),F_Y(y)),\quad x,y\in\mathbb{R},$$

where $C:[0,1]^2\to[0,1]$ is a bivariate copula (see [5], or Nelsen [7], and references therein). The copula is unique whenever F_X and F_Y are continuous.

As shown by [5], the following formula can be used to compute $GMD(X,Y)$.

Proposition 1.1. *Let X and Y be two random variables with copula C and marginal distribution functions, respectively, F_X and F_Y. Then,*

$$E|X-Y|=\int_{t\in\mathbb{R}}\left[F_X(t)+F_Y(t)-2C(F_X(t),F_Y(t))\right]dt. \tag{1}$$

Given a random vector $\vec{X}=(X_1,\dots,X_n)$ of dimension n, not necessarily with independent and identically distributed components, we will consider through the paper its order statistics $X_{(1)},\dots,X_{(n)}$, which are, for each possible value of $\vec{X}$, the permutation of the components such that $X_{(1)}\le\cdots\le X_{(n)}$.

In the case of $\vec{X}$ having independent and identically distributed components with distribution function F, it is a well-known fact that the distribution function of the order statistics $X_{(i)}$ can be easily computed with the formula

$$F_{(i)}(t)=\sum_{k=i}^{n}\binom{n}{k}F(t)^k(1-F(t))^{n-k},\quad t\in\mathbb{R}.$$

However, in many cases we will consider $\vec{X}$ to be exchangeable, i.e. such that its distribution remains invariant when changing the order of its components. This assumption is weaker than considering independent and identically distributed components, and the distribution of the order statistics becomes more complicated, and sometimes impossible to calculate explicitly.

One of the most used approaches when working with index calculated through order statistics is to consider linear combinations of them. In this work, we will focus on the case of convex linear combinations of order statistics, also known as OWA operators in the field of aggregation functions, also known as convex L-statistics (see e.g. page 33 in [8], or page 18 in Rychlik [9]). They are defined by using a weight vector, which is just a real vector $\vec{w}\in[0,1]^n$ such that $\sum_{i=1}^n w_i=1$. For some properties and applications of OWA operators, we refer, among others, to [10,11] and [12]. Then, an OWA operator is defined as $\sum_{i=1}^n w_iX_{(i)}$. One of the most remarkable example of OWA operator is the sample median, defined as $X_{(\frac{n+1}{2})}$ if n is odd and as $\frac{1}{2}X_{(\frac{n}{2})}+\frac{1}{2}X_{(\frac{n}{2}+1)}$ if n is even.

Other combinations of order statistics that we are going to consider are mixtures. In this case, an auxiliary random variable Z taking values on $\{1,\dots,n\}$ is used to choose independently of $\vec{X}$ an order statistics. Formally, the mixture is defined as $\sum_{i=1}^n \mathbb{I}_{\{Z=i\}}X_{(i)}$, where $\mathbb{I}$ denotes the indicator function, i.e., $\mathbb{I}_{\{Z=i\}}=1$ if $Z=i$ and $\mathbb{I}_{\{Z=i\}}=0$ otherwise. Notice that the mixture equals $X_{(i)}$ with probability $P(Z=i)$ and a weight vector $\vec{w}=(w_1,\dots,w_n)$ can be defined by letting $w_i=P(Z=i)$ for any $i\in\{1,\dots,n\}$. We will refer to this vector as the probability vector of the mixing variable Z. It must be pointed out that mixtures of this kind are of particular interest in the field of reliability theory, since lifetimes of coherent systems having components with exchangeable lifetimes can be always expressed in terms of mixtures of the order statistics based on the component's lifetimes. See, e.g., Section 1.3.3 in Navarro [13] for details.

In some statement the peakedness stochastic order, recalled here, is mentioned.

Definition 1. Let X and Y be two random variables that are symmetric with respect to μ_X and μ_Y, respectively. Then, X is said to be smaller than Y in the *peakedness order*, denoted as $X\le_{peak}Y$, if,

$$|X-\mu_X|\le_{st}|Y-\mu_Y|,$$

i.e., if $P(|X-\mu_X|>t)\le P(|Y-\mu_Y|>t)$ for all $t\ge0$.

Further details on the peakedness stochastic order, and on the usual stochastic order $\leq_{st}$, can be found in [14]. Note that If $\mu_X = \mu_Y = \mu$, then one has $X \leq_{peak} Y$ if and only if $F_X(t) \leq F_Y(t)$ for any $t \leq \mu$ and $F_X(t) \geq F_Y(t)$ for any $t \geq \mu$. The peakedness order is a variability order, thus $X \leq_{peak} Y$ is interpreted as X having less dispersion than Y. We will also consider the inequalities $X \leq_{a.s.} Y$ and $X =_{a.s.} Y$, which denote, respectively, $P(X \leq Y) = 1$ and $P(X = Y) = 1$. Implicitly, we are assuming that X and Y are defined in the same probability space.

We now introduce a function strictly related to the bivariate Gini Mean Difference introduced above, which will be applied in Sections 3 and 4. This function is a generalization of the measure of dependence between two random variables known as the *medial correlation coefficient* or *Blomqvist's beta*, introduced in Blomqvist [15] (see also Nelsen [7], page 182). Given X and Y, it is defined as

$$\beta = P((X - m_X)(Y - m_Y) > 0) - P((X - m_X)(Y - m_Y) < 0),$$

where m_X and m_Y are the medians of X and Y, respectively. Dealing with variables X and Y having the same median $m = m_X = m_Y$, it can be seen that the latter formula can be rewritten as

$$\beta = 2[P(X \leq m, Y \leq m) + P(X > m, Y > m)] - 1,$$

so that the Blomqvist's beta takes greater values when the random variables are more likely to be, at the same time, above of below the common median. Replacing m with any real number, we obtain the generalized Blomqvist's beta function.

Definition 2. Let X and Y be two random variables and let $t \in \mathbb{R}$. Then, the generalized Blomqvist's beta function $\beta_t(X, Y)$ is defined as

$$\beta_t(X, Y) = 2[P(X \leq t, Y \leq t) + P(X > t, Y > t)] - 1.$$

It is not hard to verify that generalized Blomqvist's beta functions satisfy properties similar to those satisfied by the bivariate Gini Mean Difference and listed in [5]. For example, the following assertions hold:

- $\beta_t(X + s, Y + s) = \beta_{t-s}(X, Y)$ for any vector (X, Y) and $t, s \in \mathbb{R}$;
- $\beta_t(\lambda X, \lambda Y) = \beta_{t/\lambda}(X, Y)$ for any vector (X, Y) and $t, \lambda \in \mathbb{R}, \lambda \neq 0$;
- $\beta_t(X, Y) = 1 - 4[F(t) - \delta(F(t))]$ for all t when X and Y have the same distribution F and where $\delta(u) = C(u, u)$ is the diagonal section of the copula C of the vector (X, Y);
- $\beta_t(X_1, Y_1) \leq \beta_t(X_2, Y_2)$ for all $t \in \mathbb{R}$ if (X_1, Y_1) and (X_2, Y_2) are in the same Frechèt class, with corresponding copulas C_1 and C_2 such that $C_1(u, v) \leq C_2(u, v)$ for all $(u, v) \in [0, 1]^2$ (i.e., the *concordance order* among copulas implies the order of the corresponding beta functions).

In particular, the beta function, aside of being a way to compare the dependence between the involved random variables when they have the same marginal distributions, is in connection with the bivariate Gini Mean Difference $GMD(X, Y)$. For it, recall the formula (1) and notice that $C(F_X(t), F_Y(t))$ is just the distribution function of (X, Y). Denoting it as F, the latter equality can be rewritten as

$$E|X - Y| = \int_{t\in\mathbb{R}} \left(F_X(t) + F_Y(t) - 2F(t, t)\right) dt.$$

Therefore, it is possible to rewrite the integrand of the expression in terms of the generalized Blomqvist's beta functions as follows.

Proposition 1.2. *Let X and Y be two random variables. Then,*

$$E|X - Y| = \frac{1}{2} \int_{t\in\mathbb{R}} \left(1 - \beta_t(X, Y)\right) dt.$$

Proof. Denote as F_X and F_Y the distribution functions of, respectively, X and Y. In addition, denote as F the distribution function of (X, Y). Similarly, denote as $\bar{F}_X$, $\bar{F}_Y$ and $\bar{F}$ the associated survival functions. Using Proposition 1.1, we obtain

$$\begin{aligned}
E|X - Y| &= \int_{t\in\mathbb{R}} \left(F_X(t) + F_Y(t) - 2F(t, t)\right) dt \\
&= \int_{t\in\mathbb{R}} \left(F_X(t) + F_Y(t) - F(t, t) - 1 + \bar{F}_X(t) + \bar{F}_Y(t) - \bar{F}(t, t)\right) dt \\
&= \int_{t\in\mathbb{R}} \left(1 - F(t, t) - \bar{F}(t, t)\right) dt = \int_{t\in\mathbb{R}} \frac{1}{2}\left(1 - \beta_t(X, Y)\right) dt,
\end{aligned}$$

which concludes the proof. □

Then, if we have the two generalized Blomqvist's beta functions that are ordered for any possible real value $t \in \mathbb{R}$, we will have inequality (in the opposite direction) for the expected absolute differences, i.e., the GMDs. This will be specially relevant in the remainder of the paper.

Corollary 1.1. *Let X_1, X_2, Y_1 and Y_2 be four random variables. Then,*

$$\beta_t(X_1, X_2) \leq \beta_t(Y_1, Y_2), \forall t \in \mathbb{R} \implies E|X_1 - X_2| \geq E|Y_1 - Y_2|.$$

We end this section introducing a result that will be useful dealing with the sample median, whose proof can be found for example in [16]. Here, the median is defined as the order statistic $x_{(\frac{n+1}{2})}$ if n is odd and as any convex combination $\lambda x_{(\frac{n}{2})} + (1-\lambda)x_{(\frac{n}{2}+1)}$, with $\lambda \in [0,1]$, if n is even (see, e.g., [17]).

Proposition 1.3. *Let $\vec{x} \in \mathbb{R}^n$. Then, the median of $\vec{x}$, denoted as $M(\vec{x})$, fulfills*

$$M(\vec{x}) \in \arg\min_{y\in\mathbb{R}} \sum_{i=1}^{n} |x_i - y|.$$

Also, we point out that in the further sections a weight vector $\vec{w} = (w_1, \dots, w_n)$ is said to be *symmetric* if $w_i = w_{n-i+1}$ for any $i \in \{1, \dots, n\}$, and that the notation $\lfloor \alpha \rfloor$ is used for the lower integer part of α, while the notation $\lceil \alpha \rceil$ is used for the upper integer part of α, for $\alpha \in \mathbb{R}$.

2. OWA operators

This section is devoted to the analysis of the effects of the weights of an OWA operator on its distance in terms of bivariate Gini Mean Difference from the input corresponding to one observation from the unknown population. In particular, given an exchangeable random vector $\vec{X}$ and two weighting vectors $\vec{w}$ and $\vec{v}$, we seek for conditions such that $E|X_k - \sum_{i=1}^n w_i X_{(i)}| \le E|X_k - \sum_{i=1}^n v_i X_{(i)}|$, where X_k is a representative of the components of the sample, thus of the underlying population X. Note that, being X_k a component of the sample, dependence exists between X_k and $\sum_{i=1}^n v_i X_{(i)}$.

The first result on this direction deals with the case where the OWA operator consists of a single order statistic. For it, a preliminary result considering the sum of the absolute differences between fixed order statistics and all the components of the random vector is needed.

Lemma 2.1. *Let $\vec{X}$ be a random vector. Then,*

$$\sum_{i=1}^{n} \left|X_i - X_{(r)}\right| \le_{a.s.} \sum_{i=1}^{n} \left|X_i - X_{(s)}\right|$$

for any $r, s \in \{1, \dots, n\}$ such that $r, s \le \frac{n+1}{2}$ and $r \ge s$, or $r, s \ge \frac{n+1}{2}$ and $r \le s$.

Proof. Denote as $S_i = X_{(i+1)} - X_{(i)}$ with $i \in \{1, \dots, n-1\}$ the differences between consecutive order statistics. Then,

$$\begin{aligned}\sum_{i=1}^{n} \left|X_i - X_{(r)}\right| &= \sum_{i=1}^{n} \left|X_{(i)} - X_{(r)}\right| = \sum_{i=1}^{r-1} \left(X_{(r)} - X_{(i)}\right) + \sum_{i=r+1}^{n} \left(X_{(i)} - X_{(r)}\right) \\ &= \sum_{i=1}^{r-1}\sum_{j=i}^{r-1} S_j + \sum_{i=r+1}^{n}\sum_{j=r}^{i-1} S_j = \sum_{i=1}^{r-1} iS_i + \sum_{i=r}^{n-1}(n-i)S_i.\end{aligned}$$

Let $r \le \frac{n+1}{2}$ and $r > 1$. Then,

$$\sum_{i=1}^{n} \left|X_i - X_{(r)}\right| - \sum_{i=1}^{n} \left|X_i - X_{(r-1)}\right| = (r-1)S_{r-1} - (n-(r-1))S_{r-1} = (2(r-1)-n)S_{r-1}.$$

Since $r \le \frac{n+1}{2}$, then $(2(r-1)-n) \le 2\left(\frac{n+1}{2} - 1\right) - n = n-1-n < 0$. So, since S_i is non-negative for any $i \in \{1, \dots, n-1\}$, it holds that $\sum_{i=1}^n |X_i - X_{(r)}| \le_{a.s.} \sum_{i=1}^n |X_i - X_{(r-1)}|$ if $r \le \frac{n+1}{2}$ and $r > 1$.

Analogously, it can be proved that $\sum_{i=1}^n |X_i - X_{(r)}| \le_{a.s.} \sum_{i=1}^n |X_i - X_{(r+1)}|$ if $r \ge \frac{n+1}{2}$ and $r < n$, thus the result holds. □

Notice that the latter statement holds for any random vector. However, in order to have the inequality with respect to the expected absolute difference, exchangeability is needed.

Theorem 2.1. *Let $\vec{X}$ be an exchangeable random vector. Then,*

$$E\left|X_k - X_{(r)}\right| \le E\left|X_k - X_{(s)}\right|$$

for any $k, r, s \in \{1, \dots, n\}$ such that $r, s \le \frac{n+1}{2}$ and $r \ge s$, or $r, s \ge \frac{n+1}{2}$ and $r \le s$.

Proof. Since the random vector is exchangeable, it is clear that $E|X_k - X_{(r)}| = E|X_j - X_{(r)}|$ for a fixed $r \in \{1, \dots, n\}$ and any $k, j \in \{1, \dots, n\}$. Then, the result is a direct consequence of Lemma 2.1. □

The interpretation of the latter result is clear. The expected absolute difference $GMD(X_k, X_{(r)})$ is smaller considering more central order statistics. In particular, one has that whenever n is odd, the sample median $X_{(\frac{n+1}{2})}$ is the OWA operator that minimizes the expected distance between any observation X_k of the sample. A similar assertion can be obtained for the case where n is even, defining the sample median as any convex combination $\lambda X_{(\frac{n}{2})} + (1-\lambda)X_{(\frac{n}{2}+1)}$, with $\lambda \in [0,1]$, between the two more central order statistics. For it, first observe that from Proposition 1.3 one immediately has the following.

Lemma 2.2. *Let $(X_1, \ldots, X_n)$ be any random vector having an even number n of components, and let $\lambda \in [0,1]$. Then,*

$$\sum_{i=1}^{n} \left| X_i - X_{(r)} \right| \geq_{a.s.} \sum_{i=1}^{n} \left| X_i - \lambda X_{(\frac{n}{2})} - (1-\lambda) X_{(\frac{n}{2}+1)} \right|$$

for any $r \in \{1, \ldots, n\}$, with almost sure equality whenever $r = \frac{n}{2}$ or $r = \frac{n}{2} + 1$.

As a direct consequence of Lemma 2.2, one gets the following, whose proof is similar to that of Theorem 2.1 and therefore omitted.

Theorem 2.2. *Let $\vec{X}$ be an exchangeable random vector having an even number n of components, and let $\lambda \in [0,1]$. Then,*

$$E\left| X_k - X_{(r)} \right| \geq E\left| X_k - \lambda X_{(\frac{n}{2})} - (1-\lambda) X_{(\frac{n}{2}+1)} \right|$$

for any $k, r \in \{1, \ldots, n\}$.

As a particular case, one gets that if $\vec{X}$ is an exchangeable random vector having an even number n of components then the expected absolute difference $GMD(X_k, (X_{(n/2)} + X_{(n/2+1)})/2)$ is smaller or equal than any expected absolute difference $GMD(X_k, X_{(r)})$, with $r \in \{1, \ldots, n\}$.

A more general result, dealing with convex combination of order statistics, can be stated under the additional assumption that the components of the sample have symmetric distributions, with median m, and the weights are symmetric. Notice that it implies, see [18] and [4], that the Orness of the OWA operator (a measure of how close the OWA operator is to maximum operator) is exactly $\frac{1}{2}$. Also in this case a preliminary result is needed.

Lemma 2.3. *Let $\vec{w}$ and $\vec{v}$ be two symmetric weight vectors such that there exist $j \in \{1, \ldots, n\}$ with $j < \frac{n+1}{2}$ and $\epsilon > 0$ fulfilling*
1) $w_i = v_i$ if $i \notin \{j, j+1\}$,
2) $w_j = v_j - \epsilon$,
and fulfilling
4) $w_{j+1} = v_{j+1} + \epsilon$, if $j \neq \frac{n-1}{2}$,
or
4′) $w_{j+1} = v_{j+1} + 2\epsilon$, if $j = \frac{n-1}{2}$.
Then, for any continuous and exchangeable random vector $\vec{X}$ such that $\vec{X} - m\vec{1} =_{st} m\vec{1} - \vec{X}$ for some $m \in \mathbb{R}$, and for any $k \in \{1, \ldots, n\}$, it holds

$$E\left| X_k - \sum_{i=1}^{n} w_i X_{(i)} \right| \leq E\left| X_k - \sum_{i=1}^{n} v_i X_{(i)} \right|.$$

Proof. We give the proof of the statement when assumption (4) holds. The proof of the statement under condition (4′) is similar and therefore omitted. Firstly, notice that, as a direct consequence of the symmetry of the involved weight vectors, we have that condition 2) implies $w_{n+1-j} = v_{n+1-j} - \epsilon$ and condition 4) implies $w_{n-j} = v_{n-j} + \epsilon$.

For any $\lambda \in \mathbb{R}^+$, consider the random vector $\vec{Y}$ defined as

$$\vec{Y} =_{st} \left[\vec{X} \mid X_{(j+1)} - X_{(j)} + X_{(n-j)} - X_{(n+1-j)} = \lambda \right].$$

Notice that, since $\vec{X}$ is exchangeable and since $X_{(j+1)} - X_{(j)} + X_{(n+1-j)} - X_{(n-j)} = \lambda$ does not depend on the order of the components of $\vec{X}$, the vector $\vec{Y}$ is exchangeable as well. Moreover, considering $m\vec{1} = (m, \ldots, m)$,

$$\begin{aligned}
\vec{Y} - m\vec{1} &=_{st} [\vec{X} - m\vec{1} \mid (X_{(j+1)} - m) - (X_{(j)} - m) + (X_{(n-j)} - m) - (X_{(n+1-j)} - m) = \lambda] \\
&=_{st} [m\vec{1} - \vec{X} \mid (m - X_{(n-j)}) - (m - X_{(n+1-j)}) + (m - X_{(j+1)}) - (m - X_{(j)}) = \lambda] \\
&=_{st} [m\vec{1} - \vec{X} \mid X_{(j+1)} - X_{(j)} + X_{(n-j)} - X_{(n+1-j)} = \lambda] =_{st} m\vec{1} - \vec{Y}.
\end{aligned}$$

Suppose that $\lambda \geq 0$. denote with Y_k the k-th component of $\vec{Y}$ and with $Y_{(i)}$ the corresponding order statistics, and decompose the expectation as

$$\begin{aligned}
&E\left| Y_k - \sum_{i=1}^{n} v_i Y_{(i)} \right| = E\left| Y_k - \sum_{i=1}^{n} w_i Y_{(i)} + \epsilon\lambda \right| \\
&= P\left(Y_k \leq \sum_{i=1}^{n} w_i Y_{(i)} \right) \left(E\left[|Y_k - \sum_{i=1}^{n} w_i Y_{(i)} + \epsilon\lambda| \middle| Y_k \leq \sum_{i=1}^{n} w_i Y_{(i)} \right] \right) \\
&\quad + P\left(Y_k \geq \sum_{i=1}^{n} w_i Y_{(i)} \right) \left(E\left[|Y_k - \sum_{i=1}^{n} w_i Y_{(i)} + \epsilon\lambda| \middle| Y_k \geq \sum_{i=1}^{n} w_i Y_{(i)} \right] \right) \\
&\geq P\left(Y_k \leq \sum_{i=1}^{n} w_i Y_{(i)} \right) \left(E\left[|Y_k - \sum_{i=1}^{n} w_i Y_{(i)}| \middle| Y_k \leq \sum_{i=1}^{n} w_i Y_{(i)} \right] - \epsilon\lambda \right) \\
&\quad + P\left(Y_k \geq \sum_{i=1}^{n} w_i Y_{(i)} \right) \left(E\left[|Y_k - \sum_{i=1}^{n} w_i Y_{(i)}| \middle| Y_k \geq \sum_{i=1}^{n} w_i Y_{(i)} \right] + \epsilon\lambda \right)
\end{aligned}$$

$$= E\left|Y_k - \sum_{i=1}^{n} w_i Y_{(i)}\right| + \epsilon\lambda\left(P(Y_k \geq \sum_{i=1}^{n} w_i Y_{(i)}) - P(Y_k \leq \sum_{i=1}^{n} w_i Y_{(i)})\right).$$

Similarly, if $\lambda < 0$,

$$E\left|Y_k - \sum_{i=1} v_i Y_{(i)}\right| \geq E\left|Y_k - \sum_{i=1}^{n} w_i Y_{(i)}\right| - \epsilon\lambda\left(P\left(Y_k \geq \sum_{i=1}^{n} w_i Y_{(i)}\right) - P\left(Y_k \leq \sum_{i=1}^{n} w_i Y_{(i)}\right)\right).$$

Then, since $\vec{Y}$ is exchangeable, continuous and symmetric, and since $\vec{w}$ is symmetric, it holds that

$$P\left(Y_k \leq \sum_{i=1}^{n} w_i Y_{(i)}\right) = P\left(Y_k \geq \sum_{i=1}^{n} w_i Y_{(i)}\right) = \frac{1}{2}.$$

Therefore, for any possible value of λ it holds $E|Y_k - \sum_{i=1} v_i Y_{(i)}| \geq E|Y_k - \sum_{i=1}^{n} w_i Y_{(i)}|$. Since the inequalities above are true for any value of λ, then denoting as G the distribution function of $X_{(j+1)} - X_{(j)} + X_{(n+1-j)} - X_{(n-j)}$ one gets

$$E\left|X_k - \sum_{i=1}^{n} w_i X_{(i)}\right| = \int_{\mathbb{R}} E\left[\left|X_k - \sum_{i=1}^{n} w_i X_{(i)}\right| \middle| X_{(j+1)} - X_{(j)} + X_{(n+1-j)} - X_{(n-j)} = \lambda\right] dG(\lambda)$$

$$\leq \int_{\mathbb{R}} E\left[\left|X_k - \sum_{i=1}^{n} v_i X_{(i)}\right| \middle| X_{(j+1)} - X_{(j)} + X_{(n+1-j)} - X_{(n-j)} = \lambda\right] dG(\lambda) = E\left|X_k - \sum_{i=1}^{n} v_i X_{(i)}\right|,$$

i.e., the assertion. □

Thus, the following general result can be obtained from the latter.

Theorem 2.3. *Let $\vec{w}$ and $\vec{v}$ be two symmetric weight vectors such that $\sum_{i=1}^{j} w_i \leq \sum_{i=1}^{j} v_i$ for any $j \leq \frac{n+1}{2}$ (or, equivalently, $\sum_{i=1}^{j} w_i \geq \sum_{i=1}^{j} v_i$ for any $j \geq \frac{n+1}{2}$). Then, for any continuous and exchangeable random vector $\vec{X}$ such that $\vec{X} - m\vec{1} =_{st} m\vec{1} - \vec{X}$ for some $m \in \mathbb{R}$, and for any $k \in \{1, \ldots, n\}$, it holds*

$$E\left|X_k - \sum_{i=1}^{n} w_i X_{(i)}\right| \leq E\left|X_k - \sum_{i=1}^{n} v_i X_{(i)}\right|.$$

Proof. Consider an even n. Consider the sequence of weight vectors $\vec{\lambda}^1, \ldots, \vec{\lambda}^{\frac{n}{2}}$ such that the ith element of the l-th vector is defined as

- $\lambda_j^l = w_j$ if $j < l$ or $j > n+1-l$,
- $\lambda_l^l = \sum_{i=1}^{l-1}(v_i - w_i) + v_l$,
- $\lambda_{n+1-l}^l = \sum_{i=n-l+2}^{n}(w_i - v_i) + v_{n+1-j}$,
- $\lambda_j^l = v_j$ if $j > l$ and $j < n+1-l$,

for any $j \in \{1, \ldots, n\}$ and $l \in \{1, \ldots, \frac{n}{2}\}$. Consider two consecutive vectors $\vec{\lambda}_l$ and $\vec{\lambda}_{l+1}$ with $l \in \{1, \ldots, \frac{n}{2} - 1\}$ and define $\epsilon = \sum_{i=1}^{l}(v_i - w_i) = \sum_{i=n-l+1}^{n}(w_i - v_i)$, which is positive by hypothesis. Then,

- $\lambda_j^l = \lambda_j^{l+1}$ if $j \notin \{l, l+1, n-l, n-l+1\}$,
- $\lambda_l^{l+1} + \epsilon = w_l + \sum_{i=1}^{l}(v_i - w_i) = \sum_{i=1}^{l-1}(v_i - w_i) + v_l = \lambda_l^l$,
- $\lambda_{l+1}^{l+1} - \epsilon = v_{l+1} + \sum_{i=1}^{l}(v_i - w_i) - \sum_{i=1}^{l}(v_i - w_i) = v_{l+1} = \lambda_{l+1}^l$,
- $\lambda_{n-l+1}^{l+1} + \epsilon = \lambda_{n-l+1}^l$ by symmetry of $\vec{\lambda}^l$ and $\vec{\lambda}^{l+1}$,
- $\lambda_{n-l}^{l+1} - \epsilon = \lambda_{n-l}^l$ by symmetry of $\vec{\lambda}^l$ and $\vec{\lambda}^{l+1}$,

for any $l \in \{1, \ldots, \frac{n}{2} - 1\}$. Therefore, it is possible to apply Lemma 2.3 to the pairs of weight vectors $\vec{\lambda}_l$ and $\vec{\lambda}_{l+1}$ for any $l \in \{1, \ldots, \frac{n}{2} - 1\}$, which implies that the sequence $E|X_k - \sum_{i=1}^{n} \lambda_i^1 X_{(i)}|, \ldots, E|X_k - \sum_{i=1}^{n} \lambda_i^{\frac{n}{2}} X_{(i)}|$ is decreasing for any $k \in \{1, \ldots, n\}$. Finally, the result holds by observing that $\vec{\lambda}^1 = \vec{v}$ and $\vec{\lambda}^{\frac{n}{2}} = \vec{w}$. The proof for an odd n is almost the same, thus omitted. □

Remark 1. Notice that the conditions for the weights in Theorem 2.3 can be restated in terms of peakedness stochastic order between the discrete variables having such weights as probability mass functions. That is, defining two symmetric variables W and V centered in $(n+1)/2$ and assuming values in $\{1, 2, \ldots, n\}$, respectively with probabilities $\vec{w}$ and $\vec{v}$, if $W \leq_{peak} V$ then $E|X_k - \sum_{i=1}^{n} w_i X_{(i)}| \leq E|X_k - \sum_{i=1}^{n} v_i X_{(i)}|$ (notice that here W and V are constructed by using the weights of an OWA operator, so they are not associated with mixtures). In words, the expected value of the difference decreases as one increases the weights assigned, in the convex linear combination, to more central order statistics.

Note that as a consequence of Theorem 2.3 one gets that, if $\vec{X}$ is a continuous and exchangeable random vector, then, for any $k \in \{1, \ldots, n\}$,

$$E|X_k - X_{(n+1)/2}| \leq E|X_k - \bar{X}_n| \quad \text{if } n \text{ is odd},$$

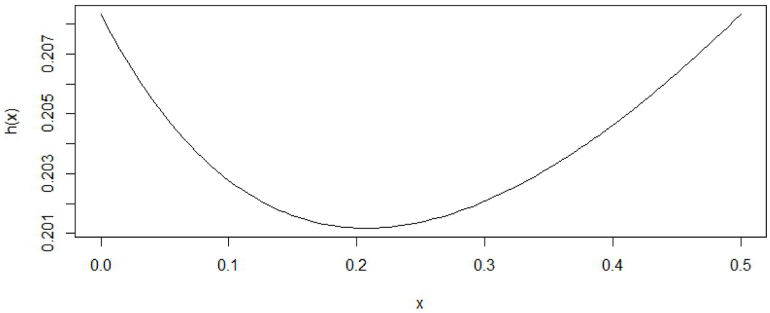

Fig. 1. Plot of the function $h(x) = \frac{1}{6}\left(1 + \frac{x^2+0.5^2}{2(x+0.5)}\right)$ for $x \in [0, 0.5]$.

and

$$E|X_k - (X_{(n/2)} + X_{(n/2+1)})/2| \leq E|X_k - \bar{X}_n| \quad \text{if } n \text{ is even,}$$

where $\bar{X}_n = \sum_{i=1}^n \frac{1}{n} X_{(i)} = \sum_{i=1}^n \frac{1}{n} X_i$ is the sample mean. It is also worth to mention that Theorem 2.3 gives conditions such that the OWA operators based on extreme values reductions are closer to the input values than OWA operators based on extreme values amplications. We refer the reader to [19] and [4] for more information in this regard.

A natural questions that arises is if the latter result holds also for not symmetric weight vectors. Unfortunately, this is not the case, as illustrated in the next example.

Example 1. Let X_1, X_2 and X_3 be iid standard uniform random variables, and consider the expected value of $E|X_1 - (w_1 X_{(1)} + w_2 X_{(2)} + w_3 X_{(3)})|$ for $w_1 + w_2 + w_3 = 1$. As proved in Appendix A, this expected value is equal to $\frac{1}{6}\left(1 + \frac{w_1^2+w_3^2}{2(w_1+w_3)}\right)$. Note that if $w_1 = w_3 = w$ then $E|X_1 - (w_1 X_{(1)} + w_2 X_{(2)} + w_3 X_{(3)})| = \frac{1}{6}\left(1 + \frac{w}{2}\right)$, which is increasing in w, confirming the assertion of Theorem 2.3. But fixing $w_3 = 0.5$ one can verify that $h(w_1) = \frac{1}{6}\left(1 + \frac{w_1^2+0.5^2}{2(w_1+0.5)}\right)$ is not monotone for $w_1 \in [0, 0.5]$ (see Fig. 1). Thus, if one considers for example the values of $w_1 = 0.3$, $w_2 = 0.2$ and $w_3 = 0.5$, the expected value is 0.20208, while if the values are $w_1 = 0.1$, $w_2 = 0.4$ and $w_3 = 0.5$, the expectation is 0.20278. Thus, even increasing the central weight, but not symmetrically, the expectation in the second case is smaller. Therefore, Theorem 2.3 is not longer true when removing the symmetry condition.

It is anyway possible to obtain valid results to compare expected absolute differences between any X_k and different OWA operators even when the weights and/or the X_i are not symmetric. An example is provided in the following Corollary 2.1, for which two preliminary results are needed.

Lemma 2.4. *Let $(X_1, \ldots, X_n)$ be a random vector of even dimension and let $\vec{w}$ and $\vec{v}$ be two weight vectors such that*
1) $v_i = \frac{w_i}{1+\delta_1+\delta_2}$ for any $i \in \{1, \ldots, \frac{n}{2}-1, \frac{n}{2}+2, \ldots, n\}$,
2) $v_{\frac{n}{2}} = \frac{w_{\frac{n}{2}}+\delta_1}{1+\delta_1+\delta_2}$,
3) $v_{\frac{n}{2}+1} = \frac{w_{\frac{n}{2}+1}+\delta_2}{1+\delta_1+\delta_2}$,
with $\delta_1, \delta_2 \geq 0$. Then it holds that

$$\sum_{j=1}^n \left| X_j - \sum_{i=1}^n w_i X_{(i)} \right| \geq_{a.s} \sum_{j=1}^n \left| X_j - \sum_{i=1}^n v_i X_{(i)} \right|.$$

Proof. Let $Y = \sum_{i=1}^n w_i X_{(i)}$. From Proposition 1.3 we have that

$$\sum_{j=1}^n \left| X_j - \sum_{i=1}^n w_i X_{(i)} \right| = \sum_{j=1}^n \left| X_j - Y \right| \geq_{a.s} \sum_{j=1}^n \left| X_j - \frac{\delta_1}{\delta_1+\delta_2} X_{(\frac{n}{2})} - \frac{\delta_2}{\delta_1+\delta_2} X_{(\frac{n}{2}+1)} \right|.$$

Observe now that the function $\psi(y) = \sum_{j=1}^n |x_j - y|$ is convex, being a sum of convex functions, so that it holds, for any value $\vec{x}$ of $\vec{X}$ and $0 \leq \lambda \leq 1$,

$$\begin{aligned}
&\psi\left(\lambda \sum_{i=1}^n w_i x_{(i)} + (1-\lambda)\left(\frac{\delta_1}{\delta_1+\delta_2} x_{(\frac{n}{2})} + \frac{\delta_2}{\delta_1+\delta_2} x_{(\frac{n}{2}+1)} \right) \right) \\
&\leq \lambda \psi\left(\sum_{i=1}^n w_i x_{(i)} \right) + (1-\lambda)\psi\left(\frac{\delta_1}{\delta_1+\delta_2} x_{(\frac{n}{2})} + \frac{\delta_2}{\delta_1+\delta_2} x_{(\frac{n}{2}+1)} \right) \leq \psi\left(\sum_{i=1}^n w_i x_{(i)} \right).
\end{aligned}$$

Denoted $\lambda = \frac{1}{1+\delta_1+\delta_2}$, let us define

$$v_i = \lambda w_i = \frac{w_i}{1+\delta_1+\delta_2} \text{ if } i \in \{1,\dots,\frac{n}{2}-1,\frac{n}{2}+2,\dots,n\},$$

$$v_{\frac{n}{2}} = \lambda w_i + (1-\lambda)\frac{\delta_1}{\delta_1+\delta_2} = \frac{w_{\frac{n}{2}}}{1+\delta_1+\delta_2} + \left(1 - \frac{1}{1+\delta_1+\delta_2}\right)\frac{\delta_1}{\delta_1+\delta_2}$$
$$= \frac{w_{\frac{n}{2}}}{1+\delta_1+\delta_2} + \frac{\delta_1+\delta_2}{1+\delta_1+\delta_2}\cdot\frac{\delta_1}{\delta_1+\delta_2} = \frac{w_{\frac{n}{2}}+\delta_1}{1+\delta_1+\delta_2},$$

and

$$v_{\frac{n}{2}+1} = \lambda w_i + (1-\lambda)\frac{\delta_2}{\delta_1+\delta_2} = \frac{w_{\frac{n}{2}+1}}{1+\delta_1+\delta_2} + \left(1 - \frac{1}{1+\delta_1+\delta_2}\right)\frac{\delta_2}{\delta_1+\delta_2}$$
$$= \frac{w_{\frac{n}{2}+1}}{1+\delta_1+\delta_2} + \frac{\delta_1+\delta_2}{1+\delta_1+\delta_2}\cdot\frac{\delta_2}{\delta_1+\delta_2} = \frac{w_{\frac{n}{2}+1}+\delta_2}{1+\delta_1+\delta_2}.$$

By the previous inequality it follows $\psi(\sum_{i=1}^n v_i x_{(i)}) \leq \psi(\sum_{i=1}^n w_i x_{(i)})$, and therefore the assertion. □

A similar statement holds for the odd case; its proof is similar to that of Lemma 2.4, and therefore omitted.

Lemma 2.5. *Let $(X_1,\dots,X_n)$ be a random vector of odd dimension and let $\vec{w}$ and $\vec{v}$ be two weight vectors such that*
1) $v_i = \frac{w_i}{1+\delta}$ for any $i \in \{1,\dots,\frac{n-1}{2},\frac{n+3}{2},\dots,n\}$,
2) $v_{\frac{n+1}{2}} = \frac{w_{\frac{n+1}{2}}+\delta}{1+\delta}$,
with $\delta \geq 0$. Then, it holds that

$$\sum_{j=1}^n \left|X_j - \sum_{i=1}^n w_i X_{(i)}\right| \geq_{a.s} \sum_{j=1}^n \left|X_j - \sum_{i=1}^n v_i X_{(i)}\right|$$

As a corollary of the two previous lemmas one gets the following statement, which does not require the weights to be symmetric. Its proof is similar to that of Theorem 2.1, i.e., based on the fact that that $E|X_j - \sum_{i=1}^n w_i X_{(i)}| = E|X_k - \sum_{i=1}^n w_i X_{(i)}|$ for any $j,k \in \{1,\dots,n\}$ when $\vec{X}$ is an exchangeable random vector, and therefore omitted.

Corollary 2.1. *Let $\vec{X}$ be an exchangeable random vector and let $\vec{w}$ and $\vec{v}$ be two weight vectors such that they satisfy the assumptions of Lemma 2.4 or of Lemma 2.5. Then for any $k \in \{1,\dots,n\}$, it holds*

$$E\left|X_k - \sum_{i=1}^n w_i X_{(i)}\right| \geq E\left|X_k - \sum_{i=1}^n v_i X_{(i)}\right|.$$

Further results can be proved for the cases where all the weights are null below or above the central position.

Theorem 2.4. *Let $\vec{w}$ and $\vec{v}$ be two weight vectors such that*
1) $w_i = 0$ for any $i > \frac{n+1}{2}$ and there exists $j < \frac{n}{2}$ and $\epsilon > 0$ such that $v_i = w_i$ if $i \notin \{j,j+1\}$, $v_j = w_j + \epsilon \leq 1$ and $v_{j+1} = w_{j+1} - \epsilon \geq 0$, or
2) $w_i = 0$ for any $i < \frac{n+1}{2}$ and there exists $j > \frac{n}{2}+1$ and $\epsilon > 0$ such that $v_i = w_i$ if $i \notin \{j-1,j\}$, $v_j = w_j + \epsilon \leq 1$ and $v_{j-1} = w_{j-1} - \epsilon \geq 0$.

Then, for any exchangeable random vector $\vec{X}$ and any $k \in \{1,\dots,n\}$,

$$E\left|X_k - \sum_{i=1}^n w_i X_{(i)}\right| \leq E\left|X_k - \sum_{i=1}^n v_i X_{(i)}\right|.$$

Proof. Suppose that condition (1) is satisfied and an even value for n. For any $\lambda \in \mathbb{R}^+$, consider the random vector $\vec{Y}$ with distribution

$$\vec{Y} =_{st} \left[\vec{X} \mid X_{(j+1)} - X_{(j)} = \lambda\right].$$

Notice that, since $\vec{X}$ is exchangeable and the condition $X_{(j+1)} - X_{(j)} = \lambda$ does not depend on the order of the components of $\vec{X}$, then $\vec{Y}$ is exchangeable as well. Then, since $\vec{Y}$ is exchangeable and $w_i = 0$ for any $i \geq \frac{n+1}{2}$, it holds

$$\sum_{i=1}^n w_i Y_{(i)} \leq_{a.s} Y_{(\frac{n}{2})}$$

from which it follows

$$P\left(Y_k \leq \sum_{i=1}^n w_i Y_{(i)}\right) \leq \frac{1}{2} = 1 - \frac{1}{2} \leq 1 - P\left(Y_k \leq \sum_{i=1}^n w_i Y_{(i)}\right) = P\left(Y_k > \sum_{i=1}^n w_i Y_{(i)}\right),$$

where Y_k denotes the k-th component of $\vec{Y}$ and $Y_{(i)}$ the corresponding order statistics.

Now, observe that

$$
\begin{aligned}
&E\left|Y_k-\sum_{i=1} v_i Y_{(i)}\right| = E\left[\left|Y_k-\sum_{i=1}^{n} w_i Y_{(i)}+\epsilon\lambda\right|\right] \\
&= P\left(Y_k \le \sum_{i=1}^{n} w_i Y_{(i)}\right) E\left[\left|Y_k-\sum_{i=1}^{n} w_i Y_{(i)}+\epsilon\lambda\right|\middle| Y_k \le \sum_{i=1}^{n} w_i Y_{(i)}\right] \\
&+ P\left(Y_k > \sum_{i=1}^{n} w_i Y_{(i)}\right) E\left[\left|Y_k-\sum_{i=1}^{n} w_i Y_{(i)}+\epsilon\lambda\right|\middle| Y_k > \sum_{i=1}^{n} w_i Y_{(i)}\right] \\
&\ge P\left(Y_k \le \sum_{i=1}^{n} w_i Y_{(i)}\right)\left(E\left[\left|Y_k-\sum_{i=1}^{n} w_i Y_{(i)}\right|\middle| Y_k \le \sum_{i=1}^{n} w_i Y_{(i)}\right]-\epsilon\lambda\right) \\
&+ P\left(Y_k > \sum_{i=1}^{n} w_i Y_{(i)}\right)\left(E\left[\left|Y_k-\sum_{i=1}^{n} w_i Y_{(i)}\right|\middle| Y_k > \sum_{i=1}^{n} w_i Y_{(i)}\right]+\epsilon\lambda\right) \\
&= E\left|Y_k-\sum_{i=1}^{n} w_i Y_{(i)}\right| + \epsilon\lambda\left[P\left(Y_k > \sum_{i=1}^{n} w_i Y_{(i)}\right)-P\left(Y_k \le \sum_{i=1}^{n} w_i Y_{(i)}\right)\right] \ge E\left|Y_k-\sum_{i=1}^{n} w_i Y_{(i)}\right|.
\end{aligned}
$$

The latter inequality is true for any value of λ, thus denoting as G the distribution function of $X_{(j+1)}-X_{(j)}$,

$$
\begin{aligned}
&E\left|X_k-\sum_{i=1}^{n} w_i X_{(i)}\right| = \int_{\mathbb{R}} E\left[\left|X_k-\sum_{i=1}^{n} w_i X_{(i)}\right|\middle| X_{(j+1)}-X_{(j)}=\lambda\right] dG(\lambda) \\
&\le \int_{\mathbb{R}} E\left[\left|X_k-\sum_{i=1}^{n} v_i X_{(i)}\right|\middle| X_{(j+1)}-X_{(j)}=\lambda.\right] dG(\lambda) = E\left|X_k-\sum_{i=1}^{n} v_i X_{(i)}\right|.
\end{aligned}
$$

The proof for case (2) and/or an odd n is almost the same, thus omitted. □

Notice that the latter result holds both for odd or even dimension. Moreover, it can be applied recursively to obtain a more general case.

Theorem 2.5. *Let $\vec{w}$ and $\vec{v}$ be two weight vectors such that*
1) $w_i = v_i = 0$ for any $i > \frac{n+1}{2}$ and $\sum_{i=1}^{j} w_i \le \sum_{i=1}^{j} v_i$ for any $j \in \{1,\dots,n\}$,
or
2) $w_i = v_i = 0$ for any $i < \frac{n+1}{2}$ and $\sum_{i=1}^{j} w_i \ge \sum_{i=1}^{j} v_i$ for any $j \in \{1,\dots,n\}$.
Then, for any exchangeable random vector $\vec{X}$ and any $k \in \{1,\dots,n\}$,

$$
E\left|X_k-\sum_{i=1}^{n} w_i X_{(i)}\right| \le E\left|X_k-\sum_{i=1}^{n} v_i X_{(i)}\right|.
$$

Proof. Consider case (1) and an even n. Consider the sequence of weight vectors $\vec{\lambda}^1,\dots,\vec{\lambda}^{\frac{n}{2}}$ such that the ith element of the lth vector is defined as

- $\lambda_j^l = w_j$ if $j < l$,
- $\lambda_l^l = \sum_{i=1}^{l-1}(v_i - w_i) + v_l$,
- $\lambda_j^l = v_j$ if $j > l$,

for any $j \in \{1,\dots,n\}$ and $l \in \{1,\dots,\frac{n}{2}\}$. Consider two consecutive vectors $\vec{\lambda}_l$ and $\vec{\lambda}_{l+1}$ and define $\epsilon = \sum_{i=1}^{l}(v_i - w_i)$, which is positive by hypothesis. Then,

- $\lambda_j^l = \lambda_j^{l+1} = w_j$ if $j < l$ or $j > l+1$,
- $\lambda_l^{l+1} + \epsilon = w_l + \sum_{i=1}^{l}(v_i - w_i) = \sum_{i=1}^{l-1}(v_i - w_i) + v_l = \lambda_l^l$
- $\lambda_{l+1}^{l+1} - \epsilon = v_{l+1} + \sum_{i=1}^{l}(v_i - w_i) - \sum_{i=1}^{l}(v_i - w_i) = v_{l+1} = \lambda_{l+1}^l$.

for any $l \in \{1,\dots,\frac{n}{2}-1\}$. Therefore, it is possible to apply Theorem 2.4 to the pairs of weight vectors $\vec{\lambda}_l$ and $\vec{\lambda}_{l+1}$ for any $l \in \{1,\dots,\frac{n}{2}-1\}$, which implies that the sequence $E|X_k - \sum_{i=1}^{n} \lambda_i^1 X_{(i)}|,\dots,E|X_k - \sum_{i=1}^{n} \lambda_i^{\frac{n}{2}} X_{(i)}|$ is decreasing for any $k \in \{1,\dots,n\}$. Finally, the result holds by noticing that $\vec{\lambda}^1 = \vec{v}$ and $\vec{\lambda}^{\frac{n}{2}} = \vec{w}$. The proof for case (2) and/or an odd n is almost the same, thus omitted. □

Example 2. As in Example 1, let X_1, X_2 and X_3 be iid standard uniform random variables, for which, letting $w_2 = 1 - w_1 - w_3$, it holds $E|X_1 - (w_1X_{(1)} + w_2X_{(2)} + w_3X_{(3)})| = \frac{1}{6}\left(1 + \frac{w_1^2+w_3^2}{2(w_1+w_3)}\right)$. If $w_3 = 0$, then $E|X_1 - (w_1X_{(1)} + w_2X_{(2)} + w_3X_{(3)})| = \frac{1}{6}\left(1 + \frac{w_1}{2}\right)$, which is clearly increasing in w_1, i.e., the expected value of the difference increases as the weight assigned to the more central order statistic decreases, confirming the statement of Theorem 2.5. But assuming $w_3 = 0.5$ so that that $E|X_1 - (w_1 X_{(1)} + w_2 X_{(2)} + w_3 X_{(3)})| = \frac{1}{6}\left(1 + \frac{w_1^2 + 0.5^2}{2(w_1+0.5)}\right)$, then the expected value is not monotone for $w_1 \in [0, 0.5]$, as seen in Fig. 1. In fact, in this case the assumptions of Theorem 2.5 are no more satisfied.

3. Mixtures

In this section we provide results dealing with comparisons in terms of bivariate Gini Mean Distances between any component X_k of the sample and different mixtures of the order statistics. In other terms, given an exchangeable random vector $\vec{X}$, we provide conditions that ensure the orderings between $GMD(X_k, Y_1)$ and $GMD(X_k, Y_2)$, where Y_1 and Y_2 are two different mixtures of the vector of order statistics $X_{(1)} \le \cdots \le X_{(n)}$. We therefore define the variables

$$Y_1 = \sum_{i=1}^{n} \mathbb{I}_{\{Z_1=i\}} X_{(i)} \quad \text{and} \quad Y_2 = \sum_{i=1}^{n} \mathbb{I}_{\{Z_2=i\}} X_{(i)}, \tag{2}$$

where Z_1 and Z_2 are two random variables independent of $\vec{X}$ and taking values on $\{1, \dots, n\}$, so that denoting

$$w_i = P(Z_1 = i) \quad \text{and} \quad v_i = P(Z_2 = i),$$

then Y_1 and Y_2 have distributions

$$F_1(t) = \sum_{i=1}^{n} w_i F_{(i)}(t) \quad \text{and} \quad F_2(t) = \sum_{i=1}^{n} v_i F_{(i)}(t), \quad t \in \mathbb{R}, \tag{3}$$

respectively.

On this aim we firstly provide sufficient conditions on the vectors of weights $\vec{w}$ and $\vec{v}$ such that $\beta_t(X_k, Y_1) \ge \beta_t(X_k, Y_2)$ and/or $\beta_t(X_k, Y_1) \le \beta_t(X_k, Y_2)$ for all t, so that, as a consequence of Corollary 1.1, one gets that $GMD(X_k, Y_1)$ and $GMD(X_k, Y_2)$ are ordered.

Note that the independence between the components of $\vec{X}$ is assumed in the first results of this section, thus they are in some sense weaker than those in the previous section, where only exchangeability is required. However, they provide stronger conclusions on the comparisons between the vectors (X_k, Y_1) and (X_k, Y_2), since they actually provide comparisons between generalized Blomqvist's beta functions $\beta_t(X_k, Y_1)$ and $\beta_t(X_k, Y_2)$ rather than only between $GMD(X_k, Y_1)$ and $GMD(X_k, Y_2)$.

Theorem 3.1. *Let $\vec{X}$ be a random vector with independent and identically distributed components, and let the variables Y_1 and Y_2 be defined as in (2), with distributions as in (3). If*

1) $\sum_{j=1}^{i} w_j \le \sum_{j=1}^{i} v_j$ for any $i \in \{1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$,
2) $\sum_{j=1}^{i} w_j \ge \sum_{j=1}^{i} v_j$ for any $i \in \{\lceil \frac{n+1}{2} \rceil, \dots, n-1\}$,

then

$$\beta_t(X_k, Y_1) \ge \beta_t(X_k, Y_2), \quad \textit{for all } t \in \mathbb{R} \textit{ and } k \in \{1, \dots, n\},$$

and

$$E|X_k - Y_1| \le E|X_k - Y_2| \quad \textit{for all} \;\; k \in \{1, \dots, n\}.$$

Proof. Let us denote with $X_{(j),n-1}$ the $j-th$ order statistic from a subset of cardinality $n-1$ of the components of $\vec{X}$. Observe that the comparison between the generalized Blomqvist's betas $\beta_t(X_k, Y_1)$ and $\beta_t(X_k, Y_2)$ is equivalent to the comparison between

$$P(X_k \le t, Y_1 \le t) + P(X_k > t, Y_1 > t)$$

and

$$P(X_k \le t, Y_2 \le t) + P(X_k > t, Y_2 > t).$$

Note that, denoting with F the distribution of X_k and with F_{Y_1} the distribution of Y_1, it holds

$$P(X_k \le t, Y_1 \le t) + P(X_k > t, Y_1 > t) = 2P(X_k \le t, Y_1 \le t) + 1 - F(t) - F_{Y_1}(t), \tag{4}$$

where

$$\begin{aligned} P(X_k \le t, Y_1 \le t) &= P\left(X_k \le t, \sum_{j=1}^{n} \mathbb{I}_{\{Z=j\}} X_{(j)} \le t\right) = \sum_{j=1}^{n} w_j P(X_k \le t, X_{(j)} \le t) \\ &= \sum_{j=1}^{n} w_j P(X_{(j)} \le t | X_k \le t) P(X_k \le t) = \sum_{j=1}^{n} w_j F(t) P(X_{(j-1):n-1} \le t) \\ &= \sum_{j=1}^{n} w_j F(t) \sum_{i=j-1}^{n-1} \binom{n-1}{i} F^i(t) \bar{F}^{n-1-i}(t) = \sum_{j=1}^{n} w_j \sum_{i=j}^{n} \binom{n-1}{i-1} F^i(t) \bar{F}^{n-i}(t). \end{aligned}$$

Then, (4) can be rewritten as

$$
\begin{aligned}
&2\sum_{j=1}^{n} w_j \sum_{i=j}^{n}\binom{n-1}{i-1}F^i(t)\bar{F}^{n-i}(t)+\bar{F}(t)-\sum_{j=1}^{n} w_j F_{(j)}(t)\\
&=2\sum_{j=1}^{n} w_j \sum_{i=j}^{n}\binom{n-1}{i-1}F^i(t)\bar{F}^{n-i}(t)+\bar{F}(t)-\sum_{j=1}^{n} w_j \sum_{i=j}^{n}\binom{n}{i}F^i(t)\bar{F}^{n-i}(t)\\
&=\bar{F}(t)+\sum_{j=1}^{n} w_j \sum_{i=j}^{n}\left[2\binom{n-1}{i-1}-\binom{n}{i}\right]F^i(t)\bar{F}^{n-i}(t)=\bar{F}(t)+\sum_{j=1}^{n} w_j \sum_{i=j}^{n}\binom{n-1}{i-1}\frac{2i-n}{i}\ F^i(t)\bar{F}^{n-i}(t)\\
&=\bar{F}(t)+\sum_{i=1}^{n}\binom{n-1}{i-1}\frac{2i-n}{i}\ F^i(t)\bar{F}^{n-i}(t)\Big(\sum_{j=1}^{i} w_j\Big).
\end{aligned}
$$

Similarly, for Y_2 we get

$$
P(X_k\le t,Y_2\le t)+P(X_k>t,Y_2>t)=\bar{F}(t)+\sum_{i=1}^{n}\binom{n-1}{i-1}\frac{2i-n}{i}\ F^i(t)\bar{F}^{n-i}(t)\Big(\sum_{j=1}^{i} v_j\Big).
$$

Then, we have

$$
\begin{aligned}
&P(X_k\le t,Y_1\le t)+P(X_k>t,Y_1>t)-P(X_k\le t,Y_2\le t)+P(X_k>t,Y_2>t)\\
&=\sum_{i=1}^{n-1}\Big(\sum_{j=1}^{i} w_j-\sum_{j=1}^{i} v_j\Big)\binom{n-1}{i-1}\frac{2i-n}{i}\ F^i(t)\bar{F}^{n-i}(t),
\end{aligned}
\tag{5}
$$

where the first sum is up to $n-1$ instead of n being both $\sum_{j=1}^n w_j$ and $\sum_{j=1}^n v_j$ equal to 1.

When n is odd, a sufficient condition such that the difference in (5) is positive, and then the corresponding difference between the generalized Blomqvist's betas $\beta_t(X_1,Y_1)-\beta_t(X_1,Y_2)$ is positive, is given by $\sum_{j=1}^i w_j\le\sum_{j=1}^i v_j$ for $i\in\{1,\dots,\frac{n-1}{2}\}$ and $\sum_{j=1}^i w_j\ge\sum_{j=1}^i v_j$ for $i\in\{\frac{n+1}{2},\dots,n-1\}$. In fact, in the first case the difference between the distribution functions of Z_1 and Z_2 is non-positive and multiplied by a non-positive coefficient giving a non-negative term in the sum, while in the second case both the terms are non-negative giving again a non-negative term.

When n is even, the term related to the index $i=\frac{n}{2}$ does not give any contribution and the sufficient conditions to have a non-negative sum are given by $\sum_{j=1}^i w_j\le\sum_{j=1}^i v_j$ for $i\in\{1,\dots,\frac{n}{2}-1\}$ and $\sum_{j=1}^i w_j\ge\sum_{j=1}^i v_j$ for $i\in\{\frac{n}{2}+1,\dots,n-1\}$.

Finally, based on Corollary 1.1 by the relation on the generalized Blomqvist's betas we get $E|X_k-Y_1|\le E|X_k-Y_2|$. □

Example 3. Let X_1, X_2 and X_3 be iid standard uniform random variables, and consider the expected value of $E|X_1-\sum_{i=1}^n \mathbb{1}_{\{Z=i\}}X_{(i)}|$ with $P(Z=i)=w_i$ and $w_1+w_2+w_3=1$. As also proved in Appendix A, this expected value is equal to $(w_1+w_3)/4+w_2/6$. Clearly, in this case the assumptions of Theorem 3.1 simply refers to the value assigned to w_2, and the expected value decreases as w_2 increases, confirming the assertion of the statement. Also note that, on the contrary of what happens considering L-Statistics, Z do not need to be symmetric, i.e., w_1 and w_3 can assume different values.

As a consequence of the previous theorem, by limiting to the cases in which Z_1 and Z_2 take just one single but different value with probability one, the variables Y_1 and Y_2 become specific order statistics and we have the following result.

Corollary 3.1. *Let $\vec{X}$ be a random vector with independent and identically distributed components. If n is odd, then*

$$
\beta_t\Big(X_k,X_{(\frac{n+1}{2})}\Big)\ge\beta_t\big(X_k,X_{(j)}\big),\quad \text{for all } t \text{ and } j\in\{1,\dots,n\}
$$

and then

$$
E\Big|X_k-X_{(\frac{n+1}{2})}\Big|\le E\Big|X_k-X_{(j)}\Big|,\quad \text{for } j\in\{1,\dots,n\}.
$$

If n is even, then

$$
\beta_t\Big(X_k,X_{(\frac{n}{2})}\Big)=\beta_t\Big(X_k,X_{(\frac{n}{2}+1)}\Big)\ge\beta_t\big(X_k,X_{(j)}\big),\quad \text{for all } t \text{ and } j\in\{1,\dots,n\}
$$

and then

$$
E\Big|X_k-X_{(\frac{n}{2})}\Big|=E\Big|X_k-X_{(\frac{n}{2}+1)}\Big|\le E\Big|X_k-X_{(j)}\Big|,\quad \text{for } j\in\{1,\dots,n\}.
$$

In the same setup of Theorem 3.1, by limiting our attention to variables Z_1 and Z_2 with mean $(n+1)/2$, we obtain a connection with the peakedness order.

Corollary 3.2. *Let $\vec{X}$ be a random vector with independent and identically distributed components. Let Z_1 and Z_2 be discrete random variables independent of $X_1,\dots,X_n$ taking values on $\{1,\dots,n\}$ with mean $(n+1)/2$. Let the variables Y_1 and Y_2 be defined as in (2), with distributions as in (3). Assume that $Z_1\le_{peak}Z_2$ Then*

$$
\beta_t(X_k,Y_1)\ge\beta_t(X_k,Y_2),\quad \text{for all } t \text{ and } k\in\{1,\dots,n\},
$$

and

$$E|X_k - Y_1| \le E|X_k - Y_2| \quad \textit{for all} \quad k \in \{1, \dots, n\}.$$

Proof. Since both the variables Z_1 and Z_2 have mean equal to $\frac{n+1}{2}$, the result follows by Theorem 3.1, whose assumptions are satisfied by the properties of the peakedeness order. □

Removing the assumption of independence, and assuming exchangeability as in the previous section, one can get results dealing with comparisons in terms only of the bivariate Gini Mean Differences, for both even or odd number of components.

Theorem 3.2. *Let $(X_1, \dots, X_n)$ be an exchangeable random vector of even dimension and let the variables Y_1 and Y_2 be defined as in (2), with distributions as in (3). Assume that*

1) $\sum_{i=1}^{\frac{n}{2}} w_i = \sum_{i=1}^{\frac{n}{2}} v_i$,
2) $\sum_{i=1}^{k} w_i \ge \sum_{i=1}^{k} v_i$ for any $k \in \{1, \dots, \frac{n}{2}\}$,
3) $\sum_{i=1}^{k} w_i \le \sum_{i=1}^{k} v_i$ for any $k \in \{\frac{n}{2}+1, \dots, n\}$.

Then, it holds that

$$E|X_k - Y_1| \ge E|X_k - Y_2| \quad \textit{for all} \quad k \in \{1, \dots, n\}.$$

Proof. Fix any $k \in \{1, \dots, n\}$. The expectations can be written as

$$E|X_k - Y_1| = \sum_{i=1}^{n} w_i E|X_k - X_{(i)}|, \qquad E|X_k - Y_2| = \sum_{i=1}^{n} v_i E|X_k - X_{(i)}|.$$

Now, define the sequence $\phi_1, \dots, \phi_{\frac{n}{2}}$ by letting

$$\phi_j = \sum_{i=j+1}^{\frac{n}{2}} (w_i - v_i) E|X_k - X_{(i)}| + \sum_{i=1}^{j} (w_i - v_i) E|X_k - X_{(j)}|.$$

if $j \in \{1, \dots, \frac{n}{2} - 1\}$ and

$$\phi_{\frac{n}{2}} = \sum_{i=1}^{\frac{n}{2}} (w_i - v_i) E\left|X_k - X_{(\frac{n}{2})}\right|.$$

Notice that $\phi_1 = \sum_{i=1}^{\frac{n}{2}} (w_i - v_i) E|X_k - X_{(i)}|$. Now, let us prove that $\phi_j \ge \phi_{j+1}$ for any $j \in \{1, \dots, \frac{n}{2} - 1\}$;

$$\begin{aligned}
\phi_j - \phi_{j+1} &= \sum_{i=j+1}^{\frac{n}{2}} (w_i - v_i) E|X_k - X_{(i)}| - \sum_{i=j+2}^{\frac{n}{2}} (w_i - v_i) E|X_k - X_{(i)}| \\
&\quad + \sum_{i=1}^{j} (w_i - v_i) E|X_k - X_{(j)}| - \sum_{i=1}^{j+1} (w_i - v_i) E|X_k - X_{(j+1)}| \\
&= (w_{j+1} - v_{j+1}) E|X_k - X_{(j+1)}| + \sum_{i=1}^{j} (w_i - v_i) E|X_k - X_{(j)}| - \sum_{i=1}^{j+1} (w_i - v_i) E|X_k - X_{(j+1)}| \\
&= \left(\sum_{i=1}^{j} (w_i - v_i) \right) \left(E|X_k - X_{(j)}| - E|X_k - X_{(j+1)}| \right).
\end{aligned}$$

The term $\sum_{i=1}^{j} (w_i - v_i)$ is positive by hypothesis (2), while the second one is positive due to Theorem 2.1. Similarly,

$$\begin{aligned}
\phi_{\frac{n}{2}-1} - \phi_{\frac{n}{2}} &= \sum_{i=\frac{n}{2}}^{\frac{n}{2}} (w_i - v_i) E|X_k - X_{(i)}| + \sum_{i=1}^{\frac{n}{2}-1} (w_i - v_i) E\left|X_k - X_{(\frac{n}{2}-1)}\right| - \sum_{i=1}^{\frac{n}{2}} (w_i - v_i) E\left|X_k - X_{(\frac{n}{2})}\right| = \\
&= \sum_{i=1}^{\frac{n}{2}-1} (w_i - v_i) \left(E\left|X_k - X_{(\frac{n}{2}-1)}\right| - E\left|X_k - X_{(\frac{n}{2})}\right| \right) \ge 0.
\end{aligned}$$

Therefore, the sequence $\phi_1, \dots, \phi_{\frac{n}{2}}$ is decreasing. Moreover, using hypothesis (1),

$$\phi_{\frac{n}{2}} = \sum_{i=1}^{\frac{n}{2}} (w_i - v_i) E\left|X_k - X_{(\frac{n}{2})}\right| = 0.$$

It is concluded that $\phi_1 = \sum_{i=1}^{\frac{n}{2}}(w_i - v_i)E|X_k - X_{(i)}| \geq 0$. Analogously, using hypothesis (3) rather than (2), it is possible to prove that $\sum_{i=\frac{n}{2}+1}^{n}(w_i - v_i)E|X_k - X_{(i)}| \geq 0$. Then,

$$\sum_{i=1}^{n} w_i E|X_k - X_{(i)}| \geq \sum_{i=1}^{n} v_i E|X_k - X_{(i)}|,$$

and the result holds. □

For the odd case, while being a little bit more difficult to prove, it is possible to drop condition 1).

Theorem 3.3. *Let $(X_1, \dots, X_n)$ be an exchangeable random vector of odd dimension and let the variables Y_1 and Y_2 be defined as in (2), with distributions as in (3). Assume that*

1) $\sum_{i=1}^{k} w_i \geq \sum_{i=1}^{k} v_i$ for any $k \in \{1, \dots, \frac{n-1}{2}\}$,
2) $\sum_{i=1}^{k} w_i \leq \sum_{i=1}^{k} v_i$ for any $k \in \{\frac{n+3}{2}, \dots, n\}$.

Then, it holds that

$$E|X_k - Y_1| \geq E|X_k - Y_2| \quad \text{for all} \;\; k \in \{1, \dots, n\}.$$

Proof. Fix any $k \in \{1, \dots, n\}$. Similarly as in the latter result, consider now the sequence $\phi_1, \dots, \phi_{\frac{n-1}{2}}$ defined as

$$\phi_j = \sum_{i=j+1}^{\frac{n-1}{2}}(w_i - v_i)E|X_k - X_{(i)}| + \sum_{i=1}^{j}(w_i - v_i)E|X_k - X_{(j)}| + d_1 E\left|X_k - X_{(\frac{n+1}{2})}\right|,$$

with $d_1 = -\sum_{i=1}^{\frac{n-1}{2}}(w_i - v_i)$. Proceeding as in the even case, the sequence $\phi_1, \dots, \phi_{\frac{n-1}{2}}$ is decreasing. Moreover,

$$\left(\sum_{i=1}^{\frac{n-1}{2}}(w_i - v_i) + d_1\right)E\left|X_k - X_{(\frac{n+1}{2})}\right| - \phi_{\frac{n-1}{2}} = \left(\sum_{i=1}^{\frac{n-1}{2}}(w_i - v_i)\right)\left(E\left|X_k - X_{(\frac{n+1}{2})}\right| - E\left|X_k - X_{(\frac{n-1}{2})}\right|\right) \leq 0.$$

Notice that, by definition of d_1, $\sum_{i=1}^{\frac{n-1}{2}}(w_i - v_i) + d_1 = 0$. Therefore, $\phi_j \geq 0$ for any $j \in \{1, \dots, \frac{n-1}{2}\}$. In particular,

$$\phi_1 = \sum_{i=1}^{\frac{n-1}{2}}(w_i - v_i)E|X_k - X_{(i)}| + d_1 E\left|X_k - X_{(\frac{n+1}{2})}\right| \geq 0.$$

Analogously, it also holds

$$\sum_{i=\frac{n+3}{2}}^{n}(w_i - v_i)E|X_k - X_{(i)}| + d_2 E\left|X_k - X_{(\frac{n+1}{2})}\right| \geq 0,$$

with $d_2 = -\sum_{i=\frac{n+3}{2}}^{n}(w_i - v_i)$. Therefore, using that $d_1 + d_2 = -\sum_{i=1, i\neq\frac{n+1}{2}}^{n}(w_i - v_i) = w_{\frac{n+1}{2}} - v_{\frac{n+1}{2}}$, it is concluded that

$$\begin{aligned} 0 &\leq \sum_{i=1}^{\frac{n-1}{2}}(w_i - v_i)E|X_k - X_{(i)}| + d_1 E|X_k - X_{(\frac{n+1}{2})}| + \sum_{i=\frac{n+3}{2}}^{n}(w_i - v_i)E|X_k - X_{(i)}| + d_2 E\left|X_k - X_{(\frac{n+1}{2})}\right| \\ &= \sum_{i=1}^{n}(w_i - v_i)E|X_k - X_{(i)}|, \end{aligned}$$

which completes the proof. □

4. Dependence between the median and the sample values

In previous sections we studied the distance of an OWA operator or a mixture of the order statistics from a single component of the initial random vector, i.e., of the sample, assuming that such components are equivalent (by imposing exchangeability or independence). In this section, we change the roles and fix a particular OWA operator, the sample median, and study its distance in terms of the bivariate Gini Mean Difference with respect to different components having different marginal distributions.

We choose the sample median since, as a consequence of the results of the latter sections, it minimizes the GMD, i.e., the expected absolute difference. In the following we will consider a random vector $\vec{X}$ in which all the components are independent and have a common median m. Intuition says that, if a component has less variability it should be closer to the central values, thus closer to the sample median, and to prove this intuition is the purpose of this section.

Let us start with an elementary lemma, whose proof directly follows by Equation (5.2.1) in [8].

Lemma 4.1. *Let $X_1, \dots, X_n$ be an odd number of independent random variables with median m. Then $X_{\left(\frac{n+1}{2}\right)}$ has median m.*

As commented before, the intuition says that smaller variability implies to be closer to the sample mean. Given two random variables X_1 and X_2 with the same median m, one can impose the condition

$$\begin{cases} F_1(t) \geq F_2(t) & \text{if } t > m, \\ F_1(t) \leq F_2(t) & \text{if } t < m, \end{cases}$$

which implies that the probability mass/density of X_2 is closer to the median m than the probability mass/density of X_1, thus X_2 has less variability.

Remark 2. Considering F_1 and F_2 as described above, the function $F_1(t) - F_2(t)$ only has one change of sign, moving from positive values for $t \leq m$ to negative values for $t \geq m$. Therefore, the latter conditions imply the increasing convex order (see Theorem 4.A.22 in [14]), the convex order when $E[X_1] = E[X_2]$ (see Theorem 3.A.44 in [14]) and the peakedness order if both X_1 and X_2 are symmetric (see Theorem 3.D.1 in [14]). It is also worth to mention that if X_1 and X_2 have the same median, then the dispersive stochastic order (see Section 3.B in [14]) trivially implies the condition above.

Under such conditions, the following result can be proved.

Theorem 4.1. *Let $X_1, \dots, X_n$ be an odd number of independent random variables with median m. Denote as F_1 and F_2 the distribution functions of, respectively, X_1 and X_2. Then,*

$$\begin{cases} F_1(t) \geq F_2(t) & \text{if } t > m \\ F_1(t) \leq F_2(t) & \text{if } t < m \end{cases} \implies \beta_t\left(X_1, X_{\left(\frac{n+1}{2}\right)}\right) \geq \beta_t\left(X_2, X_{\left(\frac{n+1}{2}\right)}\right) \quad \forall t \in \mathbb{R}$$

Proof. Consider the random vector $\vec{Y} = (X_3, \dots, X_n)$. Then, the probability $P(X_1 \leq t, X_{\left(\frac{n+1}{2}\right)} \leq t)$ can be written in terms of the distribution function of X_1, X_2 and distribution functions associated with the order statistics of $\vec{Y}$, denoted as $G_{(k)}$ with $k \in \{1, \dots, n-2\}$. In addition, denote as $G_{(0)}$ the function such that $G_{(0)}(t) = 1$ for any $t \in \mathbb{R}$, and observe that it holds

$$P(X_1 \leq t, X_{\left(\frac{n+1}{2}\right)} \leq t) = F_1(t)F_2(t)G_{\left(\frac{n-3}{2}\right)}(t) + F_1(t)(1 - F_2(t))G_{\left(\frac{n-1}{2}\right)}(t).$$

For $P(X_1 > t, X_{\left(\frac{n+1}{2}\right)} > t)$, proceed similarly but considering the survival functions. In this case, $\bar{G}_{(2)}$ when $n = 3$ denotes the constant function taking the value 1, since $\vec{Y}$ only has one component. One has

$$P(X_1 > t, X_{\left(\frac{n+1}{2}\right)} > t) = \bar{F}_1(t)\bar{F}_2(t)\bar{G}_{\left(\frac{n+1}{2}\right)}(t) + \bar{F}_1(t)(1 - \bar{F}_2(t))\bar{G}_{\left(\frac{n-1}{2}\right)}(t).$$

In addition, equivalent formulas can be derived for $(X_2, X_{\left(\frac{n+1}{2}\right)})$:

$$P(X_2 \leq t, X_{\left(\frac{n+1}{2}\right)} \leq t) = F_2(t)F_1(t)G_{\left(\frac{n-3}{2}\right)}(t) + F_2(t)(1 - F_1(t))G_{\left(\frac{n-1}{2}\right)}(t),$$
$$P(X_2 > t, X_{\left(\frac{n+1}{2}\right)} > t) = \bar{F}_2(t)\bar{F}_1(t)\bar{G}_{\left(\frac{n+1}{2}\right)}(t) + \bar{F}_2(t)(1 - \bar{F}_1(t))\bar{G}_{\left(\frac{n-1}{2}\right)}(t).$$

Now, compute the difference between the lower probabilities for each random vector.

$$\begin{aligned}
&P(X_1 \leq t, X_{\left(\frac{n+1}{2}\right)} \leq t) - P(X_2 \leq t, X_{\left(\frac{n+1}{2}\right)} \leq t) \\
&= F_1(t)F_2(t)G_{\left(\frac{n-3}{2}\right)}(t) + F_1(t)(1 - F_2(t))G_{\left(\frac{n-1}{2}\right)}(t) - F_2(t)F_1(t)G_{\left(\frac{n-3}{2}\right)}(t) - F_2(t)(1 - F_1(t))G_{\left(\frac{n-1}{2}\right)}(t) \\
&= (F_1(t)(1 - F_2(t)) - F_2(t)(1 - F_1(t)))G_{\left(\frac{n-1}{2}\right)}(t) = (F_1(t) - F_2(t))G_{\left(\frac{n-1}{2}\right)}(t).
\end{aligned}$$

The difference for the other probabilities has a similar expression:

$$P\left(X_1 > t, X_{\left(\frac{n+1}{2}\right)} > t\right) - P\left(X_2 > t, X_{\left(\frac{n+1}{2}\right)} > t\right) = \left(\bar{F}_1(t) - \bar{F}_2(t)\right)\bar{G}_{\left(\frac{n-1}{2}\right)}(t).$$

Therefore,

$$\begin{aligned}
&\beta_t\left(X_1, X_{\left(\frac{n+1}{2}\right)}\right) - \beta_t\left(X_2, X_{\left(\frac{n+1}{2}\right)}\right) = 2\left[(F_1(t) - F_2(t))G_{\left(\frac{n-1}{2}\right)}(t) + \left(\bar{F}_1(t) - \bar{F}_2(t)\right)\bar{G}_{\left(\frac{n-1}{2}\right)}(t)\right] \\
&= 2(F_1(t) - F_2(t))\left(G_{\left(\frac{n-1}{2}\right)}(t) - \bar{G}_{\left(\frac{n-1}{2}\right)}(t)\right).
\end{aligned}$$

Notice that, since $\vec{Y}$ has $n-2$ components, then $G_{\left(\frac{n-1}{2}\right)}$ and $\bar{G}_{\left(\frac{n-1}{2}\right)}$ are, respectively, the distribution and survival functions of the (sample) median of $\vec{Y}$.

If $t < m$, then $F_1(t) \leq F_2(t)$ by hyphotesis and $G_{\left(\frac{n-1}{2}\right)}(t) \leq \bar{G}_{\left(\frac{n-1}{2}\right)}(t)$ as a consequence of Lemma 4.1. Similarly, if $t > m$ then $F_1(t) \geq F_2(t)$ and $G_{\left(\frac{n-1}{2}\right)}(t) \geq \bar{G}_{\left(\frac{n-1}{2}\right)}(t)$. Finally, if $t = m$, then $F_1(t) = F_2(t)$ and $G_{\left(\frac{n-1}{2}\right)}(t) = \bar{G}_{\left(\frac{n-1}{2}\right)}(t)$. In any case, the latter difference is positive, thus the result holds. □

As a direct consequence, the ordering of the expected absolute differences is also obtained.

Corollary 4.1. *Let $X_1,\dots,X_n$ be an odd number of independent random variables with median m. Denote as F_1 and F_2 the distribution functions of, respectively, X_1 and X_2. Then,*

$$\begin{cases} F_1(t)\ge F_2(t) & \text{if } t>m \\ F_1(t)\le F_2(t) & \text{if } t<m \end{cases} \implies E\left|X_1 - X_{(\frac{n+1}{2})}\right| \le E\left|X_2 - X_{(\frac{n+1}{2})}\right|.$$

Proof. The result follows by applying Theorem 4.1 and Corollary 1.1. □

It remains to prove the even case. Unfortunately, Lemma 4.1 is not longer true when considering even random variables and $\frac{1}{2}X_{(\frac{n}{2})}+\frac{1}{2}X_{(\frac{n}{2}+1)}$ as the sample median. For example, the median of the sample median of 4 independent standard exponentially distributed random variables can be approximated by using simulation, being its value close to 0.74082. On the other hand, the median of a standard exponential random variable is $\log(2)\approx 0.6931$, thus different. Hence, the property stated in Lemma 4.1, that was crucial for the proof of Theorem 4.1, does not hold any longer for an even number of X_i. However, imposing symmetry to the random variables it is possible to prove a result involving the uniform mixture of the two central order statistics.

For it, first we need a result dealing with the uniform mixture of the central order statistics, thus dealing with the variable Y defined as

$$Y = ZX_{\left(\frac{n}{2}\right)} + (1-Z)X_{\left(\frac{n}{2}+1\right)},$$

where Z is independent of $X_1,\dots,X_n$ and such that $P(Z=0)=P(Z=1)=\frac{1}{2}$.

Theorem 4.2. *Let $X_1,\dots,X_n$ be an even number of symmetric independent random variables with median m. Consider the random variable Y to be a uniform mixture of $X_{\left(\frac{n}{2}\right)}$ and $X_{\left(\frac{n}{2}+1\right)}$. Denote as F_1 and F_2 the distribution functions of, respectively, X_1 and X_2. Then,*

$$\begin{cases} F_1(t)\ge F_2(t) & \text{if } t>m \\ F_1(t)\le F_2(t) & \text{if } t<m \end{cases} \implies \beta_t(X_1,Y)\ge \beta_t(X_2,Y) \quad \forall t\in\mathbb{R}$$

Proof. Let us start proving the result for $n=2$. Proceeding similarly as in Theorem 4.1, compute the different probabilities:

$$\begin{aligned}
&P(X_1\le t, X_{(1)}\le t) = F_1(t), \quad P(X_1\le t, X_{(2)}\le t) = F_1(t)F_2(t),\\
&P(X_1> t, X_{(1)}> t) = \bar F_1(t)\bar F_2(t), \quad P(X_1> t, X_{(2)}> t) = \bar F_1(t),\\
&P(X_2\le t, X_{(1)}\le t) = F_2(t), \quad P(X_2\le t, X_{(2)}\le t) = F_2(t)F_1(t),\\
&P(X_2> t, X_{(1)}> t) = \bar F_2(t)\bar F_1(t), \quad P(X_2> t, X_{(2)}> t) = \bar F_2(t).
\end{aligned}$$

Using that $\beta_t(X_i,Y)=\frac{1}{2}\beta_t(X_i,X_{(1)})+\frac{1}{2}\beta_t(X_i,X_{(2)})$, it is easy to see that $\beta_t(X_1,Y)=\beta_t(X_2,Y)$ for any $t\in\mathbb{R}$. Consider now the case $n\ge 4$;

$$\begin{aligned}
&P(X_1\le t, X_{\left(\frac{n}{2}\right)}\le t) = F_1(t)F_2(t)G_{(\frac{n}{2}-2)}(t) + F_1(t)(1-F_2(t))G_{(\frac{n}{2}-1)}(t),\\
&P(X_1\le t, X_{\left(\frac{n}{2}+1\right)}\le t) = F_1(t)F_2(t)G_{(\frac{n}{2}-1)}(t) + F_1(t)(1-F_2(t))G_{(\frac{n}{2})}(t),\\
&P(X_1> t, X_{\left(\frac{n}{2}\right)}> t) = \bar F_1(t)\bar F_2(t)\bar G_{(\frac{n}{2}+1)}(t) + \bar F_1(t)(1-\bar F_2(t))\bar G_{(\frac{n}{2})}(t),\\
&P(X_1> t, X_{\left(\frac{n}{2}+1\right)}> t) = \bar F_1(t)\bar F_2(t)\bar G_{(\frac{n}{2})}(t) + \bar F_1(t)(1-\bar F_2(t))\bar G_{(\frac{n}{2}-1)}(t),\\
&P(X_2\le t, X_{\left(\frac{n}{2}\right)}\le t) = F_2(t)F_1(t)G_{(\frac{n}{2}-2)}(t) + F_2(t)(1-F_1(t))G_{(\frac{n}{2}-1)}(t),\\
&P(X_2\le t, X_{\left(\frac{n}{2}+1\right)}\le t) = F_2(t)F_1(t)G_{(\frac{n}{2}-1)}(t) + F_2(t)(1-F_1(t))G_{(\frac{n}{2})}(t),\\
&P(X_2> t, X_{\left(\frac{n}{2}\right)}> t) = \bar F_2(t)\bar F_1(t)\bar G_{(\frac{n}{2}+1)}(t) + \bar F_2(t)(1-\bar F_1(t))\bar G_{(\frac{n}{2})}(t),\\
&P(X_2> t, X_{\left(\frac{n}{2}+1\right)}> t) = \bar F_2(t)\bar F_1(t)\bar G_{(\frac{n}{2})}(t) + \bar F_2(t)(1-\bar F_1(t))\bar G_{(\frac{n}{2}-1)}(t),
\end{aligned}$$

Now, notice that $\beta_t(X_i,Y)=\frac{1}{2}\beta_t(X_1,X_{(\frac{n}{2})})+\frac{1}{2}\beta_t(X_1,X_{(\frac{n}{2}+1)})$ with $i\in\{1,2\}$, so one can write

$$\beta_t(X_1,Y)-\beta_t(X_2,Y) = \frac{1}{2}(F_1(t)-F_2(t))\left(G_{(\frac{n-2}{2})}(t)+G_{(\frac{n-2}{2}+1)}(t)-\bar G_{(\frac{n-2}{2})}(t)-\bar G_{(\frac{n-2}{2}+1)}(t)\right).$$

The functions $\frac{1}{2}\left(G_{(\frac{n-2}{2})}(t)+G_{(\frac{n-2}{2}+1)}(t)\right)$ and $\frac{1}{2}\left(\bar G_{(\frac{n-2}{2})}(t)+\bar G_{(\frac{n-2}{2}+1)}(t)\right)$ are the distribution and survival function of the uniform mixture of the central order statistics of $(X_3,\dots,X_n)$, which has dimension $n-2$. Moreover, it is clear that, since $X_3,\dots,X_n$ are symmetric with median m, this mixture also has median m.

If $t<m$, then $F_1(t)\le F_2(t)$ by hyphotesis and $G_{(\frac{n-2}{2})}(t)+G_{(\frac{n-2}{2}+1)}(t)\le \bar G_{(\frac{n-2}{2})}(t)+\bar G_{(\frac{n-2}{2}+1)}(t)$. Similarly, if $t>m$ then $F_1(t)\ge F_2(t)$ and $G_{(\frac{n-2}{2})}(t)+G_{(\frac{n-2}{2}+1)}(t)\ge \bar G_{(\frac{n-2}{2})}(t)+\bar G_{(\frac{n-2}{2}+1)}(t)$. Finally, if $t=m$, then $F_1(t)=F_2(t)$ and $G_{(\frac{n-2}{2})}(t)+G_{(\frac{n-2}{2}+1)}(t)=\bar G_{(\frac{n-2}{2})}(t)+\bar G_{(\frac{n-2}{2}+1)}(t)$. In any case, the latter difference is positive, thus the result holds. □

To move from the mixture to the proper sample median, we now need a preliminary result that involves two consecutive order statistics.

Lemma 4.2. *Let $X_1, \dots, X_n$ be an even number of random variables. Consider the random variable $Y = ZX_{(j)} + (1 - Z)X_{(j+1)}$ with $j \in \{1, \dots, n-1\}$ and with Z being a random variable such that it is independent of $X_1, \dots, X_n$ and $P(Z = 1) = \lambda$, $P(Z = 0) = 1 - \lambda$. Then, for any $k \in \{1, \dots, n\}$,*

$$E|X_k - Y| = E\Big|X_k - \lambda X_{(j)} - (1-\lambda)X_{(j+1)}\Big|.$$

Proof. The expectation $E|X_k - Y|$ can be expressed as

$$E|X_k - Y| = P(Z=0)E\big[|X_k - Y| \mid Z=0\big] + P(Z=1)E\big[|X_k - Y| \mid Z=1\big] = \lambda E\Big|X_k - X_{(j)}\Big| + (1-\lambda)E\Big|X_k - X_{(j+1)}\Big|.$$

On the other hand, for any possible value of $X_1, \dots, X_n$, there exists $i \in \{1, \dots, n\}$ such that $X_k = X_{(i)}$. Therefore,

$$P\big(X_k \in \big(X_{(j)}, X_{(j+1)}\big)\big) = 0.$$

If $X_k \ge X_{(j+1)}$, then $\Big|X_k - \lambda X_{(j)} - (1-\lambda)X_{(j+1)}\Big| = X_k - \lambda X_{(j)} - (1-\lambda)X_{(j+1)} = \lambda\Big|X_k - X_{(j)}\Big| + (1-\lambda)\Big|X_k - X_{(j+1)}\Big|$. Proceeding similarly for the case $X_k \le X_{(j)}$, it holds

$$E\Big|X_k - \lambda X_{(j)} - (1-\lambda)X_{(j)}\Big| = \lambda E\Big|X_1 - X_{(j)}\Big| + (1-\lambda)E\Big|X_1 - X_{(j+1)}\Big|,$$

thus the result holds. □

Notice that the latter result is not true when considering two non-consecutive order statistics. If $\vec{X} =_{a.s.} (1,2,3,4)$, then $E|X_2 - 0.5X_{(1)} - 0.5X_{(4)}| = 0.5$, while if one considers a random variable Z such that $P(Z=0) = P(Z=1) = 0.5$, we have that $E|X_2 - ZX_{(1)} - (1-Z)X_{(4)}| = 1.5$. However, it is sufficient for the sample median in the even case.

Theorem 4.3. *Let $X_1, \dots, X_n$ be an even number of symmetric independent random variables with median m. Denote as F_1 and F_2 the distribution functions of, respectively, X_1 and X_2. Then,*

$$\begin{cases} F_1(t) \ge F_2(t) & \text{if } t > m \\ F_1(t) \le F_2(t) & \text{if } t < m \end{cases} \implies E\left|X_1 - \frac{X_{(\frac{n}{2})} + X_{(\frac{n}{2}+1)}}{2}\right| \le E\left|X_2 - \frac{X_{(\frac{n}{2})} + X_{(\frac{n}{2}+1)}}{2}\right|$$

Proof. Apply Theorem 4.2 and Corollary 1.1 to obtain the inequality for the uniform mixture of the central order statistics. Then, use Lemma 4.2 to reach the inequality of the result. □

Unfortunately, latter result is not longer true if one removes independence or the common median between the involved random variables. Let us provide an example in this regard.

Example 4. Let X_1, X_2, X_3 and X_4 be four Gaussian random variables with mean 0 and standard deviations, respectively, 1, 2, 3 and 4. By simulation, it is possible to compute the absolute difference of the components and their sample median, obtaining

$$E\left|X_1 - \frac{X_{(2)} + X_{(3)}}{2}\right| \approx 1.026, \qquad E\left|X_2 - \frac{X_{(2)} + X_{(3)}}{2}\right| \approx 1.344,$$
$$E\left|X_3 - \frac{X_{(2)} + X_{(3)}}{2}\right| \approx 1.910, \qquad E\left|X_4 - \frac{X_{(2)} + X_{(3)}}{2}\right| \approx 2.649,$$

which behavior coincides with the one stated in Theorem 4.3. However, if one introduces dependence between the random variables, the inequality is not longer true. For instance, consider the random vector $\vec{Y} = (X_1, X_4, X_4, X_4)$. In this case, we have that the median is always X_4, thus $E\left|X_4 - \frac{Y_{(2)}+Y_{(3)}}{2}\right| = 0$, while it is clear that $E\left|X_1 - \frac{Y_{(2)}+Y_{(3)}}{2}\right| > 0$. In addition, if one drops the condition of having the same median, the inequality does not hold either. Consider the random vector $\vec{Z} = (X_1 + 10, X_2, X_3, X_4)$, for which it is possible to compute by simulation

$$E\left|X_1 + 10 - \frac{Z_{(2)} + Z_{(3)}}{2}\right| \approx 8.704, \qquad E\left|X_4 - \frac{Z_{(2)} + Z_{(3)}}{2}\right| \approx 3.460.$$

5. Concluding remarks

In this paper, several results stating inequalities between the Gini Mean Difference of the components of a random vector and OWA operators and mixtures of the order statistics associated with the same random vector are provided. These results a have direct relation with different topics of applications.

First, among the possible applications of these results, we can consider their use in Statistics as a measure of distance between a single individual in a population, of which a sample is available, and a sample statistic (the OWA operator) that summarizes and represents the individuals in that population, or summarizes their centrality characteristics. In this direction, the given results provide inequalities considering two different statistics of this kind, thereby defining criteria that lead to an appropriate choice of statistics to use.

Related to mixed and coherent systems, the results considering mixtures allow to identify when the behavior of the system is similar to the one of one of its components. This could be relevant for the construction of experiments for the empirical study of reliability, since studying the behavior of an isolated component, if it is similar to the one of the complete system, is simpler and less expensive.

It is also possible to find a link between the here-presented results and the achievement of consensus in some Decision Making problems (see, for instance, García-Zamora et al. [20]). In this sense, the results in Sections 2 and 3 permit the identification of decisions that are closer to the ones initially proposed by the decision makers when we consider, respectively, a convex linear combination of the proposals and the random choice of one of them. The inequalities in Section 4 also have a simple interpretation, they allow to identify decision makers that will be further from the aggregated value, thus less happy with the final decision.

Finally, some of the inequalities can be applied for testing some complex hyphotesis. Suppose that we have a random vector $\vec{X} = (X_1, \dots, X_n)$ with independent components with common median m and that we have a collection of independent observations of $\vec{X}$. Denoting as F_1 and F_2 the distribution functions of, respectively X_1 and X_2, we might want to test the condition

$$\begin{cases} F_1(t) \geq F_2(t) & \text{if } t > m, \\ F_1(t) \leq F_2(t) & \text{if } t < m, \end{cases}$$

using the observations. This is a condition for which it does not exist a specific test in the literature. However, applying Theorem 4.1 if n is odd, or Theorem 4.2 adding the assumption of symmetry if n is even, we have that the latter condition implies an inequality between expectations of quantities that can be easily computed for each observed value of $\vec{X}$. This inequality can be easily tested by using, for instance, t-tests (see Section 10.4 in [21]) to its difference. If the inequality is rejected, so is the initial hyphotesis. This is especially relevant given the connection of the condition with different variability stochastic orders, see Remark 2.

CRediT authorship contribution statement

Juan Baz: Writing – original draft, Investigation, Conceptualization; **Francesco Buono:** Writing – original draft, Investigation, Conceptualization; **Franco Pellerey:** Writing – original draft, Methodology, Conceptualization.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Expected absolute difference of 3 uniform random variables

Let us compute $E|X_1 - w_1X_{(1)} - w_2X_{(2)} - w_3X_{(3)}|$, where X_1, X_2 and X_3 are iid standard uniform random variables, $w_1, w_2, w_3 \geq 0$ and $w_1 + w_2 + w_3 = 1$. There are three possibilities, each of them with probability $\frac{1}{3}$:

1. $X_1 = X_{(1)}$. In this case, the quantity $w_1X_{(1)} + w_2X_{(2)} + w_3X_{(3)}$ is always greater than X_1, thus $X_1 - w_1X_{(1)} - w_2X_{(2)} - w_3X_{(3)}$ is always negative. Notice that, since (X_1, X_2, X_3) is exchangeable, the vector $(X_{(1)}, X_{(2)}, X_{(3)})$ is independent of the event $X_1 = X_{(1)}$. Then, since X_1, X_2, X_3 are independent standard uniforms, $X_{(1)}$ has distribution $B(1,3)$, $X_{(2)}$ has distribution $B(2,2)$ and X_3 has distribution $B(3,1)$. Then, just use the formula for the expectation of beta distributions:

$$E\left[|X_1 - w_1X_{(1)} - w_2X_{(2)} - w_3X_{(3)}\middle| X_1 = X_{(1)}\right] =$$
$$= (w_1 - 1)E[X_{(1)}] + w_2E[X_{(2)}] + w_3E[X_{(3)}] = \frac{1}{4}(w_1 - 1) + \frac{1}{2}w_2 + \frac{3}{4}w_3.$$

2. $X_1 = X_{(2)}$. The are two subcases. Assume that $X_{(2)} \leq w_1X_{(1)} + w_2X_{(2)} + w_3X_{(3)}$, which is equivalent to $X_{(2)} \leq \frac{w_1}{1-w_2}X_{(1)} + \frac{w_3}{1-w_2}X_{(3)}$. It is clear that $\frac{w_1}{1-w_2}X_{(1)} + \frac{w_3}{1-w_2}X_{(3)}$ is smaller than $X_{(3)}$ and greater than $X_{(1)}$. Then, the integral associated with the expectation

over that part of the unit cube is:

$$\int_0^1 \int_0^x \int_z^{\frac{w_1}{1-w_2}z+\frac{w_3}{1-w_2}x} \left(w_1 z + (w_2-1)y + w_3 x\right)3!\,dy\,dz\,dx =$$

$$= 6\int_0^1 \int_0^x \left(w_1 zy + \frac{1}{2}(w_2-1)y^2 + w_3 xy\right)\Big|_{y=z}^{y=\frac{w_1}{1-w_2}z+\frac{w_3}{1-w_2}x} dz\,dx =$$

$$= 6\int_0^1 \int_0^x \left[\frac{w_1 w_3}{1-w_2}(zx - z^2) + \frac{1}{2}(w_2-1)\left(\left(\frac{w_1}{1-w_2}z + \frac{w_3}{1-w_2}x\right)^2 - z^2\right) + \frac{w_3^2}{1-w_2}(x^2 - zx)\right] dz\,dx =$$

$$= 6\int_0^1 \int_0^x \left[\frac{w_1 w_3}{1-w_2}(zx - z^2) + \frac{1}{2}(w_2-1)\left(\frac{w_1^2 z^2}{(1-w_2)^2} + \frac{2w_1 w_3 zx}{(1-w_2)^2} + \frac{w_3^2 x^2}{(1-w_2)^2} - z^2\right) + \frac{w_3^2}{1-w_2}(x^2 - zx)\right] dz\,dx =$$

$$= 6\int_0^1 \int_0^x \left[\frac{w_1 w_3 - w_1 w_3 - w_3^2}{(1-w_2)} zx + \frac{-2w_1 w_3 - w_1^2 + (1-w_2)^2}{2(1-w_2)} z^2 + \frac{-w_3^2 + 2w_3^2}{2(1-w_2)} x^2\right] dz\,dx =$$

$$= 6\int_0^1 \int_0^x \left[-\frac{w_3^2}{(1-w_2)} zx + \frac{w_3^2}{2(1-w_2)} z^2 + \frac{w_3^2}{2(1-w_2)} x^2\right] dz\,dx =$$

$$= 6\int_0^1 \left[-\frac{w_3^2}{2(1-w_2)} x^3 + \frac{w_3^3}{6(1-w_2)} x^3 + \frac{w_3^2}{2(1-w_2)} x^3\right] dx = \frac{6}{4}\left(-\frac{w_3^2}{2(1-w_2)} + \frac{w_2^2}{6(1-w_2)} + \frac{w_3^2}{2(1-w_2)}\right) =$$

$$= \frac{w_3^2}{4(1-w_2)}.$$

If $X_{(2)} \geq w_1 X_{(1)} + w_2 X_{(2)} + w_3 X_{(3)}$, proceeding similarly as above it is possible to obtain the similar value $\frac{w_1^2}{4(1-w_2)}$.

3. $X_1 = X_{(3)}$. Using the same idea as in the case $X_1 = X_{(1)}$, one obtains

$$E\left[|X_1 - w_1 X_{(1)} - w_2 X_{(2)} - w_3 X_{(3)}\middle| X_1 = X_{(3)}\right] = -\frac{1}{4}w_1 - \frac{1}{2}w_2 + \frac{3}{4}(1 - w_3).$$

Multipling by $\frac{1}{3}$ each term and summing one gets

$$E|X_1 - w_1 X_{(1)} - w_2 X_{(2)} - w_3 X_{(3)}| =$$
$$= \frac{1}{3}\left(\frac{1}{4}(w_1 - 1) + \frac{1}{2}w_2 + \frac{3}{4}w_3 + \frac{w_3^2}{4(1-w_2)} + \frac{w_1^2}{4(1-w_2)} - \frac{1}{4}w_1 - \frac{1}{2}w_2 + \frac{3}{4}(1-w_3)\right) = \frac{1}{6}\left(1 + \frac{w_1^2 + w_3^2}{2(w_1 + w_3)}\right).$$

Calculations are easier for the case of the expected absolute difference with respect to the mixture, i.e., for $E|X_1 - \mathbb{I}_{Z=i} X_{(i)}|$, where Z is a random variable independent from X_1, X_2, X_3 such that $P(Z = i) = w_i$ for any $i \in \{1, \dots, 3\}$, with $w_1 + w_2 + w_3$.

In this case, if X_1 is the minimum, the difference is 0 with the minimum, is $\frac{1}{2} - \frac{1}{4}$ with the median and $\frac{3}{4} - \frac{1}{3}$ with the maximum (by using the order between the minimum, median and maximum and the expectation of beta distributions). So, in this case, the expected absolute difference is $\frac{w_2}{4} + \frac{w_3}{2}$. If X_1 is the median, similarly, we obtain $\frac{w_1}{4} + \frac{w_3}{4}$. Finally, we also obtain, for the case X_1 being the maximum, a term $\frac{w_2}{4} + \frac{w_1}{2}$. So multiplying each case by $\frac{1}{3}$ it is obtained:

$$\frac{1}{3}\left(\frac{w_2}{4} + \frac{w_3}{2} + \frac{w_1}{4} + \frac{w_3}{4} + \frac{w_2}{4} + \frac{w_1}{2}\right) = \frac{1}{4}w_1 + \frac{1}{6}w_2 + \frac{1}{4}w_3.$$

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