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ORIGINAL ARTICLE

Polynomially oscillatory multipliers on Gelfand–Shilov spaces

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Abstract

We study continuity of the multiplier operator e^{iq} acting on Gelfand–Shilov spaces, where q is a polynomial on $\mathbf{R}^d$ of degree at least two with real coefficients. In the parameter quadrant for the spaces, we identify a wedge that depends on the polynomial degree for which the operator is continuous. We also show that in a large part of the complement region the operator is not continuous in dimension one. The results give information on well-posedness for linear evolution equations that generalize the Schrödinger equation for the free particle.

KEYWORDS

Gelfand–Shilov spaces, linear evolution equations, oscillatory multiplier operators, well-posedness

1 | INTRODUCTION

Let q be a polynomial on $\mathbf{R}^d$ with real coefficients. We consider in this paper the multiplication operator $T = T_g$ with $g(x) = e^{iq(x)}$ defined by

$$(Tf)(x) = e^{iq(x)} f(x), \quad x \in \mathbf{R}^d. \tag{1.1}$$

This operator is unitary on $L^2(\mathbf{R}^d)$ and obviously acts continuously on the Schwartz space $\mathcal{S}(\mathbf{R}^d)$ of smooth functions whose derivatives decay rapidly. The Schwartz space can equivalently be defined as the space of functions f that decay superpolynomially at infinity, plus the same condition on the Fourier transform $\widehat{f}$.

Our goal is to sort out for which parameters of Gelfand–Shilov spaces the operator T is continuous. The Gelfand–Shilov spaces are scales of spaces smaller than the Schwartz space. In fact a Gelfand–Shilov space has two parameters $\theta, s > 0$ and can be defined by the exponential decay conditions

$$\sup_{x \in \mathbf{R}^d} e^{a|x|^{\frac{1}{\theta}}} |f(x)| < \infty, \quad \sup_{\xi \in \mathbf{R}^d} e^{a|\xi|^{\frac{1}{s}}} |\widehat{f}(\xi)| < \infty, \tag{1.2}$$

for some $a > 0$. The parameter θ thus controls the decay rate of f, and the parameter s controls the decay rate of $\widehat{f}$, that is, the smoothness of f.

We use two types of Gelfand–Shilov spaces: the Roumieu spaces $\mathcal{S}_\theta^s(\mathbf{R}^d)$ for which (1.2) holds for some $a > 0$, and the Beurling spaces $\Sigma_\theta^s(\mathbf{R}^d)$ for which (1.2) holds for all $a > 0$.

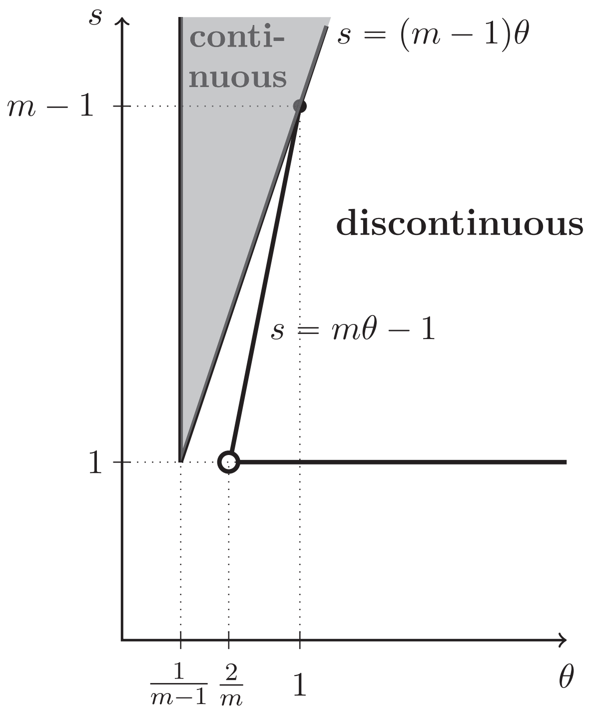

FIGURE 1 The (θ, s)-parameter quadrant with behavior of the operator T acting on Gelfand–Shilov spaces indicated when $m \geqslant 4$ and $d = 1$.

Our first result concerns sufficient conditions for continuity on Gelfand–Shilov spaces.

Theorem 1.1. *Define T by (1.1) where q is a polynomial on $\mathbf{R}^d$ with real coefficients and degree $m \geqslant 2$, and let $s, \theta > 0$.*

(i) If $s \geqslant (m-1)\theta \geqslant 1$ then T is continuous on $\mathcal{S}_\theta^s(\mathbf{R}^d)$.
(ii) If $s \geqslant (m-1)\theta \geqslant 1$ and $(\theta, s) \neq \left(\frac{1}{m-1}, 1\right)$ then T is continuous on $\Sigma_\theta^s(\mathbf{R}^d)$.

An immediate consequence is the corresponding claim for ultradistributions $\left(\mathcal{S}_\theta^s\right)'(\mathbf{R}^d)$ and $\left(\Sigma_\theta^s\right)'(\mathbf{R}^d)$, see Corollary 4.1.

Second, we prove negative results for $d = 1$ in a parameter region which is close to complementary to $s \geqslant (m-1)\theta$. The first result generalizes [1, Proposition 2].

Theorem 1.2. *Define T by (1.1) where q is a polynomial on $\mathbf{R}$ with real coefficients and degree $m \geqslant 2$, and let $s, \theta > 0$. If*

$$1 \leqslant s < \theta m - \max(\theta, 1),$$

and $\theta \geqslant 1$ if $m = 3$, then $T\mathcal{S}_\theta^s(\mathbf{R}) \nsubseteq \mathcal{S}_\theta^s(\mathbf{R})$.

Theorem 1.3. *Define T by (1.1) where q is a polynomial on $\mathbf{R}$ with real coefficients and degree $m \geqslant 2$, and let $s, \theta > 0$. If*

$$1 < s < \theta m - \max(\theta, 1),$$

and $\theta > 1$ if $m = 3$, then $T\Sigma_\theta^s(\mathbf{R}) \nsubseteq \Sigma_\theta^s(\mathbf{R})$.

Note that the statements in Theorems 1.2 and 1.3 are stronger than the lack of continuity on the spaces $\mathcal{S}_\theta^s(\mathbf{R})$ and $\Sigma_\theta^s(\mathbf{R})$, respectively. We illustrate Theorems 1.1, 1.2, and 1.3 in Figure 1.

Our results can be applied to well-posedness of the initial value Cauchy problem for linear evolution equations of the form

$$\begin{cases} \partial_t u(t,x) + \mathrm{i} p(D_x) u(t,x) &= 0, \quad x \in \mathbf{R}^d, \quad t \in \mathbf{R}, \\ u(0,\cdot) &= u_0 \end{cases} \tag{1.3}$$

where $p : \mathbf{R}^d \to \mathbf{R}$ is a polynomial with real coefficients of degree $m \geqslant 2$. This generalizes the Schrödinger equation for the free particle where $m = 2$ and $p(\xi) = |\xi|^2$. The solution operator (propagator) to (1.3) is

$$u(t,x) = \mathrm{e}^{-\mathrm{i}tp(D_x)} u_0 = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle - \mathrm{i}tp(\xi)} \widehat{u}_0(\xi) d\xi$$

for $u_0 \in \mathscr{S}(\mathbf{R}^d)$. This means that the propagator $\mathrm{e}^{-\mathrm{i}tp(D_x)} = \mathscr{F}^{-1} T \mathscr{F}$ is the conjugation by the Fourier transform of the multiplier operator $(Tf)(x) = \mathrm{e}^{-\mathrm{i}tp(x)} f(x)$ of the form (1.1). Since the Fourier transform maps the Gelfand–Shilov spaces into themselves with an interchange of indices as $\mathscr{F} : \mathcal{S}_\theta^s(\mathbf{R}^d) \to \mathcal{S}_s^\theta(\mathbf{R}^d)$ and $\mathscr{F} : \Sigma_\theta^s(\mathbf{R}^d) \to \Sigma_s^\theta(\mathbf{R}^d)$ we obtain the following consequence of Theorems 1.1, 1.2, and 1.3.

Corollary 1.4. *Let $p : \mathbf{R}^d \to \mathbf{R}$ be a polynomial with real coefficients of degree $m \geqslant 2$, consider the solution operator $e^{-itp(D_x)}$ to the Cauchy problem (1.3), let $s, \theta > 0$ and let $t \in \mathbf{R}$.*

(i) If $\theta \geqslant (m-1)s \geqslant 1$ then $e^{-itp(D_x)}$ is continuous on $\mathcal{S}_\theta^s(\mathbf{R}^d)$.
(ii) If $\theta \geqslant (m-1)s \geqslant 1$ and $(s,\theta) \neq \left(\frac{1}{m-1}, 1\right)$ then $e^{-itp(D_x)}$ is continuous on $\Sigma_\theta^s(\mathbf{R}^d)$.
(iii) If $d = 1$, $1 \leqslant \theta < sm - \max(s,1)$, $s \geqslant 1$ if $m = 3$, and $t \neq 0$ then $e^{-itp(D_x)} \mathcal{S}_\theta^s(\mathbf{R}) \nsubseteq \mathcal{S}_\theta^s(\mathbf{R})$.
(iv) If $d = 1$, $1 < \theta < sm - \max(s,1)$, $s > 1$ if $m = 3$, and $t \neq 0$ then $e^{-itp(D_x)} \Sigma_\theta^s(\mathbf{R}) \nsubseteq \Sigma_\theta^s(\mathbf{R})$.

The proof of Theorem 1.1 uses Debrouwere and Neyt's recent results [4] concerning characterization of smooth functions that acts continuously by multiplication on Gelfand–Shilov spaces, plus a result from [10]. The proofs of Theorems 1.2 and 1.3 are based on ideas from the proof of [1, Proposition 2] which concerns the multiplier function $\mathrm{e}^{-\mathrm{i}tx^2}$, together with an investigation of the polynomials p_k that appear upon differentiation as $\partial^k \mathrm{e}^{\mathrm{i}q(x)} = p_k(x)\mathrm{e}^{\mathrm{i}q(x)}$ for $k \in \mathbf{N}$, where q is a monomial.

The paper is organized as follows. In Section 2, we specify notation, conventions and background material. Then in Section 3 we work out the structure and estimates for the derivatives of exponential monomials in one variable of the form $\mathrm{e}^{\lambda x^m/m}$ for $x \in \mathbf{R}$, $m \in \mathbf{N}$, $m \geqslant 2$ and $\lambda \in \mathbf{C} \setminus \{0\}$. The results are used as tools for the negative results Theorems 1.2 and 1.3. Finally, Section 4 is devoted to the proof of Theorem 1.1 and Section 5 to the proofs of Theorems 1.2 and 1.3.

2 | PRELIMINARIES

2.1 | Notation

The floor function is denoted as $\lfloor x \rfloor$ for $x \in \mathbf{R}$. Multiindices $\alpha \in \mathbf{N}^d$ are measured with the 1-norm $|\alpha| = \sum_{j=1}^d \alpha_j$, whereas vectors $x \in \mathbf{R}^d$ are measured with the 2-norm $|x| = \left(\sum_{j=1}^d x_j^2\right)^{\frac{1}{2}}$. We use the bracket $\langle x \rangle = \left(1 + |x|^2\right)^{\frac{1}{2}}$ for $x \in \mathbf{R}^d$. Partial differential operators on $\mathbf{R}^d$ are denoted ∂^α for $\alpha \in \mathbf{N}^d$, and $D^\alpha = \mathrm{i}^{-|\alpha|} \partial^\alpha$. The normalization of the Fourier transform is

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(x) \mathrm{e}^{-\mathrm{i}\langle x,\xi\rangle} \, dx, \qquad \xi \in \mathbf{R}^d,$$

for $f \in \mathscr{S}(\mathbf{R}^d)$ (the Schwartz space), where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathbf{R}^d$. The conjugate linear action of a (ultra-)distribution u on a test function ϕ is written as (u, ϕ), consistent with the L^2 inner product $(\cdot, \cdot) = (\cdot, \cdot)_{L^2}$ which is conjugate linear in the second argument.

2.2 Gelfand–Shilov spaces

The Schwartz space $\mathcal{S}(\mathbf{R}^d)$ is the subspace of the smooth functions for which the seminorms

$$f \mapsto \sup_{x\in\mathbf{R}^d} |x^\alpha D^\beta f(x)| := C_{\alpha\beta} \tag{2.1}$$

are finite for all $\alpha, \beta \in \mathbf{N}^d$.

In this paper, we work with Gelfand–Shilov spaces and their dual ultradistribution spaces [5]. For Gelfand–Shilov spaces you impose certain restrictions of the constants $C_{\alpha\beta}$ in (2.1) which leads to spaces that are smaller than $\mathcal{S}(\mathbf{R}^d)$.

Let $\theta, s, h > 0$. The space denoted as $\mathcal{S}^s_{\theta,h}(\mathbf{R}^d)$ is the set of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}^s_{\theta,h}} \equiv \sup \frac{|x^\alpha D^\beta f(x)|}{h^{|\alpha+\beta|}\alpha!^\theta\, \beta!^s} \tag{2.2}$$

is finite, where the supremum is taken over all $\alpha, \beta \in \mathbf{N}^d$ and $x \in \mathbf{R}^d$. The function space $\mathcal{S}^s_{\theta,h}$ is a Banach space which increases with h, s and θ, and $\mathcal{S}^s_{\theta,h} \subseteq \mathcal{S}$. The topological dual $(\mathcal{S}^s_{\theta,h})'(\mathbf{R}^d)$ is a Banach space which contains the tempered distributions: $\mathcal{S}'(\mathbf{R}^d) \subseteq (\mathcal{S}^s_{\theta,h})'(\mathbf{R}^d)$.

The Beurling-type *Gelfand–Shilov space* $\Sigma^s_\theta(\mathbf{R}^d)$ is the projective limit of $\mathcal{S}^s_{\theta,h}(\mathbf{R}^d)$ with respect to h [5]. This means

$$\Sigma^s_\theta(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}^s_{\theta,h}(\mathbf{R}^d) \tag{2.3}$$

and the Fréchet space topology of $\Sigma^s_\theta(\mathbf{R}^d)$ is defined by the seminorms $\|\cdot\|_{\mathcal{S}^s_{\theta,h}}$ for $h > 0$.

We have $\Sigma^s_\theta(\mathbf{R}^d) \neq \{0\}$ if and only if $s + \theta > 1$ [8]. The topological dual of $\Sigma^s_\theta(\mathbf{R}^d)$ is the space of (Beurling type) *Gelfand–Shilov ultradistributions* [5, Section I.4.3]

$$(\Sigma^s_\theta)'(\mathbf{R}^d) = \bigcup_{h>0} (\mathcal{S}^s_{\theta,h})'(\mathbf{R}^d). \tag{2.3′}$$

The Roumieu-type Gelfand–Shilov space is the union

$$\mathcal{S}^s_\theta(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}^s_{\theta,h}(\mathbf{R}^d)$$

equipped with the inductive limit topology [9], that is the strongest topology such that each inclusion $\mathcal{S}^s_{\theta,h}(\mathbf{R}^d) \subseteq \mathcal{S}^s_\theta(\mathbf{R}^d)$ is continuous. We have $\mathcal{S}^s_\theta(\mathbf{R}^d) \neq \{0\}$ if and only if $s + \theta \geqslant 1$ [5]. The corresponding (Roumieu type) Gelfand–Shilov ultradistribution space is

$$(\mathcal{S}^s_\theta)'(\mathbf{R}^d) = \bigcap_{h>0} (\mathcal{S}^s_{\theta,h})'(\mathbf{R}^d).$$

For every $s, \theta > 0$ such that $s + \theta > 1$, and for any $\varepsilon > 0$ we have

$$\Sigma^s_\theta(\mathbf{R}^d) \subseteq \mathcal{S}^s_\theta(\mathbf{R}^d) \subseteq \Sigma^{s+\varepsilon}_{\theta+\varepsilon}(\mathbf{R}^d). \tag{2.4}$$

The dual spaces $(\Sigma^s_\theta)'(\mathbf{R}^d)$ and $(\mathcal{S}^s_\theta)'(\mathbf{R}^d)$ may be equipped with several topologies. In this paper, we use the weak* topologies on $(\Sigma^s_\theta)'(\mathbf{R}^d)$ and $(\mathcal{S}^s_\theta)'(\mathbf{R}^d)$, respectively.

The Gelfand–Shilov (ultradistribution) spaces enjoy invariance properties, with respect to translation, dilation, tensorization, coordinate transformation, and (partial) Fourier transformation. The Fourier transform extends uniquely to homeomorphisms on $\mathcal{S}'(\mathbf{R}^d)$, from $(\mathcal{S}^s_\theta)'(\mathbf{R}^d)$ to $(\mathcal{S}^\theta_s)'(\mathbf{R}^d)$, and from $(\Sigma^s_\theta)'(\mathbf{R}^d)$ to $(\Sigma^\theta_s)'(\mathbf{R}^d)$, and restricts to homeomorphisms on $\mathcal{S}(\mathbf{R}^d)$, from $\mathcal{S}^s_\theta(\mathbf{R}^d)$ to $\mathcal{S}^\theta_s(\mathbf{R}^d)$, and from $\Sigma^s_\theta(\mathbf{R}^d)$ to $\Sigma^\theta_s(\mathbf{R}^d)$, and to a unitary operator on $L^2(\mathbf{R}^d)$.

Chung, Chung, and Kim characterized in [3] the Roumieu space $\mathcal{S}_\theta^s(\mathbf{R}^d)$ as the space of functions $f \in C^\infty(\mathbf{R}^d)$ that satisfy (1.2) for some $a > 0$. It also follows that the Beurling space $\Sigma_\theta^s(\mathbf{R}^d)$ can be characterized as the space of functions $f \in C^\infty(\mathbf{R}^d)$ that satisfy (1.2) for all $a > 0$.

We need the following result which says that we may use an alternative family of seminorms instead of the seminorms (2.2) indexed by $h > 0$ for the spaces $\Sigma_\theta^s(\mathbf{R}^d)$ and $\mathcal{S}_\theta^s(\mathbf{R}^d)$. This is the family of seminorms

$$\|f\|_a \equiv \sup_{\beta \in \mathbf{N}^d,\ x \in \mathbf{R}^d} e^{a|x|^{\frac{1}{\theta}}} \beta!^{-s} a^{|\beta|} |D^\beta f(x)| \tag{2.5}$$

indexed by $a > 0$. The result may be considered quite well known but we write down a proof for the benefit of the reader.

Lemma 2.1. *Suppose $\theta, s > 0$ and $\theta + s > 1$. For any $a > 0$ there exists $C, h > 0$ such that*

$$\|f\|_a \leqslant C \|f\|_{\mathcal{S}_{\theta,h}^s}, \quad f \in \Sigma_\theta^s(\mathbf{R}^d). \tag{2.6}$$

For any $h > 0$ there exists $C, a > 0$ such that

$$\|f\|_{\mathcal{S}_{\theta,h}^s} \leqslant C \|f\|_a, \quad f \in \Sigma_\theta^s(\mathbf{R}^d). \tag{2.7}$$

Proof. Let $f \in \Sigma_\theta^s(\mathbf{R}^d)$. From (2.2) we have for any $h > 0$

$$|x^\alpha D^\beta f(x)| \leqslant \|f\|_{\mathcal{S}_{\theta,h}^s} \alpha!^\theta \beta!^s h^{|\alpha+\beta|}, \quad \alpha, \beta \in \mathbf{N}^d, \quad x \in \mathbf{R}^d.$$

This gives for any $n \in \mathbf{N}$ and any $\beta \in \mathbf{N}^d$

$$|x|^n |D^\beta f(x)| \leqslant d^{\frac{n}{2}} \max_{|\alpha|=n} |x^\alpha D^\beta f(x)| \leqslant d^{\frac{n}{2}} \|f\|_{\mathcal{S}_{\theta,h}^s} n!^\theta \beta!^s h^{n+|\beta|}, \quad x \in \mathbf{R}^d.$$

Thus for $a > 0$ we have

$$\begin{aligned}
\exp\left(\frac{a}{\theta}|x|^{\frac{1}{\theta}}\right) |D^\beta f(x)|^{\frac{1}{\theta}} &= \sum_{n=0}^\infty \frac{|x|^{\frac{n}{\theta}} |D^\beta f(x)|^{\frac{1}{\theta}} \left(d^{\frac{1}{2}} h\right)^{-\frac{n}{\theta}}}{n!} \left(\frac{a\left(d^{\frac{1}{2}} h\right)^{\frac{1}{\theta}}}{\theta}\right)^n \\
&\leqslant \|f\|_{\mathcal{S}_{\theta,h}^s}^{\frac{1}{\theta}} \beta!^{\frac{s}{\theta}} h^{\frac{|\beta|}{\theta}} \sum_{n=0}^\infty 2^{-n}, \quad x \in \mathbf{R}^d, \quad \beta \in \mathbf{N}^d,
\end{aligned}$$

provided $2a\left(d^{\frac{1}{2}} h\right)^{\frac{1}{\theta}} \leqslant \theta$. Hence

$$e^{a|x|^{\frac{1}{\theta}}} |D^\beta f(x)| \leqslant 2^\theta \|f\|_{\mathcal{S}_{\theta,h}^s} \beta!^s h^{|\beta|}$$

which gives

$$\|f\|_a \leqslant 2^\theta \|f\|_{\mathcal{S}_{\theta,h}^s}$$

provided $h \leqslant \min\left(a^{-1}, d^{-\frac{1}{2}}\left(\theta 2^{-1} a^{-1}\right)^\theta\right)$. We have shown (2.6).

In order to show (2.7) we let $f \in \Sigma_\theta^s(\mathbf{R}^d)$. From (2.6) we know that $\|f\|_a < \infty$ for any $a > 0$. Hence, we have for any $a > 0$, $\beta \in \mathbf{N}^d$ and $x \in \mathbf{R}^d$

$$\sum_{n=0}^{\infty} \frac{|x|^{\frac{n}{\theta}} |D^\beta f(x)|^{\frac{1}{\theta}}}{n!} \left(\frac{a}{\theta}\right)^n = \mathrm{e}^{\frac{a}{\theta}|x|^{\frac{1}{\theta}}} |D^\beta f(x)|^{\frac{1}{\theta}} \leqslant \|f\|_a^{\frac{1}{\theta}} \beta!^{\frac{s}{\theta}} a^{-\frac{|\beta|}{\theta}},$$

which gives

$$|x|^n |D^\beta f(x)| \leqslant \|f\|_a n!^\theta \beta!^s a^{-|\beta|} \left(\frac{\theta}{a}\right)^{\theta n}, \quad n \in \mathbf{N}, \quad \beta \in \mathbf{N}^d, \quad x \in \mathbf{R}^d,$$

and thus, using [7, Eq. (0.3.3)],

$$|x^\alpha D^\beta f(x)| \leqslant \|f\|_a \alpha!^\theta \beta!^s a^{-|\beta|} \left(\frac{d\theta}{a}\right)^{\theta|\alpha|}, \quad \alpha, \beta \in \mathbf{N}^d, \quad x \in \mathbf{R}^d.$$

From this it follows that

$$\|f\|_{S_{\theta,h}^s} \leqslant \|f\|_a, \quad f \in \Sigma_\theta^s(\mathbf{R}^d),$$

for any $h > 0$ provided $a \geqslant \max(h^{-1}, d\theta h^{-\frac{1}{\theta}})$. This proves (2.7). □

2.3 | Faà di Bruno's formula

Of the several available versions of Faà di Bruno's formula, we will use the following two. If $f, g \in C^\infty(\mathbf{R})$ then we have for any $k \in \mathbf{N}$

$$\frac{d^k}{dx^k}(f(g(x))) = \sum_{m_1+2m_2+\cdots+km_k=k} \frac{k!}{m_1! m_2! \cdots m_k!} f^{(m_1+\cdots+m_k)}(g(x)) \prod_{j=1}^{k} \left(\frac{g^{(j)}(x)}{j!}\right)^{m_j}. \tag{2.8}$$

The second version of Faà di Bruno's formula concerns $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R}^d \to \mathbf{R}$ with $f \in C^\infty(\mathbf{R})$ and $g \in C^\infty(\mathbf{R}^d)$, and reads

$$\partial^\alpha (f(g(x))) = \sum_{j=1}^{|\alpha|} \frac{f^{(j)}(g(x))}{j!} \sum_{\substack{\alpha_1+\cdots+\alpha_j=\alpha \\ |\alpha_\ell|\geq 1,\ 1\leqslant \ell \leqslant j}} \frac{\alpha!}{\alpha_1! \cdots \alpha_j!} \prod_{\ell=1}^{j} \partial^{\alpha_\ell} g(x) \tag{2.9}$$

for any $\alpha \in \mathbf{N}^d \setminus \{0\}$ [6, Eq. (2.3)]. For an even more general version of Faà di Bruno's formula, we refer the reader to [2, Proposition 4.3].

3 | DERIVATIVES OF EXPONENTIAL MONOMIALS

Let $m \in \mathbf{N}$, $m \geqslant 2$, $\lambda \in \mathbf{C} \setminus \{0\}$, and consider the function

$$g_\lambda(x) = e^{\lambda x^m / m}, \quad x \in \mathbf{R}. \tag{3.1}$$

This function can be considered a generalized Gaussian. Clearly we have for $k \in \mathbf{N}$

$$\partial^k g_\lambda(x) = p_{\lambda,k}(x)\, g_\lambda(x) \tag{3.2}$$

where $p_{\lambda,k}$ is a polynomial of degree $\deg p_{\lambda,k} = k(m-1)$. In fact we have $p_{\lambda,0}(x) = 1$, $p_{\lambda,1}(x) = \lambda x^{m-1}$, and the recursive relation

$$p_{\lambda,k+1}(x) = \lambda x^{m-1} p_{\lambda,k}(x) + p'_{\lambda,k}(x) \tag{3.3}$$

for $k \in \mathbf{N}$. When $m = 2 = -\lambda$ then $(-1)^k p_{\lambda,k}$ are the Hermite polynomials.

Lemma 3.1. *Let $m \in \mathbf{N}$, $m \geqslant 2$, $\lambda \in \mathbf{C} \setminus \{0\}$, and let g_λ be defined by (3.1) and $p_{\lambda,k}$ by (3.2) for $k \in \mathbf{N}$. Then, the polynomials $p_{\lambda,k}$ have the form*

$$p_{\lambda,k}(x) = \sum_{n=0}^{\lfloor k\frac{m-1}{m} \rfloor} \lambda^{k-n} x^{(m-1)k-nm} C_{k,n} \tag{3.4}$$

for $k \in \mathbf{N} \setminus \{0\}$, where $C_{k,n} \in \mathbf{N} \setminus \{0\}$ for $0 \leqslant n \leqslant \lfloor k\frac{m-1}{m} \rfloor$, and $C_{k,0} = 1$ for all $k \in \mathbf{N} \setminus \{0\}$.

Proof. First, we note that for $k \geqslant 1$ we have

$$\frac{k-1}{2} \leqslant \left\lfloor \frac{k}{2} \right\rfloor \leqslant \left\lfloor k\frac{m-1}{m} \right\rfloor \leqslant k-1. \tag{3.5}$$

If $k = 1$, then the sum (3.4) contains only one term of the stated form as confirmed above, with $C_{1,0} = 1$. In an induction argument, we suppose that (3.4) holds true for a fixed $k \geqslant 1$. By (3.5) we have

$$\left\lfloor (k+1)\frac{m-1}{m} \right\rfloor \geqslant \left\lfloor \frac{k+1}{2} \right\rfloor \geqslant 1$$

so the sum (3.4) with k replaced by $k+1$ does contain the term with index $n = 1$. We obtain from (3.3) and the induction hypothesis (3.4) with $C_{k,0} = 1$

$$\begin{aligned}
p_{\lambda,k+1}(x) &= \lambda x^{m-1} p_{\lambda,k}(x) + p'_{\lambda,k}(x) \\
&= \lambda^{k+1} x^{(m-1)(k+1)} + \sum_{1 \leqslant n \leqslant \lfloor k\frac{m-1}{m} \rfloor} \lambda^{k+1-n} x^{(m-1)(k+1)-nm} C_{k,n} \\
&\quad + \lambda^k x^{(m-1)k-1}(m-1)k \\
&\quad + \sum_{1 \leqslant n < k\frac{m-1}{m}} \lambda^{k-n} x^{(m-1)k-nm-1} C_{k,n}((m-1)k - nm) \\
&= \lambda^{k+1} x^{(m-1)(k+1)} + \sum_{1 \leqslant n \leqslant \lfloor k\frac{m-1}{m} \rfloor} \lambda^{k+1-n} x^{(m-1)(k+1)-nm} C_{k,n} \\
&\quad + \lambda^k x^{(m-1)(k+1)-m}(m-1)k \\
&\quad + \sum_{2 \leqslant n < k\frac{m-1}{m}+1} \lambda^{k+1-n} x^{(m-1)(k+1)-nm} C_{k,n-1}((m-1)k - (n-1)m).
\end{aligned} \tag{3.6}$$

Note that the term $\lambda^k x^{(m-1)(k+1)-m}(m-1)k$ fits into (3.4) with k replaced by $k+1$ and index $n = 1$. In the last sum, the indices n satisfy $2m \leqslant mn \leqslant (m-1)k + m - 1$ which gives $2 \leqslant n \leqslant (k+1)\frac{m-1}{m}$. Hence,

$$2 \leqslant n \leqslant \left\lfloor (k+1)\frac{m-1}{m} \right\rfloor.$$

We may conclude that all terms in (3.6) can be absorbed into formula (3.4) with k replaced by $k+1$ for certain coefficients $C_{k+1,n} \in \mathbf{N} \setminus \{0\}$ for $0 \leqslant n \leqslant \lfloor (k+1)\frac{m-1}{m} \rfloor$. In fact the coefficients $C_{k+1,n}$ are linear combinations of $\{C_{k,n}\}_{0 \leqslant n \leqslant \lfloor k\frac{m-1}{m} \rfloor}$ with positive integer coefficients. It also follows that $C_{k+1,0} = 1$. This proves the induction step which guarantees that (3.4) holds for any $k \in \mathbf{N} \setminus \{0\}$, and $C_{k,0} = 1$ for all $k \in \mathbf{N} \setminus \{0\}$. □

In order to understand more about the polynomials $p_{\lambda,k}$, we would like to gain knowledge about the coefficients $C_{k,n}$. We know from Lemma 3.1 that $C_{k,0} = 1$ for all $k \in \mathbf{N} \setminus \{0\}$.

We need a simple lemma.

Lemma 3.2. *Suppose $m \in \mathbf{N}$, $m \geqslant 2$ and $k \in \mathbf{N} \setminus \{0\}$. Then*

$$k \in m\mathbf{N} \qquad \Longrightarrow \qquad \left\lfloor (k+1)\frac{m-1}{m} \right\rfloor = \left\lfloor k\frac{m-1}{m} \right\rfloor \tag{3.7}$$

and

$$k \notin m\mathbf{N} \qquad \Longrightarrow \qquad \left\lfloor (k+1)\frac{m-1}{m} \right\rfloor = \left\lfloor k\frac{m-1}{m} \right\rfloor + 1. \tag{3.8}$$

Proof. If $k \in m\mathbf{N}$ then there exists $p \in \mathbf{N} \setminus \{0\}$ such that $k = mp$ which gives

$$k\frac{m-1}{m} = p(m-1) = \left\lfloor k\frac{m-1}{m} \right\rfloor$$

and $(k+1)\frac{m-1}{m} = p(m-1) + 1 - \frac{1}{m}$. Since $\frac{1}{2} \leqslant 1 - \frac{1}{m} < 1$ we get

$$\left\lfloor (k+1)\frac{m-1}{m} \right\rfloor = p(m-1) = \left\lfloor k\frac{m-1}{m} \right\rfloor$$

which proves (3.7).

If instead $k \notin m\mathbf{N}$ then there exist $p, q \in \mathbf{N}$ with $1 \leqslant q \leqslant m-1$ such that $k = mp + q$. Then, $(q+1)(m-1) \geqslant qm$ which yields

$$\begin{aligned}(k+1)\frac{m-1}{m} &= p(m-1) + (q+1)\frac{m-1}{m} \geqslant p(m-1) + q \\ &> p(m-1) + q\left(1 - \frac{1}{m}\right) = k\frac{m-1}{m}.\end{aligned}$$

The implication (3.8) follows. □

The following result gives a recursion formula for the coefficients $C_{k,n}$. First, note that

$$C_{2,1} = m - 1. \tag{3.9}$$

Lemma 3.3. *If $m \geqslant 2$ and $k \geqslant 2$ then*

$$C_{k+1,n} = C_{k,n} + C_{k,n-1}((m-1)k - m(n-1)), \quad 1 \leqslant n \leqslant \left\lfloor k\frac{m-1}{m} \right\rfloor. \tag{3.10}$$

If $k \geqslant 2$ and $k \notin m\mathbf{N}$, then we have also

$$C_{k+1,n} = C_{k,n-1}((m-1)k - m(n-1)), \quad n = \left\lfloor (k+1)\frac{m-1}{m} \right\rfloor. \tag{3.11}$$

Proof. We use (3.3) and (3.4). If $k \in m\mathbf{N}$, then the last term in (3.4) is constant. Thus, (3.3) gives

$$\begin{aligned} p_{\lambda,k+1}(x) = &\sum_{n=0}^{\lfloor k\frac{m-1}{m} \rfloor} \lambda^{k+1-n} x^{(m-1)(k+1)-nm} C_{k,n} \\ &+ \sum_{n=0}^{\lfloor k\frac{m-1}{m} \rfloor - 1} \lambda^{k-n} x^{(m-1)k-nm-1} C_{k,n}((m-1)k - nm) \\ = &\sum_{n=0}^{\lfloor k\frac{m-1}{m} \rfloor} \lambda^{k+1-n} x^{(m-1)(k+1)-nm} C_{k,n} \\ &+ \sum_{n=1}^{\lfloor k\frac{m-1}{m} \rfloor} \lambda^{k+1-n} x^{(m-1)(k+1)-nm} C_{k,n-1}((m-1)k - m(n-1)). \end{aligned}$$

In view of Lemma 3.2 and (3.7), this proves (3.10) when $k \in m\mathbf{N}$.

If $k \notin m\mathbf{N}$ then $\lfloor k\frac{m-1}{m} \rfloor < k\frac{m-1}{m}$. In fact if we assume $k\frac{m-1}{m} = p \in \mathbf{N}$ then we get the contradiction $k = m(k-p) \in m\mathbf{N}$. Thus, the last term in (3.4) contains a positive power of x. Thus again using (3.3) we get

$$\begin{aligned} p_{\lambda,k+1}(x) = &\sum_{n=0}^{\lfloor k\frac{m-1}{m} \rfloor} \lambda^{k+1-n} x^{(m-1)(k+1)-nm} C_{k,n} \\ &+ \sum_{n=0}^{\lfloor k\frac{m-1}{m} \rfloor} \lambda^{k-n} x^{(m-1)k-nm-1} C_{k,n}((m-1)k - nm) \\ = &\sum_{n=0}^{\lfloor k\frac{m-1}{m} \rfloor} \lambda^{k+1-n} x^{(m-1)(k+1)-nm} C_{k,n} \\ &+ \sum_{n=1}^{\lfloor k\frac{m-1}{m} \rfloor + 1} \lambda^{k+1-n} x^{(m-1)(k+1)-nm} C_{k,n-1}((m-1)k - m(n-1)). \end{aligned}$$

Again, Lemma 3.2 and (3.8) prove (3.10) when $k \notin m\mathbf{N}$, and it also shows (3.11) when $k \notin m\mathbf{N}$. □

Remark 3.4. Note that (3.9), written as $C_{2,1} = m-1 = C_{1,1} + C_{1,0}(m-1)$ using $C_{1,0} = 1$ and forcing $C_{1,1} = 0$, fits into formula (3.10) for $k = n = 1$ (without the upper bound for n). Indeed $C_{1,1}$ is not well defined due to $\lfloor \frac{m-1}{m} \rfloor = 0$.

In general, it is challenging to find explicit expressions for the coefficients $C_{k,n}$ when $n \neq 0$, but $n = 1$ is an exception.

Lemma 3.5. *If $k \geqslant 2$, then*

$$C_{k,1} = \frac{1}{2}(m-1)k(k-1). \tag{3.12}$$

Proof. We observe that $C_{2,1} = m - 1$ may be written as (3.12) for $k = 2$. If $k \geqslant 3$ then (3.5) implies $\lfloor (k-1)\frac{m-1}{m} \rfloor \geqslant 1$. From $C_{k-1,0} = 1$, Lemma 3.3 and (3.10) we get for $k \geqslant 3$ and $n = 1$

$$C_{k,1} = C_{k-1,1} + (m-1)(k-1).$$

If we assume that (3.12) holds with k replaced by $k - 1$, then we get

$$\begin{aligned} C_{k,1} &= C_{k-1,1} + (m-1)(k-1) \\ &= \frac{1}{2}(m-1)(k-1)(k-2) + (m-1)(k-1) \\ &= (m-1)(k-1)\left(\frac{1}{2}(k-2) + 1\right) \\ &= \frac{1}{2}(m-1)k(k-1). \end{aligned}$$

By induction this proves (3.12) for all $k \geqslant 2$. □

In the next proposition, we need the following lemma.

Lemma 3.6. *If $m \geqslant 2$ and $\theta \geqslant \frac{2}{m}$ then*

$$f(x) = (1+x)^{m\theta} - \left(1 - \frac{1}{m}\right)x^{m\theta-1} - 1 \geqslant 0 \tag{3.13}$$

for all $0 \leqslant x \leqslant 1$.

Proof. We have $f(0) = 0$ and $f(1) = 2^{m\theta} - 2 + \frac{1}{m} > 0$. If $0 < x < 1$ then

$$\begin{aligned} f'(x) &= m\theta(1+x)^{m\theta-1} - (m\theta-1)\left(1 - \frac{1}{m}\right)x^{m\theta-2} \\ &= m\theta(1+x)^{m\theta-2}\left(1 + x - \left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{m\theta}\right)\left(\frac{x}{1+x}\right)^{m\theta-2}\right). \end{aligned}$$

It follows that $f'(x) > 0$ for all $0 < x < 1$. The function f is thus strictly increasing in [0,1] which proves the claim (3.13) for $0 \leqslant x \leqslant 1$. □

We may now state and prove our main result on the coefficients $C_{k,n}$, which concerns a recursive bound.

Proposition 3.7. *Suppose $\lambda \in \mathbf{C} \setminus \{0\}$, $m \in \mathbf{N}$, $m \geqslant 2$, $\theta \geqslant \frac{2}{m}$, and consider the polynomials $p_{\lambda,k}$ defined by (3.2) having the form (3.4) involving the coefficients $\{C_{k,n}\} \subseteq \mathbf{N} \setminus \{0\}$. Then, we have the bound*

$$C_{k,n+1} \leqslant C_{k,n} m k^{m\theta}, \quad k \geqslant 2, \quad 0 \leqslant n \leqslant \left\lfloor k\frac{m-1}{m} \right\rfloor - 1. \tag{3.14}$$

Proof. If $k = 2$, then by (3.5) we have $\left\lfloor k\frac{m-1}{m} \right\rfloor = 1$. We have $C_{2,1} = m - 1 \leqslant C_{2,0} m 2^{m\theta} = m 2^{m\theta}$ so (3.14) is true for $k = 2$. In an induction argument, we suppose that (3.14) holds for a fixed $k \geqslant 2$.

First, Lemma 3.5 yields

$$\frac{C_{k+1,1}}{m(k+1)^{m\theta}} = \frac{(m-1)k(k+1)}{2m(k+1)^{m\theta}} = \frac{1}{2}\left(1 - \frac{1}{m}\right)\frac{k}{(k+1)^{m\theta-1}} \leqslant 1$$

due to the assumption $m\theta \geqslant 2$. Thus, (3.14) holds when k is replaced by $k + 1$ and $n = 0$.

Next, let $1 \leqslant n \leqslant \left\lfloor (k+1)\frac{m-1}{m} \right\rfloor - 1$. If $k \in m\mathbf{N}$ then by Lemma 3.2 and (3.7) we have $\left\lfloor (k+1)\frac{m-1}{m} \right\rfloor = \left\lfloor k\frac{m-1}{m} \right\rfloor$. Lemma 3.3 and (3.10) give $C_{k+1,n} \geqslant C_{k,n}$ and therefore by the induction hypothesis

$$\begin{aligned}
\frac{C_{k+1,n+1}}{C_{k+1,n}} &= \frac{C_{k,n+1} + C_{k,n}((m-1)k - mn)}{C_{k+1,n}} \\
&\leqslant \frac{C_{k,n+1}}{C_{k,n}} + (m-1)k \\
&\leqslant mk^{m\theta} + (m-1)k \\
&= m(k+1)^{m\theta} \left(\frac{k}{k+1}\right)^{m\theta} \left(1 + \left(1 - \frac{1}{m}\right) k^{1-m\theta}\right) \\
&\leqslant m(k+1)^{m\theta}
\end{aligned}$$

in the final inequality using Lemma 3.6, in the form

$$1 + \left(1 - \frac{1}{m}\right) k^{1-m\theta} \leqslant \left(\frac{k+1}{k}\right)^{m\theta}$$

for all $k \geqslant 1$. We have now shown the induction step which shows that (3.14) holds when k is replaced by $k+1$ and $0 \leqslant n \leqslant \left\lfloor (k+1)\frac{m-1}{m} \right\rfloor - 1$, provided $k \in m\mathbf{N}$.

We also need to consider $k \notin m\mathbf{N}$ for which Lemma 3.2 and (3.8) yield $\left\lfloor (k+1)\frac{m-1}{m} \right\rfloor = \left\lfloor k\frac{m-1}{m} \right\rfloor + 1$. Thus, we assume $1 \leqslant n \leqslant \left\lfloor k\frac{m-1}{m} \right\rfloor$.

If $1 \leqslant n \leqslant \left\lfloor k\frac{m-1}{m} \right\rfloor - 1$, then Lemma 3.3 and (3.10) give $C_{k+1,n} \geqslant C_{k,n}$ and thus by the induction hypothesis

$$\begin{aligned}
\frac{C_{k+1,n+1}}{C_{k+1,n}} &= \frac{C_{k,n+1} + C_{k,n}((m-1)k - mn)}{C_{k+1,n}} \\
&\leqslant \frac{C_{k,n+1}}{C_{k,n}} + (m-1)k \\
&\leqslant mk^{m\theta} + (m-1)k \\
&= m(k+1)^{m\theta} \left(\frac{k}{k+1}\right)^{m\theta} \left(1 + \left(1 - \frac{1}{m}\right) k^{1-m\theta}\right) \\
&\leqslant m(k+1)^{m\theta}
\end{aligned}$$

as before.

Finally, if $n = \left\lfloor k\frac{m-1}{m} \right\rfloor$ then again by Lemma 3.3 and (3.10) we have $C_{k+1,n} \geqslant C_{k,n}$. From (3.11), we obtain

$$\frac{C_{k+1,n+1}}{C_{k+1,n}} \leqslant \frac{C_{k+1,n+1}}{C_{k,n}} = (m-1)k - mn \leqslant m(k+1)^{m\theta}$$

again due to the assumption $m\theta \geqslant 2$. We have shown the induction step in all cases. □

As a consequence of Proposition 3.7, we have

$$C_{k,2} \leqslant m^2 k^{2m\theta} \tag{3.15}$$

for $k \geqslant 4$ and $\theta \geqslant \frac{2}{m}$. In fact (3.5) implies $\left\lfloor k\frac{m-1}{m}\right\rfloor - 1 \geqslant 1$ if $k \geqslant 4$ so Proposition 3.7 applies to $n = 0$ and $n = 1$.

In the next result, we improve the estimate (3.15).

Proposition 3.8. *Let $m \in \mathbf{N}$, $m \geqslant 2$, $\lambda \in \mathbf{C} \setminus \{0\}$ and consider the polynomials $p_{\lambda,k}$ defined by (3.1) and (3.2) having the form (3.4) involving the coefficients $\{C_{k,n}\} \subseteq \mathbf{N} \setminus \{0\}$. If $k \geqslant 4$, then*

$$C_{k,2} \leqslant \frac{1}{2} m^2 k^4. \tag{3.16}$$

Proof. From Faà di Bruno's formula (2.9), we obtain if $k \geqslant 1$

$$\begin{aligned} p_{\lambda,k}(x) = \frac{\partial^k g_\lambda(x)}{g_\lambda(x)} &= \sum_{j=1}^{k} \frac{\lambda^j}{m^j j!} \sum_{\substack{k_1+\cdots+k_j=k \\ 1\le k_\ell \le m}} \frac{k!}{k_1!\cdots k_j!} \prod_{\ell=1}^{j} \partial_x^{k_\ell} x^m \\ &= \sum_{j=1}^{k} \lambda^j x^{mj-k} \frac{k!}{m^j j!} \sum_{\substack{k_1+\cdots+k_j=k \\ 1\le k_\ell \le m}} \prod_{\ell=1}^{j} \frac{m!}{(m-k_\ell)! k_\ell!} \\ &= \sum_{n=0}^{k-1} \lambda^{k-n} x^{(m-1)k-nm} \frac{k!}{m^{k-n}(k-n)!} \sum_{\substack{k_1+\cdots+k_{k-n}=k \\ 1\le k_\ell \le m}} \prod_{\ell=1}^{k-n} \frac{m!}{(m-k_\ell)! k_\ell!}. \end{aligned} \tag{3.17}$$

The smallest power of x is $(m-1)k - (k-1)m = m - k$ when $n = k - 1$. This power seems to be negative if $k > m$ which would be absurd. But negative powers are in fact excluded by (3.17): if $k - n = 1$ then the sum over $k_1 + \cdots + k_{k-n} = k$ reduces to $k_1 = k$ and $1 \leqslant k \leqslant m$ is postulated. Thus, the smallest power of x is non-negative in (3.17).

Comparing (3.17) with (3.4) we observe that the upper limits for the summation index n seem to be possibly different. In fact by (3.5) we know that $\left\lfloor k\frac{m-1}{m}\right\rfloor \leqslant k - 1$. Suppose $n = k - 1 > \left\lfloor k\frac{m-1}{m}\right\rfloor$. Then

$$k - 2 \geqslant \left\lfloor k\frac{m-1}{m}\right\rfloor > k\frac{m-1}{m} - 1 = k - 1 - \frac{k}{m}$$

which implies $k > m$. This contradicts the conditions in the sum over $1 \leqslant k_1 = k \leqslant m$ above, which is interpreted as zero then. So, in fact (3.4) and (3.17) are identical.

The assumption $k \geqslant 4$ and (3.5) implies that $n = 2$ has a nonzero coefficient in (3.4). From the identity of (3.4) and (3.17), it follows in particular that

$$C_{k,2} = \frac{k!}{m^{k-2}(k-2)!} \sum_{\substack{k_1+\cdots+k_{k-2}=k \\ 1\leqslant k_\ell \leqslant m}} \prod_{\ell=1}^{k-2} \frac{m!}{(m-k_\ell)! k_\ell!}.$$

Since $k_\ell \geqslant 1$ for all $1 \leqslant \ell \leqslant k-2$ and $k_1 + \cdots + k_{k-2} = k$, we have

$$k_\ell \leqslant 3, \quad \ell = 1, 2, \ldots, k-2.$$

Hence,

$$C_{k,2} = \frac{k!}{m^{k-2}(k-2)!} \sum_{\substack{k_1+\cdots+k_{k-2}=k \\ 1\leqslant k_\ell \leqslant 3}} \prod_{\ell=1}^{k-2} \frac{m!}{(m-k_\ell)! k_\ell!}. \tag{3.18}$$

For $k_1, \dots, k_{k-2}$ satisfying

$$k_1 + \cdots + k_{k-2} = k \quad \text{and} \quad k_\ell \geq 1,$$

we must have $k_{\ell_1} = 3$ for some $1 \leqslant \ell_1 \leqslant k-2$, and $k_\ell = 1$ for $\ell \neq \ell_1$, or $k_{\ell_1} = k_{\ell_2} = 2$ for some $1 \leqslant \ell_1 \neq \ell_2 \leqslant k-2$, and $k_\ell = 1$ for $\ell \notin \{\ell_1, \ell_2\}$.

In the first case, we have

$$\prod_{\ell=1}^{k-2} \frac{m!}{(m-k_\ell)!k_\ell!} = \frac{m(m-1)(m-2)}{6} m^{k-3} \leqslant m^k \tag{3.19}$$

and in the second case we have

$$\prod_{\ell=1}^{k-2} \frac{m!}{(m-k_\ell)!k_\ell!} = \frac{m^2(m-1)^2}{4} m^{k-4} \leqslant m^k. \tag{3.20}$$

Combinatorics give

$$\sum_{\substack{k_1+\cdots+k_{k-2}=k \\ 1\leqslant k_\ell \leqslant 3}} \leqslant k-2 + \binom{k-2}{2} = \frac{1}{2}(k-1)(k-2). \tag{3.21}$$

Finally, we insert (3.19), (3.20), and (3.21) into (3.18) which gives

$$C_{k,2} \leqslant \frac{k!}{m^{k-2}(k-2)!} m^k \frac{1}{2}(k-1)(k-2) = \frac{1}{2} m^2 k(k-1)^2(k-2) \leqslant \frac{1}{2} m^2 k^4. \qquad \square$$

In the next result, we look at the particular case of (3.1) where $\lambda = \pm i m$, that is, $g_\lambda(x) = \mathrm{e}^{\pm \mathrm{i} x^m}$, and the corresponding polynomials $p_{\lambda,k}$ defined by (3.2).

Proposition 3.9. *Let $m \in \mathbf{N}$, $m \geqslant 2$, $\lambda = \pm i m$ and consider the polynomials $p_{\lambda,k}$ defined by (3.1) and (3.2) having the form (3.4) involving the coefficients $\{C_{k,n}\} \subseteq \mathbf{N} \setminus \{0\}$. If $\theta \geqslant \frac{2}{m}$, then there exists a sequence $\{k_j\}_{j=1}^{+\infty} \subseteq \mathbf{N}$ such that $\lim_{j\to\infty} k_j = \infty$ and*

$$\left| p_{\lambda,k_j}(k_j^\theta) \right| \geqslant \frac{1}{2} m^{k_j} k_j^{\theta k_j (m-1)}, \qquad j \in \mathbf{N} \setminus \{0\}. \tag{3.22}$$

Proof. Let $j \in \mathbf{N} \setminus \{0\}$. Due to $1 < \frac{m}{m-1} \leqslant 2$, the interval $[4j\frac{m}{m-1}, (4j+1)\frac{m}{m-1}]$ must contain positive integers. Denote by $k_j \in \mathbf{N} \setminus \{0\}$ the smallest of them. Hence, we have $4j \leqslant k_j \frac{m-1}{m} \leqslant 4j+1$, and therefore

$$4j \leqslant \left\lfloor k_j \frac{m-1}{m} \right\rfloor \leqslant 4j+1 \tag{3.23}$$

as well as

$$\left\lfloor \frac{1}{2} \left\lfloor k_j \frac{m-1}{m} \right\rfloor \right\rfloor = 2j. \tag{3.24}$$

The bound $k_j \geqslant 4j\frac{m}{m-1} > 4j$ implies that $k_j \geqslant 4$ for all $j \in \mathbf{N} \setminus \{0\}$.

For notational simplicity, we write $k = k_j$. Lemma 3.1 and (3.4) gives

$$p_{\lambda,k}(x) = (\pm 1)^k \mathrm{i}^k m^k x^{(m-1)k} \sum_{n=0}^{\lfloor k\frac{m-1}{m} \rfloor} (\pm 1)^n \mathrm{i}^{-n} m^{-n} x^{-nm} C_{k,n}$$

which in turn gives

$$\begin{aligned}\left|p_{\lambda,k}(k^\theta)\right| &= m^k k^{\theta(m-1)k}\left|\sum_{n=0}^{\lfloor k\frac{m-1}{m}\rfloor}(\pm 1)^n \mathrm{i}^{-n} m^{-n} k^{-\theta nm} C_{k,n}\right| \\ &\geqslant m^k k^{\theta(m-1)k}\left|\sum_{\substack{n=0\\ n \text{ even}}}^{\lfloor k\frac{m-1}{m}\rfloor}\mathrm{i}^{-n} m^{-n} k^{-\theta nm} C_{k,n}\right| \\ &= m^k k^{\theta(m-1)k}\left|\sum_{n=0}^{\lfloor\frac{1}{2}\lfloor k\frac{m-1}{m}\rfloor\rfloor}(-1)^n m^{-2n} k^{-2\theta nm} C_{k,2n}\right|.\end{aligned} \tag{3.25}$$

Using (3.24), and noting that $m^{-4j}k^{-4\theta jm}C_{k,4j} \geqslant 0$, we have

$$\begin{aligned}&\sum_{n=0}^{\lfloor\frac{1}{2}\lfloor k\frac{m-1}{m}\rfloor\rfloor}(-1)^n m^{-2n} k^{-2\theta nm} C_{k,2n} \geqslant \sum_{n=0}^{2j-1}(-1)^n m^{-2n} k^{-2\theta nm} C_{k,2n} \\ &= 1 - \frac{C_{k,2}}{m^2 k^{2\theta m}} \\ &\quad + \frac{C_{k,4}}{m^4 k^{4\theta m}} - \frac{C_{k,6}}{m^6 k^{6\theta m}} + \cdots + \frac{C_{k,2(2j-2)}}{m^{2(2j-2)} k^{2(2j-2)\theta m}} - \frac{C_{k,2(2j-1)}}{m^{2(2j-1)} k^{2(2j-1)\theta m}}.\end{aligned} \tag{3.26}$$

By Proposition 3.7, we have if $0 \leqslant 2n \leqslant \lfloor k\frac{m-1}{m}\rfloor - 2$,

$$C_{k,2(n+1)} \leqslant C_{k,2n} m^2 k^{2\theta m}$$

which gives

$$\frac{C_{k,2n}}{m^{2n}k^{2n\theta m}} - \frac{C_{k,2(n+1)}}{m^{2(n+1)}k^{2(n+1)\theta m}} \geqslant 0$$

for all $n \in \mathbf{N}\setminus\{0\}$ such that $0 \leqslant 2n \leqslant \lfloor k\frac{m-1}{m}\rfloor - 2$. By (3.23) this includes all $n \in \mathbf{N}\setminus\{0\}$ such that $2 \leqslant n \leqslant 2j-2$. We may conclude that

$$\sum_{n=0}^{\lfloor\frac{1}{2}\lfloor k\frac{m-1}{m}\rfloor\rfloor}(-1)^n m^{-2n} k^{-2\theta nm} C_{k,2n} \geqslant 1 - \frac{C_{k,2}}{m^2 k^{2\theta m}}. \tag{3.27}$$

Finally, the assumption $\theta \geqslant \frac{2}{m}$ and Proposition 3.8 yield

$$\frac{C_{k,2}}{m^2 k^{2\theta m}} \leqslant \frac{m^2 k^4}{2m^2 k^{2\theta m}} = \frac{1}{2}k^{2(2-\theta m)} \leqslant \frac{1}{2}$$

for all k. Combining this with (3.25) and (3.27), we obtain (3.22). □

4 | PROOF THEOREM 1.1

In [4, Theorem 5.7], the authors identify the subspace of $g \in C^\infty(\mathbf{R}^d)$ such that the multiplier operator $T_g f = fg$ is continuous on $\Sigma_\theta^s(\mathbf{R}^d)$ and on $\mathcal{S}_\theta^s(\mathbf{R}^d)$, respectively. They prove a characterization of the multiplier space formulated in terms of estimates of the form

$$|\partial^\alpha g(x)| \leqslant C h^{|\alpha|} |\alpha|!^s \mathrm{e}^{\lambda^{-\frac{1}{\theta}} |x|^{\frac{1}{\theta}}}, \quad \alpha \in \mathbf{N}^d, \quad x \in \mathbf{R}^d. \tag{4.1}$$

In fact $T_g : \mathcal{S}_\theta^s(\mathbf{R}^d) \to \mathcal{S}_\theta^s(\mathbf{R}^d)$ is continuous if and only if (4.1) holds for all $\lambda > 0$, some $C = C(\lambda) > 0$, and some $h = h(\lambda) > 0$. Moreover, $T_g : \Sigma_\theta^s(\mathbf{R}^d) \to \Sigma_\theta^s(\mathbf{R}^d)$ is continuous if and only if (4.1) holds for all $h > 0$, some $C = C(h) > 0$, and some $\lambda = \lambda(h) > 0$.

Proof of Theorem 1.1. Theorem 1.1 concerns multiplier functions $g(x) = \mathrm{e}^{\mathrm{i}q(x)}$ where q is a polynomial of degree $m \geqslant 2$, that is

$$q(x) = \sum_{|\alpha| \leqslant m} c_\alpha x^\alpha, \quad c_\alpha \in \mathbf{R}, \quad x \in \mathbf{R}^d, \tag{4.2}$$

and $c_\alpha \neq 0$ for some $\alpha \in \mathbf{N}^d$ with $|\alpha| = m$.

If $\gamma \in \mathbf{N}^d$ and $|\gamma| \leqslant m$ then

$$\partial^\gamma q(x) = \sum_{\substack{|\alpha|=m \\ \alpha \geqslant \gamma}} c_\alpha \frac{\alpha!}{(\alpha-\gamma)!} x^{\alpha-\gamma} + \text{L.O.T.} \tag{4.3}$$

Hence,

$$|\partial^\gamma q(x)| \leq C_{q,m} \max(1, |x|^{m-|\gamma|}), \tag{4.4}$$

where $C_{q,m} > 0$ is a constant depending only on m and on the coefficients of q.

From Faà di Bruno's formula (2.9), one gets for $\alpha \in \mathbf{N}^d \setminus \{0\}$

$$\partial^\alpha g(x) = \sum_{j=1}^{|\alpha|} i^j \frac{g(x)}{j!} \sum_{\substack{\alpha_1+\cdots+\alpha_j=\alpha \\ 1 \leqslant |\alpha_\ell| \leqslant m, \quad 1 \leqslant \ell \leqslant j}} \frac{\alpha!}{\alpha_1! \cdots \alpha_j!} \prod_{\ell=1}^{j} \partial^{\alpha_\ell} q(x). \tag{4.5}$$

Now, (4.4) entails if $|x| \geqslant 1$

$$\begin{aligned} |\partial^\alpha g(x)| &\leqslant \sum_{j=1}^{|\alpha|} \frac{1}{j!} \sum_{\substack{\alpha_1+\cdots+\alpha_j=\alpha \\ 1 \leqslant |\alpha_\ell| \leqslant m, \quad 1 \leqslant \ell \leqslant j}} \frac{\alpha!}{\alpha_1! \cdots \alpha_j!} \prod_{\ell=1}^{j} C_{q,m} |x|^{m-|\alpha_\ell|} \\ &\leqslant \alpha! \sum_{j=1}^{|\alpha|} \frac{C_{q,m}^j}{j!} |x|^{jm-|\alpha|} \sum_{\substack{\alpha_1+\cdots+\alpha_j=\alpha \\ 1 \leqslant |\alpha_\ell| \leqslant m, \quad 1 \leqslant \ell \leqslant j}} \frac{1}{\alpha_1! \cdots \alpha_j!}. \end{aligned} \tag{4.6}$$

On one hand, we get from [7, Eq. (0.3.15)]

$$\sum_{\substack{\alpha_1+\cdots+\alpha_j=\alpha \\ 1 \leqslant |\alpha_\ell| \leqslant m, \quad 1 \leqslant \ell \leqslant j}} \frac{1}{\alpha_1! \cdots \alpha_j!} \leqslant \binom{m+d}{m}^j \leqslant 2^{(m+d)j}. \tag{4.7}$$

On the other hand if $|x| \geqslant 1$ we have for any $\lambda > 0$, using $j \leqslant |\alpha| \leqslant jm$,

$$
\begin{aligned}
|x|^{jm-|\alpha|} &= \left(\frac{\left(\theta^{-1} \left(\lambda^{-1} |x| \right)^{\frac{1}{\theta}} \right)^{jm-|\alpha|}}{(jm-|\alpha|)!} \right)^{\theta} (jm-|\alpha|)!^{\theta} \left(\theta^{\theta} \lambda \right)^{jm-|\alpha|} \\
&\leqslant e^{\left(\lambda^{-1} |x| \right)^{\frac{1}{\theta}}} (j(m-1))!^{\theta} \left(\theta^{\theta} \lambda \right)^{jm-|\alpha|}
\end{aligned}
\tag{4.8}
$$

and, using [7, Eq. (0.3.12)],

$$
\begin{aligned}
(j(m-1))!^{\theta} &\leqslant (j(m-1))^{j(m-1)\theta} = (m-1)^{(m-1)\theta j} j^{j(m-1)\theta} \\
&\leqslant (e(m-1))^{(m-1)\theta j} j!^{(m-1)\theta}.
\end{aligned}
\tag{4.9}
$$

If $|x| \geqslant 1$, then insertion of (4.7), (4.8), and (4.9) into (4.6) gives, using the assumption $(m-1)\theta \geq 1$,

$$
\begin{aligned}
|\partial^{\alpha} g(x)| &\leqslant \alpha! \mathrm{e}^{\left(\lambda^{-1} |x| \right)^{\frac{1}{\theta}}} \sum_{j=1}^{|\alpha|} \left(C_{q,m} (e(m-1))^{(m-1)\theta} 2^{m+d} \right)^{j} \left(\theta^{\theta} \lambda \right)^{jm-|\alpha|} j!^{(m-1)\theta-1} \\
&\leqslant C^{|\alpha|} |\alpha|!^{(m-1)\theta} \mathrm{e}^{\left(\lambda^{-1} |x| \right)^{\frac{1}{\theta}}} \sum_{j=1}^{|\alpha|} \lambda^{jm-|\alpha|}
\end{aligned}
\tag{4.10}
$$

for some $C > 0$.

If $|x| \leqslant 1$ then (4.4), (4.5), (4.7), and (4.8) with $|x| = 1$, and (4.9) give for any $\lambda > 0$

$$
\begin{aligned}
|\partial^{\alpha} g(x)| &\leqslant \alpha! \sum_{j=1}^{|\alpha|} \left(C_{q,m} 2^{m+d} \right)^{j} j!^{-1} \\
&\leqslant \alpha! \sum_{j=1}^{|\alpha|} \left(C_{q,m} 2^{m+d} \right)^{j} j!^{-1} \mathrm{e}^{\lambda^{-\frac{1}{\theta}}} (j(m-1))!^{\theta} \left(\theta^{\theta} \lambda \right)^{jm-|\alpha|} \\
&\leqslant \alpha! \mathrm{e}^{\lambda^{-\frac{1}{\theta}}} \sum_{j=1}^{|\alpha|} \left(C_{q,m} (e(m-1))^{(m-1)\theta} 2^{m+d} \right)^{j} \left(\theta^{\theta} \lambda \right)^{jm-|\alpha|} j!^{(m-1)\theta-1} \\
&\leqslant C^{|\alpha|} |\alpha|!^{(m-1)\theta} \mathrm{e}^{\lambda^{-\frac{1}{\theta}}} \sum_{j=1}^{|\alpha|} \lambda^{jm-|\alpha|}
\end{aligned}
\tag{4.11}
$$

for some $C > 0$.

The estimate (4.10) for $|x| \geqslant 1$ and the estimate (4.11) for $|x| \leqslant 1$ can be combined into the estimate for $x \in \mathbf{R}^d$

$$
|\partial^{\alpha} g(x)| \leqslant \mathrm{e}^{\lambda^{-\frac{1}{\theta}}} C^{|\alpha|} |\alpha|!^{(m-1)\theta} \mathrm{e}^{\left(\lambda^{-1} |x| \right)^{\frac{1}{\theta}}} \sum_{j=1}^{|\alpha|} \lambda^{jm-|\alpha|}
\tag{4.12}
$$

for some $C > 0$.

Proof of Claim (i). The assumption $s \geqslant (m-1)\theta$ and (4.12) imply if $\lambda \geqslant 1$

$$\begin{aligned} |\partial^\alpha g(x)| &\leqslant e^{\lambda^{-\frac{1}{\theta}}} C^{|\alpha|} |\alpha|!^s e^{(\lambda^{-1}|x|)^{\frac{1}{\theta}}} \sum_{j=1}^{|\alpha|} \lambda^{|\alpha|(m-1)} \\ &\leqslant e^{\lambda^{-\frac{1}{\theta}}} \left(2C\lambda^{m-1}\right)^{|\alpha|} |\alpha|!^s e^{(\lambda^{-1}|x|)^{\frac{1}{\theta}}} \end{aligned} \tag{4.13}$$

and if $0 < \lambda < 1$

$$\begin{aligned} |\partial^\alpha g(x)| &\leqslant e^{\lambda^{-\frac{1}{\theta}}} C^{|\alpha|} |\alpha|!^s e^{(\lambda^{-1}|x|)^{\frac{1}{\theta}}} \sum_{j=1}^{|\alpha|} \lambda^{-|\alpha|} \\ &\leqslant e^{\lambda^{-\frac{1}{\theta}}} \left(2C\lambda^{-1}\right)^{|\alpha|} |\alpha|!^s e^{(\lambda^{-1}|x|)^{\frac{1}{\theta}}}. \end{aligned} \tag{4.14}$$

Combining (4.13) and (4.14) we may by the criterion in [4, Theorem 5.7] conclude that T is continuous on $\mathcal{S}_\theta^s(\mathbf{R}^d)$. Claim (i) has been proved.

Proof of Claim (ii). The assumptions imply that either $s \geqslant \theta(m-1) > 1$ or $s > \theta(m-1) \geqslant 1$. First, we suppose that $s \geqslant \theta(m-1) > 1$. Then, [10, Theorem 7.1] shows that T is continuous on $\Sigma_\theta^s(\mathbf{R}^d)$.

It remains to consider $s > \theta(m-1) \geqslant 1$. Then, $\varepsilon := s - \theta(m-1) > 0$. From (4.12) with $\lambda = 1$ we obtain for any $h > 0$

$$\begin{aligned} |\partial^\alpha g(x)| &\leqslant e\, C^{|\alpha|} |\alpha|!^{s-\varepsilon} e^{|x|^{\frac{1}{\theta}}} |\alpha| \\ &\leqslant e(2Ch)^{|\alpha|} |\alpha|!^s e^{|x|^{\frac{1}{\theta}}} \left(\frac{h^{-\frac{|\alpha|}{\varepsilon}}}{|\alpha|!} \right)^{\varepsilon} \\ &\leqslant e^{1+\varepsilon h^{-\frac{1}{\varepsilon}}} (2Ch)^{|\alpha|} |\alpha|!^s e^{|x|^{\frac{1}{\theta}}}. \end{aligned}$$

Again by the criterion in [4, Theorem 5.7] we may conclude that T is continuous on $\Sigma_\theta^s(\mathbf{R}^d)$. Claim (ii) has been proved. □

Corollary 4.1. *Define T by (1.1) where q is a polynomial on $\mathbf{R}^d$ with real coefficients and degree $m \geqslant 2$, and let $s, \theta > 0$.*

(i) If $s \geqslant (m-1)\theta \geqslant 1$, then T is continuous on $\left(\mathcal{S}_\theta^s\right)'(\mathbf{R}^d)$.
(ii) If $s \geqslant (m-1)\theta \geqslant 1$ and $(\theta, s) \neq \left(\frac{1}{m-1}, 1\right)$, then T is continuous on $\left(\Sigma_\theta^s\right)'(\mathbf{R}^d)$.

5 | PROOFS OF THEOREMS 1.2 AND 1.3

Lemma 5.1. *Let $f, g \in C^\infty(\mathbf{R})$. If there exists $A, B, a, \theta, \nu > 0$, and $s \geqslant 1$ such that*

$$|D^k f(x)| \leqslant B^k k! |f(x)| \langle x \rangle^{k \max\left(\frac{1}{\nu}-1, 0\right)}, \quad k \in \mathbf{N}, \tag{5.1}$$

and

$$|D^k (g(x) f(x))| \leqslant A B^k k!^s e^{-a|x|^{\frac{1}{\theta}}}, \quad k \in \mathbf{N}, \tag{5.2}$$

then

$$|(D^k g)(x) f(x)| \leqslant A (2B)^k k!^s e^{-a|x|^{\frac{1}{\theta}}} \langle x \rangle^{k \max\left(\frac{1}{\nu}-1, 0\right)}, \quad k \in \mathbf{N}. \tag{5.3}$$

Proof. Assumption (5.2) for $k = 0$ implies (5.3) for $k = 0$. In an induction proof, we let $k \in \mathbf{N} \setminus \{0\}$ and assume that (5.3) holds for orders smaller than k. We use

$$(D^k g)(x) f(x) = D^k(g(x) f(x)) - \sum_{n=0}^{k-1} \binom{k}{n} D^n g(x) D^{k-n} f(x)$$

(5.1), (5.2), the induction hypothesis, and $s \geqslant 1$. If $\nu \geqslant 1$, then $\max\left(\frac{1}{\nu} - 1, 0\right) = 0$ which gives

$$\begin{aligned}
|(D^k g)(x) f(x)| &\leqslant |D^k(g(x) f(x))| + \sum_{n=0}^{k-1} \binom{k}{n} |D^n g(x)|\, |D^{k-n} f(x)| \\
&\leqslant A B^k k!^s \mathrm{e}^{-a|x|^{\frac{1}{\theta}}} + \sum_{n=0}^{k-1} \binom{k}{n} |D^n g(x)|\, |f(x)|\, B^{k-n} (k-n)! \\
&\leqslant A B^k k!^s \mathrm{e}^{-a|x|^{\frac{1}{\theta}}} + \sum_{n=0}^{k-1} \binom{k}{n} A (2B)^n n!^s \mathrm{e}^{-a|x|^{\frac{1}{\theta}}} B^{k-n} (k-n)! \\
&= A B^k k!^s \mathrm{e}^{-a|x|^{\frac{1}{\theta}}} \left(1 + \sum_{n=0}^{k-1} \left(\frac{n!}{k!} \right)^{s-1} 2^n \right) \\
&\leqslant A B^k k!^s \mathrm{e}^{-a|x|^{\frac{1}{\theta}}} \left(1 + \sum_{n=0}^{k-1} 2^n \right) \\
&= A (2B)^k k!^s \mathrm{e}^{-a|x|^{\frac{1}{\theta}}}.
\end{aligned}$$

This proves the induction step, and hence (5.3), provided $\nu \geqslant 1$.

It remains to consider $0 < \nu < 1$. Then, $\max\left(\frac{1}{\nu} - 1, 0\right) = \frac{1}{\nu} - 1$ and we have

$$\begin{aligned}
|(D^k g)(x) f(x)| &\leqslant |D^k(g(x) f(x))| + \sum_{n=0}^{k-1} \binom{k}{n} |D^n g(x)|\, |D^{k-n} f(x)| \\
&\leqslant A B^k k!^s \mathrm{e}^{-a|x|^{\frac{1}{\theta}}} + \sum_{n=0}^{k-1} \binom{k}{n} |D^n g(x)|\, |f(x)|\, \langle x \rangle^{(k-n)\left(\frac{1}{\nu}-1\right)} B^{k-n} (k-n)! \\
&\leqslant A B^k k!^s \mathrm{e}^{-a|x|^{\frac{1}{\theta}}} + \sum_{n=0}^{k-1} \binom{k}{n} A (2B)^n n!^s \mathrm{e}^{-a|x|^{\frac{1}{\theta}}} \langle x \rangle^{k\left(\frac{1}{\nu}-1\right)} B^{k-n} (k-n)! \\
&= A B^k k!^s \mathrm{e}^{-a|x|^{\frac{1}{\theta}}} \left(1 + \langle x \rangle^{k\left(\frac{1}{\nu}-1\right)} \sum_{n=0}^{k-1} \left(\frac{n!}{k!} \right)^{s-1} 2^n \right) \\
&\leqslant A B^k k!^s \mathrm{e}^{-a|x|^{\frac{1}{\theta}}} \left(1 + \langle x \rangle^{k\left(\frac{1}{\nu}-1\right)} \sum_{n=0}^{k-1} 2^n \right) \\
&= A B^k k!^s \mathrm{e}^{-a|x|^{\frac{1}{\theta}}} \left(1 + \langle x \rangle^{k\left(\frac{1}{\nu}-1\right)} (2^k - 1) \right) \\
&\leqslant A (2B)^k k!^s \mathrm{e}^{-a|x|^{\frac{1}{\theta}}} \langle x \rangle^{k\left(\frac{1}{\nu}-1\right)}
\end{aligned}$$

which proves the induction step, and hence (5.3), when $0 < \nu < 1$. □

Consider the function $\langle x\rangle^t = (1+x^2)^{\frac{t}{2}}$ for $x \in \mathbf{R}$ with $t \in \mathbf{R}$. We have the following estimate for its derivatives.

Lemma 5.2. *If $t \in \mathbf{R}$, then there exists $C_t \geqslant 1$ such that for all $k \in \mathbf{N}$*

$$\left|D^k\langle x\rangle^t\right| \leqslant C_t 2^{3k} k!\langle x\rangle^{t-k}, \quad x \in \mathbf{R}. \tag{5.4}$$

Proof. First, we assume $|t| \leqslant 1$ and show

$$\left|D^k\langle x\rangle^t\right| \leqslant 2^{2k+5} k!\langle x\rangle^{t-k}, \quad x \in \mathbf{R}, \tag{5.5}$$

for all $k \in \mathbf{N}$.

The function $\langle x\rangle^t = (1+x^2)^{\frac{t}{2}}$ is a composition of $f(x) = x^{\frac{t}{2}}$ with $g(x) = 1+x^2$. The estimate (5.5) is trivial when $k=0$, and can be verified for $k=1$ and $k=2$ so we may assume that $k \geqslant 3$.

For $n \in \mathbf{N}$, we have

$$f^{(n)}(x) = \frac{t}{2}\left(\frac{t}{2}-1\right)\cdots\left(\frac{t}{2}-(n-1)\right)x^{\frac{t}{2}-n} \tag{5.6}$$

and if $m_j = 0$ for $3 \leqslant j \leqslant k$ then

$$\prod_{j=1}^{k}\left(\frac{g^{(j)}(x)}{j!}\right)^{m_j} = \left(g'(x)\right)^{m_1}\left(\frac{g''(x)}{2}\right)^{m_2} = 2^{m_1}x^{m_1}. \tag{5.7}$$

If instead $m_j > 0$ for some $3 \leqslant j \leqslant k$ then the product is zero.

Inserting (5.6) and (5.7) into Faà di Bruno's formula (2.8) yields by means of [7, Eq. (0.3.5)]

$$\begin{aligned}
\frac{\left|D^k\langle x\rangle^t\right|}{k!\langle x\rangle^{t-k}} &\leqslant \sum_{m_1+2m_2=k}\frac{1}{m_1!m_2!}\left|\frac{t}{2}\left(\frac{t}{2}-1\right)\cdots\left(\frac{t}{2}-(m_1+m_2-1)\right)\right|\langle x\rangle^{t-2(m_1+m_2)+k-t}2^{m_1}x^{m_1}\\
&\leqslant \frac{|t|}{2}\sum_{m_1+2m_2=k}\frac{1}{m_1!m_2!}\left(\frac{|t|}{2}+1\right)\cdots\left(\frac{|t|}{2}+m_1+m_2+1\right)\langle x\rangle^{-(m_1+2m_2)+k}2^{m_1}\\
&\leqslant \sum_{m_1+2m_2=k}\frac{2^{m_1-1}(m_1+m_2+2)!}{m_1!m_2!} \leqslant \sum_{m_1+2m_2=k}2^{2m_1+m_2+1}(m_2+2)(m_2+1)\\
&\leqslant \sum_{m_1+2m_2=k}2^{2m_1+3m_2+4} = \sum_{m_2=0}^{\lfloor\frac{k}{2}\rfloor}2^{2(k-2m_2)+3m_2+4} = 2^{2k+4}\sum_{m_2=0}^{\lfloor\frac{k}{2}\rfloor}2^{-m_2} \leqslant 2^{2k+5}
\end{aligned}$$

which proves (5.5) provided $|t| \leqslant 1$.

Next, we consider $|t| > 1$ and put $m = \lfloor |t| \rfloor$. Writing

$$\langle x\rangle^t = \langle x\rangle^{\pm|t|} = \langle x\rangle^{\pm(|t|-m)}\prod_{j=1}^{m}\langle x\rangle^{\pm 1}$$

we use Leibniz' rule

$$D^k\langle x\rangle^t = \sum_{k_0+k_1+\cdots+k_m=k}\frac{k!}{k_0!k_1!\cdots k_m!}D^{k_0}\langle x\rangle^{\pm(|t|-m)}\prod_{j=1}^{m}D^{k_j}\langle x\rangle^{\pm 1}.$$

Since $0 \leqslant |t| - m < 1$ we may use (5.5) which yields, using [7, Eq. (0.3.16)],

$$\begin{aligned}
\left|D^k \langle x \rangle^t\right| &\leqslant \sum_{k_0+k_1+\cdots+k_m=k} \frac{k!}{k_0! k_1! \cdots k_m!} \left|D^{k_0} \langle x \rangle^{\pm(|t|-m)}\right| \prod_{j=1}^m \left|D^{k_j} \langle x \rangle^{\pm 1}\right| \\
&\leqslant \sum_{k_0+k_1+\cdots+k_m=k} \frac{k!}{k_0! k_1! \cdots k_m!} 2^{2k_0+5} k_0! \langle x \rangle^{\pm(|t|-m)-k_0} \prod_{j=1}^m 2^{2k_j+5} k_j! \langle x \rangle^{\pm 1 - k_j} \\
&\leqslant 2^{2k+5(m+1)} k! \langle x \rangle^{t-k} \sum_{k_0+k_1+\cdots+k_m=k} \\
&= 2^{2k+5(m+1)} k! \langle x \rangle^{t-k} \binom{k+m}{m} \\
&\leqslant 2^{3k+6m+5} k! \langle x \rangle^{t-k} \leqslant 2^{3k+6|t|+5} k! \langle x \rangle^{t-k}.
\end{aligned}$$

Combined with (5.5) for $|t| \leqslant 1$ we have now shown (5.4) for all $t \in \mathbf{R}$. □

Let $\theta > 0$ and consider the function

$$f(x) = \mathrm{e}^{-\langle x \rangle^{\frac{1}{\theta}}}, \quad x \in \mathbf{R}. \tag{5.8}$$

Lemma 5.3. *If f is defined by (5.8) with $\theta > 0$ then there exists $C_\theta \geqslant 1$ such that for all $k \in \mathbf{N} \setminus \{0\}$*

$$|f^{(k)}(x)| \leqslant C_\theta^k k! f(x) \sum_{j=1}^k \frac{1}{j!} \langle x \rangle^{j\left(\frac{1}{\theta}-1\right)}.$$

Proof. Faà di Bruno's formula (2.9), Lemma 5.2 and [7, Eq. (0.3.16)] give for some $C_\theta \geqslant 1$ if $k \geqslant 1$

$$\begin{aligned}
\frac{|f^{(k)}(x)|}{k! f(x)} &= \left| \sum_{j=1}^k \frac{(-1)^j}{j!} \sum_{\substack{k_1+\cdots+k_j=k \\ k_\ell \geq 1, \quad 1 \leqslant \ell \leqslant j}} \frac{1}{k_1! \cdots k_j!} \prod_{\ell=1}^j D^{k_\ell} \langle x \rangle^{\frac{1}{\theta}} \right| \\
&\leqslant \sum_{j=1}^k \frac{1}{j!} \sum_{\substack{k_1+\cdots+k_j=k \\ k_\ell \geq 1, \quad 1 \leqslant \ell \leqslant j}} \prod_{\ell=1}^j C_\theta 2^{3k_\ell} \langle x \rangle^{\frac{1}{\theta}-k_\ell} \\
&\leqslant C_\theta^k 2^{3k} \sum_{j=1}^k \frac{1}{j!} \langle x \rangle^{\frac{j}{\theta}-k} \sum_{\substack{k_1+\cdots+k_j=k \\ k_\ell \geq 1, \quad 1 \leqslant \ell \leqslant j}} \\
&\leqslant C_\theta^k 2^{3k} \sum_{j=1}^k \frac{1}{j!} \langle x \rangle^{\frac{j}{\theta}-k} \binom{k+j-1}{j-1} \\
&\leqslant C_\theta^k 2^{3k} \sum_{j=1}^k \frac{1}{j!} \langle x \rangle^{\frac{j}{\theta}-k} 2^{k+j-1} \\
&\leqslant C_\theta^k 2^{5k} \sum_{j=1}^k \frac{1}{j!} \langle x \rangle^{j\left(\frac{1}{\theta}-1\right)}.
\end{aligned}$$

□

Corollary 5.4. *If f is defined by (5.8) with $\theta > 0$ then there exists $C_\theta \geqslant 1$ such that for all $k \in \mathbf{N}$*

$$|f^{(k)}(x)| \leqslant C_\theta^k k! f(x) \langle x \rangle^{k \max\left(\frac{1}{\theta}-1,0\right)}.$$

Proof. The estimate is trivial, when $k = 0$. Lemma 5.3 gives for some $C_\theta \geqslant 1$ and $k \geqslant 1$ with $\tau = \max\left(\frac{1}{\theta} - 1, 0\right)$

$$\begin{aligned}|f^{(k)}(x)| &\leqslant C_\theta^k k! f(x) \sum_{j=1}^{k} \frac{1}{j!} \langle x \rangle^{j\tau} \\ &\leqslant (2C_\theta)^k k! f(x) \langle x \rangle^{k\tau}.\end{aligned}$$

□

Proposition 5.5. *If $\theta > 0$ and the function f is defined by (5.8) then $f \in \mathcal{S}_\theta^1(\mathbf{R})$.*

Proof. If $\theta \geqslant 1$, then Corollary 5.4 gives

$$|f^{(k)}(x)| \leqslant C_\theta^k k! \mathrm{e}^{-\langle x \rangle^{\frac{1}{\theta}}}$$

for some $C_\theta \geqslant 1$, and thus it follows from Lemma 2.1 that $f \in \mathcal{S}_\theta^1(\mathbf{R})$.

Let $0 < \theta < 1$. We have for $x \in \mathbf{R}$ and $j \in \mathbf{N}$

$$\langle x \rangle^j = \left(\frac{\langle x \rangle^{\frac{j}{\theta}}}{j!} \right)^\theta j!^\theta \leqslant \mathrm{e}^{\theta \langle x \rangle^{\frac{1}{\theta}}} j!^\theta$$

which combined with Lemma 5.3 gives for some $C_\theta \geqslant 1$ and all $k \geqslant 1$

$$\begin{aligned}|f^{(k)}(x)| &\leqslant C_\theta^k k! f(x) \sum_{j=1}^{k} \frac{1}{j!} \langle x \rangle^{j\frac{1-\theta}{\theta}} \\ &\leqslant C_\theta^k k! f(x)\, e^{(1-\theta)\langle x \rangle^{\frac{1}{\theta}}} \sum_{j=1}^{k} j!^{-1+1-\theta} \\ &\leqslant (2C_\theta)^k k! \mathrm{e}^{-\theta \langle x \rangle^{\frac{1}{\theta}}}.\end{aligned}$$

Lemma 2.1 again shows that $f \in \mathcal{S}_\theta^1(\mathbf{R})$. □

We assume that q is a polynomial on $\mathbf{R}$ with real coefficients of degree $m \geq 2$, that is,

$$q(x) = \sum_{k=0}^{m} c_k x^k, \quad c_k \in \mathbf{R}, \quad c_m \neq 0, \quad x \in \mathbf{R}.$$

In order to show Theorems 1.2 and 1.3, we may be a rescaling argument assume that $c_m = \pm 1$, that is,

$$q(x) = \pm x^m + q_{m-1}(x), \tag{5.9}$$

where q_{m-1} is a polynomial with real coefficients of degree $\deg q_{m-1} \leqslant m - 1$.

Lemma 5.6. *Let $m \in \mathbf{N}$, $m \geqslant 2$, and let $\theta > \frac{2}{m}$. Let the polynomial q be defined by (5.9) and let $f \in C^\infty(\mathbf{R})$.*

(i) Suppose $\theta \geqslant 1$ if $m = 3$. If

$$e^{\pm i x^m} f \notin \bigcup_{1 \leqslant s < \theta m - \max(\theta,1)} S^s_\theta(\mathbf{R}) \tag{5.10}$$

then

$$e^{iq(x)} f \notin \bigcup_{1 \leqslant s < \theta m - \max(\theta,1)} S^s_\theta(\mathbf{R}). \tag{5.11}$$

(ii) Suppose $\theta > 1$ if $m = 3$. If $\frac{2}{m} < \nu < \theta$ and

$$e^{\pm i x^m} f \notin \bigcup_{1 < s < \nu m - \max(\nu,1)} \Sigma^s_\theta(\mathbf{R}) \tag{5.12}$$

then

$$e^{iq(x)} f \notin \bigcup_{1 < s < \nu m - \max(\nu,1)} \Sigma^s_\theta(\mathbf{R}). \tag{5.13}$$

Proof. We start by proving statement (i). First, we discuss the case $m = 2$ in which $q_{m-1} = q_1$ is a polynomial of degree one. By assumption $\theta > 1$. Suppose $\mathrm{e}^{\pm \mathrm{i}x^2} f \notin S^s_\theta(\mathbf{R})$ for some $1 \leqslant s < \theta$. Then, $\mathrm{e}^{\mathrm{i}q(x)} f = \mathrm{e}^{\mathrm{i}q_1(x)} \mathrm{e}^{\pm \mathrm{i}x^2} f \notin S^s_\theta(\mathbf{R})$ follows from the invariance of $S^s_\theta(\mathbf{R})$ with respect to modulation. This shows a strengthened form of statement (i).

It remains to consider the case $m \geqslant 3$, in which the assumptions imply that $(m-2)\theta \geqslant 1$. Assumption (5.10) means that $\mathrm{e}^{\pm \mathrm{i}x^m} f \notin S^s_\theta(\mathbf{R})$ for all

$$1 \leqslant s < \theta m - \max(\theta, 1).$$

In a contradictory argument, we suppose that (5.11) is not true, that is, $\mathrm{e}^{\mathrm{i}q} f \in S^s_\theta(\mathbf{R})$ for some $1 \leqslant s < \theta m - \max(\theta, 1)$.

If $(m-2)\theta \leqslant s < \theta m - \max(\theta, 1)$ then Theorem 1.1 (i) implies that $\mathrm{e}^{-\mathrm{i}q_{m-1}}$ is continuous on $S^s_\theta(\mathbf{R})$, which gives $\mathrm{e}^{\pm \mathrm{i}x^m} f = \mathrm{e}^{-\mathrm{i}q_{m-1}} \mathrm{e}^{\mathrm{i}q} f \in S^s_\theta(\mathbf{R})$. This contradicts the assumption (5.10).

If instead $1 \leqslant s < (m-2)\theta$ then $\mathrm{e}^{\mathrm{i}q} f \in S^s_\theta(\mathbf{R}) \subseteq S^{(m-2)\theta}_\theta(\mathbf{R})$. Theorem 1.1 (i) again implies that $\mathrm{e}^{-\mathrm{i}q_{m-1}}$ is continuous on $S^{(m-2)\theta}_\theta(\mathbf{R})$, which gives $\mathrm{e}^{\pm \mathrm{i}x^m} f = \mathrm{e}^{-\mathrm{i}q_{m-1}} \mathrm{e}^{\mathrm{i}q} f \in S^{(m-2)\theta}_\theta(\mathbf{R})$. This again contradicts the assumption (5.10). We have shown statement (i) for all $m \geqslant 2$.

It remains to prove statement (ii). If $m = 2$ then $q_{m-1} = q_1$ is a polynomial of degree one and $\theta > 1$. Suppose $\mathrm{e}^{\pm \mathrm{i}x^2} f \notin \Sigma^s_\theta(\mathbf{R})$ for some $1 < s < \nu$. Then, $\mathrm{e}^{\mathrm{i}q(x)} f = \mathrm{e}^{\mathrm{i}q_1(x)} \mathrm{e}^{\pm \mathrm{i}x^2} f \notin \Sigma^s_\theta(\mathbf{R})$ follows from the invariance of $\Sigma^s_\theta(\mathbf{R})$ with respect to modulation. This shows a strengthened form of statement (ii).

Let $m \geqslant 3$. The assumptions imply $(m-2)\theta > 1$. Assumption (5.12) means that $\mathrm{e}^{\pm \mathrm{i}x^m} f \notin \Sigma^s_\theta(\mathbf{R})$ for all

$$1 < s < \nu m - \max(\nu, 1).$$

Suppose that (5.13) is not true, that is, $\mathrm{e}^{\mathrm{i}q} f \in \Sigma^s_\theta(\mathbf{R})$ for some $1 < s < \nu m - \max(\nu, 1)$.

If $(m-2)\theta \leqslant s < \nu m - \max(\nu, 1)$, then Theorem 1.1 (ii) implies that $\mathrm{e}^{-\mathrm{i}q_{m-1}}$ is continuous on $\Sigma^s_\theta(\mathbf{R})$, which gives $\mathrm{e}^{\pm \mathrm{i}x^m} f = \mathrm{e}^{-\mathrm{i}q_{m-1}} \mathrm{e}^{\mathrm{i}q} f \in \Sigma^s_\theta(\mathbf{R})$. This contradicts the assumption (5.12).

If instead $1 < s < (m-2)\theta$ then $\mathrm{e}^{\mathrm{i}q} f \in \Sigma^s_\theta(\mathbf{R}) \subseteq \Sigma^{(m-2)\theta}_\theta(\mathbf{R})$. Theorem 1.1 (ii) again implies that $\mathrm{e}^{-\mathrm{i}q_{m-1}}$ is continuous on $\Sigma^{(m-2)\theta}_\theta(\mathbf{R})$, which gives $\mathrm{e}^{\pm \mathrm{i}x^m} f = \mathrm{e}^{-\mathrm{i}q_{m-1}} \mathrm{e}^{\mathrm{i}q} f \in \Sigma^{(m-2)\theta}_\theta(\mathbf{R})$. This again contradicts the assumption (5.12). We have shown statement (ii) for all $m \geqslant 2$. □

Proof of Theorem 1.2. The assumptions for Lemma 5.6 (i) are satisfied. If $m = 2$, then the assumptions for Theorem 1.2 are $1 \leqslant s < \theta$. The proof of [1, Proposition 2] shows that there exists $f \in S^1_\theta(\mathbf{R})$ such that $\mathrm{e}^{\pm \mathrm{i}x^2} f \notin S^s_\theta(\mathbf{R})$ for all $1 \leqslant s < \theta$. Lemma 5.6 (i) then implies that $\mathrm{e}^{\mathrm{i}q(x)} f \notin S^s_\theta(\mathbf{R})$ for all $1 \leqslant s < \theta$ which proves Theorem 1.2 when $m = 2$.

Suppose next that $m \geqslant 3$. The assumptions imply $(m-2)\theta \geqslant 1$. First, we show that there exists $f \in \mathcal{S}_\theta^1(\mathbf{R})$ such that $e^{\pm ix^m} f \notin \mathcal{S}_\theta^s(\mathbf{R})$ for all $s > 0$ such that $1 \leqslant s < \theta m - \max(\theta, 1)$.

In fact for the function (5.8) we have $f \in \mathcal{S}_\theta^1(\mathbf{R})$ by Proposition 5.5. In order to show $e^{\pm ix^m} f \notin \mathcal{S}_\theta^s(\mathbf{R})$ for all $s > 0$ such that $1 \leqslant s < \theta m - \max(\theta, 1)$ we argue by contradiction. Let $1 \leqslant s < \theta m - \max(\theta, 1)$ and suppose that $e^{\pm ix^m} f \in \mathcal{S}_\theta^s(\mathbf{R})$. By Lemma 2.1, the estimate (5.2) with $g(x) = e^{\pm ix^m}$ holds for some $A, B, a > 0$. By Corollary 5.4 and Lemma 5.1 with $\nu = \theta$ we may conclude that (5.3) holds. Thus, (5.3) for $x = k^\theta$ gives with $\tau = \max\left(\frac{1}{\theta} - 1, 0\right)$

$$\begin{aligned} |(D^k g)(k^\theta) f(k^\theta)| &\leqslant A(2B)^k k!^s e^{-ak} \langle k^\theta \rangle^{k\tau} \\ &\leqslant A(2B)^k k!^s e^{-ak} 2^{\frac{1}{2}k\tau} k^{k(1-\min(\theta,1))} \\ &\leqslant A(2^{1+\tau}B)^k k^{k(s+1-\min(\theta,1))} e^{-ak}, \quad k \in \mathbf{N} \setminus \{0\}. \end{aligned} \tag{5.14}$$

On the other hand, we may apply Proposition 3.9 since the assumptions imply $\theta > \frac{2}{m}$. Hence

$$|(D^k g)(k_j^\theta) f(k_j^\theta)| = \left| p_{\pm im, k_j}(k_j^\theta) \right| e^{-\langle k_j^\theta \rangle^{\frac{1}{\theta}}} \geqslant \frac{1}{2} m^{k_j} k_j^{\theta k_j (m-1)} e^{-2^{\frac{1}{2\theta}} k_j}, \quad j \in \mathbf{N} \setminus \{0\}, \tag{5.15}$$

for some sequence $\{k_j\}_{j=1}^{+\infty} \subseteq \mathbf{N}$ such that $\lim_{j \to \infty} k_j = \infty$, using $k_j^\theta \geqslant 1$.

Combining (5.15) and (5.14) yields

$$\frac{1}{2} m^{k_j} k_j^{k_j \theta(m-1)} e^{-2^{\frac{1}{2\theta}} k_j} \leqslant A(2^{1+\tau}B)^{k_j} k_j^{k_j(s+1-\min(\theta,1))} e^{-ak_j}$$

for $j \in \mathbf{N} \setminus \{0\}$ which contradicts the assumption $s < \theta m - \max(\theta, 1)$. The assumption that $e^{\pm ix^m} f \in \mathcal{S}_\theta^s(\mathbf{R})$ hence must be wrong. Thus

$$e^{\pm ix^m} f \notin \bigcup_{1 \leqslant s < \theta m - \max(\theta,1)} \mathcal{S}_\theta^s(\mathbf{R}),$$

and Lemma 5.6 (i) yields

$$Tf \notin \bigcup_{1 \leqslant s < \theta m - \max(\theta,1)} \mathcal{S}_\theta^s(\mathbf{R}).$$

Since $f \in \mathcal{S}_\theta^1(\mathbf{R}) \subseteq \mathcal{S}_\theta^s(\mathbf{R})$ when $s \geqslant 1$ we have in particular $T\mathcal{S}_\theta^s(\mathbf{R}) \nsubseteq \mathcal{S}_\theta^s(\mathbf{R})$ for any $s > 0$ such that $1 \leqslant s < \theta m - \max(\theta, 1)$. $\square$

Proof of Theorem 1.3. Pick $\nu < \theta$ such that $\nu > \frac{2}{m}$, $1 < \nu < \theta$ if $\theta > 1$, and $1 < s < \nu m - \max(\nu, 1)$. Set

$$f(x) = e^{-\langle x \rangle^{\frac{1}{\nu}}}, \quad x \in \mathbf{R}.$$

By Proposition 5.5, we have $f \in \mathcal{S}_\nu^1(\mathbf{R})$.

Let $g(x) = g_{\pm im}(x) = e^{\pm ix^m}$ and suppose that $T_g f \in \Sigma_\theta^\sigma(\mathbf{R})$ for some $\sigma > 0$ such that $1 < \sigma < \nu m - \max(\nu, 1)$. Then by Lemma 2.1 the estimate

$$|D^k(g(x)f(x))| \leqslant AB^k k!^\sigma e^{-a|x|^{\frac{1}{\theta}}}, \quad k \in \mathbf{N},$$

holds for all $B, a > 0$, and some $A > 0$ depending on B and a. By Corollary 5.4 and Lemma 5.1, we may conclude that

$$|(D^k g)(x)f(x)| \leqslant A(2B)^k k!^\sigma e^{-a|x|^{\frac{1}{\theta}}} \langle x\rangle^{k\max\left(\frac{1}{\nu}-1,0\right)}, \quad k \in \mathbf{N},$$

holds for some $A, B, a > 0$. This gives for $x = k^\nu$ with $\tau = \max\left(\frac{1}{\nu} - 1, 0\right)$

$$\begin{aligned}|(D^k g)(k^\nu)f(k^\nu)| &\leqslant A(2B)^k k!^\sigma e^{-ak^{\frac{\nu}{\theta}}} \langle k^\nu\rangle^{k\tau} \\ &\leqslant A(2^{1+\tau}B)^k k^{k(\sigma+1-\min(\nu,1))} e^{-ak^{\frac{\nu}{\theta}}}, \quad k \in \mathbf{N}\setminus\{0\}.\end{aligned} \tag{5.16}$$

Due to $\nu > \frac{2}{m}$ the assumptions for Proposition 3.9 with θ replaced by ν are satisfied. We obtain from Proposition 3.9

$$|(D^k g)(k_j^\nu)f(k_j^\nu)| = \left|p_{\pm im,k_j}(k_j^\nu)\right| e^{-\langle k_j^\nu\rangle^{\frac{1}{\nu}}} \geqslant \frac{1}{2} m^{k_j} k_j^{\nu k_j(m-1)} e^{-2^{\frac{1}{2\nu}} k_j}, \quad j \in \mathbf{N}\setminus\{0\}, \tag{5.17}$$

for some sequence $\{k_j\}_{j=1}^{+\infty} \subseteq \mathbf{N}$ such that $\lim_{j\to\infty} k_j = \infty$, using $k_j^\nu \geqslant 1$.

Combining (5.17) and (5.16) yields finally

$$\frac{1}{2} m^{k_j} k_j^{k_j\nu(m-1)} e^{-2^{\frac{1}{2\nu}} k_j} \leqslant A(2^{1+\tau}B)^{k_j} k_j^{k_j(\sigma+1-\min(\nu,1))} e^{-ak_j^{\frac{\nu}{\theta}}}$$

for $j \in \mathbf{N}\setminus\{0\}$ which contradicts the assumption $\sigma < \nu m - \max(\nu, 1)$. The assumption that $T_g f \in \Sigma_\theta^\sigma(\mathbf{R})$ hence must be wrong. Thus

$$T_g f \notin \bigcup_{1<\sigma<\nu m-\max(\nu,1)} \Sigma_\theta^\sigma(\mathbf{R}).$$

The assumptions for Lemma 5.6 (ii) are satisfied, so we obtain

$$Tf \notin \bigcup_{1<\sigma<\nu m-\max(\nu,1)} \Sigma_\theta^\sigma(\mathbf{R}).$$

Since $f \in \mathcal{S}_\nu^1(\mathbf{R}) \subseteq \Sigma_\theta^s(\mathbf{R})$ and $1 < s < \nu m - \max(\nu, 1)$, we get $T\Sigma_\theta^s(\mathbf{R}) \nsubseteq \Sigma_\theta^s(\mathbf{R})$. □

Remark 5.7. When $g(x) = g_{\pm im}(x) = e^{\pm ix^m}$ it is possible to prove the lack of continuity of the operator $T = T_g$ in Theorems 1.2 and 1.3 on $\mathcal{S}_\theta^s(\mathbf{R})$ and $\Sigma_\theta^s(\mathbf{R})$ respectively, by other means. Here, we may weaken the assumptions into $0 < s < (m-1)\theta$ and $\theta \geqslant \frac{2}{m}$. In fact we may use the criterion (4.1) and [4, Theorem 5.7].

Consider the assumptions of Theorem 1.2 relaxed into $0 < s < (m-1)\theta$ and $\theta \geqslant \frac{2}{m}$. By Proposition 3.9, we have

$$|(D^k g)(k_j^\theta)| = \left|p_{\pm im,k_j}(k_j^\theta)\right| \geqslant \frac{1}{2} m^{k_j} k_j^{\theta k_j(m-1)} \quad j \in \mathbf{N}\setminus\{0\}, \tag{5.18}$$

for some sequence $\{k_j\}_{j=1}^{+\infty} \subseteq \mathbf{N}$ such that $\lim_{j\to\infty} k_j = \infty$.

Suppose that (4.1) holds for all $\lambda > 0$, some $C = C(\lambda) > 0$, and some $h = h(\lambda) > 0$. Then, we obtain from (5.18)

$$\frac{1}{2} m^{k_j} k_j^{\theta k_j(m-1)} \leqslant Ch^{k_j} k_j!^s e^{\lambda^{-\frac{1}{\theta}} k_j} \leqslant Ch^{k_j} k_j^{sk_j} e^{\lambda^{-\frac{1}{\theta}} k_j}, \quad j \in \mathbf{N}\setminus\{0\}. \tag{5.19}$$

If $s < (m-1)\theta$ this is a contradiction. By [4, Theorem 5.7] it follows that T_g is not continuous on $\mathcal{S}_\theta^s(\mathbf{R})$.

Consider finally the assumptions of Theorem 1.3 relaxed into $0 < s < (m-1)\theta$ and $\theta \geqslant \frac{2}{m}$. Suppose that (4.1) holds for all $h > 0$, some $C = C(h) > 0$, and some $\lambda = \lambda(h) > 0$. Again the estimate (5.19) gives a contradiction if $s < (m-1)\theta$. It follows by [4, Theorem 5.7] that T_g is not continuous on $\Sigma_\theta^s(\mathbf{R})$.

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this paper as no datasets were generated or analyzed during this study.

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