Hardy spaces and Riesz transforms on a Lie group of exponential growth

HARDY SPACES AND RIESZ TRANSFORMS ON A LIE GROUP OF EXPONENTIAL GROWTH

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Abstract. Let G be the Lie group $\mathbb{R}^2 \rtimes \mathbb{R}^+$ endowed with the Riemannian symmetric space structure. Take a distinguished basis X_0, X_1, X_2 of left-invariant vector fields of the Lie algebra of G, and consider the Laplacian $\Delta = -\sum_{i=0}^{2} X_i^2$ and the first-order Riesz transforms $\mathcal{R}_i = X_i \Delta^{-1/2}$, $i = 0, 1, 2$. We first show that the atomic Hardy space H^1 in G introduced by the authors in a previous paper does not admit a characterization in terms of the Riesz transforms $\mathcal{R}_i$. It is also proved that two of these Riesz transforms are bounded from H^1 to H^1.

1. Introduction

Let G be the Lie group $\mathbb{R}^2 \rtimes \mathbb{R}^+$ where the product rule is the following:

$$(x_1, x_2, a) \cdot (x_1', x_2', a') = (x_1 + a\, x_1', x_2 + a\, x_2', a\, a')$$

for (x_1, x_2, a), $(x_1', x_2', a') \in G$. We shall denote by x the point (x_1, x_2, a), and it will be convenient to write

$$|x| = \sqrt{x_1^2 + x_2^2} \qquad \text{for} \qquad x = (x_1, x_2, a).$$

The group G is not unimodular; a right and a left Haar measure of G are given by

$$\mathrm{d}\rho(x) = a^{-1}\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}a \qquad \text{and} \qquad \mathrm{d}\lambda(x) = a^{-3}\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}a\,,$$

respectively. The modular function is thus $\delta(x) = a^{-2}$. Throughout this paper, unless explicitly stated, we use the right measure ρ on G and denote by L^p, $\|\cdot\|_p$ and $\langle \cdot, \cdot \rangle$ the L^p-space, the L^p-norm and the L^2-scalar product with respect to ρ.

The group G has a Riemannian symmetric space structure, and the corresponding metric, which we denote by d, is that of the three-dimensional hyperbolic half-space. The metric d is invariant under left translation and given by

$$\cosh r(x) = \frac{a + a^{-1} + a^{-1}|x|^2}{2}\,, \tag{1.1}$$

where $r(x) = d(x, e)$ denotes the distance of the point x from the identity $e = (0, 0, 1)$ of G. The measure of a hyperbolic ball B_r, centred at the identity and of

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radius r, behaves like

$$\lambda(B_r) = \rho(B_r) \sim \begin{cases} r^3 & \text{if } r < 1 \\ \mathrm{e}^{2r} & \text{if } r \geq 1 \,. \end{cases}$$

Thus G is a group of *exponential volume growth*, and so nondoubling. In this context, the classical Calderón–Zygmund theory and the classical definition of the atomic Hardy space (see [7, 8, 32]) do not apply. But Hebisch and Steger [13] have constructed a Calderón–Zygmund theory which applies to some nondoubling spaces, in particular to our space (G, d, ρ). The main idea is to replace the family of balls in the classical Calderón–Zygmund theory by a suitable family of parallelepipeds which we call *Calderón–Zygmund sets*. The definition appears in [13] and implicitly in [12], and reads as follows.

Definition 1.1. A Calderón-Zygmund set is a parallelepiped $P = [x_1 - L/2, x_1 + L/2] \times [x_2 - L/2, x_2 + L/2] \times [a\mathrm{e}^{-r}, a\mathrm{e}^{r}]$, where $L > 0$, $r > 0$ and $(x_1, x_2, a) \in G$ are related by

$$\begin{aligned} \mathrm{e}^2 a\, r \leq L < \mathrm{e}^8 a\, r \qquad & \text{if } r < 1\,, \\ a\, \mathrm{e}^{2r} \leq L < a\, \mathrm{e}^{8r} \qquad & \text{if } r \geq 1\,. \end{aligned}$$

The point (x_1, x_2, a) is the center of P, and we call r the parameter of P.

We let $\mathcal{P}$ denote the family of all Calderón–Zygmund sets, and observe that $\mathcal{P}$ is invariant under left translation. In [13] it is proved that every integrable function on G admits a Calderón–Zygmund decomposition involving the family $\mathcal{P}$, and that a Calderón–Zygmund theory can be developed in this context. Using the Calderón–Zygmund sets, it is natural to introduce an atomic Hardy space H^1 on the group G, as follows (see [33] for details).

For $1 < p \leq \infty$, a $(1, p)$-atom is a function A in L^1 such that

(i) A is supported in a Calderón–Zygmund set P;
(ii) $\|A\|_p \leq \rho(P)^{-1+1/p}$;
(iii) $\int A \,\mathrm{d}\rho = 0$.

The atomic Hardy space is now defined in a standard way.

Definition 1.2. The atomic Hardy space $H^{1,p}$ is the space of all functions f in L^1 which can be written as $f = \sum_j \lambda_j\, A_j$, where the A_j are $(1, p)$-atoms and λ_j are complex numbers such that $\sum_j |\lambda_j| < \infty$. We denote by $\|f\|_{H^{1,p}}$ the infimum of $\sum_j |\lambda_j|$ over such decompositions.

By [33, Theorem 2.3], for any $p \in (1, \infty)$ the space $H^{1,p}$ coincides with $H^{1,\infty}$ and their norms are equivalent. In the following we shall simply denote this space by H^1 and its norm by $\|\cdot\|_{H^1}$.

The Calderón–Zygmund theory from [13] has turned out to be useful to study the boundedness of singular integral operators related to the distinguished Laplacian on G, defined as follows.

Let X_0, X_1, X_2 denote the left-invariant vector fields

$$X_0 = a\, \partial_a, \qquad X_1 = a\, \partial_{x_1}, \qquad X_2 = a\, \partial_{x_2}\,,$$

which span the Lie algebra of G. The Laplacian $\Delta = -(X_0^2 + X_1^2 + X_2^2)$ is a left-invariant operator which is essentially selfadjoint on L^2. It is well known that the heat semigroup $\left(\mathrm{e}^{-t\Delta}\right)_{t>0}$ is given by a kernel h_t, in the sense that $\mathrm{e}^{-t\Delta} f = f * h_t$

for suitable functions f. Let $\mathcal{M}_h$ denote the corresponding heat maximal operator, defined by

$$\mathcal{M}_h f(x) = \sup_{t>0} |f * h_t(x)| \qquad \forall x \in G\,. \tag{1.2}$$

We then define the heat maximal Hardy space $H^1_{\mathrm{max,h}}$ as the space of all functions f in L^1 such that $\mathcal{M}_h f$ is in L^1, endowed with the norm

$$\|f\|_{H^1_{\mathrm{max,h}}} = \|\mathcal{M}_h f\|_1\,.$$

The authors proved in [31] that $H^1 \subset H^1_{\mathrm{max,h}}$ and that this inclusion is strict.

Similarly, one can consider a maximal Hardy space $H^1_{\mathrm{max,p}}$ on G by means of the Poisson maximal function. It also strictly contains the atomic Hardy space, see [34]. Thus there is no characterization of the atomic Hardy space by means of the heat or the Poisson maximal operator, in our setting.

We shall now consider the first-order Riesz transforms associated with Δ, defined by $\mathcal{R}_i = X_i\,\Delta^{-1/2},\ \ i = 0,1,2$. The associated Hardy space is

$$H^1_{\mathrm{Riesz}} = \{f \in L^1 : \mathcal{R}_i f \in L^1,\ \ i = 0,1,2\}\,,$$

endowed with the norm

$$\|f\|_{H^1_{\mathrm{Riesz}}} = \|f\|_1 + \sum_{i=0}^{2} \|\mathcal{R}_i f\|_1\,.$$

The authors proved in [30] that the operators $\mathcal{R}_i,\ \ i = 0,1,2$, are bounded from H^1 to L^1. Thus H^1 is included in H^1_{Riesz}. We prove the following.

Theorem 1.3. *The inclusion $H^1 \subset H^1_{\mathrm{Riesz}}$ is strict.*

Thus in our setting there is no characterization of the atomic Hardy space by means of Riesz transforms, in contrast to the situation in the Euclidean and many other cases, as we explain below.

In this paper, we also prove the following H^1-H^1 boundedness result.

Theorem 1.4. *The Riesz transforms $\mathcal{R}_i$, $i = 1,2$, are bounded from H^1 to H^1.*

This theorem is analogous to other boundedness results for Riesz transforms in different contexts known in the literature, but its proof is very different since the Hardy space H^1 has only an atomic definition in our setting. Thus the proof requires an explicit construction of the atomic decomposition of $\mathcal{R}_i A$ for an atom A. This will be based on a delicate argument of "mass transport" given in Section 4. It is still an open problem whether also $\mathcal{R}_0$ is bounded on H^1.

The relation between Hardy spaces and Riesz transforms has been studied in different settings in the literature. Here we shall only mention papers where either a characterization of an atomic Hardy space in terms of Riesz transforms or a boundedness result for Riesz transforms on an atomic Hardy space has been investigated.

In the Euclidean case, the atomic Hardy space $H^1(\mathbb{R}^n)$ can also be characterized by means of the first-order Riesz transforms, with equivalence of norms (see [32, Chapter 3]). Moreover, all Riesz transforms are bounded from $H^1(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. Analogous results have been proved on nilpotent Lie groups [16] and more generally on Lie groups of polynomial growth [28] for the first-order Riesz transforms associated with a sub-Laplacian and for the atomic Hardy space defined in this setting (see [8]). See [2, 5, 17, 27] and the references therein for boundedness results on Hardy spaces for first-order Riesz transforms on various doubling Riemannian manifolds. The relation between Riesz transforms and Hardy spaces associated with particular classes of operators has also been studied, for instance in [3, 10, 11, 25].

To the best of our knowledge, there are no results of this kind in the literature for Hardy-type spaces on nondoubling Lie groups. Results on the H^1-L^1 boundedness of Riesz transforms on some Lie groups of exponential growth can be found in [23] and [30], but it seems that the H^1-H^1 boundedness of such operators has not been investigated. Some results on the H^1-L^1 boundedness of Riesz transforms for a flow Laplacian on infinite trees equipped with flow measures of exponential growth are obtained in [15, 22, 21]; these can be thought as a discrete counterpart of the results on Lie groups mentioned above.

Finally we mention some negative results which recall in some sense the negative result in our Theorem 1.3.

On homogeneous trees, [6] says that the Hardy space defined in terms of the first-order Riesz transform associated with the standard combinatorial Laplacian does not admit an atomic decomposition. Santagati recently showed in [29] that the natural atomic Hardy space defined on homogeneous trees equipped with the canonical flow measure and the flow Laplacian does not admit a characterization in terms of the Riesz transform, proving a discrete counterpart of Theorem 1.3.

In the setting of Riemannian manifolds of exponential growth with bounded geometry and spectral gap, it was proved in [24] that the first-order Riesz transform is not bounded from the atomic Hardy space introduced in [4] into the space of integrable functions; hence it does not provide a characterization of this Hardy space. However, on the same class of manifolds, the Hardy space defined by means of the first-order Riesz transform was recently characterized as a suitably modified Hardy space of Goldberg type [19, 20, 26].

Our paper is organized as follows. Section 2 contains explicit formulas for the convolution kernels of the first-order Riesz transforms $\mathcal{R}_i$ and some of their derivatives. In Section 3, we prove Theorem 1.3, and Theorem 1.4 is proved in Section 4.

In this paper, C denotes a positive, finite constant which may vary from occurrence to occurrence and may depend on parameters according to the context. Given two positive quantities f and g, we mean by $f \lesssim g$ that there exists a constant C such that $f \leq C\,g$, and $f \sim g$ means that $g \lesssim f \lesssim g$.

2. The convolution kernels of the Riesz transforms

In this section, we write the formulas for the convolution kernels of the Riesz transforms of the first order and some of their derivatives, which were computed in [30]. First recall that the convolution of two (suitable) functions f, g on G is

$$f * g(x) = \int_G f(xy^{-1})\, g(y)\, \mathrm{d}\rho(y) \qquad \forall x \in G\,. \tag{2.1}$$

The convolution kernel from the right of the operator $\mathcal{R}_i = X_i\,\Delta^{-1/2}$ is the distribution pv k_i, where $k_i = X_i U$, and U is the convolution kernel of $\Delta^{-1/2}$ given by

$$U(x) = \frac{1}{2\pi^2}\,\delta^{1/2}(x)\,\frac{1}{r(x)\,\sinh r(x)}\,. \tag{2.2}$$

The integral kernel of $\mathcal{R}_i$ is given by the function

$$R_i(x,y) = \delta(y)\, k_i(y^{-1}x) \qquad x \neq y\,, \tag{2.3}$$

in the sense that for every $f \in C_0^\infty(G)$

$$\mathcal{R}_i f(x) = f * \mathrm{pv}\ k_i(x) = \lim_{\varepsilon \to 0^+} f * k_i^\varepsilon(x), \tag{2.4}$$

where for every $\varepsilon > 0$, $k_i^\varepsilon = k_i \chi_{B_\varepsilon^c}$ (see [30] for the details).

The explicit formulas for k_i were obtained in [30, formulae (2.7), (2.8)] by means of (2.2) and the fact that for $x \neq e$

$$X_i r(x) = \begin{cases} \frac{a - a^{-1} - a^{-1}(x_1^2 + x_2^2)}{2\,\sinh r(x)} = \frac{a - \cosh r}{\sinh r} & \text{if } i = 0 \\ \frac{x_i}{\sinh r(x)} & \text{if } i = 1, 2\,, \end{cases} \tag{2.5}$$

where $r = r(x) = r(x_1, x_2, a)$ (see [30, Lemma 2.1]).

For $i = 1, 2$ and $x \neq e$

$$2\pi^2 k_i(x) = -\, a^{-1}\, x_i\, \frac{\sinh r + r\,\cosh r}{r^2\,\sinh^3 r} \tag{2.6}$$

and

$$2\pi^2 k_0(x) = -\frac{\sinh r + r\,\cosh r}{r^2\,\sinh^3 r} + a^{-1}\,\frac{\cosh r \sinh r + r}{r^2\,\sinh^3 r}\,, \tag{2.7}$$

as verified in [30, formula (2.8)]. For all $x \neq e$ the derivative $X_1 k_1(x)$ can be computed using (2.6) and (2.5):

$$\begin{aligned} &2\,\pi^2\, X_1 k_1(x) \\ &\quad = a^{-1}\,\frac{x_1^2}{\sinh r}\,\frac{2r^2\cosh^2 r + r^2 + 2\sinh^2 r + 3r\sinh r\cosh r}{r^3 \sinh^4 r} - \frac{\sinh r + r\cosh r}{r^2 \sinh^3 r}\,. \end{aligned} \tag{2.8}$$

In particular

$$|X_1 k_1(x)| \lesssim a^{-1}\,\frac{x_1^2}{r\sinh^3 r} + \frac{1}{r\sinh^2 r} \lesssim \frac{1}{r\sinh^2 r} \qquad \forall x \in B_1^c\,, \tag{2.9}$$

the last step since $a^{-1}\,|x|^2 < 2\cosh r(x) \simeq \sinh r$ for these x.

Instead of computing the derivative $X_1 k_0(x)$ by means of formulas (2.7) and (2.5), we estimate it for $x \in B_1^c$. We need only observe that when we differentiate a power of $\cosh r$ or $\sinh r$ with respect to r, the order of magnitude does not change for $r > 1$, whereas a power of r gets smaller when differentiated. From (2.7) we then get

$$|X_1 k_0(x)| \lesssim \frac{|x_1|}{r\,\sinh^3 r} + a^{-1}\,\frac{|x_1|}{r^2\,\sinh^2 r} + a^{-1}\,\frac{|x_1|}{r\,\sinh^4 r}.$$

The last term here is no larger than the first term, since $a^{-1} < 2\cosh r \simeq \sinh r$, and we conclude that

$$|X_1 k_0(x)| \lesssim \frac{|x_1|}{r\,\sinh^3 r} + a^{-1}\,\frac{|x_1|}{r^2\,\sinh^2 r} \qquad \forall x \in B_1^c\,. \tag{2.10}$$

In the sequel we shall repeatedly use the following integration formula (see for instance [9, Lemma 1.3]): for any radial function f such that $\delta^{1/2} f$ is integrable

$$\int_G \delta^{1/2} f\,\mathrm{d}\rho = \int_0^\infty f(r)\, r\,\sinh r\,\mathrm{d}r\,. \tag{2.11}$$

3. Proof of Theorem 1.3

We recall a family of functions in the atomic Hardy space, which we introduced in [31]. Let $L > 2$ be large, and consider the rectangles $P_0 = [-1,1]\times[-1,1]\times[\frac{1}{\mathrm{e}},\mathrm{e}]$ and $P_L = (L,0,1)\cdot P_0 = [L-1,L+1]\times[-1,1]\times[\frac{1}{\mathrm{e}},\mathrm{e}]$. We then define $f_L = \chi_{P_L}-\chi_{P_0}$. Obviously f_L is a multiple of an atom and it was proved in [31] that $\|f_L\|_{H^1} \sim \log L$ for $L > 2$. We claim that there exists a positive constant C such that for $L > C$

$$\|\mathcal{R}_i f_L\|_1 \lesssim \log\log L, \qquad i = 0,1,2\,. \tag{3.1}$$

Once this has been verified, we deduce that

$$\lim_{L\to\infty} \frac{\|f_L\|_{H^1}}{\|f_L\|_{H^1_{\mathrm{Riesz}}}} = \infty\,,$$

which proves Theorem 1.3.

When we now prove (3.1), we will neglect the case $i = 2$, since it is completely analogous to $i = 1$.

Denote by $2P_0$ the rectangle $[-2,2]\times[-2,2]\times[\frac{1}{\mathrm{e}^2},\mathrm{e}^2]$ and by $2P_L$ the rectangle $(L,0,1)\cdot(2P_0)$. For $i = 0$ and 1 we shall estimate the L^1-norm of $\mathcal{R}_i f_L$ by integrating over different regions, in four steps.

Step 1. Since $\mathcal{R}_i$ is bounded on L^2 and $\rho(2P_0) = \rho(2P_L) \sim 1$, we get by applying the Cauchy–Schwarz inequality

$$\|\mathcal{R}_i f_L\|_{L^1(2P_0\cup 2P_L)} \lesssim \rho(2P_0\cup 2P_L)^{1/2}\,\|\mathcal{R}_i\|_{2\to2}\,\|f_L\|_2 \simeq 1\,. \tag{3.2}$$

Step 2. Choose a ball $B = B(e, r_B)$ with $r_B = (\log L)^2$. Then (1.1) implies that $B \supset 2P_0 \cup 2P_L$ if L is large enough. For any x

$$|\mathcal{R}_i\chi_{P_0}(x)| = \Big|\int \chi_{P_0}(y)\,k_i(y^{-1}x)\,\mathrm{d}\lambda(y)\Big| \lesssim \sup_{y\in P_0} |k_i(y^{-1}x)|\,, \tag{3.3}$$

since $\lambda(P_0)\sim 1$. If $x = (x_1,x_2,a)\in(2P_0)^c$ and $y = (y_1,y_2,b)\in P_0$, then

$$|r(y^{-1}x) - r(x)| = |d(y,x) - d(x,e)| \le d(y,e) \le C\,.$$

We will often use the following simple quotient formula:

$$y^{-1}x = (y_1,y_2,b)^{-1}(x_1,x_2,a) = (b^{-1}(x_1-y_1), b^{-1}(x_2-y_2), ab^{-1})\,. \tag{3.4}$$

Since here $ab^{-1}\sim a$ and $b^{-1}|x_1-y_1| \lesssim \max(1,|x_1|)$, applying (2.6) and (2.7) we obtain for $x\in(2P_0)^c$

$$|\mathcal{R}_1\chi_{P_0}(x)| \lesssim \sup_{y\in P_0}|k_1(y^{-1}x)| \lesssim a^{-1}\,\frac{1+|x|}{r(x)\,\sinh^2 r(x)} \tag{3.5}$$

and

$$|\mathcal{R}_0\chi_{P_0}(x)| \lesssim \sup_{y\in P_0}|k_0(y^{-1}x)| \lesssim \frac{1}{r(x)\sinh^2 r(x)} + a^{-1}\,\frac{1}{r(x)^2\sinh r(x)}\,. \tag{3.6}$$

We shall now integrate these quantities over the set $B\setminus 2P_0$. For a point $x\notin 2P_0$ it is elementary to verify that (1.1) implies $r(x) > 1$. If also $x = (x_1,x_2,a)\in B$, the same formula shows that $e^{-r_B}\le a\le e^{r_B}$ and also

$$a + a^{-1} \lesssim \cosh r(x) \simeq \sinh r(x).$$

For the expression in (3.5) we get

$$\begin{aligned}\int_{B\setminus 2P_0} a^{-1}\,\frac{1+|x|}{r(x)\sinh^2 r(x)}\,\mathrm{d}\rho(x) &\lesssim \int_{e^{-r_B}}^{e^{r_B}} \frac{da}{\log(a+a^{-1})}\int \frac{1+|x|}{(a^2+1+|x|^2)^2}\,dx\\ &\simeq \int_{e^{-r_B}}^{e^{r_B}} \frac{da}{(a+1)\log(a+a^{-1})}\\ &\lesssim \log\log e^{r_B}\\ &\lesssim \log\log L.\end{aligned}$$

The argument for the first summand in (3.6) is similar and left to the reader. For the second summand in (3.6), we use the integration formula (2.11) and the fact that $B(e,1)\subset 2P_0$. This gives

$$\int_{B\setminus 2P_0} a^{-1}\,\frac{1}{r(x)^2\sinh r(x)}\,\mathrm{d}\rho(x) \lesssim \int_1^{r_B} \frac{r\sinh r}{r^2\sinh r}\,dr \lesssim \log r_B \simeq \log\log L.$$

We conclude that for $i=1$ and $i=0$

$$\int_{B\setminus 2P_0} |\mathcal{R}_i\chi_{P_0}|\,\mathrm{d}\rho \lesssim \log\log L\,. \tag{3.7}$$

Here we observe that this inequality remains true, with the same proof, if B is replaced by the doubled ball $2B=B(e,2r_B)$.

The inequality (3.7) implies a similar estimate for $\mathcal{R}_i\chi_{P_L}$ in $B\setminus 2P_L$. Indeed, $\mathcal{R}_i\chi_{P_L}(x)=\mathcal{R}_i\chi_{P_0}(\tau_L x)$, where $\tau_L x=(-L,0,1)\cdot x$, for any x in G. Using the facts that $\tau_L P_L=P_0$ and $\tau_L B\subset 2B$ for large L, and changing variable $\tau_L x=v$, we obtain

$$\begin{aligned}\int_{B\setminus 2P_L} |\mathcal{R}_i\chi_{P_L}(x)|\,\mathrm{d}\rho(x) &= \int_{B\setminus 2P_L} |\mathcal{R}_i\,\chi_{P_0}(\tau_L x)|\,\mathrm{d}\rho(x)\\ &= \int_{\tau_L B\setminus \tau_L(2P_L)} |\mathcal{R}_i\chi_{P_0}(v)|\,\delta(-L,0,1)\,\mathrm{d}\rho(v)\\ &\le \int_{2B\setminus 2P_0} |\mathcal{R}_i\chi_{P_0}|\,\mathrm{d}\rho\\ &\lesssim \log\log L\,.\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), it now follows that

$$\int_{B\setminus(2P_0\cup 2P_L)} |\mathcal{R}_i f_L|\,\mathrm{d}\rho \lesssim \log\log L\,,\qquad i=0,1,$$

which ends Step 2.

Step 3. To deal with the complement of the ball B, we will use cancellation between the two parts of f_L. For a point $x\in B^c$, we write the convolution $f_L * k_i(x)$ as

$$\begin{aligned}f_L * k_i(x) &= \int_{P_L} k_i\big(y^{-1}x\big)\,\mathrm{d}\lambda(y) - \int_{P_0} k_i(y^{-1}x)\,\mathrm{d}\lambda(y)\\ &= \int_{P_0} \big[k_i\big(y^{-1}(-L,0,1)x\big) - k_i(y^{-1}x)\big]\,\mathrm{d}\lambda(y)\,.\end{aligned} \tag{3.9}$$

Let now $y^{-1}=(y_1,y_2,b)$ be any point in $(P_0)^{-1}$ and $x=(x_1,x_2,a)$ any point in B^c. Then $y^{-1}(-L,0,1)x=y^{-1}(-L,0,1)y\cdot y^{-1}x=(-bL,0,1)\,y^{-1}x$, and the Mean

Value Theorem implies

$$k_i\big(y^{-1}(-L,0,1)x\big)-k_i(y^{-1}x)=-bL\,\partial_1 k_i\big((s,0,1)\,y^{-1}x\big), \tag{3.10}$$

for some $s\in(-bL,0)$.

By the triangle inequality,

$$\begin{aligned}|r\big((s,0,1)\cdot y^{-1}x\big)-r(x)|&=|d\big(x,y\,\cdot(-s,0,1)\big)-d(x,e)|\\&\le d\big(y\,\cdot(-s,0,1),e\big)\\&\le d\big(y\,\cdot(-s,0,1),y\big)+d\big(y,e\big)\\&=d\big((-s,0,1),e\big)+d\big(y,e\big)\\&\lesssim \log L\,,\end{aligned}$$

where we also used (1.1). Now $r(x)>(\log L)^2$ implies $r\big((s,0,1)\,y^{-1}\,x\big)\sim r(x)$, and also $r\big((s,0,1)\,y^{-1}\,x\big)>1$, since L is large. Further,

$$\sinh r\big((s,0,1)\cdot y^{-1}x\big)\gtrsim \sinh\big(r(x)-\log L\big)\gtrsim\frac{\sinh r(x)}{L}\,. \tag{3.11}$$

In the rest of this step, we will deal only with the case $i=1$.

The estimate (2.9) now implies, since $X_1=a\partial_1$ and $b\simeq 1$,

$$\big|\partial_1 k_1\big((s,0,1)y^{-1}x\big)\big|\lesssim (ba)^{-1}\,\frac{1}{r\big((s,0,1)\,y^{-1}\,x\big)\sinh^2 r\big((s,0,1)\,y^{-1}\,x\big)}\,.$$

We deduce from (3.10) and (3.11) that for $x\in B^c$ and $y\in P_0$

$$\begin{aligned}\big|k_1\big(y^{-1}(-L,0,1)x\big)-k_1(y^{-1}x)\big|&\lesssim L\,a^{-1}\,\frac{1}{r\big((s,0,1)\,y^{-1}\,x\big)\,\sinh^2 r\big((s,0,1)\,y^{-1}\,x\big)}\\&\lesssim L^3\,a^{-1}\,\frac{1}{r(x)\sinh^2 r(x)}.\end{aligned} \tag{3.12}$$

The same bound holds for $|\mathcal{R}_1 f_L|$ in B^c because of (3.9).

By applying the integration formula (2.11), we finish Step 3 concluding that

$$\int_{B^c}|\mathcal{R}_1 f_L|\,\mathrm{d}\rho\lesssim L^3\int_{r_B}^{\infty}\frac{1}{r\,\sinh^2 r}\,r\,\sinh r\,\mathrm{d}r\lesssim\frac{L^3}{\mathrm{e}^{r_B}}\lesssim 1\,. \tag{3.13}$$

Step 4. It remains to take $i=0$ and estimate the integral of $\mathcal{R}_0 f_L$ over the complement of B. This requires some modifications from the argument in the preceding step.

We split B^c into the following three parts:

$$\begin{aligned}\Gamma_1&=\{x=(x_1,x_2,a)\in B^c:\,a<a^*\quad\text{and}\quad|x|<f(a)\}\,,\\\Gamma_2&=\{x=(x_1,x_2,a)\in B^c:\,a\ge a^*\}\,,\\\Gamma_3&=\Big\{x=(x_1,x_2,a)\in B^c:\,a<a^*,\;\;|x|\ge f(a)\Big\}\,,\end{aligned}$$

where $a^*=\mathrm{e}^{-r_B/8}$ and $f(a)=\exp(\log a^{-1})^{3/4}$.

To integrate $\mathcal{R}_0 f_L$ over Γ_1, we consider $\mathcal{R}_0\chi_{P_0}$ and $\mathcal{R}_0\chi_{P_L}$ separately, and apply (3.6). Observe that any point $x\in\Gamma_1$ satisfies $\sinh r(x)\simeq\cosh r(x)>a^{-1}(1+$

$|x|^2)/2$ and thus $r(x) \gtrsim \log a^{-1}$. Starting with $\mathcal{R}_0\chi_{P_0}$ and the last term in (3.6), we have

$$\begin{aligned}\int_{\Gamma_1} a^{-1} \frac{1}{r(x)^2 \sinh r(x)} \,\mathrm{d}\rho(x) &\lesssim \int_0^{a^*} \frac{\mathrm{d}a}{a}\, a^{-1} \int_{|x|<f(a)} \frac{\mathrm{d}x}{(\log a^{-1})^2 \; a^{-1}\,(1+|x|^2)} \\ &\simeq \int_0^{a^*} \frac{\mathrm{d}a}{a} \frac{\log f(a)}{(\log a^{-1})^2} \\ &= \int_0^{a^*} \frac{\mathrm{d}a}{a\,(\log a^{-1})^{5/4}} \lesssim 1.\end{aligned}$$

The first term in (3.6) can be treated in a similar but easier way, and so

$$\int_{\Gamma_1} \mathcal{R}_0\chi_{P_0} \,\mathrm{d}\rho \lesssim 1.$$

A translation argument like (3.8) shows that the same estimate holds for $\mathcal{R}_0\chi_{P_L}$, and thus for $\mathcal{R}_0 f_L$.

In order to estimate the integrals over Γ_2 and Γ_3, we will use (3.9) and (3.10), with $i = 0$. Combining (3.10) and (2.10), we see that

$$\left|\partial_1 k_0\big((s,0,1)y^{-1}x\big)\right| \lesssim (ba)^{-1} \frac{|s+y_1+bx_1|}{r \sinh^3 r} + (ba)^{-2} \frac{|s+y_1+bx_1|}{r^2 \sinh^2 r},$$

where $r = r\big((s,0,1)\,y^{-1}\,x\big) \gtrsim 1$. Using (3.11) and the facts that $b \simeq 1$ and $0 < s \lesssim L$, we conclude that

(3.14)
$$\left|k_0\big(y^{-1}(-L,0,1)x\big) - k_0(y^{-1}x)\right| \lesssim L^C \frac{1+|x|}{a\,r(x) \sinh^3 r(x)} + L^C \frac{1+|x|}{a^2\, r(x)^2 \sinh^2 r(x)},$$

for some C.

To integrate the first summand here, we neglect the factor $r(x)$ and apply the following estimate, valid for $r(x) > r_B$,

$$\sinh r(x) \simeq \cosh r(x) \gtrsim e^{r_B} + a + a^{-1}(1+|x|^2). \tag{3.15}$$

Further, we extend the integration to B^c which contains $\Gamma_2 \cup \Gamma_3$, and get

$$\begin{aligned}\int_{B^c} L^C \frac{1+|x|}{a\,r(x) \sinh^3 r(x)} \,\mathrm{d}\rho(x) &\lesssim L^C \int_0^\infty \frac{da}{a} \int \frac{1+|x|}{a\,\left(e^{r_B} + a + a^{-1}(1+|x|^2)\right)^3}\, dx \\ &= L^C \int_0^\infty a\, da \int \frac{1+|x|}{\left(a\, e^{r_B} + a^2 + 1 + |x|^2\right)^3}\, dx \\ &\lesssim L^C \int_0^\infty a\,\left(a\, e^{r_B} + a^2\right)^{-3/2} da \\ &\lesssim L^C \int_0^{e^{r_B}} a^{-1/2}\, e^{-3r_B/2}\, da + L^C \int_{e^{r_B}}^\infty a^{-2}\, da \\ &\lesssim L^C\, e^{-r_B} \lesssim 1\,.\end{aligned}$$

To integrate the second term in (3.14), at first over Γ_2, we use (3.15) and argue almost as above. Thus

$$\begin{aligned}\int_{\Gamma_2} L^C \frac{1+|x|}{a^2\, r(x)^2\, \sinh^2 r(x)}\, \mathrm{d}\rho(x) &= L^C \int_{a^*}^{\infty} \frac{da}{a} \int \frac{1+|x|}{\left(a\, e^{r_B} + a^2 + 1 + |x|^2\right)^2}\, dx \\ &\lesssim L^C \int_{a^*}^{\infty} \frac{1}{a\, \left(a\, e^{r_B} + a^2\right)^{1/2}}\, da \\ &\lesssim L^C \int_{a^*}^{e^{r_B}} a^{-3/2}\, e^{-r_B/2}\, da + L^C \int_{e^{r_B}}^{\infty} a^{-2}\, da \\ &\lesssim L^C\, (a^*)^{-1/2}\, e^{-r_B/2} + L^C\, e^{-r_B} \lesssim 1.\end{aligned}$$

Integrating the same term over Γ_3 in a similar way, we get at most

$$\begin{aligned}L^C \int_0^{a^*} \frac{da}{a} \int_{|x|>f(a)} \frac{1+|x|}{\left(1+|x|^2\right)^2}\, dx \lesssim L^C \int_0^{a^*} \frac{da}{a\, f(a)} &= L^C \int_0^{a^*} \frac{da}{a\, \exp(\log a^{-1})^{3/4}} \\ &= \frac{4}{3}\, L^C \int_{(\log(a^*)^{-1})^{3/4}}^{\infty} \frac{s^{1/3}}{e^s}\, ds \\ &\simeq L^C\, \frac{(\log\, (a^*)^{-1})^{1/3}}{\exp[(\log\, (a^*)^{-1})^{3/4}]} \lesssim 1\,;\end{aligned}$$

here the change of variables used is $s = (\log a^{-1})^{3/4}$.

Summing up Step 4, we conclude

$$\int_{B^c} |\mathcal{R}_0 f_L|\, \mathrm{d}\rho \lesssim 1\,.$$

The combined results of Steps 1–4 now prove (3.1) and thus also Theorem 1.3.

4. Proof of Theorem 1.4

In this section we give the proof of Theorem 1.4 only for $i = 1$, since the case $i = 2$ is completely analogous. To do so, we first prove a lemma.

Lemma 4.1. *Every atom A satisfies $\int \mathcal{R}_1 A\, \mathrm{d}\rho = 0$.*

Proof. It is enough to prove the lemma for an atom A supported in a Calderón–Zygmund set $R = [-L/2, L/2]^2 \times [\mathrm{e}^{-\alpha}, \mathrm{e}^{\alpha}]$ centred at $(0,0,1)$. For every $\tau > 0$ large and $\varepsilon > 0$ we let $\chi_{\tau,\varepsilon}$ denote the characteristic function of the set $\{x = (x_1, x_2, a) : 1/\tau < a < \tau, \quad r(x) > \varepsilon\}$. Set

$$\mathcal{R}_1^{\tau,\varepsilon} A = A * (k_1\, \chi_{\tau,\varepsilon}).$$

Then $\int \mathcal{R}_1^{\tau,\varepsilon} A\, \mathrm{d}\rho = 0$ by Fubini's theorem and the cancellation condition of the atom. We now claim that $\mathcal{R}_1^{\tau,\varepsilon} A \to \mathcal{R}_1^{\varepsilon} A$ in $L^1(\rho)$ as $\tau \to \infty$, where $\mathcal{R}_1^{\varepsilon} A = \mathcal{R}_1^{\infty,\varepsilon} A$. Indeed, for $x = (x_1, x_2, a)$

$$\mathcal{R}_1^{\tau,\varepsilon} A(x) = \int \delta(y)\, k_1(y^{-1}x)\, \chi_{\tau,\varepsilon}(y^{-1}x)\, A(y)\, \mathrm{d}\rho(y),$$

and $y = (y_1, y_2, b) \in \operatorname{supp} A$ implies $\mathrm{e}^{-\alpha} \le b \le \mathrm{e}^{\alpha}$. If $a > \mathrm{e}^{\alpha}\tau$ or $a < \mathrm{e}^{-\alpha}\tau^{-1}$, then $\mathcal{R}_1^{\tau,\varepsilon} A(x) = 0$, and if $\mathrm{e}^{\alpha}\tau^{-1} < a < \mathrm{e}^{-\alpha}\tau$ then $\mathcal{R}_1^{\tau,\varepsilon} A(x) = \mathcal{R}_1^{\varepsilon} A(x)$.

For every $\varepsilon > 0$, $\mathcal{R}_1^{\varepsilon} A = \mathcal{R}_1 A \in L^1(\rho)$ in the set $\{x : d(x, \operatorname{supp} A) > \varepsilon\}$, while $\mathcal{R}_1^{\varepsilon} A$ is given by the convolution of the atom and a bounded kernel on the compact set $\{x : d(x, \operatorname{supp} A) \le \varepsilon\}$, so that $\mathcal{R}_1^\varepsilon A$ is in $L^1(\rho)$. Hence to prove the claim it is enough to verify that

$$\int_{\mathrm{e}^{-\alpha}\tau<a<\mathrm{e}^{\alpha}\tau} |\mathcal{R}_1 A(x)| \ \mathrm{d}\rho(x) \to 0 \tag{4.1}$$

and

$$\int_{\mathrm{e}^{-\alpha}\tau^{-1}<a<\mathrm{e}^{\alpha}\tau^{-1}} |\mathcal{R}_1 A(x)| \ \mathrm{d}\rho(x) \to 0 \tag{4.2}$$

as $\tau \to \infty$.

We estimate $k_1(y^{-1}x)$, with x as in (4.1) or (4.2) and $y \in \operatorname{supp} A$. Then $|x_1 - y_1| \le C(|x_1|+1)$. Here and below constants C may depend on A, thus on L and α.

For x as in (4.1), we have $b^{-1}a \simeq \tau$ and $\cosh r(y^{-1}x) \simeq \tau^{-1}(\tau^2+|x|^2)$, and $r(y^{-1}x) \gtrsim \log\tau$.

From (2.6) we get

$$|k_1(y^{-1}x)| \lesssim \frac{|x_1|+1}{\tau}\,\frac{1}{\tau^{-2}\,(\tau^2+|x|^2)^2\,\log\tau} \lesssim \frac{\tau}{\log\tau}\,\frac{|x_1|+1}{(\tau+|x|)^4},$$

and thus

$$\int_{\mathrm{e}^{-\alpha}\tau<a<\mathrm{e}^{\alpha}\tau} \mathrm{d}\rho(x) \int \delta(y)\,|k_1(y^{-1}x)|\,|A(y)| \ \mathrm{d}\rho(y) \lesssim \frac{1}{\log\tau}, \tag{4.3}$$

which implies (4.1).

If instead x is as in (4.2), we have $b^{-1}a \simeq \tau^{-1}$ and $\cosh r(y^{-1}x) \simeq \tau(1+|x|^2)$, and $r(y^{-1}x) \gtrsim \log\tau$. Then

$$|k_1(y^{-1}x)| \lesssim \tau\,(|x_1|+1)\,\frac{1}{\tau^2\,(1+|x|^2)^2\,\log\tau} \lesssim \frac{1}{\tau\,\log\tau}\,\frac{1}{(1+|x|)^3},$$

and

$$\int_{\mathrm{e}^{-\alpha}\tau^{-1}<a<\mathrm{e}^{\alpha}\tau^{-1}} \mathrm{d}\rho(x) \int \delta(y)\,|k_1(y^{-1}x)|\,|A(y)| \ \mathrm{d}\rho(y) \lesssim \frac{1}{\tau\,\log\tau}. \tag{4.4}$$

From this, (4.2) follows, and the claim is proved. Since $\int \mathcal{R}_1^{\tau,\varepsilon} A \ \mathrm{d}\rho = 0$, this implies that $\int \mathcal{R}_1^\varepsilon A \ \mathrm{d}\rho = 0$.

Notice that $\mathcal{R}_1^\varepsilon A = \mathcal{R}_1 A$ on $\{x : d(x, \mathrm{supp}A) > 1\}$ and by (2.4), $|\mathcal{R}_1^\varepsilon A - \mathcal{R}_1 A|$ converges to 0 pointwise and is dominated by $\sup_{\varepsilon>0} |\mathcal{R}_1^\varepsilon A|$ which is square integrable and then integrable on the compact set $\{x : d(x, \mathrm{supp}A) \le 1\}$. Hence

$$\begin{aligned}\int \mathcal{R}_1 A \ \mathrm{d}\rho &= \int_{\{x:d(x,\mathrm{supp}A)>1\}} \mathcal{R}_1^\varepsilon A \,\mathrm{d}\rho + \lim_{\varepsilon\to 0} \int_{\{x:d(x,\mathrm{supp}A)\le 1\}} \mathcal{R}_1^\varepsilon A \,\mathrm{d}\rho \\ &= \lim_{\varepsilon\to 0} \int \mathcal{R}_1^\varepsilon A \,\mathrm{d}\rho = 0,\end{aligned}$$

as required.

□

Proof of Theorem 1.4. It is enough to show that

$$\sup\{\|\mathcal{R}_1 A\|_{H^1} : A \text{ is a } (1,\infty)\text{−atom}\} < \infty\,. \tag{4.5}$$

Indeed, let $f \in H^1$ and write $f = \sum_j \lambda_j A_j$, where A_j are $(1,\infty)$-atoms and $\sum_j |\lambda_j| < 2\|f\|_{H^1}$. Since by [13, Theorem 2.3] $\mathcal{R}_1$ is of weak type $(1,1)$ and the sum $\sum_j \lambda_j A_j$ is convergent in L^1,

$$\mathcal{R}_1 f = \sum_j \lambda_j \mathcal{R}_1 A_j$$

with convergence in $L^{1,\infty}$, so that (4.5) would imply

$$\|\mathcal{R}_1 f\|_{H^1} \le C \sum_j |\lambda_j| \le C\, \|f\|_{H^1}$$

and thus Theorem 1.4.

To prove (4.5) we take a $(1,\infty)$-atom A supported in a Calderón–Zygmund set $R = [-L/2, L/2]^2 \times [\mathrm{e}^{-\alpha}, \mathrm{e}^{\alpha}]$ centred at $(0,0,1)$. We call A a large atom if $\alpha \ge 20$, a small atom if $\alpha < 1$ and an intermediate atom if $1 \le \alpha < 20$.

For each $x \in G$, we have

$$|\mathcal{R}_1 A(x)| \le \int_R |A(y)|\, |k_1(y^{-1}x)|\delta(y)\,\mathrm{d}\rho(y) \le \rho(R)^{-1} \int_R |k_1(y^{-1}x)|\delta(y)\,\mathrm{d}\rho(y)\,. \tag{4.6}$$

With $y \in R$, we will use (2.6) to estimate the value of k_1 at the point $z = y^{-1}x$, in a way depending on the position of x in the group G. We first observe that (1.1) implies for a point $z = (z_1, z_2, d)$ such that $d < 1/2$

$$\sinh r(z) \sim \cosh r(z) \sim d^{-1}\,(1 + |z|^2)$$

and $r(z) \gtrsim 1$. From (2.6) we then get

$$|k_1(z)| \lesssim d^{-1}\,|z|\, \frac{1}{r(z)\,\sinh^2 r(z)} \lesssim \frac{d\,|z|}{(1+|z|^2)^2}\,. \tag{4.7}$$

On the other hand, if $d > 2$ then

$$\cosh r(z) = \frac{d + d^{-1}|z|^2}{2}\Big(1 + O\Big(\frac{1}{d^2+|z|^2}\Big)\Big)\,, \tag{4.8}$$

and the same equality holds for $\sinh r(z)$. Further,

$$r(z) \sim \log \frac{d^2+|z|^2}{d} \gtrsim \log d. \tag{4.9}$$

From (2.6) we conclude when $d > 2$

$$\begin{aligned} 2\pi^2 k_1(z) &= -d^{-1} z_1 \frac{\frac{d+d^{-1}|z|^2}{2}\Big(1+O\Big(\frac{1}{d^2+|z|^2}\Big)\Big) + r(z)\frac{d+d^{-1}|z|^2}{2}\Big(1+O\Big(\frac{1}{d^2+|z|^2}\Big)}{r^2(z)\Big(\frac{d+d^{-1}|z|^2}{2}\Big)^3\Big(1+O\Big(\frac{1}{d^2+|z|^2}\Big)\Big)} \\ &= -\frac{4dz_1}{(d^2+|z|^2)^2}\,\frac{1+r(z)}{r^2(z)}\Big(1+O\Big(\frac{1}{d^2+|z|^2}\Big)\Big)\,. \end{aligned} \tag{4.10}$$

Since A has vanishing integral, so has $\mathcal{R}_1 A$. Moreover, we shall see that $\mathcal{R}_1 A$ is integrable and has vanishing integral on each horizontal plane $\mathbb{R}^2 \times \{a\}$ for $a < \mathrm{e}^{-\alpha}$ and for $a > \mathrm{e}^{\alpha}$. We first verify that the function $x \mapsto k_1(y^{-1}x)$ has these properties for any $y = (y_1, y_2, b) \in R$.

If $x = (x_1, x_2, a)$ with $a \notin [\mathrm{e}^{-\alpha}, \mathrm{e}^{\alpha}]$ and $y \in R$, we see from (3.4) that $r(y^{-1}x) \gtrsim 1$. Then (2.6) implies that

$$\int |k_1(y^{-1}(x_1, x_2, a)|\ \mathrm{d}x_1\, \mathrm{d}x_2 < \infty\,.$$

Now $r(y^{-1}x)$ is a function of a, b and $(x_1 - y_1)^2 + (x_2 - y_2)^2$, and from (2.6) we see that $k_1(y^{-1}(x_1, x_2, a))$ is an odd function in $x_1 - y_1$. Thus

$$\int k_1(y^{-1}(x_1, x_2, a))\ \mathrm{d}x_1\, \mathrm{d}x_2 = 0. \tag{4.11}$$

Integrating this equality against $A(y)\delta(y)\ \mathrm{d}\rho(y)$, we conclude that

$$\int \mathcal{R}_1 A(x_1, x_2, a)\ \mathrm{d}x_1\, \mathrm{d}x_2 = 0, \qquad a \notin [\mathrm{e}^{-\alpha}, \mathrm{e}^{\alpha}], \tag{4.12}$$

and the integral is absolutely convergent.

We treat the three atom sizes separately.

Case I: large atom. In this case $\mathrm{e}^{2\alpha} \le L < \mathrm{e}^{8\alpha}$. We will construct an atomic decomposition of $\mathcal{R}_1 A$.

Let us first give a rough description of the idea. We will split G into slices of type $S_k = \mathbb{R}^2 \times I_k$, for disjoint intervals I_k, $k \in \mathbb{Z}$. Then (4.12) and Lemma 4.1 will imply that

$$\int_{S_k} \mathcal{R}_1 A\ \mathrm{d}\rho = 0, \tag{4.13}$$

which will make it possible to decompose $\mathcal{R}_1 A\, \chi_{S_k}$ into atoms.

For each k, the slice S_k will be split into disjoint sets $B_{k,j} = \tilde{B}_{k,j} \times I_k$, $j = 0, 1, \dots$, where the $\tilde{B}_{k,j}$, $j = 0, 1, \dots$, form a partition of $\mathbb{R}^2$ and expand exponentially towards infinity as $j \to +\infty$. We want to make each $\mathcal{R}_1 A\, \chi_{B_{k,j}}$ into an atom multiple, supported in a suitable Calderón-Zygmund set $Z_{k,j} \supseteq B_{k,j}$. The $Z_{k,j}$ will not be disjoint but increasing in j. Then

$$\mathcal{R}_1 A\, \chi_{B_{k,j}} - \int_{B_{k,j}} \mathcal{R}_1 A\ \mathrm{d}\rho\ \frac{\chi_{Z_{k,j}}}{\rho(Z_{k,j})}$$

has moment 0 and is an atom multiple. But these functions do not sum up in j to $\mathcal{R}_1 A\, \chi_{S_k}$ as we want. Instead, we will modify $\mathcal{R}_1 A\, \chi_{B_{k,j}}$ by two quantities involving $B_{k,\ell}$ for all $\ell \ge j$. This can be seen as a way of transporting the "mass" $\int_{B_{k,0}} \mathcal{R}_1 A\ \mathrm{d}\rho$ in the innermost $B_{k,0}$ step by step through $B_{k,1}, B_{k,2}, \dots$ towards infinity. At each step one leaves in $B_{k,j}$ the mass $-\int_{B_{k,j}} \mathcal{R}_1 A\ \mathrm{d}\rho$ needed there. This leads to a telescopic sum, and produces atom multiples summing up to $\mathcal{R}_1 A\, \chi_{S_k}$.

Since the slice S_0 will contain the support of A, the case $k = 0$ will require a special treatment.

Let us now go into the details of this construction. We decompose G as the union of the following three regions:

$$\begin{aligned}
\Omega_1 &= \{x = (x_1, x_2, a) \in G :\ a < \mathrm{e}^{-1-3\alpha/2}\}\,,\\
\Omega_2 &= \{x = (x_1, x_2, a) \in G :\ \mathrm{e}^{-1-3\alpha/2} \le a \le \mathrm{e}^{1+3\alpha/2}\}\,,\\
\Omega_3 &= \{x = (x_1, x_2, a) \in G :\ a > \mathrm{e}^{1+3\alpha/2}\}\,.
\end{aligned}$$

For a point x in any of these regions, we will derive size estimates of $\mathcal{R}_1 A(x)$ via estimates of the kernel $k_1(y^{-1}x)$, where $y = (y_1, y_2, b) \in R$. Observe that if $|x| > 2L$ then $L < |x - y| \sim |x|$, but if $|x| \le 2L$ then $|x - y| \le 3L$.

Let $x \in \Omega_1$, so that $b^{-1}a < 1/4$. Consider first the case $|x| > 2L$. With $z = y^{-1}x$ in (4.7), we get

$$|k_1(y^{-1}x)| \lesssim \frac{b^{-1}a\, b^{-1}|x-y|}{(1+b^{-2}|x-y|^2)^2} \lesssim \frac{a\, b^2}{|x|^3}\,.$$

We conclude from (4.6) that for $|x| > 2L$

$$|\mathcal{R}_1 A(x)| \lesssim \frac{1}{\alpha L^2} \int_{\mathrm{e}^{-\alpha}}^{\mathrm{e}^{\alpha}} \int_{|y|<L/2} \frac{ab^2}{|x|^3}\, b^{-2}\, \frac{\mathrm{d}y\ \mathrm{d}b}{b} \lesssim \frac{a}{|x|^3}\,. \tag{4.14}$$

In the case when $|x| \le 2L$, we similarly conclude from (4.7) and (4.6)

$$\begin{aligned}
|\mathcal{R}_1 A(x)| &\lesssim \frac{1}{\alpha L^2} \int_{\mathrm{e}^{-\alpha}}^{\mathrm{e}^{\alpha}} \int_{|y|<L/2} \frac{ab^{-2}\,|x-y|}{(1+b^{-2}|x-y|^2)^2}\, b^{-2} \frac{\mathrm{d}y\ \mathrm{d}b}{b} \\
&= \frac{a}{\alpha L^2} \int_{\mathrm{e}^{-\alpha}}^{\mathrm{e}^{\alpha}} \int_{|v|<3L/b} \frac{b^{-1}\,|v|}{(1+|v|^2)^2}\, \frac{\mathrm{d}v\ \mathrm{d}b}{b} \\
&\lesssim \frac{a}{\alpha L^2} \int_{\mathrm{e}^{-\alpha}}^{\mathrm{e}^{\alpha}} \frac{\mathrm{d}b}{b^2} \\
&\le \frac{a\mathrm{e}^{\alpha}}{\alpha L^2}\,.
\end{aligned} \tag{4.15}$$

Let for $k = -1, -2, \dots$

$$S_k = \mathbb{R}^2 \times \left[\mathrm{e}^{-2^{|k|}-3\alpha/2}, \mathrm{e}^{-2^{|k|-1}-3\alpha/2}\right]$$

and

$$B_{k,j} = \{x \in S_k : \, 2\mathrm{e}^{j-1}L \le |x| \le 2\mathrm{e}^{j}L\}$$

for $j = 1, 2, \dots,$ but for $j = 0$ instead

$$B_{k,0} = \{x \in S_k : \, |x| \le 2L\}.$$

Except for boundaries, S_k is the disjoint union of the $B_{k,j}$, and Ω_1 is the disjoint union of the S_k. The measure of $B_{k,j}$ is $\rho(B_{k,j}) \sim e^{2j}L^2 2^{|k|}$. For $x \in B_{k,j}$ with $j \ge 1$, (4.14) implies that

$$|\mathcal{R}_1 A(x)| \lesssim \frac{\mathrm{e}^{-2^{|k|-1}-3\alpha/2}}{e^{3j}L^3} \lesssim \frac{\mathrm{e}^{-2^{|k|-1}-\alpha/2}}{e^{3j}L^2}. \tag{4.16}$$

The last bound here holds also for $j = 0$, since (4.15) yields

$$|\mathcal{R}_1 A(x)| \lesssim \frac{\mathrm{e}^{-2^{|k|-1}-\alpha/2}}{L^2},$$

if $x \in B_{k,0}$. For $k \le -1$ and $j \ge 0$, we define

$$Z_{k,j} = [-2\mathrm{e}^{j}L, 2\mathrm{e}^{j}L]^2 \times [\mathrm{e}^{-3\cdot 2^{|k|-1}-3\alpha-j/2}, \mathrm{e}^{-2^{|k|-1}+j/2}]\,.$$

This set contains $B_{k,j}$ and is a Calderón–Zygmund set centered at $\left(0, 0, \mathrm{e}^{-2\cdot 2^{|k|-1}-3\alpha/2}\right)$, of parameter $2^{|k|-1} + 3\alpha/2 + j/2$ and of measure $\rho(Z_{k,j}) \sim \mathrm{e}^{2j}\, L^2\, (2^{|k|} + \alpha + j)$.

We define for $k \le -1$ and $j = 0, 1, \dots$

$$f_{k,j} = \rho(Z_{k,j})^{-1} \chi_{Z_{k,j}}$$

and

$$m_{k,j} = \sum_{\ell=j}^{\infty} \int_{B_{k,\ell}} \mathcal{R}_1 A \; \mathrm{d}\rho.$$

Observe that (4.12) implies (4.13) for each $k \leq -1$, which means that $m_{k,0} = 0$. For $j \geq 1$ we get from (4.16)

$$|m_{k,j}| \lesssim \sum_{\ell=j}^{\infty} \frac{\mathrm{e}^{-2^{|k|-1}-\alpha/2}\, 2^{|k|}}{\mathrm{e}^{\ell}} \sim \frac{\mathrm{e}^{-2^{|k|-1}-\alpha/2}\, 2^{|k|}}{\mathrm{e}^{j}}. \tag{4.17}$$

The functions

$$A_{k,j} = \mathcal{R}_1 A\, \chi_{B_{k,j}} - m_{k,j}\, f_{k,j} + m_{k,j+1}\, f_{k,j+1}$$

are bounded and supported in $Z_{k,j+1}$, and they are easily seen to have integrals 0. Thus they are multiples of $(1, \infty)$-atoms. Their sum over j is

$$\sum_{j=0}^{\infty} A_{k,j} = \mathcal{R}_1 A\, \chi_{S_k} - m_{k,0}\, f_{k,0} = \mathcal{R}_1 A\, \chi_{S_k}.$$

Thus we have an atomic decomposition of $\mathcal{R}_1 A\, \chi_{S_k}$. Summing over $k = -1, -2, \ldots,$ we get a decomposition of $\mathcal{R}_1 A\, \chi_{\Omega_1}$. Combining (4.16) and (4.17), and multiplying (4.16) by $\rho(Z_{k,j+1})$, we obtain

$$\|A_{k,j}\|_{H^1} \lesssim \frac{\mathrm{e}^{-2^{|k|-1}}\, \mathrm{e}^{-\alpha/2}\, (2^{|k|} + \alpha + j + 1)}{\mathrm{e}^{j}} + \frac{\mathrm{e}^{-2^{|k|-1}}\mathrm{e}^{-\alpha/2}\, 2^{|k|}}{\mathrm{e}^{j}}.$$

The right-hand side here is summable in j and k, and we conclude that

$$\|\mathcal{R}_1 A\, \chi_{\Omega_1}\|_{H^1} \lesssim 1. \tag{4.18}$$

We next consider the region Ω_3, and estimate $\mathcal{R}_1 A$ there. Let $x \in \Omega_3$ with $|x| > 2L$, and take $y = (y_1, y_2, b) \in R$. Setting $\tilde{y} = (0, 0, b)$ we decompose $\mathcal{R}_1 A$ as

$$\begin{aligned}
\mathcal{R}_1 A(x) &= \int_R A(y)[k_1(y^{-1}x)\delta(y) - k_1(\tilde{y}^{-1}x)\delta(y)]d\rho(y) \\
&\quad + \int_R A(y)[k_1(\tilde{y}^{-1}x)\delta(y) - k_1(x)]d\rho(y) \\
&= \mathcal{R}_1' A(x) + \mathcal{R}_1'' A(x)\,.
\end{aligned}$$

In this case $b^{-1}a > 2$, and from (4.8) we get for any $z = (z_1, z_2, b)$ with $|z_i| < L$

$$\cosh r(y^{-1}x) \sim \cosh r(z^{-1}x) \sim b^{-1}a + ba^{-1}b^{-2}|x - z|^2 \sim \frac{a}{b} + \frac{|x|^2}{ab}\,,$$

and also

$$r(y^{-1}x) \sim r(z^{-1}x) \sim \log\left(\frac{a}{b} + \frac{|x|^2}{ab}\right) \geq \log a - \log b \sim \log a\,,$$

since $\log a > 3\alpha/2$ and $\log b < \alpha$. By the Mean Value Theorem

$$|k_1(y^{-1}x) - k_1(\tilde{y}^{-1}x)| \leq L\, b^{-1}(|\partial_1 k_1(z^{-1}x)| + |\partial_2 k_1(z^{-1}x)|)\,,$$

for some $z = (z_1, z_2, b)$ with $|z_i| < L$. Now apply the estimate (2.9) for $X_1 k_1 = a\partial_1 k_1$ to obtain that

$$|\partial_1 k_1(x)| \lesssim \frac{1}{ar(x)\cosh^2 r(x)} \qquad \text{for} \quad r(x) > 1\,. \tag{4.19}$$

The same estimate holds for $|\partial_2 k_1|$. We then have

$$\left|k_1(y^{-1}x)-k_1(\tilde y^{-1}x)\right| \lesssim L\,b^{-1}\,\frac{1}{b^{-1}a\Big(\frac{a}{b}+\frac{|x|^2}{ab}\Big)^2\,\log a}=\frac{Lab^2}{(a^2+|x|^2)^2\log a}\,,$$

so for $|x|>2L$ and $x\in\Omega_3$
(4.20)

$$|\mathcal{R}_1'A(x)| \lesssim \frac{1}{\alpha L^2}\int_{\mathrm{e}^{-\alpha}}^{\mathrm{e}^{\alpha}}\int_{|y|<L/2}\frac{Lab^2}{(a^2+|x|^2)^2\log a}\,b^{-2}\,\frac{\mathrm{d}y\,\mathrm{d}b}{b}\,\le\,\frac{La}{(a^2+|x|^2)^2\log a}\,.$$

For $\mathcal{R}_1''A$ we use (4.10) to see that

$$2\pi^2\,k_1(\tilde y^{-1}x)\delta(y)=-\frac{4ax_1b^2}{(a^2+|x|^2)^2}\,\frac{1+r(\tilde y^{-1}x)}{r^2(\tilde y^{-1}x)}\,\Big(1+O\Big(\frac{b^2}{a^2+|x|^2}\Big)\Big)\,b^{-2}.$$

This must be compared with $2\pi^2\,k_1(x)$, which is obtained by replacing b by 1 and thus $\tilde y$ by e here. Observe that

$$\Big|\frac{1+r(\tilde y^{-1}x)}{r^2(\tilde y^{-1}x)}-\frac{1+r(x)}{r^2(x)}\Big|\lesssim\frac{|r(\tilde y^{-1}x)-r(x)|}{r(\tilde y^{-1}x)r(x)}\lesssim\frac{|\log b|}{(\log a)^2}\,,$$

where we applied the triangle inequality and (4.9). Using this and estimating the $O(\dots)$ terms, we obtain

$$\left|k_1(\tilde y^{-1}x)\delta(y)-k_1(x)\right|\lesssim\frac{a|x_1|}{(a^2+|x|^2)^2}\,\frac{1+|\log b|}{(\log a)^2}\,.$$

We then deduce that for $|x|>2L$

$$\begin{aligned}|\mathcal{R}_1''A(x)| &\lesssim \frac{1}{\alpha L^2}\,\frac{a|x_1|}{(a^2+|x|^2)^2}\,\frac{1}{(\log a)^2}\int_{|y|<L/2}\int_{\mathrm{e}^{-\alpha}}^{\mathrm{e}^{\alpha}}(1+|\log b|)\,\frac{\mathrm{d}b\,\mathrm{d}y}{b}\\ &\lesssim \frac{\alpha a|x_1|}{(a^2+|x|^2)^2\,(\log a)^2}\,.\end{aligned}\tag{4.21}$$

Take now $x\in\Omega_3$ with $|x|\le 2L$. If also $y\in R$, (4.10) implies

$$|k_1(y^{-1}x)|\lesssim\frac{ab^2|x-y|}{(a^2+|x-y|^2)^2\,\log a}\,.$$

For $\mathcal{R}_1A$ we then obtain

$$\begin{aligned}|\mathcal{R}_1A(x)| &\lesssim \frac{a}{\alpha L^2}\int_{\mathrm{e}^{-\alpha}}^{\mathrm{e}^{\alpha}}\int_{|y|<L/2}\frac{|x-y|}{(a^2+|x-y|^2)^2\,\log a}\,\frac{\mathrm{d}y\,\mathrm{d}b}{b}\\ &\le \frac{1}{\alpha L^2\log a}\int_{|v|<5L/2a}\frac{|v|}{(1+|v|^2)^2}\,dv\int_{\mathrm{e}^{-\alpha}}^{\mathrm{e}^{\alpha}}\frac{\mathrm{d}b}{b}\\ &\lesssim \frac{1}{L^2\log a}\,(\min(1,L/a))^3\,.\end{aligned}\tag{4.22}$$

From this estimate we get for $x\in\Omega_3$ and $|x|\le 2L$

$$|\mathcal{R}_1A(x)|\lesssim\begin{cases}\frac{L}{a^3\log a} & \text{if}\ \ a>L\,,\\ \frac{1}{L^2\log a} & \text{if}\ \ \mathrm{e}^{1+\frac{3}{2}\alpha}\le a\le L\,.\end{cases}\tag{4.23}$$

The atomic decomposition of $\mathcal{R}_1A$ in Ω_3 is analogous to that in Ω_1. But since the estimate (4.23) distinguishes between the cases $a>L$ and $a\le L$, we need to do the same when we define the slices S_k.

For this, we first choose an integer $\kappa > 1$ and a number $Q \in (2,3)$ such that $Q^\kappa = \log L - 3\alpha/2$. This is possible, since the equation amounts to

$$\kappa = \frac{\log(\log L - 3\alpha/2)}{\log Q}$$

and $\log L - 3\alpha/2 \geq \alpha/2 \geq 10$. One can therefore choose κ between $(\log(\log L - 3\alpha/2))/\log 3$ and $(\log(\log L - 3\alpha/2))/\log 2$, and then Q will be determined.

For $k = \kappa + 1, \kappa + 2, \dots$, we define slices

$$S_k = \mathbb{R}^2 \times [\mathrm{e}^{k-\kappa-1} L, \mathrm{e}^{k-\kappa} L].$$

But for $k = 1, \dots \kappa$, the slices are instead

$$S_k = \mathbb{R}^2 \times [\mathrm{e}^{1-Q^{\kappa-k+1}} L, \mathrm{e}^{1-Q^{\kappa-k}} L].$$

It is easily verified that $(S_k)_1^\infty$ is a partition of Ω_3, except for boundaries, and that with $x \in S_k$ the cases $k > \kappa$ and $k \leq \kappa$ correspond to $a \geq L$ and $a \leq L$, respectively.

For $k > \kappa$ we define

$$B_{k,j} = \{x \in S_k : \, 2\mathrm{e}^{k-\kappa+j-1} L \leq |x| \leq 2\mathrm{e}^{k-\kappa+j} L\}, \qquad j = 1, 2, \dots,$$

and

$$B_{k,0} = \{x \in S_k : \, |x| \leq 2\mathrm{e}^{k-\kappa} L\};$$

we also define

$$Z_{k,j} = [-2\mathrm{e}^{k-\kappa+j} L, 2\mathrm{e}^{k-\kappa+j} L]^2 \times [\mathrm{e}^{k-\kappa-1-j/2} L, \mathrm{e}^{k-\kappa+j/2} L], \qquad j = 0, 1, \dots.$$

The measures of these sets are given by

$$\rho(B_{k,j}) \sim \mathrm{e}^{2(k-\kappa+j)} L^2 \qquad \text{and} \qquad \rho(Z_{k,j}) \sim \mathrm{e}^{2(k-\kappa+j)} L^2 (j+1), \tag{4.24}$$

and $Z_{k,j}$ is a Calderón–Zygmund set, with center $(0, 0, \mathrm{e}^{k-\kappa-1/2} L)$ and parameter $(j+1)/2$.

For $k > \kappa$ and $j \geq 1$, we conclude from (4.20) and (4.21) that for $x \in B_{k,j}$

$$|\mathcal{R}_1 A(x)| \lesssim \frac{1}{L^2 \, \mathrm{e}^{3(k-\kappa)} \, \mathrm{e}^{4j} \, (k-\kappa+\alpha)} + \frac{\alpha}{L^2 \, \mathrm{e}^{2(k-\kappa)} \, \mathrm{e}^{3j} \, (k-\kappa+\alpha)^2}. \tag{4.25}$$

Observe that this holds also for $j = 0$ because of (4.23).

For $1 \leq k \leq \kappa$, we set instead

$$B_{k,j} = \{x \in S_k : \, 2\mathrm{e}^{j-1} L \leq |x| \leq 2\mathrm{e}^{j} L\}, \qquad j = 1, 2, \dots,$$

and

$$B_{k,0} = \{x \in S_k : \, |x| \leq 2L\},$$

and also

$$Z_{k,j} = [-2\mathrm{e}^j L, 2\mathrm{e}^j L]^2 \times [\mathrm{e}^{1-Q^{\kappa-k+1}-j/2} L, \mathrm{e}^{1-Q^{\kappa-k}+j/2} L], \qquad j = 0, 1, \dots.$$

The measures are now given by

$$\rho(B_{k,j}) \sim \mathrm{e}^{2j} L^2 Q^{\kappa-k} \qquad \text{and} \qquad \rho(Z_{k,j}) \sim \mathrm{e}^{2j} L^2 (Q^{\kappa-k} + j), \tag{4.26}$$

and $Z_{k,j}$ is again a Calderón-Zygmund set, with center $(0, 0, L\mathrm{e}^{1-Q^{\kappa-k}(Q+1)/2})$ and parameter $j/2 + Q^{\kappa-k}\,(Q-1)/2$.

If $1 \leq k \leq \kappa$ and $j \geq 1$, we derive from (4.20), (4.21) and the inequality $\log a > \alpha$ that

$$|\mathcal{R}_1 A(x)| \lesssim \frac{\mathrm{e}^{-Q^{\kappa-k}}}{L^2 e^{3j} \, \alpha}, \tag{4.27}$$

for any $x \in B_{k,j}$. However, when $x \in B_{k,0}$ we get from (4.23)

$$|\mathcal{R}_1 A(x)| \lesssim \frac{1}{L^2 \alpha}. \tag{4.28}$$

For $k \geq 1$ and $j \geq 0$, we let as before $f_{k,j} = \rho(Z_{k,j})^{-1} \chi_{Z_{k,j}}$ and

$$m_{k,j} = \sum_{\ell=j}^{\infty} \int_{B_{k,\ell}} \mathcal{R}_1 A \, \mathrm{d}\rho.$$

Notice that $m_{k,0} = 0$ for all $k \geq 1$, because of (4.12). From (4.25) and (4.24), we obtain for $k > \kappa$ and $j \geq 0$

$$|m_{k,j}| \lesssim \frac{1}{\mathrm{e}^{k-\kappa} \, \mathrm{e}^{2j} \, (k - \kappa + \alpha)} + \frac{\alpha}{\mathrm{e}^{j} \, (k - \kappa + \alpha)^2}. \tag{4.29}$$

For $1 \leq k \leq \kappa$ and $j \geq 1$, we similarly get from (4.27) and (4.26)

$$|m_{k,j}| \lesssim \frac{\mathrm{e}^{-Q^{\kappa-k}} \, Q^{\kappa-k}}{\mathrm{e}^{j} \, \alpha}. \tag{4.30}$$

We now come to the atoms, and define for $k \geq 1$ and $j \geq 0$

$$A_{k,j} = \mathcal{R}_1 A \, \chi_{B_{k,j}} - m_{k,j} f_{k,j} + m_{k,j+1} f_{k,j+1}. \tag{4.31}$$

Then $A_{k,j}$ is supported in the Calderón–Zygmund set $Z_{k,j+1}$ and has vanishing moment. Further

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} A_{k,j} = \mathcal{R}_1 A \, \chi_{\Omega_3}. \tag{4.32}$$

The $A_{k,j}$ are multiples of $(1, \infty)$-atoms. We estimate their norms in H^1. When $k > \kappa$ and $j \geq 0$, (4.25), (4.24) and (4.29) imply

$$\begin{aligned} \|A_{k,j}\|_{H^1} \lesssim \; & \frac{j+1}{\mathrm{e}^{k-\kappa} \, \mathrm{e}^{2j} \, (k - \kappa + \alpha)} + \frac{(j+1)\alpha}{\mathrm{e}^{j} \, (k - \kappa + \alpha)^2} \\ & + \frac{1}{\mathrm{e}^{k-\kappa} \, \mathrm{e}^{2j} \, (k - \kappa + \alpha)} + \frac{\alpha}{\mathrm{e}^{j} \, (k - \kappa + \alpha)^2}. \end{aligned}$$

Summing, one obtains

$$\Big\| \sum_{k > \kappa} \sum_{j \geq 0} A_{k,j} \Big\|_{H^1} \lesssim 1. \tag{4.33}$$

For $1 \leq k \leq \kappa$ and $j \geq 1$, we similarly see from (4.27), (4.26) and (4.30) that

$$\|A_{k,j}\|_{H^1} \lesssim \frac{\mathrm{e}^{-Q^{\kappa-k}} \, (Q^{\kappa-k} + j)}{\mathrm{e}^{j} \alpha}$$

Again, we can sum and get

$$\Big\| \sum_{1 \leq k \leq \kappa} \sum_{j \geq 1} A_{k,j} \Big\|_{H^1} \lesssim 1. \tag{4.34}$$

For $A_{k,0}$ with $1 \leq k \leq \kappa$, we have in view of (4.28), (4.26) and (4.30)

$$\|A_{k,0}\|_{H^1} \lesssim \frac{Q^{\kappa-k}}{\alpha} + \frac{\mathrm{e}^{-Q^{\kappa-k}} Q^{\kappa-k}}{\alpha},$$

and since $Q^\kappa \simeq \alpha$, this implies

$$\Big\| \sum_{1 \leq k \leq \kappa} A_{k,0} \Big\|_{H^1} \lesssim 1. \tag{4.35}$$

Summing up, we have an atomic decomposition of the restriction of $\mathcal{R}_1 A$ to Ω_3 with control of the norm.

To deal with the region Ω_2, we first derive estimates for the kernel and $\mathcal{R}_1 A$ there. For $x = (x_1, x_2, a) \in \Omega_2$ with $|x| > 2L$ and $y \in R$, one has $|x - y| \sim |x|$ and $\mathrm{e}^{-1-5\alpha/2} < b^{-1} a < \mathrm{e}^{1+5\alpha/2}$, so that $|x|^2 > 4L^2 > 4\mathrm{e}^{4\alpha} > 2(a^2 + b^2)$ and

$$\cosh r(y^{-1}x) \sim \frac{a}{b} + \frac{b}{a} + \frac{b}{a}\, b^{-2}\, |x|^2 \sim \frac{|x|^2}{ab} \qquad \text{and} \qquad r(y^{-1}x) \sim \log |x| \,.$$

This implies that by (2.6)

$$|k_1(y^{-1}x)| \lesssim \frac{b}{a}\, |x|\, b^{-1} \Big(\frac{|x|^2}{ab} \Big)^{-2} \frac{1}{\log |x|} \leq \frac{ab^2}{|x|^3}$$

for $|x| > 2L$, and by (4.6)

$$|\mathcal{R}_1 A(x)| \leq \rho(R)^{-1} \int_{\mathrm{e}^{-\alpha}}^{\mathrm{e}^{\alpha}} \int_{|y|<L/2} \frac{ab^2}{|x|^3}\, b^{-2} \frac{\mathrm{d}y\, \mathrm{d}b}{b} \lesssim \frac{a}{|x|^3}, \qquad |x| > 2L. \tag{4.36}$$

We proceed mainly as before, and define

$$S_0 = \Omega_2 = \mathbb{R}^2 \times [e^{-1-3\alpha/2}, e^{1+3\alpha/2}].$$

The argument is split into two subcases, depending on the size of $L \in [\mathrm{e}^{2\alpha}, \mathrm{e}^{8\alpha})$.
Subcase (i): $L \geq \mathrm{e}^{3\alpha}$. Here we let

$$B_{0,j} = \{x \in \Omega_2 :\ 2\mathrm{e}^{j-1} L \leq |x| \leq 2\mathrm{e}^{j} L\}, \qquad j = 1, 2, \ldots,$$

and

$$B_{0,0} = \{x \in S_0 :\ |x| \leq 2L\}.$$

Further, we define

$$Z_{0,j} = [-2\mathrm{e}^{j+2} L, 2\mathrm{e}^{j+2} L]^2 \times [\mathrm{e}^{-1-3\alpha/2-j/2}, \mathrm{e}^{1+3\alpha/2+j/2}], \qquad j = 0, 1, \ldots.$$

Then $B_{0,j} \subset Z_{0,j}$, and since $L \geq \mathrm{e}^{3\alpha}$ the $Z_{0,j}$ are Calderón–Zygmund sets, centered at $(0, 0, 1)$ and of parameter $1 + 3\alpha/2 + j/2$. The measures of these sets are given by

$$\rho(B_{0,j}) \sim \mathrm{e}^{2j} L^2 \alpha \qquad \text{and} \qquad \rho(Z_{0,j}) \sim \mathrm{e}^{2j} L^2 (\alpha + j). \tag{4.37}$$

Now (4.36) implies that for $x \in B_{0,j},\ j \geq 1$

$$|\mathcal{R}_1 A(x)| \lesssim \frac{\mathrm{e}^{3\alpha/2}}{\mathrm{e}^{3j} L^3} \tag{4.38}$$

As before, we define for $j = 0, 1, \ldots$

$$m_{0,j} = \sum_{\ell=j}^{\infty} \int_{B_{0,\ell}} \mathcal{R}_1 A \ \mathrm{d}\rho. \tag{4.39}$$

Since (4.12) and Lemma 4.1 imply $\int_{S_0} \mathcal{R}_1 A \, d\rho = 0$, so that $m_{0,0} = 0$. Further, (4.37) and (4.38) imply

$$|m_{0,j}| \lesssim \frac{\mathrm{e}^{3\alpha/2}\, \alpha}{\mathrm{e}^{j} L}, \qquad j \geq 1.$$

We again let $f_{0,j} = \rho(Z_{0,j})^{-1} \chi_{Z_{0,j}}$ for $j = 0, 1, \ldots$ and define

$$A_{0,j} = \mathcal{R}_1 A\, \chi_{B_{0,j}} - m_{0,j}\, f_{0,j} + m_{0,j+1}\, f_{0,j+1}. \tag{4.40}$$

Then

$$\sum_{j=0}^{\infty} A_{0,j} = \mathcal{R}_1 A\,\chi_{\Omega_2} - m_{0,0}\,f_{0,0} = \mathcal{R}_1 A\,\chi_{\Omega_2}.$$

Each $A_{0,j}$ is supported in $Z_{0,j+1}$ and has integral 0. Because of (4.38), $A_{0,j}$ is bounded for $j \ge 1$ and is thus a multiple of a $(1,\infty)$-atom, with

$$\|A_{0,j}\|_{H^1} \lesssim \mathrm{e}^{2j} L^2(\alpha+j)\,\frac{\mathrm{e}^{3\alpha/2}}{\mathrm{e}^{3j}L^3} + \frac{\mathrm{e}^{3\alpha/2}\,\alpha}{\mathrm{e}^{j}L} \lesssim \frac{j}{\mathrm{e}^{j}}, \qquad j \ge 1,$$

since $L \ge \mathrm{e}^{3\alpha}$ in this subcase. Thus

$$\|\sum_{1}^{\infty} A_{0,j}\|_{H^1} \lesssim 1.$$

It only remains to consider $A_{0,0}$, which need not be bounded. We use now the boundedness of $\mathcal{R}_1$ on L^2, which implies

$$\begin{aligned} \|A_{0,0}\|_2 &\lesssim \|\mathcal{R}_1 A\|_2 + |m_{0,1}|\,\rho(Z_{0,1})^{-1/2} \\ &\lesssim \|A\|_2 + \frac{\mathrm{e}^{3\alpha/2}\alpha}{L}\,\rho(Z_{0,1})^{-1/2} \lesssim \rho(Z_{0,1})^{-1/2}. \end{aligned} \tag{4.41}$$

Thus $A_{0,0}$ is a multiple of a $(1,2)$-atom.

Summing up, we conclude that $\mathcal{R}_1 A\,\chi_{\Omega_2} \in H^1$ with bounded norm. This ends Subcase (i).

Subcase (ii): $L < \mathrm{e}^{3\alpha}$. Here we divide S_0 into two slices, defining

$$S_{0-} = \mathbb{R}^2 \times [e^{-1-3\alpha/2},\, e^{\alpha/2}]$$

and

$$S_{0+} = \mathbb{R}^2 \times [e^{\alpha/2},\, e^{1+3\alpha/2}].$$

Further, we let for $j = 1, 2, \dots$

$$B_{0-,j} = \{x \in S_{0-} :\ 2\mathrm{e}^{j-1}L \le |x| \le 2\mathrm{e}^{j}L\}$$

and

$$B_{0+,j} = \{x \in S_{0+} :\ 2\mathrm{e}^{j-1}L \le |x| \le 2\mathrm{e}^{j}L\},$$

but for $j = 0$, as before,

$$B_{0\pm,0} = \{x \in S_{0\pm} :\ |x| \le 2L\}.$$

Then

$$Z_{0-,j} := [-2\mathrm{e}^{j+2}L, 2\mathrm{e}^{j+2}L]^2 \times [\mathrm{e}^{-1-3\alpha/2-j/2},\, \mathrm{e}^{\alpha/2+j/2}], \qquad j = 0, 1, \dots,$$

are Calderón–Zygmund sets of center $(0,0,\mathrm{e}^{-(\alpha+1)/2})$ and parameter $\alpha + (j+1)/2$ containing $B_{0-,j}$. Almost similarly,

$$Z_{0+,j} := [-2\mathrm{e}^{j+2}L, 2\mathrm{e}^{j+2}L]^2 \times [\mathrm{e}^{\alpha/2-j/2},\, \mathrm{e}^{1+3\alpha/2+j/2}], \qquad j = 0, 1, \dots,$$

are Calderón–Zygmund sets of center $(0,0,\mathrm{e}^{\alpha+1/2})$ and parameter $(\alpha+1+j)/2$ containing $B_{0+,j}$. The measures of these sets are

$$\rho(B_{0\pm,j}) \sim \mathrm{e}^{2j}L^2\alpha \qquad \text{and} \qquad \rho(Z_{0\pm,j}) \sim \mathrm{e}^{2j}L^2(\alpha+j).$$

For $x \in B_{0-,j}$ with $j \ge 1$, the estimate (4.36) implies

$$|\mathcal{R}_1 A(x)| \lesssim \frac{\mathrm{e}^{\alpha/2}}{\mathrm{e}^{3j}L^3} \tag{4.42}$$

and for $x \in B_{0+,j}$ with $j \geq 1$

$$|\mathcal{R}_1 A(x)| \lesssim \frac{\mathrm{e}^{3\alpha/2}}{\mathrm{e}^{3j} L^3}. \tag{4.43}$$

Proceeding as before, for $j = 0, 1, \dots$ we define $f_{0\pm,j} = \rho(Z_{0\pm,j})^{-1}\chi_{Z_{0\pm,j}}$ and

$$m_{0\pm,j} = \sum_{\ell=j}^{\infty} \int_{B_{0\pm,\ell}} \mathcal{R}_1 A \,\mathrm{d}\rho.$$

and for $j \geq 1$ also

$$A_{0\pm,j} = \mathcal{R}_1 A\, \chi_{B_{0\pm,j}} - m_{0\pm,j}\, f_{0\pm,j} + m_{0\pm,j+1}\, f_{0\pm,j+1}. \tag{4.44}$$

These $A_{0\pm,j}$ are multiples of $(1,\infty)$-atoms, because they are bounded and supported in $Z_{0\pm,j+1}$ and have vanishing integrals. Since (4.42) and (4.43) imply

$$|m_{0\pm,j}| \lesssim \frac{\mathrm{e}^{3\alpha/2}\alpha}{\mathrm{e}^j L}, \qquad j \geq 1,$$

one finds that

$$\|A_{0\pm,j}\|_{H^1} \lesssim \frac{\mathrm{e}^{3\alpha/2}(\alpha+j)}{\mathrm{e}^j L} + \frac{\mathrm{e}^{3\alpha/2}\alpha}{\mathrm{e}^j L} \lesssim \frac{j}{\mathrm{e}^j}, \qquad j \geq 1.$$

The sum of these $A_{0\pm,j}$ is thus an H^1 function, and

$$\sum_{j=1}^{\infty} A_{0\pm,j} = \mathcal{R}_1 A\, \chi_{\cup_1^\infty B_{0\pm,j}} - \int_{\cup_1^\infty B_{0\pm,j}} \mathcal{R}_1 A \,\mathrm{d}\rho.$$

These two sums form a large part of the desired atomic decomposition of $\mathcal{R}_1 A \chi_{S_0}$. What is missing is

$$\begin{aligned} &\mathcal{R}_1 A \chi_{S_0} - \sum_{j=1}^{\infty} A_{0-,j} - \sum_{j=1}^{\infty} A_{0+,j} \\ &\quad = \mathcal{R}_1 A\, \chi_{B_{0-,0}} + \mathcal{R}_1 A\, \chi_{B_{0+,0}} + \int_{\cup_1^\infty B_{0-,j}} \mathcal{R}_1 A \,\mathrm{d}\rho + \int_{\cup_1^\infty B_{0+,j}} \mathcal{R}_1 A \,\mathrm{d}\rho. \end{aligned} \tag{4.45}$$

Comparing with Subcase (i), we have $S_{0-} \cup S_{0+} = S_0$ and $B_{0-,j} \cup B_{0+.j} = B_{0,j}$ for $j = 0, 1, \dots,$ and we observed there that $\int_{S_0} \mathcal{R}_1 A \,\mathrm{d}\rho = 0$. It follows that the last two terms in the right-hand side of (4.45) sum up to $-\int_{B_{0,0}} \mathcal{R}_1 A \,\mathrm{d}\rho$, and the first two terms amount to $\mathcal{R}_1 A\, \chi_{B_{0,0}}$. This means that the expression in (4.45) coincides with the atom multiple $A_{0,0}$ from Subcase (i), and the estimate (4.41) holds in both subcases. We thus have an atomic decomposition of $\mathcal{R}_1 A\, \chi_{S_0}$ in Subcase (ii), which ends the proof of Case I in Theorem 1.4.

Case II: intermediate atom. In this case $1 \leq \alpha < 20$ and $\mathrm{e}^{2\alpha} \leq L < \mathrm{e}^{8\alpha}$. We consider A as a function supported in the larger Calderón–Zygmund set $[-\tilde{L}/2, \tilde{L}/2]^2 \times [\mathrm{e}^{-20}, \mathrm{e}^{20}]$, where $\tilde{L} = \max\{L, \mathrm{e}^{40}\}$. Considered with this support, A will be a multiple of a large atom, and the preceding argument applies.

Case III: small atom. Here $\alpha < 1$ and $\mathrm{e}^2\alpha \le L < \mathrm{e}^8\alpha$. We use a slightly different decomposition of the space into regions, as follows

$$\begin{aligned}\tilde{\Omega}_1 &= \{(x_1, x_2, a) : \ a \le \mathrm{e}^{-2}\}\,,\\ \tilde{\Omega}_2 &= \{(x_1, x_2, a) : \ \mathrm{e}^{-2} < a < \mathrm{e}^2\}\,,\\ \tilde{\Omega}_3 &= \{(x_1, x_2, a) : \ a \ge \mathrm{e}^2\}\,.\end{aligned}$$

Starting with $\tilde{\Omega}_1$, we let $x \in \tilde{\Omega}_1$ and $y = (y_1, y_2, b) \in R$. Then $b^{-1}a < e^{-1}$, and (4.7) shows that

$$|k_1(y^{-1}x)| \lesssim \frac{a}{1+|x|^3} \qquad \text{and} \qquad |\mathcal{R}_1 A(x)| \lesssim \frac{a}{1+|x|^3}\,.$$

We can construct an atomic decomposition of $\mathcal{R}_1 A\chi_{\tilde{\Omega}_1}$ using the following sets for $k < 0$ and $j \ge 0$:

$$\begin{aligned}B_{k,j} &= \{2\mathrm{e}^{j-1} \le |x| \le 2\mathrm{e}^j\} \times [\mathrm{e}^{-2^{|k|}-2}, \mathrm{e}^{-2^{|k|-1}-2}]\,, \qquad j \ge 1\,,\\ B_{k,0} &= \{|x| \le 2\} \times [\mathrm{e}^{-2^{|k|}-2}, \mathrm{e}^{-2^{|k|-1}-2}]\,,\\ Z_{k,j} &= [-2\mathrm{e}^j, 2\mathrm{e}^j]^2 \times [\mathrm{e}^{-3\cdot 2^{|k|-1}-2-j/2}, \mathrm{e}^{-2^{|k|-1}-2+j/2}], \qquad j \ge 0\,.\end{aligned}$$

The $Z_{k,j}$ are Calderón–Zygmund sets, centered at $(0, 0, \mathrm{e}^{-2\cdot 2^{|k|-1}-2})$ and with parameters $2^{|k|-1} + j/2$. The relevant measures are $\rho(B_{k,j}) \sim \mathrm{e}^{2j}2^{|k|}$ and $\rho(Z_{k,j}) \sim \mathrm{e}^{2j}(2^{|k|} + j)$. We then have in the set $B_{k,j}$

$$|\mathcal{R}_1 A| \lesssim \frac{\mathrm{e}^{-2^{|k|-1}}}{\mathrm{e}^{3j}}, \qquad k < 0, \quad j \ge 0.$$

From here, we proceed as in the case of the region Ω_1 for a large atom. The details are left to the reader, since the construction is quite similar. We will have $\|\mathcal{R}_1 A\chi_{\tilde{\Omega}_1}\|_{H^1} \lesssim 1$.

In the case of $\tilde{\Omega}_3$, we argue as we did to obtain the estimates (4.20) and (4.21), to show that for $x = (x_1, x_2, a) \in \tilde{\Omega}_3$

$$|\mathcal{R}_1 A(x)| \lesssim \frac{La}{(a^2+|x|^2)^2 \log a} + \frac{a|x_1|\alpha}{(a^2+|x|^2)^2(\log a)^2}, \qquad |x| > 2,$$

and (4.10) leads to

$$|\mathcal{R}_1 A(x)| \lesssim \frac{1}{a^3 \log a}, \qquad |x| \le 2.$$

We now construct an atomic decomposition of $\mathcal{R}_1 A\chi_{\tilde{\Omega}_3}$, using the following sets for $k \ge 1$ and $j \ge 0$:

$$\begin{aligned}B_{k,j} &= \{\mathrm{e}^{k+j+1} \le |x| \le \mathrm{e}^{k+j+2}\} \times [\mathrm{e}^{k+1}, \mathrm{e}^{k+2}]\,, \quad j \ge 1\,,\\ B_{k,0} &= \{|x| \le e^{k+2}\} \times [\mathrm{e}^{k+1}, \mathrm{e}^{k+2}]\,,\\ Z_{k,j} &= \{|x| \le \mathrm{e}^{k+j+2}\} \times [\mathrm{e}^{k+1-j/2}, \mathrm{e}^{k+2+j/2}]\,.\end{aligned}$$

Here each $Z_{k,j}$ is a Calderón–Zygmund set of center $(0, 0, \mathrm{e}^{k+3/2})$ and parameter $(j+1)/2$ containing $B_{k,j}$. The measures satisfy $\rho(B_{k,j}) \sim \mathrm{e}^{2k+2j}$ and $\rho(Z_{k,j}) \sim \mathrm{e}^{2k+2j}(j+1)$. In $B_{k,j}$ we will have

$$|\mathcal{R}_1 A| \lesssim \frac{1}{\mathrm{e}^{3k+4j}k} + \frac{1}{\mathrm{e}^{2k+3j}k^2}\,.$$

Again, we can now follow the procedure used for a large atom in the region Ω_1. In particular, f_k, $m_{k,j}$ and $A_{k,j}$ will be as there, although k is now positive and the $B_{k,j}$ and $Z_{k,j}$ are those we just defined. The relevant estimates will now be

$$|m_{k,j}| \lesssim \sum_{\ell=j}^{\infty} \left(\frac{1}{\mathrm{e}^{3k+4\ell} k} \, \mathrm{e}^{2k+2\ell} + \frac{1}{\mathrm{e}^{2k+3\ell} k^2} \, \mathrm{e}^{2k+2\ell} \right)$$
$$\lesssim \mathrm{e}^{-k-2j} + \mathrm{e}^{-j} \frac{1}{k^2}$$

and

$$\begin{aligned} &\|A_{k,j}\|_{H^1} \\ &\lesssim \frac{1}{\mathrm{e}^{3k+4j}\, k} \, \mathrm{e}^{2k+2j} \,(j+1) + \frac{1}{\mathrm{e}^{2k+3j}\, k^2} \, \mathrm{e}^{2k+2j} \,(j+1) + \mathrm{e}^{-k-2j} + \mathrm{e}^{-j} \frac{1}{k^2} \\ &\lesssim \mathrm{e}^{-k-2j} \,(j+1) + \mathrm{e}^{-j} \, \frac{j+1}{k^2}. \end{aligned}$$

These quantities are summable over $j \geq 0$ and $k \geq 1$, and so $\|\mathcal{R}_1 A \chi_{\tilde{\Omega}_3}\|_{H^1} \lesssim 1$.

To treat the region $\tilde{\Omega}_2$ we will construct a sequence of sets $B_{0,j}$ expanding from R in all three coordinates until they reach essentially unit size. Then they will expand only in the coordinates x_1, x_2. More precisely, we never let the a width of any $B_{0,j}$ be larger than $[\mathrm{e}^{-2}, \mathrm{e}^{2}]$, so that $B_{0,j}$ stays in $\tilde{\Omega}_2$. A sequence of Calderón–Zygmund sets $Z_{0,j}$ will be defined accordingly. It is not restrictive to suppose that $\alpha = 2^{j_0}$, for some integer $j_0 \leq 0$. The definition of a Calderón–Zygmund set then implies that $e^2 \leq 2^{-j_0} L < e^8$. The $B_{0,j}$ and the $Z_{0,j}$ will be defined for $j = j_0+1, j_0+2, \ldots$, as follows.

We start by setting $B_{0,j_0} = \emptyset$ and recursively for $j \geq j_0 + 1$

$$B_{0,j} = \Big(\{|x| < 2^{j-j_0-1} L\} \times [\mathrm{e}^{-\min(2,2^j)}, \mathrm{e}^{\min(2,2^j)}] \Big) \setminus B_{0,j-1}\,,$$

For $j_0 + 1 \leq j \leq 0$ (which occurs only if $j_0 < 0$), we let

$$Z_{0,j} = [-2^{j-j_0-1} L, 2^{j-j_0-1} L]^2 \times [\mathrm{e}^{-2^j}, \mathrm{e}^{2^j}]\,,$$

but for $j > 0$

$$Z_{0,j} = [-2^{j-j_0+2} L, 2^{j-j_0+2} L]^2 \times [\mathrm{e}^{-2-j/2}, \mathrm{e}^{2+j/2}]\,.$$

Then $B_{0,j} \subset Z_{0,j}$ for each $j \geq j_0 + 1$, and the $Z_{0,j}$ are Calderón–Zygmund sets centered at $(0,0,1)$ and of parameter $\min(2, 2^j) + j_+/2$. The measures of these sets are $\rho(B_{0,j}) \sim 2^{2j+j_-}$ and $\rho(Z_{0,j}) \sim 2^{2j}(2^{j_-} + j_+)$. Here $j_+ = \max(j,0)$ and $j_- = \min(j,0)$.

Suppose now that $j_0 + 2 \leq j \leq 0$. We bound $\mathcal{R}_1 A$ in the set $B_{0,j}$ by means of estimates similar to those for the Riesz transforms in the Euclidean setting. Observe first that each point $x = (x_1, x_2, a) \in B_{0,j}$ is at some distance from any point y in R, so that $r(y^{-1}x) \sim r(x)$. Then simple computations together with formulas (2.6), (2.8), (2.7) and (2.5) show that

$$\begin{aligned} |k_1(y^{-1}x)| &\lesssim r(x)^{-3}\,, \\ |X_i k_1(y^{-1}x)| &\lesssim r(x)^{-4}\,, \qquad i = 0,1,2\,. \end{aligned}$$

Notice that when $r(x)$ is small, it is essentially the Euclidean distance from x to e; indeed $r(x) \sim \sqrt{x_1^2 + x_2^2 + (\log a)^2}$ as seen from (1.1). From the Mean Value

Theorem and the fact that $\delta(y) = 1 + \mathcal{O}(\alpha)$ here, it now follows that for $x \in B_{0,j}$

$$\begin{aligned}
|\mathcal{R}_1 A(x)| &\lesssim \int_R |A(y)||k_1(y^{-1}x)\delta(y) - k_1(x)| \,\mathrm{d}\rho(y) \\
&\lesssim \int_R |A(y)||k_1(y^{-1}x) - k_1(x)| \,\mathrm{d}\rho(y) + \alpha \int_R |A(y)||k_1(y^{-1}x) \,\mathrm{d}\rho(y) \\
&\lesssim \frac{\alpha}{r(x)^4} + \frac{\alpha}{r(x)^3} \lesssim \alpha 2^{-3j} \,.
\end{aligned}$$

Suppose now instead that $j > 0$. Then (4.36) shows that for each $x \in B_{0,j}$

$$|\mathcal{R}_1 A(x)| \lesssim \frac{1}{|x|^3} \lesssim 2^{-3j} \,.$$

But we can also apply the Mean Value Theorem as in the preceding case. Then we need (2.9) to estimate $X_i k_1$ for $i = 1, 2$, and by means of (2.6) and (2.5) one can verify that the same estimate holds also for $X_0 k_1$. The result will be

$$|\mathcal{R}_1 A(x)| \lesssim \frac{\alpha}{|x|^2 \log |x|} \lesssim \frac{\alpha 2^{-2j}}{j} \,.$$

We now argue essentially as in Subcase (i) above. In particular, we define $m_{0,j}$ by (4.39) and $A_{0,j}$ by (4.40), for all $j \geq j_0 + 1$. Then $\sum_{j=j_0+1}^{\infty} A_{0,j} = \mathcal{R}_1 \chi_{\tilde{\Omega}_2}$, and $m_{0,j_0+1} = \int_{B_{\tilde{\Omega}_2}} \mathcal{R}_1 A \, d\rho = 0$.

Using these estimates for $\mathcal{R}_1 A$, we estimate $m_{0,j}$, first when $j_0 + 1 < j \leq 0$. Then

$$\begin{aligned}
|m_{0,j}| &\lesssim \sum_{\ell=j}^{\infty} \int_{B_{0,\ell}} |\mathcal{R}_1 A| \, d\rho \lesssim \sum_{\ell=j}^{0} \alpha 2^{-3\ell}\, 2^{3\ell} + \sum_{\ell=1}^{\infty} \min\left(2^{-3\ell}\, 2^{2\ell} \ell, \frac{\alpha 2^{-2\ell}}{\ell}\, 2^{2\ell} \ell\right) \\
&\lesssim \alpha\,(1 + \log 1/\alpha),
\end{aligned}$$

the last step since the number of terms in the finite sum is at most $|j_0| \lesssim 1 + \log 1/\alpha$ and the last sum is easy to control.

If $j > 0$ we have instead

$$|m_{0,j}| \lesssim \sum_{\ell=j}^{\infty} 2^{-3\ell}\, 2^{2\ell} \ell \lesssim 2^{-j} j.$$

Our estimates for $\mathcal{R}_1 A$ and $\rho(Z_{0,j})$ now show that

$$\begin{aligned}
\|A_{0,j}\|_{H^1} &\lesssim \alpha\,(1 + \log 1/\alpha)\,, \qquad j_0 + 1 < j \leq 0\,, \\
\|A_{0,j}\|_{H^1} &\lesssim 2^{-j} j \qquad j > 0\,.
\end{aligned}$$

Finally, we use the L^2-boundedness of $\mathcal{R}_1$ as in (4.41) to deduce that A_{0,j_0+1} is a multiple of a $(1,2)$-atom. We can now sum the H^1-norms of all the $A_{0,j}$ and obtain $\|\mathcal{R}_1 A\, \chi_{\tilde{\Omega}_2}\|_{H^1} \lesssim 1$.

This concludes the proof of Theorem 1.4.

□

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