Cavity-Based Approaches to Stochastic Dynamics on Sparse Graphs

Mattia Tarabolo

This thesis develops a coherent and flexible framework to study stochastic complex systems on sparse graphs through controlled small-coupling approximations of the dynamic cavity method. The aim is to extend cavity-based approaches, traditionally confined to equilibrium settings or discrete processes, to a wider class of stochastic dynamics with continuous or discrete degrees of freedom, where heterogeneity, noise, and network structure play a crucial role.

Part I introduces the mathematical foundations underpinning the whole work. It provides a detailed presentation of the cavity method for both equilibrium and dynamical systems, highlighting the difficulties that arise in its dynamic formulation and motivating the need for systematic approximations. The formal connection between the dynamic cavity equations and the path-integral representation of stochastic dynamics is established, and key tools such as the Martin-Siggia-Rose-Janssen-De Dominicis formalism, Dynamical Mean-Field Theory (DMFT), and the Plefka expansion are reviewed.

Building on this groundwork, Part II introduces the Gaussian-Expansion Cavity Method (GECaM), a systematic approximation scheme obtained from a second-order expansion of the dynamic cavity equations in the coupling strength. The method assumes a Gaussian form for the cavity marginals, parametrized by local means, correlations, and response functions, and yields a closed set of integro-differential equations valid for linearly coupled stochastic differential equations on locally tree-like graphs. For additive noise and linear drift, these equations provide an exact description of the dynamics and recover classical results from random-matrix theory. A perturbative closure scheme extends the framework to nonlinear forces and multiplicative noise. The validity of the approach is tested on models exhibiting paramagnetic-ferromagnetic phase transitions driven by cubic drift terms and on models with noise-induced transitions, for which GECaM performs excellently. Applications to the spherical two-spin model reveal how finite connectivity alters relaxation and ageing, while the extension to the random generalized Lotka-Volterra dynamics produces cavity fixed-point equations for species abundances. Solved by a population-dynamics algorithm, these equations capture how sparsity and heterogeneity reshape stability, diversity, and coexistence in large disordered ecosystems.

Part III presents the Small-Coupling Dynamic Cavity (SCDC) method, a cavity-based framework for Bayesian inference in compartmental epidemic models. Derived as a first-order expansion of the observation-reweighted dynamic cavity equations, SCDC introduces conjugate fields that encode the backward propagation of information generated by partial observations. Its efficient transfer-matrix formulation yields a computational cost linear in the epidemic duration, enabling accurate reconstruction of infection probabilities on both synthetic and real-world contact networks. The method provides a principled and interpretable approach, maintaining high accuracy even in recurrent epidemic processes where belief propagation is not applicable due to its prohibitive computational cost.