A LOWER BOUND FOR THE NUMBER OF EGYPTIAN FRACTIONS

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Abstract. An *Egyptian fraction* is a sum of the form $1/n_1 + \cdots + 1/n_r$ where $n_1, \dots, n_k$ are distinct positive integers. We prove explicit lower bounds for the cardinality of the set E_N of rational numbers that can be represented by Egyptian fractions with denominators not exceeding N. More precisely, we show that for every integer $k \geq 4$ such that $\ln_k N \geq 3/2$ it holds

$$\frac{\ln(|E_N|)}{\ln 2} \geq \Big(2 - \frac{3}{\ln_k N}\Big)\frac{N}{\ln N}\prod_{j=3}^{k} \ln_j N,$$

where $\ln_k$ denotes the k-th iterate of the natural logarithm. This improves on a previous result of Bleicher and Erdős who established a similar bound but under the more stringent condition $\ln_k N \geq k$ and with a leading constant of 1.

Furthermore, we provide some methods to compute the exact values of $|E_N|$ for large positive integers N, and we give a table of $|E_N|$ for N up to 154.

1. Introduction

An *Egyptian fraction* is a sum of the form $1/n_1 + \cdots + 1/n_r$ where $n_1, \dots, n_k$ are pairwise distinct positive integers. It is well known that every positive rational number can be written as an Egyptian fraction. In general, this representation is not unique. Several authors studied the properties of Egyptian fractions. For example, Yokota [13, 14] (see also [11]) proved that, for all positive integers $a < b$, the rational number a/b is represented by an Egyptian fraction with denominators of size $O\big(b(\ln b)^{1+\varepsilon}\big)$; while Vose [12] showed that a/b can be represented using at most $O\big(\sqrt{\ln b}\big)$ denominators. In the opposite direction of these "efficient" representations, Martin [6, 7] showed that every positive rational number can be represented using a very "dense" set of denominators. Furthermore, Croot [4] proved that all positive integers less than $\big[\sum_{n\leq N}\frac{1}{n}\big]$ can be represented using denominators up to N. We refer to the paper by Bloom and Elsholtz [3] for a recent survey on the topic of Egyptian fractions.

In this paper we are interested in the cardinality of the set

$$E_N := \Big\{\sum_{n=1}^{N}\frac{t_n}{n} \colon t_1, \dots, t_N \in \{0,1\}\Big\}, \qquad N \in \mathbb{Z}_{\geq 1}, \tag{1}$$

of Egyptian fractions that employ denominators up to N. It is clear from the definition that $|E_N|$ is increasing, and in fact it is strictly increasing, since $E_N \setminus E_{N-1}$ contains at least the N-th harmonic number.

Bleicher and Erdős [1, Th. 2 and 3] proved that

$$\alpha\frac{N}{\ln N}\prod_{j=3}^{k}\ln_j N \leq \ln(|E_N|) \leq \frac{N\ln_k N}{\ln N}\prod_{j=3}^{k}\ln_j N \tag{2}$$

where $\alpha = e^{-1}$, k is any positive integer such that $\ln_{2k} N \geq 1$, where $\ln_k$ denotes the k-th iteration of the natural logarithm. In [2, Cor. 1, 2, and 3] they improved the lower bound, increasing the value of the admissible α to $\alpha = \ln 2$ and relaxing the condition on k to $\ln_k N \geq k$. The problem of determining the size of $|E_N|$ is then also mentioned in a problem list by Erdős and Graham [5, p. 43].

In this paper, we give a new improvement to the lower bound in (2) showing that the condition on k can be further relaxed to $\ln_k N \geq 3/2$, essentially the weakest possible for a bound of this shape, while α can be taken to be $2\ln 2 - \varepsilon$ for any fixed $\varepsilon > 0$ provided that $\ln_k N$ is large enough. More specifically, we prove the following result.

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Theorem 1. *For all positive integers k and N, it holds*

$$\ln(|E_N|) \geq 2\ln 2\,\frac{N}{\ln N} \times \begin{cases} 1 & \text{if } \ln_2 N \geq 1, \\ \ln_3 N & \text{if } \ln_3 N \geq 1, \\ \Big(1 - \dfrac{3/2}{\ln_k N}\Big) \displaystyle\prod_{j=3}^{k} \ln_j N & \text{if } k \geq 4 \text{ and } \ln_k N \geq 3/2. \end{cases}$$

We note that the improvement over Bleicher and Erdős due to the weaker condition in k translates to an improvement in the order of growth of the lower bound. In fact, picking the largest admissible value of k in both bounds, Theorem 1 yields a bound that goes to infinity faster than the bound proven in [2, Cor. 3] since for extremely large values for N, larger values of k are allowed.

Further refinements to the constants in Theorem 1 are certainly possible, as our main goal was to establish the qualitative improvement resulting from relaxing the condition on k. For example, the leading factor 2 could be slightly improved by taking full advantage of the constant 2.2 appearing in (8).

As in Bleicher and Erdős' paper, the proof of the theorem proceeds by giving a lower bound on the number of N such that $|E_N| = 2|E_{N-1}|$ using a recursive approach. In particular, letting $\mathcal{U}(x)$ be the number of such occurrences with $N \leq x$, in Lemma 7 we prove $|\mathcal{U}(x)| \geq \frac{x}{\ln x}\int_1^y \frac{|\mathcal{U}(v)|}{v^2}\mathrm{d}v$ if x is large enough. Together with a bound for $\mathcal{U}(x)$ for small x, the recursive use of the afore-mentioned inequality leads to the claimed bound.

For large positive integers N, computing $|E_N|$ directly from its definition (1) is infeasible. In fact, this requires the computation of 2^N sums of rational numbers with denominators of size up to $\operatorname{lcm}\{1, 2, \dots, N\}$. In sequence A072207 of OEIS [10] it is possible to find the values of $|E_N|$ for $N = 1, \dots, 83$. In Section 3 we explain the methods that we used to compute $|E_N|$ up to $N = 154$. These values are provided in Table 2 at the end of the paper. The main difficulty to extend the computation to $N > 154$ lies in the large quantity of memory that is necessary to store the intermediate results, not in execution time.

Analysing the values of the quotient $|E_N|/|E_{N-1}|$ for the numbers that we have at our disposal, we see that most of the times $|E_N|/|E_{N-1}|$ is either equal to 2 or very close to 1, as expected since the upper-bound in (2) implies that $|E_N|/|E_{N-1}| \leq 1 + \varepsilon$ for almost all N for any fixed $\varepsilon > 0$. More precisely, let

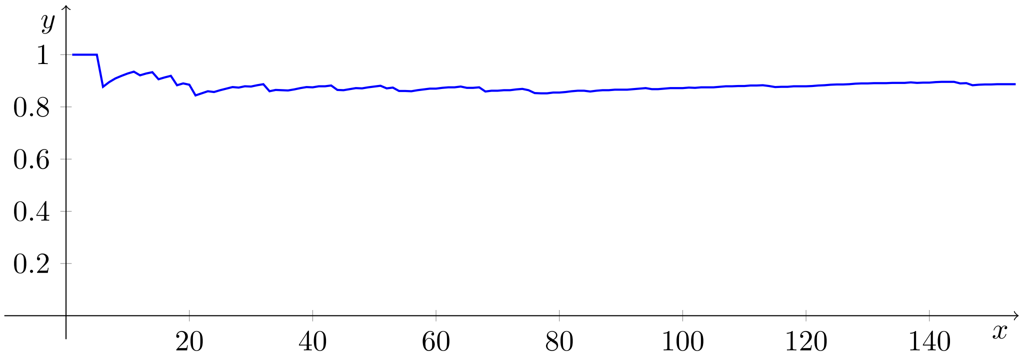
$$D(n) := \Big|\Big\{N \leq 154\colon \frac{1}{\ln 2}\ln\Big(\frac{|E_N|}{|E_{N-1}|}\Big) \in \Big(\frac{n-1}{10}, \frac{n}{10}\Big]\Big\}\Big|, \qquad n = 1, \dots, 10.$$

Table 1 shows that $D(n)$ is concentrated around 1 and 10, that is, $|E_N|/|E_{N-1}|$ is concentrated around 1 and 2. Furthermore, a direct check reveals that all but one of the cases counted in the last column of Table 1 satisfy $|E_N|/|E_{N-1}| = 2$.

n	1	2	3	4	5	6	7	8	9	10
$D(n)$	49	3	2	4	1	2	2	2	3	85

TABLE 1. The values of $D(n)$.

As an additional point in favor of our approach, we show the graph of $|\mathcal{U}(N)|\ln 2/\ln|E_N|$, for $1 \leq N \leq 154$. As one can see, the sequence does not appear to tend to 0, much more to something above 0.85. Therefore, the doublings seem to explain most of the increase of $|E_N|$.



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Notation

The letter p is reserved for prime numbers. For every finite set S, let $|S|$ be the cardinality of S. For every real number x, let $\pi(x) := |\{p \leq x\}|$ be the prime-counting function. For every integer $k \geq 0$, let e_k be the tower of k exponentials in e, that is $e_0 := 1$ and $e_k = e^{e_{k-1}}$ for $k \geq 1$. For every integer $k \geq 0$, let $\ln_k$ the k-th iteration of the natural logarithm. Note that $\ln_k e_k = 1$ and $\ln_k e_{k-1} = 0$ for every positive integer k.

2. Proof of Theorem 1

Define the following set of positive integers

$$\mathcal{U} := \left\{ N \geq 1 \colon \sum_{n=1}^{N-1} \frac{w_n}{n} \neq \frac{1}{N} \text{ for all } w_1, \dots, w_{N-1} \in \{-1, 0, +1\} \right\}.$$

Note that the set $\mathcal{U}$ is stable by divisors, i.e., if $N \in \mathcal{U}$ then every divisor of N is in $\mathcal{U}$ as well. For every real number x, let $\mathcal{U}(x) := \mathcal{U} \cap [1, x]$.

Lemma 1. *Let N be a positive integer. Then $N \in \mathcal{U}$ if and only if $|E_N| = 2|E_{N-1}|$.*

Proof. The definition of E_N yields at once $E_N = E_{N-1} \cup (E_{N-1} + 1/N)$ (setting $E_0 := \{0\}$). By the definition of $\mathcal{U}$, the union is disjoint if and only if $N \in \mathcal{U}$. Since E_{N-1} and $E_{N-1} + 1/N$ have the same cardinality, their union is disjoint if and only if $|E_N| = 2|E_{N-1}|$. □

Lemma 2. *$|E_N| \geq 2^{|\mathcal{U}(N)|}$ for every positive integer N.*

Proof. The claim follows immediately from Lemma 1. □

Thanks to Lemma 2, to produce a lower bound for $|E_N|$, it suffices to give a lower bound for $|\mathcal{U}(N)|$.

For each positive integer m, let

$$d_m := \operatorname{lcm}\{1, \dots, m\}, \qquad g_m := d_m \sum_{j=1}^{m} \frac{1}{j}.$$

We will require the following bound for $\pi(g_m)$.

Lemma 3. *For any positive integer m, we have $\pi(g_m) \leq \frac{18 \cdot 3^m}{m \ln(18 \cdot 3^m)}$.*

Proof. From [8, p. 228] we know that $d_m = \exp(\psi(m)) \le \exp(1.04m)$. On the other hand,

$$\sum_{j=1}^{m} \frac{1}{j} \le 1 + \int_1^m \frac{\mathrm{d}t}{t} = \ln(em).$$

Hence, $g_m \le \exp(1.04m)\ln(em)$. This implies, by a quick computation, that $g_m < \frac{15}{m} \cdot 3^m$ for all integers $m \ge 46$, and a direct computation shows that this bound also holds for $1 \le m \le 45$. Then, writing $u := 15 \cdot 3^m/m$ by [9, Th. 1] we have

$$\pi(g_m) \le \pi(u) \le \frac{u}{\ln u}\Big(1 + \frac{3}{2\ln u}\Big) \le \frac{18 \cdot 3^m}{m \ln(18 \cdot 3^m)}$$

for $m \ge 24$. The lemma then follows by direct verification of the cases $1 \le m \le 23$. □

We say that a prime p is *compatible* with m if p does not divide the numerator of any rational of the form $1/m - \sum_{j=1}^{m-1} w_j/j$ with $w_j \in \{-1, 0, +1\}$. Thus, if $m \notin \mathcal{U}$ then $1/m - \sum_{j=1}^{m-1} w_j/j = 0$ for some choice of w_j and no prime is compatible with m. On the other hand, if $m \in \mathcal{U}$ and $p > g_m$ then p is compatible with m. The following result shows how to generate many elements in $\mathcal{U}$ in a recursive way, starting with the evident fact that $1 \in \mathcal{U}$.

Lemma 4. *Let m be in $\mathcal{U}$ and k be any positive integer. Then $mp^k \in \mathcal{U}$, for any prime p compatible with m. In particular, this holds for any prime $p > g_m$.*

Proof. Suppose by contradiction that there exist m, p and k as above such that $N := mp^k \notin \mathcal{U}$. Hence, by definition, there exist $w_1, \dots, w_{N-1}$ in $\{-1, 0, +1\}$ such that

$$\sum_{n=1}^{N-1} \frac{w_n}{n} = \frac{1}{N} = \frac{1}{mp^k}.$$

We split the sum according to whether $p^k \mid n$ or not and multiplying by $p^k D$, where D is the denominator of $\frac{1}{m} - \sum_{j=1}^{m-1} \frac{w_{p^k j}}{j}$. We obtain

$$p^k D \sum_{\substack{n=1 \\ p^k \nmid n}}^{N-1} \frac{w_n}{n} = D\Big(\frac{1}{m} - \sum_{j=1}^{m-1} \frac{w_{p^k j}}{j}\Big).$$

The right-hand side is the numerator of $\frac{1}{m} - \sum_{j=1}^{m-1} \frac{w_{p^k j}}{j}$ and is divisible by p, since the left-hand side is. It follows that p is not compatible with m, which is a contradiction. □

Notice that, since $g_1 = 1$, the case $m = 1$ in Lemma 4 already proves that all prime powers are elements of $\mathcal{U}$.

Lemma 5. *We have*

$$\begin{aligned}(3)\quad \mathcal{U}(100) = \{&1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, 22, 23, 25, 26, 27, 29,\\ &31, 32, 34, 37, 38, 39, 41, 43, 46, 47, 49, 50, 51, 53, 57, 58, 59, 61, 62,\\ &64, 67, 69, 71, 73, 74, 79, 81, 82, 83, 86, 87, 89, 92, 93, 94, 97, 98\}.\end{aligned}$$

Proof. First, we list the numbers that belong to $\mathcal{U}(100)$ thanks to Lemma 4.

- We know that 1 is in $\mathcal{U}$.
- $g_1 = 1$, hence if $2 \le p \le 100$, $k \ge 1$ and $N = p^k$, then $N \in \mathcal{U}$.
- $g_2 = 3$, hence if $5 \le p \le 47$, $k \ge 1$ and $N = 2p^k$, then $N \in \mathcal{U}$.
- $g_3 = 11$, hence if $13 \le p \le 31$ and $N = 3p$, then $N \in \mathcal{U}$.
- 23 is compatible with 4, hence $92 = 4 \cdot 23$ is in $\mathcal{U}$.

This proves that $\mathcal{U}(100)$ contains all the numbers listed in (3). We now show that no other number belongs to $\mathcal{U}(100)$.

Every multiple of 6, 15, 20, 21, 28, 33, 35, or 44 does not belong to $\mathcal{U}$ because

$$\frac{1}{6} = \frac{1}{2} - \frac{1}{3}, \qquad \frac{1}{15} = \frac{1}{6} - \frac{1}{10}, \qquad \frac{1}{20} = \frac{1}{4} - \frac{1}{5},$$

$$\frac{1}{21} = \frac{1}{7} + \frac{1}{14} - \frac{1}{6}, \quad \frac{1}{33} = \frac{1}{6} - \frac{1}{11} - \frac{1}{22}, \quad \frac{1}{44} = \frac{1}{33} + \frac{1}{12} - \frac{1}{11}.$$

Furthermore, 52, 55, 65, 68, 76, 85, 91, and 95 do not belong to $\mathcal{U}$ because

$$\frac{1}{52}=\frac{1}{12}-\frac{1}{26}-\frac{1}{39},\qquad \frac{1}{55}=-\frac{1}{20}+\frac{1}{22}+\frac{1}{44},\quad \frac{1}{65}=-\frac{1}{10}+\frac{1}{13}+\frac{1}{26},$$
$$\frac{1}{68}=\frac{1}{4}-\frac{1}{6}-\frac{1}{17}-\frac{1}{34}+\frac{1}{51},\quad \frac{1}{76}=\frac{1}{12}-\frac{1}{19}-\frac{1}{57},\quad \frac{1}{85}=\frac{1}{10}-\frac{1}{17}-\frac{1}{34},$$
$$\frac{1}{91}=\frac{1}{42}-\frac{1}{78},\qquad \frac{1}{95}=\frac{1}{20}-\frac{1}{38}-\frac{1}{76}.$$

The claim is proved.

□

Lemma 6. *Let x be a real number. Then*

$$|\mathcal{U}(x)| \geq 2\frac{x}{\ln x} \qquad \text{if } x \geq 13, \tag{4}$$

$$|\mathcal{U}(x)| \geq \frac{137}{60}\frac{x}{\ln x} \qquad \text{if } x \geq 1000. \tag{5}$$

Proof. The explicit description of $\mathcal{U}(100)$ in Lemma 5 implies (4) for $x \leq 100$. Hence, assume that $x > 100$. Since 1, 2, 3 and 4 belong to $\mathcal{U}$, we find that all elements of the form p, $2p$ (with $p > g_3 = 3$), $3p$ (with $p > g_3 = 11$) and $4p$ (with $p > g_4 = 25$) belong to $\mathcal{U}$. In $\mathcal{U}(x)$ we thus have $\pi(x)$ elements of the first type, $\pi(x/2) - \pi(3)$ elements of the second, $\pi(x/3) - \pi(11)$ elements of the third and $\pi(x/4) - \pi(25)$ elements of the fourth, i.e., $\pi(x) + \pi(x/2) + \pi(x/3) + \pi(x/4) - 16$ elements of one of the above mentioned forms. Moreover, 1, 4, 8, 16, 32, 64, 9, 27, 81, 92, 25, 49, 50 and 98 belong to $\mathcal{U}$ and are not of the aforementioned form. Hence for $x \geq 100$, we have

$$|\mathcal{U}(x)| \geq \pi(x) + \pi\Big(\frac{x}{2}\Big) + \pi\Big(\frac{x}{3}\Big) + \pi\Big(\frac{x}{4}\Big) - 2.$$

In [9, Th. 2, Cor. 1] it is proved that $\pi(t) \geq t/\ln t$ when $t \geq 17$. Thus,

$$|\mathcal{U}(x)| \geq \frac{x}{\ln x} + \frac{x/2}{\ln(x/2)} + \frac{x/3}{\ln(x/3)} + \frac{x/4}{\ln(x/4)} - 2 \geq 2\frac{x}{\ln x},$$

where the last inequality is elementary and proves the claim.

Assume now $x \geq 1000$. Since $5 \in \mathcal{U}$, we also have elements of the form $5p$ (with $p > g_5 = 137$) in $\mathcal{U}$, and there are $\pi(x/5) - \pi(137) = \pi(x/5) - 33$ elements of this form.

Adding these terms to the previous lower bound, we find that

$$|\mathcal{U}(x)| \geq \frac{x}{\ln x} + \frac{x/2}{\ln(x/2)} + \frac{x/3}{\ln(x/3)} + \frac{x/4}{\ln(x/4)} + \frac{x/5}{\ln(x/5)} - 35 \geq \frac{137}{60}\frac{x}{\ln x},$$

where the last inequality is once again elementary and proves (5).

□

Lemma 7. *Assume $y \geq 1$ and $x \geq 18 \cdot 3^y$. Then*

$$|\mathcal{U}(x)| \geq \frac{x}{\ln x}\int_1^y \frac{|\mathcal{U}(v)|}{v^2}\mathrm{d}v. \tag{6}$$

Proof. By Lemmas 4 and 3 we have

$$|\mathcal{U}(x)| \geq \sum_{m\in\mathcal{U}(y)} \Big(\pi\Big(\frac{x}{m}\Big) - \pi(g_m)\Big) \geq \sum_{m\in\mathcal{U}(y)} \Big(\pi\Big(\frac{x}{m}\Big) - \frac{18\cdot 3^m}{m\ln(18\cdot 3^m)}\Big).$$

Since $x/m \geq x/y \geq 18 \cdot 3^y/y \geq 17$ for every $m \leq y$, we have $\pi(x/m) \geq \frac{x/m}{\ln(x/m)} \geq \frac{x/m}{\ln x}$ (by [9, Th. 2, Cor. 1]), hence

$$|\mathcal{U}(x)| \geq \sum_{m\in\mathcal{U}(y)} \Big(\frac{x}{m\ln x} - \frac{18\cdot 3^m}{m\ln(18\cdot 3^m)}\Big).$$

We write the right-hand side as a Stieltjes integral, so that by partial summation it becomes

$$\int_{1^-}^{y^+} \Big(\frac{x}{v\ln x} - \frac{18\cdot 3^v}{v\ln(18\cdot 3^v)}\Big)\mathrm{d}|\mathcal{U}(v)|$$
$$= \Big(\frac{x}{v\ln x} - \frac{18\cdot 3^v}{v\ln(18\cdot 3^v)}\Big)|\mathcal{U}(v)|\Big|_{1^-}^{y^+} - \int_1^y |\mathcal{U}(v)|\Big(-\frac{x}{v^2\ln x} - \frac{\mathrm{d}}{\mathrm{d}v}\Big(\frac{18\cdot 3^v}{v\ln(18\cdot 3^v)}\Big)\Big)\mathrm{d}v$$

$$= \Big(\frac{x}{\ln x} - \frac{18\cdot 3^y}{\ln(18\cdot 3^y)}\Big)\frac{|\mathcal{U}(y)|}{y} + \frac{x}{\ln x}\int_1^y \frac{|\mathcal{U}(v)|}{v^2}\mathrm{d}v + \int_1^y |\mathcal{U}(v)|\frac{\mathrm{d}}{\mathrm{d}v}\Big(\frac{18\cdot 3^v}{v\ln(18\cdot 3^v)}\Big)\mathrm{d}v.$$

The first term is positive, since $e < 18\cdot 3^y \le x$. Also the derivative in the last integral is positive for $v \ge 2$. Thus, assuming that $y \ge 2$ we get

$$|\mathcal{U}(x)| \ge \frac{x}{\ln x}\int_1^y \frac{|\mathcal{U}(v)|}{v^2}\mathrm{d}v + \int_1^2 |\mathcal{U}(v)|\frac{\mathrm{d}}{\mathrm{d}v}\Big(\frac{18\cdot 3^v}{v\ln(18\cdot 3^v)}\Big)\mathrm{d}v.$$

Since $|\mathcal{U}(y)| = 1$ for $y \in [1,2)$, this is

$$= \frac{x}{\ln x}\int_1^y \frac{|\mathcal{U}(v)|}{v^2}\mathrm{d}v + \frac{18\cdot 3^v}{v\ln(18\cdot 3^v)}\Big|_1^2 \ge \frac{x}{\ln x}\int_1^y \frac{|\mathcal{U}(v)|}{v^2}\mathrm{d}v,$$

which concludes the proof in this case. On the other hand, when $y \in [1,2)$ the claim states

$$|\mathcal{U}(x)| \ge \frac{x}{\ln x}\int_1^y \frac{|\mathcal{U}(v)|}{v^2}\mathrm{d}v = \frac{x}{\ln x}\Big(1 - \frac{1}{y}\Big),$$

which is true by Lemma 6. □

We want to simplify the integral recursion for $|\mathcal{U}(x)|$ described in Lemma 7. For this purpose, let G be defined by

$$G(z) := \frac{\ln x}{x}|\mathcal{U}(x)|, \quad \text{where } x := \exp(\exp(z)). \tag{7}$$

We set $y := e^{z-1/4}$, so that the condition $x = e^{(1-e^{-1/4}\ln 3)e^z}\cdot 3^y \ge 18\cdot 3^y$ and $y \ge 1$ are verified for $z \ge 3$.

Since $\ln\ln y = \ln(z - 1/4)$, we can rewrite (6) as

$$G(z) \ge \int_{-\infty}^{\ln(z-1/4)} G(w)\mathrm{d}w \quad \forall z \ge 3.$$

The part of the integral with $w \le 1$ contributes $\int_{-\infty}^1 G(w)\mathrm{d}w = \int_1^{\exp(\exp(1))} \frac{|\mathcal{U}(v)|}{v^2}\mathrm{d}v \ge 2.2$ by Lemma 5 and a direct computation. Hence,

$$G(z) \ge 2.2 + \int_1^{\ln(z-1/4)} G(w)\mathrm{d}w \quad \forall z \ge 3. \tag{8}$$

For any $k \ge 1$, let h_k be defined recursively as $h_1 := 1$ and $h_k := \exp(h_{k-1}) + 1/4$ for $k \ge 2$, and let $T_k\colon [0,+\infty) \to \mathbb{R}$ be the function defined by the identities

$$\begin{aligned} T_1(z) &:= \ln z && \text{if } z > h_1, \\ T_k(z) &:= \int_{h_{k-1}}^{\ln(z-1/4)} T_{k-1}(w)\mathrm{d}w && \text{if } z > h_k \quad \forall k \ge 2, \end{aligned}$$

and 0 everywhere else. By induction on k, one verifies that $h_k \in [e_{k-1}, e_k]$ and that each T_k is a continuous and increasing function which is zero in $[0, h_k]$ and positive in $(h_k, +\infty)$. We note that the definition of T_k with $k \ge 2$ is modeled on (8), where we removed the additive constant 2.2 that appears there, since preliminary calculations show that its presence leads to various complications that are not compensated by any truly significant improvement.

The following lower bound essentially comes from (8).

Lemma 8. *Let $k \ge 1$. Then for all $z > 0$ we have*

$$G(z) \ge 2\,T_k(z). \tag{9}$$

Proof. We first prove the statement for $k = 1$. When $z \le 1$ the claim is clear since $T_1(z)$ is zero there. By (4) we know that $G(z) \ge 2$ holds for $z \ge 1$ since $\ln\ln 13 < 1$. If $z > 3$ the recursive lower bound (8) then implies that $G(z) \ge 2.2 + \int_1^{\ln(z-1/4)} 2\,\mathrm{d}u = 2\ln(z-1/4) + 0.2 \ge 2T_1(z)$. In the intermediate ranges $[1, e]$ and $[e, 3]$ we use (4) in the first case, so that $G(z) \ge 2 = 2T_1(e) \ge 2T_1(z)$, and (5) in the second case, so that $G(z) \ge 137/60 \ge 2T_1(3) \ge 2T_1(z)$ once again.

Assume $k=2$. If $z \le h_2 = e + 1/4 = 2.96\ldots$, then $G(z) \ge 2T_2(z)$ is trivially satisfied, as the right-hand side is 0 there, and the same lower bound also holds when $z \in [h_2, 3]$ since by (4) we have $G(z) \ge 2 \ge 2T_2(3) \ge 2T_2(z)$. When $z \ge 3$ we use (8) and (9), in the case $k = 1$, obtaining

$$G(z) \ge 2.2 + 2\int_1^{\ln(z-1/4)} T_1(w)\mathrm{d}w \ge 2T_2(z).$$

Finally, assume $k \ge 3$. Once again, the bound is trivially true when $z \le h_k$. Let $z \ge h_k$. Then $z \ge 3$ so that (8) and the inductive hypothesis stating that $G(z) \ge 2T_{k-1}(z)$ give

$$G(z) \ge 2.2 + 2\int_1^{\ln(z-1/4)} T_{k-1}(w)\mathrm{d}w.$$

The function inside the integral is zero when $w < h_{k-1}$, therefore we can replace the lower integral with h_{k-1} and the claim follows by discarding the additive constant 2.2. □

Lemma 9. *Suppose that for a certain integer $k \ge 3$ there exists a positive constant $a_{k-1} \ge 1$ such that*

$$T_{k-1}(z) \ge \prod_{j=1}^{k-1} \ln_j z - a_{k-1} \prod_{j=1}^{k-2} \ln_j z \tag{10}$$

when $z \ge e_{k-1}$. Then, we have

$$T_k(z) \ge \prod_{j=1}^{k} \ln_j z - a_k \prod_{j=1}^{k-1} \ln_j z \tag{11}$$

when $z \ge e_k$, with

$$a_k := a_{k-1} + \frac{1}{e_{k-2}} + \frac{k-2}{e_{k-2}e_{k-3}}.$$

Proof. We start by noticing that the claim in (10) holds trivially in the range $z \in (e_{k-2}, e_{k-1})$. In fact, the assumption $a_{k-1} \ge 1$ implies that the right-hand side of (10) is negative whenever $z < e_{k-1}$, while T_{k-1} is always nonnegative.

Let $z \ge e_k$ and recall that $h_k \in [e_{k-1}, e_k)$, so that $z \ge h_k$. Then

$$T_k(z) = \int_{h_{k-1}}^{\ln(z-1/4)} T_{k-1}(u)\mathrm{d}u = \int_{e_{k-2}}^{\ln(z-1/4)} T_{k-1}(u)\mathrm{d}u$$

since T_{k-1} is 0 in $[e_{k-2}, h_{k-1}]$. By the assumed lower bound for T_{k-1} we have

$$T_k(z) \ge \int_{e_{k-2}}^{\ln(z-1/4)} \prod_{j=1}^{k-1} \ln_j w \,\mathrm{d}w - a_{k-1}\int_{e_{k-2}}^{\ln(z-1/4)} \prod_{j=1}^{k-2} \ln_j w \,\mathrm{d}w = A(z) - a_{k-1}B(z),$$

say. Next, we rewrite $A(z)$ as $A(z) = A_1(z) - A_2(z)$ where A_1 denotes the integral obtained by extending the upper limit of integration in A to $\ln z$, and A_2 denotes the integral over $[\ln(z-\frac{1}{4}), \ln z]$. We note that for $z \ge e_k$ one has $\ln(z-\frac14) \ge e_{k-2}$, so that every $\ln_j$ is positive and increasing over both integration ranges. Then

$$A_2(z) \le \big(\ln z - \ln(z-1/4)\big)\prod_{j=1}^{k-1} \ln_j(\ln z) = -\ln(1-1/(4z))\prod_{j=2}^{k} \ln_j z.$$

As for A_1, integration by parts (recalling that $\ln_{k-1}(e_{k-2}) = 0$) yields

$$\begin{aligned} A_1(z) &= \int_{e_{k-2}}^{\ln z} \prod_{j=1}^{k-1} \ln_j w \,\mathrm{d}w = w\prod_{j=1}^{k-1}\ln_j w\Big|_{e_{k-2}}^{\ln z} - \int_{e_{k-2}}^{\ln z} w\Big(\prod_{j=1}^{k-1}\ln_j w\Big)'\mathrm{d}w \\ &= \prod_{j=1}^{k}\ln_j z - \int_{e_{k-2}}^{\ln z}\Big(\sum_{i=2}^{k}\prod_{j=i}^{k-1}\ln_j w\Big)\mathrm{d}w \ge \prod_{j=1}^{k}\ln_j z - (\ln z - e_{k-2})\sum_{i=2}^{k}\prod_{j=i+1}^{k}\ln_j z. \end{aligned}$$

We notice that for $k \ge 3$ and $z \ge e_k$ we have the chain of elementary inequalities:

$$e_{k-2}\sum_{i=2}^{k}\prod_{j=i+1}^{k}\ln_j z \ge \prod_{j=3}^{k}\ln_j z = \frac{1}{\ln_2 z}\prod_{j=2}^{k}\ln_j z \ge -\ln(1-1/(4z))\prod_{j=2}^{k}\ln_j z \ge A_2(z).$$

Putting everything together and collecting a factor of $\prod_{j=1}^{k-1} \ln_j z$, we get

$$A(z) \geq \prod_{j=1}^{k} \ln_j z - \Big(\sum_{i=2}^{k} \ln_k z \prod_{j=2}^{i} \frac{1}{\ln_j z}\Big) \prod_{j=1}^{k-1} \ln_j z.$$

We are assuming $z \geq e_k$, and each quotient $\ln_k z/(\ln_2 z \cdots \ln_i z)$ decreases in this range, hence it can be bounded by its value at e_k, producing the bound

$$\begin{aligned}\sum_{i=2}^{k} \ln_k z \prod_{j=2}^{i} \frac{1}{\ln_j z} &\leq \sum_{i=0}^{k-2} \prod_{j=i}^{k-2} \frac{1}{e_j} = \frac{1}{e_{k-2}}\Big(1 + \frac{1}{e_{k-3}} + \frac{1}{e_{k-3}e_{k-4}} + \cdots + \frac{1}{e_{k-3}e_{k-4}\cdots e_0}\Big)\\ &\leq \frac{1}{e_{k-2}}\Big(1 + \frac{k-2}{e_{k-3}}\Big).\end{aligned}$$

Hence,

$$A(z) \geq \prod_{j=1}^{k} \ln_j z - \Big(\frac{1}{e_{k-2}} + \frac{k-2}{e_{k-2}e_{k-3}}\Big) \prod_{j=1}^{k-1} \ln_j z. \tag{12}$$

Next, we bound B. We extend its domain up to $\ln z$ by positivity of the integrand, getting

$$B(z) \leq \int_{e_{k-2}}^{\ln z} \prod_{j=1}^{k-2} \ln_j w \, \mathrm{d}w \leq \ln z \prod_{j=1}^{k-2} \ln_j(\ln z) = \prod_{j=1}^{k-1} \ln_j z. \tag{13}$$

By (12) and (13) we get the claim. □

Lemma 10. *Let $k \in \{1, 2, 3\}$ and $z \geq e_k$. Then*

$$T_k(z) \geq \prod_{j=1}^{k} \ln_j z - a_k \prod_{j=1}^{k-1} \ln_j z$$

with $a_1 = 0$, $a_2 = 1$ and $a_3 = 1.28$.

Proof. The claim for T_1 follows from its definition. Moreover, for $z > h_2$ we have

$$T_2(z) = \int_1^{\ln(z-1/4)} T_1(u)\mathrm{d}u = \int_1^{\ln(z-1/4)} \ln u \, \mathrm{d}u = (u \ln u - u)\Big|_1^{\ln(z-1/4)} \geq \ln z \ln_2 z - \ln z,$$

as one can see via elementary computations. inserting the lower bound we just proved for T_2, we get

$$T_3(z) \geq \int_{h_2}^{\ln(z-1/4)} \ln u(\ln_2 u - 1)\mathrm{d}u = \big(u \ln u \ln_2 u - u \ln u - u \ln_2 u + \mathrm{Li}(u)\big)\Big|_{h_2}^{\ln(z-1/4)},$$

for $z > h_3$ and where $\mathrm{Li}(u) := \int_2^u \frac{\mathrm{d}s}{\ln s}$. With elementary computations one proves from this bound that

$$T_3(z) \geq \ln z \ln_2 z \ln_3 z - 1.28 \ln z \ln_2 z$$

when $z \geq h_3$. □

Proof of Theorem 1. The first claim immediately follows from Lemma 2 and the lower bound (4) in Lemma 6.

The second claim follows from Lemma 2, the definition of G in (7), the case $k = 1$ of the lower bound (9) and the definition of T_1. In fact, this argument gives the claimed lower-bound already for $\ln_2 N \geq 1$ but this bound improves on the previous one for $\ln_3 N \geq 1$ only.

Finally, let $k \geq 4$. Iterating the conclusion in Lemma 9 we have (11) with

$$a_k \leq a_3 + \sum_{j=4}^{\infty} \Big(\frac{1}{e_{j-2}} + \frac{j-2}{e_{j-2}e_{j-3}}\Big) \leq a_3 + 0.12.$$

By Lemma 10 we can pick $a_3 = 1.28$, hence

$$T_k(z) \geq \Big(1 - \frac{1.4}{\ln_k z}\Big) \prod_{j=1}^{k} \ln_j z \tag{14}$$

for $z \geq e_k$, for every $k \geq 4$. By Lemma 10 the same holds for $k = 2$ and 3. By (14) (with a shift $k \to k-2$), (9), (7), and Lemma 2 we deduce that

$$\frac{\ln(|E_N|)}{\ln 2} \geq 2\Big(1 - \frac{1.4}{\ln_k N}\Big)\frac{N}{\ln N}\prod_{j=3}^{k}\ln_j N$$

for every $k \geq 4$. □

3. Computing $|E_N|$

In this section we explain how we computed $|E_N|$ for $N = 1, \ldots, 154$. For every finite set of positive integers S, let

$$E(S) := \Big\{\sum_{n\in S}\frac{t_n}{n} \colon t_n \in \{0,1\}\ \forall n \in S\Big\}$$

and

$$\mathcal{U}(S) := \Big\{N \geq 1 \colon \sum_{n\in S}\frac{w_n}{n} \neq \frac{1}{N} \text{ for all } (w_n)_{n\in S} \in \{-1,0,+1\}^{|S|}\Big\}.$$

Then it is straightforward to generalize Lemma 1 and Lemma 2 to the following results.

Lemma 11. *Let s_0 be a positive integer and let S be a finite set of positive integers not containing s_0. Then $s_0 \in \mathcal{U}(S)$ if and only if $|E(S \cup \{s_0\})| = 2|E(S)|$.*

Lemma 12. *Let S_1 and S_2 be disjoint finite set of positive integers. Then*

$$|E(S_1 \cup S_2)| \geq 2^{|\mathcal{U}(S_1)|}E(S_2).$$

Let us see how to compute $|E_{154}| = |E(\mathfrak{a})|$ for $\mathfrak{a} := \{1, \ldots, 154\}$. The computation of the values E_N for $N < 154$ is similar.

Let p^k be any prime power which is larger than $154/2 = 77$. Then the equality

$$\sum_{n\in\mathfrak{a}\setminus\{p^k\}}\frac{w_n}{n} = \frac{1}{p^k}$$

with $w_n \in \{-1,0,1\}$ is impossible. Thus, we can remove p^k from $\mathfrak{a}$, getting a shorter set $\mathfrak{a}'$, and by Lemma 11 the cardinality of the set $E(\mathfrak{a})$ is two times the cardinality of $E(\mathfrak{a}')$. Repeating this step several times we remove from the original set the numbers 128, 81, 125, 121, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151 getting a new set $\mathfrak{a}'$.

Now consider a number of the form $2p^k$ and suppose that $p^k > 154/3$. Then p^k and $2p^k$ are the unique elements in $\mathfrak{a}'$ which are divisible by p^k. Suppose moreover that $p \neq 3$, which means that it is compatible with $m = 2$, so that the equality

$$\sum_{n\in\mathfrak{a}'\setminus\{p^k,2p^k\}}\frac{w_n}{n} = \frac{1}{p^k}\Big(w + \frac{1}{2}\Big)$$

with w_n and $w \in \{-1,0,1\}$ is impossible. As a consequence, these numbers $2p^k$ can also be removed from $\mathfrak{a}'$, because their presence simply doubles the size of the set of Egyptian fractions. Repeating this step we can remove from $\mathfrak{a}'$ the numbers 128 (which has already been removed), 106, 118, 122, 134, 142, 146. Now that these numbers of the form $2p^k$ have been removed, we can also remove the corresponding p^k (applying the argument we have used as first step). Thus, we also remove 64, 53, 59, 61, 67, 71, 73, getting a new set $\mathfrak{a}''$.

Now, consider a number of the form $3p^k$ and suppose that $p^k > 154/4$. Then p^k, $2p^k$ and $3p^k$ are the unique elements in $\mathfrak{a}'$ which are divisible by p^k. Suppose moreover that $p > g_3 = 11$, so that it is compatible with $m = 3$, and the equality

$$\sum_{n\in\mathfrak{a}''\setminus\{p^k,2p^k,3p^k\}}\frac{w_n}{n} = \frac{1}{p^k}\Big(w + \frac{w'}{2} + \frac{1}{3}\Big)$$

with w_n and w, $w' \in \{-1,0,1\}$ is impossible. As above, these $3p^k$ can be removed from $\mathfrak{a}''$. In this way we remove from $\mathfrak{a}''$ the numbers 123, 129, 141. Now that these numbers of the form $3p^k$ have been removed, we can also remove the corresponding $2p^k$ (applying the argument we have used as second step), and the corresponding p^k (applying the argument we have used as first step). Thus, we also remove 82, 86, 94, 41, 43, 47, getting a new set $\mathfrak{a}'''$.

Finally, consider a number of the form $4p^k$, where $p^k > 154/5$ and $p > g_4 = 25$. As above, we can remove from $\mathfrak{a}'''$ the numbers 124 and 148, and then, in cascade, the numbers 93, 111, 62, 74, 31, 37, getting finally the set $\mathfrak{a}^{iv}$. At this point, we have $|E_{154}| = 2^{49}|E(\mathfrak{a}^{iv})|$. The set $\mathfrak{a}^{iv}$ contains only 105 elements, but it is still too large to directly compute the corresponding set of Egyptian fractions. The next remark is slightly more involved.

We subdivide $\mathfrak{a}^{iv}$ into two disjoint subsets $\mathfrak{a}_0$ and $\mathfrak{a}_1 := \{29n \colon 1 \le n \le 5\}$. Now $\mathfrak{a}_0$ has only 100 elements, and it is computationally feasible to directly compute $E_0 := E(\mathfrak{a}_0)$. We then have

$$E' := E(\mathfrak{a}^{iv}) = \bigcup_{a \in E(\mathfrak{a}_1)} (E_0 + a).$$

It is obvious that $E(\mathfrak{a}_1) = \frac{1}{29}E_5$. Let $\ell := \operatorname{lcm}(\mathfrak{a}_0)$. By construction ℓ is prime to 29. We obviously have $|E'| = |29\ell E'|$ and

$$29\ell E' = \bigcup_{a \in 29\ell E(\mathfrak{a}_1)} (29\ell E_0 + a).$$

Now, if a and $b \in 29\ell E(\mathfrak{a}_1)$ are such that a and b have different classes modulo 29, then $29\ell E_0 + a$ and $29\ell E_0 + b$ are disjoint. The elements of $29\ell E(\mathfrak{a}_1) = \ell E_5$ reduce modulo 29 in 24 classes, Among those 8 classes, the difference between the two representatives is $d_1 := 29 \cdot 37479602160$ for 2 of them and $d_2 := 29 \cdot 56219403240$ for the remaining 6. Therefore,

$$|E'| = |\ell E'| = 16|E_0| + 2|E_0 \cup (E_0 + d_1/29)| + 6|E_0 \cup (E_0 + d_2/29)|.$$

What we have gained is that $\operatorname{lcm}(\mathfrak{a}^{iv}) = 29\ell$ and, since the elements of E_0 (or E', if we had computed it) are represented by bitfields, we divide by 29 the requested memory and time for the computation. To furthermore halve the size of the sets we need to consider, we observe that any $E(S)$ has a center of symmetry (by induction because if $E \subseteq \mathbb{R}$ is symmetric, then for any $x \in \mathbb{R}$, $E \cup (E + x)$ is symmetric, with its center moved by $x/2$ with respect to the center of E).

Keeping note of the intermediate cardinalities of the sets

$$\Big\{ \sum_{n \in \mathfrak{a}_0,\ n \le N} \frac{t_n}{n} \colon t_n \in \{0,1\},\ \forall n \in \mathfrak{a}_0 \Big\},$$

and taking in consideration the cancelled elements, we can compute the cardinalities of E_N for all $N \le 154$. Notice that this works only because the multiples of 29 can be cancelled from the generating set of E_N, for all $N \le 5 \cdot 29 - 1$.

The full computation needed approximatively two hours and 300GB RAM memory on the PlaFRIM platform in Bordeaux.

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N	$\|E_N\|$	N	$\|E_N\|$	N	$\|E_N\|$
0	1	52	570733363200	104	434404550383671181312
1	2	53	1141466726400	105	435821848359665139712
2	4	54	1721081528320	106	871643696719330279424
3	8	55	1751601381376	107	1743287393438660558848
4	16	56	1767017021440	108	1754513627060579074048
5	32	57	3534034042880	109	3509027254121158148096
6	52	58	7068068085760	110	3522492005456298377216
7	104	59	14136136171520	111	7044984010912596754432
8	208	60	14245758500864	112	7068497418916307402752
9	416	61	28491517001728	113	14136994837832614805504
10	832	62	56983034003456	114	16899242066544045326336
11	1664	63	57494604873728	115	22272240164078654324736
12	1856	64	114989209747456	116	44544480328157308649472
13	3712	65	137824242237440	117	44696007986571758272512
14	7424	66	139033409748992	118	89392015973143516545024
15	9664	67	278066819497984	119	90161265693495021010944
16	19328	68	522131016253440	120	90370501722719863701504
17	38656	69	1044262032506880	121	180741003445439727403008
18	59264	70	1051387483914240	122	361482006890879454806016
19	118528	71	2102774967828480	123	722964013781758909612032
20	126976	72	2116809947873280	124	1445928027563517819224064
21	224128	73	4233619895746560	125	2891856055127035638448128
22	448256	74	8467239791493120	126	2899478785052218761412608
23	896512	75	10638462277386240	127	5798957570104437522825216
24	936832	76	17372520791408640	128	11597915140208875045650432
25	1873664	77	17522758873251840	129	23195830280417750091300864
26	3747328	78	17647454272880640	130	23285154296843172833132544
27	7494656	79	35294908545761280	131	46570308593686345666265088
28	7771136	80	35499851152097280	132	46685922910553749195849728
29	15542272	81	70999702304194560	133	46892844848672951268016128
30	15886336	82	141999404608389120	134	93785689697345902536032256
31	31772672	83	283998809216778240	135	94091474339233167820455936
32	63545344	84	285080778587504640	136	94380290287482179111878656
33	112064512	85	326182987337039872	137	188760580574964358223757312
34	224129024	86	652365974674079744	138	207845467988940343115513856
35	231010304	87	1304731949348159488	139	415690935977880686231027712
36	237031424	88	1312124045747027968	140	416574753297226313332948992
37	474062848	89	2624248091494055936	141	833149506594452626665897984
38	948125696	90	2637135095231676416	142	1666299013188905253331795968
39	1896251392	91	2653366206139990016	143	1670768625140679902219993088
40	1928593408	92	5306732412279980032	144	1674506556739064055000465408
41	3857186816	93	10613464824559960064	145	2520048783079754452174897152
42	3925999616	94	21226929649119920128	146	5040097566159508904349794304
43	7851999232	95	26280284845565280256	147	8826629661276147104436191232
44	12445024256	96	26477620983450566656	148	17653259322552294208872382464
45	12606504960	97	52955241966901133312	149	35306518645104588417744764928
46	25213009920	98	105910483933802266624	150	35407794518497650679945887744
47	50426019840	99	106363570685376200704	151	70815589036995301359891775488
48	51334348800	100	107289734959184478208	152	71018819275510828026883473408
49	102668697600	101	214579469918368956416	153	71201999904617906814717001728
50	205337395200	102	215966239954017714176	154	71356425097301949080433328128
51	410674790400	103	431932479908035428352		

TABLE 2. The first values of $|E_N|$.

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