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The Bernoulli structure of discrete distributions

Roberto Fontana* Patrizia Semeraro†

Abstract

Any discrete distribution with support on $\{0, \dots, d\}$ can be constructed as the distribution of sums of Bernoulli variables. We prove that the class of d -dimensional Bernoulli variables $\mathbf{X} = (X_1, \dots, X_d)$ whose sums $\sum_{i=1}^d X_i$ have the same distribution p is a convex polytope $\mathcal{P}(p)$ and we analytically find its extremal points. Our main result is to prove that the Hausdorff measure of the polytopes $\mathcal{P}(p), p \in \mathcal{D}_d$, is a continuous function $l(p)$ over \mathcal{D}_d and it is the density of a finite measure μ_s on \mathcal{D}_d that is Hausdorff absolutely continuous. We also prove that the measure μ_s normalized over the simplex \mathcal{D}_d belongs to the class of Dirichlet distributions. We observe that the symmetric binomial distribution is the mean of the Dirichlet distribution on \mathcal{D}_d and that when d increases it converges to the mode.

Keywords: multidimensional Bernoulli distribution; Dirichlet distribution; binomial distribution; extremal points; polytope.

MSC2020 subject classifications: 60E05; 62R01; 60A10.

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1 Introduction

Sums of Bernoulli random variables model the number of occurrences of some events within d repeated trials. The case of d independent and identically distributed Bernoulli variables, the sum of which follows the binomial distribution, is often used in modeling across different areas, such as reliability (e.g. [14]) and finance (e.g. [6]). However, the binomial distribution also arises from sums of dependent Bernoulli variables in many ways ([17]), making it a possible model even when independence cannot be assumed, [16]. Actually, any discrete distribution with support on $\{0, \dots, d\}$ can be constructed as the distribution of sums of Bernoulli variables in many ways ([6]). Formally, let $\mathcal{D}_d \subset \mathbb{R}^{d+1}$ be the d -simplex of discrete probability mass functions on $\{0, \dots, d\}$ and $\mathcal{F}_d \subset \mathbb{R}^{2^d}$ be the $2^d - 1$ -simplex of d -dimensional Bernoulli probability mass functions. For any $p \in \mathcal{D}_d$, we define the class $\mathcal{P}(p)$ of probability mass functions $\mathbf{f} \in \mathcal{F}_d$ such that if $\mathbf{X} = (X_1, \dots, X_d)$ has probability mass function \mathbf{f} then $\sum_{i=1}^d X_i$ has probability mass function p .

In [2], the author proves that, as the dimension d increases, the normalized Hausdorff measure of Bernoulli sums with distribution close to the symmetric binomial distribution $Bin(1/2, d)$ converges to one. This means that the Bernoulli sums far from the binomial distributions are rare. It can be asked whether if this is related to the Hausdorff measure of $\mathcal{P}(b(1/2))$, where $b(1/2)$ is the probability mass function of $Bin(1/2, d)$ compared to

*Politecnico di Torino, Italy. E-mail: roberto.fontana@polito.it

†Politecnico di Torino, Italy. E-mail: patrizia.semeraro@polito.it

the Hausdorff measure of $\mathcal{P}(p)$ for any other $p \in \mathcal{D}_d$. Inspired by this question we characterize the class $\mathcal{P}(p)$ for any $p \in \mathcal{D}_d$. We prove that for any $p \in \mathcal{D}_d$, $\mathcal{P}(p) \subset \mathbb{R}^{2^d-d-1}$ is a convex polytope and we analytically find its extremal points. The geometrical structure of $\mathcal{P}(p)$ allows us to analytically study many statistical properties of dependent Bernoulli trials with a given distribution of their sum.

Our main result goes a step further. We prove that the Hausdorff measures of the polytopes $\mathcal{P}(p), p \in \mathcal{D}_d$ define a continuous function $l(p)$ over \mathcal{D}_d which is the density of an Hausdorff-absolutely continuous, positive, and finite measure μ_s on \mathcal{D}_d . We also prove that the normalized measure μ_s belongs to the class of Dirichlet distributions. The Dirichlet parameters are linked to the dimension of the polytopes $\mathcal{P}(p)$. This means that we have a geometrical interpretation of the Dirichlet distribution for a specific choice of its parameters.

We observe that the symmetric binomial distribution is the mean of the Dirichlet distribution on \mathcal{D}_d and that when d increases it converges to the mode. This answers our question: we have the density $l(p)$ for any p and find that the Binomial probability mass function $b(1/2)$ is close to its maximum value. For any dimension d , given $l(p)$ we can also find the Hausdorff measure \mathcal{H}^d of a neighborhood of $\mathcal{P}(b)$ in \mathcal{D}_d , and therefore we can measure the size of probability mass functions of Bernoulli variables with symmetric binomial sums even for low dimensions when the asymptotic result in [2] does not applies.

We point out that we use the Hausdorff measure that it is the proper analytical tool to measure m -dimensional objects (polytopes and simplices) embedded in \mathbb{R}^{2^d-1} , with $m < 2^d - 1$. We remark that computations remain exactly the same if, for each m -dimensional object, we use the Lebesgue measure on \mathbb{R}^m since the Hausdorff measure and the Lebesgue measure of any m -dimensional subset of \mathbb{R}^m coincide.

This paper is organized as follows. Section 2 introduces the class $\mathcal{P}(p)$, proves that it is a polytope, find its extremal points and studies its properties. Our main result is in Section 3, where we find the Hausdorff measure of $\mathcal{P}(p)$, $p \in \mathcal{D}_d$ and we prove that it defines a density on \mathcal{D}_d . Finally, Section 4 focuses on the Binomial distribution and answers our original question.

2 The convex polytope $\mathcal{P}(p)$

Let $\mathcal{X} = \{0, 1\}^d$, we make the non-restrictive hypothesis that the set \mathcal{X} of 2^d binary vectors is ordered according to the reverse-lexicographical criterion. For example for $d = 3$, $\mathcal{X} = \{000, 100, 010, 110, 001, 101, 011, 111\}$. Let $\mathcal{X}_k = \{\mathbf{x} \in \mathcal{X} : \sum_{i=1}^d x_i = k\}$ be the subset of \mathcal{X} that contains all the $\binom{d}{k}$ binary vectors with k ones and $d - k$ zeros, $k = 0, 1, \dots, d$. We observe that \mathcal{X}_k inherits the order of \mathcal{X} . Let \mathbf{x}_k^j be the j -th element of \mathcal{X}_k , $j = 1, \dots, \binom{d}{k}$. The first element is $\mathbf{x}_k := \mathbf{x}_k^1 = (\underbrace{1, \dots, 1}_k, 0, \dots, 0)$.

Let $\mathbf{X} = (X_1, \dots, X_d)$ be a d -dimensional Bernoulli random variable with probability mass function (pmf) f . We identify f with the column vector which contains the values of f over $\mathcal{X} = \{0, 1\}^d$, by $\mathbf{f} = (f_1, \dots, f_{2^d}) = (f_{\mathbf{x}} : \mathbf{x} \in \mathcal{X}) := (f(\mathbf{x}) : \mathbf{x} \in \mathcal{X})$. Let \mathcal{F}_d the $2^d - 1$ -simplex of d -dimensional pmfs \mathbf{f} of Bernoulli vectors. In this paper we identify random variables with their distributions, therefore the notation $\mathbf{X} \in \mathcal{F}_d$ means that \mathbf{X} has pmf $\mathbf{f} \in \mathcal{F}_d$. Let \mathcal{D}_d be the d -simplex of discrete pmfs on $\{0, \dots, d\}$. The notation $D \in \mathcal{D}_d$ means that D has pmf $p = (p_0, \dots, p_d) \in \mathcal{D}_d$.

Any pmf $p \in \mathcal{D}_d$ is the distribution of the sum of the components of at least one d -dimensional Bernoulli random vector $\mathbf{X} \in \mathcal{F}_d$ (see e.g. [6]). Actually, in general, behind any discrete pmf there are infinite Bernoulli vectors $\mathbf{X} \in \mathcal{F}_d$. Formally, we define the

following map between \mathcal{F}_d and \mathcal{D}_d .

$$\begin{aligned} s : \mathcal{F}_d &\rightarrow \mathcal{D}_d \\ \mathbf{f} &\rightarrow p_{\mathbf{f}}, \end{aligned} \tag{2.1}$$

where $p_{\mathbf{f}} := s(\mathbf{f})$ is the distribution of the sum $S := \sum_{i=1}^d X_i$ and $\mathbf{X} \sim \mathbf{f}$, i.e. \mathbf{X} has pmf $\mathbf{f} \in \mathcal{F}_d$. For any $p \in \mathcal{D}_d$, we define

$$\mathcal{P}(p) = \{\mathbf{f} \in \mathcal{F}_d : p_{\mathbf{f}} = p\}. \tag{2.2}$$

The next Theorem 2.1 proves that for any choice of p , the class $\mathcal{P}(p)$ is a convex polytope and provides an analytical expression for its extremal pmfs.

Given p , we define the n -dimensional simplex

$$\{\mathbf{x} \in \mathbb{R}^{n+1} : x_j \geq 0, j = 0, \dots, n, \sum_{h=0}^n x_h = p\}, \tag{2.3}$$

whose vertices are $e_0 = (p, 0, \dots, 0), e_1 = (0, p, 0, \dots, 0), \dots, e_n = (0, \dots, 0, p)$. The length $s_{i,j}$ of the edge between e_i and e_j , with $i, j = 0, \dots, n, i \neq j$ of the simplex is defined as $s_{i,j} = \sqrt{\sum_{k=1}^n (e_{ik} - e_{jk})^2}$. Since all lengths are equal, $s_{i,j} = \sqrt{2}p$, their common value $s = \sqrt{2}p$ is called the side of the simplex. We denote the n -dimensional simplex with side $\sqrt{2}p$ by $\Delta_{n, \sqrt{2}p}$.

Theorem 2.1. For any $p \in \mathcal{D}_d$ the class $\mathcal{P}(p) = \{\mathbf{f} \in \mathcal{F}_d : p_{\mathbf{f}} = p\}$ is the convex polytope $\mathcal{P}(p) = \prod_{k=0}^d \Delta_{n_k, \sqrt{2}p_k}$, where $n_k = \binom{d}{k} - 1$ and its extremal points are

$$f^{\sigma}(\mathbf{x}) = \begin{cases} p_k & \text{if } \mathbf{x} = \mathbf{x}_k^{\sigma_k} \\ 0 & \text{otherwise,} \end{cases} \tag{2.4}$$

where $\sigma = (\sigma_0, \dots, \sigma_k, \dots, \sigma_d), \sigma_k = 1, \dots, \binom{d}{k}, k = 0, \dots, d$.

Proof. Let $p \in \mathcal{D}_d$ and let $\mathbf{X} \sim \mathbf{f} \in \mathcal{F}_d$. We have $p_{\mathbf{f}} = p$ if and only if \mathbf{f} is a positive solution of the linear system:

$$\sum_{\mathbf{x} \in \mathcal{X}_k} f(\mathbf{x}) = p_k, \quad k = 0, \dots, d. \tag{2.5}$$

Each equation of the system (2.5) defines a $\binom{d}{k} - 1$ -simplex with side $\sqrt{2}p_k$. It is well known - see [10] - that the $\binom{d}{k}$ extremal points of the simplex are $(p_k, 0, \dots, 0), (0, p_k, \dots, 0), \dots, (0, \dots, 0, p_k)$. Since $\mathcal{X}_k \cap \mathcal{X}_j = \emptyset$, for any $k \neq j$, the extremal solutions of the system are the ones in (2.4). \square

Corollary 2.2. The number n_p of extremal points of $\mathcal{P}(p)$ is $n_p = \prod_{k \in \text{Supp}(p)} \binom{d}{k}$, where $\text{Supp}(p) \subseteq \{0, \dots, d\}$ is the support of p .

Proof. The proof follows from Theorem 2.1 since $\#\mathcal{X}_k = \binom{d}{k}$. \square

From Theorem 2.1 and Corollary 2.2 it follows that for any $\mathbf{f} \in \mathcal{P}(p)$ there exist $\lambda_i \geq 0$ summing up to one such that

$$\mathbf{f} = \sum_{i=1}^{n_p} \lambda_i \mathbf{r}_i,$$

where $\mathbf{r}_i \in \mathcal{P}(p), i = 1, \dots, n_p$ are the extremal points in Equation (2.4). We call \mathbf{r}_i extremal points or extremal pmfs of $\mathcal{P}(p)$. We denote with \mathbf{R}_i a d -dimensional random variable with distribution \mathbf{r}_i .

Notice that n_p depends only on the support and not on the values of p . If $\{1, \dots, d-1\} \subseteq \text{Supp}(p)$, since $\binom{d}{0} = \binom{d}{d} = 1$ we have $n_p = \prod_{k=0}^d \binom{d}{k}$.

Example 2.3. As an illustrative example we consider the polytope $\mathcal{P}(p)$ in dimension $d = 3$ for a given $p = (p_0, p_1, p_2, p_3) \in \mathcal{D}_3$ with full support. The extremal points of $\mathcal{P}(p)$ are $n_p = \binom{3}{1} \binom{3}{2} = 9$ and they are reported in Table 1.

Table 1: Extremal pmfs of $\mathcal{P}(p)$, case $d = 3$

\mathbf{x}_1	\mathbf{x}_2	\mathbf{x}_3	\mathbf{r}_1	\mathbf{r}_2	\mathbf{r}_3	\mathbf{r}_4	\mathbf{r}_5	\mathbf{r}_6	\mathbf{r}_7	\mathbf{r}_8	\mathbf{r}_9
0	0	0	p_0	p_0	p_0	p_0	p_0	p_0	p_0	p_0	p_0
1	0	0	p_1	p_1	p_1	0	0	0	0	0	0
0	1	0	0	0	0	p_1	p_1	p_1	0	0	0
1	1	0	p_2	0	0	p_2	0	0	0	0	p_2
0	0	1	0	0	0	0	0	0	p_1	p_1	p_1
1	0	1	0	p_2	0	0	p_2	0	p_2	0	0
0	1	1	0	0	p_2	0	0	p_2	0	p_2	0
1	1	1	p_3	p_3	p_3	p_3	p_3	p_3	p_3	p_3	p_3

Remark 2.4. Theorem 2.1 can be easily generalized to any surjective map $h : \mathcal{X} \rightarrow \{0, \dots, d\}$. Let $p_f^h \in \mathcal{D}_d$ the pmf associated to $h(\mathbf{X})$ with $\mathbf{X} \sim \mathbf{f} \in \mathcal{F}_d$. For any $p \in \mathcal{D}_d$ the class $\mathcal{P}^h(p) = \{\mathbf{f} \in \mathcal{F}_d : p_f^h = p\}$ is a convex polytope and its extremal points are

$$f^\sigma(\mathbf{x}) = \begin{cases} p_y & \text{if } \mathbf{x} = \mathbf{x}_y^{\sigma_y}, \\ 0 & \text{otherwise,} \end{cases}$$

where for any $y \in \{0, \dots, d\}$, $\mathbf{x}_y^{\sigma_y}$ is the σ_y -th element of $h^{-1}(y)$. If $h(\mathbf{x}) = \sum_{i=1}^d x_i$ we have Theorem 2.1.

2.1 Moments and entropy

In [7], the authors prove that the bounds of the moments of a pmf in a convex polytope are sharp and reached on the extremal points, that in this case are explicitly known.

Proposition 2.5. Let \mathbf{X} with pmf $\mathbf{f} \in \mathcal{P}(p)$, then for any $\{j_1, \dots, j_k\} \subseteq \{1, \dots, d\}$,

$$p_d \leq E[X_{j_1} \cdots X_{j_k}] \leq \sum_{h=k}^d p_h,$$

and the bounds are sharp.

Proof. Since $X_{j_1} \cdots X_{j_k} = 1$ is contained in the event $\sum_{i=1}^d X_i \geq k$. It follows

$$E[X_{j_1} \cdots X_{j_k}] = \sum_{\{x \in \mathcal{X} : x_{j_1} \cdots x_{j_k} = 1\}} \mathbf{f}_x \leq \sum_{\{x \in \mathcal{X} : \sum_{i=1}^d x_i \geq k\}} \mathbf{f}_x = \sum_{h=k}^d p_h$$

Similarly, the event $X_1 \cdots X_d = 1$ is contained in the event $X_{j_1} \cdots X_{j_k} = 1$. It follows

$$p_d = f_{\mathbf{x}_d} \leq \sum_{\{x \in \mathcal{X} : x_{j_1} \cdots x_{j_k} = 1\}} \mathbf{f}_x = E[X_{j_1} \cdots X_{j_k}]$$

The bounds are sharp because we can consider the extremal pmf $\tilde{r}(\mathbf{x})$ defined as

$$\tilde{r}(\mathbf{x}) = \begin{cases} p_k & \text{if } \mathbf{x} = \mathbf{x}_k, \\ 0 & \text{otherwise,} \end{cases}$$

and the associated random variable \tilde{R} with pmf \tilde{r} . We have $E[\tilde{R}_1 \cdots \tilde{R}_k] = \tilde{r}(\mathbf{x}_k) + r(\mathbf{x}_{k+1}) + \dots + \tilde{r}(\mathbf{x}_{d-1}) + \tilde{r}(\mathbf{x}_d) = \sum_{h=k}^d p_h$ and $E[\tilde{R}_{d-k+1} \cdots \tilde{R}_d] = \tilde{r}(\mathbf{x}_d) = p_d$. \square

We now consider the definition of the Shannon entropy for a discrete random variable W pmf $p_w := P[W = w]$ as $H(W) = -\sum_{w \in \mathcal{W}} p_w \log p_w$ with the convention $0 \log(0) = 0$. It is easy to verify that, given a random variable $W \sim p \in \mathcal{D}_d$, the extremal random variables $\mathbf{R}_1, \dots, \mathbf{R}_{n_p}$ of the polytope $\mathcal{P}(p)$ have all the same entropy, which is equal to the entropy of W , $H(\mathbf{R}_i) = H(W), i = 1, \dots, n_p$. Moreover, we can identify the multivariate Bernoulli variables whose distributions lie within the polytope $\mathcal{P}(p)$, and which have the maximum and minimum entropy.

Proposition 2.6. *Given pmf $p \in \mathcal{D}_d$, let \mathbf{X}_M be a multivariate Bernoulli random variable with the exchangeable pmf $\mathbf{f}_M \in \mathcal{P}(p)$, whose expression is:*

$$\mathbf{f}_M(\mathbf{x}) = \begin{cases} \frac{p_k}{\binom{d}{k}} & \text{if } \mathbf{x} \in \mathcal{X}_k, k = 0, \dots, d \\ 0 & \text{otherwise,} \end{cases} \tag{2.6}$$

Then the following hold:

1. $\mathbf{X}_M = \operatorname{argmax}_{\mathbf{X} \in \mathcal{P}(p)} H(\mathbf{X})$
2. $\mathbf{R}_i = \operatorname{argmin}_{\mathbf{X} \in \mathcal{P}(p)} H(\mathbf{X}), i = 1, \dots, n_p,$

where $\mathbf{R}_i \sim r_i$ and r_i are the extremal pmfs of $\mathcal{P}(p)$, $i = 1, \dots, n_p$.

Proof. Both statements can be proved by noting that for any $\mathbf{X} \in \mathcal{F}_d$ we can express its entropy as: $H(\mathbf{X}) = -\sum_{\mathbf{x} \in \mathcal{X}} \mathbf{f}(\mathbf{x}) \log(\mathbf{f}(\mathbf{x})) = -\sum_{k=0}^d \sum_{\mathbf{x} \in \mathcal{X}_k} \mathbf{f}(\mathbf{x}) \log(\mathbf{f}(\mathbf{x}))$.

To maximize (minimize) $H(\mathbf{X})$, it is sufficient to maximize (minimize) each term $-\sum_{\mathbf{x} \in \mathcal{X}_k} \mathbf{f}(\mathbf{x}) \log(\mathbf{f}(\mathbf{x}))$, which represents the entropy restricted to $\mathcal{X}_k, k = 0, \dots, d$. It is well known that entropy is maximized by choosing a uniform distribution see, e.g., [11], and then we get \mathbf{f}_M in (2.6). The entropy is minimized by choosing a Dirac delta distribution centered at any point in $\mathcal{X}_k, k = 0, \dots, d$. \square

3 The induced measure on \mathcal{D}_d

The next Theorem 3.2 proves our main result that Bernoulli sums induce a Dirichlet distribution on the simplex \mathcal{D}_d . We need some preliminaries. Since $\mathcal{F}_d \subset \mathbb{R}^{2^d}$ is the standard 2^{d-1} -simplex, $\mathcal{D}_d \subset \mathbb{R}^d$ is the standard d -simplex and $\mathcal{P}(p) \subset \mathbb{R}^{2^d}$ is a $(2^d - d - 1)$ -convex polytope we consider the Hausdorff measure \mathcal{H}^n for any $n \in \{0, \dots, 2^{d-1}\}$. We recall that $\mathcal{H}^0(x) = 1$, for any $x \in \mathbb{R}^m, m \in \mathbb{N}$ (a standard reference for Hausdorff measures is [4]). The Corollary 3.1 finds the Hausdorff measure of $\mathcal{P}(p)$ in $\mathbb{R}^{2^d - d - 1}$ for any pmf $p \in \mathcal{D}_d$. It is well known (see, e.g., [10]) that the Hausdorff measure of the n -simplex with side $\sqrt{2}p, \Delta_{n, \sqrt{2}p} \subset \mathbb{R}^{n+1}$ is

$$\mathcal{H}^n(\Delta_{n, \sqrt{2}p}) = \frac{(\sqrt{2}p)^n \sqrt{n+1}}{n! \sqrt{2^n}} = \frac{p^n \sqrt{n+1}}{n!}. \tag{3.1}$$

We have the following corollary of Theorem 2.1.

Corollary 3.1. *For any $p = (p_0, \dots, p_d) \in \mathcal{D}_d$, it holds*

$$\mathcal{H}^{2^d - d - 1}(\mathcal{P}(p)) = \prod_{k=0}^d \mathcal{H}^{n_k}(\Delta_{n_k, \sqrt{2}p_k}),$$

where $n_k = \binom{d}{k} - 1$.

We can now prove our main result.

Theorem 3.2. Let μ_s the measure on $(\mathcal{D}_d, \mathcal{B}(\mathcal{D}_d))$, where $\mathcal{B}(\mathcal{D}_d)$ is the Borel σ -algebra on \mathcal{D}_d , induced from the function s in Equation (2.1). It holds

$$\mu_s(A) = \mathcal{H}^{2^d-1}(s^{-1}(A)) = \int_A \prod_{k=0}^d \frac{p_k^{n_k}}{n_k!} d\mathcal{H}^d(p), \quad A \in \mathcal{B}(\mathcal{D}_d), \quad (3.2)$$

where $n_k = \binom{d}{k} - 1$. Then μ_s is a positive finite measure on \mathcal{D}_d such that $\mu_s(\mathcal{D}_d) = \mathcal{H}^{2^d-1}(\mathcal{F}_d)$. The measure μ_s is absolutely continuous with respect the Hausdorff measure on \mathcal{D}_d .

Proof. We start proving the following

$$\mathcal{H}^{2^d-1}(\mathcal{F}_d) = \int_{\mathcal{D}_d} \prod_{k=0}^d \frac{1}{\sqrt{n_k+1}} \mathcal{H}^{n_k}(\Delta_{n_k, \sqrt{2}p_k}) d\mathcal{H}^d(p), \quad (3.3)$$

We explicitly build an isometry α^{ort} on \mathcal{F}_d by orthonormalizing the following transformation α

$$\begin{aligned} \alpha : \mathcal{F}_d &\rightarrow \mathcal{F}_d \\ \mathbf{f} &\rightarrow (p_k^j), \end{aligned} \quad (3.4)$$

where $k = 0, \dots, d, j = 1, \dots, \binom{d}{k}$ and

$$p_k^j = \begin{cases} \sum_{\mathbf{x} \in \mathcal{X}_k} f(\mathbf{x}), & j = 1, \\ f(\mathbf{x}_k^j), & j \neq 1. \end{cases}$$

Since the resulting isometry does not modify p_k^1 , we write $p_k = p_k^1, k = 0, \dots, d$.

Let $I_{2^d} = (i_{\mathbf{x},j})_{\mathbf{x} \in \mathcal{X}, j \in \{1, \dots, 2^d\}}$ be the 2^d - identity matrix and let $\mathbf{i}_{\mathbf{x}}$ be the row vector $\mathbf{i}_{\mathbf{x}} = (i_{\mathbf{x},j}, j = 1, \dots, 2^d)$. Let \mathbf{x}_k be the first element in the reverse lexicographic order of \mathcal{X}_k , and $\mathbf{a}_{\mathbf{x}_k} := \frac{1}{\sqrt{n_k+1}}(\mathbf{1}_{\mathcal{X}_k}(\mathbf{x}), \mathbf{x} \in \mathcal{X}), k = 1, \dots, d$, where $\mathbf{1}_B()$ is the indicator function of B .

Let $A := I_{2^d}(\mathbf{i}_{\mathbf{x}_k} \rightarrow \mathbf{a}_{\mathbf{x}_k})$ be the matrix obtained from I_{2^d} by replacing the row $\mathbf{i}_{\mathbf{x}_k}$ with the row $\mathbf{a}_{\mathbf{x}_k}, k = 1, \dots, d$. Let A^{ort} be the matrix obtained from A by the Gram-Schmidt orthonormalization process applied to the rows of A , considered from the first to the last. It holds $\mathbf{a}_{\mathbf{x}_k}^{ort} = \mathbf{a}_{\mathbf{x}_k}$. In fact, since \mathbf{x}_k is the first element in the reverse lexicographic order of \mathcal{X}_k , all the preceding rows $\mathbf{a}_{\mathbf{x}_h}$ of $A, \mathbf{x}_h < \mathbf{x}_k$, refer to \mathcal{X}_h with $\mathcal{X}_h \cap \mathcal{X}_k = \emptyset$. It follows that $\langle \mathbf{a}_{\mathbf{x}_k}, \mathbf{a}_{\mathbf{x}_h} \rangle = 0$, for any row $\mathbf{a}_{\mathbf{x}_h}$ with $\mathbf{x}_h < \mathbf{x}_k$. We also observe that if \mathbf{x}_k^2 is the second element of \mathcal{X}_k we still have $\langle \mathbf{a}_{\mathbf{x}_k^2}, \mathbf{a}_{\mathbf{x}_h} \rangle = 0$, for any row $\mathbf{a}_{\mathbf{x}_h}$ with $\mathbf{x}_h < \mathbf{x}_k$ and $\mathbf{x}_h \notin \mathcal{X}_k$. On the other hand, the product $\langle \mathbf{a}_{\mathbf{x}_k^2}, \mathbf{a}_{\mathbf{x}_k} \rangle$ is different from zero. For our purposes, it is not necessary to make explicit the result of the orthonormalization, $\mathbf{a}_{\mathbf{x}_k}^{ort}$ but it is enough to notice that the orthonormalization process of $\mathbf{a}_{\mathbf{x}_k^2}$ will produce a vector $\mathbf{a}_{\mathbf{x}_k^2}^{ort}$ with zeros in all the positions $\mathbf{x}_j \notin \mathcal{X}_k$. A similar argument holds for the subsequent rows $\mathbf{a}_{\mathbf{x}_k^j}$ of $A, j = 3, \dots, n_k + 1$. We show the matrices \mathcal{X}, A , and A^{ort} for $d = 2$ and $d = 3$. in Remark 3.3. Since A^{ort} is an orthonormal matrix the application

$$\begin{aligned} \alpha^{ort} : \mathcal{F}_d &\rightarrow \mathcal{F}_d \\ \mathbf{f} &\rightarrow \mathbf{f}^{ort} = A^{ort} \mathbf{f}, \end{aligned}$$

is an isometry. Then it holds

$$\begin{aligned} \mathcal{H}^{2^d-1}(\mathcal{F}_d) &= \int_{\mathcal{F}_d} d\mathcal{H}^{2^d-1}(\mathbf{f}) = \int_{\Delta_{2^d, \sqrt{2}}} d\mathcal{H}^{2^d-1}(\mathbf{f}) = \\ &= \int_{\Delta_{d, \sqrt{2}}} \prod_{k=0}^d \frac{1}{\sqrt{n_k+1}} \int_{\Delta_{n_k, \sqrt{2}p_k}} d\mathcal{H}^{n_k}(\mathbf{f}_{\mathcal{X}_k}^{ort}) d\mathcal{H}^d(p) \\ &= \int_{\mathcal{D}_d} \prod_{k=0}^d \frac{1}{\sqrt{n_k+1}} \mathcal{H}^{n_k}(\Delta_{n_k, \sqrt{2}p_k}) d\mathcal{H}^d(p), \end{aligned} \tag{3.5}$$

where $\mathbf{f}_{\mathcal{X}_k}^{ort} = (A^{ort} \mathbf{f}, \mathbf{f} \in \mathcal{X}_k)$, that is (3.3). Plugging Equation (3.1) in (3.5) we have

$$\mathcal{H}^{2^d-1}(\mathcal{F}_d) = \int_{\mathcal{D}_d} \prod_{k=0}^d \frac{p_k^{n_k}}{n_k!} d\mathcal{H}^d(p).$$

Thus, for any $A \in \mathcal{B}(\mathcal{D}_d)$,

$$\mu_s(A) = \mathcal{H}^{2^d-1}(s^{-1}(A)) = \int_{s^{-1}(A)} d\mathcal{H}^{2^d-1}(\mathbf{f}) = \int_A \prod_{k=0}^d \frac{p_k^{n_k}}{n_k!} d\mathcal{H}^d(p),$$

Let $l : \mathcal{D}_d \rightarrow \mathbb{R}^+$ defined by $l(p) = \prod_{k=0}^d \frac{p_k^{n_k}}{n_k!}$. The function is almost surely continuous on \mathcal{D}_d . Therefore, the measure μ_s on \mathcal{D}_d defined by Equation (2.1) is a positive, finite and Hausdorff absolutely continuous measure on \mathcal{D}_d . By construction $\mu_s(\mathcal{D}_d) = \mathcal{H}^{2^d-1}(\mathcal{F}_d)$. \square

Remark 3.3. For illustration purposes, we show the matrices \mathcal{X} , A , and A^{ort} introduced in the proof of Theorem 3.2 for $d = 2$ and $d = 3$. For $d = 2$, we have

$$\mathcal{X} = \begin{pmatrix} 00 \\ 10 \\ 01 \\ 11 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A^{ort} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

and for $d = 3$, we have

$$\mathcal{X} = \begin{pmatrix} 000 \\ 100 \\ 010 \\ 110 \\ 001 \\ 101 \\ 011 \\ 111 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

$$A^{ort} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark 3.4. We observe that Theorem 3.2 can be generalized to any surjective map h as discussed in Remark 2.4.

The function $l : \mathcal{D}_d \rightarrow \mathbb{R}^+$ defined by

$$l(p) = \prod_{k=0}^d \frac{p_k^{n_k}}{n_k!}$$

is the density of μ_s with respect the Hausdorff measure \mathcal{H}^d on \mathcal{D}_d . The following Corollary 3.5 provides a useful formula for practical computations.

Corollary 3.5. *It holds*

$$\mathcal{H}^{2^d-1}(\mathcal{F}_d) = \sqrt{2^d} \int_{\Sigma_d} \prod_{k=0}^{d-1} \frac{p_k^{n_k}}{n_k!} dp_0 \dots dp_{d-1}, \tag{3.6}$$

where $\Sigma_d = \{\mathbf{x} \in \mathbb{R}^d : x_j \geq 0, j = 0, \dots, d-1, \sum_{k=0}^{d-1} x_k \leq 1\}$, and $n_k = \binom{d}{k} - 1$.

Remark 3.6. Theorem 3.2 provides a link between the Hausdorff measure of the polytopes $\mathcal{P}(p)$ and the Hausdorff measure of \mathcal{F}_d . It holds $\mathcal{F}_d = \cup_{p \in \mathcal{D}_d} \mathcal{P}(p)$, and

$$\mathcal{H}^{2^d-1}(\mathcal{F}_d) = N \int_{\mathcal{D}_d} \mathcal{H}^{2^d-d-1}(\mathcal{P}(p)) d\mathcal{H}^d(p),$$

where $N = \prod_{k=0}^d \frac{1}{\sqrt{n_k+1}}$.

The following proposition states that the Hausdorff measure of $\mathcal{P}(p)$, $p \in \mathcal{D}_d$, induces the Dirichlet distribution on the simplex of discrete distributions \mathcal{D}_d (see [13] for an overview on the Dirichlet distribution). Let $Dirichlet(\alpha_0, \dots, \alpha_d)$ be the Dirichlet distributions with parameters $\alpha_0, \dots, \alpha_d$.

Proposition 3.7. *The density $l(p)$ normalized over the simplex \mathcal{D}_d is the Dirichlet density with parameters $\alpha_k = \binom{d}{k}$, $k = 0, \dots, d$, on the d -simplex \mathcal{D}_d .*

Proof. It is sufficient to observe $l(p) \sim \prod_{k=0}^d p_k^{n_k}$ and $n_k = \binom{d}{k} - 1$ with $n_0 = n_d = 0$. \square

Proposition 3.7 gives a geometrical interpretation for the parameters of this Dirichlet distribution as the Hausdorff dimensions of the simplexes corresponding to each p_j , $j = 0, \dots, d$.

Remark 3.8. Notice that $\mathcal{H}^{2^d-1}(\mathcal{F}_d) = \frac{\sqrt{2^d}}{(2^{d-1})!}$ is the normalizing constant for $l(p)$ to be the Dirichlet density.

The size of the class of multivariate Bernoulli distributions the sums of which have pmf close to a given $p \in \mathcal{D}_d$ depends on the behavior of $l(p)$ in a neighborhood of p . The next Corollary 3.9 explicitly provides the pmf $p^M \in \mathcal{D}_d$ that maximizes the Hausdorff measure $\mathcal{H}^{2^d-d-1}(\mathcal{P}(p))$.

Corollary 3.9. *Let $p^M = (p_0^M, \dots, p_d^M) \in \mathcal{D}$ be such that*

$$p_k^M = \frac{\binom{d}{k} - 1}{2^d - d - 1}, \quad k = 0, \dots, d$$

then $p^M = \operatorname{argmax}_{p \in \mathcal{D}_d} \mathcal{H}^{2^d-d-1}(\mathcal{P}(p))$.

Proof. The proof follows directly from Proposition 3.7 observing that p^M is the mode of the Dirichlet distribution. \square

We name p^M the maximal pmf in \mathcal{D}_d .

3.1 Measuring a neighborhood of $\mathcal{P}(p)$

This section finds the Hausdorff measure of the Bernoulli sums whose pmf is close to a given $p \in \mathcal{D}_d$ according to a given metrics d . In practice, using the measure μ_s we measure a neighborhood of a pmf $p \in \mathcal{D}_d$. This means finding the Hausdorff measure in \mathcal{F}_d of the set of multivariate Bernoulli distributions f such that p_f is close to p .

Formally, let d be a distance on \mathcal{D}_d and define a neighborhood of p in \mathcal{D}_d by

$$I_d(p, \epsilon) = \{\tilde{p} \in \mathcal{D}_d : d(\tilde{p}, p) \leq \epsilon\}, \quad \epsilon > 0,$$

and a corresponding neighborhood of $\mathcal{P}(p)$ in \mathcal{F}_d as its counter-image through the map s is (2.1)

$$I_d^{\mathcal{F}}(p, \epsilon) = s^{-1}(I_d(p, \epsilon)) = \{\tilde{f} \in \mathcal{F} : d(\tilde{p}, p) \leq \epsilon\}, \quad \epsilon > 0,$$

where $\tilde{p} = p_{\tilde{f}}$. Using Equation 3.2, the Hausdorff measure in \mathcal{F}_d of $I_d^{\mathcal{F}}(p, \epsilon)$ is the measure μ_s of $I_d(p, \epsilon)$, and this can be found by integration of $l(p)$ over $I_d(p, \epsilon)$. Following [2] we consider the maximum distance on \mathcal{D}_d and show as to estimate $I_S(p, \epsilon)$. Given two probability measures P and Q the maximum distance d_S defined by

$$d_S(\tilde{p}, p) := \max_{0 \leq k \leq d} |\tilde{p}_k - p_k|.$$

By definition of $I_S(p, \epsilon)$ it holds

$$I_S(p, \epsilon) = \{x \in \mathbb{R}^{d+1} : \sum_{i=0}^d x_i = 1, \max\{p_j - \epsilon, 0\} \leq x_j \leq \min\{p_j + \epsilon, 1\}, j = 0, \dots, d\},$$

therefore $I_S(p, \epsilon) \subseteq \mathcal{D}_d$ is a convex polytope. From Corollary 3.5 it follows that

$$\mathcal{H}^{2^d-1}(I_S^{\mathcal{F}}(p, \epsilon)) = \mu_s(I_S(p, \epsilon)) = \sqrt{2^d} \int_{\Sigma_S(p, \epsilon)} \prod_{j=0}^d \frac{p_j^{n_j}}{n_j!} dp_0 \dots dp_{d-1}, \quad (3.7)$$

where

$$\Sigma_S(p, \epsilon) = \{x \in \mathbb{R}^d : \sum_{i=0}^{d-1} x_i \leq 1, \max\{p_j - \epsilon, 0\} \leq x_j \leq \min\{p_j + \epsilon, 1\}, j = 0, \dots, d-1\}.$$

To compute $\mu_s(I_S(p, \epsilon))$ we can find an estimate $\hat{\mu}_s(I_S(p, \epsilon))$ of $\mu_s(I_S(p, \epsilon))$ by using (3.7)

$$\hat{\mu}_s^d(I_S(p, \epsilon)) = \hat{\mathcal{H}}^{2^d-1}(I_S^{\mathcal{F}}(p, \epsilon)),$$

where $\hat{\mathcal{H}}^{2^d-1}(I_S^{\mathcal{F}}(p, \epsilon))$ is an estimate of $\mathcal{H}^{2^d-1}(I_S^{\mathcal{F}}(p, \epsilon))$ computed as

$$\hat{\mathcal{H}}^{2^d-1}(I_S^{\mathcal{F}}(p, \epsilon)) = \hat{E}_U[\prod_{j=0}^d \mathcal{H}^{n_j}(\Delta_{p_j})] \mathcal{H}^d(I_S(p, \epsilon)) = \frac{\sum_{j=1}^N \prod_{j=0}^d \mathcal{H}^{n_j}(\Delta_{\hat{p}_j})}{N} \hat{\mathcal{H}}^d(I_S(p, \epsilon)),$$

where $\hat{p}_j, j = 1, \dots, N$ are uniformly extracted from $I_S(p, \epsilon)$, the expectation E_U is relative to a uniform distribution on the simplex, and $\hat{\mathcal{H}}(I_S(p, \epsilon))$ is computed using package `volesti` [5] which uses a random-walk-based method to provide uniform samples from a given convex polytope.

4 The binomial distribution

This section focuses on the Bernoulli structure behind the discrete distribution corresponding to the most important independence model: the binomial distribution.

Let $b(\theta) \in \mathcal{D}_d$ be the pmf of the binomial distribution with parameters θ and d ($B(\theta, d)$) and let $\mathcal{P}(b(\theta)) = \{f \in \mathcal{F} : p_f = b(\theta)\}$. From Theorem 2.1 its extremal points are

$$f_B^\sigma(\mathbf{x}) = \begin{cases} \binom{d}{k} \theta^k (1-\theta)^{d-k} & \text{if } \mathbf{x} = \mathbf{x}_k^{\sigma_k}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma = (\sigma_0, \dots, \sigma_k, \dots, \sigma_d)$, $\sigma_k = 1, \dots, \binom{d}{k}$, $k = 0, \dots, d$. Since the binomial distributions have full support on $\{0, \dots, d\}$ from Corollary 2.2 the number of extremal points is $n_b := n_p = \prod_{k=0}^d \binom{d}{k}$.

The class of binomial distributions describes a parametrical curve on the simplex \mathcal{D}_d , given by $b(\theta) = (b_0(\theta), \dots, b_d(\theta))$, $\theta \in [0, 1]$. The following Proposition proves that the density l restricted to the binomial class is a concave function in the parameter space $[0, 1]$ and maximal for $\theta = 1/2$, i.e. $l(\theta) = \mathcal{H}^{2^d-d-1}(\mathcal{P}(b(\theta)))$ is maximal for $\theta = 1/2$.

Proposition 4.1. *The map*

$$\begin{aligned} l : [0, 1] &\rightarrow \mathbb{R}_+ \\ \theta &\rightarrow \mathcal{H}^{2^d-d-1}(\mathcal{P}(b(\theta))), \end{aligned} \tag{4.1}$$

is a concave function in θ and

$$\operatorname{argmax}_{\theta \in [0,1]} \mathcal{H}^{2^d-d-1}(\mathcal{P}(b(\theta))) = \frac{1}{2}.$$

Proof. We have

$$\mathcal{H}^{2^d-d-1}(\mathcal{P}(b(\theta))) = \prod_{k=0}^d \frac{\binom{d}{k} \theta^k (1-\theta)^{d-k} n_k \sqrt{n_k + 1}}{n_k!},$$

thus

$$\begin{aligned} \log(\mathcal{H}^{2^d-d-1}(\mathcal{P}(b(\theta)))) &= \log \prod_{k=0}^d \frac{\binom{d}{k} \theta^k (1-\theta)^{d-k} n_k \sqrt{n_k + 1}}{n_k!} \\ &= \log \prod_{k=0}^d \frac{\binom{d}{k} n_k \sqrt{n_k + 1}}{n_k!} + \log \prod_{k=0}^d (\theta^k (1-\theta)^{d-k})^{n_k}. \end{aligned}$$

It is sufficient to find the maximum of $f(\theta) = \log \prod_{k=0}^d (\theta^k (1-\theta)^{d-k})^{n_k}$. Straightforward computations lead to

$$f'(\theta) = \sum_{k=0}^d \frac{n_k}{\theta(1-\theta)} (k - d\theta) = \sum_{k=0}^{d\theta^-} \frac{n_k}{\theta(1-\theta)} (k - d\theta) + \sum_{k=d\theta^+}^d \frac{n_k}{\theta(1-\theta)} (k - d\theta),$$

where $d\theta^-$ is the largest integer smaller than $d\theta$ and $d\theta^+$ is the smallest integer bigger than $d\theta$. $f'(\theta) = 0$ iff

$$\sum_{k=0}^{d\theta^-} \frac{n_k}{\theta(1-\theta)} (k - d\theta) = \sum_{k=d\theta^+}^d \frac{n_k}{\theta(1-\theta)} (k - d\theta)$$

and, since $n_k = n_{d-k}$ this is true iff $\theta = 1/2$. If $\theta > 1/2$ we have $f'(\theta) < 0$ and if $\theta < 1/2$ we have $f'(\theta) > 0$, and the maximum is reached on $\theta = 1/2$. \square

Remark 4.2. Notice that the symmetric binomial distribution is the mean of the Dirichlet distribution $Dirichlet(n_0 + 1, \dots, n_d + 1)$ on the simplex \mathcal{D}_d .

The following proposition proves that if the dimension d increases, the pmf $b(1/2)$ converges to the distribution p^M .

Proposition 4.3. *Let $b(1/2)$ be the pmf of the $B(1/2, d)$ and p^M is the maximal polytope pmf in \mathcal{F}_d . We have*

$$\lim_{d \rightarrow \infty} d_S(b(1/2), p^M) = 0, \quad \text{and} \quad \lim_{d \rightarrow \infty} \mathcal{H}^{2^d - d - 1} \mathcal{P}(b(1/2)) = l^M,$$

where $l^M = \mathcal{H}^{2^d - d - 1}(\mathcal{P}(p^M))$.

Proof. We have

$$|b(1/2)_k - p_k^M| = \frac{|2^d \binom{d}{k} - 1| - (2^d - d - 1) \binom{d}{k}}{2^d(2^d - d - 1)} = \frac{|(d+1) \binom{d}{k} - 2^d|}{2^d(2^d - d - 1)}.$$

Since $\binom{d}{k}$ is maximal for $k = \frac{d-1}{2}$ and $k = \frac{d+1}{2}$ if d is odd, we have

$$\max_k |b(1/2)_k - p_k^M| = \frac{|(d+1) \frac{d!}{\frac{d-1}{2}! \frac{d+1}{2}!} - 2^d|}{2^d(2^d - d - 1)},$$

that converges to 0 as d goes to ∞ . Similarly $\max_k |b(1/2)_k - p_k^M|$ converges to 0 as d goes to ∞ if d is even, since $\binom{d}{k}$ is maximal for $k = \frac{d}{2}$. Since $\lim_{d \rightarrow \infty} d_S(b(1/2), p^M) = 0$ implies $\lim_{d \rightarrow \infty} |b(1/2)_k - p_k^M| = 0$ for any $k \in \{0, \dots, d\}$. We also have $\lim_{d \rightarrow \infty} d_E(b(1/2), p^M) = 0$, where E is the usual Euclidean norm and therefore $l(b(1/2)) = \mathcal{H}^{2^d - d - 1}(\mathcal{P}(b(1/2)))$ converges to the maximum $l^M = l(p^M)$ of the density $l(p)$. \square

Since the Bernoulli distribution is close to the maximal pmf p^M the Hausdorff measure of $\mathcal{P}(b(1/2))$ is close to the maximal one both in low and high dimension. As a consequence we expect that for a given ϵ , $\mathcal{H}^{2^d - 1}(I_S^{\mathcal{F}}(b(1/2), \epsilon))$ converges to the maximal one. The following Theorem proved in [2] provides asymptotic lower bounds for the size in \mathcal{F}_d of $I_S^{\mathcal{F}}(b(1/2), \epsilon)$ and shows that its normalized Hausdorff measure goes to one when d increases.

Theorem 4.4. [2] *There exists a constant A such that for all positive integers d and all positive numbers ϵ ,*

$$\mu(I_S^{\mathcal{F}}(b(1/2), \epsilon)) \geq 1 - \frac{A\sqrt{d}}{\epsilon^2 2^{d-1}}, \quad \text{and} \quad \mu(I_{TV}^{\mathcal{F}}(b(1/2), \epsilon)) \geq 1 - \frac{Ad^{5/2}}{\epsilon^2 2^{d-1}},$$

where μ is the normalized Hausdorff measure on the probability simplex \mathcal{F}_d .

In [8] (Remark 2, Section 5) it is shown that even for moderate d , the lower bound of $\mu(I_S^{\mathcal{F}}(b(1/2), \epsilon))$ is close to one, meaning that the distributions of sums of Bernoulli random variables that are not close to the binomial $b(1/2)$ pmf are rare. Using Equations 3.6 and 3.1 we can find the Hausdorff measure of $I_S^{\mathcal{F}}(b(1/2), \epsilon)$ even for small d , where the asymptotic result can not be applied. The binomial class of pmfs $b(\theta)$ with $\theta \neq 1/2$ is not close to $b(1/2)$ even in high dimension. Notice that from Proposition 4.1 it follows that the closer θ is to $1/2$ the higher the size of the corresponding polytope $\mathcal{P}(b(\theta))$ is. We mention another important discrete distribution, the Poisson-binomial distribution. It is the law of the sum of independent and not identically distributed Bernoulli variables (see e.g. [15] and [1] for an example of its use in applications). The Poisson-binomial distribution with parameter θ , $PB(\theta)$, with $\theta = (\theta_1, \dots, \theta_d)$, is usually far from $b(1/2)$, as for example if $\sum_{i=1}^d \frac{\theta_i}{d} \neq 1/2$, where θ_i are the means of the independent Bernoulli variables. Even if $\sum_{i=1}^d \frac{\theta_i}{d} = 1/2$, [3] proved that in general $b(\theta)$ is not close to $b(1/2)$, see Remark 3) in [8].

The Shepp-Olkin entropy monotonicity conjecture proved in [12] asserts that if \mathbf{X} has independent components X_i with means θ_i , $i = 1, \dots, d$, the entropy $H(\theta)$ of their sum $S = \sum_{j=1}^d X_j$, that is a function of the parameters $\theta = (\theta_1, \dots, \theta_d)$, is non-decreasing in θ if all $\theta_j \leq 1/2$. In [9] the author proves that the binomial distribution $b(\frac{\mu}{d}) \in \mathcal{D}_d$, $\mu = \sum_{i=1}^d \theta_i$ is the maximal entropy distribution in the class of Poisson-binomial distributions $PB(\theta)$. Our Proposition 4.1 proves that the case $\theta_i = 1/2$, i.e. the symmetric binomial distribution, corresponds to the Polytope $\mathcal{P}(b(\theta))$ with maximal Hausdorff measure in the class of binomial distributions. Here, we prove that the symmetrical binomial distribution is the distribution of the sum S of the d -dimensional Bernoulli variable $\mathbf{X} = (X_1, \dots, X_d) \in \mathcal{F}_d$ with maximal entropy.

Proposition 4.5. *The multivariate Bernoulli random variable $U = (U_1, \dots, U_d) \sim \mathbf{f}_U$, where*

$$\mathbf{f}_U(\mathbf{x}) = \begin{cases} \frac{1}{2^d} & \text{if } \mathbf{x} \in \mathcal{X}, \\ 0 & \text{otherwise,} \end{cases}$$

achieves the maximum entropy within \mathcal{F}_d . The sum of its components follows a symmetric binomial distribution.

Proof. It is well known that the uniform random variable over \mathcal{X} has the highest entropy among the class \mathcal{F}_d . \square

To further investigate the parallelism with the Shepp-Olkin conjecture, studying the concavity of $l(\theta) = \mathcal{H}^{2^d-d-1}(\mathcal{P}(\theta))$ as a function of the parameters θ is on the agenda of our future research since it is an interesting and nontrivial issue. Indeed, even in dimension two, $l(\theta)$ is not concave for any $\theta \in [0, 1]^2$, e.g. for $\theta_1 = 0.1$ and $\theta_2 = 0.4$ the determinant of the Hessian matrix of $l(\theta)$ is -0.0270 , which is negative.

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