A Thermodynamically Consistent Thermal Equilibrium Gaussian White Noise Model for Nonlinear Resistors

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***Abstract*—Traditional extensions of the Nyquist-Johnson formula for thermal fluctuations in nonlinear dissipative elements have often led to thermodynamically inconsistent models and sparked long-standing debates about the proper interpretation of stochastic differential equations. In this work, we show that it is possible to derive, for each of the main stochastic interpretations, a Gaussian white-noise model for nonlinear dissipative elements at thermal equilibrium that fully complies with the fundamental principles of thermodynamics. The resulting models reproduce the Gibbs (Maxwell-Boltzmann) distribution and ensure zero-mean voltages and currents, thereby resolving the Brillouin paradox and maintaining consistency with the second law of thermodynamics. Furthermore, we demonstrate that these models satisfy additional thermodynamic requirements, including positive entropy production during transients and zero net heat exchange between dissipative elements at equilibrium.**

***Index Terms*—Gaussian white noise, thermal noise, nonlinear resistor, nonlinear noise modelling.**

I. Introduction

THE theoretical foundation of thermal noise in electronic systems dates back to the early 20th century. In the second of his *annus mirabilis* papers, Einstein described the Brownian motion of suspended particles as a diffusion process driven by random thermal agitation [1]. Shortly thereafter, in 1908, Langevin introduced a dynamic formulation that extended Newton's second law by incorporating a stochastic force term [2]. The resulting Langevin equation provided a time-domain description of Brownian motion that accounted for both the dissipative effects of viscous drag, representing a *drift* force, and a rapidly fluctuating random force representing the *diffusion* due to the thermal fluctuations. This stochastic differential equation not only reproduced Einstein's diffusive behavior in the long-time limit, but also enabled the analysis of system dynamics across a broad range of temporal and spectral scales.

These seminal contributions laid the foundation for the work of Johnson and Nyquist in the late 1920s, who studied the electrical analogue of Brownian motion in passive linear electrical networks. Johnson experimentally observed thermally induced voltage fluctuations across a resistor at thermal equilibrium [3], while Nyquist derived a theoretical explanation using principles from statistical mechanics [4]. Modeling the resistor as a dissipative element coupled to a thermal reservoir, Nyquist established a quantitative relationship between thermal fluctuations and electrical resistance, a result that remains fundamental in the modern theory of electronic noise. The so-called Johnson-Nyquist relation between the variance of the thermal fluctuations and the conductance of a dissipative element at equilibrium is the electrical equivalent of Einstein's diffusion law [1]. It may also be regarded as a special case of the famous *fluctuation-dissipation theoreom* [5].

Generalizing the Johnson-Nyquist theory to nonlinear resistive elements has proved challenging. Direct application of the classical theory to nonlinear components results in thermodynamic inconsistencies, most notably violations of energy conservation. The most well-known example is Brillouin's paradox, in which a nonlinear dissipative element, such as a diode, appears to rectify its own thermal fluctuations, generating a nonzero net average voltage or current in a thermal reservoir – effectively allowing the extraction of unlimited energy [6], [7]. This paradox triggered extensive discussions, forming part of the longstanding debate on the correct interpretation of stochastic differential equations within thermodynamic contexts [8], [9].

Over time, a prevailing view has emerged that thermodynamically consistent models incorporating Gaussian white noise cannot, in general, be formulated for nonlinear resistors [10], [11], [12]. In particular, in [13], the authors propose a set of criteria to evaluate the thermodynamic consistency of stochastic models. Their analysis leads to a somewhat surprising conclusion: standard Gaussian white noise models for nonlinear devices yield circuit behaviors that violate the laws of thermodynamics and should therefore be rejected.

However, this conclusion is based on two key implicit assumptions. The first is that internal noise generation is independent of the external load connected to the nonlin-

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ear resistor. This assumption is not shared by all authors. In fact, in [14] and [15] it is observed that the stochastic behavior of a dissipative system is inherently influenced by the thermodynamic constraints under which it operates – including the characteristics of the connected load. From this perspective, internal noise generation cannot be considered a purely intrinsic property of the component, but must instead be treated as context-dependent, reflecting the interplay between the system and its boundary conditions.

The second assumption is intrinsic to the Langevin approach: a stochastic description is obtained by starting from the macroscopic deterministic law of the noiseless system and simply adding a random fluctuating "force". For external noise – where the source is independent of the system – this procedure is often justified. However, in the case of internal noise, where fluctuations arise from interactions within the system itself, a consistent formulation requires a stochastic model that explicitly accounts for the physical origin of the noise. In such situations, there is no a priori reason to assume that the drift term coincides with the deterministic macroscopic dynamics in the absence of noise [8]. A rigorous approach would instead begin from a microscopic description that incorporates the discrete nature of charges and stochastic transitions (or random jumps), leading to a Master Equation [16]. Yet, as network complexity increases, deriving the Master Equation from first principles becomes intractable analytically and unaffordable numerically in terms of CPU time. In practice, one typically has access only to macroscopic differential equations, namely the network equations obtained from Kirchhoff's laws supplemented by the characteristic relationships of the individual components.

In a recently published work [17], the authors determine all the statistical moments (e.g., mean, variance, skewness, etc.) of the white noise physically generated within an arbitrary, potentially strongly nonlinear, dissipative device. Building on first principles of stochastic thermodynamics – the appropriate theoretical framework for describing inherently stochastic, nonlinear mesoscopic systems – they show that nonlinear resistors may indeed comply with an extended Johnson-Nyquist law, provided the noise remains white and Gaussian.

In this paper, we present an extension of the Johnson-Nyquist theory to nonlinear, passive resistors in thermal equilibrium with an external bath. Our derivation employs stochastic calculus and is based on the key assumption that noise generation can be modulated by the load connected across the resistor. For each of the main interpretations of stochastic differential equations (Itô and Stratonovich), we derive a Gaussian white-noise model for nonlinear dissipative elements and demonstrate that these models fully comply with the fundamental principles of thermodynamics. In particular, they resolve Brillouin's paradox and ensure positive entropy production during transients. Furthermore, we show that in the limiting case of a linear resistor, the model reduces to the classical Johnson-Nyquist result.

The proposed model is suitable for implementation within numerical algorithms for the simulation of nonlinear stochastic differential equations. Furthermore, an equivalent circuit representation of thermal noise in nonlinear resistors is presented, enabling straightforward integration into standard industrial circuit simulators.

The remainder of this paper is organized as follows. Section II reviews basic concepts and outlines the thermodynamic requirements that any model of thermal noise in nonlinear resistors must satisfy, and also presents a concise derivation of the Johnson-Nyquist model for linear resistors. Section III demonstrates that a state-independent Gaussian white noise model is thermodynamically inconsistent, as it predicts infinite energy extraction from a circuit at thermal equilibrium, effectively acting as a Maxwell's demon in violation of the second law of thermodynamics. The main contribution is given in Section IV, where Gaussian white noise models for nonlinear resistors are developed under both the Itô and Stratonovich interpretations. An alternative formulation, along with equivalent circuits suitable for implementation in numerical algorithms and circuit simulators, is also introduced. Section V shows that the proposed model satisfies the essential thermodynamic requirements, ensuring consistency with fundamental thermodynamic principles. Finally, Section VI is devoted to conclusions.

II. Preliminaries

A. Fundamental Definitions

The literature on noise modeling in nonlinear circuits and systems is often compromised by the misuse of terminology, particularly through the inappropriate transfer of concepts and definitions between thermodynamics and nonlinear dynamics. Here we recall some basic definitions that will be extensively used throughout the paper.

Definition 1 (Stationary process): A stochastic process $\boldsymbol{X}_t$ is *stationary* if $\boldsymbol{X}_t$ and $\boldsymbol{X}_{t+\tau}$ have the same statistical properties for all time shifts τ.

Definition 2 (Steady state): In the context of stochastic processes, a system is at steady state if its probability density function (PDF) is time-invariant, even though the individual state of the system might still fluctuate due to randomness. A steady state is reached asymptotically, implying that as time approaches infinity, the distribution of the system's states converges to the fixed, time-invariant stationary distribution:

$$\lim_{t\to+\infty} \rho(\boldsymbol{x}, t) = \rho_{\mathrm{st}}(\boldsymbol{x}). \tag{1}$$

Definition 3 (Thermal equilibrium): A system in contact with a thermal bath is said to be at *thermal equilibrium* when there is no net transfer of heat energy between them. Since thermal energy naturally flows from a hotter body to a colder one, thermal equilibrium implies that the system and the bath are at the same temperature. Consequently, the system's temperature remains constant at thermal equilibrium.

Definition 4 (Thermodynamic equilibrium): A system is said to be at thermodynamic equilibrium if it does not spontaneously change its state after it has been isolated [18]. Thermodynamic equilibrium is achieved when a system is simultaneously in thermal, mechanical, and chemical equilibrium, meaning there are no temperature differences, unbalanced forces, or ongoing chemical reactions, including phase transitions. Consequently, system's properties, such as temperature, pressure, and composition, remain constant over time, and no net exchange of energy or matter occurs with its surroundings.

Note that a system can be in a steady state that is not an equilibrium state. Moreover, certain physical quantities may exhibit stationary behavior while others do not. For example, a Brownian particle in a homogeneous gravitational field has a vertical velocity that is a stationary process, whereas its position is not [19].

B. Fundamental Requirements for a Thermodynamically Consistent Model

Any realistic model used for analysis and design must comply with the fundamental laws of physics, particularly the laws of thermodynamics. Models that violate these principles yield inaccurate predictions and exhibit unphysical behavior, and should therefore be rejected. In their foundational work, Wyatt and Coram [13] introduced three key requirements that serve as criteria for assessing whether a given model is consistent with the fundamental laws of thermodynamics. A fourth requirement was introduced in [20].

Requirement No. 1: No Isothermal Conversion of Heat Into Work [13], [20].: The second law of thermodynamics prohibits the conversion of heat into work as the sole effect in an isothermal system.

Consequently, a noisy dissipative device maintained at a constant temperature T cannot, on average, deliver net power to an external circuit. Specifically, the expected short-circuit current produced by such a device must be zero. A system that can continuously generate work from heat without an external energy input constitutes a *perpetual motion machine of the first kind*, violating the principle of energy conservation. Similarly, a system that converts heat entirely into work while interacting with only a single heat reservoir would represent a *perpetual motion machine of the second kind*, in violation of the second law of thermodynamics.

Requirement No. 2: Gibbs (Maxwell-Boltzmann) Distribution at Equilibrium [13], [20].: For a circuit composed by lossless (capacitors and inductors) and dissipative elements held at a constant temperature, the stationary distribution for inductor fluxes $\boldsymbol{\varphi}$ and capacitor charges $\boldsymbol{q}$ must have the Gibbs form

$$\rho_{\mathrm{G}}(\boldsymbol{\varphi}, \boldsymbol{q}) = A \exp\left(-\frac{1}{k\,T} E(\boldsymbol{\varphi}, \boldsymbol{q})\right) \tag{2}$$

where $E(\boldsymbol{\varphi}, \boldsymbol{q})$ is the total energy stored in the inductors and capacitors, and A is a normalization constant ensuring that the stationary distribution integrates to one over all possible states.

Requirement No. 3: Positive Entropy Production During Transient [13], [20].: The second law of thermodynamics must be satisfied during nonequilibrium transient behavior driven by the fluctuations of the dissipative elements. The total entropy of a circuit must be a nondecreasing function of time, with a maximum value corresponding to the equilibrium distribution.

Complete elimination of losses during conversion of heat into work would imply constant entropy, representing a *perpetual motion machine of the third kind.*

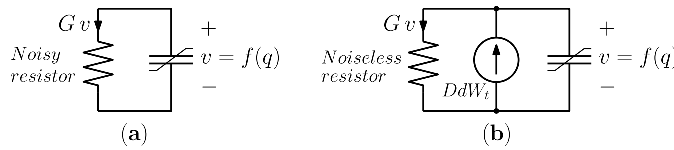

Fig. 1. **(a)** First-order circuit with noisy linear resistor. **(b)** Norton equivalent circuit for the linear resistor, with a parallel-current white Gaussian noise source added.

Requirement No. 4: No Heat Transfer Between Two Devices at Thermal Equilibrium [20].: In any circuit composed of two or more noisy devices, each in thermal contact with a reservoir at the same temperature T, and connected through a lossless lumped network, there should be no net heat transfer between the devices – that is, no device should deliver or absorb net power. However, if the devices are in contact with reservoirs at different temperatures, heat will flow from the hotter to the cooler one. The rate of this heat flow will depend on the characteristics of both the devices and the lossless network.

C. Gaussian White Model for Thermal Noise in Linear Resistors

The theory of thermal noise in linear resistors is well established. A thermodynamically consistent framework exists for general electrical circuits containing an arbitrary number of linear resistors, along with linear or nonlinear capacitors and inductors [21]. To ensure the self-consistency of this paper, we begin by analyzing a simple first-order circuit consisting of a linear resistor and a nonlinear capacitor. This example illustrates the modeling methods and techniques – largely based on those introduced in [13] – that we will later apply to develop a thermodynamically consistent model for nonlinear resistors.

Consider the first-order circuit shown in Fig. 1(a), consisting of a noisy linear resistor connected with a nonlinear, charge-controlled capacitor with characteristic $v = f(q)$. Let $p(t) = v(t)\,i(t)$ be the instantaneous absorbed power (assuming current and voltage according to the passive sign convention), then the energy stored in the capacitor is:

$$F(q) = \int f(q)\frac{dq}{dt}\,dt = \int f(q)\,dq. \tag{3}$$

Notice that, because only energy variations can be measured, the energy is always defined up to an arbitrary constant level. This corresponds to the integration constant in (3), which is set to zero for simplicity.

In addition to the standard technical conditions – ensuring the well-posedness of the problem, the existence and uniqueness of solutions for the differential equations, and the smoothness of the functions to justify the calculations we perform – we introduce the following assumption.

Assumption 1: To ensure the capacitor is physically well-defined, we assume that:

$$\lim_{q\to\pm\infty} F(q) = +\infty.$$

This assumption is physically justified, as it implies that storing an infinite amount of charge on the capacitor would

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require an infinite amount of energy. Consequently, under this condition, the Gibbs distribution tends to zero as $q \to \pm\infty$.

The state equation describing the circuit in Fig. 1(a) is obtained applying the Kirchhoff current law (KCL) and using each element's constitutive equation. Following Langevin approach, a random "force" (in this case a random current) is added to the equation. Adopting standard notation for stochastic differential equation (SDE), the state equation is:

$$dq = -G\, f(q)dt + D\, dW_t, \tag{4}$$

where G is the resistor's conductance, and D is the noise intensity. The random current is modelled as the Gaussian white noise dW_t, the "formal" derivative of a Wiener process, characterized by $\mathbb{E}[dW_t] = 0$ and $\mathbb{E}[dW_t dW_s] = \delta(t-s)$ ($\mathbb{E}[\cdot]$ denotes the expectation operator). The first term on the right hand side is called the *drift*, whereas the second term is called the *diffusion*. Fig. 1(b) shows the equivalent circuit corresponding to the SDE (4), where the noisy resistor is replaced by its Norton equivalent: a noiseless resistor in parallel with the random current source.

SDEs can be interpreted following different rules, depending on the mathematical definition adopted for stochastic integration. The two most widely used interpretations are the Itô and Stratonovich formulations. Both interpretations are mathematically rigorous and internally consistent, each with its own advantages and limitations [19], [22], [23]. The Itô interpretation is generally preferred because it favors the implementation of numerical integration schemes, and it simplifies the computation of expected values [22]. The main drawback is the need for a new set of calculus rules, known as Itô calculus. In contrast, the Stratonovich interpretation is often considered more closely aligned with the physics of natural systems, offering better physical intuition. The key advantage is the preservation of the standard calculus rules, including of course the chain rule. As a consequence, the common perception is that the Itô formulation is favored by mathematicians, while the Stratonovich approach is more widely used in physics [8].

Recently, a third interpretation, commonly referred to as the anti-Itô or Hänggi-Klimontovich interpretation, has gained increasing attention, particularly within the statistical physics community [24], [25].[1] The anti-Itô interpretation exhibits certain advantages, which will be addressed in future work.

For SDEs with constant diffusion (also known as unmodulated or additive noise), such as (4), the solution remains the same regardless of the chosen interpretation. However, in the presence of state-dependent diffusion (modulated or multiplicative noise), the solution depends on the chosen interpretation. Irrespective of the interpretation adopted, the Fokker-Planck equation (FPE) associated to the SDE (4) is:

$$\frac{\partial \rho(t,q)}{\partial t} = \frac{\partial J(t,q)}{\partial q} = \frac{\partial}{\partial q}\left[G\, f(q)\rho(t,q) + \frac{D^2}{2}\frac{\partial \rho(t,q)}{\partial q}\right] \tag{5}$$

where $\rho(t,q)$ is the probability density function and $J(t,q)$ is the probability current.

[1]In [24], the author introduced the term "backward integral." This terminology is somewhat unfortunate, as it may cause confusion with the "backward equation," which conventionally denotes the evolution equation of a probability distribution in reverse time.

At thermal equilibrium and steady state, the probability flux $J(t,q)$ is expected to be null. Accordingly, we solve the stationary FPE:

$$G\, f(q)\rho_{\rm st}(q) + \frac{D^2}{2}\frac{d\rho_{\rm st}(q)}{dq} = 0 \tag{6}$$

Using separation of variables and integrating, we obtain the stationary PDF:

$$\rho_{\rm st}(q) = A\,\exp\left(-\frac{2G}{D^2}F(q)\right) \tag{7}$$

where A is a normalization constant determined imposing that the integral of the PDF over all the possible states equals one:

$$A = \left[\int_{-\infty}^{+\infty}\exp\left(-\frac{2G}{D^2}F(q)\right)dq\right]^{-1} \tag{8}$$

According to the thermodynamic Requirement No. 2, the stationary PDF (7) must coincide with the Gibbs distribution (2) at thermal equilibrium. Comparing the two expressions for the stationary PDFs, we find the well known Johnson-Nyquist formula

$$D = \sqrt{2\,k\,T\,G} \tag{9}$$

Notice that, in general, even if the stationary PDF (7) is the Gibbs distribution, it is not Gaussian, *unless the capacitor is linear.*

Remark 1: Only the positive determination for D is considered, since the positive and negative values are equivalent. In fact, if dW_t is a Gaussian white noise, then so is $-dW_t$.

Remark 2: It is easy to see that at thermal equilibrium, the average voltage across both the capacitor and the linear resistor is zero, as a consequence of the Gibbs distribution. In fact

$$\begin{aligned}\mathbb{E}[f(q)] &= \int_{-\infty}^{+\infty} f(q)\,A\,\exp\left(-\frac{2G}{D^2}F(q)\right)dq \\ &= -k\,T\,\rho_{\rm G}(q)\Big|_{-\infty}^{+\infty} = 0\end{aligned} \tag{10}$$

as a consequence of *Assumption 1*.

D. Example 1

As an example we consider a linear resistor connected to a linear capacitor. The SDE (4) reduces to the linear SDE

$$dq = -\frac{G}{C}\,q\,dt + \sqrt{2k\,T\,G}\,dW_t, \tag{11}$$

whose solution is the Ornstein-Uhlenbeck process [22], [26]. The formal solution with deterministic initial condition $q(0) = q_0$ is:

$$\begin{aligned} q(t) &= q_0\,\exp\left(-\frac{G}{C}\,t\right) \\ &\quad + \sqrt{2k\,T\,G}\int_0^t \exp\left[-\frac{G}{C}(t-s)\right]dW_s. \end{aligned} \tag{12}$$

Fig. 2 illustrates an example of the output voltage $v(t) = q(t)/C$, corresponding to a specific realization of the Wiener process and assuming zero initial condition. The numerical solution was obtained by integrating the SDE (11) using a

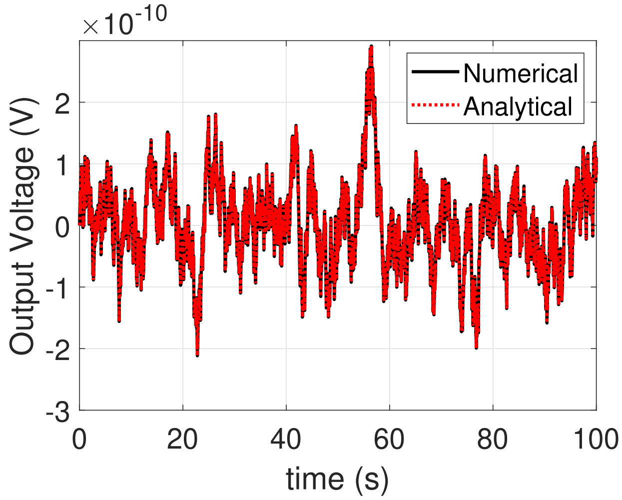

Fig. 2. An example of Ornstein-Uhlenbeck process: numerical and analytical solutions are depicted. T = 300 K.

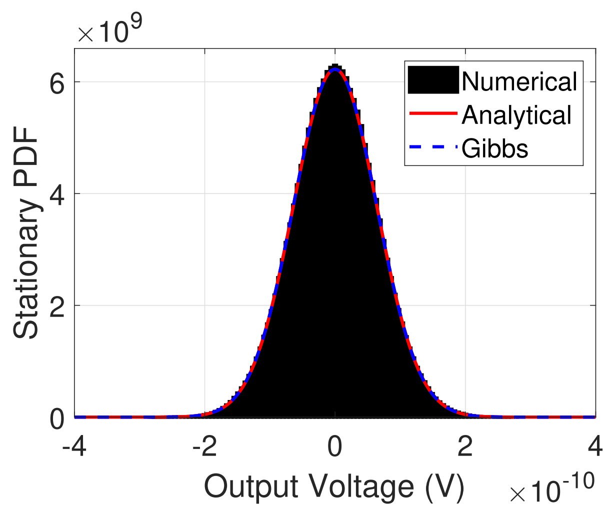

Fig. 3. Stationary PDF for the Ornstein-Uhlenbeck process, numerical and Gibbs PDFs. The analytical solution is found from (7)-(8) to coincide with the Gibbs distribution (13). T = 300 K.

stochastic Runge-Kutta method with strong order 1 convergence. A time integration step $dt = 1\,\mu$s was used, and the simulation was run for a total duration of $\Delta T = 100$ s. The choice of the time integration step is a trade-off between accuracy and computational cost: smaller steps improve trajectory resolution, while longer simulations are required to reach the stationary probability density and compute expectations. We verified that using dt in the range 10^{-7} to 10^{-5} produces negligible differences. For simplicity we assumed $G = 1$ S, and $C = 1$ F, while the temperature was set to $T = 300$ K. The numerical results are compared with the analytical solution given in (12), showing excellent agreement between the two.

Fig. 3 shows the numerical stationary PDF compared to the theoretical Gibbs distribution

$$\rho_{\mathrm{G}}(v) = \sqrt{\frac{C}{2\pi\, k\, T}} \exp\left(-\frac{1}{2k\,T}\, C\, v^2\right) \tag{13}$$

which is also the analytical solution found from (7)-(8). The numerical stationary PDF was extracted from a long-time simulation with a duration $\Delta T = 10^3$ s. The probability of observing a voltage value within the interval $[v(t), v(t) + dv(t)]$ was computed by counting the number of samples falling within that interval and dividing by the total number of samples. As expected, the empirical PDF converges towards the theoretical Gibbs distribution.

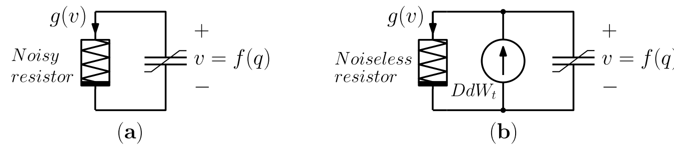

Fig. 4. **(a)** First-order circuit with noisy nonlinear resistor. **(b)** Norton equivalent circuit for the nonlinear resistor with a state independent random source.

III. Nonlinear, State-Independent Gaussian Model and the Brillouin's Paradox

As established in [13], the formulation of a state-independent white Gaussian noise model for nonlinear resistors inherently leads to thermodynamic inconsistencies. The following example illustrates the fundamental limitations of this modeling approach.

Consider the first-order circuit containing a voltage-controlled noisy nonlinear resistor with current-voltage characteristic $i = g(v)$, as depicted in Fig. 4(a), and the corresponding Gaussian white noise model in Fig. 4(b).

The state equation for the circuit now reads

$$dq = -g(f(q))dt + D\, dW_t. \tag{14}$$

Repeating the procedure of Sec. II-C, we obtain the stationary PDF:

$$\rho_{\mathrm{st}}(q) = A\, \exp\left(-\frac{2}{D^2} G(q)\right) \tag{15}$$

where

$$G(q) = \int g(f(q))\, dq. \tag{16}$$

Notice that the arbitrary integration constant in the $G(q)$ definition can be absorbed into the normalization constant A. Comparing (15) with the Gibbs distribution, we identify

$$D^2 = 2\, k\, T \frac{G(q)}{F(q)}. \tag{17}$$

Since the left-hand side is constant, the equality holds *if and only if* $G(q) \propto F(q)$, e.g. for linear resistors.

Thus, in the case of a nonlinear resistor, modeling thermal noise as a state-independent white Gaussian process leads to a stationary PDF that is different from the Gibbs distribution, violating thermodynamic Requirement No. 2. Furthermore, thermodynamic Requirement No. 1 is also violated. In fact, the expected voltage across the elements is

$$\mathbb{E}[f(q)] = \int_{-\infty}^{+\infty} f(q)\, A\, \exp\left(-\frac{2G(q)}{D^2}\right) dq \tag{18}$$

which, for general choices of $f(q)$ and $G(q)$, is not zero.

Thus, even at thermal equilibrium, the nonlinear resistor rectifies its own fluctuations, resulting in a non-zero average voltage and causing the capacitor to accumulate a finite internal charge. In principle, one could build an idealized device that periodically disconnects the capacitor from the nonlinear resistor and connects it instead to a linear resistor. In this setup, the energy stored in the capacitor would be released and dissipated by the linear resistor. Repeating this cycle indefinitely would enable continuous extraction of energy from a single thermal reservoir.[2] This paradoxical outcome is essentially what is known as Brillouin's paradox [6], [19].

Brillouin's paradox is the fundamental reason why developing a thermodynamically consistent, state-independent Gaussian white noise model (e.g., the Johnson-Nyquist model) is impossible for nonlinear resistors [13].

A. Example 2

To illustrate Brillouin's paradox, we consider an example inspired by [19]. We examine a system consisting of a linear capacitor, connected to a nonlinear resistor with current-charge characteristic $i = g(f(q)) = Z[\exp(\gamma q/C) - 1]$ A. The noise intensity is $D = \sqrt{2\,k\,T\,G_0}$, where $G_0 = dg(f(q))/df(q)|_{f(q)=0}$ is the differential conductance. It is worth noticing that the resistor is strictly passive. i.e. $g(f(q))/f(q) > 0$, for all $q \neq 0$, and $g(f(0)) = 0$ A.

The SDE (14) then becomes:

$$dq = -Z\left[\exp\left(\frac{\gamma}{C}q\right) - 1\right] dt + \sqrt{\frac{2k\,T\,Z\gamma}{C}}\,dW_t, \qquad (19)$$

while, using (15), the corresponding stationary PDF is:

$$\rho_{\rm st}(q) = A\,\exp\left[-\frac{Z}{k\,T\,G_0}\left(\frac{C}{\gamma}\exp\left(\frac{\gamma}{C}\,q\right) - q\right)\right]. \qquad (20)$$

The SDE (19) was integrated numerically using the same method and parameters described for the previous example. The parameters are set to $C = 1$ F, $Z = 1$ A and $\gamma = 5$ V^{-1}. As in the previous example, the temperature is set to $T = 300$ K. Figure 5 presents a comparison between the stationary PDF obtained from numerical simulations and the analytical expression in (20). The Gibbs distribution is also included for reference.

The numerical and analytical PDFs are clearly in excellent agreement. Furthermore, as expected, the stationary PDF does not coincide with the Gibbs distribution. In particular, the stationary PDF exhibits a pronounced negative skewness, implying that $\mathbb{E}[v] \neq 0$. As a result, the non-linear resistor rectifies its own thermal fluctuations, a clear manifestation of Brillouin's paradox.

IV. White Gaussian Noise Model for Nonlinear Resistors at Thermal Equilibrium

The conclusion that a thermodynamically consistent Gaussian white noise model for nonlinear resistors is impossible relies on a fundamental assumption: that the noise model is independent of the circuit to which the resistor is connected. For the two-terminal, voltage-controlled resistive elements introduced in Section III, this means that the current noise cannot depend on the capacitance value, or on the total charge

[2] Such an idealized experiment is one of the many manifestations of a Maxwell's demon.

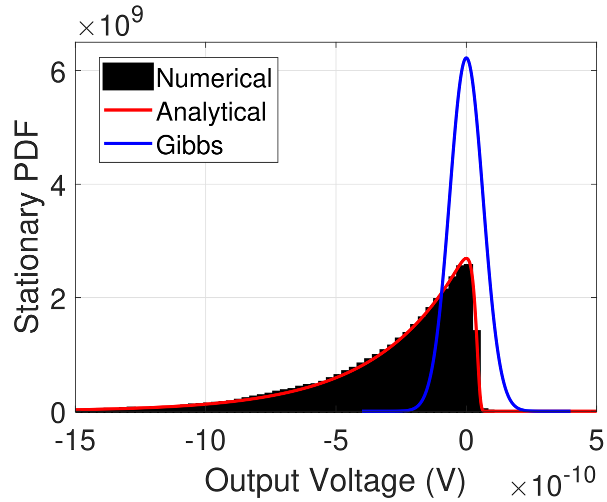

Fig. 5. Stationary PDF for the first order circuit with a nonlinear resistor and a constant noise intensity. The analytical curve corresponds to (20), while the Gibbs distribution is given by (2). $T = 300$ K.

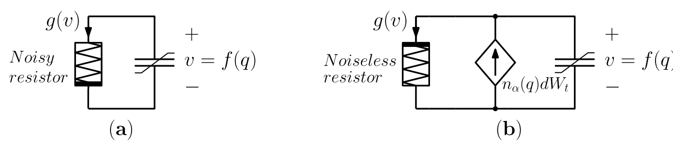

Fig. 6. **(a)** First-order circuit with noisy nonlinear resistor. **(b)** Norton equivalent circuit for the nonlinear resistor with a state dependent random source. For both circuits, $v = f(q)$ is the constitutive relation of the nonlinear capacitor.

accumulated on the capacitor plates. This assumption is a central point in [13], but it is not universally accepted. In particular, some authors have argued that noise generation in a dissipative system depends on the thermodynamic constraints under which the circuit operates, and that the load constitutes a form of constraint on the system parameters [14]. As a general consequence, a more precise statement is: a load-independent Gaussian white description of thermal noise cannot, in general, be developed for nonlinear systems [15].

In this section, we demonstrate that a thermodynamically consistent Gaussian white noise model for nonlinear resistors at thermal equilibrium can indeed be derived by relaxing the assumption of constant noise intensity. Specifically, we allow the noise intensity to be modulated by the state variable of the nonlinear resistor. From a physical standpoint, this assumption is analogous to determining the operating point of a nonlinear element, which cannot be defined without explicitly accounting for the boundary conditions under which the element functions.

In practice, we replace the constant diffusion coefficient D in (14) with a state-dependent diffusion function $n_\alpha(q)$, whose explicit form will be determined later. Consequently, the noisy nonlinear resistor in Fig. 6(a) is replaced by the Norton equivalent circuit shown in Fig. 6(b), composed by a noiseless nonlinear resistor parallel connected to a controlled random current source with amplitude $n_\alpha(q)dW_t$. Our goal is to determine the appropriate functional form of $n_\alpha(q)$ such that the resulting stochastic model satisfies thermodynamic Requirements no. 1-4.

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Using KCL, the state equation for the first-order circuit in Fig. 6(b) is

$$dq = -g(f(q))\,dt + n_\alpha(q)\,dW_t \tag{21}$$

Because noise in SDE (21) is multiplicative, the solution depends on the interpretation adopted. In what follows we shall derive the proper form for the noise modulating function for the two main interpretations, namely Itô, and Stratonovich.

Theorem 1 (Derivation of the Gaussian white noise model): Consider a first order circuit composed by a voltage controlled noisy nonlinear resistor with the voltage-current characteristic $i = g(v)$, connected to a reservoir represented by a nonlinear capacitor characterized by the voltage-charge relationship $v = f(q)$, the state equation being:

- The Itô SDE

$$dq = -g(f(q))\,dt + n_0(q)\,dW_t \tag{22}$$

with

$$n_0(q) = \left[-2\exp\left(\frac{F(q)}{kT}\right) \times \int g(f(q))\,\exp\left(-\frac{F(q)}{kT}\right) dq\right]^{1/2} \tag{23}$$

- The Stratonovich SDE

$$dq = -g(f(q))\,dt + n_{\frac{1}{2}}(q) \circ dW_t \tag{24}$$

with

$$n_{\frac{1}{2}}(q) = \left[-4\exp\left(\frac{2F(q)}{kT}\right) \times \int g(f(q))\,\exp\left(-\frac{2F(q)}{kT}\right) dq\right]^{1/2} \tag{25}$$

Then the circuit is described by the Gibbs distribution at steady-state:

$$\rho_G(q) = A\,\exp\left(-\frac{1}{kT}F(q)\right) \tag{26}$$

Proof: The FPE associated to the SDEs (22) and (24) is:

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial q}\left[\left(g(f(q)) - \alpha\, n'_\alpha(q) n_\alpha(q)\right)\rho + \frac{1}{2}\frac{\partial}{\partial q}\left(n_\alpha^2(q)\rho\right)\right]. \tag{27}$$

where $\alpha = 0$ corresponds to the Itô SDE (22), and $\alpha = 1/2$ to the Stratonovich SDE (24). At steady state a detailed balance condition is reached, which implies the zero probability flux condition. Consequently, we consider the stationary FPE:

$$\left(g(f(q)) - \frac{\alpha}{2}D'_\alpha(q)\right)\rho_{st}(q) + \frac{1}{2}\frac{d}{dq}(D_\alpha(q)\rho_{st}(q)) = 0, \tag{28}$$

where $D_\alpha(q) = n_\alpha^2(q)$ and $\rho_{st}(q)$ denotes the steady-state PDF. Expanding the derivative in (28) gives

$$g(f(q))\rho_{st}(q) + \frac{1-\alpha}{2}D'_\alpha(q)\rho_{st}(q) + \frac{D_\alpha(q)}{2}\frac{d\rho_{st}(q)}{dq} = 0. \tag{29}$$

At thermodynamic equilibrium, the steady-state PDF $\rho_{st}(q)$ must coincide with the Gibbs distribution

$$\rho_{st}(q) = \rho_G(q) = A\exp\left(-\frac{1}{kT}F(q)\right). \tag{30}$$

Substituting into (29), and taking into account that (29) must hold for all q we obtain the linear, first order ordinary differential equation (ODE):

$$\frac{1-\alpha}{2}D'_\alpha(q) - \frac{f(q)}{2kT}D_\alpha(q) + g(f(q)) = 0. \tag{31}$$

After integration, we find

$$D_\alpha(q) = \frac{2}{\alpha-1}\exp\left(\frac{F(q)}{(1-\alpha)kT}\right) \times \int g(f(q))\,\exp\left(-\frac{F(q)}{(1-\alpha)kT}\right) dq. \tag{32}$$

For the Itô interpretation, we substitute $\alpha = 0$ into (32), obtaining (23). Conversely, the Stratonovich interpretation corresponds to $\alpha = 1/2$, which gives (25). ■

Remark 3: The same consideration made in Remark 1 applies here. Only the positive determination for $n_0(q)$, and $n_{\frac{1}{2}}(q)$ is considered, because the negative one results into exactly the same model.

Remark 4: A Gaussian white noise model, valid under the Hänggi-Klimontovich (HK) interpretation, was previously derived in [20] and [27]. In the HK interpretation, the SDE describing the circuit is given by:

$$dq = -g(f(q))\,dt + n_1(q) \bullet dW_t \tag{33}$$

Substituting $\alpha = 1$ into (31), the ODE reduces to an algebraic equation, from which it follows that

$$n_1(q) = \left(2\,kT\,\frac{g(f(q))}{f(q)}\right)^{1/2} \tag{34}$$

which coincides with the results reported in [20] and [27].

The solutions of the SDEs (22), (24), and (33), along with their respective noise modulation amplitudes (23), (25), and (34), converge in probability to the same steady-state distribution – the Gibbs distribution. In this case, the SDEs share an identical drift function but differ in their diffusion terms. In the literature, two SDEs interpreted under different stochastic integration rules are considered equivalent if they yield the same path-wise solution. Any SDE can be transformed into an equivalent one with a different interpretation through the addition of a drift correction term. In the case of the Itô-Stratonovich conversion, this term is known as the Wong-Zakai drift correction [22], [24], [26]. Consequently, equivalent SDEs share the same diffusion function but differ in their drift terms. We now proceed to show that the two models (23) and (25) are indeed equivalent in this sense.

Corollary 1 (Alternative formulation for the Itô interpretation): The solution of the Itô SDE (22) with $n_0(q)$ given by (23), converges in probability to the solution of the Itô SDE:

$$dq = \left(-g(f(q)) + kT\,\frac{d}{dq}\frac{g(f(q))}{f(q)}\right)dt + \sqrt{2kT\,\frac{g(f(q))}{f(q)}}\,dW_t. \tag{35}$$

Proof: See Appendix A. ■

Corollary 2 (Alternative formulation for the Stratonovich interpretation): The solution of the Stratonovich SDE (24) with $n_{\frac{1}{2}}(q)$ given by (25), converges in probability to the solution of the Stratonovich SDE:

$$dq = \left(-g(f(q)) + \frac{kT}{2}\frac{d}{dq}\frac{g(f(q))}{(f(q)}\right)dt + \sqrt{2kT\frac{g(f(q))}{f(q)}} \circ dW_t. \tag{36}$$

Proof: The proof is completely analogous to that of corollary 1, using $\alpha = 1/2$ that corresponds to the Stratonovich interpretation. ∎

The Itô SDE (35), the Stratonovich SDE (36), and the Hänggi-Klimontovich SDE

$$dq = -g(f(q))\,dt + \sqrt{2kT\frac{g(f(q))}{f(q)}} \bullet dW_t \tag{37}$$

are equivalent in the sense that not only their solutions do converge in probability, but also that, for a given realization of the Wiener process W_t and identical initial conditions, the solution trajectories $q(t)$ produced by all three equations are identical. This equivalence can be explicitly verified by observing that each of the three SDEs can be transformed into any of the others using standard stochastic calculus transformation rules [9], [22], [26]. Because the models in all three interpretations are equivalent, we will focus hereafter only on the Itô interpretation, which is generally preferred by engineers.

Remark 5: The alternative models established in Corollaries 1 and 2 involve the inclusion of a drift correction term. The necessity of a drift correction has long been a central argument both for and against specific interpretations in the ongoing debate over the "correct" formulation of SDEs. As emphasized in [8], equations describing thermal fluctuations correspond to a microscopic description of the system, and there is no fundamental reason why the deterministic component (the drift) should coincide with the averaged macroscopic dynamics. Put more simply, if within the Langevin framework it is permissible to add to the deterministic law a fluctuating term proportional to a Gaussian white noise, then it should likewise be permissible to include a drift correction term that depends on the fluctuating variable.

The fact that consistent models can be derived within each interpretation confirms that there is no a priori "correct" interpretation of SDEs. All interpretations are valid, provided they are formulated in a mathematically consistent manner. Each offers distinct advantages and disadvantages, facilitating certain types of analysis while complicating others. Ultimately, the choice of interpretation is largely a matter of convention and analytical convenience.

In order to be consistent with established results, the proposed white Gaussian noise model must reduce to the Johnson-Nyquist formula in the case of linear resistors.

Corollary 3: The white Gaussian noise model (23), or its equivalent form (35), reduces to the Johnson-Nyquist formula in the case of linear resistors.

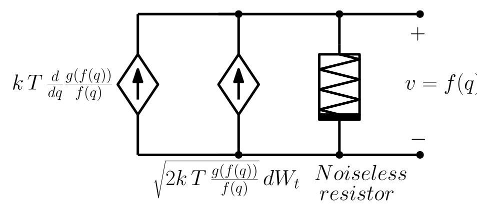

Fig. 7. Equivalent circuit for the alternative form of the white Gaussian noise model in the Itô interpretation.

Proof: For a linear resistor, we have $g(f(q)) = G\,f(q)$, where G denotes the resistor's conductance. Substituting this into (35), it is straightforward to verify that the drift correction term vanishes, while the diffusion function reduces to the Johnson-Nyquist formula.

Similarly, substituting $g(f(q)) = G\,f(q)$ into (23) and multiplying and dividing by kT, we obtain:

$$n_0(q) = \left[2kT\,G\,\exp\left(\frac{F(q)}{kT}\right) \times \int\left(-\frac{f(q)}{kT}\right)\exp\left(-\frac{F(q)}{kT}\right)dq\right]^{1/2}. \tag{38}$$

After performing the integration and setting the arbitrary integration constant to zero, we recover the Johnson-Nyquist formula. ∎

Figure 7 shows the Norton equivalent circuit for the alternative form of the white Gaussian noise model in the Itô interpretation. The equivalent circuit is composed by a noiseless nonlinear resistor parallel connected to a voltage controlled current source representing the white Gaussian random fluctuations, and a voltage controlled current source representing the drift correction term.

Remark 6: In addition to the standard existence and uniqueness conditions for strong solutions, SDEs (35), (36), and (37) show that the Gaussian white noise models are valid only for passive resistors, i.e., those satisfying condition $g(f(q))/f(q) \geq 0$. This restriction is entirely reasonable: an active resistor necessarily includes an internal power source, which must be explicitly accounted for in the derivation of a thermodynamically consistent model.

A. Example 3

We reconsider Example 2, replacing the state independent Gaussian noise model with the one described in Corollary 1, thereby obtaining the Itô SDE:

$$dq = \left[-Z\left(\exp\left(\frac{\gamma}{C}q\right)-1\right) + kT\,Z\,C\frac{\exp\left(\frac{\gamma}{C}q\right)\left(\frac{\gamma}{C}q-1\right)+1}{q^2}\right]dt + \sqrt{2kT\,Z\,C\frac{\exp\left(\frac{\gamma}{C}q\right)-1}{q}}\,dW_t. \tag{39}$$

Fig. 8 shows the stationary PDF, obtained through numerical integration of the SDE (39) for three different temperature values, ranging from 100 K to 600 K. In accordance with

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

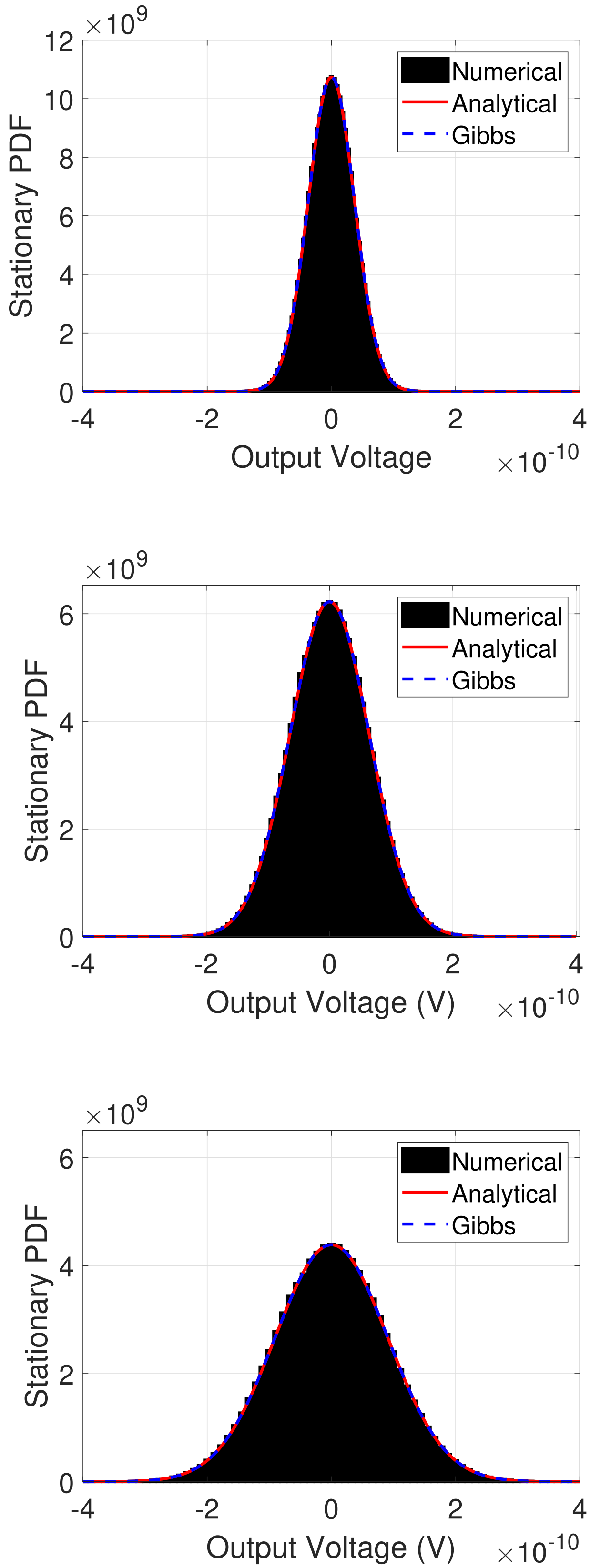

Fig. 8. Stationary PDF for the first order circuit with a nonlinear resistor and the Gaussian white noise model. Top: T = 100 K, Middle: T = 300 K, Bottom: T = 600 K.

the result of theorem 1 and corollary 1, the stationary PDF coincides with the Gibbs distribution in all cases.

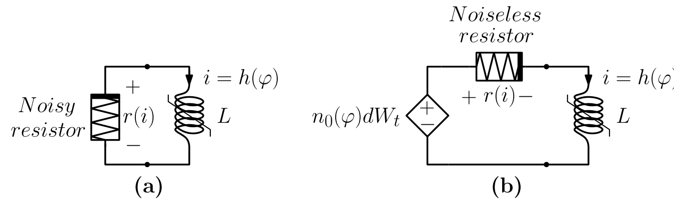

Fig. 9. **(a)** First-order circuit with noisy nonlinear resistor. **(b)** Thevenin equivalent circuit for the nonlinear resistor with a state dependent random source. For both circuits, $i = h(\varphi)$ is the constitutive relation of the nonlinear inductor.

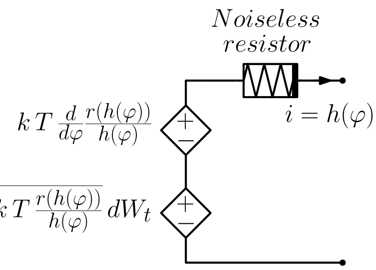

Fig. 10. Equivalent circuit for the alternative form of the white Gaussian noise model in the Itô interpretation.

B. The Dual Circuit

The same considerations as above also apply to the dual circuit, namely a first-order circuit consisting of a current-controlled noisy nonlinear resistor with voltage-current relationship $v = r(i)$, connected to a flux-controlled nonlinear inductor with characteristic $i = h(\varphi)$, as shown in Fig. 9(a). According to Theorem 1, in the Itô interpretation the noisy nonlinear resistor is equivalent to a noiseless resistor in series with a random controlled voltage source, whose voltage is given by

$$n_0(\varphi) = \left[-2\exp\left(\frac{H(\varphi)}{kT}\right) \times \int r\big(h(\varphi)\big)\,\exp\left(-\frac{H(\varphi)}{kT}\right)\,d\varphi\right]^{\frac{1}{2}}, \tag{40}$$

where

$$H(\varphi) = \int h(\varphi)\,d\varphi \tag{41}$$

The corresponding Thévenin equivalent circuit is depicted in Fig. 9(b). Alternatively, according to Corollary 1, the noisy resistor can be replaced by a noiseless resistor in series with two controlled random voltage sources, as shown in Fig. 10.

Remark 7: The alternative Gaussian white noise model introduced in Corollary 1 is well suited for implementation in numerical integration schemes for stochastic differential equations. Similarly, the equivalent circuits shown in Figs. 7 and 10 can be directly incorporated into commercial software for circuit simulation, analysis, and design. However, care must be exerted in this case as specialized algorithms for the time integration of SDEs should be properly implemented.

V. Satisfying Thermodynamic Requirements

In this section we prove that the Gaussian white noise model satisfies all four thermodynamic requirements, and therefore it is fully consistent from the physics point of view.

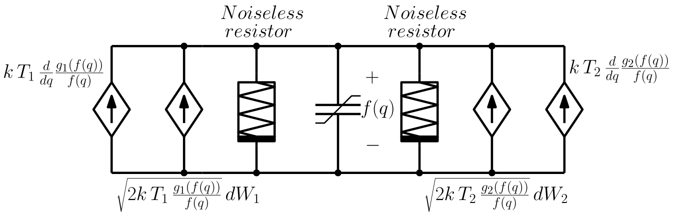

Fig. 11. Circuit used for testing thermodynamic requirement no 4, with the Norton equivalent circuits for the two nonlinear resistors.

Requirement No. 1: No Isothermal Conversion of Heat Into Work. Solving Brillouin's Paradox

Theorem 2 (Satisfying Requirement No. 1): The Gaussian white noise model (35) implies: $\mathbb{E}[f(q)] = 0$.

Proof: At steady state the solution of the Itô SDE (35) satisfies the Gibbs distribution. Therefore:

$$\mathbb{E}[f(q)] = \int_{-\infty}^{+\infty} f(q)\, A\, \exp\left(-\frac{F(q)}{kT}\right) dq$$
$$= -kT\,\rho_{\mathrm{G}}(q)\Big|_{-\infty}^{+\infty} = 0. \tag{42}$$

as a consequence of assumption 1, as required. ■

Requirement No. 2: Gibbs Distribution at Equilibrium

By construction, the Gaussian white noise model has the Gibbs distribution at thermal equilibrium and steady state, and therefore it satisfies Requirement No. 2.

Requirement No. 3: Positive Entropy Production During Transient.

We consider the circuit depicted in Fig. 7, connected to a reservoir composed by a nonlinear capacitor with characteristic $v = f(q)$, in contact with a thermal bath and at thermal equilibrium. The total entropy of the system is the sum of the entropy of the circuit, denoted by S_C, which is determined by the charge distribution, and of the entropy of the bath, denoted by S_B.

Theorem 3: For the Gaussian white noise model (35), the entropy production during a transient is positive.

Proof: See Appendix B. ■

Requirement No. 4: No Heat Transfer Between Two Devices at Thermal Equilibrium.

To verify that the proposed Gaussian white noise model satisfies thermodynamic Requirement No. 4, we consider the circuit shown in Fig. 11. The circuit consists of two nonlinear passive resistors, maintained at temperatures T_1 and T_2, respectively. Thermal fluctuations in the resistors are assumed to be statistically independent and are modeled as Wiener processes, denoted by W_1 and W_2. The resistors are interconnected via a nonlinear capacitor, which acts as an energy reservoir and is characterized by the voltage-charge relationship $v = f(q)$.

Theorem 4 (Satisfying requirement no 4): Consider a circuit consisting of two (or more) nonlinear passive resistors, respectively at temperature T_1 and T_2, each described by the Gaussian white noise model given in equation (35), and thermally coupled via a reservoir represented by a nonlinear capacitor with a constitutive relation $v = f(q)$. At thermal equilibrium, there is no net heat transfer between the resistors. The equilibrium temperature of the system corresponds to a weighted average of T_1 and T_2, and each nonlinear resistor dissipates its own thermal fluctuations.

Proof: Using the Gaussian white noise model (35), the Itô SDE describing the circuit is

$$dq = \left(-g_1(f(q)) - g_2(f(q)) + \frac{P'_{0,1}(q) + P'_{0,2}(q)}{2}\right) dt$$
$$+ \sqrt{P_{0,1}(q)}\, dW_1 + \sqrt{P_{0,2}(q)}\, dW_2 \tag{43}$$

where

$$P_{0,i}(q) = 2\,k\,T_i\,\frac{g_i(f(q))}{f(q)}, \qquad i = 1, 2. \tag{44}$$

The corresponding stationary FPE satisfies

$$\left(g_1(f(q)) + g_2(f(q)) - \frac{P'_{0,1} + P'_{0,2}}{2}\right)\rho_{\mathrm{st}}(q)$$
$$+ \frac{1}{2}\frac{d}{dq}\left[(P_{0,1}(q) + P_{0,2}(q))\rho_{\mathrm{st}}(q)\right] = 0. \tag{45}$$

Expanding the derivatives in the second term we obtain

$$\Big(g_1(f(q)) + g_2(f(q))\Big)\rho_{\mathrm{st}}(q)$$
$$+ \frac{1}{2}\left(P_{0,1}(q) + P_{0,2}\right)\frac{d\rho_{\mathrm{st}}(q)}{dq} = 0. \tag{46}$$

Again, at steady-state and at thermal equilibrium we expect $\rho_{\mathrm{st}}(q) = \rho_{\mathrm{G}}(q) = A\exp(-F(q)/(kT_{\mathrm{eq}}))$, where T_{eq} is the equilibrium temperature. Thus

$$\frac{d\rho_{\mathrm{st}}(q)}{dq} = \frac{d\rho_{\mathrm{G}}(q)}{dq} = -\frac{f(q)}{kT_{\mathrm{eq}}}\rho_{\mathrm{G}}(q) \tag{47}$$

Substituting into (46) and solving for T_{eq} we obtain:

$$T_{\mathrm{eq}} = \frac{T_1\, g_1(f(q)) + T_2\, g_2(f(q))}{g_1(f(q)) + g_2(f(q))}. \tag{48}$$

For the second part of the theorem, using Itô calculus we derive the energy equation:

$$dF(q) = f(q)\Bigg(- g_1(f(q)) - g_2(f(q)) + \frac{P'_{0,1}(q) + P'_{0,2}(q)}{2}$$
$$+ \frac{f'(q)}{2}\big(P_{0,1}(q) + P_{0,2}(q)\big)\Bigg) dt$$
$$+ \sqrt{P_{0,1}}\, dW_1 + \sqrt{P_{0,2}}\, dW_2. \tag{49}$$

Taking expectations and using again the martingale's property of Itô's integral yields the equation for the average power:

$$\mathbb{E}\left[\frac{dF(q)}{dt}\right] = -\mathbb{E}[g_1(f(q))\, f(q)] - \mathbb{E}[g_2(f(q))\, f(q)]$$
$$+ \frac{1}{2}\mathbb{E}\left[\frac{d}{dq}\Big(P_{0,1}(q)f(q) + P_{0,2}(q)f(q)\Big)\right]. \tag{50}$$

Using the passive sign convention, the first two terms on the right-hand side represent the average power absorbed by the two nonlinear resistors, respectively. In contrast, the third expectation term corresponds to the average power injected into the circuit by the thermal fluctuations associated with the resistors.

At thermal equilibrium, using (44)

$$\mathbb{E}\left[\frac{d}{dq}\Big(P_{0,i}(q)f(q)\Big)\right] = \int_{-\infty}^{+\infty} \frac{d}{dq}\Big(2\,k\,T_{\text{eq}}\,g_i(f(q))\Big)\,\rho_{\text{G}}(q)\,dq\,. \tag{51}$$

Integrating by parts and using (47)

$$\begin{aligned}\mathbb{E}\left[\frac{d}{dq}\Big(P_{0,i}(q)f(q)\Big)\right] &= 2\,k\,T_{\text{eq}}\left(g_i(f(q))\,\rho_{\text{G}}(q)\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} g_i(f(q))\frac{f(q)}{k\,T_{\text{eq}}}\,\rho_{\text{G}}(q)\,dq\right)\\ &= 2\,\mathbb{E}\left[g_i(f(q))f(q)\right]\end{aligned} \tag{52}$$

where we have again used the fact that $\rho_{\text{G}}(q)$ decays exponentially as $q \to \pm\infty$.

Equation (52) demonstrates that, at thermal equilibrium, the power injected into each resistor by its own thermal fluctuations is exactly balanced by the power dissipated within that resistor. In other words, each resistor dissipates its own internal fluctuations, and no net power is transferred between the two resistors – precisely as required by thermodynamic Requirement No. 4. ■

VI. Conclusion

This paper proposes a solution to a long-standing debate in electronic noise modeling by demonstrating that thermodynamically consistent Gaussian white noise models for nonlinear resistors at thermal equilibrium are achievable. Our work challenges the prevailing view that such models are fundamentally impossible and establishes a rigorous theoretical framework satisfying essential thermodynamic principles.

The core contribution lies in the derivation of state-dependent Gaussian white noise models for nonlinear resistors under both Itô and Stratonovich interpretations. By relaxing the assumption of load-independent noise generation, we show that noise intensity must be modulated by the circuit's state variables to maintain thermodynamic consistency. This represents a paradigm shift, implying that thermal noise may not be an intrinsic property of individual nonlinear components, but a context-dependent phenomenon governed by the thermodynamic constraints applied to the system. Explicit expressions for the modulation functions are provided for the main stochastic interpretations (Itô and Stratonovich), and we demonstrate that the two formulations are mathematically equivalent, converging to the same steady-state distributions.

We rigorously confirm that the proposed models fulfill four fundamental thermodynamic requirements: prevention of isothermal heat-to-work conversion (solving the famous Brillouin's paradox), adherence to the Gibbs distribution, positive entropy production during transients, and absence of net heat transfer between components at the same temperature. Notably, each resistor dissipates its own thermal fluctuations without net power transfer, offering key insights into energy balance mechanisms in complex nonlinear circuits.

The work provides practical tools for circuit analysis and design. The proposed models are suitable to be implemented in numerical integration solvers for stochastic differential equations. Equivalent circuit representations permit integration into commercial simulators.

This framework has broader implications for stochastic thermodynamics and nonlinear circuit theory. By showing that Gaussian white noise can be made thermodynamically consistent through state-dependent modulation, new ways emerge for modeling thermal fluctuations in passive nonlinear systems. This approach is relevant across fields from quantum electronics to neuromorphic devices, where capturing the interaction between nonlinearity and thermal noise is crucial. Furthermore, recognizing that noise generation depends on circuit-level thermodynamic constraints informs the design of noise-sensitive systems, potentially enhancing performance in precision measurements, low-noise amplifiers, and quantum devices.

Appendix A
Proof of Corollary 1

We report here the proof of corollary 1.

Proof: Rewrite (32) in the form

$$\begin{aligned}D_\alpha(q) &= 2\;kT\;\exp\left(\frac{F(q)}{(1-\alpha)kT}\right)\\ &\quad\times\int \frac{g(f(q))}{f(q)}\left(-\frac{f(q)}{(1-\alpha)kT}\right)\exp\left(-\frac{F(q)}{(1-\alpha)kT}\right)dq.\end{aligned} \tag{53}$$

Integrating by parts yields

$$\begin{aligned}D_\alpha(q) &= 2\;kT\,\frac{g(f(q))}{f(q)} - 2\;kT\;\exp\left(\frac{F(q)}{(1-\alpha)kT}\right)\\ &\quad\times\int\left(\frac{d}{dq}\frac{g(f(q))}{f(q)}\right)\exp\left(-\frac{F(q)}{(1-\alpha)kT}\right)dq.\end{aligned} \tag{54}$$

We decompose $D_\alpha(q) = P_\alpha(q) + Q_\alpha(q)$, where:

$$P_\alpha(q) = 2\;kT\,\frac{g(f(q))}{f(q)} \tag{55}$$

$$\begin{aligned}Q_\alpha(q) &= -2\;kT\;\exp\left(\frac{F(q)}{(1-\alpha)kT}\right)\\ &\quad\times\int\left(\frac{d}{dq}\frac{g(f(q))}{f(q)}\right)\exp\left(-\frac{F(q)}{(1-\alpha)kT}\right)dq.\end{aligned} \tag{56}$$

From (28) we find:

$$\begin{aligned}&\left(g(f(q)) + \frac{1-\alpha}{2}Q'_\alpha(q) + \frac{1}{2}\frac{Q_\alpha(q)}{\rho_{\text{st}}(q)}\frac{d\rho_{\text{st}}(q)}{dq}\right.\\ &\quad\left.-\frac{\alpha}{2}P'_\alpha(q)\right)\rho_{\text{st}}(q) + \frac{1}{2}\frac{d}{dq}\left(P_\alpha(q)\rho_{\text{st}}(q)\right) = 0\end{aligned} \tag{57}$$

which for Itô SDEs reduces to, being $\alpha = 0$

$$\begin{aligned}&\left(g(f(q)) + \frac{1}{2}Q'_0(q) + \frac{1}{2}\frac{Q_0(q)}{\rho_{\text{st}}(q)}\frac{d\rho_{\text{st}}(q)}{dq}\right)\rho_{\text{st}}(q)\\ &\quad+\frac{1}{2}\frac{d}{dq}\left(P_0(q)\rho_{\text{st}}(q)\right) = 0\end{aligned} \tag{58}$$

that corresponds to the Itô SDE

$$dq = \left(-g(f(q)) - \frac{1}{2}Q_0'(q) - \frac{1}{2}\frac{Q_0(q)}{\rho_{st}(q)}\frac{d\rho_{st}(q)}{dq}\right)dt + \sqrt{P_0(q)}\,dW_t. \tag{59}$$

On the other hand, for $\alpha = 0$ (56) reduces to

$$Q_\alpha(q) = -2\ kT\ \exp\left(\frac{F(q)}{kT}\right) \times \int\left(\frac{d}{dq}\frac{g(f(q))}{f(q)}\right)\exp\left(-\frac{F(q)}{kT}\right)dq \tag{60}$$

which implies

$$Q_\alpha'(q) = -2\ f(q)\exp\left(\frac{F(q)}{kT}\right) \times \int\left(\frac{d}{dq}\frac{g(f(q))}{f(q)}\right)\exp\left(-\frac{F(q)}{kT}\right)dq - 2\ kT\left(\frac{d}{dq}\frac{g(f(q))}{f(q)}\right). \tag{61}$$

Taking into account that for $\rho_{st}(q) = \rho_G(q)$

$$\frac{Q_0(q)}{\rho_G(q)}\frac{d\rho_G(q)}{dq} = -\frac{f(q)}{kT}Q_0(q), \tag{62}$$

equation (62) becomes

$$Q_0'(q) + \frac{Q_0(q)}{\rho_G(q)}\frac{d\rho_G(q)}{dq} = -2\ kT\frac{d}{dq}\frac{g(f(q))}{f(q)} \tag{63}$$

and the Itô SDE (59) finally simplifies to

$$dq = \left(-g(f(q)) + kT\frac{d}{dq}\frac{g(f(q))}{f(q)}\right)dt + \sqrt{2kT\frac{g(f(q))}{f(q)}}dW_t \tag{64}$$

as required. ■

Appendix B
Proof of Theorem 3

We report here the proof of theorem 3.

Proof: The entropy of the bath can be calculated using the thermodynamic relation

$$T\,dS_B = d\overline{Q}_B \tag{65}$$

where T is the temperature of the bath-circuit system, and $d\overline{Q}$ is the average heat absorbed by the bath. This heat is equal in magnitude and opposite in sign to the change in energy of the circuit. Using Itô's lemma, this variation can be expressed as:

$$dF = \frac{\partial F(q)}{\partial q}dq + \frac{1}{2}\frac{\partial^2 F(q)}{\partial q^2}(dq)^2 = f(q)\,dq + \frac{1}{2}f'(q)\,(dq)^2. \tag{66}$$

It is convenient to use the alternative stochastic differential form (35), which, using (55), can be rewritten as:

$$dq = \left(-g(f(q)) + \frac{1}{2}P_0'(q)\right)dt + \sqrt{P_0(q)},dW_t. \tag{67}$$

Substituting (67) into (66), and using Itô's rule

$$(dq)^2 = P_0(q)\,dt \tag{68}$$

along with (55), yields:

$$dF(q) = \left(-g(f(q))f(q) + kT\,g'(f(q))f'(q)\right)dt + f(q)\sqrt{P_0(q)}\,dW_t. \tag{69}$$

Taking expectations and using the martingale's property of Itô's integral we obtain:

$$\frac{d}{dt}\overline{F}(q) = -\int_{-\infty}^{+\infty} g(f(q))\,f(q)\rho(q,t)\,dq + kT\int_{-\infty}^{+\infty} g'(f(q))\,f(q)\rho(q,t)\,dq. \tag{70}$$

The second integral can be calculated as follows

$$\int_{-\infty}^{+\infty} g(f(q))\,\frac{\partial\rho}{\partial q}\,dq = g(f(q))\rho(q,t)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} g'(f(q))\,f'(q)\rho(q,t)\,dq. \tag{71}$$

The first term is null because, for all $t \in [0, +\infty[$:

$$\lim_{q\to\pm\infty}\rho(q,t) = \lim_{q\to\pm\infty}\frac{\partial\rho(q,t)}{\partial q} = 0. \tag{72}$$

Then

$$\frac{dS_B}{dt} = \frac{1}{T}d\overline{Q}_B = -\frac{1}{T}\frac{d\overline{F}(q)}{dt} = \frac{1}{T}\int_{-\infty}^{+\infty}\Big(g(f(q))\,f(q)\,\rho(q,t) + kT\,g(f(q))\frac{\partial\rho(q,t)}{\partial q}\Big)dq = \frac{1}{T}\int_{-\infty}^{+\infty}\left(g(f(q))\,\rho(q,t) + \frac{P_0(q)}{2}\frac{\partial\rho(q,t)}{\partial q}\right)f(q)\,dq. \tag{73}$$

The entropy of the circuit can be computed using Shannon's entropy formula [13], [28]:

$$S_C = -k\int_{-\infty}^{+\infty}\rho(q,t)\,\ln\rho(q,t)\,dq. \tag{74}$$

Then

$$\frac{dS_C}{dt} = -k\int_{-\infty}^{+\infty}\frac{\partial\rho(q,t)}{\partial t}\,(\ln\rho(q,t) + 1)\ dt. \tag{75}$$

Using the FPE associated to the SDE (67) we obtain

$$\frac{dS_C}{dt} = -k\int_{-\infty}^{+\infty}\frac{\partial}{\partial q}\left(g(f(q))\,\rho(q,t) + \frac{P_0(q)}{2}\frac{\partial\rho(q,t)}{\partial q}\right) \times (\ln\rho(q,t) + 1)\,dq. \tag{76}$$

Integrating by parts

$$\frac{dS_C}{dt} = -k\left[g(f(q))\,\rho(q,t) + \frac{P_0(q)}{2}\frac{\partial\rho(q,t)}{\partial q}\right] \times (\ln\rho(q,t) + 1)\Big|_{-\infty}^{+\infty}$$

$$- k \int_{-\infty}^{+\infty} \left[g(f(q))\rho(q,t) + \frac{P_0(q)}{2} \frac{\partial \rho(q,t)}{\partial q} \right] \times \frac{1}{\rho(q,t)} \frac{\partial \rho(q,t)}{\partial q} \, dq. \tag{77}$$

The first term on the right hand side equals zero for all t, because of (72).

Combining together (73) and (77) yields:

$$\frac{dS_B}{dt} + \frac{dS_C}{dt} = \int_{-\infty}^{+\infty} \left(g(f(q))\rho(q,t) + \frac{P_0(q)}{2} \frac{\partial \rho(q,t)}{\partial q} \right) \times \left(\frac{k}{\rho(q,t)} \frac{\partial \rho(q,t)}{\partial q} + \frac{f(q)}{T} \right) dq. \tag{78}$$

Multiplying and dividing by $g(f(q))$, and using (55)

$$\frac{dS_B}{dt} + \frac{dS_C}{dt} = \int_{-\infty}^{+\infty} \left(g(f(q))\rho(q,t) + \frac{P_0(q)}{2} \frac{\partial \rho(q,t)}{\partial q} \right)^2 \times \frac{1}{\rho(q,t)T} \frac{f(q)}{g(f(q))} \, dq. \tag{79}$$

For a passive nonlinear resistor, condition $g(f(q))/f(q) \geq 0$ holds. As a result, the entropy production is always non-negative, since it is given by the integral of a non-negative function. Therefore, Requirement No. 3 is satisfied. ■

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