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Homogenization and 3D-2D dimension reduction of a functional on manifold valued Sobolev spaces

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ABSTRACT

We study simultaneous homogenization and dimensional reduction of integral functionals for maps in manifold-valued Sobolev spaces. Due to the superlinear growth regime, we prove that the density of the Γ-limit is a tangential quasiconvex integrand represented by a cell formula.

1. Introduction and main results

The homogenization theory aims at describing the behavior of a model (either partial differential equations or energy functional) with heterogeneous coefficients that periodically oscillate on a small scale, say h. Indeed the main purpose consists of obtaining macroscopic properties of media with finely periodically distributed micro-structure, rigorously deriving these properties by means of a limiting procedure as the fine-scale h converges to zero. Many approaches have been developed in the last century: for instance asymptotic expansion methods (e.g. see A. Bensoussan, J.L. Lions and G. Papanicolaou [8], E. Sanchez-Palencia [38]) or the H-convergence methods due to F. Murat and L. Tartar [28,39,40] or the two-scale convergence [3,37], more recently re-casted in terms of a fixed functional space, i.e. within the theory of periodic unfolding (see [18,19,23,41,42]).

From the variational stand-point one is interested in the asymptotic behavior of minimizers of energy functionals depending on this small parameter h, and a crucial tool in this framework is Γ-convergence, see [20].

Another area of research in elasticity and micromagnetics is the derivation of lower dimensional theories - such as membrane, plate, string and rod models - from three-dimensional samples, i.e. one is interested in detecting a reduced model asymptotically departing from a slender body, letting the geometry of the body become singular. Again a rigorous approach is Γ-convergence and many results

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have been obtained in this context, after the pioneering papers [1,2,13,29,31,33], [15], among a wider literature, in the elastic and micromagnetic case respectively. In this paper we consider the slender domain approaching the reduced one as the heterogeneity becomes finely and finely distributed, i.e. we consider homogenization and dimensional reduction to happen simultaneously. This analysis has been performed in the realm of nonlinear heterogeneous thin structures and composites ([12,14,17,26,27,32]) or with the two-scale convergence technique [34] and for plates and rods ([36]). On the other hand, the same procedure has not been taken into account in the constrained setting, suitable to model liquid crystals, magnetostrictive or ferromagnetic materials, etc.

In the current paper we assume that the domain is an inhomogeneous cylinder, whose microstructure is assumed to be distributed with periodicity within the material described by the small parameter h comparable with the height of the domain. The equilibrium configurations are detected as minimizers of an integral functional of the form

$$\int_{\omega\times(-\frac{h}{2},\frac{h}{2})} f\left(\frac{x}{h},\nabla u\right)dx \qquad u:\omega\times\left(-\frac{h}{2},\frac{h}{2}\right)\to\mathbb{R}^3,$$

under suitable boundary conditions, where $\omega\subset\mathbb{R}^2$ is a bounded open set, and $f:\mathbb{R}^3\times\mathbb{R}^{3\times3}\to\mathbb{R}$ is a periodic integrand with respect to the first variable, and u is a manifold-valued Sobolev field, that will be specialized in the sequel.

Due to the many applications, it is worth to recall that solely the homogenization of integral functionals depending on x and ∇u and defined on manifold-valued Sobolev fields has been studied by Babadjian and Millot in [6] for $u\in W^{1,p}$ and in [5] for $u\in W^{1,1}$, we also refer to [35] for other related models. Analogously the dimensional reduction of micromagnetic and ferromagnetic energy has been studied, in several contexts, we recall [2,7,15,24,29,30] among the others.

The simultaneous homogenization and dimension reduction of an integral functional defined on real valued Sobolev functions has been studied by Braides, Fonseca and Francfort in [12] in the case $p>1$, while the case $p=1$ can be covered by the Global Method [10].

The main novelty of our contribution is to apply both homogenization and dimension reduction simultaneously and to assume the functional defined on the space of manifold-valued Sobolev functions for $p>1$. In a subsequent contribution [25] we will deal with the case $p=1$.

To conclude, we remark that other convergence regimes could be investigated. For instance, one could take the vertical thickness of the cylinder very small with respect to the periodicity, and vice-versa. A second possibility is to consider the material to be periodic also in the vertical variable. It seems that it is not easy to generalize our estimates to that cases. However, these problems are very interesting and they could be possible subject of other works.

For the sake of exposition, we focus on the model $3D-2D$ but our analysis could be extended to other dimensions, i.e. to the framework $ND-(N-d)D$. Given a Carathéodory function $f:\mathbb{R}^3\times\mathbb{R}^{3\times3}\to\mathbb{R}$ and $1<p<+\infty$ we consider the functional

$$\frac{1}{h}\int_{\omega_{,h}} f\left(\frac{x}{h},\nabla u\right)dx, \qquad u\in W^{1,p}(\omega_{,h};\mathcal{M}),$$

with $\omega_{,h}:=\omega\times\left(-\frac{h}{2},\frac{h}{2}\right)$, $h>0$, $\omega\subset\mathbb{R}^2$ open and bounded, and $\mathcal{M}$ a smooth submanifold of $\mathbb{R}^3$ without boundary. In particular, we assume that f has the following properties:

(H1) $f(\cdot,x_3,\xi)$ is 1-periodic, i.e. for every $(x_\alpha,x_3)\in\mathbb{R}^3$ and $\xi\in\mathbb{R}^{3\times3}$ it holds

$$f(x_\alpha+\mathbf{e}_i,x_3,\xi)=f(x_\alpha,x_3,\xi), \qquad \forall i=1,2$$

where $\{\mathbf{e}_1,\mathbf{e}_2\}$ is the canonical basis of $\mathbb{R}^2$;

(H2) f has p-growth: there exists $\alpha,\beta>0$ and $1<p<+\infty$ such that

$$\alpha|\xi|^p\le f(x,\xi)\le \beta(1+|\xi|^p),$$

for a.e. $x\in\mathbb{R}^3$ and for every $\xi\in\mathbb{R}^{3\times3}$.

We define the functional $\tilde I^h: L^p(\omega_{,h};\mathcal{M})\to\overline{\mathbb{R}}$

$$\tilde I^h(u):=\begin{cases}\frac{1}{h}\int_{\omega_{,h}} f\left(\frac{x}{h},\nabla u\right)dx & \text{if } u\in W^{1,p}(\omega_{,h};\mathcal{M})\\ +\infty & \text{elsewhere.}\end{cases}$$

The study of the Γ–limit of $\tilde I^h$ is equivalent to the study of the Γ–limit of the rescaled functional I^h defined as

$$I^h(u):=\begin{cases}\int_{\Omega} f\left(\frac{x_\alpha}{h},x_3,\nabla_h u\right)dx & \text{if } u\in W^{1,p}(\Omega;\mathcal{M})\\ +\infty & \text{elsewhere}\end{cases} \tag{1.1}$$

with $\Omega:=\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)=\omega_{,1}$ and $\nabla_h:=[\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{1}{h}\frac{\partial}{\partial x_3}]$. From here onward, we denote $\nabla_\alpha:=[\frac{\partial}{\partial x_1}|\frac{\partial}{\partial x_2}]$ and $\nabla_3:=\frac{\partial}{\partial x_3}$, so that $\nabla_h=[\nabla_\alpha|\frac{1}{h}\nabla_3]$. Moreover, we denote with ξ_α an element of $\mathbb{R}^{3\times2}$ and with ξ an element of $\mathbb{R}^{3\times3}$.
Our main result is the following.

Theorem 1.1. *Assume that $\mathcal{M}$ is a connected smooth manifold of $\mathbb{R}^3$ without boundary and let $f : \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ be a Carathéodory function satisfying* (H1) *and* (H2) *with $1 < p < +\infty$. Then, the $\Gamma-$limit of I^h as $h \to 0$ with respect to the strong L^p-topology is the functional $I : L^p(\omega;\mathcal{M}) \to \overline{\mathbb{R}}$ given by*

$$I(u) = \begin{cases} \displaystyle\int_\omega T f^0_{\rm hom}(u, \nabla_\alpha u)\, dx_\alpha, & \text{if } u \in W^{1,p}(\omega;\mathcal{M}) \\ +\infty & \text{elsewhere,} \end{cases} \tag{1.2}$$

with $T f^0_{\rm hom} : \mathbb{R}^3 \times \mathbb{R}^{3\times 2} \to \mathbb{R}$ defined as

$$\begin{aligned} T f^0_{\rm hom}(s,\xi_\alpha) := \liminf_{t\to+\infty} \inf_{\varphi} \Bigg\{ &\frac{1}{t^2}\int_{(tQ')_{,1}} f(x_\alpha, x_3, \xi_\alpha + \nabla_\alpha \varphi | \nabla_3 \varphi)\, dx_\alpha dx_3 : \\ &\varphi \in W^{1,\infty}((tQ')_{,1}; T_s(\mathcal{M})),\ \varphi(x_\alpha,x_3) = 0 \text{ for every } (x_\alpha,x_3) \in \partial(tQ') \times \left(-\frac{1}{2},\frac{1}{2}\right) \Bigg\}, \end{aligned} \tag{1.3}$$

where $Q' := \left(-\frac{1}{2},\frac{1}{2}\right)^2$, $(tQ')_{,1} := tQ' \times \left(-\frac{1}{2},\frac{1}{2}\right)$, and $T_s(\mathcal{M})$ denotes the tangent space to $\mathcal{M}$ in s.

We observe that, in view of $(H2)$, the same result could be achieved replacing the strong convergence in $L^p(\omega;\mathcal{M})$ by the weak convergence in $W^{1,p}(\omega;\mathcal{M})$.

The paper is organized as follows: Section 2 is devoted to fix notation and to provide preliminary results, while Section 3 is devoted to the proof of Theorem 1.1.

The $\Gamma-$liminf inequality follows by the localization and blow-up methods, while the main difficulty with the $\Gamma-$limsup inequality is due to the fact that we need to construct a manifold-valued recovery sequence. A key tool in our result is the *tangential quasiconvexification* introduced in [22]. Our integral representation result holds in $W^{1,p}(\omega;\mathcal{M})$ when $p > 1$, but we observe that (1.2) is still valid, and can be obtained with the same (even easier) techniques, when $p = 1$ and $L^p(\omega;\mathcal{M})$ strong convergence is replaced by $W^{1,1}(\omega;\mathcal{M})$, besides in this latter setting there is a lack of compactness. On the other hand, we emphasize that when $p = 1$ and $u \in W^{1,1}(\omega;\mathcal{M})$ (1.2) represents also the $\Gamma - L^1(\omega;\mathcal{M})$ strong limit of (1.1), when $h \to 0$. For this result we refer to [25].

2. Notation and preliminaries

This section is devoted to fix notation, recall previous results that will be useful in the sequel and establish properties of the energy densities appearing in the main result.

In what follows $\Omega := \omega \times (-\frac{1}{2},\frac{1}{2})$, $\omega \subset \mathbb{R}^2$ is open and bounded, and $\mathcal{M}$ is a smooth submanifold of $\mathbb{R}^3$ without boundary, further assumptions on $\mathcal{M}$ will be stated explicitly if needed. Given $s \in \mathcal{M}$, we write $T_s(\mathcal{M})$ for the tangent space to $\mathcal{M}$ in s. We denote by $\mathcal{A}(\Omega)$ the family of all open subsets of Ω and with $\mathcal{A}(\omega)$ the family of all open subset of ω. We write $B^k(s,r)$ for the closed ball in $\mathbb{R}^k$, $k \in \mathbb{N}$, of center $s \in \mathbb{R}^k$ and radius $r > 0$. Moreover, we denote by Q the cube $(-\frac{1}{2},\frac{1}{2})^3$ and with $Q(x_0,\rho)$ the rescaled and translated cube $x_0 + \rho Q$, with $x_0 \in \mathbb{R}^3$, $\rho > 0$. In a similar way, given $\nu \in \mathbb{S}^1$, Q_ν stands for the open unit square in $\mathbb{R}^2$ centered at the origin, with the two sides orthogonal to ν; we set $Q_\nu(x_0,\rho) := x_0 + \rho Q_\nu$.

Furthermore, given a set $A \subset \omega$ we denote by $A_{,h}$, with $h > 0$, the set $A \times (-\frac{h}{2},\frac{h}{2})$ and so $A_{,1} := A \times (-\frac{1}{2},\frac{1}{2})$. It follows that, in particular, $\Omega = \omega_{,1}$. We also denote by Q' the square $(-\frac{1}{2},\frac{1}{2})^2$, so $Q'_{,h} = (-\frac{1}{2},\frac{1}{2})^2 \times (-\frac{h}{2},\frac{h}{2})$ for $h > 0$, while $Q'_{,1} = (-\frac{1}{2},\frac{1}{2})^3 = Q$ and $Q_{\nu,1} = Q_\nu \times (-\frac{1}{2},\frac{1}{2})$. $\mathcal{M}(\Omega)$ is the space of real valued Radon measure in Ω with finite total variation, $\mathcal{L}^k$, is the k-dimensional Lebesgue measure, with $k \in \mathbb{N}$. Finally, given $\lambda,\mu \in \mathcal{M}(\Omega)$ we denote by $\frac{d\lambda}{d\mu}$ the Radon-Nykodým derivative of λ with respect to μ. By a generalization of Besicovitch Differentiation Theorem, see for instance [4, Proposition 2.2], there exists a Borel set E such that $\mu(E) = 0$ and $\frac{d\lambda}{d\mu}(x) = 0$ for every $x \in \operatorname{supp}\lambda \setminus E$.

Given $s \in \mathcal{M}$, we consider the orthogonal projection

$$P_s : \mathbb{R}^3 \to T_s(\mathcal{M}),$$

and we define the function $\mathbf{P}_s : \mathbb{R}^{3\times 3} \to [T_s(\mathcal{M})]^3$ as

$$\mathbf{P}_s(\xi) := \left[P_s(\xi_1)|P_s(\xi_2)|P_s(\xi_3)\right],$$

for every $\xi = [\xi_1|\xi_2|\xi_3] \in \mathbb{R}^{3\times 3}$.

For a Carathéodory function $f : \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \to \mathbb{R}$, we set

$$\bar f(x,s,\xi) := f(x, \mathbf{P}_s(\xi)) + |\xi - \mathbf{P}_s(\xi)|^p. \tag{2.1}$$

This function will play a crucial role in our subsequent analysis, since it will appear in the formulas to detect our limiting energy densities. By construction the function $\bar f : \mathbb{R}^3 \times \mathcal{M} \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ is Carathéodory, i.e. it is measurable with respect to the first variable and continuous with respect to the last two variables. Moreover, if conditions (H1) and (H2) for $1 < p < +\infty$ are satisfiedthen $\bar f$ is

1-periodic in the first variable and satisfies p-growth and p-coercivity conditions, i.e. there exists $C > 0$ such that

$$\frac{1}{C}|\xi|^p \le \bar{f}(x, s, \xi) \le C(1 + |\xi|^p), \tag{2.2}$$

for every $(s, \xi) \in \mathcal{M} \times \mathbb{R}^{3\times 3}$ and for a.e. $x \in \mathbb{R}^3$.

Following [6, Proposition 2.1], we characterize the density $Tf^0_{\rm hom}$ of the Γ-limit and we will prove some of its properties.

Proposition 2.1. *Let $1 \le p < +\infty$, for every Carathéodory function $f : \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfying conditions* (H1) *and* (H2) *the following properties hold.*

(i) *For every $s \in \mathcal{M}$ and $\xi_\alpha \in [T_s(\mathcal{M})]^2$*

$$Tf^0_{\rm hom}(s, \xi_\alpha) = \bar{f}^0_{\rm hom}(s, \xi_\alpha), \tag{2.3}$$

where $Tf^0_{\rm hom}$ is defined as in (1.3) and

$$\bar{f}^0_{\rm hom}(s, \xi_\alpha) = \lim_{t\to+\infty} \inf_{\varphi} \left\{ \frac{1}{t^2} \int_{(tQ')_{,1}} \bar{f}(x_\alpha, x_3, s, \xi_\alpha + \nabla_\alpha \varphi | \nabla_3 \varphi)\, dx_\alpha dx_3 : \right. \tag{2.4}$$
$$\left. \varphi \in W^{1,\infty}((tQ')_{,1}; \mathbb{R}^3),\ \varphi(x_\alpha, x_3) = 0 \text{ for every } (x_\alpha, x_3) \in \partial(Q') \times \left(-\frac{1}{2}, \frac{1}{2}\right) \right\}$$

(ii) *The function $Tf^0_{\rm hom}$ is tangentially quasiconvex in the second variable, i.e.*

$$Tf^0_{\rm hom}(s, \xi_\alpha) \le \int_{Q'} Tf^0_{\rm hom}(s, \xi_\alpha + \nabla_\alpha \psi) dx_\alpha,$$

for every $s \in \mathcal{M}, \xi_\alpha \in [T_s(\mathcal{M})]^2$, and $\psi \in W^{1,\infty}_0(Q'; T_s(\mathcal{M}))$. In particular, $Tf^0_{\rm hom}(s, \cdot)$ is rank one convex.

(iii) *$Tf^0_{\rm hom}$ is uniformly p-coercive and has uniform p-growth in the second variable (i.e. it satisfies inequalities as in (2.2), uniformly with respect to s). Moreover, there exists $C > 0$ such that for every $s \in \mathcal{M}$ and $\xi_\alpha, \xi'_\alpha \in [T_s(\mathcal{M})]^2$*

$$|Tf^0_{\rm hom}(s, \xi_\alpha) - Tf^0_{\rm hom}(s, \xi'_\alpha)| \le C|\xi_\alpha - \xi'_\alpha|(1 + |\xi_\alpha|^{p-1} + |\xi'_\alpha|^{p-1}). \tag{2.5}$$

Before proving our statement, it is worth observing that for every $s \in \mathcal{M}$, $\bar{f}^0_{\rm hom}(s, \cdot)$ in (2.4), is the $3D - 2D$ homogenized energy density appearing in the dimension reduction problems in the unconstrained setting and it has been introduced in [12].

Proof. We start from (i). Fix $s \in \mathcal{M}$ and $\xi_\alpha \in [T_s(\mathcal{M})]^2$. For any $t > 0$, we introduce

$$Tf_t(s, \xi_\alpha) := \inf_{\varphi} \left\{ \fint_{(tQ')_{,1}} f(y, \xi_\alpha + \nabla_\alpha \varphi | \nabla_3 \varphi) dy : \right.$$
$$\left. \varphi \in W^{1,\infty}((tQ)_{,1}; T_s(\mathcal{M})),\ \varphi(x_\alpha, x_3) = 0 \ \text{ for every } (x_\alpha, x_3) \in \partial(tQ') \times \left(-\frac{1}{2}, \frac{1}{2}\right) \right\},$$

and

$$\bar{f}_t(s, \xi_\alpha) := \inf_{\varphi} \left\{ \fint_{(tQ')_{,1}} \bar{f}(y, s, \xi_\alpha + \nabla_\alpha \varphi | \nabla_3 \varphi) dy : \right. \tag{2.6}$$
$$\left. \varphi \in W^{1,\infty}((tQ')_{,1}; \mathbb{R}^3),\ \varphi(x_\alpha, x_3) = 0 \ \text{ for every } (x_\alpha, x_3) \in \partial(tQ') \times \left(-\frac{1}{2}, \frac{1}{2}\right) \right\}.$$

Therefore, from (1.3) and (2.4), we have that $Tf^0_{\rm hom}(s, \xi_\alpha) = \liminf_{t\to\infty} Tf_t(s, \xi_\alpha)$ and $\bar{f}^0_{\rm hom}(s, \xi_\alpha) = \lim_{t\to\infty} \bar{f}_t(s, \xi_\alpha)$ provided the latter one exists.

Following a classical argument (see for instance [11, Proposition 14.4] and [16, Proposition 4.1]) which, for every $\tau > t > 0$ allows to cover $\tau Q'$ by non overlapping squares of side length t and construct suitable test functions φ_τ for the definition of $\overline{f}_\tau(s, \xi_\alpha)$ in (2.6), the existence of the following limit

$$\lim_{t\to+\infty} \bar{f}_t(s, \xi_\alpha) \qquad \text{for every } s \in \mathcal{M} \text{ and } \xi_\alpha \in [T_s(\mathcal{M})]^2,$$

is granted. Therefore, in order to conclude the proof, it is enough to show that $Tf_t(s, \xi_\alpha) = \bar{f}_t(s, \xi_\alpha)$ for every $t > 0$. For any $\varphi \in W^{1,\infty}((tQ)_{,1}; T_s(\mathcal{M}))$ with $\varphi(x_\alpha, x_3) = 0$ for every $(x_\alpha, x_3) \in \partial(tQ') \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ we have

$$\bar{f}_t(s, \xi_\alpha) \le \fint_{(tQ')_{,1}} \bar{f}(y, s, \xi_\alpha + \nabla_\alpha \varphi | \nabla_3 \varphi)\, dy = \fint_{(tQ')_{,1}} f(y, \xi_\alpha + \nabla_\alpha \varphi | \nabla_3 \varphi)\, dy,$$

since $(\xi_\alpha + \nabla_\alpha \varphi | \nabla_3 \varphi) \in [T_s(\mathcal{M})]^3$ for a.e. $y \in (tQ')_{,1}$. By taking the infimum over φ in the right hand side of the previous inequality, we obtain that

$$\bar{f}_t(s, \xi_\alpha) \le Tf_t(s, \xi_\alpha).$$

With the aim to prove the converse inequality, consider $\psi \in W^{1,\infty}((tQ')_{,1};\mathbb{R}^3))$ such that $\psi(x_\alpha, x_3) = 0$ for every $(x_\alpha, x_3) \in \partial(tQ') \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, and set

$$\tilde{\psi} = P_s(\psi).$$

It is not difficult to see that

$$\tilde{\psi} \in W^{1,\infty}((tQ')_{,1}; T_s(\mathcal{M})) \qquad \text{and} \qquad \nabla\tilde{\psi} = \mathbf{P}_s(\nabla\psi) \text{ a.e. in } (tQ')_{,1},$$

with $\tilde{\psi}(x_\alpha, x_3) = 0$ for every $(x_\alpha, x_3) \in \partial(tQ') \times \left(-\frac{1}{2}, \frac{1}{2}\right)$. Thus we get

$$\begin{aligned} Tf_t(s,\xi_\alpha) &\leq \fint_{(tQ')_{,1}} f(y, \xi_\alpha + \nabla_\alpha\tilde{\psi}|\nabla_3\tilde{\psi})\, dy \\ &= \fint_{(tQ')_{,1}} f(y, \mathbf{P}_s(\xi_\alpha + \nabla_\alpha\psi|\nabla_3\psi))\, dy \\ &\leq \fint_{(tQ')_{,1}} \bar{f}(y, s, \xi_\alpha + \nabla_\alpha\psi|\nabla_3\psi)\, dy. \end{aligned}$$

Then, the thesis follows by taking the infimum over ψ in the right hand side of the inequality.

In order to prove (ii), we observe that the classical results (see for instance [12, Theorem 4.2]) yield that $\bar{f}^0_{\hom}(s,\cdot)$ is a quasiconvex function for every $s \in \mathcal{M}$; consequently, for any $s \in \mathcal{M}, \xi_\alpha \in [T_s(\mathcal{M})]^2$ and $\varphi \in W_0^{1,\infty}(Q', T_s(\mathcal{M}))$, it holds

$$Tf^0_{\hom}(s,\xi_\alpha) = \bar{f}^0_{\hom}(s,\xi_\alpha) \leq \int_{Q'} \bar{f}^0_{\hom}(s, \xi_\alpha + \nabla_\alpha\varphi)\, dy_\alpha = \int_{Q'} Tf^0_{\hom}(s, \xi_\alpha + \nabla_\alpha\varphi) dy_\alpha.$$

This allows us to conclude that for every $s \in \mathbb{R}^3$, $Tf^0_{\hom}(s,\cdot)$ is tangentially quasiconvex. For the attainment of (iii), we observe that $Tf^0_{\hom}(s,\cdot)$ is also rank one convex as long as (2.3) holds since by (ii) $\bar{f}^0_{\hom}(s,\cdot)$ is rank one convex.

In view of assumption (H2) and the definition of $Tf^0_{\hom}$ it is possible to show that $Tf^0_{\hom}$ is p-coercive and it has uniform p-growth in the second variable, uniformly with respect to the first. Since rank one convex functions satisfying uniform p-growth and p-coercivity conditions are p-Lipschitz in view, for instance of [21, Proposition 2.32], also (2.5) holds, which concludes the proof of (iii). $\square$

3. Proof of theorem 1.1

We recall that f satisfies (H1) and (H2). We also recall that the candidate for the Γ-limit with respect to the strong L^p-topology of the family of functionals I^h in (1.1), whose localization, for every $p \geq 1$ is defined in $\mathcal{A}(\omega)$ as

$$I^h(u, A) := \begin{cases} \displaystyle\int_{A_{,1}} f\left(\frac{x_\alpha}{h}, x_3, \nabla_h u\right) dx & \text{if } u \in W^{1,p}(\Omega, \mathcal{M}) \\ +\infty & \text{elsewhere} \end{cases}$$

is the functional I given by (1.2).

For any $A \in \mathcal{A}(\omega)$ and $u \in L^p(\Omega; \mathcal{M})$, consider the functional

$$I^0(u, A) := \inf\left\{\liminf_{n\to+\infty} I^{h_n}(u_n, A) : u_n \to u \text{ in } L^p(A_{,1}; \mathcal{M})\right\}, \tag{3.1}$$

for any $u \in W^{1,p}(\omega; \mathcal{M})$ and $A \in \mathcal{A}(\omega)$, and denote $I^0(u,\omega)$ simply by $I^0(u)$ for every $u \in W^{1,p}(\omega; \mathcal{M})$.

In order to prove the Γ- limsup inequality we introduce a suitable functional that is larger than I^0 and we prove that it is the restriction to $\mathcal{A}(\omega)$ of a Radon measure absolutely continuous with respect to $\mathcal{L}^2$. Given a compact set $\mathcal{K} \subset \mathcal{M}$, and a non-relabeled subsequence $(h_{n_k})_k := (h_k)_k$ we define for $u \in W^{1,p}(\omega; \mathcal{M})$ and $A \in \mathcal{A}(\omega)$

$$\begin{aligned} I^{\{h_k\}}_{\mathcal{K}}(u, A) := \inf_{(u_k)_k} \Bigg\{ & \limsup_{k\to\infty} I^{h_k}(u_k, A) \;:\; \nabla u_k \rightharpoonup \nabla_\alpha u \text{ in } L^p(\Omega; \mathbb{R}^3), \\ & (\nabla_{h_k} u_k)_k \text{ is bounded in } L^p(\Omega, \mathbb{R}^3), \\ & u_k \to u \text{ uniformly, } u_k(x) = u(x_\alpha) \text{ if } \operatorname{dist}(u(x_\alpha), \mathcal{K}) > 1 \text{ for a.e. } x \in \Omega \Bigg\}. \end{aligned}$$

Remark 3.1. Given $A \subset \mathcal{A}(\omega)$, the set

$$V(A) := \left\{ v \in W^{1,p}(A_{,1}; \mathcal{M}) : \frac{\partial v}{\partial x_3} = 0 \text{ a.e. on } A_{,1} \right\},$$

is isomorphic to the Sobolev space $W^{1,p}(A; \mathcal{M})$.

Lemma 3.1 (Localization). *Let $p \geq 1$. For every $u \in V(\omega)$, there exists a non-relabeled subsequence $(h_k)_k$ such that the set function $I^{\{h_k\}}_{\mathcal{K}}(u,\cdot)$ is the restriction to $\mathcal{A}(\omega)$ of a Radon measure absolutely continuous with respect to $\mathcal{L}^2$.*

Proof. From the p-growth condition (H2), we obtain that, for any non-relabeled subsequence $(h_k)_k$,

$$I_{\mathcal{K}}^{\{h_k\}}(u,A) \le c\int_A (1+|\nabla_\alpha u|^p)\,dx; \tag{3.2}$$

therefore we just need to infer the existence of a suitable subsequence $(h_k)_k$ for which the trace of $I_{\mathcal{K}}^{\{h_k\}}(u,\cdot)$ is a Radon measure. This can be shown in two steps.

STEP 1: The first thing to be proved is that the following subadditivity property for the functional $I^{\{h_k\}}$

$$I_{\mathcal{K}}^{\{h_k\}}(u,A) \le I_{\mathcal{K}}^{\{h_k\}}(u,B) + I_{\mathcal{K}}^{\{h_k\}}(u,A\setminus\overline{C}), \tag{3.3}$$

holds for all $A,B,C\in\mathcal{A}(\omega)$ such that $\overline{C}\subset B\subset A$. Now, for a given $\eta>0$, there exist sequences $(u_k)_k,(v_k)_k\subset W^{1,p}(\Omega;\mathcal{M})$ such that ∇u_k and ∇v_k converge weakly to $\nabla_\alpha u$ in $W^{1,p}(\Omega;\mathbb{R}^3)$, and $(\nabla_{h_k}u_k)_k$ and $(\nabla_{h_k}v_k)_k$ are bounded in $L^p(\Omega,\mathbb{R}^3)$, $u_k(x)=v_k(x)=u(x_\alpha)$ if $\operatorname{dist}(u(x_\alpha),\mathcal{K})>1$ for a.e. $x\in\Omega$, u_k and v_k are uniformly converging to u and

$$\begin{cases}\limsup\limits_{k\to+\infty} I^{h_n}(u_k,B) \le I_{\mathcal{K}}^{\{h_n\}}(u,B)+\eta\\ \limsup\limits_{k\to+\infty} I^{h_n}(v_k,A\setminus\overline{C}) \le I_{\mathcal{K}}^{\{h_n\}}(u,A\setminus\overline{C})+\eta.\end{cases}$$

Now, let us set

$$\mathcal{K}' := \{s\in\mathcal{M} : \operatorname{dist}(s,\mathcal{K})\le 1\}.$$

We observe that $\mathcal{K}'$ is a compact subset of $\mathcal{M}$ and $u_k(x)=v_k(x)=u(x_\alpha)$ if $u(x_\alpha)\notin\mathcal{K}'$ for a.e. $x\in\Omega$. Moreover, let us fix $L:=\operatorname{dist}(C,\partial B)$, $M\in\mathbb{N}$ and define, for every $i\in\{0,\dots,M\}$

$$B_i := \left\{x_\alpha\in B : \operatorname{dist}(x_\alpha,\partial B) > \frac{iL}{M}\right\}.$$

while for every $i\in\{0,\dots,M-1\}$ set

$$S_i := B_i\setminus\overline{B_{i+1}}.$$

Consider finally, for every $i\in\{0,\dots,M-1\}$, $\zeta_i\in C_c^\infty(\Omega;[0,1])$ being a cut-off function satisfying

$$\zeta_i(x)=\zeta_i(x_\alpha)=\begin{cases}1 & \text{in } (B_{i+1})_{,1}\\ 0 & \text{in } \Omega\setminus(B_i)_{,1}\end{cases} \quad\text{and}\quad |\nabla\zeta_i| = |\nabla_\alpha\zeta_i| \le \tfrac{2M}{L}$$

By [22, Lemma 3.2 and Remark 3.3], there exist $\delta>0$, $c>0$ and a uniformly continuously differentiable mapping $\Phi : D_\delta\times[0,1]\to\mathcal{M}$, where

$$D_\delta := \{(s_0,s_1)\in\mathcal{M}\times\mathcal{M} : \operatorname{dist}(s_0,\mathcal{K}')<\delta,\ \operatorname{dist}(s_1,\mathcal{K}')<\delta,\ |s_0-s_1|<\delta\},$$

such that

$$\Phi(s_0,s_1,0)=s_0, \qquad \Phi(s_0,s_1,1)=s_1, \qquad \frac{\partial\Phi}{\partial t}(s_0,s_1,t)\le c\,|s_0-s_1|, \tag{3.4}$$

and

$$|\Phi(s_0,s_1,t)-s_0| \le c|s_0-s_1|. \tag{3.5}$$

We recall that $(u_k)_k$ and $(v_k)_k$ are uniformly converging sequences, thus we can choose k large enough such that the following conditions hold

$$\|u_k-u\|_{L^\infty(\Omega;\mathbb{R}^3)}<\delta, \qquad \|v_k-u\|_{L^\infty(\Omega;\mathbb{R}^3)}<\delta.$$

This entails that, for a.e. $x\in\Omega$, $\operatorname{dist}(v_k(x),\mathcal{K}')<\delta$ when $u(x_\alpha)\in\mathcal{K}'$. This way, we can define $w_{k,i}\in W^{1,p}(\Omega;\mathcal{M})$ as follows

$$w_{k,i}(x) := \begin{cases}\Phi(v_k(x),u_k(x),\zeta_i(x)) & \text{if } u(x)\in\mathcal{K}'\\ u(x_\alpha) & \text{if } u(x)\notin\mathcal{K}'\end{cases}$$

By exploiting the p-growth condition (H2) as well as (3.4), we are able to deduce

$$\begin{aligned}\int_{A_{,1}} f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}w_{k,i}\right)dx \le{}& \int_{B_{,1}} f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}u_k\right)dx + \int_{A_{,1}\setminus\overline{C}_{,1}} f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}v_k\right)dx\\ &+ C_0\int_{(S_i)_{,1}}(1+|\nabla_{h_k}u_k|^p+|\nabla_{h_k}v_k|^p+M^p|u_k-v_k|^p)\,dx,\end{aligned}$$

which holds for some constants $C_0>0$ independent of i,k and M. Now, if we sum up over the index $i\in\{0,\dots,M-1\}$ and we divide by M, we get

$$\begin{aligned}\frac{1}{M}\sum_{i=0}^{M-1}\int_{A_{,1}} f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}w_{k,i}\right)dx \le{}& \int_{B_{,1}} f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}u_k\right)dx + \int_{A_{,1}\setminus\overline{C}_{,1}} f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}v_k\right)dx\\ &+ \frac{C_0}{M}\int_{B_{,1}\setminus\overline{C}_{,1}}(1+|\nabla_{h_k}u_k|^p+|\nabla_{h_k}v_k|^p+M^p|u_k-v_k|^p)\,dx.\end{aligned}$$

Therefore it is possible to find some indices $i_k \in \{0, \dots, M-1\}$ such that $\overline{w}_k := w_{k,i_k}$ satisfies

$$\int_{A_{,1}} f\left(\frac{x_\alpha}{h_k}, h_k x_3, \nabla_{h_k}\overline{w}_k\right) dx \leq \int_{B_{,1}} f\left(\frac{x_\alpha}{h_k}, x_3, \nabla_{h_k} u_k\right) dx + \int_{A_{,1}\setminus\overline{C}_{,1}} f\left(\frac{x_\alpha}{h_k}, x_3, \nabla_{h_k} v_k\right) dx$$
$$+ \frac{C_0}{M}\int_{B_{,1}\setminus\overline{C}_{,1}} (1 + |\nabla_{h_k} u_k|^p + |\nabla_{h_k} v_k|^p + M^p|u_k - v_k|^p)\, dx.$$

From (3.4) and (3.5), we get that $\overline{w}_k \to u$ uniformly; moreover $\nabla\overline{w}_k \rightharpoonup \nabla_\alpha u$ in $L^p(\Omega;\mathbb{R}^3)$, $(\nabla_{h_k}\bar{w}_k)_k$ is bounded in $L^p(\Omega,\mathbb{R}^3)$, and finally $\overline{w}_k(x) = u(x_\alpha)$ whenever $\operatorname{dist}(u(x_\alpha),\mathcal{K}) > 1$ for a.e. $x \in \Omega$. Finally, we can conclude that

$$\begin{aligned}
I_{\mathcal{K}}^{\{h_k\}}(u, A) &\leq \limsup_{k\to+\infty} I^{h_k}(\overline{w}_k, A)\\
&\leq \limsup_{k\to+\infty}\left\{ I^{h_k}(u_k, B) + I^{h_k}(v_k, A\setminus\overline{C}) + \frac{C_0}{M}\int_{B_{,1}\setminus\overline{C}_{,1}} (1 + |\nabla_{h_k} u_k|^p + |\nabla_{h_k} v_k|^p + M^p|u_k - v_k|^p)\, dx \right\}\\
&\leq I_{\mathcal{K}}^{\{h_k\}}(u, B) + I_{\mathcal{K}}^{\{h_k\}}(u, A\setminus\overline{C}) + 2\eta + \frac{C_0}{M}\sup_{k\in\mathbb{N}}\int_{B_{,1}\setminus\overline{C}_1} (1 + |\nabla_{h_k} u_k|^p + |\nabla_{h_k} v_k|^p)\, dx.
\end{aligned}$$

Taking the limit first as $M \to +\infty$ and then as $\eta \to 0$, we obtain the desired subadditivity property (3.3).

STEP 2: At this point, by a standard diagonal argument, we construct a non-relabeled subsequence $h_k \to 0^+$ and a sequence $(u_k)_k \subset W^{1,p}(\Omega;\mathcal{M})$ satisfying

$$\begin{aligned}
\lim_{k\to+\infty} I^{\{h_k\}}(u_k, \omega) = \inf_{v_k}\Bigg\{ &\limsup_{k\to\infty} I^{h_k}(v_k, A) \;:\; \nabla v_k \rightharpoonup \nabla_\alpha u \text{ in } L^p(\Omega;\mathbb{R}^3),\\
&(\nabla_{h_k} v_k)_k \text{ is bounded in } L^p(\Omega,\mathbb{R}^3),\\
&v_k \to u \text{ uniformly, } v_k(x) = u(x) \text{ if } \operatorname{dist}(u(x_\alpha),\mathcal{K}) > 1 \text{ for a.e. } x\in\Omega \Bigg\}.
\end{aligned}$$

We have that

$$\lim_{k\to+\infty} I^{h_k}(u_k, \omega) = I_{\mathcal{K}}^{\{h_k\}}(u, \omega),$$

simply by construction of the sequences $(h_k)_k$ and $(u_k)_k$. By possibly extracting a further subsequence, it is possible to assume that, for some non-negative Radon measure $\mu \in \mathcal{M}(\omega)$,

$$\left(\int_{(-\frac{1}{2},\frac{1}{2})} f\left(\frac{(\cdot)_\alpha}{h_k}, x_3, \nabla_{h_k} u_k((\cdot)_\alpha, x_3)\right) dx_3\right)\mathcal{L}^2\lfloor\omega \overset{*}{\rightharpoonup} \mu \qquad \text{in } \mathcal{M}(\omega).$$

We further have, by semicontinuity, <?Qry msg="CE: Please check math symbol lfloor and fint in the equation."?>

$$\mu(\omega) \leq \lim_{k\to+\infty} I^{\{h_k\}}(u_k, \omega) = I_{\mathcal{K}}^{\{h_k\}}(u, \omega).$$

We would like to prove that

$$I_{\mathcal{K}}^{\{h_k\}}(u, A) = \mu(A) \qquad \text{for any } A \in \mathcal{A}(\omega).$$

Let us fix $A \in \mathcal{A}(\omega)$ and prove that

$$I_{\mathcal{K}}^{\{h_k\}}(u, A) \leq \mu(A) \qquad \text{for any } A \in \mathcal{A}(\omega).$$

Fix an arbitrary $\eta > 0$; exploiting (3.2), we can select $C \in \mathcal{A}(\omega)$, $C \Subset A$ such that

$$I_{\mathcal{K}}^{\{h_k\}}(u, A\setminus\overline{C}) \leq \eta.$$

Then, by (3.3), we infer that, for any $B \in \mathcal{A}(\omega)$, $C \Subset B \Subset A$,

$$I_{\mathcal{K}}^{\{h_k\}}(u, A) \leq \eta + \limsup_{k\to+\infty} I^{h_k}(u_k, B) \leq \eta + \mu(\overline{B}) \leq \eta + \mu(A).$$

By the arbitrariness of η, we come to the desired conclusion.

Conversely, for any $B \in \mathcal{A}(\omega)$, $B \Subset A$, we may deduce

$$\begin{aligned}
\mu(\omega) &\leq I_{\mathcal{K}}^{\{h_k\}}(u, \omega)\\
&\leq I_{\mathcal{K}}^{\{h_k\}}(u, A) + I_{\mathcal{K}}^{\{h_k\}}(u, \omega\setminus\overline{B})\\
&\leq I_{\mathcal{K}}^{\{h_k\}}(u, A) + \mu(\omega\setminus\overline{B})\\
&\leq I_{\mathcal{K}}^{\{h_k\}}(u, A) + \mu(\omega\setminus B) \leq I_{\mathcal{K}}^{\{h_k\}}(u, A) + \mu(\omega) - \mu(B).
\end{aligned}$$

This finally entails that

$$\mu(B) \leq I_{\mathcal{K}}^{\{h_k\}}(u, A),$$

which yields the desired conclusion by the inner regularity of μ. $\square$

Now we can prove the $\limsup$-inequality for Theorem 1.1.

Proposition 3.1 (Γ-limsup). *For every $p \geq 1$ and $u \in W^{1,p}(\omega;\mathcal{M})$ it holds*

$$I(u) \geq I^0(u),$$

where I and I_0 are defined by (1.2) and (3.1), respectively.

Proof. Let $u \in W^{1,p}(\omega;\mathcal{M})$. Consider $R > 0$ arbitrarily large, set

$$\mathcal{K} := \mathcal{M} \cap B^3(0,R),$$

and consider the sequence $(h_k)_k$ given by Lemma 3.1. It is clear that

$$I^0(u) \leq I_{\mathcal{K}}^{\{h_k\}}(u,\omega).$$

Now, we would like to show that

$$I_{\mathcal{K}}^{\{h_k\}}(u,\omega) \leq \int_\omega \left\{\chi_R(|u|)T f^0_{\mathrm{hom}}(u,\nabla_\alpha u) + \beta(1-\chi_R(|u|))(1+|\nabla_\alpha u|^p)\right\} dx_\alpha, \tag{3.6}$$

where

$$\chi_R(t) = \begin{cases} 1 & \text{for } t \leq R \\ 0 & \text{otherwise} \end{cases}$$

In order to deduce (3.6), it is enough to prove that

$$\frac{dI_{\mathcal{K}}^{\{h_k\}}(u,\cdot)}{d\mathcal{L}^2}(x_0) \leq \chi_R(|u(x_0)|)T f^0_{\mathrm{hom}}(u(x_0),\nabla u(x_0)) + \beta(1-\chi_R(|u(x_0)|)(1+|\nabla u(x_0)|^p),$$

for $\mathcal{L}^2-$a.e. $x_0 \in \omega$.

Let us consider $x_0 \in \omega$ to be a Lebesgue point of u and $\nabla_\alpha u$ such that $u(x_0) \in \mathcal{M}$, $\nabla_\alpha u(x_0) \in [T_{u(x_0)}(\mathcal{M})]^2$, and the Radon-Nykodým derivative of $I_{\mathcal{K}}^{\{h_k\}}(u,\cdot)$ with respect to the Lebesgue measure $\mathcal{L}^2$ exists. We observe that almost every point in ω satisfies these properties. Moreover let us set

$$s_0 := u(x_0) \qquad \text{and} \qquad \xi_0 := \nabla_\alpha u(x_0).$$

Assume first that $s_0 \notin \mathcal{K}$. Then, using (H2), we obtain that

$$\begin{aligned}\frac{dI_{\mathcal{K}}^{\{h_k\}}(u,\cdot)}{d\mathcal{L}^2}(x_0) &= \lim_{\rho\to 0^+}\frac{I_{\mathcal{K}}^{\{h_k\}}(u,Q'(x_0,\rho))}{\rho^2} \leq \limsup_{\rho\to 0^+}\limsup_{k\to+\infty}\rho^{-2}I^{h_k}(u,Q'(x_0,\rho)) \\ &\leq \lim_{\rho\to 0^+}\frac{\beta}{\rho^2}\int_{Q'(x_0,\rho)}(1+|\nabla_\alpha u|^p)\,dx = \beta(1+|\xi_0|^p),\end{aligned} \tag{3.7}$$

which is what we would like to prove.

If instead $s_0 \in \mathcal{K}$, then, fixed an arbitrary $0 < \eta < 1$, Proposition 2.1-(i) entails the existence of $j \in \mathbb{N}$ and $\varphi \in W^{1,\infty}((jQ')_{,1};T_{s_0}(\mathcal{M}))$ such that $\varphi(x_\alpha,x_3) = 0$ for every $(x_\alpha,x_3) \in \partial(jQ') \times (0,1)$ and such that

$$⨍_{(jQ')_{,1}} f(y,\xi_0+\nabla_\alpha\varphi(y)|\nabla_3\varphi(y))\,dy \leq T f^0_{\mathrm{hom}}(s_0,\xi_0)+\eta. \tag{3.8}$$

At this point, let us extend $\varphi((\cdot)_\alpha,x_3)$ to $\mathbb{R}^2$ by j-periodicity and define $\varphi_k(x) := \xi_0 x_\alpha + h_k\varphi\left(\frac{x_\alpha}{h_k},x_3\right)$.

Consider $\mathcal{U}$ to be an open neighborhood of $\mathcal{M}$ such that the nearest point projection $\Pi : \mathcal{U} \to \mathcal{M}$ defines a C^1-mapping; fix $\sigma,\delta_0 \in (0,1)$ such that $B^3(s_0,2\delta_0) \subset \mathcal{U}$ and consider $\delta = \delta(\sigma) \in (0,\delta_0)$ such that

$$|\nabla\Pi(s) - \nabla\Pi(s')| < \sigma \qquad \text{for all } s,s' \in B^3(s_0,\delta_0) \text{ satisfying } |s-s'| < \delta. \tag{3.9}$$

Introduce the cut-off function $\zeta \in C_c^\infty(\mathbb{R}^3;[0,1])$ as

$$\zeta(X) = \begin{cases} 1 & \text{for } X \in B^3(0,\delta/4) \\ 0 & \text{for } X \notin B^3(0,\delta/2) \end{cases} \qquad \text{with} \qquad |\nabla\zeta| \leq \frac{C}{\delta},$$

and define

$$w_k(x) := u(x_\alpha) + h_k\zeta(u(x_\alpha)-s_0)\varphi\left(\frac{x_\alpha}{h_k},x_3\right).$$

We recall that the function u has to be intended in the sense of Remark 3.1, so w_k is well defined as a function of (x_α,x_3). Let $k_0 \in \mathbb{N}$ be such that

$$\max\left\{h_k\|\varphi\|_{L^\infty((jQ')_{,1};\mathbb{R}^3)}\|\nabla\zeta\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}, \frac{2h_k\|\varphi\|_{L^\infty((jQ')_{,1};\mathbb{R}^3)}}{\delta}\right\} < 1, \tag{3.10}$$

for any $k \geq k_0$ and define, still for every $k \geq k_0$,

$$u_k(x) := \Pi(w_k(x)).$$

By (3.10), for a.e. $x \in \Omega$ and all $k \geq k_0$, we deduce that $w_k(x) \in B^3(s_0, \delta)$ when $|u(x_\alpha) - s_0| < \delta/2$, while $w_k(x) = u(x_\alpha)$ when $|u(x_\alpha) - s_0| \geq \delta/2$. Thus u_k is well defined, $(u_k)_k \subset W^{1,p}(\Omega; \mathcal{M})$, and, for a.e. $x \in \Omega$, $u_k(x) = u(x_\alpha)$ when $\mathrm{dist}(u(x_\alpha), \mathcal{K}) > 1$.

In addition to that

$$\begin{aligned}\|u_k - u\|_{L^\infty(\Omega;\mathbb{R}^3)} &= \|\Pi(w_k) - \Pi(u)\|_{L^\infty(\{|u-s_0|<\delta/2\}_{,1};\mathbb{R}^3)} \\ &\leq h_k \|\nabla \Pi\|_{L^\infty(B^3(s_0,\delta_0);\mathbb{R}^3)} \|\varphi\|_{L^\infty((jQ')_{,1};\mathbb{R}^3)} \to 0 \qquad \text{as } k \to +\infty.\end{aligned}$$

By the Chain Rule formula, we can compute

$$\begin{aligned}\nabla_{h_k} u_k(x) = \nabla \Pi(w_k(x)) \Bigg[&\nabla_\alpha u(x_\alpha) + h_k \left(\varphi\left(\frac{x_\alpha}{h_k}, x_3\right) \otimes \nabla \zeta(u(x_\alpha) - s_0) \right) \nabla_\alpha u(x_\alpha) \\ &+ h_k \zeta(u(x_\alpha) - s_0) \nabla_{h_k} \left(\varphi\left(\frac{x_\alpha}{h_k}, x_3\right)\right) \Bigg],\end{aligned}$$

which, in turn, entails

$$\begin{aligned}|\nabla_{h_k} u_k(x)| \leq \|\nabla \Pi\|_{L^\infty(B^3(s_0,\delta_0);\mathbb{R}^3)} \Bigg[&\left(1 + h_k \|\varphi\|_{L^\infty((jQ')_{,1};\mathbb{R}^3)} \|\nabla \zeta\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}\right) |\nabla_\alpha u(x_\alpha)| \\ &+ \|\nabla \varphi\|_{L^\infty((jQ')_{,1};\mathbb{R}^3)} \Bigg].\end{aligned}$$

Once more (3.10) entails that, for any $k \geq k_0$ and for some constant C_0 depending on $s_0, \xi_0, \delta_0, \eta$ and independent of x and k,

$$|\nabla_{h_k} u_k(x)| \leq C_0(|\nabla_\alpha u(x_\alpha) - \xi_0| + 1). \tag{3.11}$$

Thus, we can conclude that the sequence $(u_k)_k$ is uniformly bounded in $W^{1,p}(\Omega; \mathbb{R}^3)$ so that $\nabla u_k \rightharpoonup \nabla_\alpha u$ in $L^p(\Omega; \mathbb{R}^3)$. Moreover, we observe that $|\nabla_{h_k} u_k| \leq 2C_0$ a.e. in $\{|\nabla_\alpha u - \xi_0| < \sigma\}_{,1}$ and

$$\|\nabla_{h_k} \varphi_k\|_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} \leq |\xi_0| + \|\nabla \varphi\|_{L^\infty((jQ')_{,1};\mathbb{R}^3)}$$

Set

$$M := \max\{2C_0, |\xi_0| + \|\nabla \varphi\|_{L^\infty((jQ')_{,1};\mathbb{R}^3)}\}, \tag{3.12}$$

depending only on s_0, ξ_0, δ_0 and η, so that

$$|\nabla_{h_k} u_k| \leq M \qquad \text{and} \qquad |\nabla_{h_k} \varphi_k| \leq M \qquad \text{a.e. in } \{|\nabla_\alpha u - \xi_0| < \sigma\}_{,1}. \tag{3.13}$$

At this point, for a.e.

$$x \in \{|u - s_0| < \delta/4\}_{,1} \cap \{|\nabla_\alpha u - \xi_0| < \sigma\}_{,1},$$

we have $\zeta(u(x) - s_0) = 1$ and

$$\begin{aligned}|\nabla_{h_k} u_k(x) - \nabla_{h_k} \varphi_k(x)| \leq\ & |\nabla \Pi(w_k) \nabla_\alpha u(x_\alpha) - \xi_0| \\ &+ |h_k \nabla \Pi(w_k) \nabla_{h_k}(\varphi(x_\alpha/h_k, x_3)) - h_k \nabla_{h_k}(\varphi(x_\alpha/h_k, x_3))| \\ \leq\ & |\nabla \Pi(w_k) - \nabla \Pi(s_0)|\, |\nabla_\alpha u(x_\alpha)| + |\nabla \Pi(s_0)| |\nabla_\alpha u(x_\alpha) - \xi_0| \\ &+ |\nabla \Pi(w_k) - \nabla \Pi(s_0)|\, \|\nabla \varphi\|_{L^\infty((jQ')_{,1};\mathbb{R}^3)};\end{aligned}$$

where, in the last inequality, we have used the fact that, since $\nabla \varphi(y) \in [T_{s_0}(\mathcal{M})]^3$ for a.e. $y \in \mathbb{R}^3$, it hold $\nabla \Pi(s_0) \nabla \varphi(y) = \nabla \varphi(y)$ and $\nabla \Pi(s_0) \xi_0 = \xi_0$.

Now, taking into account (3.9) and the fact that $|w_k - s_0| < \delta$ a.e. in $\{|u - s_0| < \delta/4\}_{,1} \cap \{|\nabla u - \xi_0| < \sigma\}_{,1}$, we deduce

$$|\nabla_{h_k} u_k(x) - \nabla_{h_k} \varphi_k(x)| \leq (|\nabla_\alpha u(x_\alpha)| + |\nabla \Pi(s_0)| + \|\nabla \varphi\|_{L^\infty((jQ')_{,1};\mathbb{R}^3)})\sigma \leq C_1 \sigma, \tag{3.14}$$

a.e. in $\{|u - s_0| < \delta/4\}_{,1} \cap \{|\nabla_\alpha u - \xi_0| < \sigma\}_{,1}$, where $C_1 = C_1(s_0, \xi_0, \delta_0, \eta) > 0$ is a constant independent of σ, k and x.

Now we estimate

$$\begin{aligned}&\frac{d I_{\mathcal{K}}^{\{h_k\}}(u, \cdot)}{d\mathcal{L}^2}(x_0) \\ &= \lim_{\rho \to 0^+} \frac{I_{\mathcal{K}}^{\{h_k\}}(u, Q'(x_0, \rho))}{\rho^2} \\ &\leq \limsup_{\rho \to 0^+} \limsup_{k \to +\infty} \frac{1}{\rho^2} \int_{Q'(x_0,\rho)_{,1}} f\left(\frac{x_\alpha}{h_k}, x_3, \nabla_{h_k} u_k\right) dx\end{aligned}$$

$$\begin{aligned}
&\leq \limsup_{\rho\to 0^+}\limsup_{k\to+\infty}\frac{1}{\rho^2}\int_{Q'(x_0,\rho)_{,1}\cap\{|u-s_0|\geq\delta/4\}_{,1}} f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}u_k\right)dx\\
&\quad+\limsup_{\rho\to 0^+}\limsup_{k\to+\infty}\frac{1}{\rho^2}\int_{Q'(x_0,\rho)_{,1}\cap\{|u-s_0|<\delta/4\}_{,1}\cap\{|\nabla u-\xi_0|<\sigma\}_{,1}} f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}u_k\right)dx\\
&\quad+\limsup_{\rho\to 0^+}\limsup_{k\to+\infty}\frac{1}{\rho^2}\int_{Q'(x_0,\rho)_{,1}\cap\{|u-s_0|<\delta/4\}_{,1}\cap\{|\nabla u-\xi_0|\geq\sigma\}_1} f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}u_k\right)dx\\
&=:\ I_1+I_2+I_3.
\end{aligned}\tag{3.15}$$

The bound (3.11), the p-growth condition (H2) and the selected choice of x_0 yield

$$\begin{aligned}
I_1 &\leq C\limsup_{\rho\to 0^+}\frac{1}{\rho^2}\int_{Q'(x_0,\rho)\cap\{|u-s_0|\geq\delta/4\}}(1+|\nabla_\alpha u(x_\alpha)-\xi_0|^p)\,dx\\
&\leq C\limsup_{\rho\to 0^+}\fint_{Q'(x_0,\rho)}|\nabla_\alpha u(x_\alpha)-\xi_0|^p\,dx+\frac{4C}{\sigma}\limsup_{\rho\to 0^+}\fint_{Q'(x_0,\rho)}|u(x_\alpha)-s_0|\,dx=0,
\end{aligned}$$

while

$$\begin{aligned}
I_3 &\leq C\limsup_{\rho\to 0^+}\frac{1}{\rho^2}\int_{Q'(x_0,\rho)\cap\{|u-s_0|<\delta/4\}\cap\{|\nabla_\alpha u-\xi_0|\geq\sigma\}}(1+|\nabla_\alpha u(x_\alpha)-\xi_0|^p)\,dx\\
&\leq C\limsup_{\rho\to 0^+}\fint_{Q'(x_0,\rho)}|\nabla_\alpha u(x_\alpha)-\xi_0|^p\,dx\\
&\quad+\frac{C}{\sigma}\limsup_{\rho\to 0^+}\fint_{Q'(x_0,\rho)}|\nabla_\alpha u(x_\alpha)-\xi_0|\,dx=0,
\end{aligned}\tag{3.16}$$

Concerning integral I_2, since, for a.e. $y\in\mathbb{R}^3$, the function $f(y,\cdot)$ is continuous, it is uniformly continuous on $B^{3\times 3}(0,M)$ where $M>0$ has been introduced in (3.12).

Define

$$\omega(y,t):=\sup\{|f(y,\xi)-f(y,\xi')|\ :\ \xi,\xi'\in B^{3\times 3}(0,M)\ \text{ and }\ |\xi-\xi'|\leq t\},$$

to be the modulus of continuity of $f(y,\cdot)$ over $B^{3\times 3}(0,M)$. It is not difficult to see that $\omega(y,\cdot)$ is increasing, continuous and $\omega(y,0)=0$; on the other hand $\omega(\cdot,t)$ is measurable, since the supremum can be restricted to all admissible ξ and ξ' having rational entries, and 1-periodic.

Accounting on (3.13) and (3.14), we are able to estimate for a.e. $x\in Q'(x_0,\rho)_{,1}\cap\{|u-s_0|<\delta/4\}_{,1}\cap\{|\nabla_\alpha u-\xi_0|<\sigma\}_{,1}$

$$\left|f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}u_k(x)\right)-f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}\varphi_k(x)\right)\right|\leq\omega\left(\frac{x_\alpha}{h_k},x_3,C_1\sigma\right).$$

Integrating over $x\in Q'(x_0,\rho)_{,1}\cap\{|u-s_0|<\delta/4\}_{,1}\cap\{|\nabla_\alpha u-\xi_0|<\sigma\}_{,1}$ and taking the limit as $k\to+\infty$, we obtain

$$\begin{aligned}
&\limsup_{k\to+\infty}\frac{1}{\rho^2}\int_{Q'(x_0,\rho)_{,1}\cap\{|u-s_0|<\delta/4\}_{,1}\cap\{|\nabla_\alpha u-\xi_0|<\sigma\}_{,1}}\left|f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}u_k(x)\right)\right.\\
&\left.-f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}\varphi_k(x)\right)\right|dx\\
&\leq\limsup_{k\to+\infty}\frac{1}{\rho^2}\int_{Q'(x_0,\rho)_{,1}}\omega\left(\frac{x_\alpha}{h_k},x_3,C_1\sigma\right)dx\\
&=\int_Q\omega(y,C_1\sigma)\,dy,
\end{aligned}$$

where we have exploited the Riemann-Lebesgue Lemma and where we used the fact that $y\mapsto\omega(y,C_1\sigma)$ is a measurable 1-periodic function.

The Dominated Convergence Theorem and the fact that $\omega(y,0)=0$ for every $y\in\mathbb{R}^3$ yield

$$\lim_{\sigma\to 0^+}\int_Q\omega(y,C_1\sigma)\,dy=0.\tag{3.17}$$

Summing up,

$$I_2\leq\limsup_{\rho\to 0^+}\limsup_{k\to+\infty}\frac{1}{\rho^2}\int_{Q'(x_0,\rho)_{,1}}f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}\varphi_k(x)\right)dx+\int_{Q'}\omega(y,C_1\sigma)\,dy.\tag{3.18}$$

By the definition of φ_k and once more the Riemann-Lebesgue Lemma, we are able to deduce from (3.8) that

$$\begin{aligned}
&\limsup_{\rho\to 0^+}\limsup_{k\to+\infty}\frac{1}{\rho^2}\int_{Q'(x_0,\rho)_{,1}} f\left(\frac{x_\alpha}{h_k},x_3,\nabla_{h_k}\varphi_k(x)\right)dx\\
&=\fint_{(jQ')_{,1}} f(y,\xi_0+\nabla_\alpha\varphi(y)|\nabla_3\varphi(y))\,dy\\
&\le Tf^0_{\mathrm{hom}}(s_0,\xi_0)+\eta.
\end{aligned}\tag{3.19}$$

Finally, collecting (3.15)–(3.16), (3.18) and also (3.19), we are able to obtain

$$\frac{dI^{\{h_k\}}_{\mathcal{K}}(u,\cdot)}{d\mathcal{L}^2}(x_0)\le Tf^0_{\mathrm{hom}}(s_0,\xi_0)+\int_Q\omega(y,C_1\sigma)\,dy+\eta.$$

The thesis now follows by sending first $\sigma\to 0$ (exploiting (3.17)) and then $\eta\to 0$.

Once (3.6) is obtained, we can conclude by considering a sequence $R_j\to+\infty$ as $j\to+\infty$. By (3.7) and (3.19), since $\chi_{R_j}\to 1$ pointwise, we can deduce from Dominated Convergence Theorem together with the p-growth of Tf^0_{hom}(see Proposition 2.1(iii)) that

$$\begin{aligned}
I^0(u)&\le\limsup_{j\to+\infty}\int_\Omega\left\{\chi_{R_j}(|u|)Tf^0_{\mathrm{hom}}(u,\nabla_\alpha u)+\beta(1-\chi_{R_j}(|u|))(1+|\nabla_\alpha u|^p)\right\}dx\\
&\le\int_\Omega Tf^0_{\mathrm{hom}}(u,\nabla_\alpha u)\,dx,
\end{aligned}$$

which is what we wanted to prove. □

Now we prove the $\liminf$-inequality for Theorem 1.1, i.e. for $p>1$.

Proposition 3.2 (Γ-liminf). *For every $p>1$ and $u\in W^{1,p}(\omega;\mathcal{M})$ it holds*

$$I(u)\le I^0(u),$$

where I is defined in (1.2), while I^0 is defined in (3.1).

Proof. Before proving the result we recall that all the operations of sum and difference between the functions $u_n, v_{n,k}, v_k, w_k$ and u must be intended in the sense of Remark 3.1.

STEP 1. Fix $u\in W^{1,p}(\omega;\mathcal{M})$. We consider a recovery sequence $(u_n)_n\subset W^{1,p}(\Omega;\mathcal{M})$ related to $I^0(u,\omega)$. We define the sequence of non-negative Radon measure

$$\mu_n:=\left(\int_{(-\frac12,\frac12)} f\left(\frac{(\cdot)_\alpha}{h_n},x_3,\nabla_{h_n}u_n\right)dx_3\right)\mathcal{L}^2\lfloor\omega.$$

Up to a subsequence, there exists a Radon measure $\mu\in\mathcal{M}(\omega)$ such that $\mu_n\rightharpoonup^*\mu$ in $\mathcal{M}(\omega)$. By Lebesgue Differentiation Theorem we can split μ into the sum of two mutually disjoint non-negative Radon measure μ^a and μ^s. In particular, $\mu^a<<\mathcal{L}^2$, while μ^s is singular with respect to $\mathcal{L}^2$. By definition, $\mu^a(\Omega)\le\mu(\Omega)\le I^0(u)$ so we want to prove that

$$\frac{d\mu}{d\mathcal{L}^2}(y_0)\ge Tf^0_{\mathrm{hom}}(u(y_0),\nabla_\alpha u(y_0))\quad\text{for }\mathcal{L}^2-\text{ a.e. }y_0\in\omega.$$

Let $y_0\in\omega$ be a Lebesgue point for u and $\nabla_\alpha u$ and a point of approximate differentiability for u, i.e. such that $u(y_0)\in\mathcal{M}$ and $\nabla_\alpha u(y_0)\in[T_{u(y_0)}(\mathcal{M})]^2$, and such that the Radon-Nykodým derivative of μ with respect to $\mathcal{L}^2$ exists and it is finite. We define $s_0:=u(y_0)$ and $\xi_0:=\nabla_\alpha u(y_0)$ and we consider a vanishing sequence $(\rho_k)_k\subset(0,+\infty)$ such that $\mu(\partial Q'(y_0,\rho_k))=0$ for every $k\in\mathbb{N}$. By definition (2.1) of $\bar f$ we get

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^2}(y_0)&=\lim_{k\to+\infty}\frac{\mu(Q'(y_0,\rho_k))}{\rho_k^2}\\
&=\lim_{k\to+\infty}\lim_{n\to+\infty}\frac{\mu_n(Q'(y_0,\rho_k)_1)}{\rho_k^2}\\
&=\lim_{k\to+\infty}\lim_{n\to+\infty}\int_{Q'_{,1}} f\left(\frac{y_0+\rho_k y_\alpha}{h_n},y_3,\nabla_{h_n}u_n(y_0+\rho_k y_\alpha,y_3)\right)dy\\
&=\lim_{k\to+\infty}\lim_{n\to+\infty}\int_{Q'_{,1}}\bar f\left(\frac{y_0+\rho_k y_\alpha}{h_n},y_3,u_n(y_0+\rho_k y_\alpha,y_3),\nabla_{h_n}u_n(y_0+\rho_k y_\alpha,y_3)\right)dy\\
&=\lim_{k\to+\infty}\lim_{n\to+\infty}\int_{Q'_{,1}}\bar f\left(\frac{y_0+\rho_k y_\alpha}{h_n},y_3,s_0+v_{n,k}(y),\nabla_{\frac{h_k}{\rho_k}}v_{n,k}(y)\right)dy,
\end{aligned}$$

with $v_{n,k}(y):=\frac{(u_n(y_0+\rho_k y_\alpha,y_3)-s_0)}{\rho_k}$.

Since y_0 is a point of approximate differentiability and $u_n \to u$ in $L^p(\Omega, \mathbb{R}^3)$ it follows that

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_{Q'_{,1}} |v_{n,k}(y) - \xi_0 y_\alpha|^p dy = \lim_{k\to\infty}\int_{Q'(y_0,\rho_k)} \frac{|u(y) - s_0 - \xi_0(y_\alpha - y_0)|^p}{\rho^{2+p}} dy_\alpha = 0.$$

Therefore it is possible to find a diagonal sequence $h_k := h_{n_k} < \rho_k^2$ such that, by setting $v_k(y) := v_{n_k,k}(y)$ with $y \in \Omega$, $v_0(y_\alpha) := \xi_0 y_\alpha$ with $y_\alpha \in \omega$, then $v_k \to v_0$ in $L^p(Q'_1;\mathbb{R}^3)$ and

$$\frac{d\mu}{d\mathcal{L}^2}(y_0) = \lim_{k\to+\infty}\int_{Q'_{,1}} \bar{f}\left(\frac{y_0 + \rho_k y_\alpha}{h_k}, y_3, s_0 + \rho_k v_k(y), \nabla_{\frac{h_k}{\rho_k}} v_k(y)\right) dy. \tag{3.20}$$

At this point, we observe that $(\nabla_{h_k} v_k)_k$ is bounded in $L^p(Q'_{,1};\mathbb{R}^{3\times3})$ thanks to the coercivity condition (H2). By using the Decomposition Theorem [9, Theorem 1.1] (see also [13]), it is possible to find a sequence $(\bar{v}_k)_k \subset W^{1,\infty}(Q'_{,1};\mathbb{R}^3)$ such that $\bar{v}_k = v_0$ on a neighborhood of $\partial(Q')_{,1}$, $\bar{v}_k \to v_0$ in $L^p(Q'_1;\mathbb{R}^3)$, the sequence of gradients $(|\nabla_{h_k}\bar{v}_k|^p)_k$ is equi-integrable, and

$$\begin{aligned}
&\lim_{k\to+\infty}\int_{Q'_{,1}} \bar{f}\left(\frac{y_0 + \rho_k y_\alpha}{h_k}, y_3, s_0 + \rho_k v_k(y), \nabla_{\frac{h_k}{\rho_k}} v_k(y)\right) dy \\
&\geq \limsup_{k\to+\infty}\int_{Q'_{,1}} \bar{f}\left(\frac{y_0 + \rho_k y_\alpha}{h_k}, y_3, s_0 + \rho_k v_k(y), \nabla_{\frac{h_k}{\rho_k}} \bar{v}_k(y)\right) dy.
\end{aligned} \tag{3.21}$$

STEP 2. Let us set

$$\frac{y_0}{h_k} = m_k + s_k \qquad \text{with } m_k \in \mathbb{Z}^2 \text{ and } s_k \in [0,1)^2.$$

We can introduce

$$x_k := \frac{h_k}{\rho_k} s_k \to 0 \qquad \text{and} \qquad \delta_k := \frac{h_k}{\rho_k} \to 0.$$

We can exploit the 1-periodicity of $\bar{f}$ with respect to its first variable, (3.20) and (3.21) to get

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^2}(x_0) &\geq \limsup_{k\to+\infty}\int_{Q'_{,1}} \bar{f}\left(\frac{x_k + y_\alpha}{\delta_k}, y_3, s_0 + \rho_k v_k(y), \nabla_{\delta_k}\bar{v}_k(y)\right) dy \\
&\geq \limsup_{k\to+\infty}\int_{x_k + Q'_{,1}} \bar{f}\left(\frac{y_\alpha}{\delta_k}, y_3, s_0 + \rho_k v_k(y_\alpha - x_k, y_3), \nabla_{\delta_k}\bar{v}_k(y_\alpha - x_k, y_3)\right) dy.
\end{aligned} \tag{3.22}$$

At this point, we extend v_k as its limit (up to fixing v_k at the boundary of $\partial\omega \times (-1,1)$) and $\bar{v}_k$ by v_0 to $\mathbb{R}^2 \times (-1,1)$. As long as $x_k \to 0$, we deduce that $\mathcal{L}^3((Q'_{,1} - x_k)\Delta Q'_{,1}) \to 0$, and the equi-integrability of $(|\nabla_{h_k}\bar{v}_k|^p)_k$ together with the p-growth condition (H2) implies

$$\begin{aligned}
&\int_{Q'_{,1}\Delta(x_k + Q'_{,1})} \bar{f}\left(\frac{y_\alpha}{\delta_k}, y_3, s_0 + \rho_k v_k(y_\alpha - x_k, y_3), \nabla_{\delta_k}\bar{v}_k(y_\alpha - x_k, y_3)\right) dy \\
&\leq \beta' \int_{Q'_{,1}\Delta(Q'_{,1}) - x_k} (1 + |\nabla_{\delta_k}\bar{v}_k|^p)\, dy \to 0.
\end{aligned}$$

Therefore (3.22) entails

$$\frac{d\mu}{d\mathcal{L}^2}(y_0) \geq \limsup_{k\to+\infty}\int_{Q'_{,1}} \bar{f}\left(\frac{y_\alpha}{\delta_k}, y_3, s_0 + \rho_k w_k, \nabla_{\delta_k}\bar{w}_k\right) dy,$$

where $w_k(y) := v_k(y_\alpha - x_k, y_3)$ and $\bar{w}_k(y) := \bar{v}_k(y_\alpha - x_k, y_3)$ converge to v_0 in $L^p(Q'_{,1};\mathbb{R}^3)$, and $(|\nabla\bar{w}_k|^p)_k$ is equi-integrable as well.

STEP 3. Fixed $M > 0$, we denote by $E_{M,k}$ the set

$$E_{M,k} := \left\{x \in Q'_{,1} : |\nabla_{h_k} w_k| \leq M\right\}.$$

By Chebyschev inequality, we have that $\mathcal{L}^3(Q'_1 \setminus E_{M,k}) \leq \frac{C}{M^p}$ for some constant $C > 0$. By Scorza-Dragoni Theorem, fixed $\eta > 0$ there exists a compact set $K_\eta \subset \overline{Q'_{,1}}$ such that $\mathcal{L}^3(\overline{Q'_{,1}} \setminus K_\eta) \leq \eta$ and such that $f : K_\eta \times \mathbb{R}^{3\times3} \to \mathbb{R}$ is continuous. It follows that $\bar{f}(\cdot, s, \cdot) : K_\eta \times B^{3\times3}(0,M) \to \mathbb{R}$ is uniformly continuous for every $s \in \mathbb{R}^3$. Moreover, the function

$$\Psi_{\eta,M}(t) := \sup\left\{|f(x,\xi) - f(x,\zeta)| : x \in K_\eta,\ \xi,\zeta \in B^{3\times3}(0,M), |\xi - \zeta| \leq t\right\},$$

is continuous, takes value 0 for $t = 0$ and is bounded. By definition of $\Psi_{\eta,M}$ and $\mathbf{P}_s$ and [21, Proposition 2.32] it follows that for every $x \in K_\eta$, $\xi \in B^{3\times3}(0,M)$ and $s_1, s_2 \in \mathbb{R}^3$ holds

$$\begin{aligned}|\bar{f}(x,s_1,\xi) - \bar{f}(x,s_2,\xi)| &\le \Psi_{\eta,M}(|\mathbf{P}_{s_1}(\xi) - \mathbf{P}_{s_2}(\xi)|) + C_M|\mathbf{P}_{s_1}(\xi) - \mathbf{P}_{s_2}(\xi)| \\ &\le \Psi_{\eta,M}(M|\mathbf{P}_{s_1} - \mathbf{P}s_2|) + C_M|\mathbf{P}_{s_1} - \mathbf{P}_{s_2}| \\ &:= \tilde{\Psi}(|\mathbf{P}_{s_1} - \mathbf{P}_{s_2}|),\end{aligned}$$

where $|\mathbf{P}_{s_1} - \mathbf{P}s_2|$ denotes the operator norm of $\mathbf{P}_{s_1} - \mathbf{P}_{s_2}$. By the previous inequality, it follows that if we denote

$$K_\eta^{per} := \bigcup_{j\in\mathbb{Z}} (j + K_\eta),$$

then

$$|\bar{f}(x,s_1,\xi) - \bar{f}(x,s_2,\xi)| \le \tilde{\Psi}(|\mathbf{P}_{s_1} - \mathbf{P}s_2|),$$

for every $x \in K_\eta^{per}$, $\xi \in B^{3\times3}(0,M)$ and $s_1, s_2 \in \mathbb{R}^3$. From the previous inequality it follows that

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^2}(y_0) &\ge \limsup_{k\to+\infty} I^{h_k}(\bar{w}_k, Q_1') \\ &\ge \limsup_{k\to+\infty} \int_{E_{M,k}\cap(\delta_k K_\eta^{per})} \bar{f}\left(\frac{y_\alpha}{\delta_k}, y_3, s_0, \nabla_{\delta_k}\bar{w}_k\right) dy \\ &- \limsup_{k\to+\infty} C_M \int_{Q'_{,1}} \tilde{\Psi}_{\eta,M}(|\mathbf{P}_{s_0+\rho_k w_k(y)} - \mathbf{P}_{s_0}|) dy.\end{aligned}$$

Since $\tilde{\Psi}_{\eta,M}$ is continuous, bounded and $\tilde{\Psi}_{\eta,M}(0) = 0$ and since $|\mathbf{P}_{s_0+\rho_k w_k(y)} - \mathbf{P}_{s_0}| \to 0$ as $k \to \infty$, then the last term in the previous inequality is also 0. It follows that

$$\frac{d\mu}{d\mathcal{L}^2}(y_0) \ge \limsup_{k\to+\infty} \int_{E_{M,k}\cap\delta_k K_\eta^{per}} \bar{f}\left(\frac{y_\alpha}{\delta_k}, y_3, s_0, \nabla_{\delta_k}\bar{w}_k\right) dy.$$

From the p-growth of $\bar{f}$ and from Riemann-Lebesgue Lemma we get that

$$\begin{aligned}&\limsup_{k\to+\infty} \int_{E_{M,k}\setminus\delta_k K_\eta^{per}} \bar{f}\left(\frac{y_\alpha}{\delta_k}, y_3, s_0, \nabla_{\delta_k}\bar{w}_k\right) dy \\ &\le \limsup_{k\to+\infty} C(1+M^p)\mathcal{L}^3(Q_1' \setminus \delta_k K_\eta^{per}) \\ &= C(1+M^p)\mathcal{L}^3(Q_1' \setminus K_\eta) \\ &\le C(1+M^p)\eta.\end{aligned}$$

From the previous inequality we deduce that

$$\frac{d\mu}{d\mathcal{L}^2}(y_0) \ge \limsup_{k\to+\infty} \int_{E_{M,k}} \bar{f}\left(\frac{y_\alpha}{\delta_k}, y_3, s_0, \nabla_{\delta_k}\bar{w}_k\right) dy - C(1+M^p)\eta.$$

Since η is arbitrary, then for $\eta \to 0$ we get

$$\frac{d\mu}{d\mathcal{L}^2}(y_0) \ge \limsup_{k\to+\infty} \int_{E_{M,k}} \bar{f}\left(\frac{y_\alpha}{\delta_k}, y_3, s_0, \nabla_{\delta_k}\bar{w}_k\right) dy. \tag{3.23}$$

On the other hand, by construction, $\mathcal{L}^3(Q'_{,1} \setminus E_{M,k}) \to 0$ uniformly with respect to k as $M \to +\infty$. Since $(|\nabla_{h_k}\bar{w}_k|^p)_k$ is equi-integrable in Ω, then from the p-growth of $\bar{f}$ we get for $M \to +\infty$

$$\sup_k \int_{Q'_{,1}\setminus E_{M,k}} \bar{f}\left(\frac{y_\alpha}{\delta_k}, y_3, s_0, \nabla_{\delta_k}\bar{w}_k\right) dy \le \sup_k C \int_{Q'_{,1}\setminus E_{M,k}} \left(1 + |\nabla_{\delta_k}\bar{w}_k|^p\right) dy \to 0.$$

From this limit and from (3.23) we conclude that

$$\frac{d\mu}{d\mathcal{L}^2}(y_0) \ge \limsup_{k\to+\infty} \int_{Q'_{,1}} \bar{f}\left(\frac{y_\alpha}{\delta_k}, y_3, s_0, \nabla_{\delta_k}\bar{w}_k\right) dy.$$

Using now [12, Theorem 4.2] we get

$$\frac{d\mu}{d\mathcal{L}^2}(y_0) \ge \int_{Q'} \bar{f}^0_{\rm hom}(u(y_0), \nabla_\alpha u(y_0)) dy = \bar{f}^0_{\rm hom}(u(y_0), \nabla_\alpha u(y_0)).$$

By Proposition 2.1 it follows that

$$\bar{f}^0_{\rm hom}(u(y_0), \nabla_\alpha u(y_0)) = T f^0_{\rm hom}(u(y_0), \nabla_\alpha u(y_0)).$$

□

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