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Article

Some Further Insight into the Sturm–Liouville Theory

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Abstract

Some classical texts on the Sturm–Liouville equation $(p(x)y')' - q(x)y + \lambda\rho(x)y = 0$ are revised to highlight further properties of its solutions. Often, in the treatment of the ensuing integral equations, $\rho = const$ is assumed (and, further, $\rho = 1$). Instead, here we preserve $\rho(x)$ and make a simple change only of the independent variable that reduces the Sturm–Liouville equation to $y'' - q(x)y + \lambda\rho(x)y = 0$. We show that many results are identical with those with $\lambda\rho - q = const$. This is true in particular for the mean value of the oscillations and for the analog of the Riemann–Lebesgue Theorem. From a mechanical point of view, what is now the total energy is not a constant of the motion, and nevertheless, the equipartition of the energy is still verified and, at least approximately, it does so also for a class of complex λ. We provide here many detailed properties of the solutions of the above equation, with $\rho = \rho(x)$. The conclusion, as we may easily infer, is that, for large enough λ, locally, the solutions are trigonometric functions. We give the proof for the closure of the set of solutions through the Phragmén–Lindelöf Theorem, and show the separate dependence of the solutions from the real and imaginary components of λ. The particular case of $q(x) = \alpha\rho(x)$ is also considered. A direct proof of the uniform convergence of the Fourier series is given, with a statement identical to the classical theorem. Finally, the proof of J. von Neumann of the completeness of the Laguerre and Hermite polynomials in non-compact sets is revisited, without referring to generating functions and to the Weierstrass Theorem for compact sets. The possibility of the existence of a general integral transform is then investigated.

Keywords: anharmonic oscillators; equipartition; fourier series; integral transforms; completeness of Laguerre and Hermite polynomials

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1. Introduction

The Sturm–Liouville equation has been studied in almost every possible detail, and the several impressive properties of its solutions have been elucidated. Although a long-time established classical in the theory of differential equations, its mathematical facets, as well as applications, are diverse and still studied in recent times [1–3], with extensions also going beyond the original formulation of the problem [4–8]. A straightforward view on the classical theory is offered by the theory of integral equations. Nevertheless, we still deem it interesting to highlight some simple ideas and results that help to further clarify the simplicity of the theory. In particular, via the Implicit Function Theorem, we obtain a result that reinforces Sturm's statement about the upper and lower bounds for the number of zeros of the solutions. The validity of the equipartition of the energy is stated for $\rho = \rho(x)$. Its extension to λ complex is set in Appendix B.

The eigenfunctions $y(\lambda_n, x)$, satisfying the conditions $y(\lambda, 0) = 0,\ y'(\lambda, 0) = 1$, are similar, locally, for large λ, to $\dfrac{\sin n\pi x}{n\pi}$, with $n\pi \sim \sqrt{\lambda\rho(x_0)}$, x in the neighbourhood of x_0; see Section 3.

For the eigenvalues problem, with the exception of the non-compact sets present in Section 7, we restricted ourselves everywhere to the eigenfunctions $y(\lambda_n, x)$, $x \in [0,1]$, satisfying null boundary conditions, i.e., $y(\lambda_n, 0) = y(\lambda_n, 1) = 0$. The treatment of the other two classical boundary conditions for the eigenvalue problem is straightforward.

The Sturm–Louville equation can be reduced, via a Liouville transform, to $y'' - q(x)y + \lambda\rho y = 0$, with $\rho = const$ [9]. However, the transform is not very simple, also changing the independent variable, and the results, in turn, must be translated back to the original variables. This simplification does not compromise the generality of the treatment entirely. This is very useful for the theory, but in this way, we lose some information about the original problem, since the solutions do not maintain track of the evolution of the original function. It is not difficult to understand that, with $\rho = const$, the results will tend asymptotically to those of $y'' + \lambda\rho y = 0$. We prefer to maintain $\rho = \rho(x)$, and make only a simple transform that reduces the equation to $y'' - q(x)\,y + \lambda\rho(x)\,y = 0$, i.e., an anharmonic oscillator with a non-constant elastic parameter. We prove what can be seen and easily tested in numerical simulations. Then, having exact results, there comes the possibility to define rigorously the behaviour of a set of anharmonic oscillators. It is surprising that, even if the Sturm–Liouville equation has been extensively studied also in the complex variable, no use has been made of the holomorphic theory to obtain the closure of the set of eigenfunctions in the space of square integrable functions $L^2([0,1])$. We shall use the following notation for intervals in the real line:

$$f \in L^p([a,b]) \quad \Leftrightarrow \quad \int_a^b |f(x)|^p dx < \infty$$

(making explicit references solely to the cases $p = 1$ or $p = 2$). In Section 7, $L^2(0,\infty)$ and $L^2(-\infty,\infty)$ will refer to the square integrable functions:

$$f \in L^2(0,\infty) \quad \Leftrightarrow \quad \int_0^\infty |f(x)|^2 dx < \infty, \qquad f \in L^2(-\infty,\infty) \quad \Leftrightarrow \quad \int_{-\infty}^\infty |f(x)|^2 dx\,.$$

To recall the definition of the closure property, we proceed to briefly illustrate our findings. The solution $y(\lambda, x)$ is a twice continuously differentiable function in the variable x, i.e., $y(\lambda, x)$ is $C^2([0,1])$, and holomorphic in the parameter $\lambda \in \mathbb{C}$; see Theorem 1. The procedure, particularly simple, is the following. From the condition

$$\int_0^1 \rho(x)\varphi(x)\,y(\lambda_n, x)\,dx = 0,$$

valid for all the eigenfunctions, we deduce, through a Phragmén–Lindelöf theorem, that

$$\int_0^1 \rho(x)\varphi(x)\,y(\lambda, x)\,dx = 0$$

is identically verified for $\lambda \in \mathbb{C}$, and then it follows that the function $\varphi(x) \in L^2([0,1])$ has to be the null function. The latter implication is precisely the definition of the closure of the set of functions $\{y(\lambda_n, x)\}$ in $L^2([0,1])$. In the case of the simplest eigenvalue problem, computations are easily made and Liouville's theorem is enough to deduce the closure of the set of functions $\{\sin(n\pi x)\}$.

The closure of the Hermite and Laguerre polynomials in non-compact sets, with convenient weight functions, is well established. We give an alternative proof of it that does not use the generating functions of such polynomials, nor the Weierstrass Theorem for compact sets, as in the celebrated proof attributed to J. von Neumann.

Also, the possibility to define general integral transforms is considered, with the conclusion that, apart from the Fourier and Hankel transforms, this possibility does not exist.

The parameter λ in Sections 2 and 4 and in Appendix B is complex, while in the rest of the paper it is real and, mostly, greater than zero. To make the reading of the paper easier, some proofs are postponed to the Appendices A, B, C, D.

At various points, in addition to the p-integrable functions mentioned above, we shall use the following notations for real-valued functions on intervals I in the real line $\mathbb{R}$.

- $f(x) \in C^n(I)$ denotes an n-th times continuously differentiable function, $f : I \to \mathbb{R}$. For the particular case of the solutions $y(\lambda, x)$, where λ is a complex parameter, we shall also consider the case $f : I \to \mathbb{C}$.
- $f(x) \in BV(I)$ denotes a bounded variation function, $f : I \to \mathbb{R}$. In this paper, we shall only refer to such functions when the interval is $I = [0, 1]$.
- Absolutely continuous functions and Lipschitz continuous functions (shortly Lipschitz functions) shall also be intended for $f : I \to \mathbb{R}$, $I = [0, 1]$.

In fact, with the exception of Sections 7 and 8, when omitted, it will always be $I = [0, 1]$.

2. Present Analysis

Many classical books consider, in detail, the properties of the solutions of the Sturm–Liouville equation, for example [1,10]. The equation

$$\frac{d}{dx}\left(p(x)\, y'\right) - q(x)\, y = 0 \tag{1}$$

is considered, with the usual conditions:

$$p \in C^1([a, b]);\ p(x) > 0,\ x \in [0, 1];\ \ q \in C^0([a, b]);\ q(x) > 0,\ x \in [0, 1].$$

The operator

$$L_{SL} = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) - q(x)$$

results to be Hermitian in the space of functions

$$f \in C^0([a, b]) \cap C^2((a, b)),\ f'' \in L^2([a, b])$$

and null at the boundary.

The eigenvalue problem for the equation

$$\frac{d}{dx}\left(p(x)\, y'\right) - q(x)\, y + \lambda\rho(x)\, y = 0, \quad \rho \in C^0([a, b]),\ \rho(x)\ > 0\ (x \in [0, 1]), \tag{2}$$

with the three classical boundary conditions is similar to the elementary case

$$y'' + \lambda\, y = 0, \tag{3}$$

so nothing substantially new emerges in the apparently more general case of Equation (2).

Nonetheless, one may observe that a general linear equation of second order can always be reduced, with a change in variables [11] (p. 16), to the form

$$y'' + \alpha(x)\, y = 0.$$

The condition $q(x) > 0$ is not truly restrictive, since if $q(x)$ changes into $q(x) + C\rho(x)$, then the eigenvalues λ_n change into $\lambda_n + C$.

We start simplifying the Sturm–Liouville equation by a simple change in variables $x \mapsto z(x)$. Choosing

$$z(x) = C \int_0^x \frac{dx'}{p(x')}, \qquad \text{with } C \text{ such that } z(1) = 1, \tag{4}$$

we obtain, again using the old variables,

$$\frac{d^2y}{dx^2} - q(x)y + \lambda\rho(x)y = 0 \tag{5}$$

with the new functions $q(x)$ and $\rho(x)$ still greater than zero. Here, $z(x)$ is a monotonic increasing function of x that takes the interval $[0,1]$ into $[0,1]$. So, nothing changes substantially using this new variable.

To be definite, for the eigenvalues and eigenfunctions, we restrict ourselves to the interval $[0,1]$, with null boundary conditions. We know that any solution of Equation (5) not identically zero has only zeros of first order. We have the following:

Proposition 1. *Any solution of Equation (5) cannot have infinite zeros in the interval* $[0,1]$.

Proof. An accumulation point of zeros would be a zero of second order. □

Proposition 2. *If* $q(x) - \lambda\rho(x) > 0,\ x \in [0,1]$, *then the solutions* $y(\lambda, x)$ *cannot have other zeros in the interval* $(0,1)$ *if it is not identically zero.*

Proof. In fact, $y''(\lambda, x)$ and $y(\lambda, x)$ would have the same sign between two consecutive zeros of $y(\lambda, x)$, which is impossible. □

Proposition 3. *If* $q(x) - \lambda\rho(x) < 0,\ x \in [0,1]$, *then the solution will be a regular bell between two consecutive zeros.*

Proof. In fact $y''(\lambda, x)$ is zero only where $y(\lambda, x)$ is zero, and the local maxima of $|y'(\lambda, x)|$ will be at points where $y(\lambda, x) = 0$ (since $y'(x)$ will be a monotonic function between consecutive zeros). □

For completeness, we report on the following theorem, with the aim to obtain also bounds for the solutions.

Theorem 1. *The solutions of Equation (5) are* C^2 *in* x *and analytic for* $\lambda \in \mathbb{C}$.

Remark 1. *Then,* $y(\lambda, x) = \sum_{n=0}^{\infty} a_n(x)\lambda^n$ *must hold and the coefficients* $a_n(x)$ *turn out to be a closed set (see Theorem 8).*

Proof of Theorem 1. Equation (5), with $y(\lambda, 0) = 0,\ y'(\lambda, 0) = 1$, leads us to the integral equation

$$y(\lambda, x) = x + \int_0^x (x - x')\left[q(x') - \lambda\rho(x')\right] y(\lambda, x')dx' \tag{6}$$

$$= x + \int_0^x K_\lambda(x,\, x')y(\lambda, x')dx'$$

with the definition $K_\lambda(x, x') = (x - x')[q(x') - \lambda\rho(x')]$. Iterating the integral Equation (6), and setting $K_\lambda^{[1]}(x, x') = K_\lambda(x, x')$, the solution is given by

$$y(\lambda, x) = x + \sum_{m=1}^{n} \int_0^x K_\lambda^{[m]}(x, x')x'dx' + \int_0^x K_\lambda^{[n+1]}(x, x')y(\lambda, x')dx'$$

where, for $m > 1$,

$$K_\lambda^{[m]}(x, x') = \int_{x'}^x K_\lambda^{[m-1]}(x, x'')K_\lambda(x'', x')dx''$$

is the iterated kernel, dominated by

$$\left|K_\lambda^{[m]}(x, x')\right| < \frac{\bar{k}_\lambda^{2m}}{(m+1)!},$$

having defined $\bar{k}_\lambda^2 = \sup_x|\lambda\rho(x) - q(x)|$. The last integral, containing $y(\lambda, x')$, goes to zero in m, in any compact set of the complex plane. The solution is thus given by the series

$$y(\lambda, x) = x + \sum_{m=1}^{\infty} \int_0^x K_\lambda^{[m]}(x, x')x'dx' \tag{7}$$

uniformly convergent in any such set. This proves the holomorphy of $y(\lambda, x)$ in the whole complex plane $\mathbb{C}$. $\square$

From Equation (6),

$$|y(\lambda, x)| < x + \bar{k}_\lambda^2 \int_0^x |y(\lambda, x')dx'|$$

and, from Gronwall's inequality,

$$|y(\lambda, x)| < xe^{\bar{k}_\lambda^2 x}.$$

In Appendix B, an exponential lower bound is also found for $\lambda_R = Re\lambda < 0$ and any $\lambda_I = Im\lambda$, Equation (A11).

If we consider the integral equation

$$y'(\lambda, x) = 1 + \int_0^x \left[q(x') - \lambda\rho(x')\right]y(\lambda, x')dx', \tag{8}$$

we obtain the general bound

$$|y'(\lambda, x)| \le e^{\bar{k}_\lambda^2 x}.$$

(notice that, for the particular case λ real and $\lambda > 0$, better bounds can be easily found)

Now, we can also rewrite the solution in the following form:

$$y(\lambda, x) = \sum_{n=0}^{\infty} a_n(x)\,\lambda^n. \tag{9}$$

Substituting it into Equation (5), we have the infinite system of differential equations for the coefficients

$$a_0''(x) - q(x)a_0(x) = 0, \quad \text{with } a_0(0) = 0,\ a_0'(0) = 1, \quad a_0(x) = y(0, x), \tag{10}$$

$$a_n''(x) - q(x)a_n(x) = -\rho(x)a_{n-1}(x), \ \text{with } a_n(0) = 0,\ a_n'(0) = 0, \quad n \ge 1. \tag{11}$$

The system (10) and (11) can be solved recursively.

Note that the homogeneous equation for Equation (11) is always Equation (10). The integral equation connected to the differential Equation (11), with initial conditions

$$a_n(0) = 0, \quad a_n'(0) = 0, \qquad n \geq 1,$$

is

$$a_n(x) = \int_0^x (x - x')\left[q(x')a_n(x') - \rho(x')a_{n-1}(x')\right]dx'. \tag{12}$$

The convergence of the series (9) for every value of λ is straightforward. Further, note that the signs of $a_n(x)$ are alternating for $\rho(x) > 0$, $x \in [0,1]$: if $a_{n-1}(x) < 0$, then $a_n(x) > \int_0^x (x - x')a_n(x')dx'$ gives $a_n(x) > 0$ and, equally, if $a_{n-1}(x) > 0$, then $a_n(x) < 0$. This shows a first indication of the connection of $y(\lambda, x)$ to $\dfrac{\sin(kx)}{k}$ (and to $\dfrac{\sinh(kx)}{k}$ for $\lambda < 0$), as will be well clarified in what follows. For the definition of k see Section 3. The series expansion (9) and Equations (10) and (11) also provide, in fact, the possibility to recover, for example, the Bessel functions (not reported here).

We already noted that the zeros in x of $y(\lambda, x)$ have to be simple, otherwise $y(\lambda, x)$ is identically zero. We have the following Lemma:

Lemma 1. *If $y(\lambda, 0) = 0$, the following*

$$\frac{\partial y}{\partial \lambda}(\lambda, x)\frac{\partial y}{\partial x}(\lambda, x) - y(\lambda, x)\frac{\partial^2 y}{\partial x \partial \lambda}(\lambda, x) = \int_0^x \rho(x)y^2(\lambda, x)dx \tag{13}$$

holds, implying that, if $y(\lambda, x_i) = 0$, then

$$\frac{\partial y}{\partial \lambda}(\lambda, x_i)\,\frac{\partial y}{\partial x}(\lambda, x_i) = \int_0^{x_i} \rho(x)y^2(\lambda, x)dx\,.$$

That is, a zero of $y(\lambda, x)$ of the first order with respect to x is a zero of first order also with respect to λ.

Furthermore, if $\dfrac{\partial y}{\partial x}(\lambda, x_i^*) = 0$, then

$$-y(\lambda, x_i^*)\frac{\partial^2 y}{\partial x \partial \lambda}(\lambda, x_i^*) = \int_0^{x_i^*} \rho(x)y^2(\lambda, x)dx \tag{14}$$

Proof. From Equation (5), written for λ and μ, and $y(\lambda, 0) = 0\,, y(\mu, 0) = 0$, we deduce

$$y(\mu, x)y''(\lambda, x) - y(\lambda, x)y''(\mu, x) = (\mu - \lambda)\rho(x)y(\lambda, x)y(\mu, x) \tag{15}$$

and

$$y(\mu, x)\,y'(\lambda, x)\; - y(\lambda, x)\,y'(\mu, x)\; = (\mu - \lambda)\int_0^x \rho(x)y(\lambda, x)y(\mu, x)dx. \tag{16}$$

Subtracting and adding $y(\lambda, x)y'(\lambda, x)$, and passing to the limit $\mu \to \lambda$, the statement follows. □

Remark 2. *Equation (13) is valid also for complex λ.*

It is useful for us to underline the results with the following corollaries.

Corollary 1. *At the zeros x_i of $y(\lambda, x)$, the derivatives*

$$\frac{\partial y}{\partial x}(\lambda, x_i), \qquad \frac{\partial y}{\partial \lambda}(\lambda, x_i)$$

are inversely proportional, have the same sign, and their product has the same behavior of $y^2(\lambda, x)$.

Corollary 2. *The norm of the eigenfunctions is given by*

$$\int_0^1 \rho(x)y^2(\lambda_n, x)dx = \frac{\partial y}{\partial \lambda}\frac{\partial y}{\partial x}(\lambda_n, 1). \tag{17}$$

For example, for the Bessel functions $J_n(kx)$, which are in the lucky situation of being analytical functions of kr, with $k = \sqrt{\lambda}$, noting that the derivation is with respect to λ, we recover

$$\int_0^1 J_n(k_m^{(n)}r)^2 rdr = \frac{1}{2}(\frac{\partial J_n(z)}{\partial z})^2\Big|_{(z=k_m^{(n)})}, \tag{18}$$

with $k_m^{(n)}$ the eigenvalues of $J_n(x)$ for $m = 1, 2, \ldots$.

From Equation (16), we also recover the Orthogonality of eigenfunctions of Hermitian operators.

Theorem 2.

$$\int_0^1 \rho(x)y(\lambda_n, x)y(\lambda_m, x)dx = 0, \quad n \neq m.$$

Having established these results, we now proceed to state a qualitative property we stress in this paper.

Proposition 4. *All the zeros and all stationary points (and then every point) of the solutions* $y(\lambda, x)$ *of the Sturm–Liouville equation for* $\lambda > 0$ *are strictly decreasing functions of* λ.

Proof of Proposition 4. If (λ_i, x_i) is a zero of $y(\lambda, x)$, from the Implicit Function Theorem, being

$$\frac{\partial y}{\partial x} \neq 0, \quad \frac{\partial y}{\partial \lambda} \neq 0$$

in (λ_i, x_i), we can deduce that $y(\lambda, x(\lambda))$ is identically zero in a small neighbourhood of λ_i, with $x(\lambda)$ a regular function of λ. Then

$$\frac{dx(\lambda)}{d\lambda} = -\frac{\frac{\partial y}{\partial \lambda}}{\frac{\partial y}{\partial x}}(\lambda, x(\lambda)). \tag{19}$$

From the previous Corollary 1, we have that $\frac{\partial y}{\partial \lambda}$ and $\frac{\partial y}{\partial x}$ have the same sign at the zeros of $y(\lambda, x)$, so $\frac{dx(\lambda)}{d\lambda}$ is negative and the result follows. This result is obvious in the elementary case $y'' + k^2 y = 0$.

This proposition reinforces Sturm's result, which gives lower and upper bounds for the number of zeros in the interval $(0, 1)$.

For the stationary points of $y(\lambda, x)$, i.e., $y'(\lambda, x_i^*) = 0$, we have that, while $-y(\lambda, x_i^*)\frac{\partial^2 y}{\partial x \partial \lambda}(\lambda, x_i^*)$ is positive, $y\frac{\partial^2 y}{\partial x^2}$ is always negative for $y \neq 0$, $\lambda\rho(x) - q(x) > 0$. So, applying the Implicit Function Theorem to $y'(\lambda, x_i^*) = 0$, since $\frac{\partial^2 y}{\partial x^2}(\lambda, x_i^*)$ is different from zero, we have

$$\frac{dx^*(\lambda)}{d\lambda} = -\frac{\frac{\partial^2 y}{\partial x \partial \lambda}}{\frac{\partial^2 y}{\partial x^2}}(\lambda, x^*(\lambda)) < 0,$$

□

In the rest of the paper, when there is no possible confusion, $y'(\lambda, x)$ shall mean $\dfrac{\partial y(\lambda, x)}{\partial x}$.

For $\lambda \leq 0$, the solution with $y(\lambda, 0) = 0$ is always greater than zero. Then, from the previous proposition, also considering Sturm's Theorem, when increasing λ there exists a first value $\lambda_1 > 0$ such that $y(\lambda_1, 1) = 0$, $y(\lambda_1, x) > 0$ for $x \in (0, 1)$. Then, there exists a value λ_2 such that $y(\lambda_2, x)$ vanishes at the boundaries and has just one zero inside the interval $(0, 1)$. So, going on, we find a set of infinite eigenvalues $\{\lambda_n\}$, $\lambda_n \to \infty$, such that $y(\lambda_n, x)$ has exactly $n-1$ zeros inside the interval $(0, 1)$. If it were $\lambda_n \to \lambda^* < \infty$, then $y(\lambda^*, x)$ should have ∞ zeros inside the interval $(0, 1)$, which is impossible. Still in the light of the previous proposition, we have the following property:

Proposition 5. *There cannot be two different eigenfunctions of the Sturm–Liouville equation satisfying the same null boundary conditions, and having the same number of zeros inside the interval (0,1).*

That is, the eigenfunctions are uniquely determined, but for a constant factor, by the number of their zeros.

The existence of eigenvalues

$$\lambda_1 < \cdots < \lambda_n \to \infty$$

is easily deduced, as it is well known, also from the theory of integral equations, and from direct methods of calculus of variations [12–17], considering that the eigenvalue λ_n is the minimum of the integral

$$\int_0^1 \left[p(x) y'^2(x) + q(x)\, y^2(x) \right] dx \tag{20}$$

in the space of normalized functions, with $y(0) = y(1) = 0$ and orthogonal to the eigenfunctions

$$y_0(x),\ y_1(x), \ldots, y_{n-1}(x).$$

If we impose the condition $y'(\lambda, 1) = 0$, little changes conceptually, since between two consecutive zeros $y(\lambda, x_n) = y(\lambda, x_{n+1}) = 0$ there exists a unique point with $y'(\lambda, x^*) = 0$, and this point shifts to the left increasing λ (see Proposition 4). So, there exists $\lambda_n < \lambda_n^* < \lambda_{n+1}$ such that $y'(\lambda_n^*, 1) = 0$. If we consider the condition

$$\alpha y(\lambda, 1) + \beta y'(\lambda, 1) = 0, \quad \alpha,\ \beta \neq 0, \tag{21}$$

the function $\dfrac{y(\lambda, 1)}{y'(\lambda, 1)}$ behaves, varying λ, like the tangent function, so the condition (21) can be satisfied, for any value of α and β, for an infinite set of values of λ.

3. Some Preliminary Results

With $\rho = const$ and $k^2 = \lambda\rho$, we can give detailed information about the asymptotic behavior of $y(\lambda,x)$. For $y(\lambda, 0) = 0$ and $y'(\lambda, 0) = 1$, from the integral equation, equivalent to the differential Equation (5),

$$y(k, x) = \frac{\sin kx}{k} + \int_0^x \frac{\sin k(x - x')}{k} q(x') y(k, x') dx', \tag{22}$$

it is easy to obtain, via the Gronwall inequality, the asymptotic estimate

$$y(k, x) = O(1/k)$$

and then

$$y(k, x) = \frac{\sin kx}{k} + O(1/k^2). \tag{23}$$

Here and in the following, we shall use $f(k, x) = O(k^a)$ to denote that there exists a $k_0 > 0$ such that, for any $k > k_0$, then $C_1 k^a < \sup_x |f(k, x)| < C_2 k^a$, with C_1, C_2 positive numbers and the $\sup_x$ intended for all x in $[0, 1]$. Similarly, the same symbol shall be attributed to similar estimates of quantities independent of x and whose dependence on k may be explicit or implicit. Additionally, if $\bar{f}_k = \sup_x |f(k, x)|$ and it turns out that $\lim_{k\to\infty} k^{-a} \bar{f}_k = 0$, then we shall write $f(k, x) = o(k^a)$.

From Equation (22) it also follows that

$$y'(k, x) = \cos(k, x) + O(1/k).$$

The same conclusion (23) is obtained by defining

$$\bar{k}^2 = \lambda\rho - \alpha, \;\; \underline{k}^2 = \lambda\rho - \beta, \;\; \text{where } \alpha = \inf_x q(x), \; \beta = \sup_x q(x),$$

meaning, from Sturm's Theorem, that the square of the frequency of $y(k, x)$ is contained between these two values. For $\lambda \gg 1$, we have, for the two $k's$,

$$k \sim \sqrt{\lambda\rho} - \frac{\alpha \,(\text{or } \beta)}{2\sqrt{\lambda\rho}} \tag{24}$$

that is, asymptotically, the eigenvalues λ_n are given by

$$\sqrt{\lambda_n \rho} = n\pi + O\big(1/(n\pi)\big), \tag{25}$$

giving again for $y(k, x)$ the result (23), which we can now write as

$$y(\lambda_n, x) = \frac{\sin(n\pi x)}{n\pi} + O\Big(\frac{1}{(n\pi)^2}\Big). \tag{26}$$

This substantial identification of $y(\lambda_n, x)$ with $\dfrac{\sin(n\pi x)}{n\pi}$ suggests directly, as we shall see, that the theorem for the point convergence of the Fourier series is still valid, including obviously the completeness of the set (26).

For $\rho = \rho(x)$ and $\lambda\rho(x) - q(x) > 0, \; x \in [0, 1]$, the analysis is more complex. We can write the integral equation, for any k,

$$y(\lambda, x) = \frac{\sin kx}{k} + \int_0^x \frac{\sin k(x - x')}{k} \Big[q(x') - \lambda\rho(x') + k^2\Big] y(\lambda, x') dx'. \tag{27}$$

The comparison of Equations (22) and (27) shows the great simplification of considering $\rho = const$. Naturally, the integral Equation (27) can be useful only for $\underline{k}^2 \le k^2 \le \bar{k}^2$, where $\bar{k}^2 = \sup(\lambda\,\rho(x) - q(x)) \sim \lambda\gamma$ and $\underline{k}^2 = \inf(\lambda\rho(x) - q(x)) \sim \lambda\delta$, $\lambda > \Lambda$, with the square bracket in the integral controlled by $\bar{k}^2 - \underline{k}^2$. Here and in the following, by $\lambda > \Lambda$, we mean λ greater than any arbitrarily large number. From Proposition 7 and from Proposition A3 of Appendix C, it turns out that the two terms at the second member of (27) are of the same order.

Now, the single oscillation depends only on the local values of $q(x)$ and $\rho(x)$. Thus, for $\lambda \gg 1$, when writing the integral equation in any given interval (x_i, x_{i+1}), where $y(\lambda, x_i) = 0$, since the interval $x_{i+1} - x_i$ is at most of the order of $\dfrac{1}{\underline{k}_i}$, the extreme values of

k^2 have to be considered only on small intervals. Therefore, the local value of $\frac{\bar{k}_i^2 - \underline{k}_i^2}{\bar{k}_i}$ is dominated by a constant for $\rho(x)$, a Lipschitz function. Then, in the general case $\rho = \rho(x)$, we have

$$y(\lambda, x) = \frac{\sin \bar{k}_i(x - x_i)}{\bar{k}_i} y'(\bar{k}_i, x_i) + \int_{x_i}^{x} \frac{\sin \bar{k}_i(x - x')}{\bar{k}_i} [q(x') - \lambda\rho(x') + \bar{k}_i^2] y(\lambda, x') dx', \quad x \in [x_i, x_{i+1}], \tag{28}$$

with the first term now prevailing with respect to the integral, meaning that we cannot have that a single $\frac{\sin(kx)}{k}$ approximates $y(\lambda, x)$, but solely that it is well behaved in any small interval, where

$$y(\lambda, x) = \frac{\sin \bar{k}_i(x - x_i)}{\bar{k}_i} y'(\bar{k}_i, x_i) + O\Big(\frac{1}{\bar{k}_i^2}\Big), \quad x \in [x_i, x_{i+1}], \quad x_{i+1} - x_i = O\Big(\frac{1}{\bar{k}_i}\Big). \tag{29}$$

In any case,

$$y(\lambda, x) = O\Big(\frac{1}{\sqrt{\lambda}}\Big),$$

since $|y'(\lambda, x)|$ is bounded by a constant

$$y'(\lambda, x) = O(1),$$

as can be seen without the need to invoke the expression (28), but rather just by integrating the differential equation and considering the next Equation (34). The same bounds can be directly obtained in Appendix B, Equation (A16), for small $|\lambda_I| = |Im\lambda|$. Furthermore, we have the following Lemma:

Lemma 2. *Any derivation of $y(\lambda, x)$ with respect to x produces a factor $\sqrt{\lambda}$ in the numerator, as can be seen in Equations (27) and (28) and confirmed by the asymptotic relation $y'' = O(\lambda|y|)$. Conversely, the same equations entail that any derivation with respect to λ gives a factor $\sqrt{\lambda}$ in the denominator (consider also Lemma 1 and Corollary 1).*

Given x_0 strictly contained between two consecutive zeros x_i, x_{i+1}, the derivatives of $y(\lambda, x)$, a regular bell, are small corrections to the derivatives of $\frac{\sin\sqrt{\lambda\rho(x_0) - q(x_0)}x}{\sqrt{\lambda\rho(x_0) - q(x_0)}}$ around x_0 itself. We wrote the integral Equation (28) to stress that locally, and asymptotically, the solution is like that of the elementary case $\lambda\rho - q = const$, but with a different frequency for any single oscillation.

Equation (28) lets us also estimate the eigenvalues. Letting $k_i^2 = \lambda\rho(x_i) - q(x_i), y(\lambda_n, x_i) = 0$, we can estimate x_{i+1} from (28), and then, from Equations (25) and (26), evaluated at x_i, it has to be

$$1 = \sum_0^{n-1} (x_{i+1} - x_i) = \sum_0^{n-1} \frac{1}{k_i}(\pi + \frac{C_i}{k_i}), \quad C_i < M,$$

M connected to the Lipschitz constant. So, asymptotically, $\sum_0^{n-1} \frac{\pi}{k_i} \sim 1$, i.e.

$$\sqrt{\lambda_n} \sim \pi \sum_0^{n-1} \frac{1}{\sqrt{\rho(x_i)}}.$$

Naturally, for $\rho = const$, $\sqrt{\lambda_n \rho} \sim n\pi$ holds, in agreement with (25).

It is easy to prove directly from the differential equation the following Lemma:

Lemma 3. *The solutions $y(\lambda, x)$ of the equation*

$$y'' - q(x)y + \lambda\rho(x)y = 0, \quad \textit{with} \quad y(\lambda, 0) = 0, \quad y'(\lambda, 0) = 1,$$

initially decrease by increasing λ: if $\lambda_2 > \lambda_1$, $y(\lambda_2, x) < y(\lambda_1, x)$ holds for $x \in (0, x_{1,2})$, where $x_{1,2}$ is the first zero of $y(\lambda_2, x)$. In fact, the inequality holds in absolute value up to the intersection of the two solutions, from which point, for the needs of the following corollary, the reasoning can be reiterated.

Proof. From Equation (16),

$$y^2(\lambda_1, x)\frac{d}{dx}\frac{y(\lambda_2, x)}{y(\lambda_1, x)} = (\lambda_1 - \lambda_2)\int_0^x \rho(x')y(\lambda_1, x')y(\lambda_2, x')dx',$$

showing that $\dfrac{y(\lambda_2, x)}{y(\lambda_1, x)}$ is a decreasing function, since $y(\lambda_i, x) > 0$, starting from $\lim\limits_{x\to 0}\dfrac{y(\lambda_2, x)}{y(\lambda_1, x)} = \dfrac{y'(\lambda_2, 0)}{y'(\lambda_1, 0)} = 1$. □

Corollary 3. *If $\lambda_2 > \lambda_1$, λ_1 large, then* $\sup|y(\lambda_2, x)| < \sup|y(\lambda_1, x)|$.

Proof. Take λ_1 large, and $\lambda_2 = \lambda_1 + \epsilon$, where $\epsilon > 0$ is sufficiently small such that all the intersections of the two solutions are close to their zeros. From the previous Lemma, starting from these intersections, it follows that locally $\sup|y(\lambda_1 + \epsilon, x)| < \sup|y(\lambda_1, x)|$. The assertion is then true, for any $\lambda_2 > \lambda_1$, just going with n steps from λ_1 to λ_2. But now, the number of oscillations is different, and the statement is true only for the absolute maximum. □

Remark 3. *It can happen, for large λ_1 and λ_2 a little larger, that if $\rho(x)$ has a flat minimum, the oscillation with λ_2 can take a larger part in that interval, and the inequality can be locally reversed.*

These results can be understood from Equations (27) and (28). Obviously we can state also the following:

Corollary 4. *For $\rho_1(x) \gtrless \rho_2(x)$ and the same λ,* $\sup_x|y_2| \lessgtr \sup_x|y_1|$.

The surprising fact is that, using the apparently worse integral equation,

$$y(\lambda, x) = x + \int_0^x (x - x')[q(x') - \lambda\rho(x')]y(\lambda, x')dx', \tag{30}$$

where there is no information about the oscillatory character of the solutions for $\lambda\rho(x) - q(x) > 0$, we very easily obtain exact results, identical to those valid for the elementary case. First of all, we have that

$$\int_0^1 (1 - x)[\lambda_n\rho(x) - q(x)]y(\lambda_n, x)dx = 1,$$

just like

$$\int_0^1 (1 - x)k^2\frac{\sin kx}{k}dx = 1\,, \quad \text{for} \quad k = n\pi\,.$$

Furthermore, from

$$y'(\lambda, x) = 1 + \int_0^x [q(x') - \lambda\rho(x')]y(\lambda, x')dx',$$

taking x_n^* as a stationary point of $y(\lambda, x)$, i.e., $y'(\lambda, x_n^*) = 0$, we obtain, for any λ,

$$\int_0^{x_n^*} [\lambda\rho(x) - q(x)]y(\lambda, x)dx = 1. \tag{31}$$

So, if x_m^* is another stationary point, we have

$$\int_{x_n^*}^{x_m^*} [\lambda\rho(x) - q(x)]y(\lambda, x)dx = 0. \tag{32}$$

In particular, for the first stationary point x_1^*, similarly to

$$\int_0^{x_1^*} k^2 \frac{\sin kx}{k} dx = 1, \quad kx_1^* = \frac{\pi}{2},$$

we now have

$$\int_0^{x_1^*} [\lambda\rho(x) - q(x)]y(\lambda, x)dx = 1,$$

and then, for any semi-oscillation $x \in [x_n^*, x_{n+1}^*]$, and any $\lambda\rho(x) - q(x) > 0$, we have that the mean value is exactly zero, with the weight function $\lambda\rho(x) - q(x)$. Observe that, if $\lambda < 0$, $y(\lambda, x)$ is monotonically increasing with respect to $|\lambda|$ and x, and there are no stationary points.

From the previous formulas, we again easily obtain that $y'(\lambda, x) = O(1)$ for $\lambda > 0$. We have

$$y'(\lambda, x) = 1 + \int_0^x [q(x') - \lambda\rho(x')]y(\lambda, x')dx' = \int_{x_n^*}^x [q(x') - \lambda\rho(x')]y(\lambda, x')dx', \tag{33}$$

where x_n^* is the last stationary point before x. Now $|x - x_n^*| = O(\frac{1}{\underline{k}})$, $y(\lambda, x) = O(\frac{1}{\underline{k}})$, so the integral is of the order $O(1)$, with the local estimate $\underline{k}^2 \leq |\lambda\rho(x) - q(x)| \leq \bar{k}^2$.

Proposition 6. *From (33), since $|y'(\lambda, x)| = O(1)$, it follows that, while $|y(\lambda, x)| = O(\frac{1}{\sqrt{\lambda}})$, we have*

$$\int_0^x \rho(x')y(\lambda, x')dx' = O(\frac{1}{\lambda}), \quad \lambda > \Lambda. \tag{34}$$

That is, by virtue of the oscillatory character of the solutions, the integral goes to zero more rapidly by a factor $\frac{1}{\sqrt{\lambda}}$. The simplest example where we can observe the consequences of this oscillation is

$$\int_0^x y'(\lambda, x')dx' = y(\lambda, x),$$

with $|y'(\lambda, x)| = O(1)$ and $|y(\lambda, x)| = O(\frac{1}{\sqrt{\lambda}})$. In fact, Equation (34) is valid also for $\int_0^x \varphi(x')\rho(x')y(\lambda, x')dx'$ with $\varphi(x)$ a monotonic or bounded variation function. We state here the following proposition, whose proof will be postponed to Appendix C.

Proposition 7. *If $\varphi(x)$ is a monotonic, or a bounded variation function, then*

$$\int_0^x \rho(x')\varphi(x')y(\lambda, x')dx' = O(\frac{1}{\lambda}).$$

Knowing that for monotonic functions the derivative exists almost everywhere, and that such derivative is in L^1, this proposition corresponds to the usual integration by parts, with C^1 functions and trigonometric functions. From the density of bounded variation functions in L^1, we have then that the Riemann–Lebesgue Theorem is valid in general.

Theorem 3. *If* $\varphi \in L^1([0,1])$, *then* $\int_0^1 \rho(x)\varphi(x)y(\lambda,x)dx = o(\frac{1}{\sqrt{\lambda}})$.

We conclude this section by writing some additional simple, more useful, results ahead. We have

$$\frac{1}{2}\frac{dy'^2}{dx} = \frac{1}{2}[q(x) - \lambda\rho(x)]\frac{dy^2}{dx}. \tag{35}$$

From a mechanical point of view, with $\lambda\rho(x) - q(x) \neq cost$, what is now a total energy is no longer a constant of the motion, as we would expect.

$$\frac{d}{dx}(y'^2 + (\lambda\rho - q)y^2) = \frac{dy'^2}{dx} + (\lambda\rho - q)\frac{dy^2}{dx} + (\lambda\rho' - q')y^2) = (\lambda\rho' - q')y^2$$

is increasing or decreasing according to the sign of $\lambda\rho'(x) - q'(x) \gtrless 0$.

From (35), it follows that

$$y'^2 + [\lambda\rho(x) - q(x)]y^2 = 1 + \int_0^x [\lambda\rho'(x') - q'(x')]y^2 dx' \geq C > 0, \tag{36}$$

also asymptotically for both signs of $\rho'(x')$. The result descends directly from Equation (36) for $\rho'(x) > 0$ while, for $\rho'(x) < 0$, it follows from the equations

$$\frac{d}{dx}\left[\frac{1}{\lambda\rho - q}y'^2 + y^2\right] = -\frac{\lambda\rho' - q'}{(\lambda\rho - q)^2}y'^2, \tag{37}$$

and

$$\frac{1}{\lambda\rho(x) - q(x)}y'^2 + y^2 = \frac{1}{\lambda\rho(0) - q(0)} - \int_0^x \frac{\lambda\rho'(x') - q'(x')}{[\lambda\rho(x') - q(x')]^2}y'^2 dx'. \tag{38}$$

From the same Equations (36) and (38), changing the conditions on the sign of $\rho'(x)$, it always follows that

$$y^2(\lambda, x) \leq \frac{1}{\inf_{x' \leq x}[\lambda\rho(x') - q(x')]}, \tag{39}$$

and for the maxima of $y(\lambda, x_i^*)^2$, taken at points where $y'(\lambda, x_i^*) = 0$, we have

$$y^2(\lambda, x_i^*) > \frac{1}{\sup_{x' \leq x_i^*}[\lambda\rho(x') - q(x')]}. \tag{40}$$

Obviously, for $\rho'(x) > 0$, inf $\rho(x)$ means $\rho(0)$, and for $\rho'(x) < 0$ it means $\rho(x)$. In conclusion, Equations (39) and (40) give

$$\frac{C_2}{\lambda} \leq \sup |y|^2 \leq \frac{C_1}{\lambda}, \quad \lambda > \Lambda.$$

In the same way, we obtain

$$C_2 \leq \sup y'^2 \leq C_1.$$

The solutions of our equations exactly satisfy what we still call "*equipartition*", like in the case of the harmonic oscillator, with $\lambda_n\rho(x) - q(x)$ in the place of k_n^2.

Theorem 4 (Equipartition)**.** *It always holds that*

$$\int_0^1 y'^2(\lambda_n, x)dx = \int_0^1 [\lambda_n \rho(x) - q(x)] y^2(\lambda_n, x)dx,$$

which, for the harmonic oscillator $x'' + k^2 x = 0$, and for $k = n\pi$, reduces to $\int_0^1 x'^2 dt = \int_0^1 k^2 x^2 dt$.

Proof. It is enough to integrate

$$y(\lambda_n, x) y''(\lambda_n, x) + (-q(x) + \lambda_n \rho(x)) y(\lambda_n, x)^2 = 0.$$

□

Remark 4. *This equipartition continues to be approximately valid also for "complex potentials", for small $|\lambda_I| = |Im\lambda|$, as we will see in Appendix B.*

Using this equipartition, from Equation (36), and x^* the stationary point nearest to 1, which for $\lambda > \Lambda$ satisfies $1 - x^* = O(\frac{1}{\sqrt{\lambda}})$, it follows that

$$\int_0^1 [\lambda \rho(x') - q(x')] y^2 dx' \geq \frac{1}{2} \int_0^{x^*} \{y'^2 + [\lambda \rho(x') - q(x')] y^2\} dx > \frac{x^* C}{2},$$

from which

$$\int_0^1 \rho(x) y^2 dx > \frac{x^* C}{2\lambda} \quad (\lambda > \Lambda), \tag{41}$$

leading to the final inequality

$$\frac{1}{\int_0^1 \rho(x) y^2 dx} < C\lambda. \tag{42}$$

Since

$$y^2 = O(\frac{1}{\lambda}),$$

then

$$C_2 \lambda < \frac{1}{\int_0^1 \rho(x) y^2 dx} < C_1 \lambda.$$

Finally, from (39), we obtain

$$\frac{y(\lambda, x)^2}{\int_0^1 \rho(x) y(\lambda, x)^2 dx} < C \quad \text{and} \quad \frac{y'(\lambda, x)^2}{\int_0^1 \rho(x) y(\lambda, x)^2 dx} < C\lambda, \tag{43}$$

then:

Proposition 8. *The normalized solutions are uniformly bounded for $\lambda > \Lambda$.*

4. The Closure Property

We now pass to the closure of the set of eigenfunctions $\{y(\lambda_n, x)\}$ in the space L^2, i.e., we have to show that any function $\varphi(x) \in L^2([0,1])$, orthogonal to all the eigenfunctions, is the null function, which is expected, since we have an eigenfunction for any $n \in \mathbb{N}$ and L^2 is a separable pace. In this section, λ is considered variable on the whole complex plane.

To get acquainted with the procedure, we first consider the classical problem of

$$f(n) = \int_0^1 \varphi(x) \sin(n\pi\, x)\, dx = 0 \quad \text{for every } n \in \mathbb{N}, \quad \varphi \in L^2([0,1]).$$

We show that the entire holomorphic function $f(z) = \int_0^1 \varphi(x)\sin(z\pi\, x)dx$ is identically zero for $z \in \mathbb{C}$. We cannot use the Carlson Theorem [18] (p. 185), which says that if a holomorphic function $f(z)$ is such that $f(n) = 0$ for every $n \in \mathbb{N}$ and if, for $Re\, z \geq 0$, it holds that $|f(z)| < \exp(\pi - \epsilon)|z|$, then $f(z) \equiv 0$. In fact, we now have the opposite inequality: $|f(z)| \sim C\dfrac{|\cos \pi z|}{|\pi z|} > \exp(\pi - \epsilon)|z|$, for any $\epsilon > 0$ and $Im\, z \to \pm\infty$.

The bound is essential, e.g., barely not satisfied by $\sin(\pi z)$.

In fact, in the present case, it is easy to obtain the expected result through the Liouville Theorem. We can appreciate that

$$g(z) = \frac{\displaystyle\int_0^1 \varphi(x)\sin(zx)\, dx}{\sin(z)} \tag{44}$$

is an entire holomorphic function, since $\sin(n\pi)$ is a zero of the first order, and it is bounded for all $z \in \mathbb{C}$. Indeed, for $z = \xi + i\eta$, $|\eta| > 0$, we have

$$|g(z)|^2 \leq \frac{\displaystyle\int_0^1 \varphi(x)^2 dx \int_0^1 [\sin^2(\xi\, x) + \sinh^2(\eta\, x)]dx}{\sin^2(\xi) + \sinh^2(\eta)} < C_\varphi \frac{\displaystyle\int_0^1 [\sin^2(\xi x) + \sinh^2(\eta x)]dx}{\sin^2(\xi) + \sinh^2(\eta)} \tag{45}$$

clearly bounded for $|\xi| \leq \pi/2$. The same will be true for $z = n\pi + \xi + i\eta$, uniformly in n. It is enough to subtract $f(n)$ from the numerator of $g(z)$ and apply the same reasoning. The boundedness of $|g(z)|$ for $\eta \to \pm\infty$ is clear from (45). Then, from Liouville's Theorem,

$$\int_0^1 \varphi(x)\sin(zx)dx = C_\varphi \sin(z). \tag{46}$$

Take now $z = 2n\pi + \pi/2$, and consider the Riemann–Lebesgue Theorem. The conclusion then is that C_φ has to be zero:

$$\int_0^1 \varphi(x)\sin(zx)\, dx = 0, \quad \text{for all } z \in \mathbb{C}, \tag{47}$$

giving now $\varphi(x) = 0$ a.e. as a consequence of the inverse sine Fourier transform of $\varphi(x)$, with $\varphi(x) = 0$ for $x \in [1, \infty)$.

Remark 5. *Equation (47) means, obviously, that also the integrals* $\int_0^1 \varphi(x)\cos(n\pi x)dx$ *are identically zero for any* n.

Remark 6. *The integral* $\int_0^1 \cos(m\pi x)\sin(n\pi x)dx$ *is not a counterexample, since it is zero only for* $m + n$ *even numbers.*

For the eigenfunctions $\{y(\lambda_n, x)\}$, we cannot refer directly to Liouville's Theorem but, considering also the bounds we found in Appendix B, we can refer to the following formulation of the Phragmén–Lindelöf theorem [18] (p. 177).

Theorem 5. *If* $f(\lambda)$ *is holomorphic in the semi-plane* $Im\, \lambda \geq 0$, *if it is bounded on the real axis and*

$$|f(\lambda)| \leq C\, e^{|\lambda|^\alpha}, \quad \alpha < 1, \text{ for } \lambda_I = Im\lambda > 0$$

then $|f(\lambda)|$ *is bounded in the upper semi-plane. The same can be stated for the lower semi-plane, if* $f(\lambda)$ *is holomorphic on the full complex plane and the previous condition is satisfied.*

So the first step to the closure is the following theorem.

Theorem 6. *Given the entire holomorphic function*

$$f(\lambda) = \int_0^1 \rho(x)\varphi(x)\, y(\lambda, x)\, dx, \;\; \varphi \in L^2([0,1]),$$

if $f(\lambda_n) = 0$ for every $n \in \mathbb{N}$, then $f(\lambda)$ is identically zero:

$$f(\lambda) \equiv 0, \textit{ for all } \lambda \in \mathbb{C}.$$

Proof. Proceeding as previously, define the function

$$g(\lambda) = \frac{\int_0^1 \rho(x)\varphi(x)y(\lambda,x)dx}{y(\lambda,1)}, \tag{48}$$

which again results to be an entire holomorphic function, because the zeros in λ of $y(\lambda, x)$ are again simple. We have first to prove its boundedness on the real axis. The limitation on the compact set $|\lambda_I| \leq \epsilon$, $\lambda_R \leq \Lambda$ is obvious. For any n,

$$g(\lambda_n) = \lim_{\lambda \to \lambda_n} g(\lambda) = \frac{\int_0^1 \rho(x)\varphi(x)\frac{\partial y(\lambda_n,x)}{\partial \lambda}dx}{\frac{\partial y(\lambda_n,1)}{\partial \lambda}}$$

and

$$\left| \frac{\frac{\partial y(\lambda_n,x)}{\partial \lambda}}{\frac{\partial y(\lambda_n,1)}{\partial \lambda}} \right| = \left| \frac{\frac{\partial y(\lambda_n,1)}{\partial x}\frac{\partial y(\lambda_n,x)}{\partial \lambda}}{\int_0^1 \rho(x)y^2(\lambda_n,x)dx} \right| < C, \tag{49}$$

uniformly in n from Equation (17), from the bounds on the derivatives stated in Lemma 2, and finally from Equation (42). We have to control that the conditions of Phragmén–Lindelöf Theorem are satisfied for $|\lambda_I| \leq \epsilon, \lambda_R \to \infty$ and then for $|\lambda_I| > \epsilon$.

We know that $y(\lambda, x)$ is represented locally by Equations (27) and (28), with the integral of the order of $O(\frac{1}{\bar{k}_i^2})$, so everything goes as for Equation (44), and we can conclude that $g(\lambda)$ is uniformly bounded for $\lambda_R > 0$. For $\lambda_R\rho(x) - q(x) < 0$, $y(\lambda, x)$ is a function of exponential type, increasing in x, and then

$$|g(\lambda)| = \frac{\left|\int_0^1 \varphi(x)\rho(x)y(\lambda,x)dx\right|}{|y(\lambda,1)|} < \frac{|y(\lambda,1)|\int_0^1 |\varphi(x)|\rho(x)dx}{|y(\lambda,1)|} < C_\varphi, \tag{50}$$

so the limitation of $|g(\lambda)|$ along the full real axis is established. Also the limitation of $|g(\lambda)|$ for $\lambda_R\rho(x) - q(x) < 0$, and for any λ_I, is obvious: Equation (A9) in Appendix B says that, for any $|\lambda_I|$, $|y(\lambda, x)|$ is strictly increasing, so again Equation (50) is valid.

It remains to control $g(\lambda)$ for $|\lambda_I| > \epsilon$. At the end of Appendix B, e.g., Equation (A22), there is a bound of $g(\lambda)$, for $\lambda_R > \Lambda$ and $|\lambda_I| > \epsilon$, slightly increasing in $|\lambda|$. So we have a much better limitation than the one needed for the Phragmén–Lindelöf theorem, and $|g(\lambda)|$ is bounded on the upper semi-plane. The same result is also true for the lower semi-plane, giving us the possibility now to use the Liouville Theorem

$$g(\lambda) = C_\varphi, \quad \text{that is } \int_0^1 \rho(x)\varphi(x)y(\lambda,x)dx = C_\varphi y(\lambda,1),$$

the constant depending on $\varphi(x)$. It is easy to see now that C_φ has to be zero. Reasoning as in Equation (46), taking λ_n^* such that $y'(\lambda_n^*, 1) = 0$, $\lambda_n^* \to \infty$, we have, from Equation (40), $|y(\lambda_n^*, 1)| > \dfrac{C}{\sqrt{\lambda_n^*}}$. On the other hand, from Theorem 3,

$$\left|\int_0^1 \rho(x)\varphi(x)y(\lambda, x)dx\right| = o\Big(\frac{1}{\sqrt{\lambda}}\Big),$$

valid for any $\lambda > \Lambda$. Then the conclusion $C_\varphi = 0$ follows. □

Theorem 7. *If $\varphi \in L^2([0,1])$, from*

$$\int_0^1 \rho(x)\varphi(x)y(\lambda, x)dx = 0 \textit{ for all} \lambda \in \mathbb{C} \tag{51}$$

it follows that $\varphi(x)$ is the null function.

Proof. Now, we cannot infer directly the statement from the inverse sine Fourier transform. Here is a direct proof. Take $\lambda_I = 0$ and $\lambda_R < 0$. Denote with $\varphi^+(x)$ and $\varphi^-(x)$, respectively, the positive and negative parts of $\varphi(x)$ in the interval $[0,1]$. Since $y(\lambda, x) > 0$, we can write

$$\int_0^x \rho(x')\varphi^+(x')y(\lambda, x')dx' - \int_0^x \rho(x')\varphi^-(x')y(\lambda, x')dx' +$$

$$+ \int_x^1 \rho(x')\varphi^+(x')y(\lambda, x')dx' - \int_x^1 \rho(x')\varphi^-(x')y(\lambda, x')dx' = 0.$$

Apply to these integrals the mean value theorem. Then

$$y(\lambda, \xi_{11}(\lambda)) \int_0^x \rho(x')\varphi^+(x')dx' - y(\lambda, \xi_{12}(\lambda)) \int_0^x \rho(x')\varphi^-(x')dx' + \tag{52}$$

$$+ y(\lambda, \xi_{21}(\lambda)) \int_x^1 \rho(x')\varphi^+(x')dx' - y(\lambda, \xi_{22}(\lambda)) \int_x^1 \rho(x')\varphi^-(x')dx' = 0.$$

The function $y(\lambda, x)$ is of an exponential type for $\lambda < 0$, so it is easy to understand that, for example, $\int_x^1 \rho(x)\varphi^\pm(x)y(\lambda, x)dx \underset{\lambda \to -\infty}{\longrightarrow} \int_{1-\epsilon}^1 \rho(x)\varphi^\pm(x)y(\lambda, x)dx$, if $\int_{1-\epsilon}^1 \rho(x)\varphi^\pm(x)dx > 0$ for any small ϵ. This means that the points $\xi_{ij}(\lambda)$ approach, the extremum of the set where the function is different from zero. We need the following result. Knowing that $y(\lambda, x)$ and $y'(\lambda, x)$ are positive increasing functions for $\lambda < 0$, see Equation (30), then for $x > x_1$

$$y(\lambda, x) = y(\lambda, x_1) + (x - x_1)y'(\lambda, x_1) + \int_{x_1}^x (x - x')\big[|\lambda|\rho(x') + q(x')\big]y(\lambda, x')dx' \tag{53}$$

$$> y(\lambda, x_1) + \inf\big[|\lambda|\rho(x') + q(x')\big]y(\lambda, x_1) \int_{x_1}^x (x - x')dx'$$

from which, for $x_2 > x_1$,

$$\frac{y(\lambda, x_2)}{y(\lambda, x_1)} > 1 + \inf\big[|\lambda|\rho(x') + q(x')\big] \int_{x_1}^{x_2} (x_2 - x')dx' \underset{|\lambda| \to \infty}{\longrightarrow} \infty.$$

Suppose then that $\xi_{22}(\lambda) \to \xi_{22}^*$ is the biggest one of those limits; divide all the terms of Equation (52) by $y(\lambda, \xi_{22}(\lambda))$, and take the limit $\lambda \to -\infty$. The conclusion is that $\int_x^1 \rho(x')\varphi^-(x')dx' = 0$. The same can subsequently be obtained for $\int_x^1 \rho(x')\varphi^+(x')dx' = 0$. If it were $\xi_{22}^* = \xi_{21}^*$, it would directly be $\int_x^1 \rho(x')\varphi(x')dx' = 0$. The choice of x is arbitrary, so

$$\int_x^1 \rho(x')\varphi(x')dx' = 0, \quad \text{for any } x > 0.$$

The conclusion is

$$\varphi(x) = 0 \quad a.e.$$

□

Remark 7. *We know that if $\int_0^1 f(x)g(x)dx = 0$ for a large class of functions $g(x)$, then it is $f(x) = 0$. For that, the class of functions for example, is enough, $e^{\lambda x}$, $\lambda > 0$, as we just proved.*

The previous Theorem 7 can be rephrased in the following form:

Theorem 8. *The set of coefficients $a_n(x)$ of the series expansion (9) of $y(\lambda, x)$ form a closed set in the space L^2.*

Proof. If the integrals of the coefficients satisfy the conditions

$$\int_0^1 \rho(x)\varphi(x)a_n(x)dx = 0 \text{ for every } n \in \mathbb{N},$$

then the holomorphic function $f(\lambda) = \displaystyle\int_0^1 \rho(x)\varphi(x)y(\lambda, x)dx$ is identically zero, (and vice versa) and, $\varphi(x)$ being a zero function for Theorem 7, the conclusion follows. Lebesgue's Theorem lets us interchange the integration with the sum of the series. □

In the next section, we will obtain the same result as Theorems 6 and 7 with a very different method, in the particular case of $q(x) = \alpha\rho(x)$.

5. An Interesting Byproduct

Considering the integral Equation (6), we can rewrite Equation (51), valid for $\lambda \in \mathbb{C}$, as

$$\int_0^1 \rho(x)\varphi(x)y(\lambda, x)dx = \int_0^1 \rho(x)\varphi(x)\, x\, dx +$$

$$+ \int_0^1 \rho(x)\varphi(x)\, dx \int_0^x (x - x')q(x')y(\lambda, x')\, dx' - \lambda \int_0^1 \rho(x)\varphi(x)\, dx \int_0^x (x - x')\rho(x')y(\lambda, x')\, dx' = 0.$$

That is,

$$C_\varphi + \int_0^1 q(x')y(\lambda, x')dx' \int_{x'}^1 (x - x')\rho(x)\varphi(x)dx - \lambda \int_0^1 \rho(x')y(\lambda, x')dx' \int_{x'}^1 (x - x')\rho(x)\varphi(x)dx = 0$$

or

$$C_\varphi + \int_0^1 q(x')y(\lambda, x')\varphi_1(x')dx' - \lambda \int_0^1 \rho(x')y(\lambda, x')\varphi_1(x')dx' = 0,$$

having defined

$$C_\varphi = \int_0^x \rho(x)\varphi(x)\, x\, dx \quad \text{and } \varphi_1(x') = \int_{x'}^1 (x - x')\rho(x)\varphi(x)dx. \tag{54}$$

The conclusion is

$$C_\varphi + f(\lambda) - \lambda h(\lambda) = 0, \tag{55}$$

with the implicit definition of the functions. Note that, with $\varphi(x) \in L^2$, $\varphi_1(x) \in BV$. Easy conclusions follow if we suppose that $q(x) = \alpha\rho(x)$. In this case it has to be, for λ complex,

$$C_\varphi = (\lambda - \alpha)h(\lambda), \quad \text{and then } C_\varphi = 0,\ h(\lambda) = 0.$$

That is, we are led to the eigenvalue problem

$$y'' + \lambda\rho(x)y = 0, \qquad y(0) = y(1) = 0,$$

a little different from the classical Sturm–Liouville eigenvalue problem. We can think that asymptotically, there will not be a big difference.

It is interesting to underline that with $\rho = \rho(x)$ and $q(x) = \alpha\rho(x)$, we have the following:

Proposition 9. *For normalized solutions, $\int_0^1 y'(\lambda_n, x)y'(\lambda_m, x)dx = \sqrt{\lambda_n\lambda_m}\delta_{nm}$, showing the orthogonality of the derivatives with weight 1. See the discussion following the proof of Theorem 10. From this, we again have $y'(\lambda, x) = O(\sqrt{\lambda})$, $x \in [0, 1]$.*

The interesting result is that, from Theorem 6,

$$\int_0^1 \rho(x)\varphi(x)y(\lambda, x)dx = 0,\ \lambda \in \mathbb{C},$$

we obtain $C_\varphi = 0$ and $h(\lambda) = 0$, that is

$$\int_0^1 \rho(x)\varphi(x)\,x\,dx = 0, \quad \text{and} \int_0^1 \rho(x)\varphi_1(x)\,y(\lambda, x)\,dx = 0,\ \lambda \in \mathbb{C}. \tag{56}$$

Then, applying the result found for $\varphi(x)$ to $\varphi_1(x)$, one has

$$\int_0^1 \rho(x)\varphi_1(x)\,x\,dx = 0.$$

Note that $\varphi_1(0) = \varphi_1(1) = 0$. We now show that $\varphi(x)$ is the null function.

We are led to consider the recursive sequence, starting from $\varphi \in L^2([0, 1])$,

$$\varphi_{n+1}(x) = \int_x^1 (x' - x)\rho(x')\varphi_n(x')dx',$$

which satisfies the conditions $\displaystyle\int_0^1 \rho(x)\varphi_{n+1}(x)xdx = 0$, $\varphi_{n+1}(0) = \varphi_{n+1}(1) = 0$, and the differential equation

$${\varphi_{n+1}}''(x) = \rho(x)\varphi_n(x).$$

The sequence converges uniformly to zero, since $|\varphi_{n+1}(x)| < \dfrac{\alpha^n}{(n)!}\int_0^1 |\varphi_0(x)|dx$, where $\alpha = \sup_x \rho(x)$.

We have the following result:

Theorem 9. *If, for $\rho(x) > 0$, the sequence*

$$\varphi_{n+1}(x) = \int_x^1 (x' - x)\rho(x')\varphi_n(x')dx',$$

starting from $\varphi(x) \in L^2$, satisfies the conditions

(i) $\int_0^1 \rho(x)\varphi_n(x)xdx = 0$ for any n, and (ii) one of the functions $\varphi_n(x)$ of the sequence has a finite number of zeros, then the function $\varphi(x) \in L^2$ is a zero function.

Remark 8. *Observe that the condition (ii) is connected to the one needed for the validity of the Hankel integral transform [19].*

Remark 9. *Even if $\varphi(x)$ is identically zero in an interval $[x_1, x_2]$, the function $\varphi_1(x) = \int_x^1 (x' - x)\rho(x')\varphi_0(x')dx'$ has at most one zero in that interval, because*

$$\varphi_1'(x) = -\int_x^1 \rho(x')\varphi(x')dx'$$

is constant for $x \in [x_1, x_2]$.

Proof of Theorem 9. Starting from $\varphi(x) \in L^2$, if some $\varphi_n(x)$ has a finite number of zeros, there is an iterate φ_{n+k} without zeros, because, just looking at the differential equation and considering the convexity of the functions, we have that the number of zeros decreases in the iterates. Between the last zero of $\varphi_n(x)$ and 1, $\varphi_{n+1}(x)$ has no zeros. Furthermore, the number of further zeros of $\varphi_{n+1}(x)$ is less or equal than the number of its changes in convexity, and the latter number is equal to the number of zeros of the odd order of $\varphi_n(x)$. That is, $\varphi_{n+1}(x)$ has at least one less zero than $\varphi_n(x)$. So, if $\varphi_{n+k}(x)$ has no zeros and has to satisfy condition (i), it has to be identically zero and, tracing back, all the previous functions have to be zero. □

The conclusion is the alternative proof of the following

Proposition 10. *The sequence of eigenfunctions $y(\lambda_n, x)$ of the Sturm–Liouville equation, with $\rho(x) \neq const$, $q(x) = \alpha\rho(x)$, is closed in the space L^2 satisfying the hypothesis (ii) of the theorem.*

6. The Uniform Convergence of the Fourier Series

We give a direct proof of the following result, stronger than the completeness property and similar to the property of the trigonometric series. In this section, we take normalized solutions $y(\lambda, x)$, giving $y(\lambda, x)$ and $\dfrac{y'(\lambda, x)}{\sqrt{\lambda}}$ uniformly bounded for $\lambda > 0$.

Theorem 10. *If $\varphi(x)$, $x \in [0,1]$ is an absolutely continuous function, with $\varphi(0) = \varphi(1) = 0$ and if $\varphi' \in L^2([0,1])$ (we already know that φ' is in $L^1([0,1])$), then the Fourier series converges uniformly to $\varphi(x)$.*

See the theorem in [20], for the classic case.

Proof. We add at the moment the condition that $\varphi'(x)$, $x \in [0,1]$ is a BV function. In Appendix D, the proof will be given with the more general condition on $\varphi'(x)$. Then

$$\varphi_n = \int_0^1 \varphi(x)\rho(x)y(\lambda_n, x)dx = \frac{1}{\lambda_n}\int_0^1 \varphi(x)\left[q(x)y(\lambda_n, x) - y''(\lambda_n, x)\right]dx =$$

$$= \frac{1}{\lambda_n}\int_0^1 \varphi(x)q(x)y(\lambda_n, x)dx + \frac{1}{\sqrt{\lambda_n}}\int_0^1 \varphi'(x)\frac{y'(\lambda_n, x)}{\sqrt{\lambda_n}}dx\,.$$

The last integral goes asymptotically as $\frac{1}{\sqrt{\lambda_n}} \sim \frac{1}{n}$, according to Proposition A4 in Appendix C, where it is the unnormalized $y'(\lambda, x)$ to be uniformly bounded, here replaced by $\dfrac{y'(\lambda_n, x)}{\sqrt{\lambda_n}}$. So we have the absolute, uniform convergence for every x

$$\left|\sum_n \varphi_n y(\lambda_n, x)\right| \leq \sum_n |\varphi_n||y(\lambda_n, x)| < C\sum_n \frac{1}{\lambda_n} < \infty\,,$$

for the uniform boundedness of normalized $y(\lambda, x)$, and it is uniformly $\sum_n \varphi_n y(\lambda_n, x) = \varphi(x)$, for the closure of the set of eigenfunctions. □

It is interesting to observe that the functions $\dfrac{y'(\lambda_n, x)}{\sqrt{\lambda_n}}$ are, asymptotically, "nearly" orthonormal, and exactly orthonormal if $q(x) = \alpha\rho(x)$. In fact, we have

$$\int_0^1 \frac{y'(\lambda_n, x)}{\sqrt{\lambda_n}} \frac{y'(\lambda_m, x)}{\sqrt{\lambda_m}} dx = -\int_0^1 \frac{y(\lambda_n, x)}{\sqrt{\lambda_n}} \frac{y''(\lambda_m, x)}{\sqrt{\lambda_m}} dx =$$

$$= -\frac{1}{\sqrt{\lambda_n \lambda_m}} \int_0^1 q(x) y(\lambda_n, x) y(\lambda_m, x) dx + \sqrt{\frac{\lambda_m}{\lambda_n}} \int_0^1 \rho(x) y(\lambda_n, x) y(\lambda_m, x) dx =$$

$$= \delta_{nm} - \frac{1}{\sqrt{\lambda_n \lambda_m}} \int_0^1 q(x) y(\lambda_n, x) y(\lambda_m, x) dx.$$

The last terms are small, but there is nevertheless a problem of convergence: the sum in m of δ_{nm} gives 1 for any n, but for the last terms, the convergence has to be proved.

The closure (and completeness) of the set $\{y(\lambda_n, x)\}$ in $L^2([0, 1])$ can be stated independently through the methods of the calculus of variations [17] (p. 160). Here, we observe that the space of C^2 functions, null at the border, orthogonal to all the eigenfunctions $y(\lambda_n, x)$, is void. Otherwise, we could solve the problem of the minimum of the functional (20) in this space [13] (p. 198), and obtain an eigenfunction $y^*(x)$ that has to have a definite (finite) number of zeros inside the interval $(0, 1)$, and then has to be already in the set $\{y(\lambda_n, x)\}$, showing that the space of C^2 function, null at the border, orthogonal to all $y(\lambda_n, x)$, is void. Now, if $\psi(x)$ is an arbitrary C^2 function, null at the border, in the integral

$$\int_0^1 \rho(x) \varphi(x) \psi(x) dx$$

we can substitute to $\psi(x)$ its uniformly convergent Fourier series. So, if we have

$$\int_0^1 \rho(x) \varphi(x) y(\lambda_n, x) dx = 0, \quad \text{for every } n$$

then the given integral is identically zero for any C^2 function $\psi(x)$ null at the border, and then $\varphi(x)$ is the null function [21]. Observe that in this way, we did not give an alternative proof of completeness, since the uniform convergence comes from Theorem 10, where the closure was needed.

7. The Equations of Mathematical Physics

The equations of interest of Mathematical Physics are the equations of Legendre, of Hermite, and of Laguerre. No one of them satisfies the conditions considered in the present paper. Specifically, in the case of the Legendre equation, defined in the interval $(-1, 1)$, the function $p(x)$ is null at the border, and the equation becomes singular. In the case of the Hermite, or Laguerre equations, their sets of definitions are the axis $(-\infty, +\infty)$ or the positive real axis $(0, +\infty)$, respectively. The unbounded set is not the major difficulty. For example, [11] (p. 107), in the Laguerre equation, with $\alpha = 0$,

$$(xe^{-x}y')' + \lambda e^{-x} y = 0\,,$$

making the change in variable

$$x = \ln\left(\frac{1}{t}\right),$$

the semi-axis $(0, +\infty)$ goes to $(0, 1)$, but the new differential equation is singular at the origin

$$\frac{d}{dt}\left(\ln\left(\frac{1}{t}\right)t^2y'\right) + \lambda y = 0.$$

The problem is that in the present cases, the eigenfunctions and the corresponding eigenvalues are defined by the behavior at ∞, and not by the zeros of the solutions, so we cannot apply the present procedure. In any case, this is not a great handicap, since polynomial solutions for them are easily found.

We have the following classical theorem, of which we give a proof, without using the generating functions of the polynomials and the Weierstrass Theorem, for compact sets, as in the proof attributed to J.Von Neuman and reported in [12] (p. 95).

Theorem 11. *The generalized Laguerre and Hermite polynomials, multiplied by the corresponding weight functions are closed sets in the space $L^2(0, +\infty)$ and $L^2(-\infty, +\infty)$, respectively.*

Proof. We can proceed in the following way. Consider the "natural" weight functions e^{-x} for the integration of x^n in the set $[0, \infty)$, and e^{-x^2} in the set $(-\infty, \infty)$. For $\varphi(x) \in L^2$ we can define the holomorphic functions

$$f(z) = \int_0^\infty \varphi(x)e^{-x}e^{zx}dx \quad \text{and} \quad g(z) = \int_{-\infty}^\infty \varphi(x)e^{-x^2}e^{zx}dx, \tag{57}$$

the first one holomorphic for $Re z < 1$ and the last one for $z \in \mathbb{C}$. We have

$$f(z) = \int_0^\infty \varphi(x)e^{-x}\sum_{n=0}^\infty \frac{(zx)^n}{n!}dx = \sum_{n=0}^\infty z^n \int_0^\infty \varphi(x)e^{-x}\frac{x^n}{n!}dx,$$

$$g(z) = \int_{-\infty}^\infty \varphi(x)e^{-x^2}\sum_{n=0}^\infty \frac{(zx)^n}{n!}dx = \sum_{n=0}^\infty z^n \int_{-\infty}^\infty \varphi(x)e^{-x^2}\frac{x^n}{n!}dx\,.$$

For the interchange of the series with the integrals, and the convergence of the series, see bounds in Appendix A. Then $f(z)$ and $g(z)$ are represented, around zero, by power series, with radius $R = 1$ and $R = \infty$ respectively.

So we have that the conditions for every n,

$$\int_0^\infty \varphi(x)e^{-x}x^n dx = 0 \quad \text{and} \quad \int_{-\infty}^\infty \varphi(x)e^{-x^2}x^n dx = 0,$$

imply that $f(z)$ and $g(z)$ are identically zero, the first one also in the full semiplane $Re z < 1$. Then

$$f(ik) = \int_0^\infty \varphi(x)e^{-x}e^{ikx}dx = 0 \quad \text{and} \quad g(ik) = \int_{-\infty}^\infty \varphi(x)e^{-x^2}e^{ikx}dx = 0,$$

for any k. From the inverse of the Fourier Transform, $\varphi(x)$ is the null function. So the set of monomials $\{x^n\}$ is, with the stated weight functions, a closed (and complete) set in L^2, and so are the sets of orthogonal polynomials $L_n(x)$ and $H_n(x)$. $\square$

8. A Final Remark About the Possibility of General Integral Transform

To conclude our reasoning, we ask ourselves if it is possible to define in general an integral transform like the Hankel transform. The idea could be to repeat what is true for the Bessel functions. In that case, the eigenvalues and the eigenfunctions are defined in the interval $[0,1]$. Then, consider the orthogonal condition given, from Formula (18), by

$$\int_0^1 J_n(k_m^{(n)}r)J_n(k_{m'}^{(n)}r)rdr = \frac{1}{2}\left(\frac{\partial J_n(z)}{\partial z}\right)^2_{(z=k_m^{(n)})}\delta_{mm'},$$

where $k_m^{(n)}, k_{m'}^{(n)} \quad m, m' = 1, 2, \ldots$ are the eigenvalues of the Bessel functions of order n. In this relation, we can turn to the continuous values k, k', and can expect, correctly in this case, that the symbol of Kronecker turns into the Dirac function, also via considering that the Bessel functions are analytic in the variable kr, so we can extend the integration in r to $[0,\infty)$. Obviously, in this "transition", we need to introduce a factor of conversion from the symbol of Kronecker to the δ function. The conclusion is that we have

$$k\int_0^\infty J_n(kr)J_n(k'r)rdr = \delta(k-k')$$

and the symmetric relation

$$r\int_0^\infty J_n(kr)J_n(kr')kdk = \delta(r-r'),$$

taking us to the Hankel transform

$$\widehat{\varphi}(k) = \int_0^\infty \varphi(r)J_n(kr)rdr$$

with the anti-transform

$$\varphi(r) = \int_0^\infty \widehat{\varphi}(k)J_n(kr)kdk.$$

In this case, k and r are the respective factors of conversion.

Now we could be tempted to follow the same line of reasoning in more generality. From the orthogonal relation (and $\|\cdot\|_2$ indicating the 2-norm in $L^2([0,1])$)

$$\int_0^1 y(k_n,x)y(k_m,x)\rho(x)dx = \|y(k_n)\|_2^2\delta_{nm} \tag{58}$$

we could try to go to

$$f(k)\int_0^1 y(k,x)y(k',x)\rho(x)dx = \delta(k-k') \tag{59}$$

with $f(k)$ a suitable factor of conversion. For example, from

$$\sum_m \varphi(k_m)\int_0^1 y(k_n,x)y(k_m,x)\rho(x)dx = \varphi(k_n)\|y(k_n)\|_2^2,$$

we could try to go from the discrete variables k_n to the continuous k, obtaining

$$\sum_m \varphi(k_m)\int_0^1 y(k,x)y(k_m,x)\rho(x)dx = \varphi(k)\|y(k)\|_2^2$$

and a similar expression taking the sum with respect to n. Naturally, there lies the difficulty to go from k_n, k_m to k, k' at the same time. But there are in fact more stringent difficulties. One is the asymmetry between $x \in [0,1]$ and $k \in [0,\infty)$ (or $k \in (-\infty,\infty)$). Now, contrary to the Bessel functions and trigonometric functions, which depend on the product kr or kx, there is no possibility, in general, to go from $x \in [0,1]$ to $x \in [0,\infty)$ (or $x \in (-\infty,\infty)$). Still more importantly, to pass from (58) to (59), the integral (58) has to diverge for k_n, k_m going to k. This is possible only if, remaining with a finite interval of integration, the integrand becomes singular in the limit, as in the case of Poisson kernel. This is not the case for Equation (58).

So, unfortunately, the conclusion is that, except for the Fourier Transform and the Hankel transform, with the trigonometric functions and the Bessel functions being analytic in the product of varibles, there is no possibility to define in general other integral transforms, starting from the solutions of the Sturm–Liouville equation in the interval $[0,1]$.

9. Discussion

We saw that, apart from some technical details, the steps to obtain some properties of the Sturm–Liouville equation are very simple. First of all, the use of the Implicit Function Theorem elucidates, in a clear way, the dependence of the zeros of the solutions of Equation (5) on the positive parameter λ. Furthermore, the complex analysis takes us directly to the closure property in L^2 of the set of eigenfunctions of the Sturm–Liouville equation. In fact, retrospectively, it is natural to expect that the function

$$g(\lambda) = \frac{\int_0^1 \rho(x)\varphi(x)y(\lambda,x)dx}{y(\lambda,1)},$$

under the conditions

$$\int_0^1 \rho(x)\varphi(x)y(\lambda_n,x)dx = 0, \;\; y(\lambda_n,1) = 0,$$

would be an entire holomorphic function bounded in the complex plane $\mathbb{C}$, and then conclude that

$$\int_0^1 \rho(x)\varphi(x)y(\lambda,)dx = 0, \qquad \text{for all } \lambda \in \mathbb{C}$$

gives $\varphi(x) = 0$.

A non-trivial output of our procedure is that the closure we obtained in $L^2([0,1])$, and the uniform convergence of the Fourier series in the general case $\lambda\rho(x) - q(x),\; x \in [0,1]$, can be deduced without referring to the Weierstrass Theorem. Furthermore, an analogue of the Riemann–Lebesgue Theorem and the equipartition of the energy are valid for the functions $y(\lambda,x)$.

The principal conclusion of the paper is that we will not be wrong in identifying asymptotically, locally, the eigenfunctions of $y'' - q(x)y + \lambda\rho(x)y = 0$ with $\dfrac{\sin(n\pi x)}{n\pi}$ for $\lambda > \Lambda$, $n\pi \sim \sqrt{\lambda\rho(x_0)}$, x in the neighbourhood of x_0. Notable is the fact that some results for $y(\lambda,x)$, with $\lambda\rho(x) - q(x)$ non-constant, are exactly the same as those for $\lambda\rho - q = cost$.

In this paper, we studied in detail only a finite interval, with the first classical boundary conditions, and non-singular equations, while, e.g., in the equations connected to Mathematical Physics singular equations, or unlimited intervals, do occur. But for these equations, we have polynomial solutions. Further, we showed in a simple way how the Laguerre and Hermite polynomials, with appropriate weight functions, constitute closed sets for not bounded intervals, without referring to their generating functions or to the Weierstrass Theorem for compact sets.

Finally, we concluded negatively on the possibility of defining, in general, an integral transform, apart from the well-known Fourier and Hankel transforms.

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Appendix A

Take $f(z)$ and $g(z)$ as defined in Equation (57). Apply the Cauchy–Schwartz inequality to the function $f(z)$, and consider the following result: for $n \geq 1$,

$$\int_0^\infty \exp(-2x)x^{2n}dx = \frac{1}{2^{2n+1}}(2n)! \to \frac{1}{2\sqrt{\pi n}}(n!)^2,$$

from Stirling's formula.

As for what regards the $g(z)$ function, we have

$$\int_{-\infty}^\infty \exp(-2x^2)x^{2n}dx = \sqrt{\frac{\pi}{2}}\frac{(2n-1)!!}{2^{2n}} = \sqrt{\frac{\pi}{2}}\frac{(2n)!}{2^{3n}n!} \to \frac{1}{2^n\sqrt{n}}n!.$$

So, $f(z) = \int_0^\infty \varphi(x)e^{-x}e^{zx}dx$ and $g(z) = \int_{-\infty}^\infty \varphi(x)e^{-x^2}e^{zx}dx$, are both represented, around zero, by a power series, with radius $R = 1$ and $R = \infty$, respectively.

Appendix B

We start by considering the solutions of the Sturm–Liouville equation for a complex parameter λ. Observe that, even if, for complex λ, many results about $y(\lambda, x)$ resemble those of $\sin[(k_R + ik_I)x]$, the parallel is not absolute, since $\sin[(k_R + ik_I)x]$ is a solution of the equation $y'' + (k_R + ik_I)^2 y = 0$, where λ_R and λ_I are not independent.

To simplify notation, often we shall not insert the arguments of $y(\lambda, x)$ when they are clear from the context. Concurrently, the same letter C shall denote different constants.

Equation

$$y'' - q(x)y + (\lambda_R + i\lambda_I)\rho(x)y = 0, \tag{A1}$$

for $\lambda = \lambda_R + i\lambda_I$, translates into the system

$$\begin{aligned} y_R'' - q(x)y_R + \lambda_R\rho(x)y_R - \lambda_I\rho(x)y_I &= 0 \\ y_I'' - q(x)y_I + \lambda_R\rho(x)y_I + \lambda_I\rho(x)y_R &= 0, \end{aligned} \tag{A2}$$

from which

$$\frac{d}{dx}(y_I y_R' - y_R y_I') = \lambda_I \rho(x)|y|^2 \tag{A3}$$

and

$$y_I(x)y_R'(x) - y_R(x)y_I'(x) = \lambda_I \int_0^x \rho(x')|y(x')|^2 dx' \neq 0, \tag{A4}$$

i.e., Equation (16) for $\mu = \overline{\lambda}, y(\mu, x) = \overline{y(\lambda, x)}$. From (A4), we deduce, again, that there does not exist any $x_0 > 0$ such that $y_R(x_0) = y_I(x_0) = 0$. Furthermore, there does not exist any $x_0^* > 0$ such that $y'_R(x_0^*) = y'_I(x_0^*) = 0$. Thus, it is always $|y|^2 > 0$, and $|y'|^2 > 0$, for $\lambda_I \neq 0$. For normalized solutions,

$$y_I(1)y'_R(1) - y_R(1)y'_I(1) = \lambda_I \tag{A5}$$

holds. From (A4),

$$|y|^2 + |y'|^2 > 2|\lambda_I| \int_0^x \rho(x')|y(x')|^2 dx' \tag{A6}$$

also holds, a bound that could be further improved, considering that $|y'|^2(0) = 1$.

For $x = 1$ a stationary point (recall that for small λ_I, $|y|^2$ oscillates), and for normalized solutions, combining

$$y_I^2 y_R'^2 - 2y_I y'_R y_R y'_I + y_R^2 y_I'^2 = \lambda_I^2,$$

with the square of $y_R y_R{}' + y_I y_I{}' = 0$, it follows that

$$|y|^2|y'|^2 = \lambda_I^2. \tag{A7}$$

For any point x, any $\lambda_R + i\lambda_I$ and for non-normalized solutions, from Equation (A4) it follows that

$$|y|^2|y'|^2 \geq \lambda_I^2 \left[\int_0^x \rho(x')|y|^2(\lambda, x')dx'\right]^2, \tag{A8}$$

guaranteeing again that $|y|$ and $|y'|$ are both different from zero for $\lambda_I \neq 0$ and any $x > 0$. The system (A2) gives the following equations:

$$\frac{1}{2}\frac{d}{dx}|y|^2 = \int_0^x |y'|^2 dx' + \int_0^x [q(x') - \lambda_R \rho(x')]\,|y|^2 dx' \tag{A9}$$

and

$$\frac{1}{2}\frac{d}{dx}|y'|^2 = \frac{1}{2}[q(x) - \lambda_R \rho(x)]\frac{d}{dx}|y|^2 + \lambda_I \rho(x)(y'_R y_I - y'_I y_R) = \tag{A10}$$
$$= \frac{1}{2}[q(x) - \lambda_R \rho(x)]\frac{d}{dx}|y|^2 + \lambda_I^2 \rho(x) \int_0^x \rho(x')|y|^2 dx',$$

the first one showing explicitly the dependence on λ_R and the second one the positive contribution of λ_I^2 to $|y'|^2$. For x in the vicinity of zero, the contribution of the term proportional to λ_I^2 is not dominant, so the solution is oscillating for λ_I small, with a local frequency of the order of $\sqrt{\lambda_R \rho(x)}$.

On the contrary, an exploding behavior appears explicitly when $\lambda_R \rho(x) - q(x) < 0$, for any λ_I. Equations (A9) and (A10) show that $|y|^2$ and $|y'|^2$ are now monotonically increasing, then

$$\frac{d^2}{dx^2}|y|^2 > 2 + 2\underline{k}^2|y|^2, \quad x > 0$$

where $\underline{k}^2 = \inf[q(x) + |\lambda_R|\rho(x)]$. From this, for the null initial conditions, it follows that

$$|y|^2 > 2\int_0^x \frac{\sinh\sqrt{2}\underline{k}(x - x')}{\sqrt{2}\underline{k}} dx' = \frac{\cosh\sqrt{2}\underline{k}x - 1}{\underline{k}^2}, \tag{A11}$$

i.e., an exponential lower bound valid for any $\lambda_I \neq 0$.

Now consider $\lambda_R > 0$. The integration of (A10) gives

$$|y'|^2 = 1 + [q(x) - \lambda_R \rho(x)]|y|^2 + \int_0^x [\lambda_R \rho'(x') - q'(x')|y|^2 dx' + 2\lambda_I^2 \int_0^x \rho(x') \int_0^{x'} \rho(x'')|y|^2 dx''. \tag{A12}$$

Since $|y'|^2$ is always greater than zero, then

$$[\lambda_R\rho(x)-q(x)]|y|^2<1+\int_0^x[\lambda_R\rho'(x')-q'(x')|y|^2dx'+2\lambda_I^2\int_0^x\rho(x')\int_0^{x'}\rho(x'')|y|^2dx''. \tag{A13}$$

We can appreciate that, for high values of $|\lambda_I|$, λ_R small and for $0<x^*\leq x\leq 1$, one has

$$2[\lambda_R\rho(x)-q(x)]|y|^2<1+\int_0^x[\lambda_R\rho'(x')-q'(x')|y|^2dx'+2\lambda_I^2\int_0^x\rho(x')\int_0^{x'}\rho(x'')|y|^2dx'',$$

which means, from (A9) and (A12), that $\dfrac{1}{2}\dfrac{\partial^2|y|^2}{\partial x^2}=|y'|^2+(q(x)-\lambda_R\rho(x))|y|^2>0$, and we have no more oscillations for $x\geq x^*$.

In the extreme case of λ_R small enough such that $q(x)-\lambda_R\rho(x)>0$, it follows from Equation (A9) that $\dfrac{d|y|^2}{dx}>0$ and there will be no oscillations for any λ_I.

Define

$$R=\int_0^1\rho(x)dx.$$

For λ_I sufficiently small that $|\lambda_I|\int_0^1\rho(x)dx=|\lambda_I|R<1$, and $\lambda_R\rho-q>0$, from (A6) and (A12) it follows that

$$|y'|^2+[\lambda_R\rho(x)-q(x)]|y|^2<1+\int_0^x[\lambda_R\rho'(x')-q'(x')|y|^2dx'+|\lambda_I|R(|y|^2+|y'|^2)$$

and so

$$(1-|\lambda_I|R)|y'|^2+[\lambda_R\rho(x)-q(x)-|\lambda_I|R]|y|^2<1+\int_0^x[\lambda_R\rho'(x')-q'(x')|y|^2dx', \tag{A14}$$

that is, for any sign of $\rho'(x)$,

$$|y|^2<\frac{1}{\inf[\lambda_R\rho(x)-q(x)-|\lambda_I|R]}+C\int_0^x|y|^2dx,$$

giving in any case

$$|y|^2<\frac{C}{\inf[\lambda_R\rho(x)-q(x)-|\lambda_I|R]}\sim\frac{C}{\lambda_R}(1+\frac{|\lambda_I|}{\lambda_R}R)<\frac{C}{\lambda_R}(1+\frac{1}{\lambda_R}),\quad \lambda_R>\Lambda. \tag{A15}$$

This shows, at least as an upper bound, the contrasting contributions of λ_I and λ_R, with the increment due to λ_I rendered ineffective by the dumping due to λ_R. Obviously, for small $|\lambda_I|$, it follows that $|y|^2=O(\dfrac{1}{\lambda_R})$. From (A14) it follows that

$$|y'|^2<\frac{C}{1-\lambda_IR}(1+\frac{|\lambda_I|}{\lambda_R}R)<\frac{C}{1-\lambda_IR}(1+\frac{1}{\lambda_R}),\quad \lambda_R>\Lambda \tag{A16}$$

and again, for small λ_I and large λ_R,

$$|y'|^2<C.$$

Further, still for $|\lambda_I|$ small and large λ_R, there are points x^* near 1 where $\dfrac{d|y|^2}{dx}=0$. For those points we have the following interesting result, that we call again "*equipartition*":

Theorem A1 (Equipartition for complex λ, $|\lambda_I|$ *small*)**.** *For the points x^*, where $\dfrac{d|y|^2}{dx} = 0$, we have from Equation (A9)*

$$\int_0^{x^*} |y'|^2 dx = \int_0^{x^*} [\lambda_R \rho(x) - q(x)]|y|^2 dx. \tag{A17}$$

Then we have the following result:

Proposition A1. *For the points x^* where $\dfrac{d}{dx}|y|^2 = 0$,*

$$\frac{1}{\int_0^1 \rho(x)|y|^2 dx} < \frac{C\lambda_R}{x^*} \tag{A18}$$

holds and, for normalized solutions, from Equations (A15) and (A16), for $|\lambda_I|R \ll 1$ and $x^* \sim 1$,

$$\frac{|y|^2}{\int_0^1 \rho(x)|y|^2 dx} < C(1 + \frac{|\lambda_I|}{\lambda_R}R), \qquad \frac{|y'|^2}{\int_0^1 \rho(x)|y|^2 dx} < C\lambda_R(1 + \frac{|\lambda_I|}{\lambda_R}R), \tag{A19}$$

in agreement with Equations (43) for $\lambda_I = 0$, with the normalized solution nearly uniformly bounded.

Proof. From Equation (A12), like from Equation (36), it follows that

$$\{y'^2 + [\lambda_R \rho - q]y^2\} \geq C > 0$$

holds also asymptotically, then

$$\int_0^1 [\lambda \rho(x') - q(x')]y^2 dx' \geq \frac{1}{2}\int_0^{x^*} \{y'^2 + [\lambda \rho(x') - q(x')]y^2\} dx > \frac{x^* C}{2},$$

and Equation (A18) follows. □

For normalized solutions, for any $|\lambda_I|$ and $\lambda_R > \Lambda$, from Equation (A13)

$$|y|^2 < \frac{1}{\inf[\lambda_R \rho(x) - q(x)]} + C\int_0^x |y|^2 dx' + \frac{2\lambda_I^2 R}{\inf[\lambda_R \rho(x) - q(x)]}\int_0^1 \rho(x)|y|^2 dx,$$

$$\text{where } R = \int_0^1 \rho(x) dx\,,$$

it follows and

$$\frac{|y|^2}{\int_0^1 \rho(x)|y|^2 dx} < C + \frac{1}{\inf[\lambda_R \rho(x) - q(x)]}\left[\frac{1}{\int_0^1 \rho(x)|y|^2 dx} + 2\lambda_I^2 R\right],$$

giving

$$\frac{|y|^2}{\int_0^1 \rho(x)|y|^2 dx} < C(1 + \frac{2\lambda_I^2 R}{\lambda_R}), \quad \lambda_R > \Lambda \tag{A20}$$

not uniformly bounded, as expected. This result comes along with Equation (A18), considering the last stationary point. From Equations (A9) and (A10) it cannot always be that $\frac{d|y|^2}{dx} > 0$ for λ_R large.

Similarly we have

$$\frac{|y'|^2}{\int_0^1 \rho(x)|y|^2 dx} < C\lambda_R(1 + \frac{2\lambda_I^2 R}{\lambda_R}), \quad \lambda_R > \Lambda.$$

A final observation. From Equations (A9) and (A10) we obtain

$$\frac{d^3|y|^2}{dx^3}|y|^2 = 4[q(x) - \lambda_R \rho(x)\frac{d}{dx}|y|^2 + 2[q'(x) - \lambda_R \rho'(x)]|y|^2 + \tag{A21}$$

$$+4\lambda_I^2 \rho(x)\int_0^x \rho(x')|y|^2 dx'$$

and a similar equation for y_0^2, without the last term. We wrote an equivalent integral equation for $|y|^2 - y_0^2$. The conclusion was that, even if sup $|y|^2 >$ sup y_0^2, the inequality $|y|^2 > y_0^2$ does not always hold, depending on the values of both λ_R and λ_I. We can anticipate that $|y|^2$ may still have a component with a frequency larger than y_0^2. In fact, from Equation (A10), $\dfrac{d|y'|^2}{dx}$ becomes zero when $\dfrac{d|y|^2}{dx}$ is still positive, while, when $\dfrac{d|y|^2}{dx} = 0$, $|y'|^2$ is already increasing. A behaviour different from that of y_0^2, Equation (35), where the two derivatives are zero at the same point. This property is confirmed by the exact solutions of two simple equations:

$$y_0'' + (\lambda_R^2 - \lambda_I^2)y_0 = 0,\ \lambda_I^2 < \lambda_R^2 \quad (\text{or } \lambda_I^2 \ll \lambda_R^2), \quad \text{giving} \quad y_0^2 = \sin^2\left(\sqrt{\lambda_R^2 - \lambda_I^2}\, x\right),$$

and

$$y'' + (\lambda_R^2 - \lambda_I^2 + 2i\lambda_R\lambda_I)y = y'' + (\lambda_R + i\lambda_I)^2 y = 0$$

with

$$y = \sin((\lambda_R + i\lambda_I)x)$$

and

$$|y|^2 = \sin^2(\lambda_R x) + \sinh^2(\lambda_I x),$$

considering the periodic component of this last expression. In this case $|y|^2$ will always be greater than y_0^2, due to the interplay between the real and imaginary parts of λ.

From (A12) we have, for $\rho'(x) < 0, \lambda_R > 0$

$$|y'|^2 < 1 + 2\lambda_I^2 \int_0^x \rho(x') \int_0^{x'} \rho(x'')|y|^2 dx'',$$

(for $\rho'(x) > 0$, we have to add that term to the second member) and then, from (A8),

$$|y|^2(1) > \frac{\lambda_I^2\left[\int_0^1 \rho(x')|y|^2 dx'\right]^2}{1 + 2\lambda_I^2 \int_0^1 \rho(x')dx' \int_0^{x'} \rho(x')|y|^2 dx''}$$

showing again that, in any case, $|y|^2$ is definitely asymptotically greater than zero.

With regard to $g(\lambda)$, applying this inequality, for $|\lambda_I| > \epsilon$ and $\lambda_R > \Lambda$, to

$$|g(\lambda)|^2 < \frac{\int_0^1 \rho(x)|\varphi^2(x)|dx \int_0^1 \rho(x)|y|^2 dx}{|y|^2(1)} \quad <$$

$$< \frac{C_\varphi \int_0^1 \rho(x)|y|^2 dx \left[1 + 2\lambda_I^2 \int_0^1 \rho(x')dx' \int_0^{x'} \rho(x')|y|^2 dx''\right]}{\lambda_I^2\left[\int_0^1 \rho(x')|y|^2 dx'\right]^2}$$

we end up, from Equation (A18), with

$$|g(\lambda)|^2 < C_1 + C_2\frac{\lambda_R}{\lambda_I^2} < C_1 + C_2\frac{\lambda_R}{\epsilon^2}, \tag{A22}$$

while $|g(\lambda)|^2 < C$ if $\frac{d|y|^2}{dx} > 0$ for any x, as previously noted. The boundedness of $|g(\lambda)|$ for $|\lambda_I| \leq \epsilon$ has been duly considered after Equation (50).

Appendix C

Our first statement is the following

Proposition A2. *If $\varphi(x)$ is a monotonic bounded function, if $\lambda > 0$ and $y(\lambda, 0) = 0$, $y'(\lambda, 0) = 1$, then*

$$\int_0^x \varphi(x')\rho(x')y(\lambda, x')dx' = O(\frac{1}{\lambda}). \tag{A23}$$

Proof. It is enough to apply the following second mean value theorem.

In the following, denote with $f(a^+)$ and $f(a^-)$, respectively, the right and left limits of $f(x)$ in $x = a$.

Theorem A2 (Hobson). *If $f(x)$ is a monotonic bounded function, and $g(x) \in L^1$, then*

$$\int_a^b f(x)g(x)dx = f(a^+)\int_a^\xi g(x)dx + f(b^-)\int_\xi^b g(x)dx. \tag{A24}$$

Remark A1. *A weaker version is reported in [11] (p. 2). See also [22] (p. 169).*

So, if $\varphi(x)$ is a monotonic bounded function,

$$\int_0^x \varphi(x')\rho(x')y(\lambda, x')dx' = \varphi(0^+)\int_0^\xi \rho(x')y(\lambda, x')dx' + \varphi(x^-)\int_\xi^x \rho(x')y(\lambda, x')dx'. \tag{A25}$$

From this, and from (34), the assertion follows. □

Remark A2. *Obviously, the proposition is also true for BV functions.*

We will give a direct proof of this statement in a little different context.

Proposition A3. *If $\varphi(x)$ is an absolutely continuous function then*

$$\int_0^x \varphi(x')\rho(x')y(\lambda, x')dx' = O(\frac{1}{\lambda}).$$

Proof. We have

$$\int_0^x \varphi(x')\rho(x')y(\lambda, x')dx' = \int_0^x \varphi(x')d\,Y_\rho(\lambda, x') = \varphi(x')Y_\rho(\lambda, x')|_0^x - \int_0^x \varphi'(x')Y_\rho(\lambda, x')dx',$$

where

$$Y_\rho(\lambda, x) = \int_0^x \rho(x')y(\lambda, x')dx'.$$

The conclusion comes again from (34), since $|\varphi(x)| \leq C$, $\varphi' \in L^1([0,1])$. □

We also have:

Proposition A4.

$$\int_0^x y(\lambda, x')dx' = O(\frac{1}{\lambda}).$$

Proof. This result follows from simply integrating the integral Equation (27). □

For $\varphi(x) \in BV([0,1])$, from Hobson's Theorem we have

$$\int_0^x \varphi(x')y(\lambda,x')dx' = O\Big(\frac{1}{\lambda}\Big),$$

$$\int_0^x \varphi(x')y'(\lambda,x')dx' = O\Big(\frac{1}{\sqrt{\lambda}}\Big).$$

Remark A3. *From Equation (A24) it follows in particular that, for $\varphi(x)$ monotonic and bounded,*

$$\lim_{k\to\infty}\int_0^1 \varphi(x)\frac{\sin(kx)}{x}dx = \lim_{k\to\infty}\Big[\varphi(0^+)\int_0^{\xi}\frac{\sin(kx)}{x}dx + \varphi(1^-)\int_{\xi}^1\frac{\sin(kx)}{x}dx\Big] = \varphi(0^+)\frac{\pi}{2}.$$

It is easy to understand that, for large k, the major contribution to the integral comes from x around zero, so there is no difference, in the limit, with the integral $\int_0^1 \varphi(x)\frac{\sin(kx)}{2\sin(x/2)}dx$ and, since the limit exists, with the limit of $\int_0^1 \varphi(x)\frac{\sin((k+1/2)x)}{2\sin(x/2)}dx$. Obviously, it is also

$$\lim_{k\to\infty}\int_{-1}^0 \varphi(x)\frac{\sin(kx)}{x}dx = \varphi(0^-)\frac{\pi}{2}.$$

That is, we obtain the formulas needed for the pointwise convergence of the Fourier series for $\varphi(x) \in BV$. We note that while the integral of the Dirichlet kernel $\frac{2}{\pi}\int_0^1 \frac{\sin((n+1/2)\pi x)}{2\sin(\pi x/2)}dx = 1$ for any $n \in \mathbb{N}$, the integral $\frac{2}{\pi}\int_0^1 \frac{\sin(kx)}{x}dx$ is equal to 1 only in the limit.

Appendix D

The following theorem is presented to have the more general condition on $\varphi(x)$ of Theorem 10.

As already noted in Section 6, the key problem is that of the convergence of $\sum_{n,n'} \frac{1}{\sqrt{\lambda_n\lambda_{n'}}}\int_0^1 q(x)y(\lambda_n,x)y(\lambda_{n'},x)dx$. With $q(x) = \alpha\rho(x)$, the argument would be easily simplified. Here $f(x)$ is put in place of $\varphi'(x)$.

Theorem A3. *If $f \in L^2([0,1])$ and $\|y(\lambda_n)\|_2 = 1$, then the series*

$$\sum_n \frac{1}{\sqrt{\lambda_n}}\int_0^1 f(x)\frac{y'(\lambda_n,x)}{\sqrt{\lambda_n}}dx$$

converges absolutely.

Proof. Define

$$a_n = \frac{1}{\lambda_n^{\epsilon}}\int_0^1 f(x)\frac{y'(\lambda_n,x)}{\sqrt{\lambda_n}}dx, \qquad a_n = O\Big(\frac{1}{\lambda_n^{\epsilon}}\Big), \qquad \epsilon \quad \text{small} \tag{A26}$$

and

$$g_m(x) = \sum_{n=1}^{m} a_n\frac{y'(\lambda_n,x)}{\sqrt{\lambda_n}}.$$

From the end of Section 6, it holds that

$$0 \le \int_0^1 |f(x) - g_m(x)|^2 dx \tag{A27}$$

$$= \int_0^1 f^2(x)dx - 2\sum_{n=1}^m a_n \int_0^1 f(x)\frac{y'(\lambda_n,x)}{\sqrt{\lambda_n}}dx + \sum_{n,n'}^m a_n a_{n'}\left[\delta_{nn'} - \frac{\alpha_{nn'}}{\sqrt{\lambda_n\lambda_{n'}}}\right]$$

where

$$\alpha_{nn'} = \int_0^1 q(x)y(\lambda_n,x)y(\lambda_{n'},x)dx\,, \qquad |\alpha_{nn'}| < C,$$

and then

$$0 \le \int_0^1 f^2(x)dx - 2\sum_{n=1}^m a_n^2\lambda_n^\epsilon + \sum_{n=1}^m a_n^2 - \sum_{n,n'}^m a_n a_{n'}\frac{\alpha_{nn'}}{\sqrt{\lambda_n\lambda_{n'}}}$$

i.e.,

$$2\sum_{n=1}^m a_n^2\lambda_n^\epsilon - \sum_{n=1}^m a_n^2 + \sum_{n,n'} a_n a_{n'}\frac{\alpha_{nn'}}{\sqrt{\lambda_n\lambda_{n'}}} \le \int_0^1 f^2(x)dx.$$

Now

$$\left|\sum_{n,n'}^m a_n a_{n'}\frac{\alpha_{nn'}}{\sqrt{\lambda_n\lambda_{n'}}}\right| \le C\sum_{n,n'}^\infty \frac{|a_n a_{n'}|}{nn'} \le C\sum_n \frac{1}{n^{1+2\epsilon}}\sum_{n'}\frac{1}{n'^{(1+2\epsilon)}} \le C,$$

giving for every m

$$\sum_{n=1}^m a_n^2 - C \le \int_0^1 f^2(x)dx$$

and

$$\sum_{n=1}^\infty a_n^2 < \infty\,. \tag{A28}$$

The convergence of (A28) says that the bound (A26) is a crude bound, since, for the oscillatory character of $y'(\lambda,x)$, the integral goes to zero for $\lambda \to \infty$. From (A28)

$$\left|\frac{1}{\sqrt{\lambda_n}}\int_0^1 f(x)\frac{y'(\lambda_n,x)}{\sqrt{\lambda_n}}dx\right| =$$

$$= \frac{1}{\lambda_n^{1/2-\epsilon}}\frac{1}{\lambda_n^\epsilon}\left|\int_0^1 f(x)\frac{y'(\lambda_n,x)}{\sqrt{\lambda_n}}dx\right| \le \frac{C}{n^{1-2\epsilon}}a_n$$

and

$$\sum_n \frac{1}{\sqrt{\lambda_n}}\left|\int_0^1 f(x)\frac{y'(\lambda_n,x)}{\sqrt{\lambda_n}}dx\right| \le C\left[\sum_1^\infty \frac{1}{n^{2-4\epsilon}} + \sum_1^\infty a_n^2\right] < \infty.$$

□

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