A self-contained proof of the Alt-Caffarelli-Friedman monotonicity formula

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Abstract

The Alt-Caffarelli-Friedman monotonicity formula is a cornerstone in the theory of free boundary problems. In this note we provide a self-contained proof of this result. To prove the main stepping stone, namely the *Friedland-Hayman inequality*, we exploit a useful convexity property.

Keywords: monotonicity formula, free boundary problem, blow-up analysis, homogeneous function, variational inequality

MSC Classification 2020: 26A48; 35R35

1 Introduction

The *Alt-Caffarelli-Friedman monotonicity formula* (ACF formula) was first introduced in [4, Lemma 5.1] as a tool designed to deal with the regularity of solutions of a particular two-phase free boundary problem. A crucial issue in facing these kind of problems is to establish the optimal regularity of solutions across the free boundary. In this setting, the ACF formula is a valuable device, together with regularity techniques, to provide estimates for the behavior of the gradient of a solution of the problem in a point of the free boundary, taking into account of the contribution of the two phases.

Given the dimension n, we can depict the typical situation in the ball with radius two $B_2 \subset \mathbb{R}^n$, with center in the origin $\mathbf{0}$ (in the following the subscript of B will refer to its radius). The non negative functions $u_+, u_- \in C(B_2)$ appearing in the ACF formula need to satisfy

$$\begin{cases} \Delta u_+ \geq 0 & \text{in } \{u_+ > 0\}, \\ \Delta u_- \geq 0 & \text{in } \{u_- > 0\}, \\ u_+(x) \cdot u_-(x) = 0 & \text{for every } x \in B_2, \\ u_+(\mathbf{0}) = u_-(\mathbf{0}) = 0, \end{cases} \tag{1}$$

where the first two inequalities hold in the sense of distributions (see Figure 1).

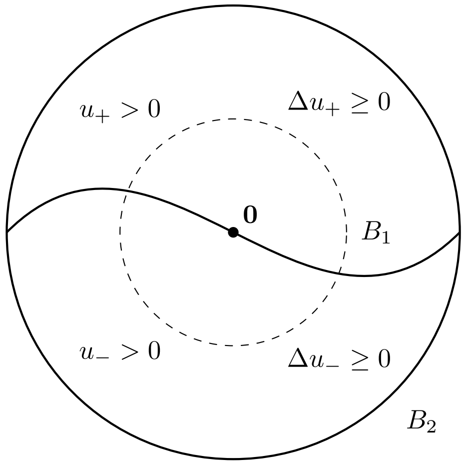

Fig. 1 A pair u_+, u_- satisfying (1) in $\mathbb{R}^2$.

The theory developed in [4] has been designed to understand the model studied in [3], which involves the irrotational flow of two ideal (incompressible) fluids. In terms of their (suitably normalized) stream functions, these properties translate into a null divergence condition.

Despite being a frequently used and very well-known object in the field of free boundary problems, in the wide literature concerning this subject, we were not able to find a self-contained comprehensive proof. This probably happened because the central fact necessary to obtain monotonicity, i.e., the *Friedland-Hayman inequality*, obtained as a corollary of [17, Theorem 3], had not at that time been demonstrated in a totally analytical way. This result is a sharp inequality concerning the growth rates of two homogeneous harmonic functions with Dirichlet boundary conditions on disjoint cones of the Euclidean space. It plays a central role in the proof of the ACF formula, since it is possible to associate u_+ and u_- with two such functions and to obtain through the inequality a lower bound on the growth rate of the ACF formula. The original proof of this powerful tool relies on a result achieved using numerical techniques, see [18], which involves *Hermite's functions*. A way to analytically complete this proof of the Friedland-Hayman inequality is provided in [13, Section 8], which, however, contains rather involved calculations.

A more detailed version of the proof of the ACF formula was given, following a different approach that exploits the one-dimensional *Gaussian measure*, in [10, Chapter 12]. Indeed, an easier proof of the two-dimensional case is provided, while in dimension greater than or equal to three a more accurate analysis and refined tools are needed. However, a part of this proof relies on the unpublished paper [6] (the contents of this work are sketched in [9, Section 2.4]).

The purpose of this paper is to give a self-contained and comprehensive proof of the ACF formula in the case $n \geq 3$, with a different approach to the Friedland-Hayman inequality. The proof we present is *not original*, its structure is the same as [22]. We exploit the content of [20] to obviate a flaw present in the proof of a convexity property. In particular instead of [22, Teorema 4.5] we use Proposition 8, which corrects this result to the extent necessary for our purposes. We have deliberately decided to avoid technicalities related to some regularity issues in order to make the presentation more immediate, while providing adequate references when necessary.

Theorem 1 (ACF formula) *Let u_+, u_- be as in (1). The function $J : (0,1) \to \mathbb{R}$, defined by*

$$J(s) = \frac{1}{s^4} \int_{B_s} \frac{|\nabla u_+(x)|^2}{|x|^{n-2}}\, dx \int_{B_s} \frac{|\nabla u_-(x)|^2}{|x|^{n-2}}\, dx \tag{2}$$

for every $s \in (0,1)$, is finite and increasing.

This result can be proved requiring weaker assumptions on the configuration described by (1), see for example [25, Theorem 1.3], where the functions are only requested to belong to the Sobolev space $H^1(B_1)$, instead of being continuous.

We recall that in [24, Theorem 2.9] the rigidity cases, namely the ones were the function appearing in the formula is constant, are analyzed (this problem was originally considered in [4, Section 6]).

Theorem 2 (Rigidity of the ACF formula) *Let u_+, u_- be as in (1) and J be as in (2). Assume that $J(R_1) = J(R_2)$ for some $0 < R_1 < R_2 < 1$, then either of the following holds:*

- *if $J(R_2) = 0$, then $u_+ \equiv 0$ in B_{R_2} or $u_- \equiv 0$ in B_{R_2};*
- *if $J(R_2) > 0$, then there exist a unit vector $\nu \in \partial B_1$ and two constants $c_+, c_- > 0$ such that*

$$u_+(x) = c_+(x \cdot \nu)^+ \quad , \quad u_-(x) = c_-(x \cdot \nu)^- \quad \text{for } x \in B_{R_2},$$

where $(x \cdot \nu)^+$ and $(x \cdot \nu)^-$ are the positive part and the negative part of the function $x \mapsto x \cdot \nu$, defined for every $x \in B_1$.

We notice that a stability result related to the ACF formula was given in [2, Theorem 1.3].

Theorem 3 (Stability of the ACF formula) *Let $\rho \in [0, 1/2]$. There exists a constant $C = C(n) > 0$ such that the following holds. Suppose that u_+, u_- satisfy (1) and let J be as in (2). Then there exist two constants $c_+, c_- > 0$ and a unit vector $\nu \in \partial B_1$ such that*

$$\int_{B_1 \setminus B_\rho} \left[(u_+(x) - c_+(x \cdot \nu)^+)^2 + (u_-(x) - c_-(x \cdot \nu)^-)^2 \right] dx \le C \log\left(\frac{J(1)}{J(\rho)} \right) \left(||u_+||_2^2 + ||u_-||_2^2 \right),$$

where we denote with $|| \cdot ||_2$ the L^2-norm of a function in B_1. Furthermore, there exists a dimensional constant $\epsilon_0 = \epsilon_0(n) > 0$ such that if the quotient $\log(J(1)/J(0^+)) < \epsilon_0$, then c_+, c_- and ν may be chosen independently from ρ.

We are aware of the fact that in the literature many other generalizations of the ACF formula are available. We have chosen to present only these two results for the sake of shortness, because they are prime examples of the so called *rigidity* and *stability* statements.

The paper is organized as follows. In Section 2 we recall some useful fact regarding changes of coordinates and rearrangements. In Section 3 we study the first Dirichlet eigenvalue on open subsets of the sphere. In Section 4 we prove a convexity property. In Section 5 we give the proof of Theorem 1. In Section 6 we present an open problem.

2 Parametrizations and rearrangements

Consider the manifold $(\mathbb{R}^n, g_{\mathbb{R}^n})$, where $g_{\mathbb{R}^n}$ denotes the (standard) *flat* Euclidean metric. We have that the $(n-1)$-dimensional sphere of unit radius ∂B_1 can be endowed with the Riemannian metric inherited from $(\mathbb{R}^n, g_{\mathbb{R}^n})$, that we call *round* and denote $g_{\partial B_1}$. So it is possible to define the notions of *Riemannian gradient* ∇_ϕ and *Laplace-Beltrami operator* Δ_ϕ on $(\partial B_1, g_{\partial B_1})$, see [11, Definitions 1 and 3, p. 2-3].

The *polar parametrization* of $\mathbb{R}^n$, with respect to the origin $\mathbf{0}$, is given by the function

$$\begin{aligned} \mathscr{P} : \mathbb{R}^+ \times \partial B_1 &\to \mathbb{R}^n \\ (r, \phi) &\mapsto x = r\phi, \end{aligned} \tag{3}$$

we call the parameters in $\mathbb{R}^+ \times \partial B_1$ *polar coordinates* and the first one *radial coordinate*. It is know that $(\mathbb{R}^n \setminus \{\mathbf{0}\}, g_{\mathbb{R}^n})$ is isometric to $(\mathbb{R}^+ \times \partial B_1, g_{\mathbb{R}^+} + r^2 g_{\partial B_1})$, see [23, Section 1.4.4]. Let F be a function with domain $\mathbb{R}^n$, we define $F_{\mathscr{P}} := F \circ \mathscr{P}$. Let $v \in L^1(\mathbb{R}^n)$ by the changes of variables formula we have

$$\int_{\mathbb{R}^n} v(x)\, dx = \int_0^{+\infty} \int_{\partial B_1} v_{\mathscr{P}}(r, \phi)\, r^{n-1}\, d\sigma_\phi\, dr, \tag{4}$$

where $d\sigma_\phi$ is $(n-1)$-dimensional Hausdorff measure on ∂B_1, see [16, Theorem 2.49]. From now on we will write $d\sigma$ instead of $d\sigma_\phi$. Let $\mathbf{r}$ be the vector of the orthonormal frame of $\mathbb{R}^+ \times \partial B_1$ corresponding to the radial coordinate. Let us also assume that v is a twice differentiable function. We can express its gradient in polar coordinates via an orthogonal decomposition (that relies on the orthonormal frame) as

$$(\nabla v)_{\mathscr{P}} = (v_{\mathscr{P}})_r \,\mathbf{r} + \frac{1}{r}\nabla_\phi(v_{\mathscr{P}}), \tag{5}$$

where the subscript r denotes the differentiation with respect to the radial coordinate, see [15, equation (1.4.6)]. Similarly its Laplacian can be written as

$$(\Delta v)_{\mathscr{P}} = (v_{\mathscr{P}})_{rr} + \frac{n-1}{r}(v_{\mathscr{P}})_r + \frac{1}{r^2}\Delta_\phi(v_{\mathscr{P}}), \tag{6}$$

see [15, Lemma 1.4.1].

The *hypershperical parametrization* of ∂B_1, with respect to the *north pole* $p := (1, 0, \ldots, 0)$, is given by the function

$$\begin{aligned} \mathscr{S} : (0,\pi)\times\mathbb{S}^{n-2} &\to \partial B_1 \\ (\theta,\xi) &\mapsto \phi = (\cos(\theta), \sin(\theta)\xi), \end{aligned} \tag{7}$$

where we define $\mathbb{S}^{n-2} := \partial B_1 \cap \{x_1 = 0\}$. We call the parameters in $(0,\pi)\times\mathbb{S}^{n-2}$

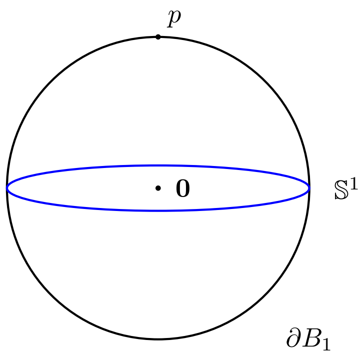

Fig. 2 The circumference of unit radius $\mathbb{S}^1$, in $\mathbb{R}^3$ (in blue).

hypershperical coordinates and the first one *colatitude coordinate*. Notice that it is possible to obtain an explicit expression for the colatitude coordinate, that from now on we will call colatitude, namely

$$\theta = \arccos(p\cdot\phi).$$

This quantity, geometrically, represents the amplitude (in radiant) of the angle between the two radii of ∂B_1 connecting the origin $\mathbf{0}$ with the points p and ϕ, respectively. Let $\theta_0 \in (0,\pi)$, we define the open set

$$\Gamma(\theta_0) := \{\phi \in \partial B_1 \,:\, 0 \le \arccos(p\cdot\phi) < \theta_0\}.$$

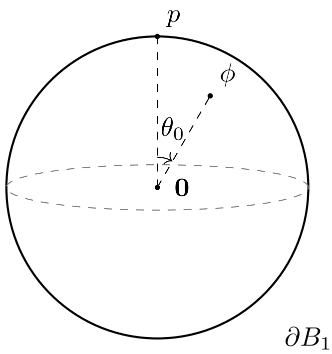

Fig. 3 A point ϕ of colatitude $\theta_0 \approx \pi/6$, in $\partial B_1 \subset \mathbb{R}^3$.

A *spherical cap* of colatitude θ_0 (with center $\psi(p)$, in $\partial B_1 \subset \mathbb{R}^n$) is a set of the type $\psi(\Gamma(\theta_0))$, where $\psi : \partial B_1 \to \partial B_1$ is an isometry of ∂B_1. Recall that the isometry group of ∂B_1 is given by *the orthogonal group* $O(n)$, that is made up of composition of the so-called *rotations* and *reflections*, see [21, Problem 5-8, p. 88].

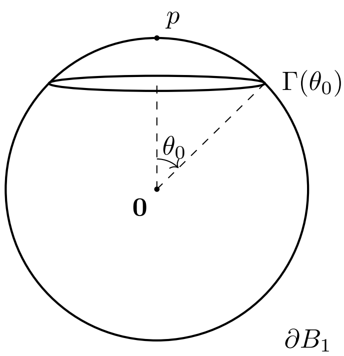

Fig. 4 The set $\Gamma(\theta_0)$ with $\theta_0 \approx \pi/4$ (in $\partial B_1 \subset \mathbb{R}^3$).

It is know that $(\partial B_1, g_{\partial B_1})$ is isometric to $((0,\pi) \times \mathbb{S}^{n-2}, g_{(0,\pi)} + (\sin\theta)^2 g_{\mathbb{S}^{n-2}})$, where $g_{\mathbb{S}^{n-2}}$ is the round metric on $\mathbb{S}^{n-2}$, see [23, Example 1.4.6]. It is possible to define the notions of Riemannian gradient ∇_ξ and Laplace-Beltrami operator Δ_ξ on $(\mathbb{S}^{n-2}, g_{\mathbb{S}^{n-2}})$. Let f be a function with domain ∂B_1, we define $f_{\mathscr{S}} = f \circ \mathscr{S}$. Let $u \in L^1(\partial B_1)$ by the changes of variables formula we have that it holds

$$\int_{\partial B_1} u(\phi)\, d\sigma = \int_0^\pi \int_{\mathbb{S}^{n-2}} u_{\mathscr{S}}(\theta, \xi) \sin^{n-2}(\theta)\, d\sigma_\xi\, d\theta, \tag{8}$$

where $d\sigma_\xi$ is $(n-2)$-dimensional Hausdorff measure on $\mathbb{S}^{n-2}$, see [15, equation (1.5.4)]. Let $\boldsymbol{\theta}$ be the vector of the orthonormal frame of $(0,\pi) \times \mathbb{S}^{n-2}$ corresponding to the colatitude coordinate. Let us also assume that u is a twice differentiable function. We can express its gradient in hypershperical coordinates via an orthogonal decomposition (that relies on the orthonormal frame) as

$$(\nabla u)_{\mathscr{S}} = (u_{\mathscr{S}})_\theta\,\boldsymbol{\theta} + \frac{1}{\sin(\theta)}\nabla_\xi(u_{\mathscr{S}}), \tag{9}$$

where the subscript θ denotes the differentiation with respect to the colatitude. Similarly its Laplace-Beltrami operator can be written as

$$(\Delta_\phi u)_{\mathscr{S}} = \frac{1}{(\sin\theta)^{n-2}}\Big((\sin\theta)^{n-2}(u_{\mathscr{S}})_\theta\Big)_\theta + \frac{1}{(\sin\theta)^2}\Delta_\xi(u_{\mathscr{S}}), \tag{10}$$

see [15, Lemma 1.4.2].

Combining (3) and (7) it is possible to find another parametrization of $\mathbb{R}^n$, given by the function

$$\begin{aligned} &\mathbb{R}^+ \times (0,\pi)\times \mathbb{S}^{n-2} \to \mathbb{R}^n \\ &\qquad (r,\theta,\xi) \mapsto \mathscr{P}(r,\mathscr{S}(\theta,\xi)) = (r\cos(\theta), r\sin(\theta)\xi). \end{aligned} \tag{11}$$

The *stereographic projection* of ∂B_1, with respect to the *south pole* $-p$, is given by the function

$$\begin{aligned} &\mathscr{X} : \partial B_1 \setminus \{-p\} \to \mathbb{R}^{n-1} \\ &\qquad \phi = (\hat\phi, \phi_n) \mapsto \hat x = \frac{\hat\phi}{1+\phi_n}. \end{aligned}$$

It is know that $(\partial B_1 \setminus \{-p\}, g_{\partial B_1})$ is isometric to $(\mathbb{R}^{n-1}, \Upsilon^2 g_{\mathbb{R}^{n-1}})$, where the analytic function $\Upsilon : \mathbb{R}^{n-1} \to \mathbb{R}$ is defined by

$$\Upsilon(\hat x) = \frac{2}{1+|\hat x|^2}$$

for every $\hat x \in \mathbb{R}^{n-1}$, see [21, Proof of Lemma 3.4, p. 37]. Let $v : \mathbb{R}^{n-1} \to \mathbb{R}$ be a function and define $v_{\mathscr{X}^{-1}} := v \circ \mathscr{X}^{-1}$. Suppose in addition that v is twice differentiable and denote Δ_Υ the Laplace-Beltrami operator induced on $\mathbb{R}^{n-1}$ by the metric $\Upsilon^2 g_{\mathbb{R}^{n-1}}$, it holds

$$\Delta_\Upsilon v = \frac{1}{\Upsilon^2}\left(\Delta v + (n-2)\nabla v \cdot \nabla \log \Upsilon\right). \tag{12}$$

It is possible to retrieve this relation by direct calculation exploiting [11, equation (33), p. 5].

In the following it will be useful to produce functions on ∂B_1 that are radially symmetric (with respect to a point of ∂B_1), namely whose superlevel sets are spherical caps with the same center, see [5, Sections 1.1, 1.2 and 7.1]. We notice that each of these functions can be obtained by as a composition of a map that is radially symmetric with respect to p and an isometry of ∂B_1. We denote $|\cdot|$, with a little abuse with respect to the norm of a vector, the $(n-1)$-dimensional Hausdorff measure on ∂B_1. We recall that, given a non-negative measurable function $u : \partial B_1 \to \mathbb{R}$, its *distribution function*, i.e., the map that describes the measure of the superlevel sets, $\mu_u : [0, \infty) \to [0, |\partial B_1|]$ is defined as

$$\mu_u(t) = |\{\phi \in \partial B_1 \, : \, u(\phi) > t\}|$$

for every $0 \leq t < +\infty$. In addition we can define its *decreasing rearrangement* $u^* : [0, |\partial B_1|] \to [0, \infty)$ as

$$u^*(s) = \inf\{t \geq 0 \, : \, \mu_u(t) \leq s\}$$

for every $s \in [0, |\partial B_1|]$. Now it is useful to introduce the map $M : \partial B_1 \to [0, |\partial B_1|]$, defined as

$$M(\phi) = |\Gamma(\arccos(p \cdot \phi))| = |\Gamma(\theta)|$$

for every $\phi \in \partial B_1$. Finally, the *symmetric decreasing rearrangement* $u^\# : \partial B_1 \to (0, +\infty)$ is defined as

$$u^\#(\phi) = (u^* \circ M)(\phi)$$

for every $\phi \in \partial B_1$ (extended by 0 where u^* is not defined). The function $u^\#$ is radially symmetric with respect to p and decreases as the colatitude increases.

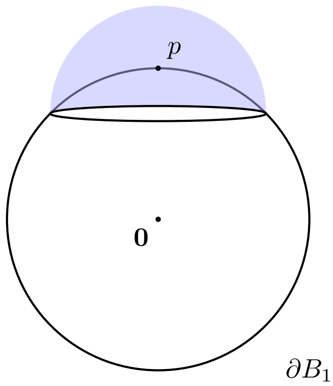

Fig. 5 The graph of a function that is radially symmetric with respect to p (in violet) decreasing as the colatitude increases.

3 The first Dirichlet eigenvalue

For the first part of this section we follow [11], where a regularity hypothesis on the boundary of a set and connectedness of the set are also required, see [11, p. 1]. Since the results and the definitions we borrow are valid even without these assumptions (with the appropriate modifications), we continue to refer to [11].

Let $\Gamma \subset \partial B_1$ be an open set, the *first Dirichlet eigenvalue of* Γ, see [11, p. 17], is

$$\lambda(\Gamma) = \min_{\substack{u \in H_0^1(\Gamma) \\ u \not\equiv 0}} \frac{\displaystyle\int_\Gamma |\nabla_\phi u(\phi)|^2 \, d\sigma}{\displaystyle\int_\Gamma u(\phi)^2 \, d\sigma}. \tag{13}$$

A function u achieving equality in (13) is called *(first Dirichlet) eigenfunction corresponding to* $\lambda(\Gamma)$ and by regularity theory (combining (12) and [7, Analyticity Theorem, p. 136]), we have that u is analytic in Γ and solves

$$-\Delta_\phi u(\phi) = \lambda(\Gamma)u(\phi) \quad \text{for every } \phi \in \Gamma. \tag{14}$$

The eigenfunctions are, as usual, extended by zero in $\partial B_1 \setminus \Gamma$. By the *Courant's nodal domain Theorem*, see [11, p. 19], an eigenfunction corresponding to $\lambda(\Gamma)$ has only a nodal domain, namely the set $\Gamma \setminus \{u = 0\}$ has only a connected component. In particular, when Γ is connected, the eigenfunction corresponding to $\lambda(\Gamma)$ is unique, up to scalar multiples (in this case the eigenvalue is called *simple*). In addition, let $\psi : \partial B_1 \to \partial B_1$ be an isometry (of ∂B_1), we notice that $\lambda(\Gamma) = \lambda(\psi(\Gamma))$.

A central role in the proof of the ACF formula will be played by the properties of the function describing the first Dirichlet eigenvalue of a spherical cap in terms of its colatitude, so we define $\lambda(\theta_0) := \lambda(\Gamma(\theta_0))$. In the case $\theta_0 = \pi/2$, by (11), we notice that the function $u : \Gamma(\pi/2) \to \mathbb{R}$, defined by

$$u_{\mathscr{S}}(\theta, \xi) = \cos(\theta),$$

for every $(\theta, \xi) \in \mathscr{S}^{-1}(\Gamma(\pi/2))$, is the restriction to the sphere ∂B_1 of the positive part of the first euclidean coordinate. In particular u is positive in $\Gamma(\pi/2)$ and vanishes on $\partial\Gamma(\pi/2)$. By (10) we have that u solves (14) with

$$\lambda(\pi/2) = n - 1, \tag{15}$$

so it is an eigenfunction corresponding to $\lambda(\pi/2)$.

We present two useful facts regarding homogeneous functions. To do so we define the *cone generated by* Γ with vertex in the origin (of $\mathbb{R}^n$) as the set

$$\{x \in \mathbb{R}^n \, : x = r\phi, \text{ with } \phi \in \Gamma \text{ and } r \in \mathbb{R}^+\}.$$

Proposition 4 *Let* $\alpha > 0$*,* u *be an eigenfunction corresponding to* $\lambda(\Gamma)$ *and* $w : \mathbb{R}^n \to \mathbb{R}$ *be the function defined by*

$$w_{\mathscr{P}}(r, \phi) = r^\alpha u(\phi),$$

for every $(r, \phi) \in \mathbb{R}^+ \times \partial B_1$*. There exists a unique positive value* $\alpha = \alpha(\Gamma)$*, called the characteristic constant of the set* Γ*, such that* w *is harmonic in the cone generated by* Γ *with vertex in the origin. This values satisfies*

$$\alpha(\Gamma) = \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda(\Gamma)} - \frac{n-2}{2}. \tag{16}$$

Proof Requiring w to be harmonic, by (6) we have

$$0 = r^{\alpha-2}\big[\alpha(\alpha-1) + \alpha(n-1)\big]u(\phi) + r^{\alpha-2}\Delta_\phi u(\phi)$$

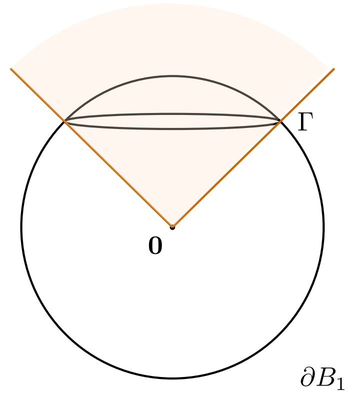

Fig. 6 The cone generated by a set $\Gamma \subset \partial B_1$ with vertex in the origin (of $\mathbb{R}^3$).

for every $(r, \phi) \in \mathbb{R}^+ \times \partial B_1$, that is equivalent by (14) to

$$\alpha^2 + (n-2)\alpha - \lambda(\Gamma) = 0,$$

giving the desired conclusion. □

As a consequence of (4) and (5) we have the following result.

Lemma 5 *Let $\alpha > 0$, $u : \partial B_1 \to \mathbb{R}$ be a differentiable function and $w : \mathbb{R}^n \to \mathbb{R}$ be the function defined by*

$$w_{\mathscr{P}}(r, \phi) = r^\alpha u(\phi)$$

for every $(r, \phi) \in \mathbb{R}^+ \times \partial B_1$. Then

$$\int_{B_1} \frac{|\nabla w(x)|^2}{|x|^{n-2}}\, dx = \frac{1}{2\alpha} \int_{\partial B_1} \left[\alpha^2 u(\phi)^2 + |\nabla_\phi u(\phi)|^2\right] d\sigma.$$

Proposition 6 *Let $\Gamma \subset \partial B_1$ be an open set, then*

$$\alpha(\Gamma) \geq \alpha(\Gamma^\#),$$

where $\Gamma^\#$ is a spherical cap with $|\Gamma^\#| \leq |\Gamma|$. In particular if Γ is a spherical cap, then an eigenfunction w corresponding to $\lambda(\Gamma)$ is radially symmetric with respect to the center of Γ. Namely there exist an isometry $\psi : \partial B_1 \to \partial B_1$ and a function $w^{\mathscr{R}} : [0, \pi) \to \mathbb{R}$ such that

$$(w \circ \psi)_{\mathscr{S}}(\theta, \xi) = w^{\mathscr{R}}(\theta),$$

for every $(\theta, \xi) \in (0, \pi) \times \mathbb{S}^{n-2}$.

Proof Let u be a non-negative eigenfunction corresponding to $\lambda(\Gamma)$ and define the open set

$$\Gamma^+ := \{\phi \in \partial B_1 \,:\, u(\phi) > 0\}.$$

We have that $|\Gamma^+| \leq |\Gamma|$ and u is a non-negative eigenfunction corresponding to $\lambda(\Gamma^+)$. Consider $u^\#$, the symmetric decreasing rearrangement of u, and define the open set

$$\Gamma^\# := \{\phi \in \partial B_1 \,:\, u^\#(\phi) > 0\}.$$

It is a spherical cap, moreover u and $u^{\#}$ are equidistributed, see [5, Proposition 1.30 (a) and p. 219], and in particular it holds $|\Gamma^{\#}| = |\Gamma^{+}|$. Therefore, by the so-called *Pólya-Szegő inequality* on the sphere, see [5, Theorem 7.4], we have $u^{\#} \in H_0^1(\Gamma^{\#})$ and from (13) we obtain

$$\lambda(\Gamma) = \lambda(\Gamma^{+}) = \frac{\displaystyle\int_{\Gamma^{+}} |\nabla_{\phi} u(\phi)|^2 \, d\sigma}{\displaystyle\int_{\Gamma^{+}} u(\phi)^2 \, d\sigma} \geq \frac{\displaystyle\int_{\Gamma^{\#}} |\nabla_{\phi} u^{\#}(\phi)|^2 \, d\sigma}{\displaystyle\int_{\Gamma^{\#}} u^{\#}(\phi)^2 \, d\sigma} \geq \lambda(\Gamma^{\#}), \tag{17}$$

by (16) we retrieve the first statement.

For the second statement we notice that if Γ is a spherical cap, then there exists an isometry $\psi : \partial B_1 \to \partial B_1$ such that $\Gamma = \psi(\Gamma^{\#})$. Since the inequalities in (17) are equalities we have that $u^{\#}$ and $u \circ \psi$ are eigenfunction corresponding to $\lambda(\Gamma^{\#})$. By the connectedness of $\Gamma^{\#}$ and the fact that $u^{\#}$ and u are equidistributed we obtain $u^{\#} = u \circ \psi$, which gives the desired thesis. □

In the following the function $V : (0, \pi) \to \mathbb{R}$ defined by

$$V(\theta) = \left(\frac{n-2}{2}\right)\left(\left(\frac{n-4}{2}\right)(\cot\theta)^2 - 1\right)$$

for every $\theta \in (0, \pi)$, where $\cot(\cdot)$ is the *cotangent function*, will play a central role. Since the minimum in (13) when Γ is a spherical cap is reached by a function that is radially symmetric with respect to the center of Γ, we can narrow down the set of optimal maps to one-variable (the colatitude) functions.

Lemma 7 *Let $\theta_0 \in (0, \pi)$, then*

$$\lambda(\theta_0) = \min_{\substack{v \in H_0^1((0,\theta_0)) \\ v \not\equiv 0}} \frac{\displaystyle\int_0^{\theta_0} \left[v'(\theta)^2 + V(\theta) v(\theta)^2\right] d\theta}{\displaystyle\int_0^{\theta_0} v(\theta)^2 \, d\theta}. \tag{18}$$

There is a unique non-negative function v normalized in $L^2((0, \theta_0))$ achieving the minimum in (18). This function satisfies

$$v(\theta) = w^{\mathscr{R}}(\theta)(\sin\theta)^{\frac{n-2}{2}}$$

for every $\theta \in [0, \theta_0]$, where w is an eigenfunction corresponding to $\lambda(\theta_0)$ and $w^{\mathscr{R}}$ is defined as in Proposition 6. Moreover it holds $v \in C^{\infty}([0, \theta_0])$, $v(0) = 0$, $v(\theta_0) = 0$ and $v(\theta) > 0$ for every $\theta \in (0, \theta_0)$.

Proof Let $v \in H_0^1((0, \theta_0))$ such that the objective function in (18) is finite. Recall that by the *Sobolev embedding Theorem* $v \in C([0, \theta_0])$, moreover it is a.e. differentiable in $(0, \theta_0)$ and by [14, Theorem 2, p. 273] we have $v(\theta_0) = 0$. Let $u : [0, \theta_0] \to \mathbb{R}$ be the function defined by

$$v(\theta) = u(\theta)(\sin\theta)^{\frac{n-2}{2}}$$

for every $\theta \in [0, \theta_0]$, we have $u(\theta_0) = 0$. Differentiating it we obtain

$$u'(\theta)(\sin\theta)^{\frac{n-2}{2}} = v'(\theta) - \left(\frac{n-2}{2}\right)\cot(\theta) v(\theta) \quad \text{for a.e. } \theta \in (0, \theta_0).$$

An integration by part yields

$$\int_0^{\theta_0} (2v'(\theta)v(\theta))\cot(\theta)\,d\theta = \int_0^{\theta_0} \frac{v(\theta)^2}{(\sin\theta)^2}d\theta,$$

therefore by combining these relations we obtain

$$\frac{\displaystyle\int_0^{\theta_0} u'(\theta)^2(\sin\theta)^{n-2}\,d\theta}{\displaystyle\int_0^{\theta_0} u(\theta)^2(\sin\theta)^{n-2}\,d\theta} = \frac{\displaystyle\int_0^{\theta_0} \left[v'(\theta)^2 + V(\theta)v(\theta)^2\right]\,d\theta}{\displaystyle\int_0^{\theta_0} v(\theta)^2\,d\theta}.$$

Let $w : \overline{\Gamma(\theta_0)} \to \mathbb{R}$ be the function, radially symmetric with respect to p, defined by

$$w_{\mathscr{S}}(\theta,\xi) = u(\theta)$$

for every $(\theta,\xi) \in [0,\theta_0] \times \mathbb{S}^{n-2}$, we have $w = 0$ on $\partial\Gamma(\theta_0)$. By (8) and (9) we obtain

$$\frac{\displaystyle\int_{\Gamma(\theta_0)} |\nabla_\phi w(\phi)|^2\,d\sigma}{\displaystyle\int_{\Gamma(\theta_0)} w(\phi)^2\,d\sigma} = \frac{\displaystyle\int_0^{\theta_0} u'(\theta)^2(\sin\theta)^{n-2}\,d\theta}{\displaystyle\int_0^{\theta_0} u(\theta)^2(\sin\theta)^{n-2}\,d\theta},$$

therefore $w \in H_0^1(\Gamma(\theta_0))$. By combining these relations and (13) we have

$$\frac{\displaystyle\int_0^{\theta_0} \left[v'(\theta)^2 + V(\theta)v(\theta)^2\right]\,d\theta}{\displaystyle\int_0^{\theta_0} v(\theta)^2\,d\theta} = \frac{\displaystyle\int_{\Gamma(\theta_0)} |\nabla_\phi w(\phi)|^2\,d\sigma}{\displaystyle\int_{\Gamma(\theta_0)} w(\phi)^2\,d\sigma} \geq \lambda(\theta_0),$$

equality holds if and only if w is an an eigenfunction corresponding to $\lambda(\theta_0)$.

Assume that w is an an eigenfunction corresponding to $\lambda(\theta_0)$, by Proposition 6 and the above relations we have

$$v(\theta) = w^{\mathscr{R}}(\theta)(\sin\theta)^{\frac{n-2}{2}} \tag{19}$$

for every $\theta \in [0,\theta_0]$. Since $\Gamma(\theta_0)$ is a connected set, the eigenvalue $\lambda(\theta_0)$ is simple and $w \neq 0$ in $\Gamma(\theta_0)$. Moreover, since $\Gamma(\theta_0)$ is a set of class C^∞, by [11, Theorem 1, p. 8] we obtain that $w \in C^\infty(\overline{\Gamma(\theta_0)})$. By imposing the conditions of non-negativity and normalization on v, using (19) and the analyticity of $\mathscr{S}$ we retrieve the desired thesis. □

4 A convexity property

This section is dedicated to the introduction of the main result exploited in the proof of the Friedland-Hayman inequality: the convexity of the one-dimensional function that describes the first eigenvalue of a spherical caps in terms of its colatitude. This fact was proved in a more general setting (and using more sophisticated tools) in [8, Corollary 1.15]. We follow the approach of [20]. It is sufficient to retrieve it in dimension greater than or equal to 5, since it will be exploited in a limiting process in Section 5.

Proposition 8 *The function $\lambda : (0,\pi) \to \mathbb{R}$, defined by (18) for every $\theta_0 \in (0,\pi)$, is twice differentiable, strictly decreasing and*

$$\lim_{\theta_0 \to 0^+} \lambda(\theta_0) = +\infty. \tag{20}$$

Moreover if $n \geq 5$, it is a strictly convex function.

Proof Let $\theta_0 \in (0,\pi)$ and denote $v = v_{\theta_0}$ the unique non-negative function normalized in $L^2((0,\theta_0))$ that achieves the minimum in (18), it holds

$$\lambda(\theta_0) = \int_0^{\theta_0} \left[v'(\theta)^2 + V(\theta)v(\theta)^2 \right] d\theta.$$

By Lemma 7 we have $v \in C^\infty([0,\theta_0])$, $v(0) = 0$, $v(\theta_0) = 0$ and $v(\theta) > 0$ for every $\theta \in (0,\theta_0)$, which yields

$$v'(0) > 0 \quad , \quad v'(\theta_0) < 0. \tag{21}$$

The function v is the unique non-negative solution of the boundary value problem

$$\begin{cases} -v''(\theta) + V(\theta)v(\theta) = \lambda(\theta_0)v(\theta) & \text{for every } \theta \in (0,\theta_0), \\ v(0) = v(\theta_0) = 0, \\ \int_0^{\theta_0} v(\theta)^2 \, d\theta = 1. \end{cases} \tag{22}$$

The first relation in (22) is the so-called *Euler-Lagrange equation* associated to (18). Consider the set

$$T = \{(\theta,\theta_0) \subset [0,\pi] \times (0,\pi) \, : \, \theta \leq \theta_0\},$$

it is possible define a composite function as

$$v : T \to \mathbb{R}, \quad (\theta,\theta_0) \mapsto v(\theta,\theta_0) = v_{\theta_0}(\theta).$$

We indicate its derivative in the first argument as v', while the one in the second argument as $\dot{v}$. This new function satisfies

$$\begin{cases} -v''(\theta,\theta_0) + V(\theta)v(\theta,\theta_0) = \lambda(\theta_0)v(\theta,\theta_0) & \text{for every } (\theta,\theta_0) \in \mathring{T}, \\ v(0,\theta_0) = v(\theta_0,\theta_0) = 0 & \text{for every } \theta_0 \in (0,\pi), \\ \int_0^{\theta_0} v(\theta,\theta_0)^2 \, d\theta = 1 & \text{for every } \theta_0 \in (0,\pi), \end{cases} \tag{23}$$

and it holds

$$\lambda(\theta_0) = \int_0^{\theta_0} \left[v'(\theta,\theta_0)^2 + V(\theta)v(\theta,\theta_0)^2 \right] d\theta \quad \text{for every } \theta_0 \in (0,\pi). \tag{24}$$

We denote by $\dot{\lambda}$ and $\ddot{\lambda}$ the first and second derivative of the function λ, respectively. In the following we will always evaluate the function v and its derivatives at a fixed second coordinate θ_0, so, whenever present, the argument of these functions will always correspond to the first one. The proof is divided in five steps.

Step 1 (regularity of λ). We claim that $\lambda \in C^\infty(0,\pi)$. By direct calculation we have $\mathscr{X}(\Gamma(\theta_0)) = B_\wedge$, where $B_\wedge$ is the ball of radius $\tan(\theta_0/2)$ in $\mathbb{R}^{n-1}$ with center in the origin. Let $\tilde{u} : \mathbb{R}^{n-1} \to \mathbb{R}$ be a solution of the eigenvalue problem

$$\begin{cases} \dfrac{1}{\Upsilon^2}\left(\Delta\tilde{u} + (n-2)\nabla\tilde{u}\cdot\nabla\log\Upsilon\right) + \lambda(\theta_0)\tilde{u} = 0 & \text{in } B_\wedge, \\ \tilde{u} = 0 & \text{on } \partial B_\wedge. \end{cases} \tag{25}$$

By (12) we notice that the function $\tilde{u}_{\mathscr{X}^{-1}}$ is an eigenfunction corresponding to $\lambda(\theta_0)$. Let u_{θ_0} be the unique non-negative solution of (25) satisfying the normalization condition

$$\int_{B_\wedge} u_{\theta_0}(\hat{x})^2 \, d\hat{x} = 1,$$

we define $\mathbf{u}_{\theta_0} := (u_{\theta_0})_{\mathscr{X}^{-1}}$. Let $\delta \in \mathbb{R}$ such that $|\delta| \ll \theta_0$, we define the function $h_\delta : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ as

$$h_\delta(\hat{x}) = \frac{\tan\left(\dfrac{\theta_0+\delta}{2}\right)}{\tan(\theta_0/2)}\hat{x}$$

for every $\hat{x} \in \mathbb{R}^{n-1}$. We notice that $h_\delta(B_\wedge) = B_\wedge^\delta$, where $B_\wedge^\delta$ is the ball of radius $\tan(\theta_0/2)+\delta$ in $\mathbb{R}^{n-1}$ with center in the origin. Since the operator in (25) has analytic coefficients, is uniformly elliptic on bounded sets, $\lambda(\theta_0)$ is simple and $B_\wedge$ is a set of class C^∞, it is possible to follow the same strategy of [19, Example 3.2, p. 33] with the family $\{h_\epsilon\}_{\epsilon\in(-\delta,\delta)}$ to conclude that $\lambda \in C^\infty(0,\pi)$.

Step 2 (regularity of v). We claim that $v \in C^\infty(T)$. Consider the set

$$\mathcal{T} = \{(\hat{x},\theta_0) \subset \mathbb{R}^{n-1} \times (0,\pi) \,:\, |\hat{x}| \le \tan(\theta_0/2)\},$$

it is possible define a composite function as

$$u : \mathcal{T} \to \mathbb{R}, \quad (\hat{x},\theta_0) \mapsto u(\hat{x},\theta_0) = u_{\theta_0}(\hat{x}).$$

Let $\underline{u}_{\theta_0} : \mathbb{R}^{n-1} \to \mathbb{R}$ be the extension of u_{θ_0} defined in [19, Example 3.2, p. 33] exploiting [19, Theorem 1.9], we have $\underline{u}_{\theta_0}(\hat{x}) = u_{\theta_0}(\hat{x})$ for every $\hat{x} \in \overline{B_\wedge}$. The construction of Step 1 allows also to conclude that for every $i \in \mathbb{N}$ the map

$$(-\delta,\delta) \to H^i(\mathbb{R}^{n-1}), \quad \epsilon \mapsto \underline{u}_{\theta_0+\epsilon} \tag{26}$$

is C^∞. We use the symbol $\dot{}$ to denote the derivative with respect to the 1-dimensional parameter used to describe the family $\{h_\epsilon\}_{\epsilon\in(-\delta,\delta)}$. By the *Sobolev embedding Theorem* the functions $\underline{u}_{\theta_0}$ and $\dot{\underline{u}}_{\theta_0}$ belong to $C^\infty(B_\wedge^\delta)$. Moreover, from the regularity of the map (26) it holds

$$\frac{\underline{u}_{\theta_0+\epsilon}(\hat{x}) - \underline{u}_{\theta_0}(\hat{x})}{\epsilon} = \dot{\underline{u}}_{\theta_0}(\hat{x}) + \tau_\epsilon(\hat{x})$$

for every $(\hat{x},\epsilon) \in B_\wedge^\delta \times (-\delta,\delta)$, where the differentiable function $\tau_\epsilon : B_\wedge^\delta \to \mathbb{R}$ satisfies $\tau_\epsilon \to 0$ in $C^1(B_\wedge^\delta)$ as $\epsilon \to 0$. Differentiating with respect to the coordinates of $\mathbb{R}^{n-1}$ and then letting $\epsilon \to 0$ we obtain

$$(\nabla \dot{\underline{u}_{\theta_0}})(\hat{x}) = \nabla \dot{\underline{u}}_{\theta_0}(\hat{x})$$

for every $\hat{x} \in B_\wedge^\delta$. We notice that a similar reasoning can also be applied to higher-order derivatives, therefore restricting ourselves to $\overline{B_\wedge}$ we obtain that $u \in C^\infty(\mathcal{T})$. As a byproduct, since $\mathscr{S}$ and $\mathscr{X}$ are analytic, we have that the function

$$\mathbf{u}^{\mathscr{R}} : T \to \mathbb{R}, \quad (\theta,\theta_0) \mapsto \mathbf{u}^{\mathscr{R}}(\theta,\theta_0) = \mathbf{u}^{\mathscr{R}}_{\theta_0}(\theta)$$

belongs to $C^\infty(T)$. Recall that by Lemma 7 we have

$$v(\theta,\theta_0) = w^{\mathscr{R}}_{\theta_0}(\theta)(\sin\theta)^{\frac{n-2}{2}} \tag{27}$$

for every $(\theta,\theta_0) \in T$, where w_{θ_0} is a non-negative eigenfunction corresponding to $\lambda(\theta_0)$. Since both $\mathbf{u}_{\theta_0}$ and w_{θ_0} are non-negative eigenfunctions corresponding to $\lambda(\theta_0)$, then they differs by a positive multiplicative factor. In particular the third condition in (22) yields

$$w^{\mathscr{R}}_{\theta_0} = \left(\int_0^{\theta_0} \mathbf{u}^{\mathscr{R}}_{\theta_0}(\theta)^2 (\sin\theta)^{n-2}\, d\theta\right)^{-1/2} \mathbf{u}^{\mathscr{R}}_{\theta_0},$$

that combined with (27) gives $v \in C^\infty(T)$.

Step 3 (monotonicity and limit behavior of λ). We claim that the function λ is strictly decreasing and that (20) holds. Differentiating the second and the third relation in (23) with respect to θ_0 we have

$$\dot{v}(0) = 0, \quad v'(\theta_0) = -\dot{v}(\theta_0)\,, \quad \int_0^{\theta_0} v(\theta)\dot{v}(\theta)\, d\theta = 0. \tag{28}$$

Differentiating (24) we obtain

$$\dot{\lambda}(\theta_0) = v'(\theta_0)^2 + \int_0^{\theta_0} \left[2v'(\theta)\dot{v}'(\theta) + 2V(\theta)v(\theta)\dot{v}(\theta)\right] d\theta,$$

integrating by parts, from the first two relations in (28), we have

$$\dot{\lambda}(\theta_0) = -v'(\theta_0)^2 + 2\int_0^{\theta_0} \dot{v}(\theta)[-v''(\theta) + V(\theta)v(\theta)]\, d\theta.$$

Thence exploiting the first relation in (22) and the third relation in (28) we find

$$\dot{\lambda}(\theta_0) = -v'(\theta_0)^2, \tag{29}$$

that combined with the second relation in (21) gives the desired strict monotonicity for the map λ. The relation (20) follows from [11, equation (36), p. 318].

Step 4 (nodal domains of $\dot{v}$). We claim that the function $\dot{v}$ has exactly two nodal domains, namely that the connected components of the set $(0,\theta_0) \setminus \{\dot{v} = 0\}$ are two. We introduce the function

$$q : (0,\theta_0) \to \mathbb{R} \quad , \quad q = \frac{\dot{v}}{v},$$

whose derivatives are

$$q' = \frac{\dot{v}'}{v} - \frac{v'}{v}q$$

and

$$\begin{aligned} q'' &= \frac{\dot{v}''}{v} - \frac{\dot{v}'v'}{v^2} - \frac{v''}{v}q + \left(\frac{v'}{v}\right)^2 q - \frac{v'}{v}q' \\ &= \frac{\dot{v}''}{v} - \frac{v''}{v}q - 2\frac{v'}{v}q'. \end{aligned}$$

By the first relation in (22) and the derivative of the first relation in (23) with respect to the second variable restricted to the set $(0,\theta_0) \times \{\theta_0\}$, namely

$$-\dot{v}''(\theta) + V(\theta)\dot{v}(\theta) = \dot{\lambda}(\theta_0)v(\theta) + \lambda(\theta_0)\dot{v}(\theta) \quad \text{for every } \theta \in (0,\theta_0),$$

it is possible to infer

$$q'' = -\dot{\lambda}(\theta_0) - 2\frac{v'}{v}q'$$

and thus

$$(v^2 q')' = -\dot{\lambda}(\theta_0)v^2.$$

In particular, by Step 3, the function $v^2 q' : [0,\theta_0] \to \mathbb{R}$ is strictly increasing, so it attains its minimum at 0. Since $v \in C^\infty(T)$, the second relation in (22) and the first relation in (28) give

$$\dot{v}'(0), v'(0) < \infty \quad \text{and} \quad \dot{v}(0) = v(0) = 0,$$

therefore we have

$$(v^2 q')(\theta) > (v^2 q')(0) = \dot{v}'(0)v(0) - \dot{v}(0)v'(0) = 0 \quad \text{for every } \theta \in (0,\theta_0).$$

So q is a strictly increasing function and thus it has at most one zero in $(0,\theta_0)$. This latter fact holds also for $\dot{v}$, since v is a positive function in $(0,\theta_0)$. By the third relation in (28) we notice that $\dot{v}$ is orthogonal to v in $L^2((0,\theta_0))$, therefore $\dot{v}$ has non-constant sign in $(0,\theta_0)$. In particular, since $\dot{v}$ is continuous, there exists a unique value $\bar{\theta} \in (0,\theta_0)$ such that $\dot{v}(\bar{\theta}) = 0$. At this point we sketch the structure of $\dot{v}$. By the first two relations in (28) and the second relation in (21) we have

$$\dot{v}(0) = 0, \quad \dot{v}(\bar{\theta}) = 0, \quad \dot{v}(\theta_0) = -v'(\theta_0) > 0,$$

so

$$\begin{aligned} \dot{v}(\theta) < 0 \quad \text{for every } \theta \in \left(0, \bar{\theta}\right), \\ \dot{v}(\theta) > 0 \quad \text{for every } \theta \in \left(\bar{\theta}, \theta_0\right), \end{aligned} \tag{30}$$

and therefore

$$\dot{v}'(0) \leq 0. \tag{31}$$

Step 5 (convexity of λ). We claim that the function λ is strictly convex if $n \geq 5$. We notice that it is possible to express the derivative of the map λ in another way. Multiplying the first relation in (22) by v' and integrating, by the third relation in (28) we find

$$-v'(\theta_0)^2 = -\int_0^{\theta_0} V(\theta) \left(v(\theta)^2\right)' \, d\theta - v'(0)^2,$$

that integrating by parts and using (29) yields

$$\dot{\lambda}(\theta_0) = \int_0^{\theta_0} V'(\theta) v(\theta)^2 \, d\theta - v'(0)^2. \tag{32}$$

Differentiating (32), by the third relation in (28), we find

$$\ddot{\lambda}(\theta_0) = 2\int_0^{\theta_0} \left[V'(\theta) - V'\left(\bar{\theta}\right)\right] v(\theta)\dot{v}(\theta) \, d\theta - 2v'(0)\dot{v}'(0) > 0.$$

The last inequality is obtained combining – recall that $\cot(\cdot)^2$ is a strictly convex function in $(0, \pi)$ – the strict convexity of V in $(0, \pi)$ for $n \geq 5$, the information contained in (30), the first relation in (21) and (31), and gives the desired strict convexity for the map λ. □

5 Proof of Theorem 1

The proof is divided in four steps. For the first two steps we follow the original strategy of [4, Lemma 5.1], while for the remaining ones, which involve the proof of the Friedland-Hayman inequality, we rely on [22, Section 4.3].

Step 1 (reduction to an inequality and finiteness). We claim that the function J is finite and to obtain its monotonicity it is sufficient to prove an inequality. Let $u \in C^2(B_2)$ be a function satisfying

$$\begin{cases} \Delta u(x) \geq 0 & \text{for every } x \in \{u > 0\}, \\ u(x) \geq 0 & \text{for every } x \in B_2, \\ u(\mathbf{0}) = 0. \end{cases} \tag{33}$$

Notice that we suppose $u \in C^2(B_2)$, this is done to minimize technical difficulties. However, aside for a passage in this Step and one at the beginning Step 2, that will be appropriately highlighted, the proof can be performed, with the assumption $u \in C(B_2)$. Below, we will repeatedly use (4) without explicitly stating it. For every $0 < s \leq 1$, when integrals over subsets of ∂B_s arise, for brevity we write u instead of $u_{\mathscr{P}}$ and omit the dependence on polar coordinates. Define the map $I : (0, 1) \to \mathbb{R}$ as

$$I(s) = \int_{B_s} \frac{|\nabla u(x)|^2}{|x|^{n-2}} \, dx$$

for every $s \in (0,1)$, when we want to emphasize the dependence on the function we write $I(\cdot, u)$ instead of $I(\cdot)$. It is possible to prove, through an approximation argument involving mollifiers, that

$$I(s) \leq \frac{\bar{C}}{s^n} \int_{B_{2s} \setminus B_s} u(x)^2 \, dx$$

for every $s \in (0,1)$, where $\bar{C}$ is a positive constant depending on n, see [10, equation (12.16)]. This shows immediately that the value $J(s)$ is finite for every $s \in (0,1)$. Since $u \in C(B_2)$ by the *Caccioppoli inequality* $u \in H^1(B_1)$. So the map I is differentiable almost everywhere, passing to polar coordinates and exploiting (5) we have

$$I'(s) = s^{2-n} \int_{\partial B_s} \left[u_r^2 + \frac{1}{s^2} |\nabla_\phi u|^2 \right] d\sigma$$

for almost everywhere $s \in (0,1)$. Differentiating (2) and evaluating it at one of these points $S \in (0,1)$ for u^+ and u^- we obtain

$$J'(S) = I(S, u_+) I(S, u_-) S^{-5} \left(S \left(\frac{I'(S, u_+)}{I(S, u_+)} + \frac{I'(S, u_-)}{I(S, u_-)} \right) - 4 \right).$$

Consider the map $u_{\boxed{S}} : B_2 \to \mathbb{R}$ defined as $u_{\boxed{S}}(x) := \frac{u(Sx)}{S}$ for every $x \in B_2$, it satisfies (33) with $u = u_{\boxed{S}}$ and we can write

$$\frac{I'(S,u)}{I(S,u)} = \frac{1}{S} \frac{\displaystyle\int_{\partial B_1} \left[((u_{\boxed{S}})_r)^2 + |\nabla_\phi (u_{\boxed{S}})|^2 \right] d\sigma}{\displaystyle\int_{B_1} \frac{|\nabla u_{\boxed{S}}(x)|^2}{|x|^{n-2}} \, dx}. \tag{34}$$

In particular to obtain the desired monotonicity it is sufficient to show that

$$\frac{I'(1,u_+)}{I(1,u_+)} + \frac{I'(1,u_-)}{I(1,u_-)} - 4 \geq 0 \tag{35}$$

for functions u_+, u_- satisfying (1).

Step 2 (reduction to the Friedland-Hayman inequality). We claim that to obtain the monotonicity of the function J it is sufficient to prove the Friedland-Hayman inequality. Define the open set

$$\Gamma = \{u > 0\} \cap \partial B_1.$$

Since (33) yields

$$\Delta(u(x)^2) \geq 2|\nabla u(x)|^2 \quad \text{for every } x \in \{u > 0\},$$

by the divergence theorem we obtain

$$I(1) \leq \frac{1}{2}\int_{B_1} \frac{\Delta u(x)^2}{|x|^{n-2}}\, dx = \int_\Gamma u u_r \, d\sigma + \frac{n-2}{2}\int_{B_1} \frac{\nabla u(x)^2 \cdot x}{|x|^n}\, dx.$$

Moreover, since $\Delta|x|^{2-n} = \bar{c}\,\delta_{\mathbf{0}}$ in the sense of distribution, where $\bar{c}$ constant depending on n and $\delta_{\mathbf{0}}$ is the *Dirac delta* evaluated at $\mathbf{0}$, and $u(\mathbf{0}) = 0$ we have

$$\int_{B_1} u(x)^2 \Delta\left(\frac{1}{|x|^{n-2}}\right) dx = 0,$$

hence an integration by part yields

$$\int_{B_1} \frac{\nabla u(x)^2 \cdot x}{|x|^n}\, dx = \int_\Gamma u^2 \, d\sigma.$$

In particular we obtain

$$I(1) \leq \int_\Gamma \left[u u_r + \frac{n-2}{2} u^2\right] d\sigma,$$

this relation can be retrieved also assuming $u \in C(B_2)$ with a little more work, exploiting an approximation argument involving mollifiers of u, see [4, Lemma 5.1]. Consequently by (34) we have

$$\frac{I'(1)}{I(1)} \geq \frac{\displaystyle\int_\Gamma \left[u_r^2 + |\nabla_\phi u|^2\right] d\sigma}{\displaystyle\int_\Gamma \left[u u_r + \frac{n-2}{2} u^2\right] d\sigma}. \tag{36}$$

Let $t \in [0,1]$ and denote $\lambda = \lambda(\Gamma)$, by (13) and the *Young's inequality* for products we have

$$\int_\Gamma \left[u_r^2 + |\nabla_\phi u|^2\right] d\sigma \geq 2\left(\int_\Gamma u_r^2 \, d\sigma\right)^{\frac{1}{2}} \left(t\lambda \int_\Gamma u^2 \, d\sigma\right)^{\frac{1}{2}} + (1-t)\lambda \int_\Gamma u^2 \, d\sigma,$$

on the other hand by *Hölder's inequality* we obtain

$$\int_\Gamma u u_r \, d\sigma + \frac{n-2}{2}\int_\Gamma u^2 \, d\sigma \leq \left(\int_\Gamma u_r^2 \, d\sigma\right)^{\frac{1}{2}} \left(\int_\Gamma u^2 \, d\sigma\right)^{\frac{1}{2}} + \frac{n-2}{2}\int_\Gamma u^2 \, d\sigma.$$

Therefore setting

$$z = \frac{\left(\displaystyle\int_\Gamma u^2\, d\sigma\right)^{\frac{1}{2}}}{\left(\displaystyle\int_\Gamma u_r^2\, d\sigma\right)^{\frac{1}{2}}},$$

it holds

$$\frac{I'(1)}{I(1)} \geq \frac{2(t\lambda)^{\frac{1}{2}} + \lambda(1-t)z}{1 + \dfrac{n-2}{2}z},$$

moreover it is possible to estimate

$$\frac{2(t\lambda)^{\frac{1}{2}} + \lambda(1-t)z}{1 + \dfrac{n-2}{2}z} \geq 2\min\left\{(t\lambda)^{\frac{1}{2}}, \frac{\lambda}{n-2}(1-t)\right\}.$$

At this point, we choose t such that these two lower bounds are equal, namely

$$t\lambda + (n-2)(t\lambda)^{\frac{1}{2}} - \lambda = 0, \tag{37}$$

this is equivalent to require

$$\sqrt{t} = \frac{\sqrt{4\lambda}}{(n-2) + \sqrt{(n-2)^2 + 4\lambda}}.$$

In particular there exists a unique such $t \in [0,1]$, therefore by (16) and (37) we have $t\lambda = \alpha(\Gamma)^2$ and

$$\frac{I'(1)}{I(1)} \geq 2\alpha(\Gamma). \tag{38}$$

Combining (35) and (38) we are reduced to show that

$$\alpha(\Gamma_+) + \alpha(\Gamma_-) \geq 2, \tag{39}$$

for any pair of disjoint open sets $\Gamma_+, \Gamma_- \subset \partial B_1$, i.e., the Friedland-Hayman inequality.

Step 3 (a monotonicity property of characteristic constants). We claim that the characteristic constant of a spherical cap of fixed colatitude is monotonically decreasing with respect to its dimension. Fix $\theta_0 \in (0,\pi)$, we denote $\Gamma_n(\theta_0)$ the spherical cap of colatitude θ_0 with center p in $\partial B_1 \subset \mathbb{R}^n$. Let $w : \mathbb{R}^n \to \mathbb{R}$ be the positive homogeneous function defined by

$$w_{\mathscr{P}}(r,\phi) = r^{\alpha(\theta_0,n)} u(\phi),$$

for every $(r,\phi) \in \mathbb{R}^+ \times \partial B_1$, where $\alpha(\theta_0,n)$ is the characteristic constant of the set $\Gamma_n(\theta_0)$, i.e, $\alpha(\theta_0,n) := \alpha(\Gamma_n(\theta_0))$, and u is a positive eigenfunction corresponding to $\lambda(\Gamma_n(\theta_0))$. By Proposition 4, the function w is harmonic in $\{w > 0\}$, this set is the cone generated by $\Gamma_n(\theta_0)$ with vertex in the origin of $\mathbb{R}^n$, in particular w satisfies (33). Recall that it is possible to embed the space $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ using the map

$$(x_1,\dots,x_n) \to (x_1,\dots,x_n,0),$$

in this way the we have $\partial B_1 \subset \mathbb{S}^n$, where $\mathbb{S}^n$ is the sphere of unit radius in $\mathbb{R}^{n+1}$ with center in the origin. We define the function $\tilde{w} : \mathbb{R}^{n+1} \to \mathbb{R}$ as

$$\tilde{w}(x_1,\dots,x_n,x_{n+1}) = w(x_1,\dots,x_n),$$

it has the same homogeneity degree of w and is harmonic in $\{\tilde{w} > 0\}$. We notice that it holds

$$\{\tilde{w} > 0\} = \{w > 0\} \times \mathbb{R},$$

and so this set is the cone generated by $(\{w > 0\} \times \mathbb{R}) \cap \mathbb{S}^n$ with vertex in the origin of $\mathbb{R}^{n+1}$.

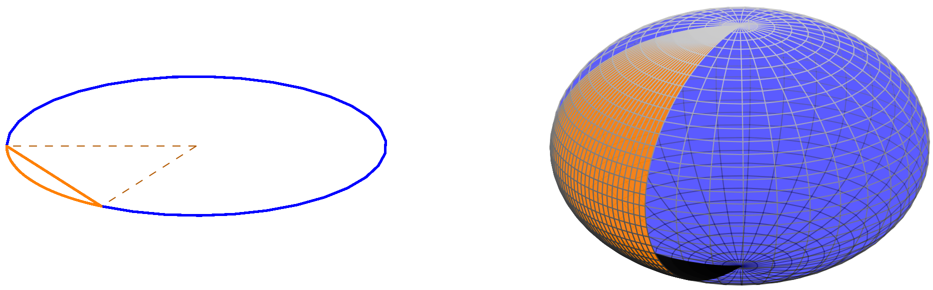

Fig. 7 The set $\partial B_1 \cap \{w > 0\}$ in the case $n = 2$ and $\theta_0 \approx \pi/6$ (in orange, on the left). The set $\mathbb{S}^n \cap \{\tilde{w} > 0\}$ in the case $n = 2$ and $\theta_0 \approx \pi/6$ (in orange, on the right).

Moreover we can write

$$\tilde{w}_{\tilde{\mathscr{S}}}(\tilde{r},\tilde{\phi}) = \tilde{r}^{\alpha(\theta_0,n)}\tilde{u}(\tilde{\phi}),$$

for every $(\tilde{r},\tilde{\phi}) \in \mathbb{R}^+ \times \mathbb{S}^n$, where $\tilde{\mathscr{S}}$ is the polar parametrization of $\mathbb{R}^{n+1}$ (and $(\tilde{r},\tilde{\phi})$ the corresponding polar coordinates) and $\tilde{u} : \mathbb{S}^n \to \mathbb{R}$ is a function. We observe that $\tilde{w}$ satisfies the analogous of (33) in $\mathbb{R}^{n+1}$, so by Step 2, see (38), we obtain

$$\frac{I'(1,\tilde{w})}{I(1,\tilde{w})} \geq 2\alpha(\theta_0, n+1),$$

on the other hand by (34) and Lemma 5 we have

$$\frac{I'(1,\tilde{w})}{I(1,\tilde{w})} = \frac{I'(1,w)}{I(1,w)} = 2\alpha(\theta_0, n),$$

which yields

$$\alpha(\theta_0,n) \geq \alpha(\theta_0,n+1). \tag{40}$$

Step 4 (the Friedland-Hayman inequality). We claim that the Friedland-Hayman inequality holds. Consider (39), by Proposition 6 it is sufficient to prove that it holds for all pairs (Γ_+, Γ_-) of disjoint spherical caps in ∂B_1. From the strict monotonicity statement of Proposition 8, (an isometry) and (16) we can take these sets to be complementary in ∂B_1, namely this is equivalent to prove that

$$\min_{\theta_0 \in (0,\pi)} \alpha(\theta_0, n) + \alpha(\pi - \theta_0, n) \geq 2.$$

By (20) and the differentiability statement of Proposition 8 there exists a value $\theta_n \in (0, \pi)$ such that the minimum is achieved. Define the function $\beta : \{m \in \mathbb{N} : m \geq 3\} \to \mathbb{R}$ as

$$\beta(n) = \alpha(\theta_n, n) + \alpha(\pi - \theta_n, n),$$

for $n \in \{m \in \mathbb{N} : m \geq 3\}$, by (40) it is monotone non-increasing.

Suppose by way of contradiction that exists an $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $\beta(n_0) < 2 - \delta$, then $\beta(n) < 2 - \delta$ for all $n \geq n_0$ (this value will be increased in the following in order to satisfy more conditions and ease the notation). Therefore by minimality of θ_n we have

$$\alpha(\theta_n, n), \alpha(\pi - \theta_n, n) < 2,$$

that by (16) gives

$$\lambda(\theta_n), \lambda(\pi - \theta_n) < 2n. \tag{41}$$

We study the behavior of the sequence

$$\left\{ \beta(n) = \left(\frac{n-2}{2}\right) \left(\sqrt{1 + \frac{4\lambda(\theta_n)}{(n-2)^2}} - 1 + \sqrt{1 + \frac{4\lambda(\pi - \theta_n)}{(n-2)^2}} - 1 \right) \right\}_n$$

as $n \to +\infty$, we can estimate

$$\beta(n) \geq \frac{\lambda(\theta_n) + \lambda(\pi - \theta_n)}{n-2} - \frac{\lambda(\theta_n)^2 + \lambda(\pi - \theta_n)^2}{(n-2)^3}$$

for all $n \geq n_0$, up to taking a bigger n_0, since the third term in the Taylor formula – the first one not appearing here – is positive. By (41) we can estimate

$$-\frac{\lambda(\theta_n)^2 + \lambda(\pi - \theta_n)^2}{(n-2)^3} \geq -\frac{c}{n},$$

where c is a positive constant (not depending on n). Consider the function $\gamma_n : (0, \pi) \to \mathbb{R}$ defined as

$$\gamma_n(\theta) := \frac{\lambda(\theta) + \lambda(\pi - \theta)}{n-2},$$

for every $\theta \in (0, \pi)$. By Proposition 8, for $n \geq 5$, γ_n is a strictly convex function, evenly symmetric with respect to the point $\pi/2$, that is its unique minimum. So by (15) we obtain

$$\gamma_n(\theta_n) \geq \gamma_n\left(\frac{\pi}{2}\right) = \frac{2n-2}{n-2} > 2$$

for all $n \geq 5$, therefore by combining the estimates above we have

$$\beta(n) > 2 - \frac{c}{n} \geq 2 - \delta,$$

for all $n \geq n_0$, up to taking a bigger n_0, a contradiction. This proves the Friedland-Hayman inequality and therefore, by Step 2, the desired monotonicity of J. $\square$

6 An open problem

We present an open question taken from [1, Introduction].

We notice that Theorems 2 and 3 describe the structure of the functions u_+, u_- satisfying (1) when the map J is constant and when the map J is near to a constant, respectively. In particular a family of functions, the one constituted by the so-called *two-planes solutions*, is central in the description of the behavior of the pair u_+, u_-. This family is the set

$$\left\{ \begin{array}{l} \ell : \mathbb{R}^n \to \mathbb{R} \\ \qquad x \mapsto c_+(x \cdot \nu)^+ + c_-(x \cdot \nu)^- \end{array} \;\middle|\; c_+, c_- > 0 \text{ and } \nu \in \partial B_1 \right\}.$$

A two-planes solution is a function made up of a pair of positive linear functions defined on two disjoint complementary half-spaces, vanishing on the common boundary that contains the origin, they satisfy (1). Let u_+, u_- be two functions satisfying (1) and

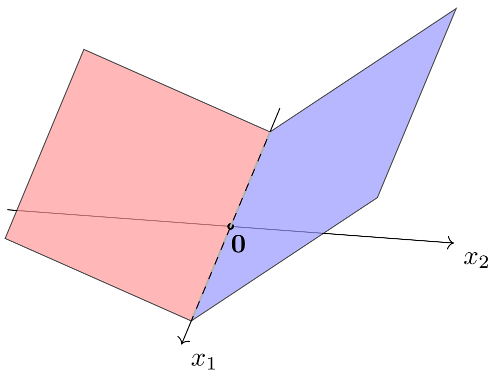

Fig. 8 An example of two-planes solution in $\mathbb{R}^2$.

$$\lim_{s \to 0^+} J(s) > 0. \tag{42}$$

Consider $\{s_k\}_k \subset (0,1)$ a *sequence of radii* decreasing to 0, and the associated *blow-up sequences*

$$\left\{\frac{u_+(s_k\,\cdot)}{s_k}\right\}_k \quad \text{and} \quad \left\{\frac{u_-(s_k\,\cdot)}{s_k}\right\}_k,$$

restricted to B_1. As in [4, Section 6], up to subsequence, it is possible to show that they converge, in an appropriate way, to the the functions

$$x \mapsto c_+(x\cdot\nu)^+ \quad \text{and} \quad x \mapsto c_-(x\cdot\nu)^-,$$

where $c_+, c_- > 0$ and $\nu \in \partial B_1$, defined for $x \in B_1$. This pair is called *blow-up limit*. Observe that the values c_+, c_-, ν, depend *a priori* on the sequence $\{s_k\}_k$. The result of the blow-up procedure described above is a pair of functions representing one of the possible behaviors that the original pair exhibits when approaching the origin. In the theory of free boundary problems, classifying the possible blow-up limits is useful to obtain information about the regularity of the free boundary, see for example [12, proof of Theorem 1.1] for a related result.

It has been shown in [1, Theorem 1.3] that it is possible to construct two functions $\tilde{u}_+$ and $\tilde{u}_-$ satisfying (1), that admit multiple different blow-up limits, depending on the chosen sequence of radii. In particular the positivity sets of these functions wrap around the origin. More precisely for all of the possible blow-up limits, $\tilde{c}_+$ and $\tilde{c}_-$ are the same, while $\tilde{\nu}$ can be different, depending on the particular subsequence of radii.

Open problem 1 ([1]) Is it possible to find a pair u_+, u_- satisfying (1) and (42) such that for all of the blow-up limits, ν is the same, while the pair (c_+, c_-) can be different, depending on the particular subsequence of radii?

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