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On a criterion for algebraic independence applied to continued fractions

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Abstract

From around 2010 onward, Elsner et al. developed and applied a method in which the algebraic independence of n quantities $x_1, \ldots, x_n$ over a field is transferred to further n quantities $y_1, \ldots, y_n$ by means of a system of polynomials in $2n$ variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$. In this paper, we systematically study and explain this criterion and its variants. Moreover, we present new results concerning its application to periodic non-regular continued fractions, namely continued fractions with real numbers as partial quotients. We show that given a continued fraction of this type, this criterion can be applied to prove that not only are the convergents algebraically independent from each other, but they are also algebraically independent from the continued fraction.

MSC: primary 11J85; secondary 11A55

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1. Introduction

The transcendence of the circular number π and of the number $e = \exp(1)$ has been known since the late 19th century, but the question of the algebraic independence of these two numbers over $\mathbb{Q}$ has still not been answered. It concerns the exclusion of the existence of a non-identical vanishing polynomial $P(X, Y)$ with rational coefficients such that $P(\pi, e) = 0$.

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The Lindemann–Weierstrass theorem (1885) [1,11], from which the transcendence of π and of e can be derived, is the beginning of a general theory on algebraic independence of complex numbers over $\mathbb{Q}$. In one of its equivalent formulations this theorem states that in the case of the *linear* independence of algebraic numbers $\alpha_1, \dots, \alpha_n$ over $\mathbb{Q}$, the numbers $e^{\alpha_1}, \dots, e^{\alpha_n}$ are *algebraically* independent over $\mathbb{Q}$ [14].

An additional significant achievement is the theorem of Gelfond–Schneider that states the transcendence of α^β when α and β are algebraic over $\mathbb{Q}$, assuming that $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$ [14]. Another important result is Baker's Theorem on linear forms of logarithms that states that given $\alpha_1, \dots, \alpha_n$ algebraic numbers different from zero such that $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over the rational numbers, then the numbers $1, \log \alpha_1, \dots, \log \alpha_n$ are linearly independent over the field of all algebraic numbers [1].

In 1916, S. Ramanujan [12] defined the series

$$S_{j+1}(x) := \frac{\zeta(-2j-1)}{2} + \sum_{n=1}^{\infty} \frac{n^{2j+1} x^n}{1-x^n},$$

where $\zeta(s)$ is the Riemann zeta function. Let

$$P(x) := -24 S_1(x), \qquad Q(x) := 240 S_3(x), \qquad R(x) := -504 S_5(x). \tag{1}$$

In 1996, Y. Nesterenko [17] proved that for every complex number x with $0 < |x| < 1$, the set

$$\{x, P(x), Q(x), R(x)\}$$

contains at least three numbers that are algebraically independent over $\mathbb{Q}$.

In terms of algebraic independence of continued fractions, Tanaka [16] gave a necessary and sufficient condition for the values of $\Theta(x, a, q)$ to be algebraically independent, where $\Theta(x, a, q)$ is a sort of q-hypergeometric series. In particular, he showed under which conditions the values of the continued fractions obtained when $x = a$, namely $\Theta(a, q)$, are algebraically dependent.

In the years since 2010, the second named author, in collaboration with the Japanese mathematicians I. Shiokawa and Sh. Shimomura, obtained numerous results on algebraic independence or dependence of number sets. They started from a set of known algebraically independent numbers and they obtained a second set of numbers, where the numbers in both sets satisfy a system of polynomial equations. Then they applied a new criterion, which allows to decide on the algebraic independence of the numbers in the second set. Different variants of this criterion with applications to certain sets of numbers have so far appeared in various publications as essential tools. To give a few examples, we refer the reader to the results in [7,8]. For more applications, see [9], Appendix A of this manuscript, or arXiv:2311.18536. However, the variants of the criterion have not yet been systematically summarized and explained in a journal (without being limited to special applications), taking into account their interrelationships. One of the goals of this paper is to fill this gap and to systematically explain this criterion and its variants. Another goal is a new application to continued fractions. Section 2 contains the statements of both the theorems regarding this criterion and the ones about applications to continued fractions. The statement of the basic criterion is in Theorem 1, whereas Theorems 2 to 4 provide variants of the criterion from Theorem 1. The applications to continued fractions can be found in Theorems 5 and 6. The first four theorems are proven in Section 3, the latter ones in Section 4.3. Before proving them, in Section 4.1 we present a method for a practical (algebraic) handling of the criterion when applied to continued fractions. In Section 4.2 we introduce an algorithm that enables us to decide on the algebraic independence of an irregular continued fraction and its convergents by applying the algebraic independence criterion. We demonstrate the application of this algorithm by an example. Finally, in the Appendix A, we present some past results obtained by applications of the algebraic independence criterion, whereas the Appendix B contains a short description of the algorithm from Section 4.2.

2. Main results

In the first part of this section, we summarize the different variants of the criterion taking into account their interrelationships. In the second part, we apply the criterion to continued fractions.

Theorem 1. *Let $\mathbb{K}$ be a field with $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$. Furthermore, it is assumed that the numbers $x_1, \ldots, x_n \in \mathbb{C}$ and $y_1, \ldots, y_n \in \mathbb{C}$ satisfy a system of equations,*

$$f_j(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0 \qquad (j = 1, \ldots, n), \tag{2}$$

where $f_j(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \in \mathbb{K}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ are polynomials for $j = 1, \ldots, n$. If the numbers $x_1, \ldots, x_n$ are algebraically independent over $\mathbb{K}$ and

$$\det_n \left(\frac{\partial f_j}{\partial X_i}(x_1, \ldots, x_n, y_1, \ldots, y_n) \right) \neq 0 \tag{3}$$

holds, then the numbers $y_1, \ldots, y_n$ are algebraically independent over $\mathbb{K}$.

Remark 1. We interpret j as the row number and i as the column number. From now on, we denote by capital letters the variables and by lowercase letters the numerical values they assume. Moreover, by $\det_k$ we mean the determinant of a $k \times k$ matrix.

Theorem 2. *Let $\mathbb{K}$ be a field with $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$. Furthermore, it is assumed that the numbers $x_1, \ldots, x_n \in \mathbb{C}$ and $y_1, \ldots, y_n \in \mathbb{C}$ satisfy a system of equations,*

$$y_j = T_j(x_1, \ldots, x_n) \qquad (j = 1, \ldots, n), \tag{4}$$

where $T_j(X_1, \ldots, X_n) \in \mathbb{K}[X_1, \ldots, X_n]$ are polynomials for $j = 1, \ldots, n$. If the numbers $x_1, \ldots, x_n$ are algebraically independent over $\mathbb{K}$ and

$$\det_n \left(\frac{\partial T_j}{\partial X_i}(x_1, \ldots, x_n) \right) \neq 0 \tag{5}$$

holds, then the numbers $y_1, \ldots, y_n$ are algebraically independent over $\mathbb{K}$.

Remark 2. Under the conditions of Theorem 2, the non-vanishing of the determinant in (5) is not only sufficient but also necessary for the algebraic independence of $y_1, \ldots, y_n$ over $\mathbb{K}$, but there are no interesting application of it.

In several applications, each of the numbers $y_1, \dots, y_n$ can be represented as the value of a *rational function* at the point $(x_1, \dots, x_n)$:

$$y_j = \frac{T_j(x_1, \dots, x_n)}{U_j(x_1, \dots, x_n)} \qquad (j = 1, \dots, n). \tag{6}$$

Here T_j and U_j are non-identical vanishing polynomials from the ring $\mathbb{K}[X_1, \dots, X_n]$. The polynomials

$$f_j(X_1, \dots, X_n, Y_j) := Y_j U_j(X_1, \dots, X_n) - T_j(X_1, \dots, X_n), \tag{7}$$

for $j = 1, \dots, n$ thus vanish at the point $(x_1, \dots, x_n, y_j)$.

Theorem 3. *Let $\mathbb{K}$ be a field with $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$. Furthermore, assume that the numbers $x_1, \dots, x_n \in \mathbb{C}$ and $y_1, \dots, y_n \in \mathbb{C}$ satisfy a system of equations,*

$$y_j = R_j(x_1, \dots, x_n) \qquad (j = 1, \dots, n),$$

where $R_j(X_1, \dots, X_n)$ are rational functions from the field extension $\mathbb{K}(X_1, \dots, X_n)$ for $j = 1, \dots, n$. If the numbers $x_1, \dots, x_n$ are algebraically independent over $\mathbb{K}$ and

$$\det{}_n \left(\frac{\partial R_j}{\partial X_i}(x_1, \dots, x_n) \right) \neq 0 \tag{8}$$

holds, then the numbers $y_1, \dots, y_n$ are algebraically independent over $\mathbb{K}$.

Theorem 4. *Let $\mathbb{K}$ be a field with $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$. Let m and n be positive integers with $1 \leq m < n$. Furthermore, assume that the numbers $x_1, \dots, x_n \in \mathbb{C}$ and $y_1, \dots, y_m \in \mathbb{C}$ satisfy a system of equations,*

$$f_j(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \qquad (j = 1, \dots, m), \tag{9}$$

where $f_j(X_1, \dots, X_n, Y_1, \dots, Y_m) \in \mathbb{K}[X_1, \dots, X_n, Y_1, \dots, Y_m]$ are polynomials for $j = 1, \dots, m$. If the numbers $x_1, \dots, x_n$ are algebraically independent over $\mathbb{K}$ and if

$$\det{}_m \left(\frac{\partial f_j}{\partial X_i}(x_1, \dots, x_n, y_1, \dots, y_m) \right) \neq 0 \tag{10}$$

holds with $1 \leqslant i, j \leqslant m$, then $y_1, \dots, y_m$ are algebraically independent over the field $\mathbb{K}(x_{m+1}, \dots, x_n)$.

The following results deal with the application of the criterion to continued fractions. We assume that the reader is familiar with the theory of ordinary (regular) continued fractions. Given a real quadratic irrational number η, its continued fraction is of the periodic form

$$\begin{aligned} \eta &= \left[a_0, a_1, \dots, a_{k-1}, \overline{a_k, \dots, a_{k+\ell-1}} \right] \\ &= \left[a_0, a_1, \dots, a_{k-1}, a_k, \dots, a_{k+\ell-1}, a_k, \dots, a_{k+\ell-1}, a_k, \dots, a_{k+\ell-1}, \dots \right], \end{aligned}$$

where $a_0 \in \mathbb{Z}$, and $a_1, \dots, a_{k+\ell}$ with $k \geqslant 0$ and $\ell \geqslant 0$ are positive integers. In this paper, we are treating continued fractions, where the partial quotients a_i are real numbers, but not all integers. In particular, we consider continued fractions of the form

$$\begin{aligned} \eta &= \left[a_0, a_1, \dots, a_{k-1}, \overline{a_k, \dots, a_{k+\ell-1}} \right] \\ &= \left[a_0, a_1, \dots, a_{k-1}, a_k, \dots, a_{k+\ell-1}, a_k, \dots, a_{k+\ell-1}, a_k, \dots, a_{k+\ell-1}, \dots \right], \end{aligned}$$

where $a_0 \in \mathbb{Z}$, and $a_1, \ldots, a_{k+\ell}$ with $k \geqslant 0$ and $\ell \geqslant 0$ are positive integers. In this paper, we call a continued fraction *non-regular* if the partial quotients a_i are real numbers, but not all integers.

In particular, we consider non-regular continued fractions of the form

$$\xi := \left[\,\overline{a_0, a_1, \ldots, a_{n-1}}\,\right]$$

with positive partial quotients $a_0, \ldots, a_{n-1}$, which are all or partly algebraically independent over the rational numbers $\mathbb{Q}$. For any continued fraction $[a_0, a_1, a_2, \ldots]$ with positive partial quotients $a_0, a_1, a_2, \ldots$, where the series $(a_\nu)_{\nu \geqslant 0}$ at some point becomes periodic, the sum $\sum_{\nu=0}^{\infty} a_\nu$ diverges towards infinity. Therefore, such a continued fraction $[a_0, a_1, a_2, \ldots]$ is convergent [3, Proposition 2.3].

Let p_m/q_m denote the convergents of ξ. They are given by the recurrence formulas

$$p_{-1} := 1\,, \quad p_0 := a_0\,, \quad p_m := a_m p_{m-1} + p_{m-2} \qquad (m \geq 1)\,, \tag{11}$$

$$q_{-1} := 0\,, \quad q_0 := 1\,, \quad q_m := a_m q_{m-1} + q_{m-2} \qquad (m \geq 1)\,. \tag{12}$$

One has

$$p_{m-1}q_m - p_m q_{m-1} = (-1)^m \qquad (m \geq 0)\,, \tag{13}$$

even if the partial quotients of ξ (and therefore the p_m and q_m, too) are not integers.

In Section 4.2 we present an algorithm and an example that allows us to decide the algebraic independence of ξ and p_n/q_n by applying the algebraic independence criterion.

Theorem 5. *Let $\mathbb{L}$ be a real finite field extension of $\mathbb{Q}$, and let α and β be two real algebraic independent numbers over $\mathbb{L}$ greater than 0. Moreover, let p_m/q_m be the convergents of $\xi := [\,\overline{\alpha, \beta}\,]$. Then, for every integer $n \geqslant 0$, the two numbers ξ and p_n/q_n are algebraically independent over the field $\mathbb{L}$.*

Remark 3. Theorem 5 no longer holds in the case of period 1, for example, when

$$y_1 = [\,\overline{\alpha}\,] = \frac{\alpha}{2} + \frac{1}{2}\sqrt{4+\alpha^2}\,.$$

In this case we may have

$$y_2 = \frac{p_0}{q_0} = \frac{\alpha}{1} = \alpha$$

so that y_1 and y_2 are algebraically dependent over $\mathbb{Q}$ by the equation

$$y_1^2 - y_1 y_2 - 1 = 0\,.$$

Remark 4. Theorem 5 can be proved with the algorithm shown in Section 4.2, but we give a simpler elementary proof in Section 4.3.

Theorem 5 shows that the two numbers $\xi = [\,\overline{\alpha, \beta}\,]$ and p_n/q_n are algebraically independent over the field $\mathbb{L}$, provided that α and β are algebraically independent. If

$$\sigma(x) := \frac{Ax+B}{Cx+D}$$

is a so-called linear fractional transformation with integers A, B, C, D and determinant $AD - BC \neq 0$, we apply Theorem 3 with

$$R_j(X_1, X_2) := \frac{AX_j + B}{CX_j + D} \qquad (j = 1, 2)$$

in order to prove the algebraic independence of the two numbers $\sigma(\xi)$ and $\sigma\big(p_n/q_n\big)$ over $\mathbb{L}$ using formula (8):

$$\det_2\Big(\frac{\partial R_j}{\partial X_i}(x_1, x_2)\Big) = \frac{(AD - BC)^2}{(C\xi + D)(Cp_n/q_n + D)} \neq 0, \qquad \text{where} \quad x_1 = \xi\,, \; x_2 = p_n/q_n\,.$$

By Theorem 3.2 in the paper [2] of Havens et al. it is shown that for $AD - BC = \pm 2$ and some additional restrictions on the regular continued fractions of a number x, the transformed convergents $\sigma(u_t/v_t)$ of x are also convergents of the transformed number $\sigma(x)$. One may pose the same problem for our irregular continued fractions from Theorem 5: what are the conditions on ξ and the linear transformation function so that the numbers $\sigma(p_n/q_n)$ are also convergents of $\sigma(\xi)$?

In the following we introduce some preliminary notations useful for the next theorem, where we derive further results regarding algebraic independence using the results obtained in Theorem 5.

Let $n \geq 2$, and let $a_0, \dots, a_{n-1}$ be positive real numbers, where $a_0 \geqslant 1$. Assume that, among $a_0, \dots, a_{n-1}$ there are at least two numbers, a_μ and a_ν with $0 \leqslant \mu < \nu \leqslant n-1$, say, which are algebraically independent over $\mathbb{Q}$. In terms of the transcendence degree, this means that

$$tr.deg\,(\mathbb{Q}(a_0, \dots, a_{n-1}) : \mathbb{Q}) \geqslant 2\,. \tag{14}$$

Next, we introduce the field $\mathbb{L}$ by the field extension

$$\mathbb{L} := \mathbb{Q}\big(a_0, \dots, a_{\mu-1}, a_{\mu+1}, \dots, a_{\nu-1}, a_{\nu+1}, \dots, a_{n-1}\big)\,.$$

Let p_m^*/q_m^* be the mth convergent of the number

$$\xi := \big[\,\overline{a_0, \dots, a_{n-1}}\,\big]\,. \tag{15}$$

Since $a_0 \geqslant 1$ and $a_k > 0$ for $k = 1, \dots, n-1$, we have

$$\frac{p_m^*}{q_m^*} > 1 \qquad (m \geqslant 1)\,. \tag{16}$$

Both, the terms p_m^* and q_m^*, depend on $a_0, \dots, a_m$ for all $m \geqslant 0$. We write

$$\begin{aligned} p_m^* &= p_m^*(x_0, \dots, x_m) \qquad \text{for} \quad x_k = a_k \quad (0 \leqslant k \leqslant m)\,,\\ q_m^* &= q_m^*(x_0, \dots, x_m) \qquad \text{for} \quad x_k = a_k \quad (0 \leqslant k \leqslant m)\,. \end{aligned}$$

Finally, we assume that

$$\det_2 \begin{pmatrix} \frac{\partial}{\partial x_\mu}\Big(\frac{p_{n-1}^* - q_{n-2}^*}{q_{n-1}^*}\Big) & \frac{\partial}{\partial x_\nu}\Big(\frac{p_{n-1}^* - q_{n-2}^*}{q_{n-1}^*}\Big) \\ \frac{\partial}{\partial x_\mu}\Big(\frac{p_{n-1}^* - q_{n-2}^*}{p_{n-2}^*}\Big) & \frac{\partial}{\partial x_\nu}\Big(\frac{p_{n-1}^* - q_{n-2}^*}{p_{n-2}^*}\Big) \end{pmatrix}_{\left(\substack{x_\nu = a_\nu \\ 0 \leqslant \nu \leqslant n-1}\right)} \neq 0\,, \tag{17}$$

where

$$\frac{p_{n-1}^*}{q_{n-1}^*} = [\,a_0, a_1, \ldots, a_{n-1}\,] \quad \text{and} \quad \frac{p_{n-2}^*}{q_{n-2}^*} = [\,a_0, a_1, \ldots, a_{n-2}\,]\,. \tag{18}$$

Now, we are ready to formulate the extension of Theorem 5.

Theorem 6. *Let all the quantities defined above satisfy the conditions in (14) to (18). Then there exist two positive real numbers α and β with the following properties. α and β are algebraically independent over $\mathbb{L}$, it is $\xi = [\,\overline{\alpha, \beta}\,]$, and for every integer $n \geqslant 0$, the two numbers ξ and p_n/q_n are algebraically independent over $\mathbb{L}$.*

Remark 5. The convergents p_m^*/q_m^* of ξ depend on $a_0, \ldots, a_{n-1}$, while the convergents p_m/q_m of ξ depend only on α and β. The convergents p_m^*/q_m^* are in general completely different from the convergents p_m/q_m, although they approximate the same number ξ. This effect occurs because we are not dealing with continued fractions whose partial quotients are natural numbers. If one tries to generalize Theorem 5 to a continued fraction with period length greater than two, our method of proof will become very complicated, but with the help of Theorem 6 we reduce such a continued fraction to one with period length two and we obtain such a result for the convergents p_m/q_m instead of p_m^*/q_m^*.

3. Proofs of the algebraic criterion and its variants

There exist many different proofs of Theorem 1 that make use of different tools. A first proof, that will be shown in this chapter, applies a result on separable algebraic field extensions from commutative algebra. This is the shortest proof, based on Lemma 1 below, and therefore we prefer to present it in this paper. Other proofs make use, respectively, of the projection theorem of Tarski–Seidenberg and the concept of semi-algebraic sets in the case of a real field $\mathbb{K}$ [15], differential forms [4] or isomorphism of fields [6, Example 1]. Finally, one last proof is an elementary one due to Kumiko Nishioka, based on the essential idea to embed a given zero of a polynomial with at least two variables into a path on which the polynomial vanishes identically [4]. In particular, the differential forms allow us to prove the inverse statement of Theorem 2. We omit the proof of the inverse statement in this paper, because the result is only of theoretical interest and has no interesting applications. The proof is given in [4].

The following lemma is essential in the proof of Theorem 1.

Lemma 1. *Let $\mathbb{L}$ denote a field, $\mathbb{F}$ a field extension of $\mathbb{L}$. We assume that $x_1, \ldots, x_n \in \mathbb{F}$ satisfy a system of equations,*

$$P_j(x_1, \ldots, x_n) = 0 \qquad (j = 1, \ldots, n)\,,$$

where $P_j(X_1, \ldots, X_n) \in \mathbb{L}[X_1, \ldots, X_n]$ are polynomials. If

$$\det_n\left(\frac{\partial P_j}{\partial X_i}(x_1, \ldots, x_n)\right) \neq 0$$

holds, then $\mathbb{L}(x_1, \ldots, x_n)$ is a separable algebraic field extension of $\mathbb{L}$.

Proof. This lemma is the Corollary of Theorem 40 in [13, p. 126]. It also results from Proposition 5.3 in [10, p. 371]. $\square$

Proof of Theorem 1. Using the polynomials $f_1, \ldots, f_n$ from Theorem 1, we define some auxiliary polynomials P_j for $j = 1, \ldots, n$ by

$$P_j(X_1, \ldots, X_n) := f_j(X_1, \ldots, X_n, y_1, \ldots, y_n) \in \mathbb{K}(y_1, \ldots, y_n)[X_1, \ldots, X_n]. \tag{19}$$

They are constructed using the variables $X_1, \ldots, X_n$, and they all vanish at the point $(x_1, \ldots, x_n)$ because of (2). The determinant condition (3) takes the form

$$\det_n\left(\frac{\partial P_j}{\partial X_i}(x_1, \ldots, x_n)\right) \neq 0. \tag{20}$$

With $\mathbb{F} = \mathbb{C}$, $\mathbb{L} = \mathbb{K}(y_1, \ldots, y_n)$ and the setting of polynomials $P_j(X_1, \ldots, X_n)$ $(j = 1, \ldots, n)$ as in (19), all the conditions of Lemma 1 including (20) are fulfilled. So, by Lemma 1, $\mathbb{L}(x_1, \ldots, x_n)$ is an algebraic field extension over $\mathbb{L}$.

We denote by $tr.deg\big(\mathbb{K}_2 : \mathbb{K}_1\big)$ the transcendence degree of the field extension $\mathbb{K}_2$ over $\mathbb{K}_1$. Then, we have

$$tr.deg\big(\mathbb{L}(x_1, \ldots, x_n) : \mathbb{L}\big) = 0 \quad \text{and} \quad tr.deg\big(\mathbb{K}(x_1, \ldots, x_n) : \mathbb{K}\big) = n,$$

where the latter due to the condition on the algebraic independence of $x_1, \ldots, x_n$ over $\mathbb{K}$ in Theorem 1. Trivially, $\mathbb{K} \subseteq \mathbb{L}$ implies that $tr.deg(\mathbb{L}(x_1, \ldots, x_n) : \mathbb{K}) \geq n$. If the fields $\mathbb{K}_1$, $\mathbb{K}_2$ and $\mathbb{K}_3$ form a field tower $\mathbb{K}_1 \subseteq \mathbb{K}_2 \subseteq \mathbb{K}_3$, we know according to the chain theorem for degrees of transcendence, that

$$tr.deg\big(\mathbb{K}_3 : \mathbb{K}_1\big) = tr.deg\big(\mathbb{K}_3 : \mathbb{K}_2\big) + tr.deg\big(\mathbb{K}_2 : \mathbb{K}_1\big). \tag{21}$$

Applying this relation to the field tower $\mathbb{K} \subseteq \mathbb{L} = \mathbb{K}(y_1, \ldots, y_n) \subseteq \mathbb{L}(x_1, \ldots, x_n)$, we obtain

$$\begin{aligned} n &\leq tr.deg\big(\mathbb{L}(x_1, \ldots, x_n) : \mathbb{K}\big) = tr.deg\big(\mathbb{L}(x_1, \ldots, x_n) : \mathbb{L}\big) + tr.deg\big(\mathbb{L} : \mathbb{K}\big) \\ &= tr.deg\big(\mathbb{L} : \mathbb{K}\big) = tr.deg\big(\mathbb{K}(y_1, \ldots, y_n) : \mathbb{K}\big) \leq n. \end{aligned}$$

Since $tr.deg\big(\mathbb{K}(y_1, \ldots, y_n) : \mathbb{K}\big) = n$, the theorem is proved. $\square$

Proof of Theorem 2. Setting

$$f_j\big(X_1, \ldots, X_n, Y_j\big) := T_j\big(X_1, \ldots, X_n\big) - Y_j \qquad (1 \leqslant j \leqslant n),$$

Theorem 2 follows immediately from Theorem 1. $\square$

Proof of Theorem 3. We denote the determinant in (8) by Δ_1 and assume first that $\Delta_1 \neq 0$. Furthermore, we express the rational functions R_j by

$$R_j(X_1, \ldots, X_n) = \frac{T_j(X_1, \ldots, X_n)}{U_j(X_1, \ldots, X_n)} \qquad (j = 1, \ldots, n),$$

using the polynomials T_j and U_j already introduced in (6). The algebraic independence of $y_1, \ldots, y_n$ over $\mathbb{K}$ can be proven with Theorem 1, where the non-vanishing of the

Jacobi determinant must be shown for the functions f_j from (7):

$$\begin{aligned}
\Delta_2 &:= \det_n\Big(\frac{\partial f_j}{\partial X_i}\Big)(x_1,\ldots,x_n,y_1,\ldots,y_n)\\
&= \det_n\Big(y_j\frac{\partial U_j}{\partial X_i}-\frac{\partial T_j}{\partial X_i}\Big)(x_1,\ldots,x_n)\\
&= \det_n\Big(\frac{T_j}{U_j}\frac{\partial U_j}{\partial X_i}-\frac{\partial T_j}{\partial X_i}\Big)(x_1,\ldots,x_n)\\
&= (-1)^n U_1\cdots U_n \det_n\left(\frac{1}{U_j^2}\Big(\frac{\partial T_j}{\partial X_i}U_j-T_j\frac{\partial U_j}{\partial X_i}\Big)\right)(x_1,\ldots,x_n)\\
&= (-1)^n U_1\cdots U_n \det_n\Big(\frac{\partial R_j}{\partial X_i}\Big)(x_1,\ldots,x_n)\\
&= (-1)^n U_1\cdots U_n(x_1,\ldots,x_n)\Delta_1\,.
\end{aligned}$$

Since $\Delta_1 \neq 0$, we have $\Delta_2 \neq 0$. The $x_1,\ldots,x_n$ are considered as algebraically independent over $\mathbb{K}$ so that the polynomial $Q_1\cdots Q_n$ does not vanish identically. Then, $U_1\cdots U_n(x_1,\ldots,x_n) \neq 0$ is guaranteed. Theorem 1 thus proves the algebraic independence of $y_1,\ldots,y_n$ over $\mathbb{K}$. $\square$

Proof of Theorem 4. In addition to the equations in (9), we introduce the polynomials

$$f_j(X_j,Y_j) := X_j - Y_j \in \mathbb{K}[X_j,Y_j] \qquad (j=m+1,\ldots,n)$$

so that for $y_j := x_j$ $(j=m+1,\ldots,n)$ the polynomial f_j vanishes at the point (x_j,y_j). Theorem 1 can now be applied for the proof of the algebraic independence of $y_1,\ldots,y_n$ over $\mathbb{K}$. We compute the Jacobi determinant of the functions $f_1,\ldots,f_n$ at the position $(x_1,\ldots,x_n,y_1,\ldots,y_n)$ and apply the condition from (10):

$$\begin{aligned}
&\det_n\Big(\frac{\partial f_j}{\partial X_i}(x_1,\ldots,x_n,y_1,\ldots,y_n)\Big)\\
&= \det_n\begin{pmatrix}
\frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_m} & \frac{\partial f_1}{\partial X_{m+1}} & \cdots & \frac{\partial f_1}{\partial X_n}\\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots\\
\frac{\partial f_m}{\partial X_1} & \cdots & \frac{\partial f_m}{\partial X_m} & \frac{\partial f_m}{\partial X_{m+1}} & \cdots & \frac{\partial f_m}{\partial X_n}\\
0 & \cdots & 0 & 1 & \cdots & 0\\
\vdots & \cdots & \vdots & \cdots & \cdots & \vdots\\
0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}(x_1,\ldots,x_n,y_1,\ldots,y_m)\\
&= \det_n\begin{pmatrix}
\frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_m}\\
\vdots & \cdots & \vdots\\
\frac{\partial f_m}{\partial X_1} & \cdots & \frac{\partial f_m}{\partial X_m}
\end{pmatrix}(x_1,\ldots,x_n,y_1,\ldots,y_m) \quad \neq 0\,.
\end{aligned}$$

Thus, according to Theorem 1 and to $y_j = x_j$ for $j = m+1, \ldots, n$, the algebraic independence of the numbers $y_1, \ldots, y_m, x_{m+1}, \ldots, x_n$ over $\mathbb{K}$ is proven. This implies

$$tr.deg\big(\mathbb{K}(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n) : \mathbb{K}\big) = n\,.$$

Taking into account the presupposed algebraic independence of $x_1, \ldots, x_n$ over $\mathbb{K}$, we know that the equation $tr.deg\big(\mathbb{K}(x_{m+1}, \ldots, x_n) : \mathbb{K}\big) = n - m$ holds. Again we make use of the chain rule from (21), applied to the fields $\mathbb{K} \subseteq \mathbb{K}(x_{m+1}, \ldots, x_n) \subseteq \mathbb{K}(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n)$. Thus we obtain

$$tr.deg\big(\mathbb{K}(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n) : \mathbb{K}(x_{m+1}, \ldots, x_n)\big) = m\,,$$

which is the statement of Theorem 4. □

4. Proofs and discussion of the application of the criterion to continued fractions

4.1. A lemma for the practical handling of the determinant condition in the theorems on algebraic independence

The aim of this short section is to show the logical structure when applying a criterion of independence. The proofs of the algebraic independence of a set of numbers are by contradiction and require some additional calculations when eliminating parameters, which are accomplished with the help of resultants. This method will be presented here. In particular, we want to show the method used later to get a contradiction in the proof of Theorem 5.

Let X_1, X_2, Y_1, Y_2 be variables and x_1, x_2, y_1, y_2 be real numbers, where x_1 and x_2 are algebraically independent over a finite field extension $\mathbb{L}$ of $\mathbb{Q}$. Let us consider three polynomials, such that

$$\begin{aligned} P_1\big(X_1, X_2, Y_1\big) &\in \mathbb{L}\big[X_1, X_2, Y_1\big]\,,\\ P_2\big(X_1, X_2, Y_2\big) &\in \mathbb{L}\big[X_1, X_2, Y_2\big]\,,\\ D\big(X_1, X_2, Y_1, Y_2\big) &\in \mathbb{L}\big[X_1, X_2, Y_1, Y_2\big]\,. \end{aligned}$$

We require that

$$\begin{aligned} P_1(x_1, x_2, y_1) &= 0\,,\\ P_2(x_1, x_2, y_2) &= 0\,. \end{aligned} \tag{22}$$

At this point, assume that

$$D(x_1, x_2, y_1, y_2) = 0\,. \tag{23}$$

Later, in our application, we require that D is not zero. As a first step, we consider $\overline{D}(Y_1, Y_2) := D(x_1, x_2, Y_1, Y_2)$ and $\overline{P}_1(Y_1) := P_1(x_1, x_2, Y_1)$ as polynomials from the polynomial ring $\mathbb{Q}^*(x_1, x_2)[Y_1, Y_2]$. By (22) and (23), the two polynomials $\overline{D}(Y_1, Y_2)$ and $\overline{P}_1(Y_1)$ have a common root at $(Y_1, Y_2) = (y_1, y_2)$, and thus their resultant with respect to Y_1 vanishes for $Y_2 = y_2$. Therefore, on the one side, we have for the polynomial $\overline{P}_3(Y_2)$ defined by

$$\overline{P}_3(Y_2) := \mathrm{Res}_{Y_1}\big(\overline{D}(Y_1, Y_2),\ \overline{P}_1(Y_1)\big) = \mathrm{Res}_{Y_1}\big(D(x_1, x_2, Y_1, Y_2)\,,\, P_1(x_1, x_2, Y_1)\big)$$

the property

$$\overline{P}_3(y_2) = 0, \tag{24}$$

on the other side, it is

$$\overline{P}_3(Y_2) \in \mathbb{L}(x_1, x_2, Y_2). \tag{25}$$

As a second step, we consider $\overline{P}_2(Y_2) := P_2(x_1, x_2, Y_2)$ as a polynomial from the polynomial ring $\mathbb{Q}^*(x_1, x_2)[Y_2]$ over the field $\mathbb{L}(x_1, x_2)$. Together with (24) and (25), we obtain for the polynomial $\overline{P}_4(X_1, X_2)$ defined by

$$\overline{P}_4(X_1, X_2) := \mathrm{Res}_{Y_2}\big(\overline{P}_3(Y_2), \overline{P}_2(Y_2)\big) \in \mathbb{L}\big[X_1, X_2\big]$$

the property

$$\overline{P}_4(x_1, x_2) = 0.$$

In order to obtain a contradiction to the algebraic independence of x_1 and x_2, we need an argument that $\overline{P}_4(X_1, X_2)$ does not vanish identically. For this we have the following lemma. If it can be applied, the contradiction shows that (23) does not hold.

Lemma 2. *Let α_1 and α_2 be two real numbers, which are not necessarily algebraically independent over $\mathbb{L}$. Set $X_1 = \alpha_1$ and $X_2 = \alpha_2$ so that we have for the above polynomials*

$$\begin{aligned} P_1(\alpha, \beta, Y_1) &= \overline{P}_1(Y_1), \\ P_2(\alpha, \beta, Y_2) &= \overline{P}_2(Y_2), \\ D(\alpha, \beta, Y_1, Y_2) &= \overline{D}(Y_1, Y_2), \end{aligned}$$

and all these polynomials are considered as polynomials over the field $\mathbb{L}(\alpha_1, \alpha_2)$. If

$$\gamma := Res_{Y_2}\left(Res_{Y_1}\big(\overline{D}(Y_1, Y_2), \overline{P}_1(Y_1)\big), \overline{P}_2(Y_2)\right)$$

is a non-vanishing number from the field $\mathbb{L}(\alpha_1, \alpha_2)$, then the polynomial $\overline{P}_4(X_1, X_2)$ does not vanish identically.

Proof. Since there is no difference whether the resultants is computed with respect to Y_1 and Y_2 with unspecified parameters X_1 and X_2, and then assigning special values to these parameters, or whether assigning these values in the involved polynomials already at the beginning before the resultant calculations, then the proof follows. □

4.2. *An algorithm for the algebraic independence of ξ and p_n/q_n*

Let $P_0, \dots, P_\rho \in \mathbb{Q}[X_1, X_2]$ and $T_1, \dots, T_w \in \mathbb{Q}[X_1, X_2]$ be polynomials, where $\rho \geqslant 0$ and $w \geqslant 2$. Let $\alpha, \beta \in \mathbb{R}$ be two algebraically independent numbers over the field $\mathbb{Q}$ of rationals. We consider the number ξ defined by the continued fraction expansion

$$\xi := \left[P_0(\alpha, \beta), \dots, P_\rho(\alpha, \beta), \overline{T_1(\alpha, \beta), \dots, T_w(\alpha, \beta)} \right]. \tag{26}$$

We denote the convergents of ξ by p_n/q_n. What follows is the description of an algorithm which allows to prove the algebraic independence over $\mathbb{Q}$ of any pair ξ and p_n/q_n with

$n \geqslant 0$. In Appendix B, we summarize the algorithm in a short step-by-step for the convenience of the reader.

The convergents of the number

$$\eta := \left[\,\overline{T_1(\alpha,\beta),\ldots,T_w(\alpha,\beta)}\,\right] \tag{27}$$

are denoted by p_n^*/q_n^*. By the recurrence formulas for numerators and denominators of convergents it is clear that

$$p_n,\, p_n^*,\, q_n,\, q_n^* \in \mathbb{Q}[\alpha,\beta] \tag{28}$$

for all $n \geqslant -1$. From (27), we have the identity

$$\eta = \frac{p_{w-1}^*\eta + p_{w-2}^*}{q_{w-1}^*\eta + q_{w-2}^*}\,,$$

which is equivalent with

$$q_{w-1}^*\eta^2 + \left(q_{w-2}^* - p_{w-1}^*\right)\eta - p_{w-2}^* = 0\,. \tag{29}$$

From (26) and (27), we obtain

$$\xi = \left[\,P_0(\alpha,\beta),\ldots,P_\rho(\alpha,\beta),\eta\,\right] = \frac{p_\rho\eta + p_{\rho-1}}{q_\rho\eta + q_{\rho-1}}\,,$$

or

$$\left(q_\rho\xi - p_\rho\right)\eta + \left(q_{\rho-1}\xi - p_{\rho-1}\right) = 0\,. \tag{30}$$

Solving (30) for η, substituting into (29) and clearing denominators, we finally have the equation

$$q_{w-1}^*\left(q_{\rho-1}\xi - p_{\rho-1}\right)^2 - \left(q_{w-2}^* - p_{w-1}^*\right)\left(q_\rho\xi - p_\rho\right)\left(q_{\rho-1}\xi - p_{\rho-1}\right) - p_{w-2}^*\left(q_\rho\xi - p_\rho\right)^2 = 0\,. \tag{31}$$

By (28), it is clear that the left-hand side of (31) is a polynomial in $x_1 := \alpha$, $x_2 := \beta$ and $y_1 := \xi$.

Next, we apply Corollary 1 in [5] and set[1]

$$\begin{aligned}
r &:= w\,, \qquad i \in \{\rho+1,\rho+2,\ldots,\rho+w\}\,,\\
z_n &\in \{p_{wn+i}, q_{wn+i}\}\,,\\
T(a) &:= T_{w\{\frac{a-\rho-1}{w}\}+1}\left(\left\lceil\frac{a-\rho}{w}\right\rceil\right)\\
&= T_{w\{\frac{a-\rho-1}{w}\}+1}\\
&= T_{1+(a-\rho-1 \bmod w)} \quad (a \in \mathbb{N})\,,\\
M &:= \big((n-1)w + i + 2\big) \bmod w = (i+2) \bmod w\,.
\end{aligned}$$

[1] There is a small error in Corollary 1 and Theorem 1: the condition on i should be read as $0 \leqslant \rho < i < \rho + r + 1$. Then we have $\rho + 1 \leqslant i \leqslant \rho + r$.

Let a and l be positive integers. Define $D_0(a) := 1$, and

$$D_l(a) := \begin{vmatrix} -T(a) & -1 & 0 & & & & \\ 1 & -T(a+1) & -1 & & & & \\ 0 & 1 & -T(a+2) & & & & \\ & & & \ddots & & & \\ & & & & -T(a+l-3) & -1 & 0 \\ & & & & 1 & -T(a+l-2) & -1 \\ & & & & 0 & 1 & -T(a+l-1) \end{vmatrix}.$$

It follows for the determinants D_{w-1} in Corollary 1 that $D_{w-1}(M-w) = D_{w-1}(M) \neq 0$. The linear recursion formula in Corollary 1 therefore simplifies to

$$(-1)^{w-1} z_n + \big(D_w(M-w) + D_{w-2}(M+1)\big) z_{n-1} - z_{n-2} = 0,$$

or, for $z_n = p_{wn+\rho+t}$ with $t = 1, 2, \ldots, w$ and $n \geqslant 2$,

$$p_{wn+\rho+t} = (-1)^w \big(D_w(M-w) + D_{w-2}(M+1)\big) p_{w(n-1)+\rho+t} + (-1)^{w-1} p_{w(n-2)+\rho+t}. \tag{32}$$

The denominators $z_n = q_{wn+\rho+t}$ of the leaping convergents satisfy exactly the same linear recursion formula. Hence, there is a polynomial $P(X_1, X_2)$ depending at most on w, ρ and t (but not on n) such that

$$P(\alpha, \beta) = (-1)^w \big(D_w(M-w) + D_{w-2}(M+1)\big)$$

and, by (32),

$$\left.\begin{aligned} p_{wn+\rho+t} &= P(\alpha, \beta) p_{w(n-1)+\rho+t} + (-1)^{w-1} p_{w(n-2)+\rho+t}, \\ q_{wn+\rho+t} &= P(\alpha, \beta) q_{w(n-1)+\rho+t} + (-1)^{w-1} q_{w(n-2)+\rho+t}. \end{aligned}\right\} \tag{33}$$

Set

$$\left.\begin{aligned} U_1 &= U_1(\alpha, \beta) := \tfrac{1}{2} P(\alpha, \beta), \\ U_2 &= U_2(\alpha, \beta) := \tfrac{1}{4} P^2(\alpha, \beta) + (-1)^{w-1}; \\ V_{1,p} &= V_{1,p}(\alpha, \beta) := \frac{p_{w+\rho+t} - U_1 p_{\rho+t} + \sqrt{U_2} p_{\rho+t}}{2\sqrt{U_2}}, \\ V_{2,p} &= V_{2,p}(\alpha, \beta) := \frac{-p_{w+\rho+t} + U_1 p_{\rho+t} + \sqrt{U_2} p_{\rho+t}}{2\sqrt{U_2}}; \\ V_{1,q} &= V_{1,q}(\alpha, \beta) := \frac{q_{w+\rho+t} - U_1 q_{\rho+t} + \sqrt{U_2} q_{\rho+t}}{2\sqrt{U_2}}, \\ V_{2,q} &= V_{2,q}(\alpha, \beta) := \frac{-q_{w+\rho+t} + U_1 q_{\rho+t} + \sqrt{U_2} q_{\rho+t}}{2\sqrt{U_2}}. \end{aligned}\right\} \tag{34}$$

Then, we have for every integer $n \geqslant 0$ the two formulas

$$\left.\begin{aligned} p_{wn+\rho+t} &= V_{1,p}\big(U_1 + \sqrt{U_2}\big)^n + V_{2,p}\big(U_1 - \sqrt{U_2}\big)^n, \\ q_{wn+\rho+t} &= V_{1,q}\big(U_1 + \sqrt{U_2}\big)^n + V_{2,q}\big(U_1 - \sqrt{U_2}\big)^n. \end{aligned}\right\} \tag{35}$$

Note that for $n = 0$ and $n = 1$ we have the identities

$$p_{\rho+t} = V_{1,p} + V_{2,p} \qquad \text{and} \qquad p_{w+\rho+t} = V_{1,p}\big(U_1 + \sqrt{U_2}\big) + V_{2,p}\big(U_1 - \sqrt{U_2}\big),$$

and two similar formulas for the leaping denominators.

In the first formula of (35), we now expand the two bracketed expressions using the binomial formula:

$$\begin{aligned} p_{wn+\rho+t} &= V_{1,p}\sum_{k=0}^{n}\binom{n}{k}U_1^{n-k}U_2^{k/2} + V_{2,p}\sum_{k=0}^{n}\binom{n}{k}U_1^{n-k}(-1)^k U_2^{k/2} \\ &= \left(V_{1,p}+V_{2,p}\right)\sum_{\substack{0\leqslant k\leqslant n\\ k\equiv 0 \pmod 2}}\binom{n}{k}U_1^{n-k}U_2^{k/2} \\ &\quad + \left(V_{1,p}-V_{2,p}\right)\sum_{\substack{0\leqslant k\leqslant n\\ k\equiv 1 \pmod 2}}\binom{n}{k}U_1^{n-k}U_2^{1/2}U_2^{(k-1)/2} \\ &= p_{\rho+t}\sum_{\substack{0\leqslant k\leqslant n\\ k\equiv 0 \pmod 2}}\binom{n}{k}U_1^{n-k}U_2^{k/2} \\ &\quad + \left(p_{w+\rho+t}-U_1 p_{\rho+t}\right)\sum_{\substack{0\leqslant k\leqslant n\\ k\equiv 1 \pmod 2}}\binom{n}{k}U_1^{n-k}U_2^{(k-1)/2} \\ &\in \mathbb{Q}\big[\alpha,\beta\big]. \end{aligned} \tag{36}$$

Similarly, we have

$$\begin{aligned} q_{wn+\rho+t} &= q_{\rho+t}\sum_{\substack{0\leqslant k\leqslant n\\ k\equiv 0 \pmod 2}}\binom{n}{k}U_1^{n-k}U_2^{k/2} \\ &\quad + \left(q_{w+\rho+t}-U_1 q_{\rho+t}\right)\sum_{\substack{0\leqslant k\leqslant n\\ k\equiv 1 \pmod 2}}\binom{n}{k}U_1^{n-k}U_2^{(k-1)/2} \\ &\in \mathbb{Q}\big[\alpha,\beta\big]. \end{aligned} \tag{37}$$

The parameter

$$y_2 := \frac{p_{wn+\rho+t}}{q_{wn+\rho+t}}$$

is now added to the previous parameters $x_1 := \alpha$, $x_2 := \beta$ and $y_1 := \xi$ already introduced above. We now insert the formulas from (36) and (37) into the equation

$$y_2 q_{wn+\rho+t} - p_{wn+\rho+t} = 0$$

and obtain

$$\begin{aligned}
&\Bigg(q_{\rho+t} \sum_{\substack{0\leqslant k\leqslant n\\ k\equiv 0 \pmod 2}} \binom{n}{k} U_1^{n-k}U_2^{k/2} + \big(q_{w+\rho+t} - U_1 q_{\rho+t}\big) \sum_{\substack{1\leqslant k\leqslant n\\ k\equiv 1 \pmod 2}} \binom{n}{k} U_1^{n-k}U_2^{(k-1)/2} \Bigg)\, y_2 \\
&- \Bigg(p_{\rho+t} \sum_{\substack{0\leqslant k\leqslant n\\ k\equiv 0 \pmod 2}} \binom{n}{k} U_1^{n-k}U_2^{k/2} + \big(p_{w+\rho+t} - U_1 p_{\rho+t}\big) \sum_{\substack{1\leqslant k\leqslant n\\ k\equiv 1 \pmod 2}} \binom{n}{k} U_1^{n-k}U_2^{(k-1)/2} \Bigg) . \\
= \;& \big(y_2 q_{\rho+t} - p_{\rho+t}\big) \sum_{\substack{0\leqslant k\leqslant n\\ k\equiv 0 \pmod 2}} \binom{n}{k} U_1^{n-k}U_2^{k/2} \\
&+ \big((y_2 q_{w+\rho+t} - p_{w+\rho+t}) - U_1(y_2 q_{\rho+t} - p_{\rho+t})\big) \sum_{\substack{1\leqslant k\leqslant n\\ k\equiv 1 \pmod 2}} \binom{n}{k} U_1^{n-k}U_2^{(k-1)/2} = 0 .
\end{aligned} \tag{38}$$

Finally, we perform the following substitutions in the two Eqs. (31) and (38):

$$\alpha \to X_1 , \quad \beta \to X_2 , \quad \xi \to Y_1 , \quad y_2 \to Y_2 .$$

Then, the left-hand sides of the formulas (31) and (38) represent two polynomials $P_1(X_1, X_2, Y_1) \in \mathbb{L}[X_1, X_2, Y_1]$ and $P_2(X_1, X_2, Y_2) \in \mathbb{L}[X_1, X_2, Y_2]$, respectively, to which the method described in Chapter 4 can be applied to prove the algebraic independence of $y_1 = \xi$ and $y_2 = p_{wn+\rho+t}/q_{wn+\rho+t}$ for $n \geqslant 0$ and $1 \leqslant t \leqslant w$. For this purpose, we have to compute the determinant

$$D(X_1, X_2, Y_1, Y_2) := \begin{vmatrix} \frac{\partial P_1}{\partial X_1} & \frac{\partial P_1}{\partial X_2} \\ \frac{\partial P_2}{\partial X_1} & \frac{\partial P_2}{\partial X_2} \end{vmatrix} , \tag{39}$$

and then we proceed as described in Section 4.1.

It remains to prove the algebraic independence of ξ and p_n/q_n for $n = 0, 1, \ldots, \rho$. For any such n, we express p_n and q_n in terms of α and β using the recurrence formulas $p_m = P_m(\alpha, \beta)p_{m-1} + p_{m-2}$ with $p_{-1} = 1$, $p_0 = P_0(\alpha, \beta)$ and $q_m = P_m(\alpha, \beta)q_{m-1} + q_{m-2}$ with $q_{-1} = 0$, $q_0 = 1$, respectively. Then, again by setting $y_2 = p_n/q_n$, we have $y_2 q_n - p_n = 0$, so that we have the polynomial $P_2(X_1, X_2, Y_2) = U(X_1, X_2)Y_2 - V(X_1, X_2)$ with certain polynomials $U(X_1, X_2), V(X_1, X_2) \in \mathbb{Q}[X_1, X_2]$ instead of (38). The polynomial $P_1(X_1, X_2, Y_1)$ is the same as already used above. Then, we proceed again as described by the method introduced in Section 4.1.

We demonstrate the application of the algorithm using the following example. Let α and β be two algebraically independent real numbers, and let $T_1(X_1, X_2)$, $T_2(X_1, X_2)$ and $T_3(X_1, X_2)$ be three polynomials from $\mathbb{Q}[X_1, X_2]$ specified later and taking positive values for $X_1 = \alpha$ and $X_2 = \beta$. We consider the continued fraction expansion

$$\xi = \big[0, \overline{T_1(\alpha, \beta), T_2(\alpha, \beta), T_3(\alpha, \beta)} \big] , \tag{40}$$

by choosing $P_0(\alpha, \beta) = 0$ with $\rho = 0$ and $w = 3$ in (26). We want to show that ξ and p_n/q_n are algebraically independent for all convergents p_n/q_n with $n \equiv 0 \pmod 3$ and $n \geqslant 3$; the cases with $n \equiv 1, 2 \pmod 3$ can be treated similarly. For simplicity, we write T_j instead of $T_j(\alpha, \beta)$ for $j = 1, 2, 3$.

In (29), we have

$$p_1^* = T_1T_2 + 1\,, \quad p_2^* = T_1T_2T_3 + T_1 + T_3\,, \quad q_1^* = T_2\,, \quad q_2^* = T_2T_3 + 1\,,$$

so that (31) becomes the formula

$$\big(1 + T_1T_2\big)\xi^2 + \big(T_1T_2T_3 + T_1 - T_2 + T_3\big)\xi - T_2T_3 - 1 = 0\,. \tag{41}$$

Because of this identity, we introduce the polynomial

$$P_1\big(X_1, X_2, Y_1\big) := \big(1 + T_1T_2\big)Y_1^2 + \big(T_1T_2T_3 + T_1 - T_2 + T_3\big)Y_1 - \big(1 + T_2T_3\big)\,, \tag{42}$$

where $T_j = T_j(X_1, X_2)$ for $j = 1, 2, 3$. The recurrence relations (33) are the same for $t = 1, 2, 3$; for $z_{3n+t} = p_{3n+t}$ and $z_{3n+t} = q_{3n+t}$ we have the common formula

$$z_{3n+t} - \big(T_1T_2T_3 + T_1 + T_2 + T_3\big)z_{3(n-1)+t} - z_{3(n-2)+t} = 0 \qquad (t \in \{1, 2, 3\},\ n \geqslant 2)\,.$$

For $t = 3$ we have

$$z_{3(n+1)} - \big(T_1T_2T_3 + T_1 + T_2 + T_3\big)z_{3n} - z_{3(n-1)} = 0\,. \tag{43}$$

This equation is also valid for $n = 1$ since all the subscripts are nonnegative. Set

$$P(\alpha, \beta) := T_1T_2T_3 + T_1 + T_2 + T_3\,.$$

Then, the quantities in (34) are given by

$$\begin{aligned}
U_1 &= \frac{T_1T_2T_3 + T_1 + T_2 + T_3}{2}\,,\\
U_2 &= \frac{(T_1T_2T_3 + T_1 + T_2 + T_3)^2}{4} + 1\,,\\
V_{1,p} &= \frac{T_2T_3 + 1}{2\sqrt{1 + P^2/4}}\,,\\
V_{2,p} &= -\frac{T_2T_3 + 1}{2\sqrt{1 + P^2/4}}\,,\\
V_{1,q} &= \frac{1}{2} + \frac{T_1T_2T_3 + T_1 - T_2 + T_3}{4\sqrt{1 + P^2/4}}\,,\\
V_{2,q} &= \frac{1}{2} - \frac{T_1T_2T_3 + T_1 - T_2 + T_3}{4\sqrt{1 + P^2/4}}\,.
\end{aligned}$$

Next, (35) gives the two formulas

$$\left.\begin{aligned}
p_{3n} &= V_{1,p}\left(\tfrac{P}{2} + \sqrt{\tfrac{P^2}{4} + 1}\right)^n + V_{2,p}\left(\tfrac{P}{2} - \sqrt{\tfrac{P^2}{4} + 1}\right)^n,\\
q_{3n} &= V_{1,q}\left(\tfrac{P}{2} + \sqrt{\tfrac{P^2}{4} + 1}\right)^n + V_{2,q}\left(\tfrac{P}{2} - \sqrt{\tfrac{P^2}{4} + 1}\right)^n.
\end{aligned}\right\} \tag{44}$$

In formula (38) we may choose $t = 0$ so that $p_{\rho+t} = 0$. We obtain

$$y_2 \cdot \sum_{\substack{0 \leqslant k \leqslant n\\ 2|k}} \binom{n}{k} U_1^{n-k}U_2^{k/2} + \big(y_2q_3 - p_3 - U_1y_2\big) \cdot \sum_{\substack{0 \leqslant k \leqslant n\\ 2\nmid k}} \binom{n}{k} U_1^{n-k}U_2^{(k-1)/2} = 0\,,$$

which motivates the definition of the polynomial $P_2(X_1, X_2, Y_2)$ given by

$$P_2(X_1, X_2, Y_2) := Y_2 \cdot \sum_{\substack{0 \leqslant k \leqslant n \\ 2|k}} \binom{n}{k} U_1(X_1, X_2)^{n-k} U_2(X_1, X_2)^{k/2} +$$

$$+ \Big(\frac{Y_2(T_1T_2T_3 + T_1 - T_2 + T_3)}{2} - T_2T_3 - 1 \Big) \cdot \sum_{\substack{0 \leqslant k \leqslant n \\ 2 \nmid k}} \binom{n}{k} U_1(X_1, X_2)^{n-k} U_2(X_1, X_2)^{(k-1)/2} . \quad (45)$$

Later, we replace Y_2 by $y_2 = p_{3n}/q_{3n}$. In order to proceed with the algorithm, the polynomials T_1, T_2 and T_3 must be specified. We choose

$$T_1(X_1, X_2) := X_1 ,$$
$$T_2(X_1, X_2) := X_2 ,$$
$$T_3(X_1, X_2) := X_1X_2 .$$

Then, we have in (40), $\xi = \big[0, \overline{\alpha, \beta, \alpha\beta}\,\big] = \frac{1}{\alpha + \frac{1}{\beta + \frac{1}{\alpha\beta + \xi}}}$, which yields

$$\xi = -\frac{\alpha^2\beta^2 + \alpha\beta + \alpha - \beta}{2(\alpha\beta + 1)} + \sqrt{\frac{(\alpha^2\beta^2 + \alpha\beta + \alpha - \beta)^2}{4(\alpha\beta + 1)^2} + \frac{\alpha\beta^2 + 1}{\alpha\beta + 1}} . \quad (46)$$

This formula can also be obtained from (41). Moreover, we have

$$P(X_1, X_2) = X_1^2X_2^2 + X_1 + X_2 + X_1X_2 ,$$
$$U_1(X_1, X_2) = \frac{1}{2}\big(X_1^2X_2^2 + X_1 + X_2 + X_1X_2\big) ,$$
$$U_2(X_1, X_2) = 1 + \frac{1}{4}\big(X_1^2X_2^2 + X_1 + X_2 + X_1X_2\big)^2 .$$

The determinant in (39) simplifies to

$$P_3(X_1, X_2, Y_1, Y_2) = \frac{\partial P_1}{\partial X_1}\frac{\partial P_2}{\partial X_2} - \frac{\partial P_1}{\partial X_2}\frac{\partial P_2}{\partial X_1} \quad (47)$$
$$= \big(X_2Y_1^2 + (2X_1X_2^2 + X_2 + 1)Y_1 - X_2^2 \big)\frac{\partial P_2}{\partial X_2} - \big(X_1Y_1^2 + (2X_1^2X_2 + X_1 - 1)Y_1 - 2X_1X_2 \big)\frac{\partial P_2}{\partial X_1} ,$$

where

$$\frac{\partial P_2}{\partial X_1} =$$
$$Y_2 \cdot \sum_{\substack{0 \leqslant k \leqslant n \\ 2|k}} \binom{n}{k} (n-k)U_1^{n-k-1}\frac{\partial U_1}{\partial X_1}U_2^{k/2} + \frac{k}{2}U_1^{n-k}U_2^{k/2-1}\frac{\partial U_2}{\partial X_1}$$
$$+ \left(\frac{(2X_1X_2^2 + X_2 + 1)Y_2}{2} - X_2^2 \right) \cdot \sum_{\substack{0 \leqslant k \leqslant n \\ 2 \nmid k}} \binom{n}{k} U_1^{n-k}U_2^{(k-1)/2}$$

$$+\left(\frac{\left(X_1^2X_2^2+X_1-X_2+X_1X_2\right)Y_2}{2}-X_1X_2^2-1\right)\cdot\sum_{\substack{0\leqslant k\leqslant n\\ 2\nmid k}}\binom{n}{k}\Big((n-k)U_1^{n-k-1}\frac{\partial U_1}{\partial X_1}U_2^{(k-1)/2}$$

$$+\frac{(k-1)}{2}U_1^{n-k}U_2^{(k-3)/2}\frac{\partial U_2}{\partial X_1}\Big) \tag{48}$$

and

$$\frac{\partial P_2}{\partial X_2}=$$

$$Y_2\cdot\sum_{\substack{0\leqslant k\leqslant n\\ 2|k}}\binom{n}{k}(n-k)U_1^{n-k-1}\frac{\partial U_1}{\partial X_2}U_2^{k/2}+\frac{k}{2}U_1^{n-k}U_2^{k/2-1}\frac{\partial U_2}{\partial X_2}$$

$$+\left(\frac{\left(2X_1^2X_2+X_1-1\right)Y_2}{2}-2X_1X_2\right)\cdot\sum_{\substack{0\leqslant k\leqslant n\\ 2\nmid k}}\binom{n}{k}U_1^{n-k}U_2^{(k-1)/2}$$

$$+\left(\frac{\left(X_1^2X_2^2+X_1-X_2+X_1X_2\right)Y_2}{2}-X_1X_2^2-1\right)\cdot\sum_{\substack{0\leqslant k\leqslant n\\ 2\nmid k}}\binom{n}{k}\Big((n-k)U_1^{n-k-1}\frac{\partial U_1}{\partial X_2}U_2^{(k-1)/2}$$

$$+\frac{(k-1)}{2}U_1^{n-k}U_2^{(k-3)/2}\frac{\partial U_2}{\partial X_2}\Big)\,. \tag{49}$$

As described in Section 4.1, we may choose special values for X_1 and X_2, namely,

$$X_1 = 1 \quad\text{and}\quad X_2 = -1\,.$$

Then we have $U_1(X_1, X_2) = 0$ and $U_2(X_1, X_2) = 1$ and, additionally,

$$\frac{\partial U_1}{\partial X_1} = 1\,,\quad \frac{\partial U_1}{\partial X_2} = \frac{\partial U_2}{\partial X_1} = \frac{\partial U_2}{\partial X_2} = 0\,.$$

Case 1. $n \equiv 0 \bmod 2$ and $n \geqslant 2$.

With the special values for X_1 and X_2, we obtain from (48) and (49),

$$\begin{aligned}\frac{\partial P_2}{\partial X_1} &= n(Y_2-2)\,,\\ \frac{\partial P_2}{\partial X_2} &= 0\,;\\ \overline{D}(Y_1,Y_2) &= n(Y_1^2-2Y_1+2)\big(2-Y_2\big)\,.\end{aligned} \tag{50}$$

Moreover, for an even n, the polynomials P_1 and P_2 take the special forms

$$\begin{aligned}\overline{P}_1(Y_1) &= 2(Y_1-1)\,,\\ \overline{P}_2(Y_2) &= Y_2\,.\end{aligned}$$

We compute the number γ in Lemma 2:

$$\gamma = \mathrm{Res}_{Y_2}\left(\mathrm{Res}_{Y_1}\big(n(Y_1^2-2Y_1+2)\big(2-Y_2\big),\ 2(Y_1-1)\big),\ Y_2\right) = -8n \neq 0\,. \tag{51}$$

Case 2. $n \equiv 1 \bmod 2$ and $n \geqslant 1$.

Substituting the special values for X_1 and X_2 into (48) and (49), we now have

$$\begin{aligned}\frac{\partial P_2}{\partial X_1} &= (n+1)Y_2 - 1\,,\\ \frac{\partial P_2}{\partial X_2} &= 2 - Y_2\,;\\ \overline{D}(Y_1,Y_2) &= (Y_2-2)(Y_1-1)^2 - \big((n+1)Y_2-1\big)(Y_1^2-2Y_1+2)\,.\end{aligned} \tag{52}$$

By the assumption of Case 2, it is

$$\begin{aligned}\overline{P}_1(Y_1) &= 2(Y_1-1)\,,\\ \overline{P}_2(Y_2) &= Y_2-2\,.\end{aligned}$$

Using these polynomials and (52), the number γ in Lemma 2 becomes

$$\gamma \;=\; 8n+4 \;\neq\; 0\,. \tag{53}$$

Theorem 1 and Lemma 2 together with (51) and (53) prove the algebraic independence of $y_1 = \xi$ and $y_2 = p_{3n}/q_{3n}$ over $\mathbb{Q}$ for all $n \geq 1$.

Case 3. $n = 0$.

Since $p_0/q_0 = 0$ is a rational number, $y_1 = \xi$ and $y_2 = p_0/q_0$, are obviously algebraically dependent over the rationals $\mathbb{Q}$. This completes our example for the algorithm. $\diamondsuit$

4.3. Proofs of Theorems 5 and 6 by application of the algebraic independence criterion

Proof of Theorem 5. Let

$$y_1 \;:=\; [\,\overline{\alpha,\beta}\,] \;=\; \alpha + \frac{1}{\beta+\frac{1}{y_1}} \;=\; \frac{(\alpha\beta+1)y_1+\alpha}{\beta y_1+1}\,.$$

Then we have

$$\beta y_1^2 - \alpha\beta y_1 - \alpha \;=\; 0\,. \tag{54}$$

This implies

$$y_1 \;=\; \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4}+\frac{\alpha}{\beta}}\,, \tag{55}$$

From [5, Corollary 1], we obtain three-term recurrence formulas for leaping convergents of $y_0 = \xi$ with indices modulo 2. For $n \geqslant 2$ we have

$$\left.\begin{aligned} p_{2n} &= (\alpha\beta+2)p_{2n-2} - p_{2n-4}\,,\\ q_{2n} &= (\alpha\beta+2)q_{2n-2} - q_{2n-4}\,;\end{aligned}\right\} \tag{56}$$

and for $n \geqslant 1$,

$$\left.\begin{aligned} p_{2n+1} &= (\alpha\beta+2)p_{2n-1} - p_{2n-3}\,,\\ q_{2n+1} &= (\alpha\beta+2)q_{2n-1} - q_{2n-3}\,.\end{aligned}\right\} \tag{57}$$

The characteristic polynomial $P(x)$ of all four recursion formulas in (56) and (57) is

$$P(x) = x^2 - (\alpha\beta + 2)x + 1 .$$

Its roots t_1, t_2 can be easily calculated:

$$t_1 = \frac{\alpha\beta + 2}{2} + \sqrt{\frac{(\alpha\beta + 2)^2}{4} - 1} = \frac{\alpha\beta + 2}{2} + \frac{1}{2}\sqrt{\alpha\beta(\alpha\beta + 4)}, \tag{58}$$

and, similarly,

$$t_2 = \frac{\alpha\beta + 2}{2} - \frac{1}{2}\sqrt{\alpha\beta(\alpha\beta + 4)}. \tag{59}$$

We have the initial values

$$\left.\begin{array}{lcl} p_{-1} & = & 1 , \\ p_0 & = & \alpha , \\ p_1 & = & \alpha\beta + 1 , \\ p_2 & = & \alpha^2\beta + 2\alpha , \\ p_3 & = & \alpha^2\beta^2 + 3\alpha\beta + 1 . \end{array}\right\} \tag{60}$$

Applying formula (55) we get

$$\frac{\xi}{\alpha} = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\alpha\beta}}.$$

From formulas (56) and (60), it follows $p_{2n}/q_{2n} = \alpha R_{2n}(\alpha\beta)$, where $R_{2n}(x)$ is a rational function. Then

$$\left(\frac{\xi}{p_{2n}/q_{2n}} - \frac{1}{2R_{2n}(\alpha\beta)}\right)^2 = \left(\frac{1}{4} + \frac{1}{\alpha\beta}\right)\frac{1}{R_{2n}(\alpha\beta)^2},$$

$$\frac{p_{2n}/q_{2n}}{\alpha} = R_{2n}(\alpha\beta) = R_{2n}(R(2\xi\alpha^{-1} - 1)),$$

with $R(x) = 4/(x^2 - 1)$, and hence

$$tr.deg(\mathbb{Q}^*(\alpha, \beta, \xi, p_{2n}/q_{2n}) : \mathbb{Q}^*(\xi, p_{2n}/q_{2n})) = 0.$$

By $tr.deg(\mathbb{Q}^*(\alpha, \beta, \xi, p_{2n}/q_{2n}) : \mathbb{Q}^*) \geq tr.deg(\mathbb{Q}^*(\alpha, \beta) : \mathbb{Q}^*) = 2$, where

$$\begin{aligned} & tr.deg(\mathbb{Q}^*(\alpha, \beta, \xi, p_{2n}/q_{2n}) : \mathbb{Q}^*) \\ & = tr.deg(\mathbb{Q}^*(\alpha, \beta, \xi, p_{2n}/q_{2n}) : \mathbb{Q}^*(\xi, p_{2n}/q_{2n})) + tr.deg(\mathbb{Q}^*(\xi, p_{2n}/q_{2n}) : \mathbb{Q}^*) \\ & tr.deg(\mathbb{Q}^*(\xi, p_{2n}/q_{2n}) : \mathbb{Q}^*), \end{aligned}$$

it follows that $tr.deg(\mathbb{Q}^*(\xi, p_{2n}/q_{2n}) : \mathbb{Q}^*) = 2$. Similarly $tr.deg(\mathbb{Q}^*(\xi, p_{2n+1}/q_{2n+1}) : \mathbb{Q}^*) = 2$. This completes the proof of Theorem 5. $\square$

Proof of Theorem 6. From (15), we have the identity

$$\xi = \frac{p_{n-1}^*\xi + p_{n-2}^*}{q_{n-1}^*\xi + q_{n-2}^*},$$

which can be rearranged to the quadratic equation

$$q_{n-1}^*\xi^2 - (p_{n-1}^* - q_{n-2}^*)\xi - p_{n-2}^* = 0.$$

Solving this equation for the positive number ξ yields

$$\xi = \frac{p_{n-1}^* - q_{n-2}^*}{2q_{n-1}^*} + \sqrt{\frac{(p_{n-1}^* - q_{n-2}^*)^2}{4q_{n-1}^{*2}} + \frac{p_{n-2}^*}{q_{n-1}^*}} = R_1 + \sqrt{R_2}, \tag{61}$$

where

$$R_1 := \frac{p_{n-1}^* - q_{n-2}^*}{2q_{n-1}^*} \qquad \text{and} \qquad R_2 := R_1^2 + \frac{p_{n-2}^*}{q_{n-1}^*}.$$

Set

$$\alpha := \frac{p_{n-1}^* - q_{n-2}^*}{q_{n-1}^*} \qquad \text{and} \qquad \beta := \frac{p_{n-1}^* - q_{n-2}^*}{p_{n-2}^*}. \tag{62}$$

We obtain from (16) for $m = n-1 \geqslant 1$ that $p_{n-1}^* > q_{n-1}^* \geqslant q_{n-2}^*$. Therefore, α and β are positive real numbers. From the combination of Theorems 3 and 4, due to the assumed nonvanishing of the determinant in (17), we have the algebraic independence of the two numbers α and β over $\mathbb{L}$.

Since

$$R_1 = \frac{\alpha}{2} \qquad \text{and} \qquad R_2 = \frac{\alpha^2}{4} + \frac{\alpha}{\beta},$$

we obtain from the Eqs. (55) and (61) that

$$\xi = R_1 + \sqrt{R_2} = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \frac{\alpha}{\beta}} = y_1 = [\,\overline{\alpha, \beta}\,]. \tag{63}$$

The remaining statement in Theorem 6 on the algebraic independence of ξ and p_n/q_n over $\mathbb{Q}^*$ for $n \geqslant 0$ follows from Theorem 5. □

4.4. *Some supplementary results and their proofs*

From the formulas (15), (18), (62) and (63) we obtain an explicit formula to transform the regular continued fraction $\xi = [b_0, \overline{b_1, \ldots, b_n}]$ of a quadratic irrational number ξ into a non-regular continued fraction with a small preperiod and a period of length one. Here, b_0 and $b_\nu \geqslant 1$ for $1 \leqslant \nu \leqslant n$ are integers.

For this result, we additionally need an identity between a periodic regular continued fraction with rational partial quotients and a non-regular continued fraction with integer denominators and numerators.

Let a/b and c/d be rationals with $a, c \in \mathbb{Z} \setminus \{0\}$ and $b, d \in \mathbb{N}$. Then, we have

$$\left[\overline{\frac{a}{b}, \frac{c}{d}}\right] = \frac{a}{b} + \cfrac{1}{\frac{c}{d} + \cfrac{1}{\frac{a}{b} + \cfrac{1}{\frac{c}{d} + \ddots}}} = \frac{a}{b} + \cfrac{d}{c + \cfrac{bd}{a + \cfrac{bd}{c + \cfrac{bd}{\ddots}}}}$$

$$= \frac{a}{b} + \cfrac{d}{c + \left[\overline{\frac{bd}{a} + \frac{bd}{c}}\right]} = \frac{a}{b} + \left[\frac{d}{c} + \overline{\frac{bd}{a} + \frac{bd}{c}}\right].$$

In the special case of $a = c$, this identity becomes the simple form

$$\left[\overline{\frac{a}{b}, \frac{a}{d}}\right] = \frac{a}{b} + \left[\frac{d}{a} + \overline{\frac{bd}{a}}\right]. \tag{64}$$

Shifting the index in (18), we have with

$$\frac{p_n^*}{q_n^*} = [b_1, \dots, b_n] \quad \text{and} \quad \frac{p_{n-1}^*}{q_{n-1}^*} = [b_1, \dots, b_{n-1}]$$

the equation

$$[\overline{b_1, \dots, b_n}] = \left[\overline{\frac{p_n^* - q_{n-1}^*}{q_n^*}, \frac{p_n^* - q_{n-1}^*}{p_{n-1}^*}}\right], \tag{65}$$

which results from (15), (62) and (63). Combining (64) and (65), we obtain

$$\xi = [b_0, \overline{b_1, \dots, b_n}] = b_0 + [0, \overline{b_1, \dots, b_n}] = b_0 + \frac{1}{[\overline{b_1, \dots, b_n}]}$$

$$\overset{(65)}{=} b_0 + \frac{1}{\left[\overline{\frac{p_n^*-q_{n-1}^*}{q_n^*}, \frac{p_n^*-q_{n-1}^*}{p_{n-1}^*}}\right]} \overset{(64)}{=} b_0 + \frac{1}{\frac{p_n^*-q_{n-1}^*}{q_n^*} + \left[\frac{p_{n-1}^*}{p_n^*-q_{n-1}^*} + \overline{\frac{p_{n-1}^* q_n^*}{p_n^*-q_{n-1}^*}}\right]}$$

$$= b_0 + \frac{q_n^*}{p_n^* - q_{n-1}^* + \left[\frac{p_{n-1}^* q_n^*}{p_n^*-q_{n-1}^*} + \overline{\frac{p_{n-1}^* q_n^*}{p_n^*-q_{n-1}^*}}\right]}$$

$$= b_0 + \left[\frac{q_n^*}{p_n^* - q_{n-1}^*} + \overline{\frac{p_{n-1}^* q_n^*}{p_n^* - q_{n-1}^*}}\right]$$

$$= b_0 + \cfrac{q_n^*}{p_n^* - q_{n-1}^* + \cfrac{p_{n-1}^* q_n^*}{p_n^*-q_{n-1}^* + \cfrac{p_{n-1}^* q_n^*}{p_n^*-q_{n-1}^* + \ddots}}}.$$

For example,

$$\sqrt{19} = [4, \overline{2, 1, 3, 1, 2, 8}] = 4 + \cfrac{117}{312 + \cfrac{4563}{312 + \cfrac{4563}{312 + \ddots}}} = 4 + \left[\frac{117}{312} + \overline{\frac{4563}{312}}\right],$$

since

$$\frac{p_6}{q_6} = \big[2, 1, 3, 1, 2, 8\big] = \frac{326}{117} \quad \text{and} \quad \frac{p_5}{q_5} = \big[2, 1, 3, 1, 2\big] = \frac{39}{14}\,.$$

Example 1. An interesting special case of Theorem 6 is given when the period of ξ is composed of only two algebraically independent numbers α^* and β^* over $\mathbb{Q}$. For this, we consider the following two examples.

(i) Let $\xi := [\,\overline{\alpha^*, \beta^*, \beta^*, \alpha^*}\,]$.

With $n = 4$ we have by $a_0 = a_3 = \alpha$ and $a_1 = a_2 = \beta$. Then, we obtain with (62),

$$\alpha := \frac{p_3^* - q_2^*}{q_3^*} = \frac{\alpha^{*2}\beta^{*2} + \alpha^{*2} - \beta^{*2} + 2\alpha^*\beta^*}{\alpha^*\beta^{*2} + \alpha^* + \beta^*} = \frac{p_3^* - q_2^*}{p_2^*} =: \beta\,.$$

Due to $\alpha = \beta$, the algebraic independence of these two quantities is not given; the determinant in (17) vanishes. Theorem 6 is not applicable to this situation.

(ii) Next, let $\xi := [\,\overline{\alpha^*, \beta^*, \beta^*}\,]$.

With $n = 3$ we obtain

$$\alpha := \frac{p_2^* - q_1^*}{q_2^*} = \frac{\alpha^*\beta^{*2} + \alpha^*}{\beta^{*2} + 1} = \alpha^*\,,$$
$$\beta := \frac{p_2^* - q_1^*}{p_1^*} = \frac{\alpha^*\beta^{*2} + \alpha^*}{\alpha^*\beta^* + 1}\,.$$

The determinant in (17) takes the value

$$\det\begin{pmatrix} \frac{\partial}{\partial x_0}\,(x_0) & \frac{\partial}{\partial x_1}\,(x_0) \\ \frac{\partial}{\partial x_0}\left(\frac{x_0x_1^2+x_0}{x_0x_1+1}\right) & \frac{\partial}{\partial x_1}\left(\frac{x_0x_1^2+x_0}{x_0x_1+1}\right) \end{pmatrix}_{\left(\begin{smallmatrix} x_0 = \alpha^* \\ x_1 = \beta^* \end{smallmatrix}\right)} = \frac{\alpha^*\big(\alpha^*\beta^{*2} - \alpha^* + 2\beta^*\big)}{\big(\alpha^*\beta^* + 1\big)^2}\,,$$

which does not vanish by the algebraic independence of α^* and β^* over $\mathbb{L}$. Thus, Theorem 6 is applicable with

$$\xi = \big[\,\overline{\alpha^*, \beta^*, \beta^*}\,\big] = \left[\,\overline{\alpha^*, \frac{\alpha^*(\beta^{*2} + 1)}{\alpha^*\beta^* + 1}}\,\right].$$

We complete the application of the algebraic independence criterion to non regular continued fractions by the following proposition, for which we provide two different proofs.

Proposition 1. *Let $n \ge 1$ and let $a_0, \ldots, a_{n-1}$ be real algebraic independent numbers greater than 1. Then the convergents*

$$\frac{p_m}{q_m} = [a_0, a_1, \ldots, a_m] \qquad (0 \le m \le n-1) \tag{66}$$

of the continued fraction $[\overline{a_0, a_1, \ldots, a_{n-1}}]$ *are algebraically independent over the field* $\mathbb{Q}$ *of rational numbers.*

Proof with Theorem 3. We may consider p_m/q_m as a rational function formed by integer polynomials at the places $a_0, \ldots, a_m$. We compute these rational functions using the recurrence formulas (11) and (12). We have for $m = 0$ and $m = 1$

$$\frac{p_0}{q_0} = a_0\,, \qquad \frac{p_1}{q_1} = \frac{a_0 a_1 + 1}{a_1}\,, \tag{67}$$

for $m \geq 2$

$$\frac{p_m}{q_m} = \frac{a_m p_{m-1} + p_{m-2}}{a_m q_{m-1} + q_{m-2}}\,. \tag{68}$$

Note that by (66) the four numbers p_{m-1}, q_{m-1}, p_{m-2} and q_{m-2} do not depend on a_μ for $\mu = m+1, \ldots, n$. Since p_m/q_m is a rational function $R_m(X_0, \ldots, X_m)$ at the places $a_0, \ldots, a_m$, we have[2]

$$y_j := \frac{p_j}{q_j} = R_j\big(a_0, a_1, \ldots, a_j\big) \qquad (j = 0, \ldots, n-1)\,.$$

In order to prove the algebraic independence of $y_0, \ldots, y_{n-1}$ over $\mathbb{Q}$ we apply Theorem 3

$$\det_n\left(\frac{\partial R_j}{\partial X_i}(a_0, \ldots, a_{n-1})\right) = \begin{vmatrix} \dfrac{\partial R_0}{\partial X_0} & \cdots & \dfrac{\partial R_0}{\partial X_{n-1}} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial R_{n-1}}{\partial X_0} & \cdots & \dfrac{\partial R_{n-1}}{\partial X_{n-1}} \end{vmatrix} (a_0, \ldots, a_{n-1})$$

$$= \begin{vmatrix} \dfrac{\partial R_0}{\partial X_0} & 0 & 0 & \cdots & 0 \\ \dfrac{\partial R_1}{\partial X_0} & \dfrac{\partial R_1}{\partial X_1} & 0 & \cdots & 0 \\ \dfrac{\partial R_2}{\partial X_0} & \dfrac{\partial R_2}{\partial X_1} & \dfrac{\partial R_2}{\partial X_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial R_{n-1}}{\partial X_0} & \dfrac{\partial R_{n-1}}{\partial X_1} & \dfrac{\partial R_{n-1}}{\partial X_2} & \cdots & \dfrac{\partial R_{n-1}}{\partial X_{n-1}} \end{vmatrix} (a_0, \ldots, a_n) = \prod_{j=0}^{n-1} \frac{\partial R_j}{\partial X_j}(a_0, \ldots, a_j)\,. \tag{69}$$

From (67) we obtain

$$\frac{\partial R_0(a_0)}{\partial X_0} = 1 = \frac{1}{q_0^2}\,, \qquad \frac{\partial R_1(a_0, a_1)}{\partial X_1} = \frac{a_0 a_1 - (a_0 a_1 + 1)}{a_1^2} = -\frac{1}{a_1^2} = -\frac{1}{q_1^2}\,. \tag{70}$$

[2] We change the index from m to j to adjust to the notation in Theorem 3.

Similarly, using the quotient rule for the derivative in (68), we get for $j \geq 2$, respecting the remark made about (68)

$$\begin{aligned}\frac{\partial R_j(a_0,\dots,a_j)}{\partial X_j} &= \frac{p_{j-1}(a_j q_{j-1}+q_{j-2})-(a_j p_{j-1}+p_{j-2})q_{j-1}}{(a_j q_{j-1}+q_{j-2})^2}\\ &= \frac{p_{j-1}q_{j-2}-p_{j-2}q_{j-1}}{q_j^2}\\ &\overset{(13)}{=} \frac{(-1)^j}{q_j^2}\,. \end{aligned}\tag{71}$$

With (70) and (71) the formula (69) changes into

$$\det_n\left(\frac{\partial R_j}{\partial X_i}(a_0,\dots,a_{n-1})\right) = \prod_{j=0}^{n-1}\frac{(-1)^j}{q_j^2} = (-1)^{(n-1)n/2}\prod_{j=0}^{n-1} q_j^{-2}\,. \tag{72}$$

At this point, we insert a brief consideration of the nonvanishing of the q_j for $j = 0,\dots,n-1$. It is clear that $q_0 = 1$ and $q_1 = a_1 > 1$ do not vanish. By induction on the index m, it then follows from the recurrence formula (12) that all denominators q_m are positive and thus nonzero.

The determinant in (72) does not vanish either, and with Theorem 3 the proof of our Proposition 1 is completed.

Proof without Theorem 3. We would like to underline that there is another way to prove Proposition 1 with the use of algebra notions. In particular, since the degree of transcendence of the field $\mathbb{Q}(a_0,\dots,a_{n-1})$ over $\mathbb{Q}$ is n, we have a chain of transcendental field extensions

$$\mathbb{Q} \subset \mathbb{Q}(a_0) \subset \mathbb{Q}(a_0,a_1) \subset \cdots \subset \mathbb{Q}(a_0,\dots,a_{n-2}) \subset \mathbb{Q}(a_0,\dots,a_{n-1}).$$

From the recurrence formulas of p_m and q_m we know for the generic convergents p_m/q_m that

$$c_m := \frac{p_m}{q_m} \in \mathbb{Q}(a_0,\dots,a_m)\setminus\mathbb{Q}(a_0,\dots,a_{m-1}) \qquad (0 \leq m \leq n).$$

By induction, we will show that

$$tr.deg\left(\mathbb{Q}(c_0,\dots,c_\nu):\mathbb{Q}\right) = \nu+1 \qquad (0 \leq \nu \leq n). \tag{73}$$

For $\nu = 0$ it follows that $tr.deg\left(\mathbb{Q}(c_0):\mathbb{Q}\right) = tr.deg\left(\mathbb{Q}(a_0):\mathbb{Q}\right) = 1$. Let us suppose that Eq. (73) is true for $0 \leq \nu \leq n-2$ and we want to prove that

$$tr.deg\left(\mathbb{Q}(c_0,\dots,c_{\nu+1}):\mathbb{Q}\right) = \nu+2.$$

Let us assume the contrary, so there exists a polynomial $P(X_0,\dots,X_{\nu+1}) \in \mathbb{Q}[X_0,\dots,X_{\nu+1}]\setminus\{0\}$ such that $P(c_0,\dots,c_{\nu+1}) = 0$. From the algebraic independence of $c_0,\dots,c_\nu$, it follows that the degree of the polynomial P with respect to the variable $X_{\nu+1}$ is greater than 0, otherwise we would have $P(c_0,\dots,c_\nu,0) = 0$, which is impossible. Now, let us pose the polynomial $\bar{P}(X_{\nu+1}) := P(c_0,\dots,c_\nu,X_{\nu+1}) \in \mathbb{Q}(c_0,\dots,c_\nu)[X_{\nu+1}]$. From the consideration above, the leading coefficient of $\bar{P}$ is a non-vanishing polynomial from

$\mathbb{Q}[X_0,\ldots,X_\nu]$, and when computed for $c_0,\ldots,c_\nu$, it does not vanish. In other words,

$$\bar{P}[X_{\nu+1}] = P(c_0,\ldots,c_\nu,X_{\nu+1}) \in \mathbb{Q}(a_0,\ldots,a_\nu)[X_{\nu+1}] \setminus \{0\}. \tag{74}$$

From $P(c_0,\ldots,c_{\nu+1}) = 0$ it should follow that $\bar{P}(c_{\nu+1}) = P(c_0,\ldots,c_\nu,c_{\nu+1}) = 0$, but this is not possible by (74) and the fact that $c_{\nu+1} = p_{\nu+1}/q_{\nu+1}$ is transcendental over $\mathbb{Q}(a_0,\ldots,a_\nu)$. $\square$

CRediT authorship contribution statement

Gessica Alecci: Writing – original draft. **Carsten Elsner:** Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Past results obtained with the algebraic independence criterion and its variants

We summarize some results that have been obtained with the algebraic independence criterion and its variants. It is only a small selection of scattered published results, but it is intended to demonstrate the spectrum of applications of the method. An extended version of this Appendix with more references can be found on arXiv:2311.18536.

An application of Theorem 1 can be found in [8]. Let $j \geqslant 0$ be an integer. There are 16 families of q-series like the three families

$$\begin{aligned} A_{2j+1}(q) &= \sum_{n=1}^{\infty} \frac{n^{2j+1}q^{2n}}{1-q^{2n}}, \\ B_{2j+1}(q) &= = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^{2j+1}q^{2n}}{1-q^{2n}}, \\ C_{2j+1}(q) &= \sum_{n=1}^{\infty} \frac{n^{2j+1}q^{n}}{1-q^{2n}}, \end{aligned} \tag{75}$$

which are generated from the Fourier expansion of the Jacobian elliptic functions. In [8, Theorem 2] sets of three such algebraically independent q-series for an algebraic number q with $0 < |q| < 1$ are characterized. For example, $A_{2j+1}(q)$, $B_{2j+1}(q)$ and $C_{2j+1}(q)$ are algebraically independent over $\mathbb{Q}$.

Here there are some applications of Theorem 2.

- Let q be an algebraic number with $0 < |q| < 1$. Then the three numbers $A_1(q)$, $A_{2i+1}(q)$ and $A_{2j+1}(q)$ from the series in (75) with $1 \leqslant i < j$ and $(i,j) \neq (1,3)$ are algebraically independent over $\mathbb{Q}$. For $A_1(q)$, $A_3(q)$ and $A_7(q)$ there is the algebraic relation $A_7(q) = A_3(q) + 120A_3^2(q)$ [6, Theorem 2].

- Let F_n denote the Fibonacci numbers with $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1}+F_{n-2}$ for all $n \geq 2$. Let s_1, s_2, s_3 be distinct positive integers. It follows from [7, Theorem 1.1] that the three numbers

$$\zeta_{Fib}(2s_i) := \sum_{n=1}^{\infty} \frac{1}{F_n^{2s_i}} \qquad (i = 1, 2, 3) \tag{76}$$

are algebraically independent over $\mathbb{Q}$ if and only if at least one of s_1, s_2, s_3 is even.
- In the PhD thesis [15] the previous result on the series given in (76) is generalized to reciprocal sums of sequences of integers satisfying a linear three-term recurrence formula, see [15, Theorem 5.3].

The following example is an application of Theorem 4. We investigate the values of the two series

$$y_1 := \zeta_{Fib}(4) = \sum_{n=1}^{\infty} \frac{1}{F_n^4} = 2.076730850\ldots ,$$

$$y_2 := \zeta_{Fib}(8) = \sum_{n=1}^{\infty} \frac{1}{F_n^8} = 2.004061286\ldots .$$

One can express y_1 and y_2 each by three parameters which are associated with the complete elliptic integrals of the first and second kind,

$$K(k) := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad E(k) := \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt ,$$

and with the underlying modulus k. This modulus is given by the uniquely determined real number k in the interval $[0, 1]$ such that

$$-\frac{2}{\pi} \log\Big(\frac{\sqrt{5}-1}{2} \Big) = \frac{K\big(\sqrt{1-k^2}\big)}{K(k)} .$$

Moreover, we introduce three parameters x_1, x_2, x_3 by

$$x_1 := \frac{2K(k)}{\pi} = 3,264710703\ldots ,$$

$$x_2 := \frac{2E(k)}{\pi} = 0,637448893\ldots ,$$

$$x_3 := k = 0,999718575\ldots .$$

It is known that the three parameters x_1, x_2, x_3 are algebraically independent over $\mathbb{Q}$. Furthermore, there are two explicitly given polynomials $f_i(X_1, X_2, X_3, Y_i) \in \mathbb{Q}[X_1, X_2, X_3, Y_i]$ such that $f_i(x_1, x_2, x_3, y_i) = 0$ for $i = 1, 2$. We apply Theorem 4 with $n = 3$, $m = 2$ and we obtain that $\zeta_{Fib}(4)$ and $\zeta_{Fib}(8)$ are algebraically independent over the field $\mathbb{Q}(k)$.

Appendix B. An overview on the algorithm from Section 4.2

Input:

1. integers $\rho \geqslant 0$, $w \geqslant 2$, and t with $1 \leqslant t \leqslant w$
2. polynomials $P_0(X_1, X_2), \ldots, P_\rho(X_1, X_2)$ and $T_1(X_1, X_2), \ldots, T_\omega(X_1, X_2)$ from $\mathbb{Q}[X_1, X_2]$, taking positive values for positive numbers X_1, X_2
3. parameters α, β.

Question:

Let $n \geqslant 0$ be an arbitrary integer and let $\alpha, \beta > 0$ be any pair of algebraically independent numbers. Are the two numbers $y_1 = \xi$ given by (26) and the convergent of ξ given by

$$y_2 = \frac{p_{wn+\rho+t}}{q_{wn+\rho+t}},$$

algebraically independent?

Step 1. Compute the convergents $p_{\rho-1}/q_{\rho-1}$, p_ρ/q_ρ, $p_{\rho+t}/q_{\rho+t}$ and $p_{w+\rho+t}/q_{w+\rho+t}$ of ξ (given by (26)) and the convergents p^*_{w-2}/q^*_{w-2} and p^*_{w-1}/q^*_{w-1} of η (given by (27)).

Comment: All the numerators and denominators appear as polynomials in α and β.

Step 2. Substitute numerators and denominators from Step 1 into (31).

Replace α by X_1, β by X_2, and ξ by Y_1. This gives the polynomial $P_1(X_1, X_2, Y_1)$.

Step 3. Compute the determinants $D_w(M - w)$ and $D_{w-2}(M + 1)$ using the formula for $D_\ell(a)$ with

$$T(a) := T_{1+(a-\rho-1 \bmod w)}(\alpha, \beta) \quad \text{and} \quad M := (\rho + t + 2) \bmod w\,.$$

Comment: $D_w(M - w)$ and $D_{w-2}(M + 1)$ are polynomials in α, β.

Step 4. Set

$$P(\alpha, \beta) := (-1)^w \big(D_w(M - w) + D_{w-2}(M + 1)\big)\,.$$

Step 5. Use the expression $P(\alpha, \beta)$ from Step 4 to compute U_1 and U_2 given by the first two formulas in (34).

Step 6. Substitute the expressions for U_1 and U_2 obtained in Step 5 as well as numerators and denominators obtained in Step 1 into the two formulas in (36) and (37):

$$p_{wn+\rho+t} = p_{\rho+t} \sum_{\substack{0 \leqslant k \leqslant n \\ k \equiv 0 \pmod 2}} \binom{n}{k} U_1^{n-k} U_2^{k/2} + (p_{w+\rho+t} - U_1 p_{\rho+t}) \sum_{\substack{0 \leqslant k \leqslant n \\ k \equiv 1 \pmod 2}} \binom{n}{k} U_1^{n-k} U_2^{(k-1)/2},$$

$$q_{wn+\rho+t} = q_{\rho+t} \sum_{\substack{0 \leqslant k \leqslant n \\ k \equiv 0 \pmod 2}} \binom{n}{k} U_1^{n-k} U_2^{k/2} + (q_{w+\rho+t} - U_1 q_{\rho+t}) \sum_{\substack{0 \leqslant k \leqslant n \\ k \equiv 1 \pmod 2}} \binom{n}{k} U_1^{n-k} U_2^{(k-1)/2}.$$

Comment: After the substitution process, the formulas for $p_{wn+\rho+t}$ and $q_{wn+\rho+t}$ are again polynomials in α, β, depending on an integer $n \geqslant 0$.

Step 7. Substitute the expressions for $p_{wn+\rho+t}$ and $q_{wn+\rho+t}$, respectively, obtained in Step 6 into the formula

$$y_2 q_{wn+\rho+t} - p_{wn+\rho+t}\,.$$

Replace α by X_1, β by X_2, and y_2 by Y_2. This gives the polynomial $P_2(X_1, X_2, Y_2)$.

Step 8. Compute the polynomial $D(X_1, X_2, Y_1, Y_2)$ defined by the determinant in (39) and by partial derivatives of the polynomials $P_1(X_1, X_2, Y_1)$ and $P_2(X_1, X_2, Y_2)$ with respect to X_1 and X_2.

Step 9. Find real numbers α_1, α_2 such that the number given by

$$\mathrm{Res}_{Y_2}\left(\mathrm{Res}_{Y_1}\left(D(\alpha_1, \alpha_2, Y_1, Y_2),\ P_1(\alpha_1, \alpha_2, Y_1)\right),\ P_2(\alpha_1, \alpha_2, Y_2)\right)$$

does not vanish.
Comment: It may happen that one needs different pairs $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ depending on residue classes to which n belongs (we distinguished in the above example whether n is even or odd).

Step 10. If such numbers α_1, α_2 exist satisfying the condition in Step 9, the algorithm terminates and gives a positive answer to the question from the beginning of the algorithm.
If such numbers α_1, α_2 cannot be found, the question remains open. $\Diamond$

For the proof of the algebraic independence of ξ and any number from the set $\left\{p_k/q_k : k = 0, 1 \ldots, \rho\right\}$, it suffices to apply Theorem 1 on the polynomial $P_1(X_1, X_2, Y_1)$ obtained in Step 2 and on the polynomial $P_2(X_1, X_2, Y_2)$ obtained from the term $y_2 q_k - p_k$, when p_k and q_k are expressed explicitly as polynomials in α and β. Again, α, β and y_2 must be replaced as in Step 7.

Data availability

No data was used for the research described in the article.

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