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Ranks of tensors: geometry and applications

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Abstract
In this article, we briefly survey some of the recent results in the geometry of tensors with a focus on bounds for various important notions of ranks. This is written in honor of Edoardo Ballico, who made important contributions to this field, on the occasion of his 70th birthday.

Keywords Tensors · Secant varieties of Segre–Veronese varieties · X-ranks · Strength · Slice rank · Multiplicativity · Asymptotic ranks

Mathematics Subject Classification 14N07 · 15A69 · 14N05

1 Introduction

Tensors are mathematical objects which are ubiquitous in the sciences. Since they are multidimensional matrices, they naturally appear in several areas of applied mathematics such as algebraic statistics [24, 111], computational complexity [88, 117], data compression [84], machine deep learning [83], quantum information theory [15, 48, 66] and signal processing [73, 106].

Tensors are naturally elements of (the projectivization of) tensor products of vector spaces. This standpoint allows to look at them by means of algebraic geometry. Given any projective variety $X \subset \mathbb{P}^n$, one defines the geometric notion of X-*rank* for any point $p \in \mathbb{P}^n$: that is the smallest cardinality of a set of points on X whose linear span contains p. For example, if X is the set of rank-one matrices, i.e., the two-factor *Segre variety*, then X-rank coincides exactly with the usual matrix rank. Analogously, one can define the rank with respect to general Segre varieties and obtain the *tensor rank*. Quoting Ballico's words: *"in the applications one mainly needs the cases in which X is either a Veronese embedding of a projective space or a Segre embedding of a multiprojective space. We feel that the general case gives a treasure of new projective geometry"* [11]. In this survey, we will meet several types of ranks of

In honor of Edoardo Ballico, on the occasion of his 70th birthday, for his inspiring influence and guidance

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tensors defined with respect to different algebraic varieties. Besides *tensor rank*, we will see, for example, *slice rank* and *strength* which have interesting connections with areas of combinatorics and commutative algebra.

Once X-rank is defined for any algebraic variety X, it is natural to study geometric properties of the set of tensors sharing the same rank. In the case of matrices, it is well-known that the Zariski closure of the set of rank-r matrices is equal to the algebraic variety of matrices with rank at most r which is defined by the vanishing of $(r+1)$-minors. The case of higher-order tensors is much more challenging and interesting. There are examples of one-parameter families of rank-r higher-order tensors whose limit is of rank strictly larger than r. The Zariski closure of the set of points having X-rank at most r is called *r-th secant variety* of X which, as already quoted, *"gives a treasure of new projective geometry"*. For example, this makes the problem of finding a stratification of secant varieties with respect to the ranks a very intriguing and challenging problem. Also, as a consequence of the failure of semicontinuity of X-ranks, it was introduced the concept of *border rank* which nowadays plays a pivotal role in the field. Indeed, for example, one of the guiding problems of the tensor community is the computation of tensor rank and border rank of the matrix multiplication tensor: these are, up to a constant, equal to the asymptotic number of arithmetic operations (digits multiplications) needed to optimally compute the product of two matrices; see [117] or the extensive discussions in [35, 88].

Structure of the paper. The geometry behind tensor ranks have inspired the introduction of X-ranks, where X is any projective variety over an arbitrary field. The case of tensors is recovered when X is the Segre variety and the case of symmetric tensors (or homogeneous polynomials) is when X is the Veronese variety. We discuss results in the direction of X-rank in Sect. 2. We naturally delve into more detail when X is a Segre–Veronese variety or the Grassmannian and analyse what is known about their secants, including very recent results and still wide open conjectures.

In Sect. 3, we discuss two important instances of X-ranks: strength and slice ranks. The first one is connected to the work of Ananyan and Hochster and the rich theory of polynomial functors. In the setting of homogeneous polynomials, slice rank is a more restrictive concept than strength with an important geometric interpretation: it is the codimension of the largest linear subspace contained in the zero locus of the polynomial. Slice rank was first introduced for tensors by Sawin and Tao: their aim was to recast in its language deep combinatorial results. We highlight both of these connections.

Finally, in Sect. 4, we present results on multiplicativity properties of border tensor ranks, on values of (border) subranks and their asymptotic relatives. Multiplicativity and asymptotic ranks are motivated by quantum information theory, computational complexity and again by combinatorics.

We strongly believe that this is a story of profound connections among scientific fields, driven by tensors. We envision much more progress in the development of these links in the future.

2 *X*-ranks and secant varieties

Let $\mathbb{K}$ be an algebraically closed field. Let $X \subset \mathbb{P}^n$ be an irreducible algebraic variety.

2.1 Definitions

First we recall the basic definitions of the geometric perspective on tensors and tensor decompositions. As already mentioned, these are X-rank and secant varieties.

Definition 2.1 (X*-rank*) Given $p \in \mathbb{P}^n$, the X*-rank* of p is the minimum integer r such that there exists a set of points of X of cardinality r and not smaller whose linear span contains the point p, i.e.,

$$\mathrm{rk}_X(p) := \min\{r \ : \ \exists x_1, \dots, x_r \in X \text{ s.t. } p \in \langle x_1, \dots, x_r \rangle\}.$$

Definition 2.2 (r*-th secant variety*) The r*-th secant variety* of X is the Zariski closure of the set of points of X-rank at most r, i.e.,

$$\sigma_r(X) := \overline{\{p \ : \ \mathrm{rk}_X(p) \leq r\}}.$$

In other words, this is the Zariski closure of the union of all secant linear spaces spanned by r points of X.

Example 2.3 In the tensor space $\mathbb{P}(V_1 \otimes \cdots \otimes V_d)$, where the V_i's are finite dimensional $\mathbb{K}$-vector spaces, the *Segre variety* is the variety of decomposable (or rank-one) tensors, i.e., $X = \mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_d) = \{v_1 \otimes \cdots \otimes v_d \ : \ v_i \in V_i\}$. The X-rank of any point $[T] \in \mathbb{P}(V_1 \otimes \cdots \otimes V_d)$ corresponds to *tensor rank* [80], i.e., the smallest length of a decomposition of the tensor T as a linear combination of rank-one tensors. The case $d = 2$ corresponds to the case of matrices. Similarly, if we consider the *Veronese variety* $X = \mathrm{Ver}_d(\mathbb{P}V) \subset \mathbb{P}(\mathrm{Sym}^d V)$, i.e., the variety of decomposable symmetric tensors or, equivalently, the variety of powers of linear forms in the space of degree-d polynomials, then the X-rank of any symmetric tensor $[T] \in \mathbb{P}(\mathrm{Sym}^d V)$ corresponds to the classical *Waring rank* whose definition dates back to Clebsch, Sylvester and others [60, 112], see also [82]. In this case, a Waring decomposition of a degree-d homogeneous polynomial $[T]$ as a sum of degree-d powers of linear forms corresponds to a set of points on the Veronese variety whose linear span contains the point $[T]$.

We recall later other instances of X-rank when dealing with tensors and homogeneous polynomials.

2.2 Generic ranks and defectiveness of projective varieties

Computing ranks of given points is a very difficult challenge. A first approach is to relax the question by looking at generic elements. This turns out to be a geometric problem related to the computation of dimensions of secant varieties.

Remark 2.4 Given any projective variety $X \subset \mathbb{P}^n$, we have that:

(1) if X is linear, then $\sigma_k(X) = X$ for any k;
(2) if $\sigma_k(X) = \sigma_{k+1}(X)$ then $\sigma_k(X)$ is a linear space.

From this, we may observe that, if $X \subset \mathbb{P}^n$ is nondegenerate, i.e., it is not contained in any proper linear subspace, then the chain of inclusions of secant varieties eventually fills the ambient space, namely $X \subsetneq \sigma_2(X) \subsetneq \cdots \subsetneq \sigma_g(X) = \mathbb{P}^n$.

This leads to the following definition.

Definition 2.5 (*Generic X-rank*) Let $X \subset \mathbb{P}^n$ be a nondegenerate projective variety. The *generic X-rank* is the minimum integer r such that $\sigma_r(X) = \mathbb{P}^n$. It is denoted by rk_X°.

Question 1 What is the generic X-rank?

The classical approach to the study of generic X-ranks is through the computation of dimensions of secant varieties. One might expect to determine dimensions by a simple parameter count.

Definition 2.6 Let $X \subset \mathbb{P}^n$. The *expected dimension* of the r-th secant variety is given by

$$\exp.\dim \sigma_r(X) = \min\{n, r\dim(X) + r - 1\}.$$

From this, the *expected generic X-rank* is

$$\exp.\mathrm{rk}_X^\circ = \left\lceil \frac{n+1}{\dim(X)+1} \right\rceil.$$

Remark 2.7 The expected dimension is an upper bound for the actual dimension but several examples are known in which the inequality is strict. For example, $\sigma_2(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)) \subset \mathbb{P}^8$ is expected to fill the ambient space, but it is the hypersurface defined by the vanishing determinant since it corresponds the variety of (3×3)-matrices of rank at most 2. In general, two-factor Segre varieties have always dimensions strictly smaller than the expected, see [87, Exercise 5.3.2-(6)].

Definition 2.8 (*Defective varieties*) A variety $X \subset \mathbb{P}^n$ is said to be *r-defective* if $\dim \sigma_r(X) < \exp.\dim \sigma_r(X)$. In general, we say that X is *defective* if it is r-defective for some r.

A very classical problem in algebraic geometry since the XIX century is the following.

Question 2 Provide a classification of defective varieties.

The approach of Question 2 followed two paths. On one side, the classification of low-dimensional defective varieties: curves are never defective [101]; a classification of defective surfaces [100, 115] and three-folds [104] was established; the classification of four-folds was recently completed in the preprint [47], completing the work started in [105]. On the other hand, as already mentioned, a full answer to Question 2 implies a full answer to Question 1. Because of these venerable roots, the classification of defective varieties of decomposable tensors such as symmetric tensors (Veronese varieties), arbitrary tensors (Segre varieties), partially-symmetric tensors (Segre–Veronese varieties) or skew-symmetric tensors (Grassmannians) is of great interest.

We try to give a picture the current state-of-the-art about Question 2 in the case of varieties of decomposable tensors.

Veronese varieties

A list of examples of defective Veronese varieties were known since the end of XIX century, see [39]. However, we had to wait almost 100 years to have a complete classification establishing that such examples were the only ones.

Theorem 2.9 (Alexander–Hirschowitz [6]) *The Veronese variety* $\mathrm{Ver}_d(\mathbb{P}^n)$ *is r-defective if and only if:*

- $d = 2$ *and* $2 \leq r \leq n$*;*
- $n = 2, d = 4, r = 5$*;*
- $n = 3, d = 4, r = 9$*;*
- $n = 4, (d, r) \in \{(3, 7), (4, 14)\}$.

The usual approach to the classification problem is by multiple induction on dimensions and degrees. Base cases are therefore crucial. For example, Alexander and Hirschowitz breakthrough was to introduce a powerful method, called *Horace differential method*, which allowed them to deal with the base case of cubics ($d = 3$) [6]. Such result gave a new impulse to the whole area and, since then, an extensive literature has been devoted to Question 2 for other varieties of decomposable tensors. It is worth recalling that the study of dimensions of secant varieties of polarized varieties, such as Veronese, Segre and Segre–Veronese varieties, is equivalent to interpolation problems regarding dimensions of linear systems of hypersurfaces with prescribed singularities, for more on this see [41].

Segre and Segre–Veronese varieties

Recall the definition of *Segre–Veronese variety*.

Definition 2.10 Let $\mathbf{d} = (d_1, \ldots, d_k), \mathbf{n} = (n_1, \ldots, n_k)$ be vectors of positive integers. Then, the *Segre–Veronese variety* $\mathrm{Seg}_{\mathbf{n}}^{\mathbf{d}}$ is the image of the embedding

$$\mathbb{P}(\mathbb{K}^{n_1+1}) \times \cdots \times \mathbb{P}(\mathbb{K}^{n_k+1}) \to \mathbb{P}(\mathrm{Sym}^{d_1}\mathbb{K}^{n_1+1} \otimes \cdots \otimes \mathrm{Sym}^{d_k}\mathbb{K}^{n_k+1})$$
$$([v_1], \ldots, [v_k]) \mapsto [v_1^{\otimes d_1} \otimes \cdots \otimes v_k^{\otimes d_k}].$$

The case $k = 1$ is the *Veronese variety* while the case $d_i = 1$, for all i, is the *Segre variety*.

Denote by $\mathbf{1}^k = (1, \ldots, 1)$ the multiindex of all 1's.

As already mentioned, the classical approach to Question 2 is by multiple induction on dimensions and degrees. Therefore, it is crucial to deal with the cases with low-dimensions and low-degrees which are, usually, also the most difficult as they include many defective cases.

The classification of Segre–Veronese varieties of products of lines $\mathrm{Seg}_{\mathbf{1}^k}^{\mathbf{d}}$ is completely known. Catalisano, Geramita and Gimigliano proved that the Segre varieties of products of $\mathbb{P}^1$'s are defective only for $k = 4$ [52]. Laface and Postinghel completed the classification to any degree [93].

Theorem 2.11 (Catalisano–Geramita–Gimigliano [52]; Laface-Postinghel [93]) *The Segre–Veronese variety* $\mathrm{Seg}_{\mathbf{1^k}}^{\mathbf{d}}$ *is* r*-defective if and only if:*

- *for* $\mathbf{d} = \mathbf{1}^k, k = 4, r = 3$*;*
- *for* $\mathbf{d} = (2, 2a), r = 2a + 1$*;*
- *for* $\mathbf{d} = (1, 1, 2a), r = 2a + 1$*;*
- *for* $\mathbf{d} = (2, 2, 2), r = 7$.

Another extremal case is given by two-factor Segre–Veronese varieties ($k = 2$). The case of two factors Segre varieties is known to be always defective as recalled in Remark 2.7.

In [2], the authors worked out the details of the inductive step of a possible proof of non-defectiveness. As already mentioned, the base cases are usually the more difficult ones since they present defective cases and therefore need different methods. In the case of two-factor Segre–Veronese varieties, a list of defective cases is known in all bidegrees $\mathbf{d} \in$

$\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 2a)\}$, see [2, Table 1]. In [75], it was proved that for bidegrees $\mathbf{d} \in \{(3, 3), (3, 4), (4, 4)\}$ there are no defective cases. This offered the necessary base cases for the inductive proof of [2]. Summarizing, we got the following.

Theorem 2.12 (Abo-Brambilla [2] and Galuppi-Oneto [75]) *The Segre–Veronese varieties* $\mathrm{Seg}^{(d,e)}_{(m,n)}$ *are never defective if* $d \geq 3$ *and* $e \geq 3$.

It is worth mentioning that the case of bidegrees $(1, d)$ is particularly interesting since it is strongly related to the concept of *simultaneous rank*.

Definition 2.13 (*Simultaneous rank*) Let $X \subset \mathbb{P}^n$ be nondegenerate and let $\mathcal{F} = \{p_0, \dots, p_m\} \in \mathbb{P}^n$. The *simultaneous* X*-rank* of $\mathcal{F}$ is the minimum cardinality of points on X containing all of the p_i's, i.e.,

$$\mathrm{rk}_X(\mathcal{F}) = \min\{r \ : \ \exists x_1, \dots, x_r \in X \text{ s.t. } p_i \in \langle x_1, \dots, x_r\rangle \ \forall i \in \{0, \dots, m\}\}.$$

In the case of Veronese varieties, this was already studied by Terracini [114]. For a more recent literature, we refer to [5, 45, 65, 72].

It is known that if $X \subset \mathbb{P}(\mathbb{K}^{n+1})$ and $Y = \mathbb{P}(\mathbb{K}^{m+1}) \times X \subset \mathbb{P}(\mathbb{K}^{m+1} \otimes \mathbb{K}^{n+1})$ is the Segre product, the Y-rank of $q = \sum_{i=0}^m x_i \otimes p_i$, where $\{x_0, \dots, x_m\}$ are linearly independent, is equal to the simultaneous X-rank of $\{p_0, \dots, p_m\}$, see [37, Theorem 2.5] or [76, Lemma 2.4]. In particular, the generic Y-rank corresponds to the simultaneous X-rank of set of generic points.

Therefore, the simultaneous rank of symmetric matrices, i.e., the simultaneous rank of quadratic forms, corresponds to the case $\mathrm{Seg}^{(1,2)}_{(m,n)}$. In this case, a list of defective cases are known:

(1) for $(m, n) = (2, 2a + 1)$, $\mathrm{Seg}^{(1,2)}_{(m,n)}$ is r-defective for $r = 3a + 2$ [45];
(2) for $(m, n) = (4, 3)$, $\mathrm{Seg}^{(1,2)}_{(m,n)}$ is 6-defective [99].

The proof of the non-defectiveness for $(m, n) = (1, n)$ is given in [45]. The fact that the defective case (1) is the only one for $(m, n) = (1, n)$ is given in [1]. The cases $m = n$ and $m = n - 1$ are completely classified in [4]. Additionally, in [1, Corollary 1.2] it is given a nearly optimal result of nondefectivity of $\mathrm{Seg}^{(1,2)}_{(m,n)}$ for n sufficiently large. For other bidegrees, we refer to [2, Table 1] for a list of known defective cases and to [2, Table 2] for partial nondefectivity results.

For higher dimensional Segre varieties other partial results are known. For example, $\mathrm{Seg}_{(2,2,2)}$ is r-defective only for $r = 4$ [50] while $\mathrm{Seg}_{(n,n,n)}$ is never defective for $n \geq 3$; for arbitrary many factors a partial nondefectivity results for $\mathrm{Seg}_{\mathbf{n}}$ were provided in [91]. Other defective cases of Segre varieties are known: $(\mathbf{n}, r) \in \{((2, 3, 3), 5), ((2, 2a, 2a), 3a + 1), ((1, 1, n, n), 2n + 1)\}$ [9, 50] and the so-called *unbalanced case* $n_k > \prod_{i=1}^{k-1}(n_i + 1) - \prod_{i=1}^{k-1} n_i$ [50]. In [9], the authors gave a complete classification of Segre varieties being r-defective for $r \leq 6$.

For Segre–Veronese varieties with more than two factors, the current state-of-the-art result was provided by Ballico in [13].

Theorem 2.14 (Ballico [13]) *Let* $k \geq 3$. *Fix integers* $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ *and* $\mathbf{d} = (d_1, d_2, \dots, d_k) \in \mathbb{N}^k$ *such that* $d_1 \geq 3$, $d_2 \geq 3$ *and* $d_i \geq 2$ *for all* $i \in \{3, \dots, k\}$.

In [12], Ballico proved a general result of nondefectivity of $X \times \mathbb{P}^1$ embedded by $\mathcal{O}(1, 2)$ by assuming nondefectivity of X. An analogous result for $X \times \mathbb{P}^2$ embedded by $\mathcal{O}(1, 2)$ was obtained in [18]. In this way, a version of Theorem 2.14 for $n_i \in \{1, 2\}$ for $i = 3, \dots, k$ was obtained from Theorem 2.12. These results for factors in low-dimensions together with Theorem 2.12 for low number of factors served as bases cases for an inductive proof of Theorem 2.14 when $d_i \geq 3$ [3]. At the same time, Ballico managed to allow smaller degrees obtaining Theorem 2.14 [13].

A crucial tool in Ballico's approach was a generalization of a general result due to Ådlandsvik [119, Corollary 2.2] in a form slightly stronger than the one that was recently proved in [26]. Ådlandsvik proved that if X is r-defective then $\sigma_{r+\dim(X)-1}(X)$ is a cone; i.e., if X defective and none of its secant varieties are cones, then r-defectiveness implies that $\mathrm{rk}^\circ_X \leq r + \dim(X) - 1$. In the recent pre-print [26], a similar result was obtained for projective varieties closed under group actions.

Theorem 2.15 (Blomenhofer–Casarotti [26]) *Let G be a group and X be an irreducible affine cone in an irreducible G-module $\mathcal{L}$. If X is closed under the G-action then X is not r-defective for all $r \leq \frac{\dim(\mathcal{L})}{\dim(X)} - \dim(X)$. Moreover,*

$$\frac{\dim(\mathcal{L})}{\dim(X)} \leq \mathrm{rk}^\circ_X \leq \frac{\dim(\mathcal{L})}{\dim(X)} + \dim(X).$$

Note that all varieties of tensors that we have been considering satisfy the assumptions of Theorem 2.15. For example, if we consider $X = \mathrm{Seg}^{\mathbf{d}}_{\mathbf{n}} \subset \mathbb{P}^N$ where $N = \prod_{i=1}^k \binom{n_i+d_i}{d_i} - 1$, then

$$\frac{N+1}{n_1 + \ldots + n_k + 1} \leq \mathrm{rk}^\circ_X \leq \frac{N+1}{n_1 + \ldots + n_k + 1} + n_1 + \cdots + n_k + 1.$$

Grassmannians

We consider the Grassmannian $\mathrm{Gr}(k, n)$ of k-dimensional projective linear spaces in $\mathbb{P}^n$. The $k = 1$ case, i.e., Grassmannian of lines, is always defective until it fills the ambient space and $\mathrm{rk}^\circ_{\mathrm{Gr}(1,n)} = \left\lfloor \frac{n+1}{2} \right\rfloor$ [118]. For Grassmannians of higher dimensional linear spaces there is a list of r-defective cases which is conjectured to be complete [30, 51].

Conjecture 2.16 The Grassmannian $\mathrm{Gr}(k, n)$ is r-defective if and only if

$$(k, n, r) \in \{(2, 6, 3), (3, 7, 3), (3, 7, 4), (2, 8, 4)\}.$$

The conjecture has been supported by computational approach in the case $n \leq 15$ [30, 96] or for $r \leq 12$ [40]. As mentioned, Theorem 2.15 implies that Conjecture 2.16 holds for

$$r \leq \frac{\dim \bigwedge^d \mathbb{K}^{n+1}}{\dim \mathrm{Gr}(k, n) + 1} - \dim \mathrm{Gr}(k, n) + 1$$

2.3 Identifiability

Another very important concept when dealing with tensor decomposition is the one of *identifiability*.

Definition 2.17 Let $X \subset \mathbb{P}^n$. Let $p \in \mathbb{P}^n$ be such that $\mathrm{rk}_X(p) = r$. Then, p is said to be *identifiable* if there a unique set $\{x_1, \dots, x_r\} \subset X$ such that $p \in \langle x_1, \dots, x_r \rangle$.

This notion is particularly useful when dealing with applications. Indeed, in applications we often deal with the inverse problem of recovering a minimal decomposition of a given tensor. If the tensor is identifiable, we know that the problem is well-posed.

A question having roots in the birational geometry of secant varieties is the following.

Question 3 Is the generic point $p \in \sigma_r(X)$ identifiable with respect to $X \subset \mathbb{P}^n$?

Definition 2.18 (*Generic identifiability*) Let $X \subset \mathbb{P}^n$. We say that X is *generically r-identifiable* if the general element of $\sigma_r(X)$ is identifiable. In particular, if $r = \mathrm{rk}^\circ_X$ we say that X is *generically identifiable*.

Geometrically, Question 3 is interpreted through the *abstract secant variety*. This is

$$\mathrm{abs}.\sigma_r(X) = \overline{\{(x_1, \dots, x_r, p) \ : \ p \in \langle x_1, \dots, x_r \rangle\}} \subset X^{(r)} \times \mathbb{P}^n$$

where $X^{(r)}$ denotes r-tuples of points up to symmetry. The r-th abstract secant variety has dimension $r\dim(X) + r - 1$ and it projects to the r-th secant variety $\sigma_r(X)$ in the last factor. A positive answer to Question 3 is equivalent to have that such projection is dominant and generically one-to-one.

In the case of Veronese varieties, a list of generically identifiable cases were classically known since Sylvester and Hilbert. Only recently it was proven that these are the only ones.

Theorem 2.19 (Galuppi–Mella [74]) *The Veronese variety* Ver^n_d *is generically r-identifiable if and only if* $(n, d, r) \in \{(1, 2a - 1, a), (3, 3, 5), (2, 5, 7)\}$.

In [42], Bronowski conjectured that generic identifiability of $X \subset \mathbb{P}^n$, where $n = r(\dim(X) + 1) - 1$, is related to the birationality of the linear projection of X from the $(r - 1)$-th *tangential projection*, i.e., the projection from the linear span of $r - 1$ generic tangent spaces of X. It is known that generic identifiability implies the birationality of the $(r - 1)$-th tangential projection [64]. The conjecture is known to be true for example for curves [59] and smooth surfaces [64]. A counterexample has been recently presented in the pre-print [97, Theorem 1.3].

In the same paper [97], the authors established also a criterion for sub-generic identifiability extending the main result of [62]: it was proven that, under mild assumptions, non r-defectiveness implies generic $(r - 1)$-identifiability, see [62, Theorem 1.5]. In view of the extensive literature to prove non-defectiveness of varieties of decomposable tensors, as presented in Sect. 2.2, [62, Theorem 1.5] allows us to deduce many identifiability results.

Another interesting point of view studies the so-called *Terracini loci*, i.e., the locus of $(x_1, \dots, x_r, p) \in \mathrm{abs}.\sigma_r(X) \subset X^{(r)} \times \mathbb{P}^n$ where the differential of the projection on the factor drops rank. Once generic identifiability is established, the study of Terracini loci allows to understand the points that should be avoided as the inverse problem of retrieving a minimal decomposition is not well-defined. Terracini loci of projective varieties were defined by Ballico and Chiantini in [25]. Recent results in this direction can be found in [22, 27, 78], but we refer to the chapter [85] for a more extensive description of this question.

2.4 Maximum ranks

In the Sect. 2.2, we approached Question 1 on the generic X-rank. Another interesting direction deals with *maximum rank*.

Definition 2.20 (*Maximum X-rank*) Let $X \subset \mathbb{P}^n$. The *maximum X-rank* is the minimum integer r such that $\mathrm{rk}_X(p) \le r$ for every $p \in \mathbb{P}^n$. It is denoted by $\mathrm{rk}^{\max}_X$.

Question 4 What is the maximum X-rank?

As recalled in Example 2.3, in the case of X being the Segre product of two projective spaces, i.e., the variety of rank-one matrices, the X-rank corresponds with the classical matrix rank. In this case, it is well-known that the generic rank is equal to the maximum rank. This is far from being true in the case of tensors. Indeed, X-ranks are generically not lower-semicontinuous which means that, even if $\sigma_{\mathrm{rk}_X^\circ}(X) = \mathbb{P}^n$ we cannot conclude that *all* tensors have rank at most the generic one.

Example 2.21 The monomial $m = x^2y$ has Waring rank 3, i.e., $\mathrm{rk}_X(m) = 3$ where $X = \mathrm{Ver}_3(\mathbb{P}^1) \subset \mathbb{P}^3$ is the rational normal curve, but $m = \lim_{t\to 0} \frac{1}{3t}[x^3 - (x - ty)^3]$. One can easily check that $\dim \sigma_2(X) = 3$. More generally, Palatini showed that curves are never defective. In particular, the generic X-rank is 2 while the maximum rank is actually 3.

Remark 2.22 (Border X-rank) The latter example is the source of deep questions and phenomena in the theory. For example, the membership $p \in \sigma_r(X)$ does *not* imply that $\mathrm{rk}_X(p) \leq r$. For this reason, one introduces the *border X-rank*, i.e., $\underline{rk}_X(p) = \min\{r \ : \ p \in \sigma_r(X)\}$, which is intensively studied in the recent literature. This is especially due to the connection with the complexity of matrix multiplication which goes back to the works of Strassen [108, 109] and Bini [36]: indeed, tensor and border tensor rank of the matrix multiplication tensor are, up to a constant, equal to the asymptotic number of arithmetic operations (digits multiplications) needed to optimally compute the product of two matrices; see [117, §12.3.2] or the extensive discussions in [35, 88]. Establishing membership to secant varieties is a very challenging question and, for this reason, it is worth mentioning the recent breakthrough due to Buczyńska and Buczyński [14] where a method to establish lower bounds on border tensor rank was proposed and that might lead to new results on this problem.

A very general upper bound on the maximum X-rank with respect to algebraic varieties in characteristic zero is given by the following, see e.g. [95, Proposition 5.1].

Theorem 2.23 *Let $X \subset \mathbb{P}^n$ be irreducible and nondegenerate over an algebraically closed field $\mathbb{K}$ of characteristic zero. Then*

$$rk_X^{\max} \leq n + 1 - \dim(X).$$

This inequality fails in positive characteristic.

Example 2.24 Let $\mathrm{char}(\mathbb{K}) = 2$. Let $X \subset \mathbb{P}^2$ be a smooth conic and $p \in \mathbb{P}^2$ its *strange point*: this is such that every line passing through p is tangent to X. Thus $\mathrm{rk}_X(p) = 3$.

However, Ballico proved that, in arbitrary characteristic, the following bound holds, see [10].

Theorem 2.25 (Ballico [10]) *Let $X \subset \mathbb{P}^n$ be irreducible and nondegenerate over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. Then*

$$\mathrm{rk}_X^{\max} \leq n + 2 - \dim(X).$$

Blekherman and Teitler [44] found a clever general inequality between maximum and generic X-ranks which, in general, provide better upper bounds.

Theorem 2.26 (Blekherman–Teitler [44]) *Let $X \subset \mathbb{P}^n$ defined over an algebraically closed field $\mathbb{K}$. Then*

$$r_X^{\max} \leq 2\mathrm{rk}_X^\circ.$$

Proof Let U be a Zariski open set of points of X-rank exactly rk_X°. Let $p \in U$ and let $q \in \mathbb{P}^n$ be arbitrary points. The projective line $\langle p, q\rangle$ intersects U and so intersects it in infinitely many points. Let $p' \in U \cap \langle p, q\rangle$ be one of these points. Hence $q \in \langle p, p'\rangle$ and so $\mathrm{rk}_X(q) \leq \mathrm{rk}(p) + \mathrm{rk}(p') = 2r_{gen}$. Since q is arbitrary, the statement follows. □

It is worth mentioning also the following generalization of Theorem 2.23, see [44].

Theorem 2.27 (Blekherman–Teitler [44]) *Let $X \subset \mathbb{P}^n$ be defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let $c = \mathrm{codim}(\sigma_r(X))$ and s be the maximum rank of points on $\sigma_r(X)$. Then, $r_{max} \leq \max\{s, (c+1)r\}$.*

Proof Let $q \in \mathbb{P}^n$. If $q \in \sigma_r(X)$, then $\mathrm{rk}_X(q) \leq s$. If $q \notin \sigma_r(X)$, then a general projective linear subspace of dimension c passing through q is spanned by its intersection with $\sigma_r(X)$ and is reduced by Bertini's theorem. This shows that q is in the projective span of $c+1$ general points on $\sigma_r(X)$, which each have rank r. □

Remark 2.28 The case $r = s = 1$ in Theorem 2.27 is Theorem 2.23. The case $c = 1$ and $r = r_X^\circ - 1$ is a useful bound whenever the last nontrivial secant variety is a hypersurface.

2.5 High rank loci

In Example 2.21, we have seen an example where the maximum rank is higher than the generic rank. Also in the case of three variables, monomials offer examples of symmetric tensors whose symmetric rank is strictly larger than the general one, see [46]. However, in general, it is extremely difficult to exhibit elements having X-rank strictly larger than the generic rank. This is the case even for the most studied case where X is a variety of decomposable tensors like a Veronese variety or a Segre variety. So, even the following question is very challenging.

Question 5 Is the maximum X-rank strictly larger than the generic X-rank?

Motivated by this question, we recall the *high rank loci* [34].

Definition 2.29 (*Higher rank loci*) Given $X \subset \mathbb{P}^n$, the *r-th higher rank locus* is the Zariski closure of the set of points in $\mathbb{P}^n$ of X-rank equal to r, i.e.,

$$W_r = \overline{\{p \ : \ \mathrm{rk}_X(p) = r\}}.$$

Remark 2.30 For $r \leq \mathrm{rk}_X^\circ$ we have $W_r = \sigma_r(X)$. Instead, for $r > \mathrm{rk}_X^\circ$, we have $W_r \neq \sigma_r(X) = \mathbb{P}^n$.

A general relation between higher rank loci is given by the following result [34]. Here, we denote by $J(X, Y)$ the *join variety*, i.e., the Zariski closure of all lines $\langle x, y\rangle$ where $x \in X$ and $y \in Y$.

Theorem 2.31 (Buczyński–Han–Mella–Teitler [34]) *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $X \subset \mathbb{P}^n$ be nondegenerate. Then, for each r such that $\mathrm{rk}_X^\circ + 1 \leq r \leq \mathrm{rk}_X^{\max}$, one has $J(W_r, X) \subseteq W_{r-1}$. In particular, we have the sequence of strict inclusions*

$$W_{\mathrm{rk}_X^{\max}} \subsetneq W_{\mathrm{rk}_X^{\max}-1} \subsetneq \cdots \subsetneq W_{\mathrm{rk}_X^\circ+1} \subsetneq W_{\mathrm{rk}_X^\circ} = \mathbb{P}^n.$$

It is generally hard to check whether the higher rank loci are nonempty. From the latter result, one might wonder examples where the inclusion $J(W_r, X) \subset W_{r-1}$ is strict. An example is provided by the following [23].

Theorem 2.32 (Ballico–Bernardi–Ventura [23]) *Let $X = \mathrm{Ver}_4(\mathbb{P}^2)$ be the Veronese surface of plane quartics and let W_7 be the maximum X-rank locus. Then one has the strict inclusion $J(W_7, X) \subsetneq W_6 = \sigma_6(X) = \mathbb{P}^{14}$.*

2.6 Typical ranks over the reals

So far, we have discussed mostly secant varieties and X-ranks of algebraic varieties defined over algebraically closed fields. It is of particular interest also to consider the case over real numbers. Indeed, most applications of tensor decompositions are over real numbers or even nonnegative numbers.

Let $X \subset \mathbb{P}^n_{\mathbb{R}}$ be a nondegenerate real variety such that its real points are dense in X.

Definition 2.33 (*Typical rank*) An integer r is a *typical X-rank* if the set of points whose X-rank is r is full-dimensional in the real Euclidean topology, i.e., it contains an open Euclidean ball.

These were introduced by Comon and Ottaviani [63]. In sharp contrast to the case of an algebraically closed field, there can be several typical ranks. These are of interest for applications, because if in the vicinity of a point the rank does not change, then rank is numerically stable under small perturbations in the data. A first observation about typical ranks is that they are at least the generic rank of the complexification of the variety.

Proposition 2.34 (Blekherman–Teitler [44]) *Let $X_{\mathbb{C}}$ be the complexification of X. Then, the lowest typical X-rank is the generic $X_{\mathbb{C}}$-rank.*

Since we have several typical ranks, it is natural to ask how such ranks distribute.

Theorem 2.35 (Bernardi-Blekherman-Ottaviani [19]) *Let $X \subset \mathbb{P}^n_{\mathbb{R}}$ be a nondegenerate real variety. Then any X-rank between the lowest typical rank and the highest typical rank is also typical.*

Since the minimum typical X-rank is given by the the generic X-rank, the difficult challenge is about the maximum typical rank.

Question 6 What is the largest typical X-rank?

Note that a general version of Theorem 2.26 for real varieties holds as well, i.e., also over the real numbers the maximum X-rank is at most twice the X-generic rank [44].

Even in the case of Veronese varieties $X = \mathrm{Ver}_d(\mathbb{P}^n)$ the maximum typical X-rank is not known in general. The case of binary forms ($n = 1$) is solved: the minimum typical rank is equal to $\lceil \frac{d+1}{2} \rceil$, which is the generic rank while the maximum typical rank is equal to d [29, 103].

Already the case of ternary forms is unsolved and even in the case of cubic ternary forms we do not know the maximum typical rank. In [19], it was proved that typical ranks for ternary cubics are between 6 and 8. However, we do not know if the largest does occur.

Question 7 Let $X = \mathrm{Ver}_4(\mathbb{P}^2_{\mathbb{R}}) \subset \mathbb{P}^{14}_{\mathbb{R}}$. The maximum typical X-rank of a real ternary quartic form is at most 8. Do there exist real quartics whose real Waring rank is 8?

The regions of typical ranks are separated by codimension one topological Euclidean boundaries. Their Zariski closures are the real algebraic boundaries, which are typically very complicated algebraic hypersurfaces; see e.g. [94] for the instructive case of real binary forms, whose typical ranks were originally determined by Blekherman [38]. For foundational results over the reals that inspired more later work see [43].

3 Strength and slice rank

We focus now on another type of rank for homogeneous polynomials and tensors.

3.1 Definitions and first remarks

In the space of degree-d homogeneous polynomials, we call *strength* the rank with respect to the variety of reducible forms [53].

Let $\mathbb{K}$ be an arbitrary field and let V be an $(n+1)$-dimensional $\mathbb{K}$-vector space.

Definition 3.1 (*Strength*) Let $F \in \mathrm{Sym}^d V$ be a homogeneous polynomial (or *forms*) of degree $d > 1$. The *strength* of F, denoted $\mathrm{str}_{\mathbb{K}}(F)$, is the minimal integer r such that

$$F = \sum_{i=1}^{r} G_i H_i,$$

where $1 \leq \deg(G_i), \deg(H_i) \leq d-1$ for all i.

The *strength* of a polynomial was introduced in the work of Ananyan and Hochster [7] where they employed it to prove *Stillman's conjecture* regarding the *projective dimension* of ideals in polynomial rings: the projective dimension *does not* depend on the number of variables of the generators but only on their degrees. Thereafter, Erman, Sam and Snowden [70] constructed some subtle limits of polynomial rings that are polynomial rings in infinitely many variables (called *big polynomial rings*), where elements of infinite strength behave like independent variables. As a byproduct, they gave another beautiful proof of Stillman's conjecture.

Another related type of rank is *slice rank*. This is the rank with respect to forms admitting a linear factor. Sometimes it is referred to as *symmetric* slice rank to tell it apart from a different slice rank for arbitrary tensors that we recall later in Sect. 3.4.

Definition 3.2 (*Symmetric slice rank*) The *(symmetric) slice rank* of F, denoted $\mathrm{sl}_{\mathbb{K}}(F)$, is the minimal integer r such that

$$F = \sum_{i=1}^{r} G_i H_i,$$

where $\deg(G_i) = 1$ for all i.

Remark 3.3 Unraveling definitions, one finds the following:

(i) $\mathrm{str}_{\mathbb{K}}(F) \leq \mathrm{sl}_{\mathbb{K}}(F) \leq n+1$.
(ii) $\mathrm{sl}_{\mathbb{K}}(F) \leq r$ if and only if there exists a linear projective subspace $W \subset X_F := \{F = 0\}$ with $\mathrm{codim}_{\mathbb{P}V}(W) \leq r$.

Remark 3.4 Let $F_{n-r}(X_F)$ be the *Fano scheme* whose closed points are the linear projective subspaces of codimension r contained in the hypersurface defined by F. Then,

$$\mathrm{sl}_{\mathbb{K}}(F) = \min\{r \mid F_{n-r}(X_F) \neq \emptyset\}.$$

Example 3.5 (Quadrics) Let $\mathrm{char}(\mathbb{K}) \neq 2$ and let $F \in \mathrm{Sym}^2 V$ be a quadratic form. Then $\mathrm{sl}_{\mathbb{K}}(F) = \mathrm{str}_{\mathbb{K}}(F) = \lceil \mathrm{rk}(F)/2 \rceil$, where $\mathrm{rk}(F)$ is the rank of F as a symmetric bilinear form.

Example 3.6 (Fermat hypersurface) Let $F = x_0^d + \cdots + x_n^d \in \mathrm{Sym}^d V$. Then we have that $\mathrm{str}_{\mathbb{K}}(F) \leq \mathrm{sl}_{\mathbb{K}}(F) \leq \lceil (n+1)/2 \rceil$.

Remark 3.7 Let $\mathrm{Sing}(X_F)$ be the singular locus of X_F. If $\mathrm{codim}_{\mathbb{P}V}(\mathrm{Sing}(X_F)) \geq 2h+1$, then $\mathrm{str}_{\mathbb{K}}(F) \geq h+1$: this quite easy to prove and sometimes goes under the name of *Ananyan-Hochster bound* as it was observed in [7]. When $\mathrm{char}(\mathbb{K})$ does not divide the degree d, this bound implies that $\mathrm{str}_{\mathbb{K}}(F) = \mathrm{sl}_{\mathbb{K}}(F) = \lceil (n+1)/2 \rceil$, where F is the degree-d Fermat polynomial from Example 3.6.

Strength and slice rank may be considered over any field $\mathbb{K}$. Unlike matrix rank, and similarly to tensor rank, strength and slice rank are sensible to field extensions.

Question 8 (Adiprasito–Kazhdan–Ziegler [8]) Let $F \in \mathrm{Sym}^d V$ be any degree d form over a field $\mathbb{K}$. Does there exist a function K_d such that

$$\mathrm{str}_{\mathbb{K}}(F) \leq K_d \cdot \mathrm{str}_{\overline{\mathbb{K}}}(F)?$$

Note that K_d should not depend on F nor on the dimension of V^*, but only on d.

Theorem 3.8 (Lampert [86]) *Question 8 has a positive answer for any number field* $\mathbb{K}$.

3.2 Slice rank and strength are generically equal

We consider now Question 1 in the case of strength and slice rank. Let $\mathbb{K} = \overline{\mathbb{K}}$ with $\mathrm{char}(\mathbb{K}) = 0$. Let V a $\mathbb{K}$-vector space of dimension $n+1$.

Theorem 3.9 *Let $n \geq 1$ and $d \geq 3$. The generic slice rank in $P(\mathrm{Sym}^d V)$ is*

$$\mathrm{sl}^\circ(n,d) = \min \left\{ r \mid r(n+1-r) \geq \binom{d+n-r}{d} \right\}.$$

This is proven showing that $F_{n-r}(X_F) \neq \emptyset$, where X_F is a general degree d hypersurface, see e.g. [79, Theorem 12.8]. In [53, Remark 7.7] it was conjectured that the generic slice rank and the generic strength should coincide. This equality was proved by our joint work with Ballico and Bik [21].

Theorem 3.10 (Ballico–Bik–Oneto–Ventura [21]) *For $n \geq 1$ and $d \geq 2$, generic slice rank and generic strength coincide.*

In order to prove the latter theorem, we actually proved a conjecture suggested in [53, Remark 7.7]. Let $X_{\mathrm{red}} = \bigcup_{j=1}^{\lfloor d/2 \rfloor} X_{(j,d-j)} \subset \mathbb{P}(\mathrm{Sym}^d V^*)$ be the variety of reducible forms where $X_{(j,d-j)}$ is the variety of forms of type GH with $\deg(G) = j$, $\deg(H) = d-j$. The r-th secant variety $\sigma_r(X_{\mathrm{red}})$ has also a lot of irreducible components: these correspond to all possible joins among r components of X_{red}, possibly taken with repetitions. In [53, Remark 7.7] it was suggested that $\sigma_r(X_{(1,d-1)})$ is the largest component of $\sigma_r(X_{\mathrm{red}})$.

Theorem 3.11 (Ballico–Bik–Oneto–Ventura [21]) *For each integer $r \geq 1$*, $\dim \sigma_r(X_{\mathrm{red}}) = \dim \sigma_r(X_{1,\mathrm{d}-1})$.

The latter theorem immediately implies Theorem 3.10.

The ingredients of the proof are very interesting on their own right. We leverage a description of Hilbert schemes of complete intersections and employ combinatorial techniques on power series.

3.3 Strength is not semicontinuous

In Example 2.21, we have noticed that tensor rank is not lower-semicontinuous, making the geometric study of tensor ranks much more difficult and intriguing than the case of matrices. It is natural to ask whether strength and slice rank are also semicontinuous.

Remark 3.12 (Slice rank is lower-semicontinuous) We have described in Remark 3.4 the slice rank of a polynomial in terms of Fano schemes. In particular, we may consider the incidence variety

$$\{(H, F) \;:\; H \subset X_F\} \subset \mathrm{Gr}(n-r, n) \times \mathrm{Sym}^d(V^*).$$

The projection on the first factor gives the Fano scheme of X_F. By Remark 3.3, the projection on the second factor ends up in $\sigma_r(X_{(1,d-1)})$ and, since it is a projection from a a Grassmannian, this tells us that the image of such projection is Zariski closed. This allows us to deduce that $\sigma_r(X_{(1,d-1)})$ is given by all forms having slice rank at most r. See [79, Example 12.5].

Instead, strength is not lower-semicontinuous.

Theorem 3.13 (Ballico–Bik–Oneto–Ventura [20]) *Let $\mathbb{K}$ be an algebraically closed field of any characteristic. There exist forms of degree $d \geq 4$ that are limits of forms of strength ≤ 3 but still have strength 4. In particular, bounded strength is not a Zariski-closed condition.*

Example 3.14 Our construction proving Theorem 3.13 goes as follows. Let $F = x^2w+y^2g+u^2p+v^2q \in \mathrm{Sym}^4(\mathbb{K}^n)$, where x, y, u, v are linear forms and w, g, p, q are quadratic forms. Let

$$H(t) = \frac{1}{t}\left[(x^2+tg)(y^2+tw) - (u^2-tq)(v^2-tp) - (xy+uv)(xy-uv)\right].$$

For $t \neq 0$, we see that $\mathrm{str}_{\mathbb{K}}(H(t)) \leq 3$. On the other hand, it is clear that $\lim_{t\to 0} H(t) = F$ and, therefore, $[F] \in \sigma_r(X_{\mathrm{red}})$. However, we employed the theory of *polynomial functors* developed in [31] to deduce that F has actually strength 4 when the number of variables is sufficiently large.

The asymptotic nature of our construction leaves open the following question.

Question 9 What is the smallest number of variables where the form F constructed in Example 3.14 has strength 4?

Moreover, our result deals with the set of forms of strength at most 3 but leaves open the case of strength-two forms.

Question 10 Is the set of forms with strength at most 2 always closed?

Remark 3.15 Question 10 might actually have a positive answer. It is easy to note that if $X = X_{(j,d-j)}$ then the set of forms having X-rank at most two is Zariski closed. Indeed, it is well-known that the second secant variety is given by the union of the set of rank-two elements, i.e., points on secant lines, and the tangential variety, i.e., points on tangent lines. It is straightforward to observe that the tangent line at $[GH] \in X_{(j,d-j)}$ is given by forms of type $GH' + G'H$ which are also of X-rank. The only components of X_{red} that are left out by this observation are the join varieties $J(X_{(a,d-a)}, X_{(b,d-b)})$ for $a \neq b$. In this case, we should understand the limit of lines $\langle x, y\rangle$, with $x \in X_{(a,d-a)}$, $y \in X_{(b,d-b)}$ when we let x and y collapse together on a point of intersection.

Remark 3.16 Despite being non lower-semicontinuous, the maximum strength is equal to the generic strength. Indeed,

$$\mathrm{str}^\circ(n,d) \le \mathrm{str}^{\max}(n,d) \le \mathrm{sl}^{\max}(n,d) = \mathrm{str}^\circ(n,d)$$

where the inequalities follow by definitions and the last equality follows from Remark 3.12. By Theorem 3.10, this is actually a chain of equalities.

3.4 The tensor slice rank and the cap-set problem

Here and in the next section we recall the *slice rank* for tensors. This appeared in different areas of mathematics.

Definition 3.17 (*Tensor slice rank*) Let $T \in V_1 \otimes \cdots \otimes V_k$. The *(tensor) slice rank* of T is the minimal integer r such that

$$T = \sum_{i_1=1}^{r_1} u_{i_1} \otimes v_{i_1} + \sum_{i_2=r_1+1}^{r_2} u_{i_2} \otimes v_{i_2} + \cdots + \sum_{i_k=r_{k-1}+1}^{r} u_{i_k} \otimes v_{i_k},$$

where $u_{i_\ell} \in V_\ell$ and $v_{i_\ell} \in V_1 \otimes \cdots \otimes \widehat{V_\ell} \otimes \cdots \otimes V_k$ is a $(k-1)$-tensor. This is denoted $\mathrm{sl}_{\mathbb{K}}(T)$.

A tensor T of slice rank-one is of the form $u \otimes T'$, where u is a vector.

Remark 3.18 If $T \in \mathrm{Sym}^d V$ we may consider both symmetric slice rank (Definition 3.2) and tensor slice rank (Definition 3.17). These two definitions are completely different. Consider for example the monomial $m = x^2 y \in \mathrm{Sym}^2\mathbb{K}^2$ which clearly has symmetric slice rank equal to one as it is irreducible. If we interpret it as a general tensor we write $m = \frac{1}{3}(x\otimes x\otimes y + x\otimes y\otimes x + y\otimes x\otimes x) \in \mathbb{K}^2\otimes\mathbb{K}^2\otimes\mathbb{K}^2$ which has tensor slice rank equal to 2 since we can write $m = \frac{1}{3}(x\otimes(x\otimes y + y\otimes x) + y\otimes x\otimes x)$ but we clearly cannot do better.

Slice rank of 3-tensors has been successfully employed in *additive combinatorics*. It was explicitly introduced by Sawin and Tao [107] in order to provide a reformulation of a polynomial type method employed by Ellenberg and Gijswijt [69], which was inspired by the work of Croot, Lev, and Pach [61]. Ellenberg and Gijswijt were able to obtain exponentially small bounds for the *cap set problem* on arithmetic progressions in $\mathbb{F}_3^n$.

A subset $A \subset \mathbb{F}_3^n$ contains an arithmetic progression of length 3 (**3AP**) if there exist $x, y, z \in A$ (not all equal!) such that $x+y+z=0$. Equivalently, if A contains $x, x+a, x+2a$ for some $a \in \mathbb{F}_3^n \setminus \{0\}$. If A contains no such elements, then it is said to be a *cap set*. An important result of Meshulam and Roth showed that if $A \subset \mathbb{F}_3^n$ with $|A| \ge \delta \cdot 3^n$, where $\delta = O(1/n)$, then A contains a **3AP**; see e.g. [77, §8.3].

Thus the *capset problem* is to find the size of the largest possible cap set $A \subset \mathbb{F}_3^n$. Clearly one has $|A| \le 3^n$ and the previously known bounds were still functions of 3^n. Ellenberg–Gijswijt proved the following exponentially better bound. We will sketch the simple and beautiful proof of this result due to Tao [113] (here we recall a slightly weaker upper bound as in [77, §8.3 and Lemma 2.6]). We believe including it here could serve to sparkle further interest in the emerging applications of tensors to combinatorics, which is a vibrant recent trend of research.

Theorem 3.19 (Ellenberg–Gijswijt) *Let $A \subset \mathbb{F}_3^n$ be a cap set. Then*

$$|A| \leq 3 \cdot 3^n \cdot (\exp(-n/36)) = O((2.917)^n).$$

Proof Let $A \subset \mathbb{F}_3^n$ be a cap set. If $x, y, z \in A$ satisfy $x + y + z = 0$, then $x = y = z$. Let A be a finite set and let $g : A \times A \times A \to \mathbb{F}_3$ be a function such that $g(x, y, z) \neq 0$ if and only if $x = y = z$. The function g can be regarded as a tensor in $V_1 \otimes V_2 \otimes V_3$, where V_i is the vector space of functions $A \to \mathbb{F}_3$. Then the slice rank of g is $|A|$. (This is analogous to a diagonal matrix and its rank.)

Let $A \subset \mathbb{F}_3^n$ be a cap set and let $f : A \times A \times A \to \mathbb{F}_3$ be the polynomial function

$$f(x, y, z) = \prod_{i=1}^{n} \left(1 - (x_i + y_i + z_i)^2\right).$$

Note that the right-hand-side is 1 if $x + y + z = 0$, and 0 otherwise. So f agrees with the function $f(x, y, z) = 1$ if and only if $x = y = z$, and 0 otherwise. Regarding f as a tensor in $V_1 \otimes V_2 \otimes V_3$, it has slice rank equal to $|A|$.

Write f as a linear combination of $x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} z_1^{c_1} \cdots z_n^{c_n}$, with $\sum_{i=1}^n a_i + b_i + c_i \leq 2n$ and $0 \leq a_i, b_i, c_i \leq 2$. Partition these monomials into 3 sets. In the first set, we have monomials of degree $\leq 2n/3$ in $x_1, \ldots, x_n$. Similarly for the other two sets.

Let M be the number of sequences in $\{0, 1, 2\}^n$ adding up to at most $2n/3$. This is the same as the number of sequences in $\{-1, 0, 1\}^n$ adding up to at least $n/3$. Each set contributes at most M to the slice rank of f. Thus $|A| \leq 3 \cdot M$.

Let $X_1, \ldots, X_n$ be random independent variables, uniformly distributed in $\{-1, 0, 1\}$. Denote by $\Pr[A]$ the probability of an event A to take place. By a variant of Markov's inequality, one finds that $\Pr[\sum_{i=1}^n X_i \geq n/3] \leq \exp(-n/36)$ and so $M = 3^n \Pr[\sum_{i=1}^n X_i \geq n/3] \leq 3^n \exp(-n/36)$. $\square$

Question 11 As Tao suggested [113], a natural follow-up question is whether similar tensor rank inspired polynomial methods may be effective in the context of length four progressions.

Inspired by this tremendous success, besides slice rank, several other ranks are of interest in additive combinatorics. Among the most interesting ones are *analytic rank* and the *partition rank* with several interesting applications; see the works of Lovett [92] and Naslund [98], along with the references therein.

3.5 The asymptotic tensor slice rank and Shannon's entropy

When looking at the *asymptotic* behaviour of the slice rank of tensor powers, *Shannon's entropy* of a random variable naturally appears. This is one of the fundamental notions in quantum information theory and very much related to Boltzmann's entropy in statistical physics.

Definition 3.20 (*Shannon's entropy*) Given a random variable X taking values in a finite set χ and distributed according to $p : \chi \to [0, 1]$, its *Shannon's entropy* is $H(X) = -\sum_{x \in \chi} p(x) \log_2(p(x))$.

Definition 3.21 (*Kronecker products*) Let $T \in V_1 \otimes \cdots \otimes V_k$ and $T' \in W_1 \otimes \cdots \otimes W_k$ be two order-k tensors. The *Kronecker product* $T \boxtimes T'$ is the tensor product between T and T' when regarded as a tensor in $(V_1 \otimes W_1) \otimes \cdots (V_k \otimes W_k)$. The *d-th Kronecker power* of T is the Kronecker product of T with itself d times, i.e. $T^{\boxtimes d} \in V_1^{\otimes d} \otimes \cdots \otimes V_k^{\otimes d}$.

As $d \to \infty$, Sawin and Tao [107] proved the following estimate on the slice rank of the d-th Kronecker power of a tensor involving a remarkable connection to Shannon's entropy.

Given a basis $\{e_{(i_1,\dots,i_k)} : i_j \in \{0,\dots,n_j\}\}$ of $V_1 \otimes \cdots \otimes V_k$, we call *support* of $T \in V_1 \otimes \cdots \otimes V_k$ the subset $\Gamma \subset S = \prod_{j=1}^k \{0,\dots,n_j\}$ of multi-indices for which T has a nonzero coordinate with respect to the fixed basis.

Theorem 3.22 (Sawin-Tao [107]) *Let $T \in V_1 \otimes \cdots \otimes V_k$ be a tensor with support Γ. Then*

$$\mathrm{sl}_{\mathbb{K}}(T^{\boxtimes d}) \leq 2^{(H_\Gamma + o(1))d}, \ \text{as } d \to \infty,$$

where $H_\Gamma = \sup_{(X_1,\dots,X_k)} \min\{H(X_1),\dots,H(X_k)\}$ and $(X_1,\dots,X_k)$ ranges over the random variables taking values in Γ. Let $\Gamma_{max} \subset \Gamma$ be the set of maximal elements of Γ under the componentwise total ordering $\preceq$ on S, and let $H_{\Gamma_{max}} = \sup_{(X_1,\dots,X_k)} \min\{H(X_1),\dots,H(X_k)\}$, where $(X_1,\dots,X_k)$ ranges over the random variables taking values in Γ_{max}. Then

$$\mathrm{sl}_{\mathbb{K}}(T^{\boxtimes d}) \geq 2^{(H_{\Gamma_{max}} + o(1))d}, \ \text{as } d \to \infty.$$

In particular, when Γ is antichain under $\preceq$ then $\mathrm{sl}_{\mathbb{K}}(T^{\boxtimes d}) = 2^{(H_\Gamma + o(1))d}$ as $d \to \infty$.

Theorem 3.22 is a result about the *asymptotic slice rank* of a tensor T.

Definition 3.23 Let $T \in V_1 \otimes \cdots \otimes V_k$. The *asymptotic tensor slice rank* of T is

$$\underset{\sim}{\mathrm{sl}}(T) = \lim_{d\to\infty} (\mathrm{sl}_{\mathbb{K}}(T^{\boxtimes d}))^{1/d}$$

Remark 3.24 Note that the asymptotic tensor slice rank is well-defined by Fekete's lemma.

The second part of Theorem 3.22 may be rephrased as saying that whenever a tensor T is *oblique* [89], we can determine its asymptotic slice rank. For this class of tensors, this rank coincides with the asymptotic subrank, which we introduce in Sect. 4.5.

Asymptotic ranks are a major theme in recent years in the field of tensors. These results are partly motivated by quantum information and partly by the deep theory of Strassen's spectral points [66]. A further connection is between slice rank and its asymptotic version and *stability* in geometric invariant theory (GIT), a theory of quotients in the framework of algebraic geometry.

Recall that $T \in V_1 \otimes \cdots \otimes V_k$ is an *unstable point* under the action of $G = \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_k)$ whenever the origin 0 belongs to the orbit closure $\overline{G \cdot T}$.

Theorem 3.25 (Bürgisser–Garg–Oliveira–Walter–Wigderson [33]) *Let $T \in (\mathbb{C}^n)^{\otimes k}$ and let $d \geq 1$. Then T is unstable if and only if $T^{\boxtimes d}$ is unstable. Moreover, $\underset{\sim}{\mathrm{sl}}(T) < n$ if and only if $T^{\boxtimes d}$ is unstable for some $d \geq 1$.*

4 Multiplicativity of ranks and asymptotic ranks

4.1 Motivation from quantum information theory

We shall only mention quantum mechanics and quantum information theory *en passant*. For a basic quantum mechanics book one may look at the superb classic [71]. For quantum information theory with a mathematical and tensor inclination we refer to the recent textbook [90].

In quantum mechanics, any isolated physical system (e.g. a single particle not interacting with other particles) is modeled by a complex Hilbert space $\mathcal{H}$ equipped with a hermitian product, called the *state space*. The system is completely described by a single vector $|\psi\rangle \in \mathcal{H}$. A physical state of the system $|\psi\rangle = \sum_i \alpha_i |i\rangle$, where $|i\rangle$ are basis vectors and $\alpha_i \in \mathbb{C}$, has the property that $|\alpha_i|^2$ is the probability of *measuring* that the system is in state $|i\rangle$. Because of this property, scaling a physical state by $e^{i\theta}$ does not give an observable modification of the system. Hence states are regarded up to this multiplication. Moreover, on the space $\mathcal{H}$ one has a *hermitian* (or a *self-adjoint* in the infinite-dimensional case) operator H that describes the *deterministic evolution* of the system: this is the Schödinger equation. Usually H is called the *Hamiltonian* of the system $\mathcal{H}$.

A simple example is when $\mathcal{H} = \mathbb{C}^2$: this is the space representing quantum mechanical situations with two eigenstates of the Hamiltonian. Examples of this are spin states of an electron, say in a magnetic field, or the description of a molecule of ammonia [71, Chapter 6].

From the perspective of quantum information theory, this vector space is an important building block. Let us see why. Let $|0\rangle, |1\rangle \in \mathbb{C}^2$ be the basis vectors. They can be interpreted as representing the classical bits of information, 0 and 1. We move from these two bits to a space of *qubits*. The set of qubits is the set of physical states in $\mathbb{C}^2$ in the given basis, i.e. $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$. Thus the set of qubits may be identified with the sphere $\mathbb{S}^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$.

Recall that every physical state in quantum mechanics is defined up to the multiplication by $e^{i\theta}$. So the space of qubits is identified with $\mathbb{S}^3/\mathbb{S}^1 \cong \mathbb{S}^2$, called the *Bloch sphere*. This topological quotient is the result of the famous *Hopf fibration*. (In algebraic geometry, instead of thinking of the Bloch sphere, one usually look at another topological incarnation, i.e., the projective line $\mathbb{P}^1(\mathbb{C}^2) = \mathbb{P}^1$).

Qubits have an immense importance in science and technology and they are the building blocks of the architecture of a *quantum computer*, which we may simply think of as a circuit where unitary operations are performed on qubits. The power of qubits and quantum information has been proven to be remarkable, for instance leading to efficient probabilistic algorithms like Grover's search algorithm or Shor's factorization algorithm [90, §3.6], that have no known classical counterpart.

In this field, the main concern is to study the nature, the resources, and the obstacles behind communicating qubits. So it is often of interest to study situations where two or more isolated systems interact, e.g. two labs A and B exchange information. Each of them shall have its own Hilbert space, $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The *composite system* of the two will be by definition $\mathcal{H}_A \otimes \mathcal{H}_B$. The tensor product space in quantum information theory is the source of one of the most important phenomena in physics: the *entanglement*. This accounts for several non classical situations we witness in the quantum world. Very loosely speaking, it turns out that a measure of entanglement of a state in a composite system is measured by its tensor rank.

Definition 4.1 A state $|\psi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ is called *entangled* if $\mathrm{rk}(|\psi\rangle) > 1$.

In view of measuring the degree of entanglement encapsulated in a state, it is important to understand multiplicativity properties of the tensor rank under the Kronecker power $|\psi\rangle^{\boxtimes d}$ of a state $|\psi\rangle$.

4.2 *W*-states and submultiplicativity of tensor rank

W-states are a particular class of tensors which are central in quantum information theory.

Definition 4.2 (W-*state*) Let $\{x, y\}$ be a basis of $\mathbb{C}^2$. The W-*state* is, up to scalar, the tensor

$$W_k = y \otimes x \otimes \cdots \otimes x \otimes x \ + \ x \otimes y \otimes \cdots \otimes x \otimes x \ + \ldots + \ x \otimes x \otimes \cdots \otimes x \otimes y \in \mathrm{Sym}^k \mathbb{C}^2.$$

In quantum information literature, it is denoted $W_k = |10\cdots00\rangle + |01\cdots00\rangle + \cdots + |00\cdots01\rangle$, where $x = |0\rangle$, $y = |1\rangle$ and their tensor product is written as the corresponding list of 0's and 1's.

Note that the W-state is symmetric and, up to scalar, it corresponds exactly to the monomial $x^{k-1}y$. We have already seen that these monomials play a crucial role in the geometry of secant varieties: indeed, they are elements on tangent lines to the degree-k rational normal curve and are the easiest examples of tensors having border rank strictly smaller than the rank, see Example 2.21.

On the other hand, they are central also in quantum information. In the case of order-three tensors, i.e., three body systems, we have two entangled classes: one is the one of W-states and the other is the one of *GHZ*-type which corresponds to rank-2 cubic $x^3 + y^3$ [68]. For higher degrees the situation is much more complicated and generalizations of W-states corresponding to arbitrary monomials need to be considered [81].

W-states offer also a first example of failure of multiplicativity of tensor rank.

Example 4.3 (Rank submultiplicativity) It is easy to see that $\mathrm{rk}(W_k) = k$. In [58, Proposition 14], it was proved that $\mathrm{rk}(W_3 \otimes W_3) \leq 8 < 9 = \mathrm{rk}(W_3)^2$. Indeed, for any $c \in \mathbb{C} \smallsetminus \{0\}$ then

$$W_3 + cy^{\otimes 3} = \frac{1}{2\sqrt{c}}\left((x + \sqrt{c}y)^{\otimes 3} - (x - \sqrt{c}y)^{\otimes 3}\right).$$

Then, by expanding $W_3 \otimes W_3$, one obtains

$$W_3^{\otimes 2} = \left(W_3 + y^{\otimes 3}\right)^{\otimes 2} - \left(W_3 + \frac{1}{2}y^{\otimes 3}\right) \otimes y^{\otimes 3} - y^{\otimes 3} \otimes \left(W_3 + \frac{1}{2}y^{\otimes 3}\right).$$

Since the first summand expands in the sum of four rank-one tensors by the previous observation, we deduce that $W_3^{\otimes 2}$ has rank at most 8. It was then proved that indeed $\mathrm{rk}(W_3^{\otimes 2}) = 2$ [49].

This example was extended in the following upper bound on the *partially symmetric rank* of $W_3^{\otimes k}$, i.e., it rank with respect to the Segre–Veronese variety $\mathrm{Seg}^{\mathbf{3}^k}_{\mathbf{1}^k}$.

Theorem 4.4 (Ballico–Bernardi–Christandl–Gesmundo [16]) *For any k, let $X = \mathrm{Seg}^{\mathbf{3}^k}_{\mathbf{1}^k}$. Then,*

$$\mathrm{rk}_X(W_3^{\otimes k}) \leq (2+k)2^{k-1}.$$

An upper bound on partially-symmetric tensor of tensor products of arbitrary W-states is proved.

Theorem 4.5 (Ballico–Bernardi–Christandl–Gesmundo [16]) *For $k \geq 3$ and $d_1, \ldots, d_k \geq 3$, let $X = \mathrm{Seg}^{(d_1,\ldots,d_k)}_{\mathbf{1}^k}$. Then,*

$$\mathrm{rk}_X(W_{d_1} \otimes \cdots \otimes W_{d_k}) \leq 2^{k-1}(d_1 + \cdots + d_k).$$

It is natural therefore to explicitly state the following challenging question.

Question 12 What is the partially-symmetric rank of $W_{d_1} \otimes \cdots \otimes W_{d_k}$ for any $d_1, \ldots, d_k \geq 3$?

In [54, Proposition 3.1], an example of submultiplicativity of border rank was also presented.

Example 4.6 (Border rank submultiplicativity) Let A, B, C 3-dimensional vector spaces. Let $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}, \{c_1, c_2, c_3\}$ basis of A, B, C, respectively. Consider

$$\begin{aligned} T &= a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3 \\ &\quad + \sum_i a_i \otimes \sum_i b_i \otimes \sum_i c_i + 2(a_1 + a_2) \otimes (b_1 + b_3) \otimes (c_2 + c_3) * \in A \otimes B \otimes C. \end{aligned}$$

Then, $\underline{rk}(T) = 5$: the upper bound is obvious, the lower bound is proved in [54, Proposition 3.1] by flattening method.

On the other hand, for $Z = (a_1 + a_2) \otimes (b_1 + b_3) \otimes (c_2 + c_3)$: it is immediate to see that $\underline{rk}(T - 2Z) \leq \mathrm{rk}(T - 2Z) \leq 4$, but it is also possible to show that also $\underline{rk}(T - Z) = \mathrm{rk}(T - Z) = 4$. With this observation, it is possible to conclude that $\underline{rk}(T^{\otimes 2}) \leq 24 < 25 = \underline{rk}(T)^2$ by writing

$$T^{\otimes 2} = (T - 2Z)^{\otimes 2} + (T - Z) \otimes 2Z + 2Z \otimes (T - Z).$$

These examples opened up a systematic study of submultiplicativity of tensor rank and, more in general, of X-ranks. In particular, we formulate the following problem.

Question 13 Let $X_1 \subset \mathbb{P}V_1$, $X_2 \subset \mathbb{P}V_2$ be two algebraic varieties and let $p_1 \in \mathbb{P}V_1$, $p_2 \in \mathbb{P}V_2$. In $\mathbb{P}(V_1 \otimes V_2)$ we look at the Segre product $X_1 \times X_2$ and to the point $p_1 \otimes p_2 \in$. Find conditions which guarantee that $\mathrm{rk}_{X_1 \times X_2}(p_1 \otimes p_2) < \mathrm{rk}_{X_1}(p_1)\mathrm{rk}_{X_2}(p_2)$ or $\underline{rk}_{X_1 \times X_2}(p_1 \otimes p_2) < \underline{rk}_{X_1}(p_1)\underline{rk}_{X_2}(p_2)$.

In [54, Section 4], the general geometric construction behind Example 4.6 is described introducing the *multidrop line* for an algebraic variety $X \subset \mathbb{P}^n$: a special secant line whose existence implies the existence of a point p such that $\underline{rk}_X(p) = k + 1$ and $\underline{rk}_{X^k}(p^{\otimes k}) < (k + 1)^2$.

In our joint work with Ballico, Bernardi and Gesmundo, we investigate geometric conditions guaranteeing the existence of multidrop lines. Moreover, we observe how the existence of *multisecant lines* or *multisecant linear spaces* are key tools for submultiplicativity. Recall that a *multisecant line* $\mathbb{P}(L) \subset \mathbb{P}^n$ to X is a line such that the support of $X \cap \mathbb{P}(L)$ contains at least three distinct points.

For example, we characterize the cases in which the submultiplicativity of ranks as in Question 13 holds under the assumption that $\mathrm{rk}_{X_1}(p_1) = \mathrm{rk}_{X_2}(p_2) = 2$.

Theorem 4.7 (Ballico–Bernardi–Gesmundo–Oneto–Ventura [17]) *For $i = 1, 2$, let $X_i \subseteq \mathbb{P}(V_i)$ be nondegenerate irreducible varieties and let $p_i \in \mathbb{P}(V_i)$ such that $\mathrm{rk}_{X_i}(p_i) = 2$. Then, $3 \leq \mathrm{rk}_{X_1 \times X_2}(p_1 \otimes p_2) \leq 4$. Moreover, for $a_1, a_2, a_3 \in X_1$ and $b_1, b_2, b_3 \in X_2$, the following are equivalent:*

(i) $\mathrm{rk}_{X_1 \times X_2}(p_1 \otimes p_2) = 3$ *with* $p_1 \otimes p_2 \in \langle a_1 \otimes b_1, a_2 \otimes b_2, a_3 \otimes b_3 \rangle$*;*
(ii) *The linear spaces $\mathbb{P}(L_1) = \langle a_1, a_2, a_3 \rangle$ and $\mathbb{P}(L_2) = \langle b_1, b_2, b_3 \rangle$ are multisecant lines to X_1 and X_2, respectively, where the a_i's and the b_i's are all distinct; moreover, if $\varphi \colon \mathbb{P}(L_1) \to \mathbb{P}(L_2)$ is the unique linear map such that $\varphi(a_j) = b_j$, then $\varphi(p_1) = p_2$.*

Corollary 4.8 *Let $X \subset \mathbb{P}^n$ be a variety. Let $p \in \mathbb{P}^n$ such that $\mathrm{rk}_X(p) = 2$. Then, $3 \leq \mathrm{rk}_{X^2}(p^{\otimes 2}) \leq 4$ and, moreover, $\mathrm{rk}_{X^2}(p^{\otimes 2}) = 3$ if and only if p lies on a secant line.*

A submultiplicativity result also under the existence of a k-dimensional *multisecant linear spaces* are given in [17]. Recall that a k-dimensional linear space $\mathbb{P}(W)$ is multisecant if the support of $X \cap \mathbb{P}(W)$ contains at least $k+2$ distinct points.

In view of the examples provided in [17], we propose the following conjecture. For example, this is proven to be true for binary forms and cubic ternary forms with respect to Veronese varieties.

Conjecture 4.9 *Let $X \subset \mathbb{P}^n$ be a nondegenerate variety over an an algebraically closed field and let $p \in \mathbb{P}^n$. If $\underline{rk}_X(p) < \mathrm{rk}_X(p)$ then $\mathrm{rk}_{X^{\times 2}}(p \otimes p) < \mathrm{rk}_X(p)^2$.*

4.3 Asymptotic ranks and subrank

A recent reference for the asymptotic geometry of tensors is the research monograph [89]. In the following, let $\mathbb{K}$ be an arbitrary field: we mention when more restrictive assumptions will be needed. Let V_i's be finite dimensional $\mathbb{K}$-vector spaces.

Definition 4.10 (*Restrictions and degenerations*) Let $T, T' \in V_1 \otimes \cdots \otimes V_k$. T' is a *restriction* of T if there exist endomorphisms $\phi_i \in \mathrm{End}(V_i)$ such that

$$T' = \phi_1 \otimes \cdots \otimes \phi_k(T),$$

here the tensor product of ϕ's acts naturally on T. The tensor T' is a *degeneration* of T if

$$T' = \lim_{t \to 0} \phi_1(t) \otimes \cdots \otimes \phi_k(t)(T),$$

for $\phi_i(t) \in \mathrm{End}(V_i) \otimes_{\mathbb{K}} \mathbb{K}(t)$.

Definition 4.11 (*Unit tensor*) Fixed basis $\{e^i_j\}_j \subset V_i$ for $1 \leq i \leq k$, the *unit tensor* of rank r is $U_k(r) = \sum_{i=1}^r e^1_i \otimes \cdots \otimes e^k_i$.

Remark 4.12 Using the terminology of restrictions and degenerations, we may reformulate the definitions of tensor rank and border tensor rank from Sect. 2 as follows:

$$\mathrm{rk}(T) = \min\{r \mid T \text{ is a restriction of } U_k(r)\}.$$
$$\underline{rk}(T) = \min\{r \mid T \text{ is a degeneration of } U_k(r)\}.$$

Definition 4.13 (*Tensor subrank*) The *tensor subrank* of T is

$$Q(T) = \max\{r \mid U_k(r) \text{ is a restriction of } T\}.$$

The *tensor border subrank* of T is

$$\underline{Q}(T) = \max\{r \mid U_k(r) \text{ is a degeneration of } T\}.$$

Remark 4.14 One has $U_k(r)^{\boxtimes d} = U_k(r^d)$ and hence

$$\underline{rk}(T^{\boxtimes d}) \leq \underline{rk}(T)^d,$$
$$\underline{Q}(T^{\boxtimes d}) \geq \underline{Q}(T)^d.$$

Strassen's laser method is a method to give an upper bound on the *matrix multiplication exponent*, i.e., the exponent ω showing up in the arithmetic complexity $\mathcal{O}(n^{\omega})$ for performing the multiplication of two matrices of size $n \times n$. The core of the method is to find a good candidate tensor T admitting a degeneration of some of its Kronecker powers $T^{\boxtimes k}$ to a matrix multiplication tensor for some n. In this context, Strassen introduced the *asymptotic ranks* [110].

Definition 4.15 (*Asymptotic border rank and subrank*) The *asymptotic border rank* of T is

$$\underset{\sim}{\mathrm{rk}}(T) = \lim_{d\to\infty} \underline{rk}(T^{\boxtimes d})^{1/d}.$$

The *asymptotic border subrank* of T is

$$\underset{\sim}{\mathrm{Q}}(T) = \lim_{d\to\infty} \underline{Q}(T^{\boxtimes d})^{1/d}.$$

Remark 4.16 Again, these are well-defined by Fekete's lemma.

Remark 4.17 A tensor $T \in (\mathbb{K}^n)^{\otimes k}$ is said to be *concise* if there is not a proper subspace $V_1 \otimes \cdots \otimes V_k \subset (\mathbb{K}^n)^{\otimes k}$ such that $T V_1 \otimes \cdots \otimes V_k$. For a concise tensor $T \in (\mathbb{K}^n)^{\otimes k}$, by Definition 4.15 one derives that the following inequalities hold:

$$Q(T) \le \underline{Q}(T) \le \underset{\sim}{\mathrm{Q}}(T) \le n \le \underset{\sim}{\mathrm{rk}}(T) \le \underline{rk}(T) \le \mathrm{rk}(T)$$

Example 4.18 When $T = U_k(r)$, then all these ranks are equal to r.

Example 4.19 Let $T = W_k \in (\mathbb{C}^2)^{\otimes k}$ be the W-state tensor. Then $Q(W_k) = \underline{Q}(W_k) = 1$ and $\underset{\sim}{\mathrm{rk}}(W_k) = \underline{rk}(W_k) = 2$. Moreover $\underset{\sim}{\mathrm{Q}}(W_k) = 2^{H(1/k)}$ [66, Corollary 3.24], where $H(1/k) = -1/k\log(1/k) - (1 - 1/k)\log(1 - 1/k)$ is the Shannon's entropy introduced before applied to the uniform distribution on the set $[k]$.

4.4 Subrank and border subrank of generic tensors

The nature of subrank is rather different than the one of rank. In this section, we assume $\mathbb{K}$ to be infinite.

For instance the generic border subrank is different from the generic subrank. In their recent pre-print, Biaggi, Chang, Draisma and Rupniewski proved that the border subrank in $(\mathbb{K}^n)^{\otimes 3}$ is at least $\lfloor\sqrt{4n}\rfloor - 3$, which is bigger than the generic subrank for n large enough [28]. Moreover, they found the asymptotic growth of the border subrank.

Theorem 4.20 (Biaggi–Chang–Draisma–Rupniewski [28]) *The generic border subrank of tensors in* $(\mathbb{K}^n)^{\otimes k}$ *is* $O(n^{1/(k-1)})$.

For the generic subrank, Pielasa, Šafránek and Shatsila recently found the exact values for any format, thus establishing a conjecture that Derksen, Makam and Zuiddam formulated in the three factor case [67].

Theorem 4.21 (Pielasa–Šafránek–Shatsila [102]) *Let* $k \ge 3$. *For any* $n_1, \ldots, n_k$, *the generic subrank in* $\mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_k}$ *is*

$$\min\left\{n_1, \ldots, n_k, \lfloor(n_1 + \cdots + n_k - k + 1)^{1/(k-1)}\rfloor\right\}.$$

In particular, in $(\mathbb{K}^n)^{\otimes 3}$, *the generic subrank is* $\lfloor\sqrt{3n-2}\rfloor$, *which the value expected in* [67].

Natural geometric questions related to these questions are the following

Question 14 Let $X_r^{(n_1,\dots,n_k)}$ be the locus of tensors $T \in \mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_k}$ with $\underline{Q}(T) \geq r$. What is the dimension of $X_r^{(n_1,\dots,n_k)}$?

There are partial results on Question 14 for $k = 3$ and $r = n_1 = n_2 = n_3$ due to Chang [56]. However, establishing sharp estimates seems hard.

Question 15 What is the exact value of the generic border subrank in $\mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_k}$?

4.5 Asymptotic gaps

Let $\mathbb{K}$ be an arbitrary field. A recent progress in understanding asymptotic subranks concerns where and how the possible values are located on the real line.

Theorem 4.22 (Christandl–Gesmundo–Zuiddam [55]) *Let $T \in V_1 \otimes \cdots \otimes V_k$ with $k \geq 3$. Then exactly one of the following holds:*

(i) $\underset{\sim}{Q}(T) = 1$*;*
(ii) $\underset{\sim}{Q}(T) \geq 2^{H(1/k)}$.

Hence there is a remarkable *gap* in the possible values of $\underset{\sim}{Q}(T)$ as T varies in the tensor space.

Note that the definition of border subrank may be formulated as follows. Let $G = \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_k)$; then T degenerates to S if and only if $\overline{G \cdot T} \supseteq \overline{G \cdot S}$. Now, by definition one finds that the border subrank of T satisfies $\underline{Q}(T) \geq r$ if and only if the orbit closure $\overline{G \cdot T}$ contains the orbit closure $\overline{G \cdot U_k(r)}$. This implies that then $\underline{Q}(T) \geq \underline{Q}(S)$ and so $\underset{\sim}{Q}(T) \geq \underset{\sim}{Q}(S)$, whenever T degenerates to S. In order to prove Theorem 4.22, the authors prove the next result using induction and looking at a degeneration in a Grassmannian.

Theorem 4.23 *If all flattenings of T have rank at least 2, then T degenerates to W_k, the W-state.*

For 3-tensors, they can say more and their result reads as follows.

Theorem 4.24 (Christandl–Gesmundo–Zuiddam [55]) *Let $T \in V_1 \otimes V_2 \otimes V_3$. Then, exactly one of the following holds:*

(i) $\underset{\sim}{Q}(T) = 1$*;*
(ii) $\underset{\sim}{Q}(T) \geq 2^{H(1/3)} \approx 1.89$.
(iii) $\underset{\sim}{Q}(T) \geq 2$*;*

Theorems 4.24 and 4.22 actually apply to a larger class of functions on tensors than simply the asymptotic subrank: these are called *restriction normalized monotone functions*; see [55, §1.3]. As a consequence, they are applicable to a wide rage of parameters showing up in combinatorics and quantum information theory, such as the asymptotic slice rank and Naslund's partition rank [98] we mentioned before.

Remark 4.25 Inspired by the celebrated graph minor theorem of Robertson and Seymour, which says that finite graphs are well-quasi-ordered by the minor order, Blatter, Draisma and Rupniewski [32, Corollary 1.4.3] proved that for any real-valued restriction monotone (such as the asymptotic subrank) f on k-tensors over a *finite field* $\mathbb{K}$, the image of f is a *well-ordered set*. This means that any subset of the image of f has a smallest element. Equivalently, every strictly decreasing sequence of elements of the image of f terminates after finitely many steps. For asymptotic rank, this was recently extended to any field in [57]: the authors show that the property of having asymptotic rank at most some real number is a Zariski-closed property over an arbitrary field. This result implies that over any field each asymptotic rank cannot be an accumulation point *from the right* and so the set of the asymptotic ranks is well-ordered over any field. However, there could still be accumulation points from the left.

Inspired by the results featured in the above remark, we ask the following.

Question 16 Over an infinite or a finite field, what is the topology of the spectral set consisting of the values $\underset{\sim}{\mathrm{Q}}(T)$ or of $\underset{\sim}{\mathrm{rk}}(T)$, as T varies?

We finally recall a very compelling conjecture on what should be the asymptotic subrank of a monomial, also showing how little is known about this quantity and so how much room there is for new methods and ideas in front of us.

Conjecture 4.26 (Christandl–Vrana [116]) *Let M be the symmetric tensor in $(\mathbb{C}^n)^{\otimes k}$ corresponding to the monomial $x_1^{d_1} \cdots x_n^{d_n}$, where $k = d_1 + \cdots + d_n$. This is called* Dicke state. *Then*

$$\underset{\approx}{Q}(M) = 2^{H(d_1/k,\ldots,d_n/k)},$$

where H is Shannon's entropy, i.e. $H(d_1/k, \ldots, d_n/k) = -\sum_{i=1}^{n}(d_i/k)\cdot\log(d_i/k)$. The statement is known to hold when $n = 2$ or when $d_i = 1$ for all $1 \le i \le n$.

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