Border apolarity and varieties of sums of powers

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Tomasz Mańdziuk and Emanuele Ventura

Abstract

We study border varieties of sums of powers ($\underline{\mathrm{VSP}}$'s for short), recently introduced by Buczyńska and Buczyński, parameterizing border rank decompositions of a point (e.g. of a tensor or a homogeneous polynomial) with respect to a smooth projective toric variety. Their importance stems from the role of border tensor rank in theoretical computer science, especially in the estimation of the exponent of matrix multiplication. We compare $\underline{\mathrm{VSP}}$'s to other well-known loci in the Hilbert scheme, parameterizing scheme-theoretic versions of decompositions. We introduce the notion of border identifiability and provide sufficient criteria for its appearance, relying on the Maclagan-Smith multigraded regularity. We link border identifiability to wildness of points. Finally, we determine $\underline{\mathrm{VSP}}$'s in several instances, in the contexts of tensors and homogeneous polynomials. These include concise 3-tensors of minimal border rank and in particular of border rank three, answering a question of Buczyńska and Buczyński.

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Keywords: Toric variety, Cox ring, Tensors, Forms, Border rank, Border variety of sums of powers, Apolarity, Multigraded Hilbert scheme, Multigraded Castelnuovo-Mumford regularity.

AMS Mathematical Subject Classification 2020: Primary 14C05; Secondary 14M25, 15A69, 68Q17.

1 Introduction

The last two decades have witnessed steady progress on the theory and applications of tensor and Waring ranks. Perhaps the strongest driving forces behind these fast developments were, on one hand, uncovering the rich geometry of special projective varieties and, on the other, exploiting the fundamental connection between tensors and theoretical computer science. This second link is very deep and goes back to the works of Strassen [42, 41] and Bini [1]: tensor and border tensor rank of the matrix multiplication tensor are, up to a constant, equal to the asymptotic number of arithmetic operations (digits multiplications) needed to optimally compute the product of two matrices; see [43, §12.3.2] or the extensive discussions in [28] and [7].

On the geometric and algebraic side, there have been intense research efforts to understand the subtleties of tensor and Waring ranks, entailing secant varieties [38, Chapter 1] and Macaulay's theory of apolarity and inverse systems [24, §1.1].

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Strikingly, scheme-theoretic versions of ranks have been an important tool to understand the previous ranks. These schematic ranks take into account more general zero-dimensional schemes, besides the reduced ones featured in the tensor and Waring ranks. The latter more general framework naturally leads to new notions: the *smoothable rank* and the *cactus rank*, originally called *scheme length* [24, Definition 5.1]. We recall their definitions in §2.

In a 2019 groundbreaking work, Buczyńska and Buczyński [4] introduced a new method for border estimation, called *border apolarity*, see Theorem 2.15 below. This result opened up a way to potentially overcome the well-known *barriers* affecting vector bundle methods for lower bounds on border ranks [17, 13]. These barriers are naturally explained in the context of schematic ranks, as cactus rank tends to be much lower than border rank (but not always!), whereas vector bundle methods, such as flattening constructions, give lower bounds on the former.

The strong effectiveness of the new method was demonstrated by Conner, Harper and Landsberg [10] who proved the following lower bounds on the border rank of rectangular matrix multiplication (of indicated sizes): $\underline{\mathbf{rk}}(M_{\langle 2,n,n\rangle}) \geq n^2 + 1.32n$ and $\underline{\mathbf{rk}}(M_{\langle 3,n,n\rangle}) \geq n^2 + 1.6n$. This is remarkable as the previously known bounds had no linear terms in n.

As fundamental aspect of their border apolarity theorem, Buczyńska and Buczyński discovered an *algebraic* way to describe and parameterize border rank decompositions with respect to a smooth toric projective variety. This gives rise to the central geometric objects lurking in the theory, that are the main characters of our article: the *border varieties of sums of powers* (called $\underline{\mathrm{VSP}}$'s for short); see Definition 2.16. As yet, to the best of our knowledge, only few examples and results about these varieties are known; see e.g. [10] and [23]. More recently, Jelisiejew, Ranestad and Schreyer [27] studied loci inside multigraded and usual Hilbert schemes that are linked to $\underline{\mathrm{VSP}}$'s of quadratic forms; see Remark 3.7. Our main motivation is then to start a systematic study of these interesting objects that could shed light on the nature of border rank. We believe this has the potential to reverberate in explicit and powerful results in the theory of computation, as border varieties of sums of powers are arguably the ultimate geometric objects governing border rank phenomena, that are of great relevance in algebraic complexity.

Main results.

The original set-up of border apolarity is for smooth projective toric varieties and their (finitely generated and multigraded) Cox rings. We make use of the multigraded regularity of Maclagan and Smith, which extends the classical Castelnuovo-Mumford regularity to Cox rings, *both* as a framework and as a tool for border apolarity.

Let $S = S[X]$ be the Cox ring of a smooth projective toric variety X which is multigraded by $\mathbb{Z}^s$ for some s. One defines a similarly graded dual ring T; those are equipped with a pairing given by differentiation of S on T. Inside the Haiman-Sturmfels multigraded Hilbert scheme $\mathrm{Hilb}_S^{h_{r,X}}$, Buczyńska and Buczyński look at the irreducible component defined by all the limits of all saturated ideals of r points with generic Hilbert function $h_{r,X}$. This projective variety is called $\mathrm{Slip}_{r,X}$. For $F \in T$, all the ideals J in the following closed locus

$$\underline{\mathrm{VSP}}(F,r) = \left\{ J \in \mathrm{Slip}_{r,X} \text{ such that } J \subset \mathrm{Ann}(F) \right\}$$

govern the border decompositions of a homogeneous element $F \in T$ (see Theorem 2.15). One natural approach is contrasting the latter with well-known loci attached to F: $\mathrm{VSP}(F,r)$ and $\mathrm{VPS}(F,r)$; see Definition 3.1 and Definition 3.2. In the description of the relationships amongst $\underline{\mathrm{VSP}}$, VPS and VSP there are several subtleties lurking. First of all, VPS could be non closed, a fact leading to perhaps unintuitive phenomena at the boundary, informally known as *bad limits* in [36]; see Remark 3.6. We give a sufficient condition for the closedness of VPS in Proposition 3.15.

Although there is a proper surjective morphism $\phi_{r,X} : \mathrm{Slip}_{r,X} \to \mathrm{Hilb}^r_{sm}(X)$, this *does not* usually descent to a map from $\underline{\mathrm{VSP}}(F,r)$ to $\mathrm{VPS}(F,r)$. However, one has that $\phi^{-1}_{r,X}(\mathrm{VPS}(F,r) \cap \mathrm{Hilb}^r_{sm}(X)) \subset \underline{\mathrm{VSP}}(F,r)$ (Lemma 3.4). When equality holds, we say that $\underline{\mathrm{VSP}}(F,r)$ is *of fiber type*. A special role in the theory is played by homogeneous elements $F \in T$ such that $\mathrm{srk}(F) > \underline{\mathbf{rk}}(F)$; these are called *wild*. Proposition 3.17 proves that whenever $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F))$ is of fiber type, then F cannot be wild. Wild elements have the remarkable property that their border varieties of sums of powers $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F))$ do *not* have points corresponding to saturated ideals.

Given X, we fix an embedding of $X \subset \mathbb{P}(T_{\mathbf{v}})$ for some multidegree $\mathbf{v} \in \mathbb{Z}^s$. One may wonder how $\underline{\mathrm{VSP}}(F,r)$ behaves as we move inside $\sigma_r(X)$. Proposition 3.9 states there exists a dense open set $W \subset \sigma_r(X)$ such that if $F \in W$, then $\underline{\mathrm{VSP}}(F,r)$ contains a saturated ideal (a similar conclusion holds when we search for a radical ideal). A corollary to this result (Corollary 3.12) is that whenever $\sigma_r(X)$ is nondefective or fills up without excess the ambient space, then for a general $F \in \sigma_r(X)$, $\underline{\mathrm{VSP}}(F,r)$ contains *only* saturated ideals. However without these assumptions, this might fail as observed in Remark 3.14.

The openness of the locus of saturated ideals in $\mathrm{Hilb}^h_S(X)$ [26, 5], [26, Proposition 3.9] and its slight generalization in our Theorem 2.26 all suggest that we should compare $\underline{\mathrm{VSP}}$, VPS and VSP from a birational perspective. Relying on these results we prove the following.

Theorem (Theorem 3.19)**.** *Let $F \in T_{\mathbf{v}}$ be a homogeneous polynomial of degree $\mathbf{v}$ and r be a positive integer. Then $\phi_{r,X}$ induces a bijection between the set of those irreducible components of the closure of $\mathrm{VPS}(F,r) \cap \mathrm{Hilb}^r_{sm}(X)$ that contain a scheme with the generic Hilbert function $h_{r,X}$ and the set of those irreducible components of $\underline{\mathrm{VSP}}(F,r)$ that contain a saturated ideal. Under this bijection, the irreducible components in correspondence are birational.*

We say that $F \in T_{\mathbf{v}}$ is *border identifiable* if $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F))$ is a single point. We establish a criterion for border identifiability employing the machinery of multigraded regularity of Maclagan and Smith. Let $\mathcal{K} = \mathbb{N}\mathbf{c}_1 + \cdots + \mathbb{N}\mathbf{c}_l$ be the integral nef cone of X. Our main theorem here reads as follows:

Theorem (Theorem 4.2)**.** *Let $X \subset \mathbb{P}(T_{\mathbf{v}})$ and $r = \underline{\mathbf{rk}}_X(F)$ be the border rank of $F \in T_{\mathbf{v}}$. Suppose that there exists $\mathbf{u} \in \mathcal{K}$ such that*

$$\mathrm{HF}(S/\mathrm{Ann}(F), \mathbf{u}) = \mathrm{HF}(S/\mathrm{Ann}(F), \mathbf{u} + \mathbf{c}_1 + \cdots + \mathbf{c}_l) = r.$$

If there exists a B-saturated ideal $I \in \underline{\mathrm{VSP}}(F,r)$, then $\underline{\mathrm{VSP}}(F,r) = \{I\}$.

For $X = \mathbb{P}^n$, $d = 2s+1$ and $r = \binom{n+s}{s}$, this implies that a general $F \in \sigma_r(\nu_d(\mathbb{P}^n))$ is border identifiable (Corollary 4.4).

A fundamental playground for border apolarity is the challenging world of tensors. These constitute one of the first motivations behind the very conception of border apolarity. Matrix multiplication is a 3-tensor, so this class of tensors is particularly important – besides just being the next case after matrices – and already encapsulate richness of structure in sharp contrast with matrices. Let $X = \mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$, S be its Cox ring and T be its dual. Then $T_{\mathbf{1}} \cong \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, where $\mathbf{1} = (1,1,1)$. We focus on concise minimal border rank tensors, i.e., $F \in T_{\mathbf{1}}$ with $\underline{\mathbf{rk}}(F) = m$ and that are not annihilated by any multigraded linear polynomial. Our result in this direction is very much related to work of Jelisiejew, Landsberg and Pal [25] on this class of tensors.

Theorem (Theorem 5.3). *Let $F \in T_{\mathbf{1}}$ be concise and of minimal border rank, i.e., $\underline{\mathbf{rk}}(F) = m$. Let $I = (\mathrm{Ann}(F)_{(1,1,0)}) + (\mathrm{Ann}(F)_{(1,0,1)}) + (\mathrm{Ann}(F)_{(0,1,1)}) \subset S$ and $K = (I \colon B^\infty)$. Then the following statements hold:*

(i) If $\mathrm{HF}(S/I, \mathbf{1}) \neq m$, then F is wild.

(ii) If $\mathrm{HF}(S/I, \mathbf{1}) = m$, then F is not wild if and only if $I_{(a,b,c)} = K_{(a,b,c)}$ for every $(a, b, c) \in \mathcal{S}$, where $\mathcal{S} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}$.

Note that statement (i) is [25, Theorem 9.2]. The corollary to this is that: if F is a nonwild concise minimal border rank tensor in $T_{\mathbf{1}}$, then $\underline{\mathrm{VSP}}(F, m) = \{K\}$ where K is the saturation of $I = (\mathrm{Ann}(F)_{(1,1,0)}) + (\mathrm{Ann}(F)_{1,0,1}) + (\mathrm{Ann}(F)_{(0,1,1)})$ (Corollary 5.4). In other words, much of the complexity of concise minimal border rank 3-tensors is due to wild tensors, at least from the perspective of $\underline{\mathrm{VSP}}$'s.

Minimal border rank tensors $F \in T_{\mathbf{1}}$ with $m = 3$ are classified [6, Theorem 1.2]. Therefore, in this case, we can actually improve the previous result to an explicit description of all $\underline{\mathrm{VSP}}(F, 3)$ for all such F's. We prove the ensuing.

Theorem (Theorem 5.5). *Let $X = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ and let F be a border rank three concise tensor in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \cong T_{\mathbf{1}}$. The variety $\underline{\mathrm{VSP}}(F, 3)$ is either a single point, or $\underline{\mathrm{VSP}}(F, 3) \cong \mathbb{P}^3$ when F is wild.*

This answers a question of Buczyńska and Buczyński [4, §5.2] about the geometry of $\underline{\mathrm{VSP}}(F, 3)$.

As mentioned, whenever F is wild, $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F))$ consists only of nonsaturated ideals. Does the converse hold? We give a negative answer to this question, providing a monomial counterexample; see Example 5.7. Elaborating more on wildness, we prove a result that is similar in the spirit to that for tensors explained above. When $X = \mathbb{P}^n$ and $d = 3$ or $d \geq n + 2$, we prove that for a minimal border rank $F \in T_d$ one has $\mathrm{Hess}(F) \neq 0$ if and only if $\underline{\mathrm{VSP}}(F, n + 1)$ consists of a unique saturated ideal. When this holds, the unique saturated ideal is $(\mathrm{Ann}(F)_2)$ (Corollary 5.9). This follows from our Theorem 4.2 on multigraded regularity and [23, Theorem 4.9], which characterizes wildness for minimal border rank forms.

We investigate binary forms, proving that $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F))$ are either one point or $\mathbb{P}^1$ (Proposition 6.1). Leveraging the classifications of ternary cubic forms and reducible cubic forms, we describe $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F))$ for each of these classes in Theorem 6.3 and in §6.2.2. Note that reducible cubics are particularly interesting in algebraic complexity theory, as they correspond to symmetrizations of the big and small Coppersmith-Winograd tensors.

Following the terminology of [37, §1], a *nondegenerate* ternary form of degree $d = 2p-2$ is a form $F \in T_d$ such that $\mathrm{Ann}(F)_{\leq p-1} = 0$. We show that, for $2 \leq p \leq 5$ and a general nondegenerate form F of degree $d = 2p - 2$, we have an isomorphism between $\mathrm{VPS}(F, r_p) = \mathrm{VSP}(F, r_p)$ and $\underline{\mathrm{VSP}}(F, r_p)$ given by $\phi_{r_p,\mathbb{P}^2}$ with $r_p = \binom{p+1}{2} = \underline{\mathbf{rk}}(F)$ (Theorem 6.9).

In §6.4, we exhibit an example of a reducible $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F))$. This addresses and solves the question whether $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F))$ can be positive dimensional and reducible even in $\mathbb{P}^2$. Note that this is not the case when $X = \mathbb{P}^1$. Finally, in §6.5, Proposition 6.12 describes an example of $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F))$ that is isomorphic to the Schubert variety $\Sigma_1 \subset \mathbb{G}(3, 5)$.

Organisation of this paper.

In §2, we discuss preliminaries. We recall toric varieties and their Cox rings where apolarity will take place, the notions of ranks we shall be concerned with, and multigraded regularity of Maclagan and Smith. In §3, we recall the definitions of the loci we are interested in and show all the results based on comparing $\underline{\mathrm{VSP}}$, VPS and VSP. In §4, we prove Theorem 4.2 and corollaries thereof. In §5, we show Theorem 5.3 and its Corollary 5.4. In §6 we describe the $\underline{\mathrm{VSP}}$'s for binary forms (§6.1), cubic ternary forms and reducible cubics (§6.2). In §6.3, we prove Theorem 6.9. In §6.4 and §6.5, we give instances when $\underline{\mathrm{VSP}}$ is reducible and when it is a Schubert variety, respectively.

Acknowledgements.
We thank the organizers of the semester "Tensors: geometry, complexity and quantum entanglement" at the University of Warsaw and at the Institute of Mathematics of the Polish Academy of Sciences, held in the Fall of 2022, for their warm hospitality and support. We thank J. Buczyński, A. Conca, H. Huang, J. Jelisiejew, A. Massarenti, G. Ottaviani, and K. Ranestad for very useful discussions. The first author is a member of TensorDec laboratory of the Mathematical Department of Trento. The second author is a member of the GNSAGA group of INdAM. The preparation of this article was partially carried out within the following PRIN2022 projects funded by the European Union Next Generation EU: *Applied Algebraic Geometry of Tensors* (CUP E53C24002330001, PRIN 2022 - Prot. n. 2022NBN7TL). We acknowledge the invaluable help of the algebra software `Macaulay2` [20]. We thank an anonymous referee for useful suggestions.

Conflict of interests. The authors state that there is no conflict of interests.

2 Cox rings, ranks and border apolarity

2.1 Cox rings and Picard groups

For details on toric varieties, their combinatorial construction and their properties we refer to the textbooks by Cox, Little and Schenck [12] and by Fulton [16]. Let $X = X_\Sigma$ be a d-dimensional complex smooth projective toric variety corresponding to the fan $\Sigma \subset N \cong \mathbb{Z}^d$ with $n+1$ rays. The group of Cartier divisors or Picard group $\mathrm{Pic}(X)$ is isomorphic to $\mathbb{Z}^s$ because X is smooth [16, §3.4], where $s = n + 1 - d$.

The *Cox ring* of X is the polynomial ring $S[X] = \mathbb{C}[y_0, \dots, y_n]$ with a $\mathrm{Pic}(X) \cong \mathbb{Z}^s$-grading defined by $\deg(y_i) = \mathbf{a}_i \in \mathbb{Z}^s$ for $0 \le i \le n$ [12, §5.2], where $\mathbf{a}_i \in \mathbb{Z}^s \cong \mathrm{Pic}(X)$ is the class of the torus invariant divisor corresponding to ith ray in Σ. Alternatively, one may write $S[X] = \bigoplus_{D \in \mathrm{Pic}(X)} H^0(\mathcal{O}_X(D))$ [11, Proposition 1.1], where each summand is the $\mathbb{C}$-vector space of global sections of the corresponding Cartier divisor. In the following, we shall identify Cartier divisors with integral vectors. Every Cox ring $S[X]$ has a *positive grading*, i.e., the only monomial of degree $\mathbf{0} \in \mathbb{Z}^s$ is 1. The combinatorics of the fan Σ encodes the *irrelevant ideal* B.

Example 2.1. The first example is when $X = \mathbb{P}^d$ and $d = n$. In this case, $S[X] = \mathbb{C}[y_0, \dots, y_n]$ with the standard $\mathbb{Z}$-grading given by $\deg(x_i) = 1$. The ideal B is $(y_0, \dots, y_n) \subset S[X]$.

The irrelevant ideal B allows one to obtain a quotient construction of X as a geometric quotient [12, Theorem 5.1.11]. This quotient construction generalizes to provide a toric ideals-subscheme correspondence between $S[X]$ and X. More generally, Cox shows that there is a categorical equivalence between coherent $\mathcal{O}_X$-modules and finitely generated $\mathbb{Z}^s$-graded S-modules *modulo B-torsion* [11, Proposition 3.3]. For a homogeneous ideal $I \subset S[X]$, being B-torsion free means that $I = (I :$

$B^\infty) = \{g \in S[X] \mid \exists N \in \mathbb{N} : gB^N \subset I\}$. For ideal sheaves on X, the Cox correspondence is as follows: there is a bijection between $\mathbb{Z}^s$-homogeneous ideal $I \subset S[X]$ that are B-saturated and closed subschemes $Z \subset X$. From the fan $\Sigma \subset \mathbb{Z}^s$, one can construct another semigroup, called $\mathcal{K}$. The points of the cone $\mathcal{K} \otimes_{\mathbb{Z}} \mathbb{R}$ correspond to numerically effective (nef) $\mathbb{R}$-divisors on X, and it is called the *nef cone* of X. When X is projective, this is a pointed s-dimensional cone [12, Theorem 6.3.22], where s as above is the Picard rank of X. The nef cone is the closure of the so-called *Kähler* or *ample cone* of X. Note that a Cartier divisor on X is nef if and only if it is globally generated [12, Theorem 6.3.12]. So the points in $\mathcal{K}$ correspond to divisors with nonzero global sections.

2.2 Toric apolarity and ranks

For this part of the exposition, we follow [4, §3.1]. Given the $\mathbb{Z}^s$-graded Cox ring $S[X] = \mathbb{C}[y_0, \ldots, y_n]$, define the dual graded polynomial ring $T[X]$ defined as

$$T[X] = \bigoplus_{D \in \mathbb{Z}^s} H^0(\mathcal{O}_X(D))^* \cong \mathbb{C}[x_0, \ldots, x_n].$$

The ring $T[X]$ is a graded $S[X]$ module with the differential graded action induced by the formula $y_i \circ x_j = \delta_{i,j}$. As mentioned before, we identify Cartier divisors with integral vectors $\mathbf{v}$, writing $S_{\mathbf{v}}$ and $T_{\mathbf{v}}$ for the degree $\mathbf{v}$ summands of these rings.

Example 2.2. When $X = \mathbb{P}^n$, $S[X] = \mathbb{C}[y_0, \ldots, y_n]$ and its dual graded ring $T[X] = \mathbb{C}[x_0, \ldots, x_n]$ are standard $\mathbb{Z}$-graded rings and the action described above is the classical *Macaulay apolarity action* [24, §1.1].

Example 2.3. When $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, $S[X] = \mathbb{C}[y_{1,0}, \ldots, y_{1,n_1}] \otimes \cdots \otimes \mathbb{C}[y_{s,0}, \ldots, y_{s,n_s}]$; its dual graded ring $T[X]$ has a similar description. The grading is given by $\deg(y_{i,j}) = \mathbf{e}_i \in \mathbb{Z}^s$, the ith standard basis vector. When $s = 2$ or 3, we shall use α_i, β_j and γ_k to denote the generators of the corresponding Cox ring.

The elements of T_d from Example 2.2 are homogeneous polynomials of degree d and are classically called *forms* (or *symmetric tensors*). When $n = 2$, they are *binary* forms; when $n = 3$, they are *ternary* forms. The elements of $T_{\mathbf{1}} \cong \mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_s+1}$ from Example 2.3, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^s$, are called *tensors.*

In the rest of the article, once the toric variety X is fixed, we drop this symbol from the notations $S[X]$ and $T[X]$. For a homogeneous ideal $J \subset S$, let $J_{\mathbf{v}}$ denote the complex finite dimensional vector space consisting of its degree $\mathbf{v}$ homogeneous elements. For $J \subset T$, let $\overline{J}$ denote its saturation with respect to B, i.e.,

$$\overline{J} = (J : B^\infty) \subset S.$$

The *multigraded Hilbert function* of a homogeneous $J \subset S$ is the numerical function $\mathrm{HF}(S/J, \cdot) : \mathbb{Z}^s \to \mathbb{N}$ defined by

$$\mathrm{HF}(S/J, \mathbf{v}) = \dim_{\mathbb{C}} S_{\mathbf{v}} - \dim_{\mathbb{C}} J_{\mathbf{v}}.$$

A numerical function $h : \mathbb{Z}^s \to \mathbb{N}$ is said to be an *admissible multigraded Hilbert function* if there exists an ideal $J \subset S$ such that $\mathrm{HF}(S/J, \cdot) = h$. Given an element $0 \neq F \in T_{\mathbf{v}}$, denote $[F] \in \mathbb{P}(T_{\mathbf{v}})$ to be the corresponding point in projective space.

Definition 2.4 (Apolar ideals). Let $F \in T_{\mathbf{v}}$. Then its *apolar* or *annihilator ideal* is

$$\mathrm{Ann}(F) = \{\psi \in S \mid \psi \circ F = 0\} \subset S.$$

This is a $\mathbb{Z}^s$-homogeneous ideal.

Given X, we fix an ample line bundle $\mathcal{L} = \mathcal{O}_X(D)$ on X, whose divisor corresponds to a vector $\mathbf{v} \in \mathbb{Z}^s$, embedding X in the projective space $\mathbb{P}(H^0(\mathcal{L})^*) = \mathbb{P}(T_{\mathbf{v}})$. Let $Z \subset X \subset \mathbb{P}(T_{\mathbf{v}})$ be a closed subscheme. Then its *projective span* $\langle Z \rangle$ the smallest linear space containing Z. Equivalently, $\langle Z \rangle = \mathbb{P}((S_{\mathbf{v}}/I_{Z,\mathbf{v}})^*) \subset \mathbb{P}(T_{\mathbf{v}})$, where $I_{Z,\mathbf{v}}$ is the degree $\mathbf{v}$ homogeneous summand of the B-saturated ideal I_Z of Z.

Definition 2.5 (Rank). The X-rank of $[F] \in \mathbb{P}(T_{\mathbf{v}})$ is the minimal integer $r \geq 1$ such that there exist r points $p_1, \ldots, p_r \in X \subset \mathbb{P}(T_{\mathbf{v}})$ such that $[F] \in \langle p_1, \ldots, p_r \rangle$. Equivalently, let Z be the smooth scheme consisting of the union of the points p_i. Then $[F] \in \langle p_1, \ldots, p_r \rangle$ is equivalent to the condition $I_Z \subset \mathrm{Ann}(F)$. This is the same as the classical *apolarity lemma* [24, Lemma 1.15], but here X is any smooth projective toric variety. This equivalence is then called *multigraded apolarity.*

Given a homogeneous ideal $I \subset S$ and $F \in T_{\mathbf{v}}$, the following equivalence is well-known and very useful to put apolarity to work.

Proposition 2.6 ([18, Proof of Theorem 1.4])**.** $I \subset \mathrm{Ann}(F) \iff I_{\mathbf{v}} \subset \mathrm{Ann}(F)_{\mathbf{v}}$.

Definition 2.7 (Border rank). For a point $[F] \in \mathbb{P}(T_{\mathbf{v}})$, the X-*border rank* of F is the minimal integer $r \geq 1$ such that $[F] \in \sigma_r(X)$, the r-th secant variety of $X \subset \mathbb{P}(T_{\mathbf{v}})$. When X and its embedding are fixed, the border rank of F is denoted $\underline{\mathbf{rk}}_X(F)$ (or simply $\underline{\mathbf{rk}}(F)$, whenever dropping X should not cause confusion).

Definition 2.8 (Smoothable rank). The X-*smoothable rank* of $[F] \in \mathbb{P}(T_{\mathbf{v}})$ is the minimal integer $r \geq 1$ such that there exists a finite scheme $Z \subset X$ of length r which is *smoothable* (in X) and $[F] \in \langle R \rangle$. Equivalently, there exists a finite smoothable scheme $Z \subset X$ of length r whose B-saturated ideal satisfies $I_Z \subset \mathrm{Ann}(F) \subset S$. This is in analogy with the classical Apolarity lemma cited above, when $X = \mathbb{P}^n$ and Z is a smooth finite scheme. When X and its embedding are fixed, the smoothable rank of F is denoted $\mathrm{srk}_X(F)$ (or simply $\mathrm{srk}(F)$, whenever dropping X should not cause confusion).

Remark 2.9. Smoothable and border ranks satisfy $\underline{\mathbf{rk}}(F) \leq \mathrm{srk}(F)$; see [3, §2.1]. Equality holds for general points of any secant variety.

Buczyńska and Buczyński introduced the notion of *wildness*:

Definition 2.10 (Wildness). An element $F \in T_{\mathbf{v}}$ is *wild* if $\mathrm{srk}(F) > \underline{\mathbf{rk}}(F)$.

Definition 2.11 (Cactus rank). The X-*cactus rank* of $[F] \in \mathbb{P}(T_{\mathbf{v}})$ is the minimal integer $r \geq 1$ such that there exists a finite scheme $Z \subset X$ of length r such that $[F] \in \langle Z \rangle$. Equivalently, there exists a finite scheme $Z \subset X$ of length r whose B-saturated ideal satisfies $I_Z \subset \mathrm{Ann}(F)$. When X and its embedding are fixed, the cactus rank of F is denoted $\mathrm{crk}_X(F)$ (or simply $\mathrm{crk}(F)$, whenever dropping X should not cause confusion).

We finish off this subsection with the notions of conciseness and minimal border rank elements. In the following, recall that $\mathbf{a}_i = \deg y_i$.

Definition 2.12 (Conciseness). A homogeneous $F \in T_{\mathbf{v}}$ is said to be *concise* whenever $\mathrm{Ann}(F)_{\mathbf{a}_i} = 0$ for all $0 \leq i \leq n$.

The following lemma is again well-known and motivates the next definition.

Lemma 2.13. *Let $F \in T_{\mathbf{v}}$ be concise. Then $\underline{\mathbf{rk}}(F) \geq \max_{i\in[n]}\{\dim_{\mathbb{C}} T_{\mathbf{a}_i}\}$.*

Definition 2.14. Let $F \in T_{\mathbf{v}}$ be concise. If $\underline{\mathbf{rk}}(F) = \max_{i\in[n]}\{\dim_{\mathbb{C}} T_{\mathbf{a}_i}\}$, then F is said to be of *minimal border rank*.

2.3 Multigraded Hilbert schemes and border apolarity

Given an admissible numerical function $h : \mathrm{Pic}(X) = \mathbb{Z}^s \to \mathbb{N}$, let Hilb^h_S be the scheme represent the functor whose points are the homogeneous ideals $I \subset S$ with $\mathrm{HF}(S/I, \cdot) = h$. The functor in question is indeed representable and the scheme Hilb^h_S is the *Haiman-Sturmfels multigraded Hilbert scheme*. This natural object was introduced by Haiman and Sturmfels [22].

Since S is the Cox ring of X and hence is positively graded, Hilb^h_S is a projective scheme for any Hilbert function h. We usually ignore the scheme structure of Hilb^h_S and look at the underlying reduced scheme $(\mathrm{Hilb}^h_S)_{\mathrm{red}}$. A closed point of Hilb^h_S corresponds to an ideal $I \subset S$ with Hilbert function h: we express this membership in a set-theoretic fashion as $I \in \mathrm{Hilb}^h_S$.

For any integer $r \geq 0$, define the numerical function $h_{r,X} : \mathrm{Pic}(X) \to \mathbb{N}$ to be

$$h_{r,X}(\mathbf{v}) = \min\{r, \dim H^0(D)\} = \min\{r, \dim_{\mathbb{C}} S_{\mathbf{v}}\},$$

where the Cartier divisor D is identified with $\mathbf{v}$. The function $h_{r,X}$ is the *generic Hilbert function of r points on X* [4, §3.2]. By [4, Lemma 3.9] the equality $\mathrm{HF}(S/I_Z, \cdot) = h_{r,X}$ holds for a very general collection of r points $Z \subset X$. Buczyńska-Buczyński proved that $\mathrm{Hilb}^{h_{r,X}}_S$ contains a unique irreducible component, called $\mathrm{Slip}_{r,X}$, that is the closure of the locus of all $I \in \mathrm{Hilb}^{h_{r,X}}_S$ that are the B-saturated ideals of r distinct points in X [4, Proposition 3.13].

We are now ready to state the following groundbreaking result of Buczyńska-Buczyński [4, Theorem 3.15], which is an effective tool at our disposal to estimate border ranks of forms or tensors; as already mentioned, a major achievement of this method was demonstrated in [10].

Theorem 2.15 (Border apolarity of Buczyńska-Buczyński). *Keep the notation from above and let $F \in T_{\mathbf{v}}$. Then the following assertions are equivalent:*

(i) The X-border rank of F satisfies $\underline{\mathbf{rk}}_X(F) \leq r$;

(ii) there exists a homogeneous ideal $I \subset \mathrm{Ann}(F) \subset S$ which lies in the irreducible component $\mathrm{Slip}_{r,X}$.

One of the main contributions of Buczyńska-Buczyński is the realization of a (projective and so topologically compact) parameter space of border decompositions of a given $F \in T_{\mathbf{v}}$. These are the *border varieties of sums of powers*, the main characters of this article.

Definition 2.16. Let $h_{r,X}$ be the generic Hilbert function and $F \in T_{\mathbf{v}}$. The *border variety of sums of r powers* $\underline{\mathrm{VSP}}(F, r)$ is the closed subvariety of $\mathrm{Slip}_{r,X}$ defined as follows:

$$\underline{\mathrm{VSP}}(F, r) = \left\{J \in \mathrm{Slip}_{r,X} \text{ such that } J \subset \mathrm{Ann}(F)\right\}.$$

Note that this is a closed subscheme of the projective scheme $\mathrm{Slip}_{r,X}$ and so projective. Although the definition is for an arbitrary $r \in \mathbb{N}$, particularly interesting and meaningful is $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F))$.

Definition 2.17 (Border identifiability). Let $F \in T_{\mathbf{v}}$. We say that F is *border identifiable* if $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F))$ is a single point.

2.4 Multigraded regularity and Hilbert schemes of toric varieties

This is a technical section where we collect useful material on Maclagan-Smith multigraded regularity and the theory of Hilbert schemes of toric varieties. Recall that X is a smooth projective complex toric variety with Cox ring S, a graded ring graded by $\mathrm{Pic}(X) \cong \mathbb{Z}^s$, with irrelevant ideal $B \subseteq S$. The semigroup $\mathcal{K}$ denotes the points in $\mathbb{Z}^s$ corresponding to nef divisors on X; let us fix a minimal generating set $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_l\}$ of $\mathcal{K}$, i.e., $\mathcal{K} = \mathbb{N}\mathbf{c}_1 + \cdots + \mathbb{N}\mathbf{c}_l$.

The multigraded Castelnuovo-Mumford regularity of Maclagan and Smith [30, Definition 1.1] has the following definition.

Definition 2.18 (Multigraded regularity). Let M be a graded S-module and fix $\mathcal{C}$ as above. Given $\mathbf{m} \in \mathrm{Pic}(X) \cong \mathbb{Z}^s$, one says that M is $\mathbf{m}$-regular if its graded local cohomology module satisfies the vanishing $H^i_B(M)_{\mathbf{p}} = 0$ for all of the following i and $\mathbf{p}$:

(i) $i \geq 1$ and $\mathbf{p}$ is of the form $\mathbf{p} = \mathbf{m} - \lambda_1\mathbf{c}_1 - \cdots - \lambda_l\mathbf{c}_l + \mathbf{u}$, where the coefficients $\lambda_j \in \mathbb{N}$ satisfy $\lambda_1 + \cdots + \lambda_l = i - 1$ and $\mathbf{u} \in \mathcal{K}$.

(ii) $i = 0$ and $\mathbf{p}$ is of the form $\mathbf{p} = \mathbf{m} + \mathbf{c}_j + \mathbf{u}$, where $1 \leq j \leq l$ and $\mathbf{u} \in \mathcal{K}$.

Define the *multigraded regularity* of M, $\mathrm{reg}(M) \subset \mathrm{Pic}(X) \cong \mathbb{Z}^s$, to be the set

$$\mathrm{reg}(M) = \{\mathbf{m} \in \mathbb{Z}^s \mid M \text{ is } \mathbf{m}\text{-regular}\}.$$

In [31, Theorem 4.11] Maclagan and Smith obtained the following universal bound on the multigraded regularity.

Theorem 2.19 (Maclagan-Smith). *Let X be a smooth projective toric variety and Z be a subscheme of X with multigraded Hilbert polynomial P. Then there exists $\mathbf{k} \in \mathcal{K}$ such that $\dim_{\mathbb{C}}(S/I)_{\mathbf{u}} = P(\mathbf{u})$ for every $\mathbf{u} \in \mathbf{k} + \mathcal{K}$ and every B-saturated ideal I with S/I having the multigraded Hilbert polynomial P.*

As a consequence of this result, the parameter space $\mathrm{Hilb}^P(X)$ of closed subschemes of X with fixed multigraded Hilbert polynomial P exists and is isomorphic to the multigraded Hilbert scheme $\mathrm{Hilb}^P_{S_{\mathbf{k}+\mathcal{K}}}$ parameterizing all $\mathrm{Pic}(X)$-graded vector subspaces K of $S|_{\mathbf{k}+\mathcal{K}}$ that are closed under multiplication by those monomials for which the product remains in $\mathbf{k} + \mathcal{K}$ and such that $\dim_{\mathbb{C}}(S/K)_{\mathbf{u}} = P(\mathbf{u})$ for every $\mathbf{u} \in \mathbf{k} + \mathcal{K}$. See [31, Theorem 6.1]. Note that $\mathrm{Hilb}^P_{S_{\mathbf{k}+\mathcal{K}}}$ requires a slightly more general notion of a multigraded Hilbert scheme, than the one we recalled above. See [22, §6.1] and [31, §6] for more details.

Theorem 2.20 (Maclagan-Smith). *There exists $\mathbf{k} \in \mathcal{K}$ such that $\mathrm{Hilb}^P_{S_{\mathbf{k}+\mathcal{K}}}$ and $\mathrm{Hilb}^P(X)$ are isomorphic.*

Theorem 2.21. *Let $S = S[X]$ be the Cox ring of a smooth projective toric variety with $\mathrm{Pic}(X) \cong \mathbb{Z}^s$. Let $h : \mathbb{Z}^s \to \mathbb{N}$ be an admissible Hilbert function that coincides for all $\mathbf{v} \in \mathbb{Z}^s$ sufficiently far from the boundary of $\mathcal{K}$ with a polynomial $P(\mathbf{t}) \in \mathbb{Q}[t_1, \dots, t_s]$. Then there exists a morphism*

$$\psi_{\mathbf{k}+\mathcal{K}} : \mathrm{Hilb}^h_S \longrightarrow \mathrm{Hilb}^P(X).$$

Proof. Let $\tau_{\mathbf{k}+\mathcal{K}}$ be the isomorphism of schemes $\mathrm{Hilb}^P_{S_{\mathbf{k}+\mathcal{K}}} \cong \mathrm{Hilb}^P(X)$ from Theorem 2.20. Then $\psi_{\mathbf{k}+\mathcal{K}} = \tau_{\mathbf{k}+\mathcal{K}} \circ \mathrm{Pr}_{|\mathbf{k}+\mathcal{K}}$, where $\mathrm{Pr}_{|\mathbf{k}+\mathcal{K}}$ is the projection of an ideal $I \in \mathrm{Hilb}^h_S$ to the graded vector space spanned by the degree $\mathbf{k}+\mathcal{K}$ pieces of I. This gives the desired morphism. □

A slight modification of the proof of [22, Proposition 1.6] gives the following description of the tangent space to $\mathrm{Hilb}^P_{S_{\mathbf{k}+\mathcal{K}}}$.

Proposition 2.22. *The tangent space at a graded vector space $I_{|\mathbf{k}+\mathcal{K}} \in \mathrm{Hilb}^P_{S_{\mathbf{k}+\mathcal{K}}}$ to the scheme $\mathrm{Hilb}^P_{S_{\mathbf{k}+\mathcal{K}}}$ is isomorphic to the space of degree zero F-module homomorphisms*

$$\mathrm{Hom}_F(I_{|\mathbf{k}+\mathcal{K}}, S_{|\mathbf{k}+\mathcal{K}}/I_{|\mathbf{k}+\mathcal{K}})_0,$$

i.e., $\mathbb{C}$-linear maps ϕ preserving the degree such that for every $\mathbf{v}, \mathbf{u} \in \mathbf{k}+\mathcal{K}$, every monomial G in $S_{\mathbf{v}-\mathbf{u}}$ and every element $x \in I_{\mathbf{u}}$ we have $\phi(Gx) = G\phi(x)$.

By the description of the tangent space at $I_{|\mathbf{k}+\mathcal{K}}$ to $\mathrm{Hilb}^P_{S_{k+\mathcal{K}}}$ given in Proposition 2.22 and by the well-known description of the tangent space at I to Hilb^h_S (see [22, Proposition 1.6]) we obtain the following.

Lemma 2.23. *The tangent map of the morphism $\mathrm{Pr}_{|\mathbf{k}+\mathcal{K}} : \mathrm{Hilb}^h_S \longrightarrow \mathrm{Hilb}^P_{S_{\mathbf{k}+\mathcal{K}}}$ at a point I is the map of vector spaces*

$$(d\mathrm{Pr}_{|\mathbf{k}+\mathcal{K}})_I : \mathrm{Hom}_S(I, S/I)_0 \longrightarrow \mathrm{Hom}_F(I_{|\mathbf{k}+\mathcal{K}}, S_{|\mathbf{k}+\mathcal{K}}/I_{|\mathbf{k}+\mathcal{K}})_0$$

given by restriction of the corresponding S-module map $\phi \in \mathrm{Hom}_S(I, S/I)_0$.

The next ingredient we are going to need is the locus of saturated ideals inside the multigraded Hilbert scheme Hilb^h_S.

Definition 2.24 (Saturable locus). The subset of closed points of Hilb^h_S corresponding to B-saturated ideals $I \subset S$ is denoted $\mathrm{Hilb}^{h,\mathrm{sat}}_S$. Its closure is the *saturable locus.*

Theorem 2.25 (Buczyńska-Buczyński [5], Jelisiejew-Mańdziuk [26]). *Let h be an admissible Hilbert function for S. Then the locus $\mathrm{Hilb}^{h,\mathrm{sat}}_S$ is a Zariski open subset of the multigraded Hilbert scheme Hilb^h_S.*

The following is a slight generalization of [26, Proposition 3.9] to the case of smooth projective toric varieties.

Theorem 2.26. *Let h be the Hilbert function of the quotient algebra of a B-saturated ideal of S and P be the corresponding multigraded Hilbert polynomial. Then the projection morphism $\psi_{\mathbf{k}+\mathcal{K}}$: $\mathrm{Hilb}^h_S \longrightarrow \mathrm{Hilb}^P(X)$ restricts to a locally closed immersion $\mathrm{Hilb}^{h,\mathrm{sat}}_S \to \mathrm{Hilb}^P(X)$.*

Proof. Since $\tau_{\mathbf{k}+\mathcal{K}}$ is an isomorphism, it is enough to prove that the restriction of $\mathrm{Pr}_{|\mathbf{k}+\mathcal{K}}$ to $\mathrm{Hilb}^{h,\mathrm{sat}}_S$ is a locally closed immersion. By Theorem 2.25 the subset $U = \mathrm{Hilb}^{h,\mathrm{sat}}_S$ is Zariski open. Let Z be the complement of $U \subset \mathrm{Hilb}^h_S$. Then the restriction of $\mathrm{Pr}_{|\mathbf{k}+\mathcal{K}}$ to U gives a map

$$\mathrm{Pr}_{U|\mathbf{k}+\mathcal{K}} : U \longrightarrow \mathrm{Hilb}^P_{S_{\mathbf{k}+\mathcal{K}}} \setminus \mathrm{Pr}_{|\mathbf{k}+\mathcal{K}}(Z).$$

We prove that $\mathrm{Pr}_{U|\mathbf{k}+\mathcal{K}}$ is a closed immersion. By [19, Proposition 12.94], it is enough to show that

- $\mathrm{Pr}_{U|\mathbf{k}+\mathcal{K}}$ is injective on closed points;
- the tangent map to $\mathrm{Pr}_{U|\mathbf{k}+\mathcal{K}}$ at every closed point is injective.

Let $J \in U$ be in the fiber of $I_{|\mathbf{k}+\mathcal{K}}$ for some $I \in U$. Then we have the inclusion of ideals $(I_{|\mathbf{k}+\mathcal{K}}) \subset J = \overline{J}$. Hence $\overline{(I_{|\mathbf{k}+\mathcal{K}})} = \overline{I} = I \subset \overline{J} = J$. Since I and J have the same Hilbert function h, we conclude that $I = J$.

Let $\phi \in \ker((d\mathrm{Pr}_{U|\mathbf{k}+\mathcal{K}})_J)$ where $(d\mathrm{Pr}_{U|\mathbf{k}+\mathcal{K}})_J$ is the tangent map at J, featured in Lemma 2.23, and let $\mathbf{u} \in \mathbb{Z}^s$. Then there exists $\mathbf{v} \in \mathcal{K}$ such that $\mathbf{u}+\mathbf{v} \in \mathbf{k}+\mathcal{K}$, because the cone $\mathcal{K} \otimes_{\mathbb{Z}} \mathbb{R}$ is full-dimensional in $\mathrm{Pic}(X)$ [12, Proposition 6.3.24]. Let $h \in S_{\mathbf{v}}$ be a nonzerodivisor on S/I. By [30, Proposition 3.1] this exists in every degree $\mathbf{v} \in \mathcal{K}$ because $I = \overline{I}$. For every $g \in I_{\mathbf{u}}$, we have $\phi(hg) = 0$, because the restriction of ϕ is the zero map. Since ϕ is an S-module map, this implies $h\phi(g) = 0$. Hence $\phi(g) = 0$ for all $g \in I_{\mathbf{u}}$, for h is nonzerodivisor. Thus ϕ is the zero map itself, establishing the desired injectivity. □

Similarly to the case when $X = \mathbb{P}^n$, we have the following, whose proof is completely analogous and we omit it.

Proposition 2.27. *Let $h = h_{r,X}$ be the generic Hilbert function and P be the constant polynomial $P(\mathbf{t}) = r$. Then morphism $\psi_{\mathbf{k}+\mathcal{K}} : \mathrm{Hilb}^h_S \longrightarrow \mathrm{Hilb}^r(X)$ descends to a surjective morphism*

$$\phi_{r,X} : \mathrm{Slip}_{r,X} \longrightarrow \mathrm{Hilb}^r_{sm}(X),$$

where $\mathrm{Hilb}^r_{sm}(X)$ is the smoothable component.

3 $\underline{\mathrm{VSP}}$ versus VSP versus VPS

As in §2, let X be a smooth projective toric variety over $\mathbb{C}$. Let $S = S[X]$ be its Cox ring and let T be its graded dual ring. Both these rings are graded by $\mathrm{Pic}(X) \cong \mathbb{Z}^s$. In this section, we shall be concerned with some more notions of varieties of sums of powers that will be contrasted to the Buczyńska-Buczyński border varieties of sums of powers $\underline{\mathrm{VSP}}(F, r)$.

3.1 Basic definitions and first properties

Definition 3.1. Let $F \in T_{\mathbf{v}}$. The *(open) variety of sums of r powers* (VSP^0) of F in the Hilbert scheme $\mathrm{Hilb}^r(X)$ is defined set-theoretically as follows:

$$\mathrm{VSP}^0(F, r) = \{Z \mid \mathrm{length}(Z) = r \text{ and } I_Z \subset \mathrm{Ann}(F) \text{ is radical}\}.$$

The *(closed) variety of sums of r powers* (VSP) of F is its Zariski closure in $\mathrm{Hilb}^r(X)$. Note that $\mathrm{VSP}(F, r)$ is then a reduced closed subscheme of the irreducible smoothable component $\mathrm{Hilb}^r_{sm}(X)$.

The name is inherited from the classical case: $X = \mathbb{P}^n$ and $F \in T_d$, a degree d form. Then the smooth zero-dimensional scheme Z spanning F may be identified with a presentation of F as a sum of powers of linear forms. Although for another toric variety this might be meaningless, we shall keep this suggestive terminology.

Definition 3.2. Let $F \in T_{\mathbf{v}}$. The *variety of apolar schemes of length r to F* is

$$\mathrm{VPS}(F, r) = \{Z \in \mathrm{Hilb}^r(X) \mid I_Z \subset \mathrm{Ann}(F)\}\,.$$

So $\mathrm{VSP}^0(F, r) \subset \mathrm{VPS}(F, r) \cap \mathrm{Hilb}^r_{sm}(X) \subset \mathrm{VPS}(F, r)$. It is important to stress that $\mathrm{VPS}(F, r)$ is *not* a closed subset of the Hilbert scheme in general; see Remark 3.6 and [27, Example 1.1].

Remark 3.3. Let $\mathrm{Gor}^r(X) \subset \mathrm{Hilb}^r(X)$ be the Gorenstein locus of the Hilbert scheme, whose closed points are the zero-dimensional Gorenstein schemes embedded in X. Note that, when $r = \mathrm{crk}(F)$ is the cactus rank of F, [2, Lemma 2.3] implies that $\mathrm{VPS}(F, r) \subset \mathrm{Gor}^r(X)$.

Lemma 3.4. *We have* $\phi^{-1}_{r,X}(\mathrm{VPS}(F, r) \cap \mathrm{Hilb}^r_{sm}(X)) \subseteq \underline{\mathrm{VSP}}(F, r)$.

Proof. Let $I \in \mathrm{Slip}_{r,X}$ be such that $\phi_{r,X}(I) \in \mathrm{VPS}(F, r)$. This means that $\overline{I} \subseteq \mathrm{Ann}(F) \subset S$. So $I \subseteq \mathrm{Ann}(F)$ as well and hence $I \in \underline{\mathrm{VSP}}(F, r)$. □

The lemma motivates the following definition.

Definition 3.5 ($\underline{\mathrm{VSP}}$ **of fiber type).** A border variety of sums of powers $\underline{\mathrm{VSP}}(F, r)$ is said to be *of fiber type* if equality in Lemma 3.4 holds true.

Remark 3.6. A $\underline{\mathrm{VSP}}(F, r)$ need not be of fiber type. If the equality in Lemma 3.4 holds we get $\mathrm{VPS}(F, r) \cap \mathrm{Hilb}^r_{sm}(X) = \phi_{r,X}(\underline{\mathrm{VSP}}(F, r))$. Since the right-hand side is closed, in the circumstance that $\mathrm{VPS}(F, r) \cap \mathrm{Hilb}^r_{sm}(X)$ is not closed, equality cannot hold. [27, Example 1.1] shows this failure for a quadric in four variables.

Remark 3.7. The loci studied extensively in [27] for full-rank quadrics are related to ours, although they are different. Let $X = \mathbb{P}^n$, $T = \mathbb{C}[x_0, \ldots, x_n]$ and $Q \in T_2$ be a full-rank quadric. Our $\underline{\mathrm{VSP}}(Q, n + 1)$ embeds in the *Jelisiejew-Ranestad-Schreyer variety* $\mathrm{VPS}(Q, H)$ (using the notation of *loc. cit.*), which in turn sits inside the multigraded Hilbert scheme $\mathrm{Hilb}_S^{h_{n+1,\mathbb{P}^n}}$. Moreover, the image of their $\mathrm{VPS}^{good}(Q, H)$ under $\phi_{n+1,\mathbb{P}^n}$ is our $\mathrm{VPS}(Q, n+1)$, sitting inside $\mathrm{Hilb}^{n+1}(\mathbb{P}^n)$. When $n + 1 \leq 13$, their variety $\mathrm{VSP}^{sbl}(Q, H)$ coincides with $\underline{\mathrm{VSP}}(Q, n + 1)$ [27, Corollary 3.9].

Remark 3.6 shows that generally there is no *natural* map between $\underline{\mathrm{VSP}}(F, r)$ and $\mathrm{VPS}(F, r)$, which is perhaps one of the sources of complication and interest of the theory.

It is now time for a simple example, which leads to some generalizations; see Theorem 4.2 and Corollary 5.9.

Example 3.8. Let $X = \mathbb{P}^n$ and let $d = 3$. Let $F \in T_d$ be a concise nonwild form with $\mathrm{srk}(F) = \underline{\mathbf{rk}}(F) = \dim_{\mathbb{C}} T_1 = n + 1$. Then $\phi^{-1}_{n+1,\mathbb{P}^n}(\mathrm{VPS}(F, n + 1) \cap \mathrm{Hilb}^r_{sm}(\mathbb{P}^n)) = \underline{\mathrm{VSP}}(F, n + 1)$ is a single point. Moreover, if $\mathrm{rk}(F) = \underline{\mathbf{rk}}(F) = n + 1$ then $\mathrm{VSP}(F, n + 1) = \mathrm{VPS}(F, n + 1) \cap \mathrm{Hilb}^r_{sm}(\mathbb{P}^n)$.

Proof. There exists a saturated ideal $I = I_Z \subset \mathrm{Ann}(F)$ of a smoothable scheme Z of length $n + 1$. This ideal has necessarily the generic Hilbert function $h_{n+1,\mathbb{P}^n}$. Since I is the saturated ideal of a smoothable scheme and has the generic Hilbert function, $I \in \underline{\mathrm{VSP}}(F, n + 1)$. In fact, such an I is unique and $I = (\mathrm{Ann}(F)_2)$. For the last sentence, the equality follows from noting that the scheme Z above must be smooth. □

Let $\mathbb{G}(\mathbb{P}^{h(\mathbf{v})-1}, \mathbb{P}(T_{\mathbf{v}}))$ be the Grassmannian of $(h(\mathbf{v})-1)$-dimensional linear spaces in $\mathbb{P}(T_{\mathbf{v}})$. There exists a morphism $\rho\colon \mathrm{Slip}_{r,X} \to \mathbb{G}(\mathbb{P}^{h(\mathbf{v})-1}, \mathbb{P}(T_{\mathbf{v}}))$ [4, Lemma 3.16] defined by $\rho(I) = I_{\mathbf{v}}^{\perp}$, the degree $\mathbf{v}$ summand of the annihilator of I. Given an embedding of our toric variety $X \subset \mathbb{P}(T_{\mathbf{v}})$, let $\sigma_r(X)$ be the r-th secant variety of X inside $\mathbb{P}(T_{\mathbf{v}})$.

Proposition 3.9. *There exists a Zariski open dense subset $W_{sat} \subseteq \sigma_r(X)$ such that, if $[F] \in W_{sat}$, then $\underline{\mathrm{VSP}}(F, r)$ contains a B-saturated ideal. There exists a Zariski open dense subset W_{rad} such that if $[F] \in W_{rad}$, then $\underline{\mathrm{VSP}}(F, r)$ contains a radical ideal.*

Proof. Let
$$\mathcal{V} \subseteq \mathbb{G}(\mathbb{P}^{h(\mathbf{v})-1}, \mathbb{P}(T_{\mathbf{v}})) \times \mathbb{P}(T_{\mathbf{v}})$$
be the universal subbundle on the Grassmannian. Let $\mathcal{U} \subseteq \mathrm{Slip}_{r,X} \times \mathbb{P}(T_{\mathbf{v}})$ be the pullback of $\mathcal{V}$ along the morphism ρ. More explicitly,
$$\mathcal{U} = \{(I, F) \mid I \in \mathrm{Slip}_{r,X}, [F] \in \mathbb{P}(T_{\mathbf{v}}), F \in \rho(I)\}.$$
Then $\sigma_r(X)$ is the image of the projection π_2 onto the second factor of $\mathcal{U}$ [4, Lemma 3.16].

Let U be the locus in $\mathrm{Slip}_{r,X}$ consisting of B-saturated ideals. This is the intersection of $\mathrm{Slip}_{r,X}$ with the open locus $\mathrm{Hilb}_S^{h_{r,X},sat}$ and therefore U is open in $\mathrm{Slip}_{r,X}$. Consider the restriction $\mathcal{U}_{|U}$ and its image $\pi_2(\mathcal{U}_{|U}) \subseteq \sigma_r(X)$. If $(I, F) \in \mathcal{U}_{|U}$, then $F \in (I^{\perp})_{\mathbf{v}}$ and so $I_{\mathbf{v}} \subset \mathrm{Ann}(F)_{\mathbf{v}}$. From Proposition 2.6 we obtain $I \subset \mathrm{Ann}(F)$. Therefore $\underline{\mathrm{VSP}}(F, r)$ contains a B-saturated ideal. By Chevalley's theorem, $\pi_2(\mathcal{U}_{|U})$ is a constructible subset of $\sigma_r(X)$. Since $\mathcal{U}_{|U}$ is dense in $\mathcal{U}$, $\pi_2\colon \mathcal{U} \to \sigma_r(X)$ is surjective and $\pi_2\left(\overline{\mathcal{U}_{|U}}\right) = \overline{\pi_2(\mathcal{U}_{|U})}$, it follows that $\pi_2(\mathcal{U}_{|U})$ is a dense subset. Since the latter set is constructible, it contains an open dense subset W_{sat} with the claimed property.

Let U' be the preimage $\phi_{r,X}^{-1}(V)$, where V is the open set that appeared in the proof of Proposition 2.27. The set U' is open and nonempty. The rest of the argument exhibiting the existence of such W_{rad} is completely analogous as the previous paragraph and we omit it. □

Remark 3.10. Let $X = \mathbb{P}^n$ be equipped with the Veronese embedding $\nu_d(\mathbb{P}^n) \subset \mathbb{P}(T_d)$. If $n, d \geq 1$ and $r \leq 2$, then the claimed open subset W_{sat} in Proposition 3.9 coincides with $\sigma_r(\nu_d(\mathbb{P}^n))$ and if $r = 2$ and $d \geq 3$ it strictly contains W_{rad}.

Proof. The case $r = 1$ or $d = 1$ is clear, so let $r = 2$ and $d \geq 2$. The locus of points of border Waring rank equal to 2 is set-theoretically the union $\sigma_2(\nu_d(\mathbb{P}^n)) \setminus \nu_d(\mathbb{P}^n) = \sigma_2^0(\nu_d(\mathbb{P}^n)) \cup \tau(\nu_d(\mathbb{P}^n))$, where $\sigma_2^0(\nu_d(\mathbb{P}^n))$ is the locus of points of Waring rank equal to 2 and $\tau(\nu_d(\mathbb{P}^n))$ is the tangential variety. Up to the action of $\mathrm{PGL}(n+1, \mathbb{C})$ on $\mathbb{P}^n$, we may assume that: if $[F] \in \sigma_2^0(\nu_d(\mathbb{P}^n))$, then $F = x_0^d + x_1^d$; if $[F] \in \tau(\nu_d(\mathbb{P}^n))$, then $F = x_0 x_1^{d-1}$. In the first case the ideal $(y_0 y_1, y_2, \ldots, y_n) \subset \mathrm{Ann}(F)$ and in the second the ideal $(y_0^2, y_2, y_3, \ldots, y_n) \subset \mathrm{Ann}(F)$ are saturated ideals with the generic Hilbert function of two points in $\mathbb{P}^n$. Since $\mathrm{Hilb}^2(\mathbb{P}^n)$ is irreducible, both ideals are in $\mathrm{Slip}_{2,\mathbb{P}^n}$ which shows that $W_{sat} = \sigma_2(\nu_d(\mathbb{P}^n))$. If $d \geq 3$ these are the only ideals in $\underline{\mathrm{VSP}}(F, 2)$. The latter ideal is not radical, so $W_{rad} \neq \sigma_2(\nu_d(\mathbb{P}^n))$. □

Corollary 3.11. *The set of all $[F] \in \sigma_r(X)$ for which there exists a nonsaturated (respectively nonradical) ideal in $\underline{\mathrm{VSP}}(F, r)$ is closed.*

Proof. In the notation of Proposition 3.9, let $\mathcal{Z}$ (respectively $\mathcal{Z}'$) be the complement of $\mathcal{U}_{|U}$ (respectively $\mathcal{U}_{|U'}$) in $\mathcal{U}$. It is closed and so is the locus of $[F] \in \sigma_r(X)$ in the image of $\mathcal{Z}$ (respectively $\mathcal{Z}'$) under π_2. □

Corollary 3.12. *The following statements hold true:*

(i) If there exists an $[F] \in \sigma_r(X)$ *such that* $\underline{\mathrm{VSP}}(F, r)$ *consists only of* B*-saturated (respectively radical) ideals, then the same is true for a general element of this secant variety.*

(ii) If $\sigma_r(X) \subsetneq \mathbb{P}(T_{\mathbf{v}})$ *is nondefective or* $\sigma_r(X) = \mathbb{P}(T_{\mathbf{v}})$ *and* $r(\dim X + 1) = \dim_{\mathbb{C}} T_{\mathbf{v}}$*, then for a general element* $[F] \in \sigma_r(X)$*, one has that* $\underline{\mathrm{VSP}}(F, r)$ *consists only of* B*-saturated and radical ideals.*

Proof. (i) Let $\mathcal{Y} = \pi_2(\mathcal{Z})$ and $\mathcal{Y}' = \pi_2(\mathcal{Z}')$ be the closed subsets introduced in Corollary 3.11. If $\sigma_r(X) \setminus \mathcal{Y} \neq \emptyset$ (respectively $\sigma_r(X) \setminus \mathcal{Y}' \neq \emptyset$), then $\mathcal{Y}$ (respectively $\mathcal{Y}'$) is a proper subset, so its complement is open and dense.
(ii) Let $d = \dim X$. The dimension of $\mathcal{Z}$ and $\mathcal{Z}'$ is at most $(rd-1)+(r-1) = r(d+1)-2$. Therefore, the dimension of $\mathcal{Y}$ and $\mathcal{Y}'$ is at most $r(d+1)-2$. If $\sigma_r(X) \subsetneq \mathbb{P}(T_{\mathbf{v}})$ and is nondefective then its dimension is the expected count of parameters $r(d+1)-1$. Hence, in both cases described in the statement, the closed subsets $\mathcal{Y}$ and $\mathcal{Y}'$ are proper subsets of the secant variety. □

Remark 3.13. Let $X = \mathbb{P}^n$. Then a radical ideal $I \neq B = (y_0, \ldots, y_n) \subset S$ is B-saturated. However, this generally fails for other toric varieties. For instance, let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $B = (\alpha_0\beta_0, \alpha_0\beta_1, \alpha_1\beta_0, \alpha_1\beta_1) \subset S = \mathbb{C}[\alpha_0, \alpha_1, \beta_0, \beta_1]$. The ideal $I = (\alpha_0\beta_0, \alpha_0\beta_1, \beta_0 - \beta_1) \neq B$ is radical but not B-saturated. Its saturation $\overline{I} = (\alpha_0, \beta_0 - \beta_1)$ is the radical B-saturated ideal corresponding to the point $([0:1],[1:1]) \in X$.

Remark 3.14. Consider again the case $X = \mathbb{P}^n$ equipped with the Veronese embedding ν_d. If $\sigma_r(\nu_d(\mathbb{P}^n))$ fills the ambient space and $r(n+1) > \binom{n+d}{d}$ or $\sigma_r(\nu_d(\mathbb{P}^n))$ is defective, then it is possible that the $\underline{\mathrm{VSP}}$ of a general element of it consists only of saturated ideals as in Corollary 3.12. However, it is not always the case.

Let $n = 2, d = 2$ and $r = 3$ and take a full-rank (equivalently, concise) quadric Q. We show that $\underline{\mathrm{VSP}}(Q, 3)$ does not contain a nonsaturated ideal. Assume that $I \subset \mathrm{Ann}(Q) \subset S$ is a nonsaturated ideal. Then there exists $\ell \in \overline{I}_1$ and hence $\ell \cdot S_1 \subset I \subset \mathrm{Ann}(Q)$. It follows that $\ell \in \mathrm{Ann}(Q)$ which contradicts the assumption that Q is full-rank.

On the other hand, [27, Corollary 3.10] shows that for a general (and hence, *for every*, by Corollary 3.11) quadric Q in four variables there exists a nonsaturated ideal in $\underline{\mathrm{VSP}}(Q, 4)$.

Now we proceed to the defective cases of quadrics of rank n on $\mathbb{P}^n$. We may change coordinates so that $Q = x_0^2 + \cdots + x_{n-1}^2$. If $n = 3$ and $I \in \underline{\mathrm{VSP}}(Q, 3)$, then $I = (y_3) + K^e$ where the second summand is the extension of $K \subseteq \mathbb{C}[y_0, y_1, y_2]$, which is an apolar ideal of $Q \in \mathbb{C}[x_0, x_1, x_2]$ and it has Hilbert function $h_{3,\mathbb{P}^2}$. By the second paragraph, K is saturated and thus so is I.

Let $n = 4$ and let $K \in \underline{\mathrm{VSP}}(Q, 4) \subseteq \mathrm{Hilb}_S^{h_{4,\mathbb{P}^3}}$ be a nonsaturated ideal. Then $I = (y_4) + K^e$ is a nonsaturated apolar ideal to Q, having the generic Hilbert function $h_{4,\mathbb{P}^4}$. Since $K \in \mathrm{Slip}_{4,\mathbb{P}^3}$, it follows from [9, Proposition 3.1] that $I \in \mathrm{Slip}_{4,\mathbb{P}^4}$. We showed that for a general (and hence, for every) quadric Q in 5 variables with $\mathrm{rk}(Q) = 4$ there exists a nonsaturated ideal in $\underline{\mathrm{VSP}}(Q, 4)$.

Using a similar construction as in [4, Lemma 3.16] and Proposition 3.9, we give a sufficient condition for the closedness of the loci $\mathrm{VPS}(F, r)$. Recall that the closedness of $\mathrm{VPS}(F, r) \cap \mathrm{Hilb}^r_{sm}(X)$ is a necessary condition in order to have a $\underline{\mathrm{VSP}}(F, r)$ of fiber type.

Proposition 3.15. *Let* $r \in \mathbb{N}$*. Then there exists* $\mathbf{k} \in \mathrm{Pic}(X) = \mathbb{Z}^s$ *such that for every* $\mathbf{v} \in \mathbf{k} + \mathcal{K}$ *and every* $F \in T_{\mathbf{v}}$*, one has that* $\mathrm{VPS}(F, r)$ *is closed (possibly empty).*

Proof. By Theorem 2.19 applied to the constant Hilbert polynomial $P = r$, there exists $\mathbf{k} \in \mathbb{Z}^s$ such that for any $\mathbf{v} \in \mathbf{k} + \mathcal{K}$ and any scheme $Z \in \mathrm{Hilb}^r(X)$, one has $\mathrm{HF}(S/I_Z, \mathbf{v}) = r$. There exists a natural morphism from $\mathrm{Hilb}^r_{S_{\mathbf{k}+\mathcal{K}}}(X)$ to the Grassmannian $\mathbb{G}(\mathbb{P}^{r-1}, \mathbb{P}(T_{\mathbf{v}}))$. Composing this map with the isomorphism established in Theorem 2.20 gives the morphism $\delta : \mathrm{Hilb}^r(X) \to \mathbb{G}(\mathbb{P}^{r-1}, \mathbb{P}(T_{\mathbf{v}}))$ defined by $Z \mapsto (I_Z)^{\perp}_{\mathbf{v}}$. As in Proposition 3.9, let $\mathcal{V}$ be the tautological subbundle on the Grassmannian, $\mathcal{V} \subseteq \mathbb{G}(\mathbb{P}^{r-1}, \mathbb{P}(T_{\mathbf{v}})) \times \mathbb{P}(T_{\mathbf{v}})$. Let $\mathcal{W}$ be the pullback of $\mathcal{V}$ along the morphism δ. Then $\mathcal{W} = \{(Z, F) \mid Z \in \mathrm{Hilb}^r(X), [F] \in \mathbb{P}(T_{\mathbf{v}}), (I_Z)_{\mathbf{v}} \subset \mathrm{Ann}(F)_{\mathbf{v}}\} \subseteq \mathrm{Hilb}^r(X) \times \mathbb{P}(T_{\mathbf{v}})$. Let π_1 and π_2 be the first and second projections of the latter product. Let $[F] \in \mathbb{P}(T_{\mathbf{v}})$. Then $\mathrm{VPS}(F, r) = \pi_1(\pi_2^{-1}(F) \cap \mathcal{W})$. Note that $\mathcal{W}$ is closed, because $\mathcal{V}$ is. The latter statement can be checked locally on the Grassmannian. Hence $\pi_2^{-1}(F) \cap \mathcal{W}$ is closed. Since π_1 is a proper map, $\mathrm{VPS}(F, r)$ is closed. □

The closedness of $\mathrm{VPS}(F, r)$ prevents the existence of *bad limits*, in the terminology of [36].

Corollary 3.16. *Let $X = \mathbb{P}^n$ and let $F \in T_d$. Then, for $d \geq r - 1$, $\mathrm{VPS}(F, r)$ is closed.*

Proposition 3.17. *Assume that $F \in T_{\mathbf{v}}$ has border rank r. If $\underline{\mathrm{VSP}}(F, r)$ is of fiber type, then F is not wild.*

Proof. If $\underline{\mathrm{VSP}}(F, r)$ is of fiber type, then $\phi_{r,X}(\underline{\mathrm{VSP}}(F, r)) = \mathrm{VPS}(F, r) \cap \mathrm{Hilb}^r_{sm}(X)$ since $\phi_{r,X}$ is surjective. By the border apolarity Theorem 2.15, $\underline{\mathrm{VSP}}(F, r)$ is nonempty. Hence $\mathrm{VPS}(F, r) \cap \mathrm{Hilb}^r_{sm}(X)$ is nonempty and thus $\mathrm{srk}(F) \leq r$. On the other hand, we have $\mathrm{srk}(F) \geq \underline{\mathbf{rk}}(F) = r$. Hence $\underline{\mathbf{rk}}(F) = \mathrm{srk}(F)$. □

Corollary 3.18. *Let $\mathbf{k}$ be as in Theorem 2.19 applied to the constant Hilbert polynomial $P = r$. If $F \in T_{\mathbf{v}}$ for some $\mathbf{v} \in \mathbf{k} + \mathcal{K}$, then $\underline{\mathrm{VSP}}(F, r)$ is of fiber type. In particular, if $\underline{\mathbf{rk}}(F) = r$, then F is not wild.*

Proof. By Proposition 3.4, in order to prove that $\underline{\mathrm{VSP}}(F, r)$ is of fiber type, it is sufficient to show that $\phi_{r,X}(\underline{\mathrm{VSP}}(F, r)) \subseteq \mathrm{VPS}(F, r) \cap \mathrm{Hilb}^r_{sm}(X)$. Let $I \in \underline{\mathrm{VSP}}(F, r)$. We need to show that $\overline{I} \subset \mathrm{Ann}(F)$. By Proposition 2.6 it is enough to establish that $\overline{I}_{\mathbf{v}} \subseteq \mathrm{Ann}(F)$. By the definition of $\mathbf{k}$ we have $\overline{I}_{\mathbf{v}} = I_{\mathbf{v}}$ and hence $\overline{I}_{\mathbf{v}} \subseteq \mathrm{Ann}(F)$ which shows that $\underline{\mathrm{VSP}}(F, r)$ is of fiber type. If $\underline{\mathbf{rk}}(F) = r$, then F is not wild by Proposition 3.17. □

3.2 Birational results

Theorem 3.19. *Let $F \in T_{\mathbf{v}}$ and r be a positive integer. Then $\phi_{r,X}$ induces a bijection between the set of those irreducible components of the closure of $\mathrm{VPS}(F, r) \cap \mathrm{Hilb}^r_{sm}(X)$ that contain a scheme with the generic Hilbert function $h_{r,X}$ and the set of those irreducible components of $\underline{\mathrm{VSP}}(F, r)$ that contain a B-saturated ideal. Under this bijection, the irreducible components in correspondence are birational.*

Proof. Let $\mathcal{X}_1, \ldots, \mathcal{X}_s$ be the irreducible components of $\underline{\mathrm{VSP}}(F, r)$ that contain a B-saturated ideal and $\mathcal{X} = \bigcup_{i=1}^s \mathcal{X}_i$. Since the set of B-saturated ideals is open in $\mathrm{Slip}_{r,X}$ there are ideals $J_1, \ldots, J_s$ such that $J_i \in \mathcal{X}_j$ if and only if $i = j$. Let $\mathcal{Y}_1, \ldots, \mathcal{Y}_t$ be the irreducible components of the closure of $\mathrm{VPS}(F, r) \cap \mathrm{Hilb}^r_{sm}(X)$ that contain a scheme with the generic Hilbert function $h_{r,X}$ and $\mathcal{Y} = \bigcup_{i=1}^t \mathcal{Y}_i$. Let $U \subset \mathrm{Hilb}^r_{sm}(X)$ be the locus of schemes with the generic Hilbert function. It is the complement of the image of the closed subset $\mathrm{Slip}_{r,X} \setminus \mathrm{Hilb}^{h_{r,X},sat}_S$ under the proper map $\phi_{r,X}$. Hence U is open. Therefore, there are schemes $Z_1, \dots, Z_t$ with generic Hilbert functions such that $Z_i \in \mathcal{Y}_j$ if and only if $i = j$. Let $i \in \{1, \dots, s\}$. The image $\phi_{r,X}(\mathcal{X}_i \cap \mathrm{Hilb}_S^{h_{r,X},sat})$ is an irreducible subset of $\mathrm{VPS}(F,r) \cap \mathrm{Hilb}^r_{sm}(X)$, hence $\phi_{r,X}(\mathcal{X}_i) \subseteq \mathcal{Y}_j$ for some j.

On the other hand, for each j, the dense open subset $\mathcal{Y}_j \cap U$ of $\mathcal{Y}_j$ is contained in $\phi_{r,X}(\mathcal{X})$, therefore, each $\mathcal{Y}_j$ is the union $\bigcup_{i=1}^s \phi_{r,X}(\mathcal{X}_i) \cap \mathcal{Y}_j$. In particular, there exists i such that $\phi_{r,X}(\mathcal{X}_i)$ contains the generic point of $\mathcal{Y}_j$. Since $\phi_{r,X}(\mathcal{X}_i)$ is closed we get that $\mathcal{Y}_j \subset \phi_{r,X}(\mathcal{X}_i)$. By the above there exists k such that $\phi_{r,X}(\mathcal{X}_i) \subseteq \mathcal{Y}_k$ and we obtain $Z_j \in \mathcal{Y}_k$. Hence $j = k$ and we conclude that $\phi_{r,X}(\mathcal{X}_i) = \mathcal{Y}_j$.

Next we claim that for every i there exists j such that $\phi_{r,X}(\mathcal{X}_i) = \mathcal{Y}_j$. Let $\mathcal{Y}_j$ be such that $\phi_{r,X}(\mathcal{X}_i) \subset \mathcal{Y}_j$. By the above, there exists k such that $\phi_{r,X}(\mathcal{X}_k) = \mathcal{Y}_j$. In particular, $\phi_{r,X}(J_i) = \phi_{r,X}(K)$ for some B-saturated ideal $K \in \mathcal{X}_k$. Since $\phi_{r,X}$ is injective when restricted to B-saturated ideals we obtain $J_i = K \in \mathcal{X}_k$ and hence $i = k$.

We showed that $\phi_{r,X}$ induces a bijection between irreducible components of $\mathcal{X}$ and $\mathcal{Y}$. Furthermore, if $\phi_{r,X}(\mathcal{X}_i) = \mathcal{Y}_j$ then the induced bijective map $\phi_{r,X} \colon \mathcal{X}_i \cap \mathrm{Hilb}_S^{h_{r,X},sat} \to \mathcal{Y}_j \cap U$ is a locally closed immersion by Theorem 2.26. Hence, it is a surjective closed immersion with reduced target and thus an isomorphism. It follows that the morphism $\mathcal{X}_i \to \mathcal{Y}_j$ is birational. □

Note that the locus of elements $[F] \in \sigma_r(X) \subset \mathbb{P}(T_{\mathbf{v}})$ such that there exists a smoothable scheme Z with $I_Z \subseteq \mathrm{Ann}(F)$ and $\mathrm{HF}(S/I_Z) = h_{r,X}$ is dense by Proposition 3.9. Therefore, the sets of the irreducible components under bijection explained in Theorem 3.19 are nonempty for a general $[F] \in \sigma_r(X)$.

Proposition 3.20. *Let $F \in T_{\mathbf{v}}$ and r be a positive integer. Then* $\mathrm{VSP}(F,r)$ *is the union of those irreducible components of the closure of* $\mathrm{VPS}(F,r)$ *that contain a smooth scheme.*

Proof. We claim that the locus $W \subset \mathrm{Hilb}^r_{sm}(X)$ consisting of smooth schemes is a Zariski dense open subset. It is enough to show that the locus of smooth schemes is open in $\mathrm{Hilb}^r(X)$. Let $\mathcal{U} \subset \mathrm{Hilb}^r(X) \times X$ be the universal family and $\pi \colon \mathcal{U} \to X$ be the projection. The locus $U \subset \mathcal{U}$ of those x such that the fiber of π over $\pi(x)$ is smooth is open by [21, Theorem 12.1.6]. Therefore, its image W under π is open since π is flat and locally of finite presentation and thus, open by [19, Theorem 14.35]. One has $\mathrm{VSP}^0(F,r) = \mathrm{VPS}(F,r) \cap W$. Therefore, the irreducible components of $\mathrm{VSP}(F,r)$ are exactly those irreducible components of the closure of $\mathrm{VPS}(F,r)$ that have nonempty intersection with W. □

Using similar arguments as the ones in Theorem 3.19 yields the following.

Theorem 3.21. *Let $F \in T_{\mathbf{v}}$ and r be a positive integer. Then $\phi_{r,X}$ induces a bijection between the set of irreducible components of* $\mathrm{VSP}(F,r) \subset \mathrm{Hilb}^r_{sm}(X)$ *that contain a scheme with the generic Hilbert function $h_{r,X}$ and the set of irreducible components of* $\underline{\mathrm{VSP}}(F,r)$ *containing a B-saturated radical ideal. Under this bijection, the irreducible components in correspondence are birational.*

By Proposition 3.9 the bijection from Theorem 3.21 is nontrivial for a general $[F] \in \sigma_r(X)$.

4 Border identifiability and multigraded regularity

We establish a criterion for border identifiability employing multigraded regularity, defined in §2.4. With this notion of regularity, Maclagan and Smith proved the following.

Proposition 4.1 ([30, Proposition 6.7]). *Let $Z \subset X$ be a zero-dimensional subscheme of length r and let I_Z be its B-saturated ideal in the Cox ring S. Then $\mathbf{m} \in \operatorname{reg}(S/I_Z)$ if and only if the space of forms vanishing on Z has codimension r in the space of forms of multidegree $\mathbf{m}$ in S.*

We are now ready to state our criterion to decide border identifiability.

Theorem 4.2. *Let $X \subset \mathbb{P}(T_{\mathbf{v}})$ and $r = \underline{\mathbf{rk}}_X(F)$ be the border rank of $F \in T_{\mathbf{v}}$. Suppose that there exists $\mathbf{u} \in \mathcal{K}$ such that*

$$\operatorname{HF}(S/\operatorname{Ann}(F), \mathbf{u}) = \operatorname{HF}(S/\operatorname{Ann}(F), \mathbf{u} + \mathbf{c}_1 + \cdots + \mathbf{c}_l) = r.$$

If there exists a B-saturated ideal $I \in \underline{\mathrm{VSP}}(F, r)$, then $\underline{\mathrm{VSP}}(F, r) = \{I\}$.

Proof. Let $\mathbf{e} = \mathbf{c}_1 + \cdots + \mathbf{c}_l$. We claim that for every choice of $\varepsilon_i \in \{0, 1\}$ for $i \in \{1, 2, \ldots, l\}$ we have $\operatorname{HF}(S/I, \mathbf{u} + \varepsilon_1\mathbf{c}_1 + \cdots + \varepsilon_l\mathbf{c}_l) = r$. Indeed, since I is B-saturated, for each i there is a nonzerodivisor on S/I of degree $\mathbf{c}_i$ [30, Proposition 3.1]. Thus $\operatorname{HF}(S/I, \mathbf{v} + \mathbf{c}_i) \geq \operatorname{HF}(S/I, \mathbf{v})$ for every $\mathbf{v} \in \mathbb{Z}^s$. Our claim follows from the assumed equality of Hilbert functions. By Proposition 4.1, we have $\mathbf{u} + \varepsilon_1\mathbf{c}_1 + \cdots + \varepsilon_l\mathbf{c}_l \in \operatorname{reg}(S/I)$. We claim that $\mathbf{u} + \mathbf{e} \in \operatorname{reg}(I)$. Since $I \subseteq S$ and S is an integral domain, we have $H^0_B(I)_{\mathbf{p}} = 0$ for every $\mathbf{p} \in \mathbb{Z}^s$. According to Definition 2.18, we need to show that $H^i_B(I)_{\mathbf{p}} = 0$ for every $i \geq 1$ and all $\mathbf{p} \in \bigcup(\mathbf{u} + \mathbf{e} - \lambda_1\mathbf{c}_1 - \cdots - \lambda_l\mathbf{c}_l + \mathcal{K})$ where the union is over all $\lambda_1, \ldots, \lambda_l \in \mathbb{N}$ such that $\lambda_1 + \cdots + \lambda_l = i - 1$. [30, Corollary 3.6] states that if $\mathbf{u} \in \mathcal{K}$, then the local cohomology group $H^i_B(S)_{\mathbf{u}}$ vanishes. Now, using the long exact sequence of local cohomology groups associated to the short exact sequence $0 \to I \to S \to S/I \to 0$ we conclude that for every $\mathbf{u} \in \mathcal{K}$ and every $i \in \mathbb{N}$ we have

$$H^i_B(S/I)_{\mathbf{u}} \cong H^{i+1}_B(I)_{\mathbf{u}}. \tag{1}$$

Assume first that $i = 1$ and $\mathbf{w} \in \mathcal{K}$. Since $\lambda_j \in \mathbb{N}$, the only zero linear combination is when they are all zero; so we need to show that $H^1_B(I)_{\mathbf{u}+\mathbf{e}+\mathbf{w}} = 0$. By (1) we have $H^1_B(I)_{\mathbf{u}+\mathbf{e}+\mathbf{w}} \cong H^0_B(S/I)_{\mathbf{u}+\mathbf{e}+\mathbf{w}}$ which is zero since $\mathbf{u} \in \operatorname{reg}(S/I)$ and $\mathbf{e}+\mathbf{w}-\mathbf{c}_1 \in \mathcal{K}$. Let $\mathbf{w} \in \mathcal{K}$, $i \geq 2$ and $\lambda_1, \ldots, \lambda_l$ be nonnegative integers whose sum is $i - 1$. We have to prove that $H^i_B(I)_{\mathbf{u}+\mathbf{e}-\lambda_1\mathbf{c}_1-\cdots-\lambda_l\mathbf{c}_l+\mathbf{w}} = 0$. Equivalently, by (1), this amounts to check $H^{i-1}_B(S/I)_{\mathbf{u}+\mathbf{e}-\lambda_1\mathbf{c}_1-\cdots-\lambda_l\mathbf{c}_l+\mathbf{w}} = 0$. Up to permuting the vectors $\mathbf{c}_j$, we may assume that $\lambda_1 \geq 1$. We have

$$\mathbf{u} + \mathbf{e} - \lambda_1\mathbf{c}_1 - \cdots - \lambda_l\mathbf{c}_l + \mathbf{w} = (\mathbf{u} + \mathbf{e} - \mathbf{c}_1) - (\lambda_1 - 1)\mathbf{c}_1 - \cdots - \lambda_l\mathbf{c}_l + \mathbf{w}$$

which shows that $H^{i-1}_B(S/I)_{\mathbf{u}+\mathbf{e}-\lambda_1\mathbf{c}_1-\cdots-\lambda_l\mathbf{c}_l+\mathbf{w}} = 0$ since $\mathbf{u} + \mathbf{e} - \mathbf{c}_1 \in \operatorname{reg}(S/I)$.

Let $J \in \underline{\mathrm{VSP}}(F, r)$. Since $\operatorname{Ann}(F)_{\mathbf{u}+\mathbf{e}} = I_{\mathbf{u}+\mathbf{e}}$, we have $J_{\mathbf{u}+\mathbf{e}} = I_{\mathbf{u}+\mathbf{e}}$. It follows that J contains $(I_{\mathbf{u}+\mathbf{e}})$. The latter ideal is equal to $(I_{|\mathbf{u}+\mathbf{e}+\mathcal{K}})$ by [30, Theorem 5.4] since $\mathbf{u} + \mathbf{e} \in \operatorname{reg}(I)$. Due to the equality of Hilbert functions we have $J_{|\mathbf{u}+\mathbf{e}+\mathcal{K}} = I_{|\mathbf{u}+\mathbf{e}+\mathcal{K}}$. Hence, by [30, Lemma 6.8] we obtain $\overline{J} = \overline{I} = I$. In particular, $J \subseteq I$ which implies that $I = J$. □

This result immediately yields the following corollary, which we also prove in an alternative elementary way.

Corollary 4.3. *Let $X = \mathbb{P}^n$ and $S = \mathbb{C}[y_0, \ldots, y_n]$ with $\deg(y_i) = 1$. Let $F \in T_d$ with $\underline{\mathbf{rk}}(F) = r$ and suppose there exists $a \in \mathbb{Z}$ such that $\operatorname{HF}(S/\operatorname{Ann}(F), a) = \operatorname{HF}(S/\operatorname{Ann}(F), a + 1) = r$. If $\underline{\mathrm{VSP}}(F, r)$ contains a saturated ideal I, then $\underline{\mathrm{VSP}}(F, r) = \{I\}$.*

Alternate proof. For every $J \in \underline{\mathrm{VSP}}(F,r)$ we have $I_a = J_a$ and $I_{a+1} = J_{a+1}$. In particular, $(I_{a+1}) \subseteq J$. We claim that $(I_{a+1}) = I_{\geq a+1}$. Indeed, if c is the smallest integer such that $\mathrm{HF}(S/I, c) = r$, then by [24, Theorem 1.69] the ideal I has no minimal generators in degree greater than $c+1$. The claim follows since $c \leq a$. The ideals I and J have the same Hilbert function so $I = \overline{I} = \overline{J}$. Hence $J = I$ by the equality of their Hilbert functions. □

Corollary 4.4. *Let $d = 2s+1$ and $r = \binom{n+s}{s}$. Then a general $[F] \in \sigma_r(\nu_d(\mathbb{P}^n))$ is border identifiable.*

Proof. By [24, Lemma 1.17], a general $[F] \in \sigma_r(\nu_d(\mathbb{P}^n))$ has apolar ideal $\mathrm{Ann}(F)$ with Hilbert function

$$\mathrm{HF}(S/\mathrm{Ann}(F), k) = \min\{\dim_{\mathbb{C}} S_k, \dim_{\mathbb{C}} S_{d-k}\}, \text{ for } 0 \leq k \leq d.$$

Thus $\mathrm{HF}(S/\mathrm{Ann}(F), s) = \mathrm{HF}(S/\mathrm{Ann}(F), s+1) = r$. Proposition 3.9 and Corollary 4.3 prove the statement. □

Lemma 4.5. *Let $F \in T_{\mathbf{v}}$ and suppose that $\max\{\mathrm{HF}(S/\mathrm{Ann}(F), \mathbf{v}) \mid \mathbf{v} \in \mathbb{Z}\} = r$. Let $\mathbf{b}_1, \ldots, \mathbf{b}_k$ be the degrees of the minimal generators of the irrelevant ideal B. If there exist $\mathbf{0} \neq \mathbf{u}, \mathbf{u}' \in \mathbb{Z}^s$, with $\mathbf{u}' - \mathbf{u} - \mathbf{b}_i \in \mathcal{K}$ for every $i = 1, 2, \ldots, k$ such that $\mathrm{HF}(S/\mathrm{Ann}(F), \mathbf{u}) < \dim_{\mathbb{C}} S_{\mathbf{u}} \leq r$ and $\mathrm{HF}(S/\mathrm{Ann}(F), \mathbf{u}') = r$, then $\underline{\mathrm{VSP}}(F,r)$ consists only of nonsaturated ideals.*

Proof. By assumption, there exists $0 \neq \omega \in \mathrm{Ann}(F)_{\mathbf{u}}$. For every $I \in \underline{\mathrm{VSP}}(F,r)$ and every $i \in \{1, \ldots, k\}$ we have $S_{\mathbf{u}'-\mathbf{u}-\mathbf{b}_i} S_{\mathbf{b}_i} \omega \in \mathrm{Ann}(F)_{\mathbf{u}'} = I_{\mathbf{u}'}$. Let $f_i \in S_{\mathbf{u}'-\mathbf{u}-\mathbf{b}_i}$ be a nonzerodivisor on $S/\overline{I}$ (see [30, Proposition 3.1]). We conclude that $S_{\mathbf{b}_i}\omega \in \overline{I}$ for every i and therefore $\omega \in \overline{I}$. In particular $I \neq \overline{I}$. □

Example 4.6. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ with Cox ring $S = \mathbb{C}[\alpha_0, \alpha_1, \beta_0, \beta_1]$, where $\deg(\alpha_0) = \deg(\alpha_1) = (1,0)$ and $\deg(\beta_0) = \deg(\beta_1) = (0,1)$. Let $T = \mathbb{C}[a_0, a_1, b_0, b_1]$ be the graded dual ring. Consider $F = a_0^4 b_1^4 + a_1^4 b_0^4 + a_1^4 b_1^4 \in T_{(4,4)}$ and let $r = 3$. One verifies that $\mathrm{HF}(S/\mathrm{Ann}(F), (2,0)) = 2 < 3 = \dim_{\mathbb{C}} S_{(2,0)}$, but $\mathrm{HF}(S/\mathrm{Ann}(F), (3,1)) = 3$. Finally observe that $(\alpha_0\alpha_1, \alpha_0\beta_0, \beta_0\beta_1)$ is a radical and B-saturated ideal of S contained in $\mathrm{Ann}(F)$ with Hilbert polynomial 3. Hence $\underline{\mathrm{VSP}}(F,3)$ is nonempty and by Lemma 4.5 it consists only of nonsaturated ideals.

5 Minimal border rank and wildness

5.1 Concise minimal border rank tensors

We start with recalling a useful result about multigraded Hilbert functions of zero-dimensional schemes inside products of projective spaces.

Proposition 5.1 ([40, Proposition 1.9], [33, Lemma 4.24])**.** *Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ and let $Z \subset X$ be a zero-dimensional scheme with B-saturated ideal I_Z. Let $\mathbf{e}_j \in \mathbb{N}^s \subset \mathrm{Pic}(X)$ be the j-th standard basis vector. Then:*

(i) for all $\mathbf{v} \in \mathbb{N}^s$ and all $1 \leq j \leq s$, one has $\mathrm{HF}(S/I_Z, \mathbf{v}) \leq \mathrm{HF}(S/I_Z, \mathbf{v} + \mathbf{e}_j)$;

(ii) if $\mathrm{HF}(S/I_Z, \mathbf{v}) = \mathrm{HF}(S/I_Z, \mathbf{v} + \mathbf{e}_j)$, then $\mathrm{HF}(S/I_Z, \mathbf{v} + \mathbf{e}_j) = \mathrm{HF}(S/I_Z, \mathbf{v} + 2\mathbf{e}_j)$;

(iii) $\mathrm{HF}(S/I_Z, \mathbf{v}) \leq \mathrm{length}(Z)$ for all $\mathbf{v} \in \mathbb{N}^s$.

Let $X = \mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$, S be its Cox ring and T the graded dual of S. Then $T_{\mathbf{1}} \cong \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, where $\mathbf{1} = (1,1,1)$.

Lemma 5.2. *Let K be a B-saturated ideal of S such that $\mathrm{HF}(S/K,(a,b,c)) = m$ for every $(a,b,c) \in \{0,1\}^3$ with $a+b+c \in \{1,2\}$. If $\mathrm{HF}(S/K,\mathbf{1}) = m$, then S/K has generic Hilbert function $h_{m,X}$.*

Proof. Let $(a,b,c) \in \mathbb{N}^3 \setminus \{(0,0,0)\}$. We need to show that $\mathrm{HF}(S/K,(a,b,c)) = m$. Assume first that only one of $\{a,b,c\}$ is positive. We may and do assume that it is a. From $\mathrm{HF}(S/K,(0,1,0)) = \mathrm{HF}(S/K,(1,1,0)) = m$ we conclude using Proposition 5.1(ii) that $\mathrm{HF}(S/K,(a,1,0)) = m$. From Proposition 5.1(i), we derive

$$m = \mathrm{HF}(S/K,(1,0,0)) \leq \mathrm{HF}(S/K,(a,0,0)) \leq \mathrm{HF}(S/K,(a,1,0)) = m,$$

which implies that $\mathrm{HF}(S/K,(a,0,0)) = m$.

In what follows, we repeatedly use Proposition 5.1(i) and (ii). Assume that $a,b > 0$ and $c = 0$. We have already established that $\mathrm{HF}(S/K,(a,0,0)) = \mathrm{HF}(S/K,(a,1,0)) = m$. Thus, we obtain $\mathrm{HF}(S/K,(a,b,0)) = m$.

Finally, assume that $a,b,c > 0$. It is enough to show that $\mathrm{HF}(S/K,(a,b,1)) = m$ to conclude that $\mathrm{HF}(S/K,(a,b,c)) = m$. We have $\mathrm{HF}(S/K,(0,b,1)) = m$, so it is sufficient to show that $\mathrm{HF}(S/K,(1,b,1)) = m$. This follows since $\mathrm{HF}(S/K,(1,0,1)) = \mathrm{HF}(S/K,\mathbf{1}) = m$. □

Theorem 5.3. *Let $F \in T_{\mathbf{1}}$ be concise and of minimal border rank, i.e., $\underline{\mathbf{rk}}(F) = m$. Let $I = (\mathrm{Ann}(F)_{(1,1,0)}) + (\mathrm{Ann}(F)_{(1,0,1)}) + (\mathrm{Ann}(F)_{(0,1,1)}) \subset S$ and $K = \overline{I}$. Then the following statements hold:*

(i) If $\mathrm{HF}(S/I,\mathbf{1}) \neq m$, then F is wild.

(ii) If $\mathrm{HF}(S/I,\mathbf{1}) = m$, then F is not wild if and only if $I_{(a,b,c)} = K_{(a,b,c)}$ for every $(a,b,c) \in \mathcal{S}$, where $\mathcal{S} = \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$.

Proof. (i) Since $\underline{\mathbf{rk}}(F) = m$, it follows from the border apolarity Theorem 2.15 (see also [4, Theorem 5.5]) that $\mathrm{HF}(S/I,\mathbf{1}) \geq m$. If $\mathrm{HF}(S/I,\mathbf{1}) > m$, then, in the terminology of [25], the tensor F is not 111-sharp. Therefore, it is wild by [25, Theorem 9.2].

(ii) We then assume that F is 111-sharp, i.e., $\mathrm{HF}(S/I,\mathbf{1}) = m$. Suppose first that F is not wild. According to Definition 2.10, this implies that its cactus rank satisfies $\mathrm{crk}(F) \leq \mathrm{srk}(F) = \underline{\mathbf{rk}}(F) = m$. By the cactus apolarity lemma (see Definition 2.11), there is a B-saturated homogeneous ideal $J \subseteq \mathrm{Ann}(F) \subset S$ having the multigraded Hilbert polynomial equal to $\mathrm{crk}(F)$. By Proposition 5.1(iii), for every $(a,b,c) \in \mathbb{N}^3$ we have $\mathrm{HF}(S/J,(a,b,c)) \leq \mathrm{crk}(F)$. If $(a,b,c) \in \{0,1\}^3$ with $a+b+c \in \{1,2\}$, then from

$$m = \mathrm{HF}(S/\mathrm{Ann}(F),(a,b,c)) \leq \mathrm{HF}(S/J,(a,b,c)) \leq \mathrm{crk}(F) \leq m$$

we conclude that $\mathrm{crk}(F) = m$ and $J_{(a,b,c)} = \mathrm{Ann}(F)_{(a,b,c)} = I_{(a,b,c)}$. Hence $I \subseteq J$ and therefore $K \subseteq J$. Using Proposition 5.1(i)(iii) we get the inequalities

$$m = \mathrm{HF}(S/J,(1,1,0)) \leq \mathrm{HF}(S/J,\mathbf{1}) \leq \mathrm{crk}(F) = m,$$

from which we conclude that $\mathrm{HF}(S/J,\mathbf{1}) = m$. It follows that $I_{(a,b,c)} = J_{(a,b,c)}$ for every $(a,b,c) \in \mathcal{S}$, which implies that for any such (a,b,c) we have also $I_{(a,b,c)} = K_{(a,b,c)}$.

For the converse, assume that $I_{(a,b,c)} = K_{(a,b,c)}$ for every $(a,b,c) \in \mathcal{S}$. By Lemma 5.2, the algebra S/K has generic Hilbert function $h_{m,X}$.

Let $J \in \underline{\mathrm{VSP}}(F,m)$. We have $J_{(a,b,c)} = I_{(a,b,c)}$ for every $(a,b,c) \in \mathcal{S}$. Therefore, $I \subseteq J$ and so $K = \overline{I} \subset \overline{J}$. However, K and J have the same multigraded Hilbert polynomial, so $\overline{J} = \overline{K} = K$. Hence $J \subseteq \overline{J} = K$. Since these two have the same Hilbert function, $J = K$. This implies that $\underline{\mathrm{VSP}}(F,m) = \{K\}$. In particular, $\mathrm{Proj}(S/K)$ is a smoothable scheme. It follows that F has minimal smoothable rank, i.e., $\mathrm{srk}(F) = m = \underline{\mathbf{rk}}(F)$. Thus, F is not wild. ☐

Corollary 5.4. *If F is a nonwild concise minimal border rank tensor in $T_{\mathbf{1}}$, then $\underline{\mathrm{VSP}}(F,m) = \{K\}$ where K is the saturation of $I = (\mathrm{Ann}(F)_{(1,1,0)}) + (\mathrm{Ann}(F)_{1,0,1}) + (\mathrm{Ann}(F)_{(0,1,1)})$.*

Proof. By Theorem 5.3, $I_{(a,b,c)} = K_{(a,b,c)}$ for every $(a,b,c) \in \mathcal{S}$. Therefore, $\underline{\mathrm{VSP}}(F,m) = \{K\}$ by the argument presented in the last paragraph of the proof of Theorem 5.3. ☐

5.2 Tensors of minimal border rank three

In this section we answer a question posed by Buczyńska and Buczyński [4, §5.2] finding the description of $\underline{\mathrm{VSP}}(F,3)$ for all minimal border rank three tensors F. Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ and let $\mathbf{1} = (1,\ldots,1) \in \mathbb{Z}^s$, which corresponds to a very ample line bundle D. Then the closed embedding induced by $\mathcal{L} = \mathcal{O}_X(D)$ is the *Segre embedding* $\mathrm{Seg}(X) \subset \mathbb{P}(T_{\mathbf{1}})$. Given $[G] \in \mathrm{Seg}(X)$, denote by $\widehat{T}_{[G]}\mathrm{Seg}(X)$ the tangent space at G to the affine cone of $\mathrm{Seg}(X)$.

Theorem 5.5. *Let $X = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ and let F be a border rank three concise tensor in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \cong T_{\mathbf{1}}$. The variety $\underline{\mathrm{VSP}}(F,3)$ is a single point, unless $F = G' + H'$ where $H' \in \widehat{T}_{[H]}\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$, $G' \in \widehat{T}_{[G]}\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$ with $[G],[H]$ being two distinct points on a line in $\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$. In the latter case, F is wild and $\underline{\mathrm{VSP}}(F,3)$ is isomorphic to $\mathbb{P}^3$.*

Proof. By [6, Theorem 1.2], it is sufficient to consider the case when F is one of the four tensors whose normal forms (i)–(iv) are classified by Buczyński and Landsberg [6, pages 477–478]. Let I be the ideal generated by the degree $(1,1,0),(1,0,1)$ and $(0,1,1)$ parts of $\mathrm{Ann}(F)$ and let $K = \overline{I}$. If F is one of the tensors described in (i)–(iii) we conclude using Theorem 5.3 that F is not wild. It follows from Corollary 5.4 that $\underline{\mathrm{VSP}}(F,3) = \{K\}$.

Assume that F has normal form (iv), i.e., $F = a_2 \otimes b_1 \otimes c_2 + a_2 \otimes b_2 \otimes c_1 + a_1 \otimes b_1 \otimes c_3 + a_1 \otimes b_3 \otimes c_1 + a_3 \otimes b_1 \otimes c_1$. It follows from Theorem 5.3 that F is wild (this was already known, see [3, proof of Proposition 2.4]). We show that $\underline{\mathrm{VSP}}(F,3) \cong \mathbb{P}^3$. Let $\alpha_i, \beta_j, \gamma_k$ be the generators of S dual to a_i, b_j, c_k respectively.

As before, we consider the ideal $I = (\mathrm{Ann}(F)_{(1,1,0)}) + (\mathrm{Ann}(F)_{(1,0,1)}) + (\mathrm{Ann}(F)_{(0,1,1)})$. We have $\mathrm{HF}(S/I,(2,1,0)) = 3$ and $(I\colon (\beta_1,\beta_2,\beta_3))_{(2,0,0)} = \langle \alpha_1\alpha_3, \alpha_2\alpha_3, \alpha_3^2\rangle$. It follows that if $J \in \underline{\mathrm{VSP}}(F,3)$, then $J_{(2,0,0)} \subseteq \langle \alpha_1\alpha_3, \alpha_2\alpha_3, \alpha_3^2\rangle$. Since the latter vector subspace has codimension 3 in $S_{(2,0,0)}$, we conclude that the containment is an equality. Therefore, $J \supseteq I'$ where

$$I' = I + (\alpha_1\alpha_3, \alpha_2\alpha_3, \alpha_3^2).$$

Furthermore, $\mathrm{HF}(S/I',(3,0,0)) = 4$. Therefore, any ideal $J \in \underline{\mathrm{VSP}}(F,3)$ is such that there exists a unique cubic $C_J \in \langle \alpha_1^3, \alpha_1^2\alpha_2, \alpha_1\alpha_2^2, \alpha_2^3\rangle$ up to scaling that is the generator of the vector space $(J/(\alpha_3))_{(3,0,0)}$. Such a cubic may be regarded as a point $C_J \in \mathbb{P}(\langle \alpha_1^3, \alpha_1^2\alpha_2, \alpha_1\alpha_2^2, \alpha_2^3\rangle)$. This defines a morphism

$$\psi_{(3,0,0)} : \underline{\mathrm{VSP}}(F,3) \longrightarrow \mathbb{P}(\langle \alpha_1^3, \alpha_1^2\alpha_2, \alpha_1\alpha_2^2, \alpha_2^3\rangle) \cong \mathbb{P}^3,$$

$$J \longmapsto [C_J].$$

This morphism is clearly injective on $\mathbb{C}$-points. It is also surjective on $\mathbb{C}$-points. To see this, first notice that $\psi_{(3,0,0)}$ is projective and so closed. Thus it is enough to show its image contains a general point of $\mathbb{P}(\langle \alpha_1^3, \alpha_1^2\alpha_2, \alpha_1\alpha_2^2, \alpha_2^3\rangle)$.

Let $\ell_1 = e_1a_1 + e_2a_2$ and $\ell_2 = f_1a_1 + f_2a_2$ be two linearly independent vectors in $\mathbb{C}^3$ and let $\ell_3 \in \langle \ell_1, \ell_2\rangle$. Up to scaling ℓ_1 and ℓ_2, we may assume $\ell_3 = \ell_1 + \ell_2$. We now exhibit a minimal border rank decomposition for F utilising ℓ_1 and ℓ_2. For any $t \in \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$, consider the expression

$$G_t = \frac{1}{t}\left[(ta_3 - \ell_3)\otimes b_1 \otimes c_1 + \ell_1 \otimes (b_1 + t\cdot\delta_2 b_2 + t\cdot \delta_3 b_3)\otimes (c_1 + t\cdot\rho_2 c_2 + t\cdot\rho_3 c_3)\right] +$$

$$+\frac{1}{t}\left[\ell_2 \otimes (b_1 + t\cdot\tau_2 b_2 + t\cdot\tau_3 b_3)\otimes (c_1 + t\cdot \eta_2 c_2 + t\cdot\eta_3 c_3)\right].$$

Given $f_1, f_2, e_1, e_2 \in \mathbb{C}$, one has that $\lim_{t\to 0} G_t = F$ if and only if there exist complex values for the parameters $\delta_2, \delta_3, \ldots, \eta_2, \eta_3$ such that the following relations are satisfied

$$\begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix} \cdot \begin{pmatrix} \delta_2 & \delta_3 & \rho_2 & \rho_3 \\ \tau_2 & \tau_3 & \eta_2 & \eta_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Since ℓ_1 and ℓ_2 are linearly independent, such parameters' values do exist and are unique. Using `Macaulay2`, we find that this border rank decomposition corresponds to an ideal $J \in \underline{\mathrm{VSP}}(F,3)$ whose unique generator modulo the ideal (α_3) in degree $(3,0,0)$ is the cubic

$$C_J = (e_2\alpha_1 - e_1\alpha_2)(f_2\alpha_1 - f_1\alpha_2)((e_1+f_1)\alpha_2 - (e_2+f_2)\alpha_1).$$

Up to scaling, this cubic form is a general point in $\mathbb{P}(\langle \alpha_1^3, \alpha_1^2\alpha_2, \alpha_1\alpha_2^2, \alpha_2^3\rangle) \cong \mathbb{P}^3$, because any two of the three linear forms are linearly independent. Hence $\psi_{(3,0,0)}$ is surjective. Since the morphism $\psi_{(3,0,0)}$ is bijective on $\mathbb{C}$-points and the target is a normal scheme, Zariski's Main Theorem implies that $\psi_{(3,0,0)}$ is an isomorphism of $\mathbb{C}$-schemes. □

5.3 Wildness

A special and simple feature of the structure of $\underline{\mathrm{VSP}}$'s for wild elements (see Definition 2.10) is that their closed points are nonsaturated ideals.

Lemma 5.6. *Let $F \in T_{\mathbf{v}}$ be wild with $\underline{\mathbf{rk}}(F) = r$. Then any ideal $J \in \underline{\mathrm{VSP}}(F, r)$ is nonsaturated.*

Proof. On the contrary, suppose there exists a B-saturated ideal $J \in \underline{\mathrm{VSP}}(F, r)$. By Proposition 2.27 the scheme Z defined by J is smoothable. One has $\mathrm{srk}(F) \leq \mathrm{length}(Z) = r$, a contradiction. □

The previous lemma cannot be used as a criterion for wildness, because its converse is false even for $X = \mathbb{P}^2$. The following example illustrates this.

Example 5.7. Let $F = x_0x_1^2x_2^3 \in T_6$. Then $\underline{\mathbf{rk}}(F) = \mathrm{srk}(F) = \mathrm{crk}(F) = 6$. In particular, F is not wild. However, $\underline{\mathrm{VSP}}(F, 6)$ consists of a unique nonsaturated ideal.

Proof. Since $\mathrm{Hilb}^r(\mathbb{P}^2)$ is irreducible, it follows that $\mathrm{srk}(F) = \mathrm{crk}(F)$. We have $\mathrm{Ann}(F) = (y_0^2, y_1^3, y_2^4)$ and its Hilbert function is $\mathrm{HF}(S/\mathrm{Ann}(F)) = 1\ \ 3\ \ 5\ \ 6\ \ 5\ \ 3\ \ 1$. As a result, we get $6 \leq \underline{\mathbf{rk}}(F) \leq \mathrm{srk}(F) = \mathrm{crk}(F)$. From the containment $(y_0^2, y_1^3) \subset \mathrm{Ann}(F) \subset S$ we deduce that $\mathrm{crk}(F) = 6$ and therefore $\underline{\mathbf{rk}}(F) = \mathrm{srk}(F) = \mathrm{crk}(F) = 6$. Any ideal $J \in \underline{\mathrm{VSP}}(F, 6)$ satisfies $J_{\leq 2} = 0$ and $J_{\geq 3} = (y_0^2, y_1^3)_{\geq 3}$ and hence $(y_0^2, y_1^3)_{\geq 3}$ is a unique such ideal. Note that it is not saturated. □

In order to find an equivalence between the presence of nonsaturated ideals in $\underline{\mathrm{VSP}}$ and wildness of F, we then have to impose further assumptions.

Proposition 5.8. *Let $X = \mathbb{P}^n$. Let $F \in T_d$ be such that there exists $a \in \mathbb{Z}$ with $\mathrm{HF}(S/\mathrm{Ann}(F), a) = \mathrm{HF}(S/\mathrm{Ann}(F), a+1) = r = \underline{\mathbf{rk}}(F)$, and, for all $k < a$, one has $\mathrm{HF}(S/\mathrm{Ann}(F), k) = h_{r,\mathbb{P}^n}(k)$. Then F is wild if and only if $\underline{\mathrm{VSP}}(F, r)$ contains a nonsaturated ideal. Furthermore, in that case $\underline{\mathrm{VSP}}(F, r)$ consists only of nonsaturated ideals.*

Proof. If F is wild, then $\underline{\mathrm{VSP}}(F, r) \neq \emptyset$ contains only nonsaturated ideals by Lemma 5.6. For the converse, since $\underline{\mathrm{VSP}}(F, r)$ contains a nonsaturated ideal, by Theorem 4.2 it cannot contain any saturated ideal. Suppose now by contradiction that $\mathrm{srk}(F) = \underline{\mathbf{rk}}(F) = r$. Then there exists a B-saturated ideal $J \subset \mathrm{Ann}(F)$ of a smoothable scheme with Hilbert polynomial equal to r. By assumption on the Hilbert function of $\mathrm{Ann}(F)$, J has the generic Hilbert function $h_{r,\mathbb{P}^n}$. Hence $J \in \mathrm{Slip}_{r,\mathbb{P}^n}$. Therefore $J \in \underline{\mathrm{VSP}}(F, r)$, a contradiction. Thus F is wild. □

This result along with a characterization of wildness for minimal border rank forms in terms of Hessians yields the following corollary.

Corollary 5.9. *Let $X = \mathbb{P}^n$. Let $d = 3$ or $d \geq n+2$ and $F \in T_d$ be a minimal border rank concise form, i.e., $\underline{\mathbf{rk}}(F) = n+1$. Then $\mathrm{Hess}(F) \neq 0$ if and only if $\underline{\mathrm{VSP}}(F, n+1)$ consists of a unique saturated ideal. When this holds, the unique saturated ideal is $(\mathrm{Ann}(F)_2)$.*

Proof. By [23, Theorem 4.9] $\mathrm{Hess}(F) \neq 0$ if and only if F is not wild. We claim that in both cases considered in the statement $\mathrm{HF}(S/\mathrm{Ann}(F), 2) = n+1$. If $d = 3$ it is clear. If $d \geq n+2$ we obtain the claim by using Macaulay's bound applied to $\mathrm{HF}(S/\mathrm{Ann}(F), d-2)$ and the symmetry of the Hilbert function of $\mathrm{Ann}(F)$. If F is not wild, then $\underline{\mathrm{VSP}}(F, n+1) \neq \emptyset$ contains only saturated ideals by Proposition 5.8. Since, in particular, $\underline{\mathrm{VSP}}(F, n+1)$ contains a saturated ideal, then by Corollary 4.3 it contains a unique one. For the converse, suppose $\underline{\mathrm{VSP}}(F, n+1)$ contains a unique saturated ideal. Again by Proposition 5.8, F is not wild. □

6 Botany of forms and their $\underline{\mathrm{VSP}}$'s

In this section, we assume $X = \mathbb{P}^n$, so S and T are polynomial rings with the standard grading. The surjective morphism $\phi_{r,\mathbb{P}^n} : \mathrm{Slip}_{r,\mathbb{P}^n} \to \mathrm{Hilb}^r_{sm}(\mathbb{P}^n)$ is given on closed points by $J \mapsto \mathrm{Proj}(S/J)$.

6.1 Binary forms

Here let $n = 1$. It follows from [22, Lemma 4.1] that $\mathrm{Hilb}_S^{h_{r,\mathbb{P}^1}}$ and $\mathrm{Hilb}^r(\mathbb{P}^1)$ represent the same functor. In particular, $\mathrm{Hilb}_S^{h_{r,\mathbb{P}^1}} \cong \mathrm{Hilb}^r(\mathbb{P}^1) \cong \mathbb{P}^r$ where the latter isomorphism is well-known [15, Proposition 7.3.3 and Example 7.1.3].

Proposition 6.1. *Let $F \in T_d$ be a binary form with border rank $\underline{\mathbf{rk}}(F) = r$. Then $r = \mathrm{crk}(F) = \mathrm{srk}(F)$ and $\mathrm{VPS}(F, r) \cong \underline{\mathrm{VSP}}(F, r)$ where the isomorphism is induced by the isomorphism $\phi_{r,\mathbb{P}^1}$ and either $\mathrm{VPS}(F, r) \cong \mathbb{P}^1$ or it is a point. If $\mathrm{VPS}(F, r) \cong \mathbb{P}^1$, then d is even. Moreover, if $\mathrm{rk}(F) = r$ then $\mathrm{VSP}(F, r) \cong \mathrm{VPS}(F, r)$.*

Proof. By a theorem of Sylvester, for any $F \in T_d$ its annihilator is a complete intersection $\mathrm{Ann}(F) = (g_1, g_2) \subset S$ with $\deg(g_1) \leq \deg(g_2)$ and $\deg(g_1) + \deg(g_2) = d + 2$. It is clear that $r := \min\{\deg(g_1), \deg(g_2)\} = \deg(g_1) = \mathrm{crk}(F)$. Since $\mathrm{Hilb}^r(\mathbb{P}^1)$ is irreducible, one has $\mathrm{crk}(F) = \mathrm{srk}(F)$. By the border apolarity Theorem 2.15, we have $\underline{\mathbf{rk}}(F) = r$ as well and the equality is proven.

Either $\deg(g_1) = \deg(g_2)$ or not. In the first case, $\dim_{\mathbb{C}} \mathrm{Ann}(F)_{\deg(g_1)} = 2$ and so any $I = (g) \subset \mathrm{Ann}(F)$ with $\deg(g) = \deg(g_1) = r$ is a point in $\mathrm{VPS}(F, r)$. We have a morphism

$$\mathbb{P}^1 = \mathbb{P}(\mathrm{Ann}(F)_{\deg(g_1)}) \longrightarrow \mathrm{VPS}(F, r)$$

defined on closed points by $g \mapsto \mathrm{Proj}(S/(g))$, which is an isomorphism. Note that $d = 2(\deg(g_1) - 1)$ is even. In the second case, the only point of $\mathrm{VPS}(F, r)$ is given by $I = (g_1)$.

If $\mathrm{rk}(F) = r$, then $\mathrm{Ann}(F)_{\deg(g_1)}$ contains a square-free form. If we are in the first case, then VSP^0 which is set-theoretically given by all the square-free forms inside the pencil $\mathbb{P}(\mathrm{Ann}(F)_{\deg(g_1)})$, is dense in $\mathbb{P}(\mathrm{Ann}(F)_{\deg(g_1)})$. If we are in the second case, then the unique point of $\mathrm{VPS}(F, r)$ is a radical ideal. □

6.2 Cubic forms

6.2.1 Ternary cubics

Proposition 6.2. *If F is a plane cubic such that $\mathrm{Ann}(F)$ is generated by three quadrics, then $\underline{\mathrm{VSP}}(F, 4) \cong \mathbb{P}^2$.*

Proof. Let $\mathrm{Ann}(F) = (q_1, q_2, q_3)$. The quadrics q_1, q_2, q_3 form a regular sequence. Let q' and q'' be two linear combinations of q_1, q_2, q_3 such that $\dim_{\mathbb{C}}\langle q', q''\rangle = 2$. Then q' and q'' form a regular sequence, i.e., the ideal they generate $I = (q', q'')$ has codimension 2 and it is saturated. Therefore, any choice of a 2-dimensional subspace generates a saturated ideal with Hilbert function $h_{4,\mathbb{P}^2}$. Hence $\underline{\mathrm{VSP}}(F, 4) \cong \mathbb{G}(2, 3) \cong \mathbb{P}^2$. □

The $\mathrm{PGL}(3, \mathbb{C})$-orbits of ternary cubic forms with their ranks and border ranks are reported in [29, Theorem 8.1].

Theorem 6.3. *Let F be any ternary cubic form. Then either $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F)) \cong \mathbb{P}^2$ or it is a point.*

Proof. It is sufficient to prove the result for the normal forms appearing in [29, Theorem 8.1]. When $\underline{\mathbf{rk}}(F) \leq 2$, F is a binary form and we apply Proposition 6.1. For the cases when $\underline{\mathbf{rk}}(F) = 3$ (i.e., minimal border rank), we use Corollary 5.9 for $n = 2, d = 3$. In the cases $\underline{\mathbf{rk}}(F) = 4$ (i.e., generic border rank), we employ Proposition 6.2. □

6.2.2 Reducible cubics

Concise reducible cubic forms in $T = T[X]$ for $X = \mathbb{P}^n$ fall in the following four classes up to the $\mathrm{PGL}(n + 1, \mathbb{C})$-action [8]:

(i) $A = x_0(x_0^2 + x_1^2 + \cdots + x_n^2)$;

(ii) $B = x_0(x_1^2 + x_2^2 + \cdots + x_n^2)$;

(iii) $C = x_0(x_0x_1 + x_2^2 + \cdots + x_n^2)$;

(iv) monomials (which are all nonconcise if and only if $n \geq 3$).

So the fourth cases appear only for $n \leq 2$, and hence their $\underline{\mathrm{VSP}}$'s are described by Theorem 6.3. In this subsection we focus on cases (i)–(iii). The cubic form B is the symmetrization of the *small Coppersmith-Winograd tensor* $T_{cw,n}$. The cubic form C is the symmetrization of the *big Coppersmith-Winograd tensor* $T_{CW,n-1}$.

Proposition 6.4. *Let $n \geq 2$ and let $B \in T_3$ be the symmetrization of the small Coppersmith-Winograd tensor. Then $\underline{\mathrm{VSP}}(B, n+2)$ is $\mathbb{P}^2$ for $n = 2$ and a single point for $n \geq 3$.*

Proof. By [28, Proposition 3.4.9.1] we have $\underline{\mathbf{rk}}(B) = n+2$. For $n = 2$, the statement is a special case of Proposition 6.2. Let $n \geq 3$ and let $I \subset \mathrm{Ann}(B)$ be an ideal with generic Hilbert function $h_{n+2,\mathbb{P}^n}$. Then I_2 has codimension one in $\mathrm{Ann}(B)_2$. Write $\mathrm{Ann}(B) = (y_0^2, y_iy_j, y_i^2 - y_n^2 \mid 1 \leq i < j \leq n)$. Fix a graded lexicographic monomial ordering with $y_0 \succ \cdots \succ y_n$.

We now prove that $y_0^2 \notin J = \mathrm{in}_{\succ}(I)$. If not, then $J_2 = \langle y_0^2, m \in \mathcal{M} \rangle$, where $\mathcal{M}$ is the set of the initial monomials of the generators of $\mathrm{Ann}(B)$ except one. Suppose $y_iy_j \notin \mathcal{M}$ with $i \neq j$. Then $y_\ell^2 \in J$ for all $0 \leq \ell \leq n-1$. If $i, j \neq n$, then the quotient S_k/J_k for $k \geq 4$ is spanned by at most two monomials: $y_0y_n^{k-1}$ and y_n^k. Thus $\mathrm{HF}(S/I, k) \leq 2$ for $k \geq 4$, a contradiction. If $y_iy_n \notin \mathcal{M}$, then the quotient S_k/J_k for $k \geq 3$ is spanned by at most four monomials: $y_0y_iy_n^{k-2}, y_iy_n^{k-1}, y_0y_n^{k-1}$ and y_n^k. Thus $\mathrm{HF}(S/I, k) \leq 4 < n+2$ for $k \geq 3$, a contradiction. If $y_i^2 \notin \mathcal{M}$ for some $0 < i < n$, then the quotient S_k/J_k for $k \geq 3$ is spanned by at most four monomials: $y_0y_i^{k-1}, y_0y_n^{k-1}, y_i^k, y_n^k$. Thus $\mathrm{HF}(S/I, k) \leq 4 < n+2$ for $k \geq 3$, a contradiction.

Therefore $y_0^2 \notin J$. This implies that $I = (y_iy_j, y_i^2 - y_n^2)$, which is the unique point of $\underline{\mathrm{VSP}}(F, n+2)$. Indeed, it is easy to check that the basis of the quotient S_k/I_k for $k \geq 3$ is spanned by $y_0^k, y_0^{k-1}y_1, \ldots, y_0^{k-1}y_n, y_0^{k-2}y_1^2$. □

Proposition 6.5. *Let $n \geq 2$ and let $C \in T_3$ be the symmetrization of the big Coppersmith-Winograd tensor. Then $\underline{\mathrm{VSP}}(C, n+1)$ is a single point.*

Proof. By [28, Exercise 3.4.9.3] we have $\underline{\mathbf{rk}}(C) = n+1$, i.e., C has minimal border rank. Corollary 5.9 shows that $\underline{\mathrm{VSP}}(F, n+1)$ contains a single point which is the saturated ideal $(\mathrm{Ann}(C)_2)$. □

Proposition 6.6. *Let $n \geq 2$ and let $A \in T_3$ be as above. Then $\underline{\mathrm{VSP}}(A, n+2)$ is $\mathbb{P}^2$ for $n = 2$ and a single point for $n \geq 3$.*

Proof. For $n = 2$, this follows from Proposition 6.2. Assume that $n \geq 3$. Since $\mathrm{Ann}(F)$ is generated by quadrics we have $\underline{\mathbf{rk}}(A) \geq n+2$. On the other hand, the completely analogous proof to the one for the cubic B shows that $I = (y_iy_j, y_i^2 - y_n^2)$ is the unique apolar ideal of A with generic Hilbert function $h_{n+2,\mathbb{P}^n}$. Furthermore, it is in $\mathrm{Slip}_{n+2,\mathbb{P}^n}$. We conclude that $\underline{\mathbf{rk}}(A) = n+2$ and $\underline{\mathrm{VSP}}(A, n+2) = \{I\}$. □

6.3 Ternary forms of low even degree

Let $X = \mathbb{P}^2, S = \mathbb{C}[y_0, y_1, y_2]$ and $T = \mathbb{C}[x_0, x_1, x_2]$. In this section, by a *nondegenerate* ternary form of degree $d = 2p - 2$, we mean a ternary form $F \in T_d$ such that $\mathrm{Ann}(F)_{\leq p-1} = 0$, following the terminology of [37, §1]. There exists a dense set of such forms, so this is a generality-type assumption.

Proposition 6.7. *Let F be a nondegenerate ternary form of degree $d = 2p - 2$ with $p \geq 2$, then for every choice of $p + 1$ linearly independent forms of degree p in* $\mathrm{Ann}(F)$ *there is no linear form that divides all of them.*

Proof. Suppose by contradiction this is not the case. Then we have $\langle g_0, \ldots, g_p \rangle \subset \mathrm{Ann}(F')$ where g_i are linearly independent forms of degree $p - 1$ and $F' = l \circ F$ is a form of degree $2p - 3$. The Hilbert function of the quotient $S/\mathrm{Ann}(F')$ must satisfy:

$$\mathrm{HF}(S/\mathrm{Ann}(F'), p-2) = \mathrm{HF}(S/\mathrm{Ann}(F'), p-1) \leq \binom{p+1}{2} - (p+1) < \binom{p}{2}.$$

Hence we have a form g of degree $p - 2$ in $\mathrm{Ann}(F')$. This implies that the degree $(p - 1)$ form $l \cdot g$ is in $\mathrm{Ann}(F)$, which is impossible for a nondegenerate ternary form of degree $2p - 2$. □

Lemma 6.8. *Suppose that every ideal $I_Z \subset \mathrm{Ann}(F)$ of a length r subcheme Z has generic Hilbert function $h_{r,X}$ and the corresponding subscheme is smoothable and that any ideal $J \subset \mathrm{Ann}(F)$ with generic Hilbert function is B-saturated and belongs to* $\mathrm{Slip}_{r,X}$. *Then $\phi_{r,X}$ induces an isomorphism* $\underline{\mathrm{VSP}}(F, r) \cong \mathrm{VPS}(F, r)$.

Proof. By Theorem 2.26 the morphism $\phi_{r,X}$ restricts to a (bijective) locally closed immersion $\phi_{r,X}^{-1}(\mathrm{VPS}(F, r)) \to \mathrm{VPS}(F, r)$. Therefore, we have a surjective closed immersion $\underline{\mathrm{VSP}}(F, r) \to \mathrm{VPS}(F, r)$. Since the latter scheme is reduced, we claim that this is an isomorphism. Indeed, this can be checked locally on the target, so we need to prove that the only ideal $I \subset A$ such that the map of schemes $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ is bijective is $I = 0$. Any such I is contained in the intersection of all the prime ideals of A, hence it is the zero ideal as A is reduced. □

Theorem 6.9. *Let F be a general nondegenerate ternary form of degree $d = 2p - 2$ with $2 \leq p \leq 5$, we have an isomorphism between* $\mathrm{VPS}(F, r_p) = \mathrm{VSP}(F, r_p)$ *and* $\underline{\mathrm{VSP}}(F, r_p)$ *given by $\phi_{r_p,\mathbb{P}^2}$ with $r_p := \binom{p+1}{2} = \underline{\mathbf{rk}}(F)$.*

Proof. The equality $\mathrm{VPS}(F, r_p) = \mathrm{VSP}(F, r_p)$ follows from [37, Theorem 1.7 and Lemma 1.8]. By the assumption on F, the ideal of every zero-dimensional scheme in the variety of apolar schemes $\mathrm{VPS}(F, r_p) = \mathrm{VSP}(F, r_p)$ must have the generic Hilbert function. Then, by definition, the preimage under $\phi_{r_p,\mathbb{P}^2}$ of each of these is a single ideal in $\underline{\mathrm{VSP}}(F, r_p)$. Now, we will show that every ideal $J \subset \mathrm{Ann}(F)$ with generic Hilbert function is saturated and so lies in $\phi_{r_p,\mathbb{P}^2}^{-1}(\mathrm{VPS}(F, r_p))$. By Lemma 6.8 this proves the statement.

Suppose that J is not saturated and let K be its saturation. By the first part of the proof, K is not contained in $\mathrm{Ann}(F)$. Therefore, $\mathrm{HF}(S/K, 2p - 2) < r_p$, otherwise we would have $K_{2p-2} = J_{2p-2} \subset \mathrm{Ann}(F)$ and hence $K \subset \mathrm{Ann}(F)$, by Proposition 2.6, a contradiction.

We claim that there is a linear form ℓ dividing every form in K_{2p-2}. To prove the claim, one employs Macaulay's bound to list all the possible Hilbert functions of K. Then for each of them we check utilizing [39, Theorem 7.3.4] that a linear form ℓ as above exists. Let ℓ' be a linear form such that $\dim_{\mathbb{C}}\langle \ell, \ell' \rangle = 2$. Suppose that $F \in J_p$, then $(\ell')^{p-2}F \in J_{2p-2} \subset K_{2p-2}$. Hence ℓ divides $(\ell')^{p-2}F$ and thus it divides F. This contradicts Proposition 6.7. □

Remark 6.10. The varieties $\mathrm{VSP}(F, r_p)$ are very interesting. Mukai [34, 35] and Ranestad-Schreyer [37, Theorem 1.7] proved that: for $p = 2$, $\underline{\mathrm{VSP}}(F, 3) \cong \mathrm{VSP}(F, 3)$ is a Fano threefold of index 2 and degree 5 in $\mathbb{P}^6$; for $p=3$, $\underline{\mathrm{VSP}}(F,6) \cong \mathrm{VSP}(F,6)$ is a Fano threefold of index 1 and degree 22 in $\mathbb{P}^{13}$; for $p=4$, $\underline{\mathrm{VSP}}(F,10) \cong \mathrm{VSP}(F,10)$ is a $K3$ surface of genus 20 of degree 38 canonically embedded in $\mathbb{P}^{20}$; for $p=5$, $\underline{\mathrm{VSP}}(F,15) \cong \mathrm{VSP}(F,15)$ consists of 16 reduced points.

6.4 A reducible $\underline{\mathrm{VSP}}$ of a ternary form

The $\underline{\mathrm{VSP}}$ might be reducible, as we saw for nondegenerate ternary forms of degree 8 in §6.3; see also [23, Theorem 7.9] for examples of positive dimensional reducible $\underline{\mathrm{VSP}}$ of wild forms. We exhibit an example of a positive dimensional reducible $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F))$ when F is a ternary form. This addresses and solves the question whether $\underline{\mathrm{VSP}}(F, \underline{\mathbf{rk}}(F))$ can be positive dimensional and reducible even in $\mathbb{P}^2$. Note that this is not the case when $X = \mathbb{P}^1$. Let $X = \mathbb{P}^2$ and $F = x_0^{11} + x_1^5 x_2^6 \in T_{11}$. It is easy to check that $\underline{\mathbf{rk}}(F) = 7$. Let $J = (y_0y_1, y_0y_2, y_1^6) \subset \mathrm{Ann}(F)$.

Proposition 6.11. *We have* $\underline{\mathrm{VSP}}(F,7) = \phi^{-1}_{7,\mathbb{P}^2}(\mathrm{Proj}(S/J))$ *and it is reducible.*

Proof. We have $\mathrm{Ann}(F)_6 = J_6$ and $\mathrm{HF}(S/J,6) = 7$. Therefore, the saturation of every $[I] \in \underline{\mathrm{VSP}}(F,7)$ is J. From Lemma 3.4 we obtain $\underline{\mathrm{VSP}}(F,7) = \phi^{-1}_{7,\mathbb{P}^2}(\mathrm{Proj}(S/J))$.

If $I \in \underline{\mathrm{VSP}}(F,7)$ then $(y_0^2y_1, y_0^2y_2)_4 \subseteq I_4$ by [26, Example 4.2]. Since $\mathrm{HF}(S/J,4) = 6$ and I has generic Hilbert function, one has $I_4^\perp = J_4^\perp + \mathbb{C}\omega$ where $\omega = ax_0x_1^3 + bx_0x_1^2x_2 + cx_0x_1x_2^2 + dx_0x_2^3$ for some $a,b,c,d \in \mathbb{C}$ not all zero. Note that $\partial\omega/\partial x_0 \in J_3^\perp$.

For every $[\omega] \in \mathbb{P}\langle x_0x_1^3, x_0x_1^2x_2, x_0x_1x_2^2, x_0x_2^3\rangle$ there is an element $I \in \mathrm{Hilb}_S^{h_{7,\mathbb{P}^2}}$ with $I_4^\perp = J_4^\perp + \mathbb{C}\omega$. Indeed, it is enough to choose a 7-dimensional subspace of T_3 that contains $J_3^\perp + \langle \partial\omega/\partial x_1, \partial\omega/\partial x_2\rangle$. The ideal I constructed in this way is uniquely determined by $I_{\geq 4}$ if and only if $\dim_{\mathbb{C}} J_3^\perp + \dim_{\mathbb{C}}\langle \partial\omega/\partial x_1, \partial\omega/\partial x_2\rangle = 7$, which is equivalent to the condition that $\langle \partial\omega/\partial x_1, \partial\omega/\partial x_2\rangle$ is 2-dimensional. Therefore, if it is nonempty, the locus in $\underline{\mathrm{VSP}}(F,7)$ of ideals corresponding to such ω is open and irreducible and so its closure V_J is an irreducible component of $\underline{\mathrm{VSP}}(F,7)$. Observe that by construction, every element I in V_J satisfies $(y_0^2y_1, y_0^2y_2) \subseteq I$.

Let $\prec$ be the lexicographic monomial order with $y_0 \prec y_1 \prec y_2$. Consider the ideal $K = (y_0y_2^2, y_0^2y_2, y_0^2y_1, y_0y_1^3, y_1^6)$ corresponding to $\omega = x_0x_1^2x_2$. We claim that it belongs to V_J. To justify this, we need to show that K is in $\mathrm{Slip}_{r,\mathbb{P}^2}$. This follows from the fact that it is the initial ideal of the saturated ideal $L = (y_0y_2^2 + y_1^3, y_0^2y_2, y_0^2y_1)$.

Consider the ideal $K' = (y_0y_2^2, y_0y_1y_2, y_0^2y_2, y_0^2y_1^2, y_0^3y_1, y_0y_1^4, y_1^6)$. It is not in V_J but it is apolar to F. It is sufficient to show that it is in $\mathrm{Slip}_{7,\mathbb{P}^2}$ to conclude that $\underline{\mathrm{VSP}}(F,7)$ is reducible. This follows from the fact that it is the initial ideal of the saturated ideal $L' = (y_0^2y_2 + y_0y_1^2, y_0y_1y_2 + y_1^3, y_0y_2^2 + y_1^2y_2 + y_0^2y_1)$. □

6.5 A Schubert variety as $\underline{\mathrm{VSP}}$

Let $\mathbb{G}(k,n) = \mathbb{G}(\mathbb{P}^{k-1}, \mathbb{P}(V))$ be the Grassmannian of k-dimensional subspaces of an n-dimensional vector space V. Fix a complete flag $\mathcal{V} : 0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$, where $\dim(V_i) = i$, and a sequence $\mathbf{a} = (a_1, \ldots, a_k)$ of integers with $n-k \geq a_1 \geq \cdots \geq a_k \geq 0$. For such a sequence $\mathbf{a}$, the *Schubert variety* $\Sigma_{\mathbf{a}}(\mathcal{V}) \subset \mathbb{G}(k,n)$ is the closed subset

$$\Sigma_{\mathbf{a}}(\mathcal{V}) = \{\Lambda \in \mathbb{G}(k,n) \mid \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i \text{ for all } i\}.$$

The definition of $\Sigma_{\mathbf{a}}(\mathcal{V})$ depends on the specific flag. However, its class in the cohomology ring of $\mathbb{G}(k,n)$ does not. Hence, whenever the flag is fixed, the Schubert variety is denoted $\Sigma_{\mathbf{a}}$.

The Schubert varieties $\Sigma_{\mathbf{a}}$ are irreducible, rational and of codimension $\sum_{i=1}^{k} a_i$ inside $\mathbb{G}(k,n)$ [14, Theorem 4.1]. In the next result, we show an instance of $\underline{\mathrm{VSP}}$ that is a Schubert variety. When $\mathbf{a} = (a)$, then we write $\Sigma_a = \Sigma_{\mathbf{a}}$.

Proposition 6.12. *Let $X = \mathbb{P}^2$ and let $F = x_0^{11} + x_0x_1^6x_2^4 + x_2^{11} \in T_{11}$. Then $\underline{\mathbf{rk}}(F) = 12$ and $\underline{\mathrm{VSP}}(F,12)$ is isomorphic to the Schubert variety $\Sigma_1 \subset \mathbb{G}(3,5)$.*

Proof. We have $\mathrm{HF}(5, S/\mathrm{Ann}(F)) = \mathrm{HF}(6, S/\mathrm{Ann}(F)) = 12$, therefore $\underline{\mathbf{rk}}(F) \geq 12$. On the other hand, the ideal $J = (y_0^2y_1, y_0^2y_2, y_1y_2^5, y_0y_2^5)$ is a saturated ideal contained in $\mathrm{Ann}(F)$ that defines a length 12 subscheme of $\mathbb{P}^2$. So $\mathrm{crk}(F) \leq 12$. Since $X = \mathbb{P}^2$, $\mathrm{crk}(F) = \mathrm{srk}(F) \geq \underline{\mathbf{rk}}(F) = 12$. Hence $\underline{\mathbf{rk}}(F) = 12$.

Assume that $I \in \underline{\mathrm{VSP}}(F,12)$. Then $I_6 = \mathrm{Ann}(I)_6$ which equals J_6. Since J is generated in degrees at most 6 and its quotient algebra has Hilbert function 12 in all degrees larger than 4 we get $I_{\geq 6} = J_{\geq 6}$. In particular, I belongs to the fiber of $\phi_{12,\mathbb{P}^2}\colon \mathrm{Slip}_{12,\mathbb{P}^2} \to \mathrm{Hilb}^{12}(\mathbb{P}^2)$ over $\mathrm{Proj}(S/J)$. It follows from Lemma 3.4 that $\underline{\mathrm{VSP}}(F,12)$ is in fact equal to this fiber.

Recall we have the morphism $\psi_{12,\mathbb{P}^2} : \mathrm{Hilb}_S^{h_{12,\mathbb{P}^2}} \to \mathrm{Hilb}^{12}(\mathbb{P}^2)$ (whose restriction on $\mathrm{Slip}_{12,\mathbb{P}^2}$ is $\phi_{12,\mathbb{P}^2}$). Let $\mathcal{F}$ be its fiber over $\mathrm{Proj}(S/J)$. We have proven that $\underline{\mathrm{VSP}}(F,12)$ is the intersection of $\mathrm{Slip}_{12,\mathbb{P}^2}$ with $\mathcal{F}$.

Since $h_{12,\mathbb{P}^2}(3) = \dim_{\mathbb{C}} S_3$ and $h_{12,\mathbb{P}^2}(d) = 12 = \mathrm{HF}(S/J,d)$ for every $d > 4$, the closed points of $\mathcal{F}$ are parameterized by the choices of subspaces of J_4 of suitable dimension. We have $\dim_{\mathbb{C}} S_4 = 15$ and $\mathrm{HF}(S/J,4) = 10$, so these may be identified with the closed points of $\mathbb{G}(\mathbb{P}^2, \mathbb{P}(J_4)) = \mathbb{G}(3,5)$. To simplify notation, in what follows we identity $\mathcal{F}$ with $\mathbb{G}(3,5)$.

Consider the full flag $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_5 = J_4$ where V_i is spanned by i smallest monomials in J_4 in the graded reverse lexicographic order with $y_0 \prec y_1 \prec y_2$. There are 10 monomial ideals in $\mathcal{F}$ corresponding to the choices of 3 out of 5 monomials in J_4. Since we chose an ordering of monomials in J_4, the elements of $\mathcal{F}$ correspond to full-rank 3×5-matrices. More precisely, given $I \in \mathcal{F}$, the (i,j)th entry of the matrix corresponding to I is the coefficient in front of the jth largest monomial in J_4 of the ith minimal generator of I of degree 4.

For a sequence of integers $1 \leq i_1 < i_2 < i_3 \leq 5$, let $\mathcal{F}_{(i_1,i_2,i_3)}$ denote the locally closed subset of $\mathcal{F}$, whose corresponding matrices can be expressed in the reduced row echelon form with coefficients 1 at entries (s, i_s) for $s = 1,2,3$ and zeroes for every entry (s,t) with $s \in \{1,2,3\}$ and $1 \leq t < i_s$. These sets cover $\mathcal{F}$ and furthermore, we claim that the matrices whose row spans define an element in the Schubert variety Σ_1 are precisely the matrices that belong to one of the sets $\mathcal{F}_{(i_1,i_2,i_3)}$ with $(i_1,i_2,i_3) \neq (1,2,3)$. Indeed, by the choice of our flag and the definition of Schubert varieties we have

$$\Sigma_1 = \{I_4 \in \mathbb{G}(3,5) \mid I_4 \cap \langle y_0^3y_1, y_0^3y_2\rangle \neq \{0\}\}.$$

This is the union of those $\mathcal{F}_{(i_1,i_2,i_3)}$ for which $\{4,5\} \cap \{i_1,i_2,i_3\} \neq \emptyset$.

To show that $\mathcal{F} \cap \mathrm{Slip}_{12,\mathbb{P}^2}$ coincides with Σ_1, we proceed in two steps. First, we prove that $\mathcal{F}_{(1,2,3)}$ is disjoint from $\mathrm{Slip}_{12,\mathbb{P}^2}$. Let I be any element in $\mathcal{F}_{(1,2,3)}$. By our choice of the monomial order, we have $K = \mathrm{in}_{\prec}(I) = (y_0^2y_1^2, y_0^2y_1y_2, y_0^2y_2^2) + J_{\geq 5}$ and it is enough to show that K is not in $\mathrm{Slip}_{12,\mathbb{P}^2}$. By construction, $\overline{K} = J$. We compute that $\mathrm{Ext}^1_S(J/K, S/J)_0 = 0$. By [26, Theorem 3.4], K is not in $\mathrm{Slip}_{12,\mathbb{P}^2}$. Therefore $\underline{\mathrm{VSP}}(F,12) = \mathcal{F} \cap \mathrm{Slip}_{12,\mathbb{P}^2} \subseteq \Sigma_1$.

Σ_1 is closed, irreducible and of codimension one in $\mathcal{F}$. Therefore, by [32, Lemma 2.6] to derive the desired equality $\underline{\mathrm{VSP}}(F,12) = \Sigma_1$, it is sufficient to find a point in $\mathcal{F} \cap \mathrm{Slip}_{12,\mathbb{P}^2}$ such that the dimension of its tangent space to $\mathrm{Hilb}_S^{h_{12,\mathbb{P}^2}}$ is $25 = \dim \mathrm{Slip}_{12,\mathbb{P}^2} + 1$. Let $\widehat{K} = (y_0^2y_1^2, y_0^2y_2^2, y_0^3y_2) + J_{\geq 5}$.

We compute that $\dim_{\mathbb{C}} \operatorname{Hom}_S(\widehat{K}, S/\widehat{K})_0 = 25$ so we are left with showing that $\widehat{K}$ is in $\mathrm{Slip}_{12,\mathbb{P}^2}$. Let $\widehat{I} = (y_0^2y_1^2 + y_0y_2^3, y_0^3y_2, y_0^2y_2^2, y_0^4y_1, y_1y_2^5, y_0y_2^5)$. Its initial ideal with respect to graded reverse lexicographic order with $y_0 \succ y_1 \succ y_2$ is $\widehat{K}$ so it is sufficient to observe that $\widehat{I}$ is in $\mathrm{Slip}_{12,\mathbb{P}^2}$. This follows from [32, Theorem 1.3] since the saturation of $\widehat{I}$ differs from $\widehat{I}$ only in degree 4. □

7 References

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