Para-Kähler immersions in para-Kähler space forms

Results Math (2026) 81:22

1422-6383/26/010001-24
published online December 19, 2025
https://doi.org/10.1007/s00025-025-02562-8

Results in Mathematics

Para-Kähler Immersions in Para-Kähler Space Forms

Gianni Manno and Filippo Salis

Abstract. A classical problem addressed, among others, by Calabi, is that to characterize non-isometric Kähler manifolds immersed in a finite-dimensional Kähler space form. In this paper we address the same problem in the para-Kähler context and provide necessary and sufficient conditions for the existence of para-Kähler immersions in para-Kähler space forms. As a consequence, we prove that, in general, a local para-Kähler immersion cannot be globally extended, even if it is defined on a simply connected para-Kähler manifold. Finally, we classify para-Kähler immersions between para-Kähler space forms.

Mathematics Subject Classification. 53C15, 53C42.

Keywords. para-Kähler immersion, para-Kähler space forms, Calabi's diastasis, rigidity, global extendability.

1. Introduction

1.1. The Context and Description of the Problem

An *almost para-complex manifold* is a $2n$-dimensional manifold M provided with a field of endomorphisms $\mathcal{T}$ such that $\mathcal{T}^2 = 1$, having eigenvalues 1 and -1, whose associated eigendistributions are n-dimensional. If, furthermore, such distributions are integrable, which is equivalent to requiring the vanishing of the Nijenhuis tensor of $\mathcal{T}$, an almost para-complex manifold is called a *para-complex manifold*. A *para-Kähler manifold* is a para-complex manifold with a pseudo-Riemannian metric g against which the field of endomorphism $\mathcal{T}$ is parallel, or, equivalently, the 2-form $\omega = g\,(\mathcal{T}(\cdot),\,\cdot\,)$ is symplectic: in this case, the aforementioned distributions give a couple of Lagrangian foliations. These are important aspects for which para-Kähler geometry and, more in general, para-complex geometry is an area of research rather active, as evidenced by lots of papers in this field. In this regard, the survey [5] shows a large spectrum of applications of this geometry, mainly focused on actions of Lie groups on the aforementioned manifolds, together with a historical introduction. One can also consult [1–3] for further study and their bibliography for further readings. For different applications, one can see [7] and references therein, where, among other things, it was shown a direct relation between Lagrangian submanifolds of para-Kähler manifolds and the Monge-Kantorovich mass transport problem.

The formal analogy with the Kähler geometry makes it possible to formulate problems also in the para-Kähler context by translating the original ones. For instance, a classical problem in Kähler geometry is the characterization of holomorphic and isometric immersion into complex space forms, i.e. into Kähler manifolds of constant holomorphic sectional curvature (see [4] and for a modern introduction to this subject [8]). In particular, Calabi provided in [4] a distinguished function, called *diastasis*, whose Taylor's expansion facilitated the study of the aforementioned immersions, allowing the author to achieve some remarkable results. The same problem can be formulated in the para-Kähler case, i.e., to characterize para-holomorphic and isometric immersions, called *para-Kähler immersions*, into *para-Kähler space forms*, i.e., para-Kähler manifolds having constant para-holomorphic sectional curvature (namely the para-complex counterpart of the holomorphic sectional curvature).

Since analyticity of holomorphic functions plays a key role in the aforementioned Calabi's work, his techniques cannot be applied in the considered context. Indeed, even though its formal similarity with complex geometry, para-complex geometry is quite different, since para-holomorphic functions are not, in general, analytic but only $\mathcal{C}^\infty$-smooth. Therefore, even if inspired by the Calabi's work, in order to achieve necessary and sufficient conditions for the existence of local para-Kähler immersions into para-Kähler space forms, we need to develop some different ideas, which we will explain more in details in the next section.

1.2. Description of the Paper and Main Results

In Sect. 2, after recalling some fundamental aspects of para-complex geometry, we introduce the diastasis function for para-Kähler manifolds inspired by a Calabi's idea. We end the section with the description of para-Kähler manifolds of constant para-holomorphic curvature.

Section 3 is mainly devoted to find necessary and sufficient conditions for the existence of para-Kähler immersions into para-Kähler space forms. In particular, the main achievement of the Sect. 3.1 is Theorem 14 where, essentially, we prove that, if it exists, a para-Kähler immersion is locally unique up to an isometry of the ambient space. The example of para-Kähler manifold described in Sect. 3.3 helps to better illustrate such local character of para-Kähler immersions. Moreover, differently from the Kähler case, this example shows that, in general, local para-Kähler immersions into para-Kähler space forms and defined on a simply connected manifold, cannot be globally extended. In Sect. 3.2, taking into account the results obtained so far, we arrive to the main achievements of the paper, i.e. Theorem 17. More precisely, we found algebraic necessary and sufficient conditions for the existence of para-Kähler immersions in para-Kähler space forms. Such conditions are expressed in terms of a suitable constructed function and its derivatives and, being algebraic, they can be straightforwardly computed.

Finally, in Sect. 4, by means of the results obtained in the previous sections, we get a complete classification of para-Kähler immersions between para-Kähler space forms.

Notation

A multi-index $I=(i_1,\ldots,i_n)$ is an element of $\mathbb{N}^n$ and its length $|I|$ is defined as the number $|I|:=\sum_{k=1}^n i_k$. If $(x_1,\ldots,x_n)$ are local coordinates, we define the derivative operators $\frac{\partial^{|I|}}{\partial x^I}$ as follows:

$$\frac{\partial^{|I|}}{\partial x^I}:=\frac{\partial^{|I|}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}\,.$$

We fix a total order on $\mathbb{N}^n$ such that $I_0=0\in\mathbb{N}^n$ and $|I_i|\leq|I_{i+1}|$ for any $i\in\mathbb{N}$. This obviously induces a total order on the set of the aforementioned derivative operators $\frac{\partial^{|I|}}{\partial x^I}$. Once a total order on $\mathbb{N}^n$ is fixed, one can construct a bijection ι between any subset $\mathcal{I}\subset\mathbb{N}^n$ with finite cardinality and $\{1,\ldots,\#\mathcal{I}\}\subset\mathbb{N}$ preserving the total orders:

$$\iota: I\in\mathcal{I}\longrightarrow\iota(I)\in\{1,\ldots,\#\mathcal{I}\}. \tag{1}$$

2. Basics of Para-Complex Geometry

2.1. Para-Holomorphic Functions

The 2-dimensional algebra over $\mathbb{R}$ of *para-complex numbers* $\mathbb{D}$ is generated by 1 and τ, where

$$\tau^2=1\,.$$

In analogy with the complex numbers, we are going to adopt the notation used in [7]: each $z\in\mathbb{D}$ can be written as

$$z=x+\tau y\,,$$

and we are going to refer to x and y as the *real* and *imaginary part* of z, respectively. In analogy with the complex numbers, we define the conjugate of z

$$\bar{z}=x-\tau y$$

and

$$|z|^2=z\bar{z}=x^2-y^2.$$

For later purposes, it is useful to introduce also another coordinate system on $\mathbb{D}$, described as follows. We switch the basis $(1,\tau)$ with $(e,\bar{e})$, where

$$e=\frac{1}{2}\left(1-\tau\right) , \qquad \bar{e}=\frac{1}{2}\left(1+\tau\right) ,$$

and we are going to say that (u,v) are *null-coordinates* of z if $z=ue+v\bar{e}$.

Now we can translate on $\mathbb{D}^n$ what we said about $\mathbb{D}$. In particular, for any $z,w\in\mathbb{D}^n$, we define

$$\langle z,w\rangle=\sum_{i=1}^{n} z_i\bar{w}_i$$

and

$$\|z\|^2:=\sum_{i=1}^{n}|z_i|^2.$$

Definition 1. A function

$$\begin{aligned} F: \quad U\subseteq\mathbb{D}^n &\longrightarrow \mathbb{D} \\ (z_1,\dots,z_n) &\longmapsto g(x_1,y_1,\dots,x_n,y_n)+\tau\ h(x_1,y_1,\dots,x_n,y_n), \end{aligned}$$

where $z_i=x_i+\tau y_i$, is called *para-holomorphic* if and only if g and h are smooth and

$$\frac{\partial F}{\partial \bar{z}_i}:=\frac{1}{2}\left(\frac{\partial g}{\partial x_i}-\frac{\partial h}{\partial y_i}\right)+\frac{\tau}{2}\left(\frac{\partial h}{\partial x_i}-\frac{\partial g}{\partial y_i}\right)=0 \tag{2}$$

for any $1\le i\le n$.

The number n stands for the *"para-complex" dimension* of $\mathbb{D}^n$.

In analogy to the complex setting, the differential operator $\frac{\partial}{\partial z_i}$ is defined by

$$\frac{\partial F}{\partial z_i}:=\overline{\left(\frac{\partial \bar{F}}{\partial \bar{z}_i}\right)}.$$

Remark 2. Let $F:U\subseteq\mathbb{D}^n\to\mathbb{D}$ be a para-holomorphic function. By considering the null-coordinates $(\xi,\eta)=(\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n)$ on $\mathbb{D}^n$ and by writing F as

$$F(\xi_1 e+\eta_1\bar{e},\dots,\xi_n e+\eta_n\bar{e})=u(\xi_1,\eta_1,\dots,\xi_n,\eta_n)\ e+v(\xi_1,\eta_1,\dots,\xi_n,\eta_n)\ \bar{e} ,$$

where u and v are real functions on an open subset of $\mathbb{R}^{2n}$, we straightforwardly get

$$\frac{\partial F}{\partial \bar{z}_i}=\frac{\partial u}{\partial \eta_i}e+\frac{\partial v}{\partial \xi_i}\bar{e} .$$

Therefore, F is para-holomorphic if and only if u is independent of $(\eta_1,\dots,\eta_n)$ and v is independent of $(\xi_1,\dots,\xi_n)$.

Definition 3. A function

$$\begin{array}{rccc} F: & U \subseteq \mathbb{D}^n & \longrightarrow & \mathbb{D}^m \\ & z=(z_1,\dots,z_n) & \longmapsto & \big(f_1(z),\dots,f_m(z)\big) \end{array}$$

is para-holomorphic if and only if each component f_i is para-holomorphic.

2.2. Para-Kähler Manifolds, Diastasis and Para-Kähler Space Forms

The condition of integrability of almost para-complex structures, that we wrote in the Introduction, is equivalent to the existence of a para-holomorphic atlas. More precisely, we give the following definition.

Definition 4. A smooth manifold M^n of para-complex dimension n is called *para-complex* if it admits an atlas of para-holomorphic coordinates $(z_1,\dots,z_n)$, such that the transition functions are para-holomorphic.

Differently from what we did in the Introduction, for our purposes, below we define para-Kähler manifolds by introducing local potentials.

Definition 5. A *para-Kähler manifold* of para-complex dimension n is a para-complex manifold M endowed with a symplectic form ω (called *para-Kähler form*) such that, for any point $p \in M$, there exists an open neighborhood $U \ni p$ and a smooth function $\Phi : U \to \mathbb{R}$ (called *para-Kähler potential*) satisfying

$$\omega|_U = \frac{\tau}{2}\partial\bar{\partial}\Phi := \frac{\tau}{2}\sum_{i,j=1}^{n}\frac{\partial^2\Phi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.$$

Remark 6. To a given para-Kähler manifold (M,ω) of para-complex dimension n it is associated a pseudo-Riemannian metric g on M. Indeed, if ω admits a para-Kähler potential Φ in an open set $U \subset M$, the restriction of g to U can be defined as the real part of

$$\sum_{i,j=1}^{n}\frac{\partial^2\Phi}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j.$$

Let U be an open subset of a para-Kähler manifold (M,ω), where it is defined a local potential Φ. We assume that U can be covered by a system of null-coordinates

$$(\xi,\eta) = (\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n).$$

Moreover, up to shrinking U, we can also assume that it splits as a product

$$U = \Omega \times \Omega.$$

We define the *diastasis function*

$$D : U \times U = \Omega \times \Omega \times \Omega \times \Omega \to \mathbb{R}$$

as

$$D\,(\xi,\eta,\zeta,\lambda) = \Phi(\xi,\eta) - \Phi(\zeta,\eta) - \Phi(\xi,\lambda) + \Phi(\zeta,\lambda). \tag{3}$$

Nevertheless, even if the previous definition is given in local coordinates, it gives rise to a well defined function on a neighborhood of the diagonal of $M \times M$. More precisely, we have the following proposition.

Proposition 7. *The diastasis function is a function defined in a neighborhood of the diagonal of the product manifold $M \times M$. In particular, it does not depend on the choice of the para-Kähler potential.*

Proof. Let $(p, p) \in M \times M$. Let Φ and $\tilde{\Phi}$ be two local potentials defined on the same open subset $U \ni p$ of a para-Kähler manifold. By definition of potential (see Definition 5) we have

$$\partial\bar{\partial}(\Phi - \tilde{\Phi}) = 0,$$

namely, if we fix some null-coordinates (ξ, η) on U,

$$\frac{\partial^2(\Phi - \tilde{\Phi})}{\partial\xi_i \partial\eta_j} = 0, \qquad \forall 1 \leq i, j \leq n.$$

Hence, there exist two functions $F,\ G \in \mathcal{C}^\infty(\Omega)$ such that

$$\tilde{\Phi}(\xi, \eta) = \Phi(\xi, \eta) + F(\xi) + G(\eta).$$

Our statement follows by the definition of diastasis function, see (3). □

Remark 8. Let (M, ω) be a para-Kähler manifold with local potential Φ, defined on an open subset $U \subseteq M$. Then the function

$$\begin{aligned} D_p : U &\longrightarrow \quad \mathbb{R} \\ q &\longmapsto D(p, q) \end{aligned}$$

is also a local potential for (M, ω).

In complete analogy with the case of Kähler manifolds, one can define the para-holomorphic sectional curvature (see for instance [6]) and, so, the para-Kähler space forms.

Definition 9. A *para-Kähler space form* is a para-Kähler manifold with constant para-holomorphic sectional curvature. We denote with $\mathcal{S}_c^N$ an N-dimensional simply connected para-complex manifold that can be endowed with a para-Kähler form ω_c whose associated pseudo-Riemannian metric g_c is complete and has constant para-holomorphic sectional curvature equal to c.

Proposition 10 *([6] Prop.* 3.11*). Two complete and simply connected para-Kähler space forms with the same para-holomorphic sectional curvature are para-holomorphically isometric.*

By [6], we have that an open subset of a para-Kähler space form is para-holomorphically isometric to an open subset of one of the subsequent models, according to their (constant) para-holomorphic sectional curvature. Therefore, since we mainly interested in local para-Kähler immersions into open subsets of para-Káhler space forms, we are going to assume that $\left(\mathcal{S}_c^N, \omega_c\right)$ is one of the following models.

Flat case: The model of the flat para-Kähler space

$$\left(\mathcal{S}_0^N, \omega_0\right) = \left(\mathbb{D}^N, \tau\partial\bar{\partial}\|z\|^2\right),$$

whose potential Φ reads in null-coordinates $(\xi_1, \dots, \xi_N, \eta_1, \dots, \eta_N)$ as

$$\Phi = 4\sum_{i=1}^{N} \xi_i \eta_i = D_0(\xi, \eta), \tag{4}$$

where D_0 is defined in Remark 8, is an example of homogeneous space with respect to its para-holomorphic isometry group. Such group consists of translations and $\mathbb{D}$-unitary transformations $z \in \mathbb{D}^N \to Az \in \mathbb{D}^N$, where A belongs to the $\mathbb{D}$-*unitary group*

$$\mathrm{U}_N(\mathbb{D}) = \{A \in \mathbb{D}^{N,N} \mid \|Aw\|^2 = \|w\|^2 \ \forall w \in \mathbb{D}^N\}.$$

Non-Flat Cases: Similarly to the real and complex setting, the para-complex projective space $\mathbb{DP}^N$ can be defined as the quotient of

$$\{Z \in \mathbb{D}^{N+1} \mid \|Z\|^2 > 0\}$$

under the equivalence relation given by $Z \sim W$ if and only if there exists $\alpha \in \mathbb{D}$ such that $Z = \alpha W$ with $|\alpha|^2 > 0$.

Our model of non-flat para-Kähler space form will be

$$(\mathcal{S}_c^N, \omega_c) = \left(\mathbb{DP}^N, \frac{4\tau}{c}\partial\bar{\partial}\log\|Z\|^2\right).$$

In null-coordinates $(\xi_1, \dots, \xi_N, \eta_1, \dots, \eta_N)$ of the affine chart $\mathcal{U}_\alpha$:=$\{[Z_0, \dots, Z_N] \in \mathbb{DP}^N \mid |Z_\alpha|^2 \neq 0\}$, where $\alpha = 1, \dots, n$, i.e., $\xi_i e + \eta_i \bar{e} = \frac{Z_i}{Z_\alpha}$ for any $i \neq \alpha$, the potential Φ is equal to

$$\Phi = \frac{8}{c}\log\left(1 + 2\sum_{i=1}^{N} \xi_i \eta_i\right) = D_0(\xi, \eta). \tag{5}$$

Since the action of $\mathrm{U}_{N+1}(\mathbb{D})$ passes to the quotient, we can easily see that these para-Kähler space forms are homogeneous with respect to the action of their para-holomorphic isometry groups.

3. Para-Kähler Immersions and Full Para-Kähler Immersions in Space Forms

Definition 11. Let (S, θ) and (M, ω) be two para-Kähler manifolds. A *para-Kähler immersion* of (S, θ) into (M, ω) is a para-holomorphic immersion $f : S \to M$ such that $f^*(\omega) = \theta$.

If h and g are, respectively, the pseudo-Riemannian metric associated to θ and ω, then we also have that $f^*(g) = h$.

Proposition 12 *(Hereditary Property). Let (S,θ) and (M,ω) be two para-Kähler manifolds and let*

$$f : (S,\theta) \to (M,\omega)$$

be a para-Kähler immersion. Let D^S and D^M be the diastasis functions of S and M, respectively. If $p \in S$ and if D^S_p (cfr. Remark 8) is defined on an open subset $U \ni p$ of S, then

$$D^S_p(q) = D^M_{f(p)}\left(f(q)\right), \qquad \forall q \in U.$$

Proof. If Φ^S is a potential for (S,θ) around p and Φ^M is a potential for (M,ω) around $f(p)$, then, by taking into account that $f^*(\omega) = \theta$, we have that

$$\partial\bar{\partial}\left(\Phi^M \circ f - \Phi^S\right) = 0.$$

Therefore, the statement of the proposition follows from the diastasis' definition, see (3). □

Diastasis function, together with its properties, in particular the hereditary one, will be used to characterize para-Kähler Einstein manifolds admitting a para-Kähler immersion into a para-Kähler space form, following the ideas present in [9–11].

3.1. Rigidity of Full Para-Kähler Immersions

In the Kähler case, a Kähler immersion of a Kähler manifold into a complex space form is full if such space form has the smallest dimension. An important property of such immersions is that their restriction to any open subset of the Kähler manifold is still full. On the contrary, this property does not hold true in the para-Kähler context. For this reason, we need to distinguish two kinds of full para-Kähler immersions: weakly and strongly ones.

Definition 13. A para-Kähler immersion $f : (M^n,\omega) \to (\mathcal{S}^N_c,\omega_c)$ is said to be

- *weakly full* if and only if (M,ω) does not admit a para-Kähler immersion into any $(\mathcal{S}^{N^*}_c,\omega_c)$ with $N^* < N$;
- *strongly full* if and only if the restriction of f to any open subset of M is weakly full.

Indeed, the restriction of a weakly full immersion to an open subset might be not weakly full (see the example described in Section 3.3).

Theorem 14 *(Rigidity). Let f and g be two weakly full para-Kähler immersions from an open subset U of a para-Káhler manifold (M^n,ω) into $(\mathcal{S}^{N_1}_c,\omega_c)$ and $(\mathcal{S}^{N_2}_c,\omega_c)$, respectively. Let also assume that the diastasis function is defined on $U \times U$. Then, $N_1 = N_2$ and f differs from g for a para-holomorphic isometry of the ambient space.*

Proof. We fix a para-holomorphic coordinate system

$$z = (z_1, \dots, z_n)$$

centred at an arbitrary chosen point $p \in U$. Let firstly consider the case $c = 0$.

Being $\mathcal{S}_0^N$ a homogeneous space with respect to the action of its para-holomorphic isometry group, we can assume without loss generality that $f(p) = g(p) = 0$. Since f and g are para-holomorphic, $f(U)$, as well as $g(U)$, cannot be a real subspace of $\mathbb{D}^N$. Indeed, if $f(z) = \overline{f(z)}$ for any z, then f would be a constant by (2).

We now choose s points $p_1, \dots, p_s \in U$ different from p and we consider s real constants α_i such that

$$\sum_{i=1}^{s} \alpha_i f(p_i) = 0,$$

from which

$$\sum_{i,j=1}^{s} \alpha_i \alpha_j f(p_i) \overline{f(p_j)} = 0.$$

In view of Proposition 12, by considering the definition of diastasis (3) and by taking into account (4), we obtain that $D^U(p_i, p) = \|f(p_i)\|^2$. Hence, the previous equality can be written as

$$\sum_{i=1}^{s} \alpha_i^2 D^U(p_i, p) + \sum_{i<j} \alpha_i \alpha_j \left(D^U(p_i, p) + D^U(p_j, p) - D^U(p_i, p_j) \right) = 0.$$

Then, it follows that the maximal number of linearly independent vectors $f(p_i)$ depends only on the diastasis function, not on the immersion. Therefore, since both immersions f and g are weakly full, the dimensions of the ambient spaces need to be equal. Furthermore, in view again of Proposition 12, we have that

$$D^{\mathcal{S}_0^N}_{f(p)}(f(q)) = D^{\mathcal{S}_0^N}_{g(p)}(g(q)),$$

namely

$$\|f(q)\|^2 = \|g(q)\|^2, \qquad \forall q \in U\,.$$

By considering that $f(U)$ and $g(U)$ span respectively two $\mathbb{D}$-linear spaces having the same dimension, then such immersions differ from each other for a $\mathbb{D}$-unitary transformation.

Now, let us consider the case $c \neq 0$.

Being $\mathcal{S}_c^N$ a homogeneous space with respect to the action of its para-holomorphic isometry group, we can assume without loss generality that $f(p) = g(p) = [1 : 0 : \dots : 0]$ and that their images are contained in the affine chart $\{[Z_0, \dots, Z_N] \in \mathbb{DP}^N \mid Z_0 \neq 0\}$. Hence, our immersions can be computed from

$$\tilde{f}, \tilde{g} : (M^n, \omega) \to \left(\mathbb{D}^{N+1}, \frac{4\tau}{c} \partial\bar{\partial} \log \|Z\|^2 \right)$$

by considering the canonical projection. More precisely, in view of Proposition 12, by taking into account the definition of diastasis (3) and (5), we have that

$$D^U(p,q) = \frac{4}{c}\log\left(\frac{\|\tilde f(p)\|^2\ \|\tilde f(q)\|^2}{\left|\langle \tilde f(p), \tilde f(q)\rangle\right|^2}\right).$$

Our statement follows by considering very similar arguments we adopted (above) in the proof for the flat case. □

In Sect. 3.3, it is described an example of para-Kähler manifold that better illustrates the local character of para-Kähler immersions. To understand this example, one needs only Proposition 15 below.

3.2. Characterization of Full Para-Kähler Immersions

To start with, below we give another application of Proposition 12, which will be useful for characterizing weakly full para-Kähler immersions into a para-Kähler space form of para-holomorphic curvature $c \in \mathbb{R}$, through a suitable function H_c of the diastasis.

If

$$f : (M^n, \omega) \to (\mathcal{S}^N_c, \omega_c), \qquad n \le N,$$

is a para-Kähler immersion reading locally as (cfr. Remark 2 and Definition 3)

$$f(\xi_1 e + \eta_1 \bar e, \dots, \xi_n e + \eta_n \bar e) = (u_1(\xi)e + v_1(\eta)\bar e, \dots, u_N(\xi)e + v_N(\eta)\bar e)\,, \quad (6)$$

where $(\xi,\eta) = (\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n)$ and $(u,v) = (u_1,\dots,u_N,v_1,\dots,v_N)$ are null-coordinates on an open subset $U = \Omega\times\Omega \subseteq M^n$ and $\mathcal{S}^N_c$, respectively, then we have

$$D_0^{M^n}(\xi e + \eta\bar e) = D_{f(0)}^{\mathcal{S}^N_c}(ue + v\bar e).$$

Since, up to change of coordinates, we can assume that $f(0) = 0$, i.e., in coordinates,

$$u(0) = 0, \qquad v(0) = 0, \quad (7)$$

then we get, in view of (4) and (5), that

$$D_0^{M^n}(\xi e + \eta\bar e) = \begin{cases} 4\sum_{i=1}^N u_i(\xi)v_i(\eta) & \text{if } c = 0;\\ \frac{8}{c}\log\left(1 + 2\sum_{i=1}^N u_i(\xi)v_i(\eta)\right) & \text{if } c \ne 0.\end{cases} \quad (8)$$

Therefore, the function $H_c : U = \Omega\times\Omega \subseteq M^n \to \mathbb{R}$ defined by

$$H_c(\xi,\eta) := \begin{cases} \frac{1}{4}D_0^{M^n}(\xi e + \eta\bar e) & \text{if } c = 0;\\ \frac{1}{2}\exp\left(\frac{c}{8}D_0^{M^n}(\xi e + \eta\bar e)\right) - \frac{1}{2} & \text{if } c \ne 0,\end{cases} \quad (9)$$

reads as

$$H_c(\xi,\eta) = \sum_{\alpha=1}^N u_\alpha(\xi)v_\alpha(\eta). \quad (10)$$

Vice versa, given a para-Kähler manifold (M^n,ω), an arbitrary point $p \in M$ and a system of null-coordinates (ξ,η) centered at p, whenever the condition (10) holds true, with H_c defined by (9), there exists a neighborhood of p

that can be para-Kähler immersed into $(\mathcal{S}_c^N, \omega_c)$ via a para-Káhler immersion reading as (6). In fact, being the diastasis function D_0 a para-Kähler potential (see Remark 8), we notice that

$$\det\left(\frac{\partial^2 H_c}{\partial \xi_i \partial \eta_j}(0,0)\right)_{1\le i,j\le n} \neq 0.$$

Therefore, by taking into account (9), we get that the Jacobians of both $(u_1,\ldots,u_N)$ and $(v_1,\ldots,v_N)$ need to have maximal rank in a neighborhood of p. To sum up we have the following proposition.

Proposition 15. *Let U be an open subset of para-Kähler manifold (M^n,ω) where it is defined a system of null-coordinates (ξ,η). Then, there exists a weakly full para-Kähler immersion $f:(U,\omega|_U)\to(\mathcal{S}_c^N,\omega_c)$ if and only if the function H_c defined by (9) reads as (10) with the smallest possible N.*

Below, in Theorem 17, we provide some more practical criteria to verify if the conditions of the previous proposition hold true. To this aim, we need to introduce the following definition.

Definition 16. A *generalized Wronskian* of $u_1(\xi),\ldots,u_N(\xi)$ is any determinant of the type

$$\det\begin{pmatrix} \mathcal{D}_0(u_1) & \ldots & \mathcal{D}_0(u_N)\\ \vdots & & \vdots \\ \mathcal{D}_{N-1}(u_1) & \ldots & \mathcal{D}_{N-1}(u_N)\end{pmatrix},$$

where $\mathcal{D}_j$ denotes a partial derivative $\frac{\partial^{|I|}}{\partial \xi^I}$ where $|I|\le j$. In particular, $\mathcal{D}_0=\mathrm{id}$. A *generalized $r\times r$ sub-Wronskian* of $u_1,\ldots,u_N$ is a Wronskian of a subset of $u_1,\ldots,u_N$ containing r elements. A point p where all the functions $\{u_1,\ldots,u_N\}$ are defined, is said to be of *order* $\mathrm{ord}(p)\in\mathbb{N}$, if any $r\times r$ sub-Wronskian vanishes at p for $r>\mathrm{ord}(p)$.

Theorem 17. *Let p be a point of a para-Kähler manifold (M^n,ω). Let $U=\Omega\times\Omega$ be a neighborhood of p where it is defined a system of null-coordinates (ξ,η) centered at p. Let the diastasis function be defined on $U\times U$. Then necessary conditions for the existence of a weakly full para-Kähler immersion*

$$f:(U,\omega|_U)\to(\mathcal{S}_c^N,\omega_c)$$

are the following:
There exist two sets $\mathcal{I},\mathcal{J}\subset\mathbb{N}^n$ of multi-indices, containing 0 and having finite cardinality, for which $\forall\, K\in\mathbb{N}^n$ there exist smooth functions $a_K^I, b_K^J:\Omega\to\mathbb{R}$, where $I\in\mathcal{I}$, $J\in\mathcal{J}$, $K\in\mathbb{N}^n$ and two smooth functions a,b not identically zero in a neighborhood of 0 with $a(0)=b(0)=0$, such that

$$a(\xi)\,\frac{\partial^{|K|}H_c}{\partial\xi^K}(\xi,\eta)+\sum_{I\in\mathcal{I}}a_K^I(\xi)\,\frac{\partial^{|I|}H_c}{\partial\xi^I}(\xi,\eta)\equiv 0 \tag{11}$$

and

$$b(\eta)\ \frac{\partial^{|K|}H_c}{\partial\eta^K}(\xi,\eta)+\sum_{J\in\mathcal{J}}b_K^J(\eta)\ \frac{\partial^{|J|}H_c}{\partial\eta^J}(\xi,\eta)\equiv 0 \tag{12}$$

where the function H_c is defined by (9). The above conditions are sufficient by assuming that either

$$\Omega'=\left\{\xi\in\Omega\mid a(\xi)\neq 0\right\}, \tag{13}$$

or $\left\{\eta\in\Omega\mid b(\eta)\neq 0\right\}$, is a connected open dense subset of Ω. This leads actually to a strongly full para-Kähler immersion.

Proof. By differentiating (10), we get

$$\begin{pmatrix}\mathcal{D}_0(u_1) & \dots & \mathcal{D}_0(u_N)\\ \vdots & & \vdots\\ \mathcal{D}_{N-1}(u_1) & \dots & \mathcal{D}_{N-1}(u_N)\\ \mathcal{D}_j(u_1) & \dots & \mathcal{D}_j(u_N)\end{pmatrix}\begin{pmatrix}v_1\\ \vdots\\ v_N\end{pmatrix}=\begin{pmatrix}\mathcal{D}_0(H_c)\\ \vdots\\ \mathcal{D}_{N-1}(H_c)\\ \mathcal{D}_j(H_c)\end{pmatrix} \tag{14}$$

where the subscript j is arbitrary chosen in $\mathbb{N}$. Since (14) can be seen as a linear system in $(v_1,\dots,v_N)$, it follows, from its compatibility condition, that

$$\det\begin{pmatrix}\mathcal{D}_0(u_1) & \dots & \mathcal{D}_0(u_N) & \mathcal{D}_0(H_c)\\ \vdots & & \vdots & \vdots\\ \mathcal{D}_{N-1}(u_1) & \dots & \mathcal{D}_{N-1}(u_N) & \mathcal{D}_{N-1}(H_c)\\ \mathcal{D}_j(u_1) & \dots & \mathcal{D}_j(u_N) & \mathcal{D}_j(H_c)\end{pmatrix}=0 \qquad \forall(\xi,\eta)\in\Omega\times\Omega. \tag{15}$$

We notice that the first row of (15) is equal to 0 at $\xi=0$ by (7) and by definition of H_c.

We have now to consider two cases: the first is when there exists a generalized Wronskian which does not vanish identically on Ω and the second when all generalized Wronskians vanish identically on Ω.

The case when there exists a generalized Wronskian which does not vanish identically on Ω. In this case, the condition (11) follows straightforwardly from the Laplace's expansion of (15) w.r.t. the last column, by suitably choosing the operators $\mathcal{D}_0,\dots,\mathcal{D}_{N-1}$ so that the corresponding Wronskian of $u_1(\xi),\dots,u_N(\xi)$ does not vanish identically on Ω.

The case when all generalized Wronskians vanish identically on Ω. Firstly, we notice that we cannot have open subsets of Ω consisting only of zero-order points. Indeed, if we assume the existence of a subset $\tilde{\Omega}\subseteq\Omega$ of such type, all the restrictions of the function $u_1,\dots,u_N$ to $\tilde{\Omega}$ would be identically zero. Being these functions the components of an immersion, we would get a contradiction. Now, we pick a point $p\in\Omega$ of maximal order. Hence, up to renaming the functions u_i, we have

$$\det\begin{pmatrix}\mathcal{D}_0(u_1) & \dots & \mathcal{D}_0(u_{\mathrm{ord}(p)})\\ \vdots & & \vdots\\ \mathcal{D}_{\mathrm{ord}(p)-1}(u_1) & \dots & \mathcal{D}_{\mathrm{ord}(p)-1}(u_{\mathrm{ord}(p)})\end{pmatrix}(p)\neq 0.$$

By taking into account the system

$$\begin{pmatrix} \mathcal{D}_0(u_1) & \dots & \mathcal{D}_0(u_N) \\ \vdots & & \vdots \\ \mathcal{D}_{\mathrm{ord}(p)-1}(u_1) & \dots & \mathcal{D}_{\mathrm{ord}(p)-1}(u_N) \\ \mathcal{D}_j(u_1) & \dots & \mathcal{D}_j(u_N) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} \mathcal{D}_0(H_c) \\ \vdots \\ \mathcal{D}_{\mathrm{ord}(p)-1}(H_c) \\ \mathcal{D}_j(H_c) \end{pmatrix},$$

where the subscript j is arbitrary chosen in $\mathbb{N}$, we have, up to renaming u_i, that

$$\det \begin{pmatrix} \mathcal{D}_0(u_1) & \dots & \mathcal{D}_0(u_{\mathrm{ord}(p)}) & \mathcal{D}_0(H_c) \\ \vdots & & \vdots & \vdots \\ \mathcal{D}_{\mathrm{ord}(p)-1}(u_1) & \dots & \mathcal{D}_{\mathrm{ord}(p)-1}(u_{\mathrm{ord}(p)}) & \mathcal{D}_{\mathrm{ord}(p)-1}(H_c) \\ \mathcal{D}_j(u_1) & \dots & \mathcal{D}_j(u_{\mathrm{ord}(p)}) & \mathcal{D}_j(H_c) \end{pmatrix} = 0 \qquad \forall (\xi, \eta) \in \Omega \times \Omega.$$

The condition (11) follows from the Laplace's expansion w.r.t. the last column of the previous determinant.

With a similar reasoning, we get (12).

Now we prove that condition stated in the Theorem are also sufficient under the hypothesis that Ω', given by (13), be a connected open dense subset of Ω. This will be done in several steps.

Step 1. To start with, in this step, we define a distinguished set of multi-indices (that we are going to call $\mathcal{I}$ with slight abuse of notation), whose cardinality is strictly related to the dimension of the ambient space form.

Let $\mathcal{I} = \bigcup_{i \in \mathbb{N}} \mathcal{I}_i \subseteq \mathbb{N}^n$, where

- $\mathcal{I}_0 = \{I_0\}$,
- $\mathcal{I}_{i+1} = \mathcal{I}_i \cup \{I_{i+1}\}$ if

$$\sum_{I \in \mathcal{I}_i \cup \{I_{i+1}\}} c_I(\xi) \frac{\partial^{|I|} H_c}{\partial \xi^I}(\xi, \eta) \equiv 0$$

holds true if and only if all the $c_I \in \mathcal{C}^\infty(\Omega)$ vanish identically on Ω. Otherwise $\mathcal{I}_{i+1} = \mathcal{I}_i$.

Therefore, in view of the conditions (11) and by construction, $\#\mathcal{I} \in \mathbb{N}$. Let

$$N = \#\mathcal{I}.$$

Below we are going to show the existence of a strongly full para-Kähler immersion

$$f : (M^n, \omega) \to (\mathcal{S}_c^N, \omega_c).$$

Step 2. In this step, we will study a particular system of first order PDEs whose compatibility is strictly related to the existence of a para-Kähler immersion (6).

We define $e_k \in \mathbb{N}^n$ as follows:

$$e_k = (0, \dots, 0, 1, 0 \dots, 0), \quad \text{where 1 is at the } k\text{-th position.}$$

The condition (11) reads in particular as

$$a(\xi)\,\frac{\partial^{|I|+1}H_c}{\partial\xi^{I+e_k}}(\xi,\eta)+\sum_{J\in\mathcal{I}}a^{J}_{I+e_k}(\xi)\,\frac{\partial^{|J|}H_c}{\partial\xi^{J}}(\xi,\eta)\equiv 0, \tag{16}$$

for any $I\in\mathcal{I}$ and for $1\leq k\leq n$. The functions

$$A_k{}^J_I:=\frac{a^J_{I+e_k}}{a}$$

are defined on the open non-empty subset Ω', see (13). Now we are going to study the following system of first order PDEs,

$$\frac{\partial U_{IJ}}{\partial\xi_k}-\sum_{L\in\mathcal{I}}A_k{}^L_J\,U_{IL}=0\,,\quad k=1,\ldots,n\,, \tag{17}$$

where U_{IJ} are N^2 unknown functions defined on Ω', whose solutions will allow us to prove that H_c reads as (10). This will be done in Step 3.

By recalling the definition (1) of ι, we can define the $N\times N$ matrices U and A_k, for any $k\in\{1,\ldots,n\}$, as the matrices whose entries are, respectively, $U_{\iota(I)\iota(J)}$ and $A_k{}^{\iota(I)}_{\iota(J)}$, where $(I,J)\in\mathcal{I}\times\mathcal{I}$:

$$U=\left(U_{\iota(I)\iota(J)}\right)\,,\quad A_k=\left(A_k{}^{\iota(I)}_{\iota(J)}\right)\,. \tag{18}$$

Keeping in mind this notation, system (17) assumes the more concise form:

$$\frac{\partial U}{\partial\xi_k}-U\cdot A_k=0\,,\quad k=1,\ldots,n\,. \tag{19}$$

We can successfully study the above system thanks to the following lemma, that is a classical result coming from the Frobenius Theorem on the complete integrability of vector distributions [12].

Lemma 18. *Let $\xi=(\xi_1,\ldots,\xi_n)$ be coordinates on a connected open subset $\mathcal{W}$ and $V=(V_1,\ldots,V_m)$, where $V_i\in\mathcal{C}^\infty(\mathcal{W})$, be a row vector. The following Cauchy problem*

$$\begin{cases}\frac{\partial V}{\partial\xi_k}+V\cdot B_k=0\\[2ex] V(\xi_0)=V_0\end{cases} \tag{20}$$

where $B_k=\left(B_k{}^i_j\right)$ with $B_k{}^i_j\in\mathcal{C}^\infty(\mathcal{W})$ and $\xi_0\in\mathcal{W}$, such that the compatibility conditions

$$\frac{\partial B_h}{\partial\xi_k}+B_h\cdot B_k-\frac{\partial B_k}{\partial\xi_h}-B_k\cdot B_h=0\,,\quad\forall\,k,h\,, \tag{21}$$

are satisfied, admits a unique solution defined on $\mathcal{W}$.

Proof. We only notice that (20) is a Pfaffian system of m unknown functions V_i and n independent variables ξ_k, i.e., any solution $V = V(\xi)$ annihilates the m differential 1-forms

$$\rho_i = dV_i + \sum_{a=1}^{m}\sum_{k=1}^{n} B_{k}{}_{i}^{a} V_a \, d\xi_k \,, \quad i = 1, \ldots, m\,.$$

The system $\{\rho_i = 0\}_{i=1,\ldots,m}$ has constant maximal rank equal to m and, since its (Frobenius) compatibility conditions

$$d\rho_i = \sum_{j=1}^{m} \rho_{ij} \wedge \rho_j \,, \quad i = 1, \ldots, m$$

where ρ_{ij} are suitable 1-differential forms, are satisfied if and only if (21) holds, the assertion of the lemma follows. □

Now, going back to system (17), since it is of type (20) (with some obvious substitutions), in order to apply Lemma 18 to such system, we shall show that its compatibility conditions are satisfied. To this aim, in view of (16), we have

$$\begin{aligned}\frac{\partial}{\partial \xi_h}\frac{\partial^{|I|+1} H}{\partial \xi^{I+e_k}} &= -\sum_{J\in\mathcal{I}} \frac{\partial A_{k}{}_{I}^{J}}{\partial \xi_h}\frac{\partial^{|J|} H}{\partial \xi^{J}} - \sum_{J\in\mathcal{I}} A_{k}{}_{I}^{J}\frac{\partial^{|J|+1} H}{\partial \xi^{J+e_h}}\\ &= \sum_{J\in\mathcal{I}}\left(-\frac{\partial A_{k}{}_{I}^{J}}{\partial \xi_h} + \sum_{L\in\mathcal{I}} A_{k}{}_{I}^{L} A_{h}{}_{L}^{J}\right)\frac{\partial^{|J|} H}{\partial \xi^{J}}\end{aligned} \tag{22}$$

and

$$\frac{\partial}{\partial \xi_k}\frac{\partial^{|I|+1} H}{\partial \xi^{I+e_h}} = \sum_{J\in\mathcal{I}}\left(-\frac{\partial A_{h}{}_{I}^{J}}{\partial \xi_k} + \sum_{L\in\mathcal{I}} A_{h}{}_{I}^{L} A_{k}{}_{L}^{J}\right)\frac{\partial^{|J|} H}{\partial \xi^{J}}\,. \tag{23}$$

By subtracting (23) to (22) and by taking into account the construction of $\mathcal{I}$, we get

$$\frac{\partial A_{h}{}_{I}^{J}}{\partial \xi_k} - \frac{\partial A_{k}{}_{I}^{J}}{\partial \xi_h} + \sum_{L\in\mathcal{I}} A_{h}{}_{L}^{J} A_{k}{}_{I}^{L} - \sum_{L\in\mathcal{I}} A_{k}{}_{L}^{J} A_{h}{}_{I}^{L} = 0\,,$$

that are exactly the compatibility conditions of system (17).

Let us adopt the notation (18)-(19). We underline that if a solution $U = U(\xi)$ of system (19) is such that $\det(U(\xi')) \neq 0$ for some $\xi' \in \Omega'$ (see (13)), then $\det(U(\xi)) \neq 0$ for all $\xi \in \Omega'$. This follows from the following consideration. If, by contradiction, there existed $\xi_0 \in \Omega'$ such that $\det(U(\xi_0)) = 0$, then it would exist a (row) vector $v \neq 0$ such that $v \cdot U(\xi_0) = 0$. Therefore, the function $v \cdot U : \xi \mapsto v \cdot U(\xi)$ would be a solution of the following Cauchy problem

$$\begin{cases} \frac{\partial}{\partial \xi_k}(v \cdot U) - (v \cdot U) \cdot A_k = 0 \\ \\ (v \cdot U)(\xi_0) = 0 \end{cases}$$

implying, in view of Lemma 18, $(v \cdot U)(\xi) = 0$ for all $\xi \in \Omega'$, in particular $(v \cdot U)(\xi') = v \cdot U(\xi') = 0$, leading to $\det(U(\xi')) = 0$, a contradiction.

Step 3. In this step, we will show that the equality (10) holds true on $\Omega' \times \Omega$.

To this aim, we fix $\xi_0 \in \Omega'$ and a matrix $U_0 \in \mathbb{R}^{N,N}$ with $\det U_0 \neq 0$. Taking into account the last part of Step 2 and Lemma 18, there exists a unique solution $U = U(\xi)$, defined on Ω', to system (19) such that $U(\xi_0) = U_0$. By recalling the definition (1) of ι, if

$$v_{\iota(I)} = \sum_{J\in\mathcal{I}} U_{IJ} \frac{\partial^{|J|} H}{\partial \xi^J}, \tag{24}$$

then $\frac{\partial v_{\iota(I)}}{\partial \xi_k} \equiv 0$ for any $I \in \mathcal{I}$ and for any $1 \leq k \leq n$. In fact, by (16) and (17)

$$\begin{aligned}\frac{\partial v_{\iota(I)}}{\partial \xi_k} &= \sum_{J\in\mathcal{I}} \frac{\partial U_{IJ}}{\partial \xi_k} \frac{\partial^{|J|} H}{\partial \xi^J} + \sum_{J\in\mathcal{I}} U_{IJ} \frac{\partial^{|J+1|} H}{\partial \xi^{J+e_k}} \\ &= \sum_{J\in\mathcal{I}} \sum_{L\in\mathcal{I}} A_k{}^L_J \, U_{IL} \frac{\partial^{|J|} H}{\partial \xi^J} - \sum_{J\in\mathcal{I}} \sum_{L\in\mathcal{I}} A_k{}^L_J \, U_{IL} \frac{\partial^{|J|} H}{\partial \xi^J} = 0\,.\end{aligned}$$

Therefore, $v_{\iota(I)} = v_{\iota(I)}(\eta) \in \mathcal{C}^\infty(\Omega)$ for any $I \in \mathcal{I}$. Furthermore, we notice that, by definition of H_c, we have $\frac{\partial^{|J|} H_c}{\partial \xi^J}(\xi, 0) = 0$ for any $J \in \mathbb{N}^n$ and for any $\xi \in \Omega$. Hence, in view of (24), we have

$$v_i(0) = 0, \qquad \forall i = 1\ldots, N.$$

Keeping in mind the notation (18) and let

$$u(\xi) = (1, 0, \ldots, 0) \cdot U^{-1}$$

where $(1, 0, \ldots, 0)$ is a $1 \times N$ matrix, we have, by construction,

$$\sum_{i=1}^N u_i(\xi) v_i(\eta) = H(\xi, \eta), \qquad \forall (\xi, \eta) \in \Omega' \times \Omega. \tag{25}$$

Step 4. In this step, we are going to extend (25) to $\Omega \times \Omega$ when the linear closure of $(v_1, \ldots, v_N)\,(\Omega)$ has dimension N. Under such assumption, there exist some $\eta_1, \ldots, \eta_N \in \Omega$ such that

$$\det \begin{pmatrix} v_1(\eta_1) & \ldots & v_1(\eta_N) \\ \vdots & & \vdots \\ v_N(\eta_1) & \ldots & v_N(\eta_N) \end{pmatrix} \neq 0.$$

By considering (25), we have

$$\begin{aligned}(u_1, \ldots, u_N)\,(\xi) &= \big(H_c(\xi, \eta_1), \ldots, H_c(\xi, \eta_N)\big) \\ &\quad \times \begin{pmatrix} v_1(\eta_1) & \ldots & v_1(\eta_N) \\ \vdots & & \vdots \\ v_N(\eta_1) & \ldots & v_N(\eta_N) \end{pmatrix}^{-1}, \qquad \forall \xi \in \Omega'.\end{aligned}$$

Therefore, $u_1, \ldots, u_N$ admit a smooth extension to Ω. Thus, we have that (25) holds true on the whole $\Omega \times \Omega$. Moreover, since $H_c(0, \eta) \equiv 0$ by construction, we have that $(u_1, \ldots, u_N)(0) = 0$.

Step 5. Finally, we show that the linear closure of $(v_1, \ldots, v_N)(\Omega)$ cannot have dimension less than N. Indeed, if we assume by contradiction the existence of some $\eta_1, \ldots, \eta_\rho \in \Omega$ such that

$$\operatorname{rank} \begin{pmatrix} v_1(\eta_1) & \ldots & v_1(\eta_\rho) \\ \vdots & & \vdots \\ v_N(\eta_1) & \ldots & v_N(\eta_\rho) \end{pmatrix} = \rho \tag{26}$$

and

$$\operatorname{rank} \begin{pmatrix} v_1(\eta_1) & \ldots & v_1(\eta_\rho) & v_1(\eta) \\ \vdots & & \vdots & \vdots \\ v_N(\eta_1) & \ldots & v_N(\eta_\rho) & v_N(\eta) \end{pmatrix} = \rho, \qquad \forall \eta \in \Omega, \tag{27}$$

then, by considering the construction (24) of the functions v (recall that U is an invertible matrix) and by taking into account (27), we have that

$$\operatorname{rank} \begin{pmatrix} \frac{\partial^{|I_{k_1}|} H_c}{\partial \xi^{I_{k_1}}}(\xi, \eta_1) & \ldots & \frac{\partial^{|I_{k_1}|} H_c}{\partial \xi^{I_{k_1}}}(\xi, \eta_\rho) & \frac{\partial^{|I_{k_1}|} H_c}{\partial \xi^{I_{k_1}}}(\xi, \eta) \\ \vdots & & \vdots & \vdots \\ \frac{\partial^{|I_{k_N}|} H_c}{\partial \xi^{I_{k_N}}}(\xi, \eta_1) & \ldots & \frac{\partial^{|I_{k_N}|} H_c}{\partial \xi^{I_{k_N}}}(\xi, \eta_\rho) & \frac{\partial^{|I_{k_N}|} H_c}{\partial \xi^{I_{k_N}}}(\xi, \eta) \end{pmatrix} = \rho, \qquad \forall \eta \in \Omega, \ \forall \xi \in \Omega',$$

where $\mathcal{I} = \{I_{k_1}, \ldots, I_{k_N}\}$. It follows that

$$\det \begin{pmatrix} \frac{\partial^{|I_{k_1}|} H_c}{\partial \xi^{I_{k_1}}}(\xi, \eta_1) & \ldots & \frac{\partial^{|I_{k_1}|} H_c}{\partial \xi^{I_{k_1}}}(\xi, \eta_\rho) & \frac{\partial^{|I_{k_1}|} H_c}{\partial \xi^{I_{k_1}}}(\xi, \eta) \\ \vdots & & \vdots & \vdots \\ \frac{\partial^{|I_{k_{\rho+1}}|} H_c}{\partial \xi^{I_{k_{\rho+1}}}}(\xi, \eta_1) & \ldots & \frac{\partial^{|I_{k_{\rho+1}}|} H_c}{\partial \xi^{I_{k_{\rho+1}}}}(\xi, \eta_\rho) & \frac{\partial^{|I_{k_{\rho+1}}|} H_c}{\partial \xi^{I_{k_{\rho+1}}}}(\xi, \eta) \end{pmatrix} = 0, \quad \forall \eta \in \Omega, \ \forall \xi \in \Omega'.$$

By the Laplace's expansion of the previous determinant w.r.t the last column, we get

$$\sum_{i=1}^{\rho+1} c_i(\xi) \frac{\partial^{|I_{k_i}|} H_c}{\partial \xi^{I_{k_i}}}(\xi, \eta) = 0, \qquad \forall (\xi, \eta) \in \Omega' \times \Omega,$$

where

$$c_i : \Omega \to \mathbb{R}$$

denotes the cofactor correspondent to i-th entry of the last column. By taking into account (26) and (24), not all c_i can be identically zero. Since we are assuming that all the indices I_{k_i} belong to $\mathcal{I}$, we get a contradiction to the construction of $\mathcal{I}$, cfr. Step 1.

Step 6.

We notice that the para-Kähler immersion obtained by means the previous steps is strongly full. In fact, with a reasoning similar the one adopted in Step 5, we can prove that the dimension of the linear closure $(v_1,\ldots,v_N)(\tilde\Omega)$ is N for any open subset $\tilde\Omega$ of Ω. Moreover, also the dimension of the linear closure $(u_1,\ldots,u_N)(\tilde\Omega)$ is N for any open subset $\tilde\Omega$ of Ω. Indeed, by differentiating (10) w.r.t. ξ^{I_k} for any $I_k\in\mathcal{I}$, we get a linear system whose compatibility condition reads as

$$\operatorname{rank}\begin{pmatrix} \frac{\partial^{|I_{k_1}|}u_1}{\partial\xi^{I_{k_1}}} & \cdots & \frac{\partial^{|I_{k_1}|}u_N}{\partial\xi^{I_{k_1}}} \\ \vdots & & \vdots \\ \frac{\partial^{|I_{k_N}|}u_1}{\partial\xi^{I_{k_N}}} & \cdots & \frac{\partial^{|I_{k_N}|}u_N}{\partial\xi^{I_{k_N}}} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \frac{\partial^{|I_{k_1}|}u_1}{\partial\xi^{I_{k_1}}} & \cdots & \frac{\partial^{|I_{k_1}|}u_N}{\partial\xi^{I_{k_1}}} & \frac{\partial^{|I_{k_1}|}H_c}{\partial\xi^{I_{k_1}}} \\ \vdots & & \vdots & \vdots \\ \frac{\partial^{|I_{k_N}|}u_1}{\partial\xi^{I_{k_N}}} & \cdots & \frac{\partial^{|I_{k_N}|}u_N}{\partial\xi^{I_{k_N}}} & \frac{\partial^{|I_{k_N}|}H_c}{\partial\xi^{I_{k_N}}} \end{pmatrix}$$

on $\Omega\times\Omega$.

Hence, if the left side of the previous equality is at most equal to $\rho<N$ on $\tilde\Omega\subset\Omega$, then we have

$$\det\begin{pmatrix} \frac{\partial^{|I_{k_1}|}u_1}{\partial\xi^{I_{k_1}}} & \cdots & \frac{\partial^{|I_{k_1}|}u_N}{\partial\xi^{I_{k_1}}} & \frac{\partial^{|I_{k_1}|}H_c}{\partial\xi^{I_{k_1}}} \\ \vdots & & \vdots & \vdots \\ \frac{\partial^{|I_{k_{\rho+1}}|}u_1}{\partial\xi^{I_{k_{\rho+1}}}} & \cdots & \frac{\partial^{|I_{k_{\rho+1}}|}u_N}{\partial\xi^{I_{k_{\rho+1}}}} & \frac{\partial^{|I_{k_{\rho+1}}|}H_c}{\partial\xi^{I_{k_{\rho+1}}}} \end{pmatrix} = 0, \qquad \forall\eta\in\Omega,\ \forall\xi\in\tilde\Omega.$$

Since we can assume w.l.g. that not all the $(\rho\times\rho)$-minors of the previous matrix vanish identically on $\tilde\Omega$, the Laplace's expansion w.r.t. the last column leads us to a contradiction to the construction of $\mathcal{I}$, cfr. Step 1. □

To conclude, we provide an example of some para-Kähler metrics satisfying the connectedness and density conditions of (13).

Example 19. Let $(\xi,\eta)=(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_n)$ be a system of null-coordinates on an n-dimensional para-Kähler manifold, such that the restriction $D_0(\xi,\eta)$ of its diastasis function to a neighborhood $\Omega\times\Omega$ of the center of the coordinate system, is related (cfr. (9)) to the function H_c reading as

$$H_c(\xi,\eta)=\sum_{i=1}^n\xi_i\eta_i+\left(\xi_1^2+\ldots+\xi_n^2\right)^h\left(\eta_1^2+\ldots+\eta_n^2\right)^k,$$

where h and k denote two arbitrary positive integer numbers. Via straightforward computations, we get that

$$\left(\eta_1^2+\ldots+\eta_n^2\right)^k\frac{\partial^{|I|}H_c}{\partial\eta^I}+a_I(\eta)\left(\sum_{i=1}^n\eta_i\frac{\partial H_c}{\partial\eta_i}-H_c\right)=0$$

holds true for any $I\in\mathbb{N}^n$ such that $|I|\geq 2$ (after suitably choosing the smooth functions $a_I:\Omega\to\mathbb{R}$). Therefore, the set Ω', defined by (13), can be obtained from Ω by removing only the projection on Ω of the center of the coordinates system.

3.3. On the Local Character and Global Extendability of Para-Kähler Immersions

As proved by Calabi in [4], if an open subset of a connected Kähler manifold can be holomorphically and isometrically immersed (i.e. Kähler immersed) into a complex space form, then any point of such manifold admits a neighborhood that can be Kähler immersed in the same ambient space. Moreover, any Kähler immersion into a complex space form defined on an open subset of a simply connected Kähler manifold can be extended to an immersion defined on the whole manifold.

As the following example shows, such global aspects of local Kähler immersions are not shared with their para-Kähler counterparts.

Indeed, below we are going to show that there exists a para-Kähler structure ω on $\mathbb{D} = \mathbb{D}^1$ for which, for any point $z_0 \in \mathbb{D}$, there exist neighborhoods of z_0 covering $\mathbb{D}$, such that they are weakly full para-Kähler immersed in flat para-Kähler space forms of different dimensions. As a consequence, it turns out that $(\mathbb{D}, \omega)$ cannot be globally para-Kähler immersed into any flat para-Kähler space form.

Let $i \in \mathbb{N}$ and let $u_i : \mathbb{R} \to \mathbb{R}$ be the smooth (non-analytic in i) function defined by

$$u_i(\xi) = \begin{cases} \exp\left(-\frac{1}{(\xi - i)^{i+1}}\right) & \text{if } \xi > i \\ 0 & \text{if } \xi \leq i \end{cases}$$

We consider the one dimensional para-complex manifold $\mathbb{D}$ endowed with the symplectic form associated to the (global) potential

$$\begin{array}{rccl} \Phi : & \mathbb{D} & \longrightarrow & \mathbb{R} \\ & (\xi e + \eta \bar{e}) & \longmapsto & \xi\eta + \sum_{i=0}^{+\infty} u_i(\xi)\eta^{2i+3}, \end{array}$$

namely, the 2-form reading in null-coordinates (ξ, η) as

$$\omega = \frac{1}{2}\left(1 + \sum_{i=0}^{+\infty}(2i+3)u_i'(\xi)\eta^{2i+2}\right) d\xi \wedge d\eta,$$

where

$$u_i'(\xi) = \begin{cases} \frac{\exp\left(-\frac{1}{(\xi - i)^{i+1}}\right)}{(\xi - i)^{i+2}} & \text{if } \xi > i \\ 0 & \text{if } \xi \leq i \end{cases}$$

Let

$$\mathcal{U}_i := \{\xi e + \eta \bar{e} \in \mathbb{D} \mid (\xi, \eta) \in (-\infty, i+1) \times \mathbb{R}\}, \quad i \in \mathbb{N}.$$

Since

$$\Phi\big|_{\mathcal{U}_i}(\xi, \eta) = \xi\eta + \sum_{j=0}^{i} u_j(\xi)\eta^{2j+3},$$

in view of (9), there exists a weakly full para-Kähler immersion of $\left(\mathcal{U}_i, \omega\big|_{\mathcal{U}_i}\right)$ into $(\mathbb{D}^{i+1}, \omega_0)$.

Furthermore, by considering that $\mathbb{D} = \cup_{i\in\mathbb{N}} \mathcal{U}_i$, we conclude that $(\mathbb{D}, \omega)$ cannot be globally para-Kähler immersed into any flat para-Kähler space form.

4. Para-Kähler Space Forms that Admit a Para-Kähler Immersion into Another

As an application of Theorem 17, in the present section we classify para-Kähler immersions between para-Kähler space forms. In particular, we are going to prove the following theorem.

Theorem 20. *A para-Kähler space form $(\mathcal{S}^n_c, \omega_c)$ can be locally para-Kähler immersed into $(\mathcal{S}^N_b, \omega_b)$, where $N \geq n$, if and only if either $c = b = 0$ or $b/c \in \mathbb{Z}^+$. Such local immersions can be extended to the whole manifold. Moreover, if $N = n$ (when $c = b = 0$), or if $N = \binom{n+\frac{b}{c}}{n} - 1$ (when $\frac{b}{c} \in \mathbb{Z}^+$), these immersions are strongly full.*

Proof. Since para-Kähler space forms are homogeneous with respect to the action of their para-holomorphic isometry group (see the end of Section 2.2), the existence (or non-existence) of a para-Kähler immersion defined on a neighborhood of an arbitrary chosen point implies the existence (or non-existence) of a para-Kähler immersion defined on a neighborhood of any other point. Therefore, we can study, without loss of generality, only local para-Kähler immersions defined either on a neighborhood of $0 \in \mathcal{S}^n_0$ or on a neighborhood of $[1, 0 \ldots, 0] \in \mathcal{S}^n_c$, $c \neq 0$. Let us fix a null coordinate system (ξ, η) around such point. Keeping in mind the definitions (4), (5) and (9), we compute the function H_b in order to apply Theorem 17. Notice that a flat complex space form $(\mathcal{S}^n_0, \omega_0)$ can be trivially para-Kähler immersed by inclusion into $(\mathcal{S}^N_0, \omega_0)$, hence only the following cases will be taken into account:

- $c \neq 0$ and $b = 0$;
- $c = 0$ and $b \neq 0$;
- $c \neq 0$ and $b \neq 0$.

Below we treat the above cases separately.

• If $c \neq 0$ and $b = 0$, by taking into account (9) and that the diastasis $D_0^{\mathcal{S}^n_c}$ is given by (5), we have that

$$H_0(\xi, \eta) = \frac{2}{c} \log\left(1 + 2\sum_{i=1}^{n} \xi_i \eta_i\right).$$

By differentiating the previous equation with respect to ξ_1, we get, for any $k \in \mathbb{Z}^+$,

$$\frac{\partial^k H_0}{\partial \xi_1^k} = \frac{(-2)^{k+1}(k-1)!}{c\left(1+2\sum_{i=1}^n \xi_i \eta_i\right)^k}\ \eta_1^k .$$

Let now assume that the condition (11) of Theorem 17 holds true for any $\mathcal{I} \subset \mathbb{N}^n$ such that $\#\mathcal{I} = N+1$. In particular, if

$$\mathcal{I} = \{(k, 0, \ldots, 0) \mid k = 1, \ldots, N+1\}, \tag{28}$$

we have that

$$\sum_{k=1}^{N+1} a_k(\xi) \frac{\partial^k H_0}{\partial \xi_1^k}(\xi,\eta) = \frac{\sum_{k=1}^{N+1} (-2)^{k+1}(k-1)!\left(1+2\sum_{i=1}^n \xi_i\eta_i\right)^{N+1-k} a_k(\xi)\eta_1^k}{c\left(1+2\sum_{i=1}^n \xi_i\eta_i\right)^{N+1}} \equiv 0.$$

Since the numerator in the above equality is a polynomial in $\eta_1, \ldots, \eta_n$ and the coefficient of its monomial in η_1 of degree 1 is $4a_1(\xi)$, it follows that $a_1(\xi)$ needs to be identically zero. By means of similar considerations, we get also that $a_k(\xi) \equiv 0$ for any $k = 2, \ldots, N+1$. In view of Theorem 17, we conclude that there are no local para-Kähler immersions of a non-flat para-Kähler space form into a flat one.

• If $c = 0$ and $b \neq 0$, by taking into account (9) and that the diastasis $D_0^{\mathcal{S}_0^n}$ is given by (4), we have that

$$H_b(\xi,\eta) = \frac{1}{2}\exp\left(\frac{b}{2}\sum_{i=1}^n \xi_i\eta_i\right) - \frac{1}{2}.$$

By differentiating the previous equation with respect to ξ_1, we get, for any $k \in \mathbb{Z}^+$,

$$\frac{\partial^k H_b}{\partial \xi_1^k} = \frac{b^k}{2^{k+1}}\exp\left(\frac{b}{2}\sum_{i=1}^n \xi_i\eta_i\right)\ \eta_1^k .$$

Now we assume that the condition (11) of Theorem 17 holds true for any $\mathcal{I} \subset \mathbb{N}^n$ such that $\#\mathcal{I} = N+1$. In particular, if we consider the subset $\mathcal{I}$ of multi-indices (28), we have that

$$\sum_{k=1}^{N+1} a_k(\xi)\frac{\partial^k H_b}{\partial \xi_1^k}(\xi,\eta) = \exp\left(\frac{b}{2}\sum_{i=1}^n \xi_i\eta_i\right)\sum_{k=1}^{N+1}\frac{b^k a_k(\xi)}{2^{k+1}}\eta_1^k \equiv 0,$$

that implies $a_k(\xi) \equiv 0$ for any $k = 1, \ldots, N+1$. In view of Theorem 17, we conclude that there are no local para-Kähler immersions of a flat para-Kähler space forms into a non-flat one.

• If $c \neq 0$ and $b \neq 0$, by taking into account (9) and that the diastasis $D_0^{\mathcal{S}_c^n}$ is given by (5), we have that

$$H_b(\xi,\eta) = \frac{1}{2}\left(1+2\sum_{i=1}^n \xi_i\eta_i\right)^{\frac{b}{c}} - \frac{1}{2}.$$

By differentiating the previous equation with respect to ξ_1, we get, for any $k \in \mathbb{Z}^+$,

$$\frac{\partial^k}{\partial \xi_1^k} H_b = 2^{k-1} \prod_{j=0}^{k-1} \left(\frac{b}{c} - j \right) \left(1 + 2 \sum_{i=1}^{n} \xi_i \eta_i \right)^{\frac{b}{c} - k} \eta_1^k .$$

Now we assume that the condition (11) of Theorem 17 holds true for any $\mathcal{I} \subset \mathbb{N}^n$ such that $\#\mathcal{I} = N + 1$. In particular, if we consider the subset $\mathcal{I}$ of multi-indices (28), we have that

$$\sum_{k=1}^{N+1} a_k(\xi) \frac{\partial^k H_b}{\partial \xi_1^k}(\xi, \eta) = \left(1 + 2 \sum_{i=1}^{n} \xi_i \eta_i \right)^{\frac{b}{c} - N - 1} \sum_{k=1}^{N+1} a_k(\xi) 2^{k-1} \prod_{j=0}^{k-1} \left(\frac{b}{c} - j \right) \left(1 + 2 \sum_{i=1}^{n} \xi_i \eta_i \right)^{N+1-k} \eta_1^k \equiv 0.$$

Since the above equality can be seen as a polynomial in $\eta_1, \ldots, \eta_n$ and the coefficient of its monomial in η_1 of degree 1 is $\frac{b}{c} a_1(\xi)$, it follows that $a_1(\xi)$ needs to be identically zero. If $\frac{b}{c} \notin \mathbb{Z}^+$, then we get, by means of similar considerations, that also $a_k(\xi) \equiv 0$ for any $k = 2, \ldots, N + 1$. Otherwise, if $\frac{b}{c} \in \mathbb{Z}^+$, then H_b reads as (10), so, in view of Proposition 15 and Theorem 17, there exists, for any $i = 0, \ldots, n$, a strongly full local para-Kähler immersion

$$f_i : \left(\mathcal{U}_i = \{ [Z] \in \mathbb{DP}^n \mid |Z_i|^2 \neq 0 \}, \omega_c \right) \to \left(\mathcal{S}_b^{\binom{n+\frac{b}{c}}{n} - 1}, \omega_b \right) .$$

Indeed, we need at least $\binom{n+\frac{b}{c}}{n} - 1$ monomials to describe the polynomial H_b.

Finally, we notice that it follows from the Rigidity Theorem 14 the existence of a para-holomorphic isometry F of the ambient space such that $f_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = F \circ f_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$. Hence, we have that any local immersion f_i can be extended to a global immersion. □

Acknowledgements

G. Manno gratefully acknowledges support by "Finanziamento alla Ricerca" 53_RBA17MANGIO and 53_RBA21MANGIO, and PRIN project 2022 "Real and Complex Manifolds: Topology, Geometry and holomorphic dynamics" (code 2022AP8HZ9). The second author is supported by the "Starting Grant" under the contact number 53_RSG22SALFIL. Both authors are members of GNSAGA of INdAM.

Data Availability Statement The authors declare that no associated data is included.

Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

Open access funding provided by Politecnico di Torino within the CRUI-CARE Agreement. G. Manno gratefully acknowledges support by "Finanziamento alla Ricerca" `53_RBA17MANGIO` and `53_RBA21MANGIO`, and PRIN project 2022 "Real and Complex Manifolds: Topology, Geometry and holomorphic dynamics" (code `2022AP8HZ9`). The second author is supported by the "Starting Grant" under the contact number `53_RSG22SALFIL`. Both authors are members of GNSAGA of INdAM.

References

[1] D. V. Alekseevsky, *Pseudo-Kähler and para-Kähler symmetric spaces*. Handbook of pseudo-Riemannian geometry and supersymmetry, 703–729, IRMA Lect. Math. Theor. Phys., 16, Eur. Math. Soc., Zürich, 2010

[2] Alekseevsky, D.V., Medori, C., Tomassini, A.: Homogeneous para-Kähler Einstein manifolds. Russ. Math. Surv. **64**, 1–43 (2009)

[3] Alekseevsky, D.V., Medori, C., Tomassini, A.: Maximally homogeneous para-CR manifolds. Ann. Global Anal. Geom. **30**(1), 1–27 (2006)

[4] Calabi, E.: Isometric Imbedding of Complex Manifolds. Ann. of Math. **58**(1), 1–23 (1953)

[5] Cruceanu, V., Fortuny, P., Gadea, P.M.: A survey on paracomplex geometry. Rocky Mountain J. Math. **26**, 83–115 (1996)

[6] P. M. Gadea, A. Montesinos Amilibia, *Spaces of constant para-holomorphic sectional curvature*. Pacific J. Math. 136 (1989) 85-101

[7] Harvey, F.R., Lawson, H.B.: Split special Lagrangian geometry in Metric and differential geometry. Progr. Math. **297**, 43–89 (2012)

[8] A. Loi, M. Zedda *Kähler immersions of Kähler manifolds into complex space forms*, Lecture notes of the Unione Matematica Italiana **23**, Springer (2018)

[9] Manno, G., Salis, F.: 2-dimensional Kähler-Einstein metrics induced by finite dimensional complex projective spaces. New York J. Math. **28**, 420–432 (2022)

[10] G. Manno, F. Salis, *Toric para-Kähler-Einstein manifolds immersed in para-Kähler space forms*, J. Geom. Phys., 218 (2025) Paper No. 105688, 16pp

[11] G. Manno, F. Salis, T^n*-invariant* Kahler-*Einstein manifolds immersed in complex projective spaces*, preprint, arXiv: 2407.12685

[12] Warner, F.W.: Foundations of Differentiable Manifolds and Lie Groups, GTM, vol. 94. Springer, New York, NY (1983)

Gianni Manno and Filippo Salis
Dipartimento di Scienze Matematiche "G. L. Lagrange"
Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino
Italy
e-mail: giovanni.manno@polito.it;
filippo.salis@polito.it

Received: April 23, 2025.
Accepted: November 10, 2025.