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Toric para-Kähler-Einstein manifolds immersed in para-Kähler space forms ☆

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ABSTRACT

A classical and long-staying problem addressed, among others, by Calabi and Chern, is that to find a complete list of mutually non-isometric Kähler-Einstein manifolds immersed in a finite-dimensional Kähler space form. We address the same problem in the para-Kähler context and, then, we find a list of mutually non-isometric toric para-Kähler manifolds analytically immersed in a finite-dimensional para-Kähler space form.

1. Introduction

1.1. Description of the problem and main results

An *almost para-complex manifold* is a $2n$-dimensional manifold M provided with a field of endomorphisms $\mathcal{T}$ such that $\mathcal{T}^2 = 1$, having eigenvalues 1 and -1, whose associated eigendistributions are n-dimensional. An almost para-complex manifold whose the aforementioned distributions are integrable, is called a *para-complex manifold*. A *para-Kähler manifold* is a para-complex manifold endowed with a symplectic form ω such that $g = \omega(\mathcal{T}(\cdot), \cdot)$ is a pseudo-Riemannian metric. A para-Kähler manifold having constant para-holomorphic sectional curvature is said *para-Kähler space form*: if the curvature is zero then it is called *flat* otherwise *non-flat*.

The formal analogy with the Kähler geometry makes it possible to state problems, originally formulated in the Kähler context, also in the para-Kähler case. For instance, a classical problem in Kähler geometry is the characterization of holomorphic and isometric immersions into Kähler space forms, i.e., into Kähler manifolds of constant holomorphic sectional

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curvature (see [4] and for a modern introduction to this subject [9]). This problem can be stated in a unified way, including also the para-Kähler case, as follows:

General problem:
To classify (para-)Kähler manifolds that can be (para-)Kähler immersed into (para-)Kähler space forms.

In the Kähler case, despite E. Calabi found in [4] some criteria that allow, at least from a theoretical viewpoint, to treat the above problem, a satisfactory classification is far to be obtained as the problem remains too underdetermined. In fact, even in special cases of great interest, such as *Kähler-Einstein* manifolds (see e.g. [7–9,12,13] for more details), a complete classification is still unknown. More precisely, one can ask to find Kähler-Einstein manifolds that can be Kähler immersed into Kähler space forms. This problem is indeed still open only when the ambient space has positive holomorphic sectional curvature: in this case, the Kähler-Einstein manifolds are called *projectively induced*. Even if one restricts to the class of *toric* projectively induced Kähler-Einstein manifolds, the problem of their characterization is only partially solved, see e.g. [2,11,12].

In the para-Kähler case, as in the Kähler setting, the above general problem is actually very challenging. The recent paper [10], where necessary and sufficient conditions for the existence of para-Kähler immersions in para-Kähler space forms have been found, is a first step in addressing the issue. In fact, one of the main difficulties one meets in facing the aforementioned problem, is that in the para-complex context, unlike the complex one, para-holomorphic functions are not, in general, analytic but only $\mathcal{C}^\infty$-smooth.

Therefore, we are going to focus our attention to the case in which the para-Kähler immersions are analytic and the para-Kähler metrics admit symmetries similar to the toric ones. More precisely, in this paper, we will study the following problem:

Problem 1. Classify toric para-Kähler-Einstein manifolds admitting an analytic para-Kähler immersion into a para-Kähler space form.

Another disadvantage in the para-Kähler context, unlike the Kähler one, is the absence of a notion of Bochner's coordinates, which makes the study of Problem 1 more involved. We overcome such problem by a case by case analysis.

Our main results are contained in the following theorems. Theorem 1.1 concerns Ricci-flat para-Kähler manifolds, that turn out to be the only para-Kähler-Einstein manifold embeddable into a flat para-Kähler space form, whereas Theorem 1.2 concerns the embeddability of toric non-flat para-Kähler manifolds into non-flat para-Kähler space forms.

Theorem 1.1. *Flat para-Kähler space forms are the only toric Ricci-flat para-Kähler manifolds that can be analytically para-Kähler immersed into a para-Kähler space form. In particular, they can be para-Kähler immersed only into another flat para-Kähler space form. Moreover, toric Ricci-flat para-Kähler manifolds are the only para-Kähler-Einstein manifolds that can be analytically para-Kähler immersed into a flat para-Kähler space form.*

Let $\mathbb{DP}^N$ be projective space constructed on the algebra of para-complex number $\mathbb{D}$ (cf. Section 3), endowed with the para-Kähler counterpart of the Fubini-Study metric, namely the para-Kähler metric g^N_{pFS} admitting

$$\log\left(|Z_0|^2 + \ldots + |Z_N|^2\right)$$

as para-Kähler potential, where $(Z_0, \ldots, Z_N)$ are homogeneous coordinates and $|\cdot|^2$ denotes the para-complex modulus.

Theorem 1.2. *The toric para-Kähler-Einstein manifolds $\mathbb{DP}^{n_1} \times \ldots \times \mathbb{DP}^{n_k}$ with the para-Kähler metric*

$$\frac{K}{h} \bigoplus_{i=1}^{k} (n_i + 1)\, g^{n_i}_{pFS},$$

where h denotes the greater common divisor between $\{n_1 + 1, \ldots, n_k + 1\}$, can be analytically para-Kähler immersed for any $K \in \mathbb{Z}^+$ into $\left(\mathbb{DP}^N, g^N_{pFS}\right)$ with N large enough.
If the dimension $\sum_{i=1}^k n_i$ is less or equal to 2, they are the only ones.

1.2. Description of the paper

In Section 2, we recall basic notions concerning para-complex geometry by introducing the algebra of para-complex numbers $\mathbb{D}$ and its cartesian product $\mathbb{D}^n$.

In Section 3, we focus our attention to para-Kähler manifolds, in particular, those of Einstein type. We indeed characterize them in terms of suitable Monge-Ampère equations and consider more closely the distinguished class of toric ones. Then, after introducing the diastasis function (a distinguished para-Kähler potential) in the para-Kähler context in a way similar to that performed in [4], we characterize para-Kähler space forms, i.e., para-Kähler manifolds with constant para-holomorphic sectional curvature, in terms of such function.
Section 4 is dedicated to the proof of Theorems 1.1 and 1.2. More precisely, in the beginning of the section, we prove a preliminary result telling that, if a toric para-Kähler manifold can be para-Kähler and analytically immersed into a para-Kähler space form, then it admits a polynomial (or a logarithm of a polynomial) potential (that turns out to be a diastasis' function). This allows to reformulate Problem 1 in terms of existence of particular solutions of a distinct Monge-Ampère equation.
In Section 4.1, based on the aforementioned reformulation, we prove Theorem 1.1.
In Section 4.2 we focus our attention to the proof of Theorem 1.2. As a first step, we refine Problem 1 as we are interested in a particular class of para-Kähler immersions, i.e., those into non-flat para-Kähler space forms. As a second step, we prove the technical Lemma 4.4, that is crucial for proving the second part of Theorem 1.2, which is addressed by separately analyzing the case of one-dimensional manifolds (Section 4.2.1) and the case of two-dimensional ones (Section 4.2.2). The validity of the first part of the theorem is shown by means of explicit computations presented in Section 4.2.3.

Notation. A multi-index $I=(i_1,\dots,i_n)$ is an element of $\mathbb{N}^n$ and its length $|I|$ is defined as the number $|I|:=\sum_{k=1}^n i_k$. If $(x_1,\dots,x_n)$ are local coordinates, we define the derivative operators $\frac{\partial^{|I|}}{\partial x^I}$ as follows:

$$\frac{\partial^{|I|}}{\partial x^I}:=\frac{\partial^{|I|}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}\,.$$

2. Basics of para-complex geometry

2.1. Para-holomorphic functions

The 2-dimensional algebra over $\mathbb{R}$ of *para-complex numbers* $\mathbb{D}$ is generated by 1 and τ, where

$$\tau^2=1\,.$$

In analogy with the complex numbers, we are going to adopt the notation used in [6]: each $z\in\mathbb{D}$ can be written as

$$z=x+\tau y\,,$$

and we are going to refer to x and y as the *real* and *imaginary part* of z, respectively. In analogy with the complex numbers, we define the conjugate of z

$$\bar z=x-\tau y$$

and

$$|z|^2=z\bar z=x^2-y^2.$$

It will be useful to introduce also another coordinate system on $\mathbb{D}$, described as follows. We switch the basis $(1,\tau)$ with $(e,\bar e)$, where

$$e=\frac{1}{2}(1-\tau)\,,\qquad \bar e=\frac{1}{2}(1+\tau)\,,$$

and we are going to say that (u,v) are the *null-coordinates* of z if $z=ue+v\bar e$. Note that null-coordinates are uniquely determined once we fix a coordinate system $z=x+\tau y$.

Now we can translate on $\mathbb{D}^n$ what we said about $\mathbb{D}$. In particular, for any $z,w\in\mathbb{D}^n$, we define

$$\langle z,w\rangle=\sum_{i=1}^n z_i\bar w_i$$

and

$$\|z\|^2:=\sum_{i=1}^n |z_i|^2.$$

Definition 2.1. A function

$$\begin{array}{rccl} F: & U \subseteq \mathbb{D}^n & \longrightarrow & \mathbb{D} \\ & (z_1,\dots,z_n) & \longmapsto & g(x_1,y_1,\dots,x_n,y_n)+\tau\, h(x_1,y_1,\dots,x_n,y_n), \end{array}$$

where $z_i = x_i + \tau y_i$, is called *para-holomorphic* if and only if g and h are smooth and

$$\frac{\partial F}{\partial \bar z_i} := \frac{1}{2}\left(\frac{\partial g}{\partial x_i} - \frac{\partial h}{\partial y_i}\right) + \frac{\tau}{2}\left(\frac{\partial h}{\partial x_i} - \frac{\partial g}{\partial y_i}\right) = 0$$

for any $1 \le i \le n$.

The number n stands for the *"para-complex" dimension* of $\mathbb{D}^n$. In analogy to the complex setting, the differential operator $\frac{\partial}{\partial z_i}$ is defined by

$$\frac{\partial F}{\partial z_i} := \overline{\left(\frac{\partial \bar F}{\partial \bar z_i}\right)}.$$

Remark 2.2. Let $F : U \subseteq \mathbb{D}^n \to \mathbb{D}$ be a para-holomorphic function. By considering the null-coordinates

$$(\xi,\eta) = (\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n)$$

on $\mathbb{D}^n$, that, as for $n = 1$, are uniquely determined once a coordinate system $z_i = x_i + \tau y_i$ is fixed, F can be written as follows:

$$F(\xi_1 e + \eta_1 \bar e, \dots, \xi_n e + \eta_n \bar e) = u(\xi_1,\eta_1,\dots,\xi_n,\eta_n)\, e + v(\xi_1,\eta_1,\dots,\xi_n,\eta_n)\, \bar e\,,$$

where u and v are real functions on an open subset of $\mathbb{R}^{2n}$. We then straightforwardly get

$$\frac{\partial F}{\partial \bar z_i} = \frac{\partial u}{\partial \eta_i} e + \frac{\partial v}{\partial \xi_i} \bar e\,.$$

Therefore, F is para-holomorphic if and only if u is independent of $(\eta_1,\dots,\eta_n)$ and v is independent of $(\xi_1,\dots,\xi_n)$.

Definition 2.3. A function

$$\begin{array}{rccc} F: & U \subseteq \mathbb{D}^n & \longrightarrow & \mathbb{D}^m \\ & z = (z_1,\dots,z_n) & \longmapsto & \big(f_1(z),\dots,f_m(z)\big) \end{array}$$

is para-holomorphic if and only if each component f_i is para-holomorphic.

3. Para-Kähler and para-Kähler-Einstein manifolds

As we said in the introduction, a para-complex manifold is an almost para-complex manifold such that eigendistributions of the almost para-complex structure $\mathcal{T}$ are integrable: this is the same to require the vanishing of the Nijenhuis tensor associated to $\mathcal{T}$. An equivalent definition is the following.

Definition 3.1. A smooth manifold M^n of para-complex dimension n is called *para-complex* if it admits an atlas of para-holomorphic coordinates $(z_1,\dots,z_n)$, such that the transition functions are para-holomorphic.

Below we shall give a slightly different definition of para-Kähler manifold (respect to that given in the introduction) which takes into consideration the concept of para-Kähler potential, that will be crucial for our purposes. We then clarify, in the Remark 3.3, the relationship between para-Kähler potential and para-Kähler metric.

Definition 3.2. A *para-Kähler manifold* of para-complex dimension n is a para-complex manifold M endowed with a symplectic form ω (called *para-Kähler form*) such that, for any point $p \in M$, there exists an open neighborhood $U \ni p$ and a smooth function $\phi : U \to \mathbb{R}$ (called *para-Kähler potential*) satisfying

$$\omega|_U = \frac{\tau}{2}\partial\bar\partial\phi$$

where

$$\partial\bar\partial\phi := \sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_i \partial \bar z_j} dz_i \wedge d\bar z_j\,.$$

Remark 3.3. To a given para-Kähler manifold (M, ω) of para-complex dimension n and para-complex structure $\mathcal{T}$, it is associated the pseudo-Riemannian metric $g = \omega(\mathcal{T}(\cdot), \cdot)$ on M. If $\phi : U \subset M \to \mathbb{R}$ is a para-Kähler potential of ω, the restriction of g to U can be obtained as the real part of

$$\sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j \,,$$

i.e., in null coordinates (ξ, η), the above pseudo-Riemannian metric reads as

$$\sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial \xi_i \partial \eta_j} d\xi_i \otimes d\eta_j \,. \tag{1}$$

Definition 3.4. A para-Kähler manifold (M, ω) is called *para-Kähler-Einstein* if the associated pseudo-Riemannian metric g is Einstein, namely, if the Ricci tensor of g is proportional to g via a constant $\lambda \in \mathbb{R}$:

$$\mathrm{Ric}(g) = \lambda g \,. \tag{2}$$

Let $\left(U, \frac{\tau}{2}\partial\bar{\partial}\phi\right)$ be a para-Kähler-Einstein manifold with Einstein constant equal to λ. Then, after a straightforward computation (see Section 5 of [1] for detailed computations of the Ricci tensor's components), we have that condition (2) is expressed by

$$\partial\bar{\partial}\left(\log\left|\det\left(\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}\right)\right| + \frac{\lambda}{2}\phi\right) = 0 \,, \quad 1 \le i, j \le n \,. \tag{3}$$

Hence, by taking into account the system of null coordinates (ξ, η), equation (3) translates into

$$\left|\det\left(\frac{\partial^2 \phi}{\partial \xi_i \partial \eta_j}\right)(\xi, \eta)\right| = e^{-\frac{\lambda}{2}\phi(\xi,\eta) + F(\xi) + G(\eta)} \,, \quad 1 \le i, j \le n \,, \tag{4}$$

where F and G are smooth functions.

3.1. Toric para-Kähler manifolds

Let $U \subseteq \mathbb{D}^n$ be an open neighborhood of 0, where it is defined a para-Kähler potential reading as

$$\phi(|z_1|^2, \ldots, |z_n|^2).$$

Hence, if $\xi = (\xi_1, \ldots, \xi_n)$ and $\eta = (\eta_1, \ldots, \eta_n)$ are the null-coordinates, we have that

$$\phi(|z_1|^2, \ldots, |z_n|^2) = \phi(\xi_1\eta_1, \ldots, \xi_n\eta_n). \tag{5}$$

A potential of the above form is called *toric*. A para-Kähler manifold with a toric-invariant potential ϕ is called a *toric para-Kähler manifold*. In this case, up to rescaling the coordinates (ξ, η), we can assume, w.l.o.g., that

$$\left|\det\left(\frac{\partial^2 \phi}{\partial \xi_i \partial \eta_j}\right)(\xi, 0)\right| = \left|\det\left(\frac{\partial^2 \phi}{\partial \xi_i \partial \eta_j}\right)(0, \eta)\right| = 1 \,, \quad 1 \le i, j \le n \,. \tag{6}$$

On account of (6), by evaluating (4) at $\eta = 0$ and recalling (5), we get that F in (4) is constant and, in particular, it is equal to $\frac{\lambda}{2}\phi(0) - G(0)$. With a similar reasoning, we can also prove that G in (4) is constant. Therefore, by replacing ϕ with

$$\Phi(\xi, \eta) = \phi(\xi_1\eta_1, \ldots, \xi_n\eta_n) - \phi(0), \tag{7}$$

so that $\Phi(0, 0) = 0$, equation (4) can be put in the following form:

$$\left|\det\left(\frac{\partial^2 \Phi}{\partial \xi_i \partial \eta_j}\right)\right| = e^{-\frac{\lambda}{2}\Phi} \,, \quad 1 \le i, j \le n \,; \tag{8}$$

Remark 3.5. We notice that, by introducing the new set of variables

$$y_i = \log|\xi_i \eta_i|$$

and the unknown function

$$\tilde{\Phi} = \Phi - \frac{2}{\lambda}\sum_{i=1}^{n} y_i,$$

the Monge-Ampère equation (8) reads as

$$\left|\det\left(\frac{\partial^2\tilde{\Phi}}{\partial y_i \partial y_j}\right)\right| = e^{-\frac{\lambda}{2}\tilde{\Phi}}, \quad 1 \le i, j \le n.$$

Such extensively studied equation could provide inputs for new strategies to tackle Problem 1, as in the past for the Kähler counterpart (see e.g. [3]).

3.2. Diastasis

Let U be an open subset of a para-Kähler manifold (M, ω) of para-Kähler dimension n, where it is defined a local potential Φ. We assume that U can be covered by a system of null-coordinates

$$(\xi, \eta) = (\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n).$$

Up to shrinking U, we can also assume that it splits as a product

$$U = \Omega \times \Omega.$$

According to [10], we define the *diastasis function*

$$\mathrm{D} : U \times U = \Omega \times \Omega \times \Omega \times \Omega \to \mathbb{R}$$

as

$$\mathrm{D}(\xi, \eta, \zeta, \lambda) = \Phi(\xi, \eta) - \Phi(\zeta, \eta) - \Phi(\xi, \lambda) + \Phi(\zeta, \lambda).$$

Proposition 3.6 *([10]). The diastasis is a function defined in a neighborhood of the diagonal of the product manifold $M \times M$. In particular, it is independent of the choice of the para-Kähler potential.*

We define

$$\mathrm{D}_p(q) := \mathrm{D}(q, p).$$

In particular, taking into account (7), we get

$$\mathrm{D}_0(\xi, \eta) = \mathrm{D}(\xi, \eta, 0, 0) = \phi(\xi, \eta) - \phi(0, 0) = \Phi(\xi, \eta). \tag{9}$$

3.3. Para-Kähler space forms

In complete analogy with the case of Kähler manifolds, one can define the para-holomorphic sectional curvature (see for instance [5]).

Definition 3.7. A *para-Kähler space form* is a para-Kähler manifold with constant para-holomorphic sectional curvature. We denote by $\mathcal{S}_c^N$ an N-dimensional simply connected para-complex manifold that can be endowed with a para-Kähler form ω_c whose associated pseudo-Riemannian metric g_c is complete and has constant para-holomorphic sectional curvature equal to c.

Proposition 3.8 *([5] Prop. 3.11). Two complete and simply connected para-Kähler space forms with the same para-holomorphic sectional curvature are para-holomorphically isometric.*

By [5], we have that an open subset of a para-Kähler space form is para-holomorphically isometric to an open subset of one of the subsequent models, according to their (constant) para-holomorphic sectional curvature. Therefore, since we are mainly interested in local para-Kähler immersions into open subsets of para-Kähler space forms, we are going to assume that $\left(\mathcal{S}_c^N, \omega_c\right)$ is one of the following models.

Model for the flat case: The model of the flat para-Kähler space form is

$$\left(\mathcal{S}_0^N, \omega_0\right) = \left(\mathbb{D}^N, \tau\,\partial\bar{\partial}\|z\|^2\right),$$

whose potential reads in null-coordinates $(\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N)$ as

$$4\sum_{i=1}^{N} \xi_i \eta_i = \mathrm{D}_0(\xi, \eta),$$

where D_0 is defined by (9), leading, via formula (1), to the metric

$$4\sum_{i,j=1}^{N} d\xi_i \otimes d\eta_j\,.$$

Models for the non-flat cases: Similarly to the real and complex setting, the para-complex projective space $\mathbb{DP}^N$ can be defined as the quotient of

$$\{Z \in \mathbb{D}^{N+1} \mid \|Z\|^2 > 0\}$$

under the equivalence relation given by $Z \sim W$ if and only if there exists $\alpha \in \mathbb{D}$ such that $Z = \alpha W$ with $|\alpha|^2 > 0$. Then, our model of non-flat para-Kähler space form will be

$$(\mathcal{S}_c^N, \omega_c) = \left(\mathbb{DP}^N, \frac{4\tau}{c}\partial\bar{\partial}\log\|Z\|^2\right).$$

In null-coordinates $(\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N)$ of the affine chart $\mathcal{U}_\alpha := \{[Z_0, \ldots, Z_N] \in \mathbb{DP}^N \mid |Z_\alpha|^2 \neq 0\}$, where $\alpha = 1, \ldots, n$, i.e., $\xi_i e + \eta_i \bar{e} = \frac{Z_i}{Z_\alpha}$ for any $i \neq \alpha$, the potential is equal to

$$\frac{8}{c}\log\left(1 + 2\sum_{i=1}^{N}\xi_i\eta_i\right) = \mathrm{D}_0(\xi, \eta), \tag{10}$$

where D_0 is defined by (9), leading, via formula (1), to the metric $\frac{8}{c}g_{pFS}^N$, with g_{pFS}^N the para-Fubini-Study metric defined after Theorem 1.1.

4. Para-Kähler immersions of toric para-Kähler-Einstein manifolds in para-Kähler space forms

According to Section 3.2, let $U = \Omega \times \Omega$ be an open subset of a para-Kähler manifold of para-complex dimension equal to n. As shown in the Section 3 of [10], when $(U, \frac{\tau}{2}\partial\bar{\partial}\phi)$ admits a para-Kähler immersion into the para-Kähler space form $\mathcal{S}_c$, the diastasis function D_0 reads as

$$\mathrm{D}_0(\xi, \eta) = \begin{cases} 4\displaystyle\sum_{i=1}^{N} u_i(\xi)v_i(\eta) & \text{if } c = 0; \quad \text{(a)} \\ \dfrac{8}{c}\log\left(1 + 2\displaystyle\sum_{i=1}^{N} u_i(\xi)v_i(\eta)\right) & \text{if } c \neq 0, \quad \text{(b)} \end{cases} \tag{11}$$

where $u_i : \Omega \to \mathbb{R}$ and $v_i : \Omega \to \mathbb{R}$ are suitable smooth functions.

As we said in the introduction, one of the main differences one meets in the para-complex context, unlike the complex one, is that para-holomorphic functions are not, in general, analytic, but only $\mathcal{C}^\infty$-smooth. This makes the para-complex geometry much less rigid with respect to the complex one. Taking this into account, we restrict our attention to the case of analytic immersion and toric-invariant para-Kähler potential, hence the study of Problem 1. In fact, as we shall see below, in this case the analyticity condition implies polynomiality. More precisely we have the following lemma.

Lemma 4.1. *Let ϕ be a toric-invariant para-Kähler potential. If u_i and v_i are some analytic functions such that*

$$\sum_{i=1}^{N} u_i(\xi)v_i(\eta) = \phi(\xi_1\eta_1, \ldots, \xi_n\eta_n), \tag{12}$$

where $N \in \mathbb{N}$ is the smallest possible, then they are polynomials. Hence, ϕ is a polynomial in the variables

$$x_i = \xi_i\eta_i\,. \tag{13}$$

Proof. By differentiating (12) and evaluating at $\xi = 0$, we obtain the following (compatible) linear system

$$\begin{pmatrix} \frac{\partial^{|I_1|}u_1}{\partial \xi^{I_1}}(0) & \dots & \frac{\partial^{|I_1|}u_N}{\partial \xi^{I_1}}(0) \\ \vdots & & \vdots \\ \frac{\partial^{|I_N|}u_1}{\partial \xi^{I_N}}(0) & \dots & \frac{\partial^{|I_N|}u_N}{\partial \xi^{I_N}}(0) \end{pmatrix} \begin{pmatrix} v_1(\eta) \\ \vdots \\ v_N(\eta) \end{pmatrix} = \begin{pmatrix} \eta^{I_1} \frac{\partial^{|I_1|}\phi}{\partial x^{I_1}}(0) \\ \vdots \\ \eta^{I_N} \frac{\partial^{|I_N|}\phi}{\partial x^{I_N}}(0) \end{pmatrix}, \tag{14}$$

where $I_i \in \mathbb{N}^n$ are arbitrary chosen multi-indices. As a first step, we notice that there exist some $I_1, \dots, I_N \in \mathbb{N}^n$ such that

$$\operatorname{rank} \begin{pmatrix} \frac{\partial^{|I_1|}u_1}{\partial \xi^{I_1}}(0) & \dots & \frac{\partial^{|I_1|}u_N}{\partial \xi^{I_1}}(0) \\ \vdots & & \vdots \\ \frac{\partial^{|I_N|}u_1}{\partial \xi^{I_N}}(0) & \dots & \frac{\partial^{|I_N|}u_N}{\partial \xi^{I_N}}(0) \end{pmatrix} > 0.$$

Indeed, if, on the contrary, the previous matrix had rank equal to 0 for any $I_1, \dots, I_N$, then, in view of the analyticity of $u_1, \dots, u_N$, we would have

$$u_1 = \ldots = u_N \equiv 0$$

and $\phi \equiv 0$ would not be a potential of a symplectic form.

Now, if

$$\operatorname{rank} \begin{pmatrix} \frac{\partial^{|I_1|}u_1}{\partial \xi^{I_1}}(0) & \dots & \frac{\partial^{|I_1|}u_N}{\partial \xi^{I_1}}(0) \\ \vdots & & \vdots \\ \frac{\partial^{|I_N|}u_1}{\partial \xi^{I_N}}(0) & \dots & \frac{\partial^{|I_N|}u_N}{\partial \xi^{I_N}}(0) \end{pmatrix} = N$$

for some $I_1, \dots, I_N \in \mathbb{N}^n$, then, being $(v_1, \dots, v_N)$ a solution of the linear system (14), $v_1, \dots, v_N$ are polynomials.
To conclude, we take into account the case in which

$$\operatorname{rank} \begin{pmatrix} \frac{\partial^{|I_1|}u_1}{\partial \xi^{I_1}}(0) & \dots & \frac{\partial^{|I_1|}u_N}{\partial \xi^{I_1}}(0) \\ \vdots & & \vdots \\ \frac{\partial^{|I_N|}u_1}{\partial \xi^{I_N}}(0) & \dots & \frac{\partial^{|I_N|}u_N}{\partial \xi^{I_N}}(0) \end{pmatrix} < N$$

for any $I_1, \dots, I_N \in \mathbb{N}^n$. In view of what we said above, we can assume that

$$\operatorname{rank} \begin{pmatrix} \frac{\partial^{|I_1|}u_1}{\partial \xi^{I_1}}(0) & \dots & \frac{\partial^{|I_1|}u_\rho}{\partial \xi^{I_1}}(0) \\ \vdots & & \vdots \\ \frac{\partial^{|I_\rho|}u_1}{\partial \xi^{I_\rho}}(0) & \dots & \frac{\partial^{|I_\rho|}u_\rho}{\partial \xi^{I_\rho}}(0) \end{pmatrix} = \rho > 0 \tag{15}$$

for some suitable $I_1, \dots, I_\rho \in \mathbb{N}^n$ and

$$\operatorname{rank} \begin{pmatrix} \frac{\partial^{|I_1|}u_1}{\partial \xi^{I_1}}(0) & \dots & \frac{\partial^{|I_1|}u_{\rho+1}}{\partial \xi^{I_1}}(0) \\ \vdots & & \vdots \\ \frac{\partial^{|I_\rho|}u_1}{\partial \xi^{I_\rho}}(0) & \dots & \frac{\partial^{|I_\rho|}u_{\rho+1}}{\partial \xi^{I_\rho}}(0) \\ \frac{\partial^{|J|}u_1}{\partial \xi^{J}}(0) & \dots & \frac{\partial^{|J|}u_{\rho+1}}{\partial \xi^{J}}(0) \end{pmatrix} = \rho \tag{16}$$

for any $J \in \mathbb{N}^n$. It follows from the Laplace's expansion w.r.t. the last row of the matrix in (16), and by considering also (15), that

$$\frac{\partial^{|J|}u_{\rho+1}}{\partial \xi^J}(0) = -\frac{K_1}{K_{\rho+1}} \frac{\partial^{|J|}u_1}{\partial \xi^J}(0) - \ldots - \frac{K_\rho}{K_{\rho+1}} \frac{\partial^{|J|}u_\rho}{\partial \xi^J}(0), \qquad \forall J \in \mathbb{N}^n,$$

where K_i denotes the algebraic complement of i-th element of the last row of the matrix in (16). Since

$$u_{\rho+1}(\xi) = \sum_{J \in \mathbb{N}^n} \frac{\partial^{|J|}u_{\rho+1}}{\partial \xi^J}(0) \, \frac{\xi^J}{J!} = -\frac{K_1}{K_{\rho+1}} u_1(\xi) - \ldots - \frac{K_\rho}{K_{\rho+1}} u_\rho(\xi)\,,$$

by taking suitable linear combinations $\tilde{v}_1, \ldots, \tilde{v}_\rho$ of $v_1, \ldots, v_n$, we have

$$\sum_{i=1}^{\rho} u_i \tilde{v}_i = \phi.$$

Hence, N in (12) would not be the smallest integer, as assumed.
The same reasoning can be applied to the functions u_i. $\square$

In view of Lemma 4.1 and by taking into account (11a)-(11b), we straightforwardly get the following proposition.

Proposition 4.2. *Let ϕ be a toric-invariant para-Kähler potential defined on a open neighborhood $U \subseteq \mathbb{D}^n$ of the origin. Let $(U, \frac{\tau}{2}\partial\bar{\partial}\phi)$ be a para-Kähler manifold admitting an analytic para-Kähler immersion into the para-Kähler space form $\mathcal{S}_c$. Recalling that $x_i = \xi_i \eta_i$ (cf. (13)), the diastasis function $\mathrm{D}_0 : U \to \mathbb{R}$ reads as*

$$\mathrm{D}_0(x_1, \ldots, x_n) = \begin{cases} \displaystyle\sum_{0<|I|\le d} a_I x^I & \text{if } c = 0; \quad \text{(a)} \\ \displaystyle\frac{8}{c}\log\left(1 + \sum_{0<|I|\le d} a_I x^I\right) & \text{if } c \neq 0, \quad \text{(b)} \end{cases} \tag{17}$$

where $d \in \mathbb{Z}^+$, $a_I \in \mathbb{R}$ and $I \in \mathbb{N}^n$.

To sum up, by taking into account (8) and (9), we have that Problem 1 can be reformulated as follows:

Reformulation of Problem 1. *Find all the solutions of type (17a)-(17b) of the following Monge-Ampère equation:*

$$\left|\det\left(\frac{\partial^2 \mathrm{D}_0}{\partial x_i \partial x_j} x_i + \frac{\partial \mathrm{D}_0}{\partial x_j}\delta_{ij}\right)\right| = e^{-\frac{\lambda}{2}\mathrm{D}_0}, \quad 1 \le i, j \le n\,. \tag{18}$$

4.1. Proof of Theorem 1.1

In view of the above Reformulation of Problem 1, in order to get Theorem 1.1, it will be enough to prove the following properties:

- there are no solutions of type (17a) of the Monge-Ampère equation (18) with $\lambda \neq 0$;
- there are no solutions of type (17b) of the Monge-Ampère equation (18) with $\lambda = 0$;
- P is a solution of type (17a) of the Monge-Ampère equation (18) with $\lambda = 0$, if and only if $\deg P = 1$ and the product of the coefficients of the linear terms is ± 1.

Let $x = (x_1, \ldots, x_n)$ and let $P(x)$ be a polynomial.

Concerning the first point, let us assume by contradiction that P is a solution of the Monge-Ampère equation (18). We immediately notice that the left hand side of the equation is a polynomial, while the right hand side is a polynomial only when P is constant. Nevertheless, when P is constant, the left hand side is identically zero, leading to a contradiction.

Concerning the second point, if we assume by contradiction that $k \log P$ is a solution of the Monge-Ampère equation (18) with $\lambda = 0$, then we straightforwardly get that

$$\frac{\det\left[\left(P\frac{\partial^2 P}{\partial x_\alpha \partial x_\beta} - \frac{\partial P}{\partial x_\alpha}\frac{\partial P}{\partial x_\beta}\right)x_\alpha + P\frac{\partial P}{\partial x_\alpha}\delta_{\alpha\beta}\right]_{1\le\alpha,\beta\le n}}{P^{n-1}} = \pm\left(\frac{1}{k}\right)^n P^{-n+1}.$$

We notice that both sides of the previous equality are polynomials. In particular, the degree of left hand side cannot be greater that $(n+1)d - n$. Hence, by comparing the degrees of both sides, we get a contradiction. Indeed,

$$(n+1)d - n \ge (n+1)d.$$

Concerning the third point, let P be a polynomial such that

$$\det\left(\frac{\partial^2 P}{\partial x_i \partial x_j} x_i + \frac{\partial P}{\partial x_j}\delta_{ij}\right)_{1\le i,j\le n} = \pm 1.$$

By evaluating the previous equality on the line $x_2 = \ldots = x_n = 0$, we easily get

$$\left(x_1 \frac{\partial^2 P}{\partial x_1^2}\Big|_{(x_1,0)} + \frac{\partial P}{\partial x_1}\Big|_{(x_1,0)} \right) \prod_{i=2}^{n} \frac{\partial P}{\partial x_i}\Big|_{(x_1,0)} = \pm 1. \tag{19}$$

By taking into account that each term of the previous product is a polynomial, we get that each term needs to be constant. Therefore, we have in particular that

$$\frac{\partial^2 P}{\partial x_1 \partial x_j}\Big|_{(x_1,0)} = 0$$

and

$$x_1 \frac{\partial^2 P}{\partial x_1^2}\Big|_{(x_1,0)} + \frac{\partial P}{\partial x_1}\Big|_{(x_1,0)} = k_1,$$

with $k_1 \in \mathbb{R} \setminus \{0\}$. By solving the previous equations and keeping in mind that $P(x_1, 0)$ is a polynomial, we obtain

$$P(x) = k_0 + k_1 x_1 + \sum_{i=2}^{n} x_i \, Q_i(x_2, \ldots, x_n),$$

where Q_i are polynomials. Moreover, by means of a similar reasoning, namely by restricting the Monge-Ampère equation (19) to the other coordinate axes, we finally get that

$$P(x) = k_0 + \sum_{i=1}^{n} k_i x_i,$$

with

$$\prod_{i=1}^{n} k_i = \pm 1.$$

4.2. Proof of Theorem 1.2

In order to prove Theorem 1.2, we have to study the case that has remained out the discussion of Section 4.1, i.e., to study solutions of type (17b) of the Monge-Ampère equation (18) when $\lambda \neq 0$. Before doing it, below we give a remark that will be important to give a further reformulation of Problem 1 in the case under consideration. Then, after proving the technical Lemma 4.4, the second part of Theorem 1.2 will be essentially proved in Sections 4.2.1 and 4.2.2, according to the dimension of the toric para-Kähler manifolds. Section 4.2.3 will complete the proof.

Remark 4.3. If $\mathrm{D}_0 = \frac{8}{c} \log Q$ is a solution of type (17b) of the Monge-Ampère equation (18), then the polynomial

$$Q(x) = 1 + \sum_{0<|I|\le d} a_I x^I, \tag{20}$$

is a solution of the PDE

$$\frac{\det\left[\left(Q \frac{\partial^2 Q}{\partial x_\alpha \partial x_\beta} - \frac{\partial Q}{\partial x_\alpha}\frac{\partial Q}{\partial x_\beta}\right) x_\alpha + Q \frac{\partial Q}{\partial x_\alpha} \delta_{\alpha\beta}\right]_{1\le\alpha,\beta\le n}}{Q^{n-1}} = \pm \left(\frac{c}{8}\right)^n Q^{n+1-\frac{4\lambda}{c}}.$$

After the change of coordinates $(x_1, \ldots, x_n) \to \frac{8}{c}(x_1, \ldots, x_n)$, the previous equation reads as

$$\frac{\det\left[\left(Q \frac{\partial^2 Q}{\partial x_\alpha \partial x_\beta} - \frac{\partial Q}{\partial x_\alpha}\frac{\partial Q}{\partial x_\beta}\right) x_\alpha + Q \frac{\partial Q}{\partial x_\alpha} \delta_{\alpha\beta}\right]_{1\le\alpha,\beta\le n}}{Q^{n-1}} = \pm Q^{n+1-\frac{4\lambda}{c}}. \tag{21}$$

Taking into account that the left hand side of (21) is a polynomial, it straightforwardly follows that $\frac{4\lambda}{c} \in \mathbb{Q}$. Moreover, by comparing the degrees of both sides of (21), we get

$$(n+1)d - n \ge \left(n + 1 - \frac{4\lambda}{c}\right) d.$$

Hence,

$$\frac{4\lambda}{c} \in \mathbb{Q}^+.$$

Let us assume that $\frac{4\lambda}{c} = \frac{s}{q}$, where $\gcd(q,s) = 1$. Being the left hand side of (21) a polynomial, Q is forced to be the q-th power of another polynomial R, namely

$$Q\,(x_1,\ldots,x_n) = R\left(\frac{x_1}{q},\ldots,\frac{x_n}{q}\right)^q .$$

Since the constant term of Q is equal to 1, we notice, instead, that the constant term of R can be equal either to 1 or -1. Moreover, one can easily check that $R(x_1,\ldots,x_n)$ is a solution to the equation

$$\frac{\det\left[\left(R\frac{\partial^2 R}{\partial x_\alpha \partial x_\beta} - \frac{\partial R}{\partial x_\alpha}\frac{\partial R}{\partial x_\beta}\right)x_\alpha + R\frac{\partial R}{\partial x_\alpha}\delta_{\alpha\beta}\right]_{1\le\alpha,\beta\le n}}{R^{n-1}} = \pm R^{n+1-s}. \tag{22}$$

Furthermore, if a polynomial $R(x_1,\ldots,x_n)$ is a solution of (22) for some $s \in \mathbb{N}$, then

$$P(x_1,\ldots,x_n) = R\left(\frac{x_1}{s},\ldots,\frac{x_n}{s}\right)^s$$

is a polynomial solution to the equation

$$\frac{\det\left[\left(P\frac{\partial^2 P}{\partial x_\alpha \partial x_\beta} - \frac{\partial P}{\partial x_\alpha}\frac{\partial P}{\partial x_\beta}\right)x_\alpha + P\frac{\partial P}{\partial x_\alpha}\delta_{\alpha\beta}\right]_{1\le\alpha,\beta\le n}}{P^{n-1}} = \pm P^n. \tag{23}$$

We notice that the constant term of P is equal to 1 or -1.

In view of the above remark, the Reformulation of Problem 1 can be refined, in the case considered in the present section, as follows.

Refinement of Problem 1. *Find all polynomials P^k with constant term equal to 1, with $k \in \mathbb{Q}^+$ and such that P is a polynomial solution, having constant term equal to ± 1, to the Monge-Ampère equation (23).*

Indeed, by considering (13) and (17b), we can associate to such P^k the diastasis' function of a para-Kähler metric admitting a para-Kähler immersion into $(\mathbb{DP}^N, g^N_{pFS})$, where N need to be at least equal to the number of monomials forming P^k (see [10]):

$$\log P(\xi_1\eta_1,\ldots,\xi_n\eta_n)^k. \tag{24}$$

Being P a solution of the Monge-Ampère equation (23), it necessarily follows that, for any aforementioned admissible k, any diastasis' function (24) gives rise to a para-Kähler-Einstein metric which is para-Kähler immersed into $(\mathbb{DP}^N, g^N_{pFS})$.

This can be achieved, in the case when the dimension n is equal either to 1 (see Section 4.2.1) or 2 (see Section 4.2.2), by using the following lemma, that holds in any dimension. A discussion of the cases with dimension $n \ge 3$ is contained in Section 4.2.3.

Lemma 4.4. *The restriction $p(t)$ on a coordinate axis of a polynomial solution, having constant term equal to ± 1, to the Monge-Ampère equation (23), reads as:*

$$p(t) = \pm\left(1 + \frac{t}{r}\right)^k ,$$

where $k \in \mathbb{Z}^+$ and $r \in \mathbb{R}$.

Proof. Being $p(t)$ be the restriction on the i-th coordinate axis (i.e. the line $x_j = 0,$ for $j \ne i$) of a polynomial solution P to the Monge-Ampère equation (23), we have that

$$\left(\left(p(t)\; p''(t) - p'(t)^2\right)t + p(t)\; p'(t)\right)q(t) = \pm p(t)^n, \tag{25}$$

where the $q(t)$ denotes the restriction on the i-th coordinate axis of $\prod_{j\ne i}\frac{\partial P}{\partial x_j}$.

Let $\{-r_1,\ldots,-r_R\}$ be all the (possibly complex) distinct roots of p, namely

$$p(t) = A\prod_{i=1}^{R}(t + r_i)^{k_i}, \tag{26}$$

with

$$r_i - r_j \neq 0, \qquad \forall i \neq j. \tag{27}$$

Via some straightforward computations, the equation (25) reads as

$$\left(\sum_{i=1}^{R} k_i r_i \prod_{\substack{j=1 \\ j\neq i}}^{R} (t+r_j)^2 \right) q(t) = \pm A^{n-2} \prod_{i=1}^{R} (t+r_i)^{k_i(n-2)+2}. \tag{28}$$

It easily follows that any root of $q(t)$ needs to be also a root of $p(t)$, namely

$$q(t) = \pm \frac{A^{n-2}}{B} \prod_{i=1}^{R} (t+r_i)^{h_i},$$

for some suitable $h_i \in \mathbb{N}$ and $B \in \mathbb{R} \setminus \{0\}$. Hence, the equality (28) reads as

$$\sum_{i=1}^{R} k_i r_i \prod_{\substack{j=1 \\ j\neq i}}^{R} (t+r_j)^2 = B \prod_{i=1}^{R} (t+r_i)^{k_i(n-2)-h_i+2}.$$

If we assume the existence of an index i such that $k_i(n-2) - h_i + 2 \neq 0$, then, by evaluating the previous equality at $-r_i$, we get

$$-k_i r_i \prod_{\substack{j=1 \\ j\neq i}}^{R} (r_j - r_i)^2 = 0.$$

Nevertheless, it would contradict (27). Therefore,

$$q(t) = \pm \frac{A^{n-2}}{B} \prod_{i=1}^{R} (t+r_i)^{k_i(n-2)+2} \tag{29}$$

and

$$\sum_{i=1}^{R} k_i r_i \prod_{\substack{j=1 \\ j\neq i}}^{R} (t+r_j)^2 - B = 0. \tag{30}$$

Let now consider (30) as a linear system in the variables $k_1, \ldots, k_R$.
If $R = 1$, such system consists of just one equation, which has a unique solution:

$$k_1 = \frac{B}{r_1}.$$

If $R \geq 2$, it cannot be compatible for any t. Indeed, being the left hand side of (30) a polynomial in t of degree $2R-2$, in particular its first R coefficients of higher order have to vanish. Therefore, $k_1, \ldots, k_R$ need to satisfy a homogeneous system, whose determinant of the matrix of coefficients can be easily computed:

$$R! \prod_{i=1}^{R} r_i \prod_{1 \leq i < j \leq R} (r_i - r_j).$$

In view of (27), such determinant is always different from zero. Therefore, the system admits only the trivial solution, leading to a contradiction, since k_i, for any i, represents the multiplicity of a root of a polynomial, so it should be positive.

To conclude, since we are assuming that the constant term $\varepsilon = A r_1^{k1}$ of p, see (26), is equal to ± 1, by taking into account (26) and (29), we have that

$$p(t) = A(t+r_1)^{k_1} = \varepsilon \sum_{i=0}^{k_1} \binom{k_1}{i} t^i r_1^{-i} = \varepsilon \left(1 + \frac{t}{r_1}\right)^{k_1}$$

and

$$q(t) = \pm \varepsilon^{n-2} \frac{r_1}{k_1} \left(1 + \frac{t}{r_1}\right)^{k_1(n-2)+2}. \quad \square \tag{31}$$

4.2.1. Case $n = 1$

In the present section, we shall consider the Refinement of Problem 1 at page 11 when $n = 1$ and solve it. In view of Lemma 4.4, polynomial solutions to (23), that we have been studying, need to read as

$$P(x) = \varepsilon \left(1 + \frac{x}{r}\right)^k,$$

where $\varepsilon = \pm 1$. Moreover, after some straightforward computations on the Monge-Ampère equation (23) when $n = 1$, namely

$$\left(PP'' - (P')^2\right)x + PP' = \pm P,$$

we get, in particular, that $k = 2$ and $r = \pm 2$, i.e.,

$$P(x) = \varepsilon \left(1 \pm \frac{x}{2}\right)^2.$$

In view of the Refinement of Problem 1 at page 11 we have to take into account the polynomials

$$P(x) = \left(1 \pm \frac{x}{2}\right)^K, \qquad \text{with } K \in \mathbb{Z}^+.$$

By considering (13) and (17b), we can associate to the previous polynomials, the following two families of diastasis functions:

$$\log\left(1 + \frac{\xi\eta}{2}\right)^K \qquad \text{and} \qquad \log\left(1 - \frac{\xi\eta}{2}\right)^K.$$

We immediately see that the para-holomorphic change of coordinates $(\xi, \eta) \to (-\xi, \eta)$ transforms the first diastasis' function into the second, therefore the corresponding para-Kähler metrics, via formula (1), are isometric. In particular, they are isometric to the metric having the first function as a potential:

$$\frac{2K}{(\xi\eta + 2)^2} d\xi d\eta.$$

By considering the further change of coordinates $(\xi, \eta) \to (2\xi, 2\eta)$, we see that the previous metric is

$$K\, g^1_{pFS} = \frac{2K}{(1 + 2\xi\eta)^2} d\xi d\eta.$$

Theorem 1.2 when $n = 1$ is thus proved.

4.2.2. Case $n = 2$

In the present section, we are going to find, when $n = 2$, all polynomial solutions of type (20) of the Monge-Ampère equation (23), namely all the polynomials in two variables with constant term equal to 1 solving the following Monge-Ampère equation:

$$\frac{\det\left[\left(P\frac{\partial^2 P}{\partial x_\alpha \partial x_\beta} - \frac{\partial P}{\partial x_\alpha}\frac{\partial P}{\partial x_\beta}\right)x_\alpha + P\frac{\partial P}{\partial x_\alpha}\delta_{\alpha\beta}\right]_{1 \le \alpha, \beta \le 2}}{P} = \pm P^2. \tag{32}$$

More precisely, this section is devoted to prove the following proposition.

Proposition 4.5. *The only polynomials in two variables of type (20) solving the Monge-Ampère equation (32) are*

$$\varepsilon\left(1 + \frac{1}{r}x_1 \pm \frac{r}{9}x_2\right)^3 \tag{33}$$

and

$$\varepsilon\left(1 + \frac{1}{r}x_1\right)^2\left(1 \pm \frac{r}{4}x_2\right)^2, \tag{34}$$

where $r \neq 0$ is an arbitrary real number and $\varepsilon = \pm 1$.

The proof follows from the following two lemmas.

Lemma 4.6. *An arbitrary polynomial solution, with constant term equal to* ± 1*, of the Monge-Ampère equation* (32)*, satisfies one and only one of the following initial conditions on the coordinate axis* $x_2 = 0$*:*

$$\begin{gathered} P(x_1,0) = \varepsilon\left(1+\tfrac{x_1}{r}\right)^2, \ \tfrac{\partial P}{\partial x_2}(x_1,0) = \pm\tfrac{r}{2}\left(1+\tfrac{x_1}{r}\right)^2 \\ \textit{or} \\ P(x_1,0) = \varepsilon\left(1+\tfrac{x_1}{r}\right)^3, \ \tfrac{\partial P}{\partial x_2}(x_1,0) = \pm\tfrac{r}{3}\left(1+\tfrac{x_1}{r}\right)^2, \end{gathered} \tag{35}$$

where $\varepsilon = \pm 1$.

Proof. Let P be a solution, whose constant term is equal to ± 1, of (32). By Lemma 4.4, we have that

$$P(x_1,0) = \varepsilon\left(1+\frac{x_1}{r_1}\right)^{k_1}, \qquad P(0,x_2) = \varepsilon\left(1+\frac{x_2}{r_2}\right)^{k_2}, \tag{36}$$

for suitable $k_1, k_2 \in \mathbb{Z}^+$ and $r_1,\ r_2 \in \mathbb{R}$. Moreover, by (31),

$$\frac{\partial P}{\partial x_2}(x_1,0) = \pm\frac{r_1}{k_1}\left(1+\frac{x_1}{r_1}\right)^2, \qquad \frac{\partial P}{\partial x_1}(0,x_2) = \pm\frac{r_2}{k_2}\left(1+\frac{x_2}{r_2}\right)^2. \tag{37}$$

From the comparison of the derivative w.r.t. x_1 of the first equality of (37) and the derivative w.r.t. x_2 of the second equality of (37), in particular by considering their evaluation at $(0,0)$, we obtain

$$k_1 = k_2$$

and

$$\frac{\partial P}{\partial x_2}(x_1,0) = \sigma\frac{r_1}{k_1}\left(1+\frac{x_1}{r_1}\right)^2, \qquad \frac{\partial P}{\partial x_1}(0,x_2) = \sigma\frac{r_2}{k_1}\left(1+\frac{x_2}{r_2}\right)^2 \tag{38}$$

where $\sigma = \pm 1$. Moreover, from the comparison of the second equality of (38) and the derivative w.r.t. x_1 of the first equality of (36), in particular by considering their evaluation at $(0,0)$, we get

$$r_2 = \varepsilon\sigma\frac{k_1^2}{r_1}.$$

Therefore, the polynomial P can be written as:

$$\begin{aligned} &\varepsilon\left(1+\frac{x_1}{r_1}\right)^{k_1} + \varepsilon\left(1+\varepsilon\sigma\frac{r_1x_2}{k_1^2}\right)^{k_1} - \varepsilon + \varepsilon\frac{k_1}{r_1}\left(1+\varepsilon\sigma\frac{r_1x_2}{k_1^2}\right)^2 x_1 + \sigma\frac{r_1}{k_1}\left(1+\frac{x_1}{r_1}\right)^2 x_2 \\ &\qquad - \varepsilon\frac{k_1}{r_1}x_1 - \sigma\frac{r_1}{k_1}x_2 - \sigma\frac{2}{k_1}x_1x_2 + x_1^2x_2^2\,\eta(x_1,x_2), \end{aligned} \tag{39}$$

where η is a polynomial.

By putting (39) in (32), by differentiating both sides of the equation by $\frac{\partial^2}{\partial x_1\partial x_2}$ and by evaluating at $(0,0)$, we straightforwardly get the following Diophantine equations:

$$k_1^2 - 5k_1 + 6 = 0$$

and

$$3k_1^2 - k_1 + 6 = 0.$$

By solving the first equation, we obtain

$$k_1 = 2, \quad \text{or} \quad k_1 = 3,$$

while the second equation does not admit any real solution. Hence, by considering (36) and (38), we get our statement. □

Since each solution (33)-(34) satisfies the correspondent initial condition (35), we conclude the proof of Proposition 4.5 by proving the following lemma.

Lemma 4.7. *If there exists a polynomial solution of type* (20) *to* (32) *satisfying one of the initial conditions* (35)*, then it is unique.*

Proof. Let F be a function whose zero defines the PDE (32), i.e.,

$$F := \frac{\det\left[\left(P\frac{\partial^2 P}{\partial x_\alpha \partial x_\beta} - \frac{\partial P}{\partial x_\alpha}\frac{\partial P}{\partial x_\beta}\right)x_\alpha + P\frac{\partial P}{\partial x_\alpha}\delta_{\alpha\beta}\right]_{1\leq\alpha,\beta\leq 2}}{P} \mp P^2.$$

Then, from a straightforward computation, we get the following formula:

$$\frac{\partial^h F}{\partial x_2^h}(x_1,0) = \left((h+1)\left(P\frac{\partial^2 P}{\partial x_1^2}x_1 - \left(\frac{\partial P}{\partial x_1}\right)^2 x_1 + P\frac{\partial P}{\partial x_1}\right)\frac{\partial^{h+1} P}{\partial x_2^{h+1}} + T^h\right)(x_1,0)\,, \tag{40}$$

where $T^h(x_1,0)$ is a polynomial expression in x_1, $P(x_1,0)$ and derivatives of P up to order $h+2$ (computed in $(x_1,0)$), that does not contain $\frac{\partial^{h+1}P}{\partial x_2^{h+1}}(x_1,0)$, $\frac{\partial^{h+2}P}{\partial x_2^{h+2}}(x_1,0)$ or $\frac{\partial^{h+2}P}{\partial x_1\partial x_2^{h+1}}(x_1,0)$. If P is a polynomial solution of type (20) to (32) satisfying one of the initial conditions (35), i.e., $P(x_1,0)=\varepsilon\left(1+\frac{x_1}{r}\right)^k$ for a suitable integer k, hence we have

$$\left(P\frac{\partial^2 P}{\partial x_1^2}x_1 - \left(\frac{\partial P}{\partial x_1}\right)^2 x_1 + P\frac{\partial P}{\partial x_1}\right)(x_1,0) = \frac{k}{r}\left(\frac{x_1}{r}+1\right)^{2k-2} \not\equiv 0.$$

By considering formula (40) when $h=1$, we realize that initial conditions (35) uniquely determine $\frac{\partial^2 P}{\partial x_2^2}(x_1,0)$, from which one obtains $\frac{\partial^{2+h}P}{\partial x_1^h\partial x_2^2}(x_1,0)$ for every $h\in\mathbb{N}$. By iteration, we get the whole Taylor expansion of P on the line $x_2=0$. Therefore, we get the statement of the lemma. □

By taking into account suitable para-holomorphic change of coordinates and in view of the Refinement of Problem 1 at page 11 we have to take into account only the powers of the polynomials of Proposition 4.5 with $\varepsilon=1$:

$$\left(1+\frac{\xi_1\eta_1}{2}\right)^K\left(1+\frac{\xi_2\eta_2}{2}\right)^K, \qquad \left(1+\frac{\xi_1\eta_1}{3}+\frac{\xi_2\eta_2}{3}\right)^K,$$

for any $K\in\mathbb{Z}^+$. By considering (13) and (17b), we can associate to the aforementioned polynomials the following two families of diastasis functions:

$$K\log\left[\left(1+\frac{\xi_1\eta_1}{2}\right)\left(1+\frac{\xi_2\eta_2}{2}\right)\right] \quad \text{and} \quad K\log\left(1+\frac{\xi_1\eta_1}{3}+\frac{\xi_2\eta_2}{3}\right).$$

Note that the metric we obtain via formula (1) from the first family is a metric on $\mathbb{DP}^1\times\mathbb{DP}^1$, whereas the second family gives a metric on $\mathbb{DP}^2$ (see the end of Section 3).

Theorem 1.2 when $n=2$ is thus proved.

4.2.3. Case $n\geq 3$

In this section we shall show how to prove the first part of Theorem 1.2. We shall discuss only the case $n=3$ as the multi-dimensional one is a straightforward generalization of it. It will be enough to consider, taking into account (13), only the following polynomial solutions to equation (23):

$$\left(1+\frac{\xi_1\eta_1}{2}\right)^2\left(1+\frac{\xi_2\eta_2}{2}\right)^2\left(1+\frac{\xi_3\eta_3}{2}\right)^2, \quad \left(1+\frac{\xi_1\eta_1}{2}\right)^2\left(1+\frac{\xi_2\eta_2}{3}+\frac{\xi_3\eta_3}{3}\right)^3,$$
$$\left(1+\frac{\xi_1\eta_1}{4}+\frac{\xi_2\eta_2}{4}+\frac{\xi_3\eta_3}{4}\right)^4. \tag{41}$$

In view of (17b) and of the Refinement of Problem 1 at page 11, the above polynomials lead to diastasis functions (in particular, potentials) that, via formula (1), give the para-Kähler metrics we are looking for. More precisely, the first polynomial leads to the metric on $\mathbb{DP}^1\times\mathbb{DP}^1\times\mathbb{DP}^1$, the second one to metric on $\mathbb{DP}^1\times\mathbb{DP}^2$ and the third one to metric on $\mathbb{DP}^3$.

Data availability

No data was used for the research described in the article.

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