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STABILITY OF SWITCHED AFFINE SYSTEMS: ARBITRARY AND DWELL-TIME SWITCHING

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Abstract. The dynamical behavior of switched affine systems is known to be more intricate than that of the well-studied switched linear systems, essentially due to the existence of distinct equilibrium points for each subsystem. First, under arbitrary switching rules, the stability analysis must be generally carried out with respect to a compact set with non-empty interior rather than to a singleton. We provide a novel proof technique for existence and outer approximation of attractive invariant sets of a switched affine system, under the hypothesis of global uniform stability of its linearization. On the other hand, considering dwell-time switching signals, forward invariant sets need not exist for this class of switched systems, even for stable ones. Hence, more general notions of stability/boundedness are introduced and studied, highlighting the relations of these concepts to the uniform stability of the linear part of the system under the same class of dwell-time switching signals. These results reveal the main differences and specificities of switched affine systems with respect to linear ones, providing a first step for the analysis of switched systems composed by subsystems *not* sharing the same equilibrium. Numerical methods based on linear matrix inequalities and sum-of-squares programming are presented and illustrate the developed theory.

1. Introduction. Switched dynamical systems provide a mathematical model for a large class of phenomena and have been the subject of intense analysis in the past decades; for an overview, see the monographs [1, 2]. In this framework, given M vector fields $f_1, \dots, f_M : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we consider the differential equation

$$(1.1) \quad \dot{x}(t) = f_{\sigma(t)}(x(t)),$$

where $\sigma : \mathbb{R}_+ \rightarrow \{1, \dots, M\}$ is a so-called *switching signal*, which selects, at each instant of time, one subsystem among f_1, \dots, f_M to be active, i.e., to define how the state $x(t)$ evolves. One of the first but yet challenging problems studied in the literature is the *stability* analysis of (1.1), specifically for some particular classes of subsystems f_1, \dots, f_M and/or some particular classes of switching signals $\sigma : \mathbb{R}_+ \rightarrow \{1, \dots, M\}$.

Regardless the specific framework (linear or non-linear subsystems, arbitrary or restricted switching signals, discrete or continuous-time evolution), a classical assumption in switched systems literature consists in considering subsystems that *share the same equilibrium* at the origin, i.e., $f_1(0) = \dots = f_M(0) = 0$; the stability analysis is then carried out with respect to this common equilibrium (this is for instance the case when the subsystems f_i are linear functions). This assumption, which is reasonable in several practical contexts, simplifies the analysis, and leads to elegant and handy Lyapunov characterizations of (several kinds of) stability notions, for an overview we refer to [1, 2, 3, 4, 5] and references therein. On the other hand it is often fruitful or even necessary to consider subsystems which *do not* share the same equilibrium, leading to a greater modeling capacity but paying the price of a more convoluted stability analysis. A natural and interesting case is provided by switched *affine* systems, thus considering maps $f_1, \dots, f_M : \mathbb{R}^n \rightarrow \mathbb{R}^n$ in (1.1) which are affine, i.e., $f_i(x) = A_i x + b_i$, for all $i \in \{1, \dots, M\}$, for some given matrices and vectors $A_i \in \mathbb{R}^{n \times n}$ and $b_i \in \mathbb{R}^n$.

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43 Switched affine models are often adopted in dynamical system analysis and con-
 44 trol, mostly in the form of piecewise affine systems, to characterize or approximate
 45 complex non-linear behaviors, see [6, 7]. In this case, the switching events are state-
 46 dependent and subsystems are selected according to which operating region the state
 47 trajectory lies in. Beyond that, the use of affine models is also recurrent in power elec-
 48 tronics [8, 9], as several power converters can be conveniently represented by them. In
 49 this setting, the switching signal is generally a control variable that must be designed
 50 to guide the state vector toward a user-defined goal.

51 In our work, instead, we focus on the continuous-time switched affine systems with
 52 switching signals that are considered to be unknown exogenous disturbances/inputs.
 53 In practice, this encompasses many situations of interest such as switching linear
 54 systems undergoing additive disturbances, and/or faulty switching controllers (i.e.,
 55 when a switched controller malfunctions and the switching becomes arbitrary). The
 56 stability analysis is then carried out in two distinct cases:

- 57 • the *arbitrary switching case*, in which no further assumption is made on the switch-
 58 ing signals;
- 59 • the *dwell-time switching case*, where it is assumed the existence of a *dwell-time* $\tau >$
 60 0, which represents the minimal time interval between two consecutive switching
 61 instants.

62 In both of these cases, in analyzing the stability of a given switched affine system,
 63 our approach requires to consider, as a first step, the corresponding *linearized* switched
 64 system. More precisely, supposing that subsystems $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ in (1.1) are affine
 65 and defined by matrices and vectors $(A_i, b_i) \in \mathbb{R}^{n \times n} \times \mathbb{R}^n$ with $i \in \{1, \dots, M\}$, we
 66 consider the corresponding switched linear system

$$67 \quad \dot{x}(t) = A_{\sigma(t)}x(t)$$

68 as a surrogate in some of our analyses. This is motivated by the fact that stability
 69 (with respect to the origin) of switched linear systems is a fundamental and mature
 70 topic in switched systems literature, see for example [1, 10, 11] and references therein.
 71 Thus, considering the linearization of system (1.1), we rely on well-known converse
 72 Lyapunov results for switched *linear* systems, see [12, 13] for the arbitrary switching
 73 case, and [4, 14] for the dwell-time case. Assuming that the corresponding linearized
 74 system is stable, the aforementioned converse Lyapunov theorems provide us with the
 75 existence of (multiple) Lyapunov functions for the linearized system. Therefore, we
 76 generalize these results to cope with stability/boundedness properties for the original
 77 switched affine system. As said, since the affine subsystems do not, in general, share
 78 the same equilibrium, we need to consider tailored concepts of stability.

79 In the arbitrary switching case, under the hypothesis that the corresponding lin-
 80 earized system is stable, we prove the existence of a compact forward invariant attrac-
 81 tor and that all the trajectories exponentially converge to it. The proposed analysis
 82 is inspired by the results presented in [15] for the discrete-time setting. The existence
 83 of minimal forward invariant set was already proven, via a different proof technique,
 84 in [16]; our approach allows us to prove that the minimal forward invariant set is *ex-*
 85 *ponentially attractive* (which was only conjectured in [16]), and enables the proposal
 86 of algorithmic schemes to approximate this set. Indeed, some numerical techniques
 87 based on semidefinite and sum-of-squares optimization are proposed in order to pro-
 88 vide outer approximations of the compact attractor. We further show that, if the
 89 linearized system admits diverging solutions, no forward invariant set exists.

90 When considering switching signals with minimum dwell-time, we prove that the
 91 solutions of switched affine systems are ultimately bounded, i.e., all the solutions

92 eventually enter a compact set without leaving it afterwards, under the assumption
 93 that the linearized system is stable for the corresponding class of dwell-time signals.
 94 On the other hand, we also prove that, due to the presence of multiple equilibria, a
 95 compact forward invariant set *does not* exist, in general, unlike the arbitrary switch-
 96 ing case previously studied. We thus provide tailored definitions characterizing the
 97 asymptotic behavior of switched affine systems under dwell-time assumption, allow-
 98 ing us to underline the peculiarity of this setting with respect to the case of arbitrary
 99 switching signals. It is noteworthy that the stability of switched affine systems un-
 100 dergoing dwell-time switching has not been addressed in the literature to date, to the
 101 best of our knowledge. Numerical outer approximations of the attracting/bounding
 102 regions are provided, by means of semidefinite programming.

103 This manuscript is organized as follows: In Section 2 we present the consid-
 104 ered framework along with the necessary definitions and we recall the foundational
 105 Lyapunov results concerning switched linear systems. In Section 3 we study the
 106 arbitrary switching case, proving the existence of a forward invariant compact attrac-
 107 tor, together with a numerical scheme to approximate it; the analysis for dwell-time
 108 switching signals is then developed in Section 4, underlining the subtleties and the
 109 differences with respect to the general arbitrary switching case. Section 5 closes the
 110 manuscript providing some concluding remarks together with possible future direc-
 111 tions of research.

112 **Notation:** Given $M \in \mathbb{N}$, we define $\langle M \rangle := \{1, \dots, M\}$. The symbols $\mathbb{R}_{>0}$ and
 113 $\mathbb{R}_{\geq 0}$ denote the sets positive and nonnegative real numbers, respectively. With the
 114 notation $G : A \rightrightarrows B$ we denote a *set-valued map* between A and B . Given a set
 115 $S \subset \mathbb{R}^n$, the symbols \bar{S} , ∂S and $\text{Int}(S)$ denote the *closure*, *boundary* and *interior* of
 116 S , respectively; $\text{co}(S)$ denotes its *convex hull*.

117 **2. Preliminaries.** Given $M \in \mathbb{N}$, consider $\mathcal{F} := \{(A_i, b_i)_{i \in \langle M \rangle} \mid A_i \in \mathbb{R}^{n \times n}, b_i \in$
 118 $\mathbb{R}^n\}$ and the continuous-time switched system

$$119 \quad (2.1) \quad \dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)},$$

120 where the *switching signals* σ are selected, in general, among the set \mathcal{S} defined by

$$121 \quad (2.2) \quad \mathcal{S} := \{\sigma : \mathbb{R}_{\geq 0} \rightarrow \langle M \rangle \mid \sigma \text{ piecewise constant}\}.$$

122 We recall that a function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \langle M \rangle$ is said to be *piecewise constant* if it has a
 123 finite number of discontinuity points in any bounded subinterval of $\mathbb{R}_{\geq 0}$. Moreover,
 124 without loss of generality, we suppose that signals $\sigma \in \mathcal{S}$ are right-continuous.

125 Stability analysis of (2.1) under *arbitrary switching signals*, can be equivalently
 126 tackled studying the differential inclusion

$$127 \quad (2.3) \quad \dot{x} \in F(x) := \text{co}\{A_i x + b_i, i \in \langle M \rangle\}.$$

128 Indeed, it can be proven that the set of solutions to (2.1) is dense in the set of solutions
 129 of (2.3), see for example [17], [2, Section 2] and [18, Theorem 2, pag. 124].

130 It turns out that the stability/asymptotic properties of (2.1) are closely related
 131 to the properties of the corresponding *linearized* switched system given by

$$132 \quad (2.4) \quad \dot{x}(t) = A_{\sigma(t)}x(t).$$

133 From now on, we call (2.4) the *linearization of system* (2.1). Given any $\sigma \in \mathcal{S}$, let us
 134 call $\Psi_\sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ the flow map of (2.1), i.e.,

$$135 \quad \Psi_\sigma(t, x) := \text{solution to (2.1), starting at } x(0) = x, \text{ evaluated at time } t \in \mathbb{R}_{\geq 0}.$$

136 We have that solutions of (2.1), by variation of constants formula, are given by

$$137 \quad (2.5) \quad \Psi_\sigma(t, x) = \Phi_\sigma(t, 0)x + \int_0^t \Phi_\sigma(t, s)b_{\sigma(s)} ds,$$

138 where $\Phi_\sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n \times n}$ is the state-transition matrix of the linearization (2.4),
139 see, for example, [19].

140 We introduce the following classical definitions of stability for the linearized sys-
141 tem (2.4), which will be used in subsequent sections to characterize the behavior
142 of (2.1).

143 *Definition 2.1.* Given any subset of switching signals $\tilde{\mathcal{S}} \subseteq \mathcal{S}$ we say that the
144 linearized system (2.4) is *uniformly globally asymptotically stable (UGAS)* on $\tilde{\mathcal{S}}$ if
145 there exists a class \mathcal{KL} function¹ β such that

$$146 \quad |\Phi_\sigma(t, 0)x| \leq \beta(|x|, t) \quad \forall x \in \mathbb{R}^n, \quad \forall t \in \mathbb{R}_{\geq 0}, \quad \forall \sigma \in \tilde{\mathcal{S}}.$$

147 We recall here a classical Lyapunov converse result for linear switching systems under
148 arbitrary switching rules.

149 **LEMMA 2.2** (Theorem 1 in [13]). *The linearized switching system (2.4) is UGAS*
150 *on \mathcal{S} if and only if there exist a norm $v : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and a scalar $\kappa > 0$ such that*

$$151 \quad (2.6) \quad v(\Phi_\sigma(t, 0)x) \leq e^{-\kappa t}v(x), \quad \forall x \in \mathbb{R}^n, \quad \forall t \in \mathbb{R}_{\geq 0}, \quad \forall \sigma \in \mathcal{S}.$$

152 *This in particular implies that UGAS of (2.4) on \mathcal{S} is equivalent to exponential*
153 *stability, i.e., the function $\beta \in \mathcal{KL}$ in Definition 2.1 can be chosen of the form*
154 *$\beta(a, t) = Me^{-\kappa t}a$ for some $M > 0$.*

155 The proof of this Lemma is provided in [13, Theorem 1], see also [12, Theorem III.1].

156 It is usual in the switched systems setting to refine the analysis only focusing on
157 *subclasses* of \mathcal{S} . One of the most common subclass is given by the set of dwell-time
158 switching signals, introduced in the seminal paper [10]. Formally, given a $\tau > 0$,
159 $\mathcal{S}_{\text{dw}}(\tau)$ denotes the class of *dwell-time switching signals* defined by

$$160 \quad (2.7) \quad \mathcal{S}_{\text{dw}}(\tau) := \{ \sigma \in \mathcal{S} \mid t_k^\sigma - t_{k-1}^\sigma \geq \tau, \quad \forall t_k^\sigma > 0 \},$$

161 where $\{t_k^\sigma\}$ denotes the set of time instants at which σ is discontinuous, and by
162 convention, $t_0^\sigma = 0$ for all $\sigma \in \mathcal{S}$. This class can be intuitively seen as the set of “slow”
163 switching signals, i.e., signals for which two distinct switching events cannot occur on
164 time intervals smaller than the given threshold $\tau > 0$.

165 As for the arbitrary switching signals case (Lemma 2.2), the stability of (2.4)
166 under any dwell time class $\mathcal{S}_{\text{dw}}(\tau)$ can also be characterized via a converse Lyapunov
167 result, this time involving *multiple Lyapunov norms*, as recalled in the following state-
168 ment.

169 **LEMMA 2.3** ([14, 4]). *Given $\tau > 0$, the linearized system in (2.4) is UGAS on*
170 *$\mathcal{S}_{\text{dw}}(\tau)$ if and only if there exist norms $v_1 \dots v_M : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and $\kappa > 0$ such that*

$$171 \quad (2.8) \quad v_i(e^{A_i t} x) \leq e^{-\kappa t} v_i(x), \quad \forall x \in \mathbb{R}^n, \quad \forall t \in \mathbb{R}_{\geq 0}, \quad \forall i \in \langle M \rangle.$$

$$172 \quad (2.9) \quad v_i(e^{A_i \tau} x) \leq e^{-\kappa \tau} v_j(x), \quad \forall x \in \mathbb{R}^n, \quad \forall (i, j) \in \langle M \rangle^2.$$

174 *In particular, for any $\tau > 0$, UGAS of (2.4) on $\mathcal{S}_{\text{dw}}(\tau)$ is equivalent to exponential*
175 *stability.*

¹A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class \mathcal{KL} if $\beta(0, s) = 0$ for all s , $\beta(\cdot, s)$ is strictly increasing for all s , and $\beta(r, \cdot)$ is decreasing and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$, for all r .

176 The proof of this Lemma can be found in [14] and [4, Corollary 6.5].

177 The supremum over $\kappa \in \mathbb{R}_{>0}$ for which the norm(s) as in Lemma 2.2 and
 178 Lemma 2.3 can be found represents the best exponential decay rate for system (2.4)
 179 on \mathcal{S} and $\mathcal{S}_{\text{dw}}(\tau)$, respectively; its opposite $-\kappa$ is also called the (*maximal*) *Lyapunov exponent*
 180 of (2.4) on \mathcal{S} and $\mathcal{S}_{\text{dw}}(\tau)$, respectively. We decided to keep this sign
 181 convention (considering the best decay rate instead of the Lyapunov exponent) for
 182 notational simplicity. The interested reader can find further discussion concerning the
 183 computation of the Lyapunov exponent for example in [2], [14] and references therein.

184 **3. Arbitrary Switching.** In this section we first study the behavior of the
 185 switched affine system (2.1) under arbitrary switching rules, i.e., considering switching
 186 signals in the class \mathcal{S} given in (2.2). Since, in general, the subsystems of (2.1) do not
 187 share a common equilibrium, we analyze asymptotic properties with respect to *sets*.
 188 More specifically, under the hypothesis that the linearized system (2.4) is UGAS on
 189 \mathcal{S} we provide a proof of existence (and numerical approximations) of the minimal
 190 forward invariant set. We first recall some definitions, characterizing properties of
 191 *sets* with respect to solutions of (2.1).

192 *Definition 3.1.* Given any subset of switching signals $\tilde{\mathcal{S}} \subseteq \mathcal{S}$ and a compact set
 193 $C \subset \mathbb{R}^n$, we say that:

194 1. C is *forward invariant* for (2.1) on $\tilde{\mathcal{S}}$ if, for all $x \in C$, all $\sigma \in \tilde{\mathcal{S}}$, and all $t \in \mathbb{R}_{\geq 0}$,
 195 it holds that

$$196 \quad \Psi_{\sigma}(t, x) \in C.$$

197 2. A forward invariant set C is *minimal* if, for every forward invariant set D , it
 198 holds that $C \subseteq D$.

199 3. C is *attractive* for (2.1) on $\tilde{\mathcal{S}}$ if for every $x \in \mathbb{R}^n$ and every $\sigma \in \tilde{\mathcal{S}}$, we have

$$200 \quad |\Psi_{\sigma}(t, x)|_C \rightarrow 0, \quad \text{as } t \rightarrow +\infty,$$

201 where $|z|_C := \min_{y \in C} \{|x - y|\}$ denotes the distance between a point $z \in \mathbb{R}^n$ and
 202 the compact set C , with respect to the Euclidean norm.

203 4. C is *exponentially stable* for (2.1) on $\tilde{\mathcal{S}}$ if there exist a $M > 0$ and a $\kappa > 0$ such
 204 that, for all $x \in \mathbb{R}^n$, all $\sigma \in \tilde{\mathcal{S}}$ and all $t \in \mathbb{R}_{\geq 0}$, it holds that

$$205 \quad |\Psi_{\sigma}(t, x)|_C \leq M e^{-\kappa t} |x|_C.$$

206

207 If a minimal forward invariant set C exists, minimality ensures its uniqueness. More-
 208 over, it is easy to see that exponential stability implies forward invariance and at-
 209 tractiveness. For further discussions on stability notions with respect to sets, we refer
 210 to [20]. In the following statement we prove a crucial property of forward invariant
 211 sets.

212 **LEMMA 3.2.** *For any $t \in \mathbb{R}_{\geq 0}$, any $x, y \in \mathbb{R}^n$, any $\sigma \in \mathcal{S}$, and any $\lambda \in [0, 1]$ we*
 213 *have*

$$214 \quad \Psi_{\sigma}(t, \lambda x + (1 - \lambda)y) = \lambda \Psi_{\sigma}(t, x) + (1 - \lambda) \Psi_{\sigma}(t, y).$$

215 *In particular given any set of signals $\tilde{\mathcal{S}} \subseteq \mathcal{S}$, if $C \subset \mathbb{R}^n$ is forward invariant for (2.1)*
 216 *on $\tilde{\mathcal{S}}$, so is $\text{co}\{C\}$.*

217 *Proof.* For the first part of the statement, using (2.5), we compute:

$$\begin{aligned}
\Psi_\sigma(t, \lambda x + (1 - \lambda)y) &= \Phi_\sigma(t, 0)(\lambda x + (1 - \lambda)y) + \int_0^t \Phi_\sigma(t, s)b_{\sigma(s)} ds \\
&= \lambda \left(\Phi_\sigma(t, 0)x + \int_0^t \Phi_\sigma(t, s)b_{\sigma(s)} ds \right) + \\
&\quad + (1 - \lambda) \left(\Phi_\sigma(t, 0)y + \int_0^t \Phi_\sigma(t, s)b_{\sigma(s)} ds \right) \\
&= \lambda \Psi_\sigma(t, x) + (1 - \lambda) \Psi_\sigma(t, y).
\end{aligned}$$

219 For the second part, consider any $\tilde{\mathcal{S}} \subseteq \mathcal{S}$ and suppose that C is a forward invariant
220 set for (2.1) on $\tilde{\mathcal{S}}$. Consider any $x, y \in C$, any $\sigma \in \tilde{\mathcal{S}}$, any $\lambda \in [0, 1]$ and any $t \in \mathbb{R}_{\geq 0}$,
221 then

$$222 \quad \Psi_\sigma(t, \lambda x + (1 - \lambda)y) = \lambda \Psi_\sigma(t, x) + (1 - \lambda) \Psi_\sigma(t, y) \in \text{co}\{C\},$$

223 since by forward invariance of C , we have $\Psi_\sigma(t, x), \Psi_\sigma(t, y) \in C$. \square

224 **3.1. Minimal Forward Invariant and Exponentially Attractive Set.** In
225 this subsection, we prove the existence of minimal forward invariant and attractive
226 sets for (2.1) on \mathcal{S} , i.e., under arbitrary switching sequences. We also show that
227 the minimal forward invariant set is exponentially stable, with the same exponential
228 decay rate of the linearized system. For that, we first need a “theoretic outer bound”,
229 ensuring that, under some hypotheses, forward invariant sets do exist.

230 **PROPOSITION 3.3.** *Consider system (2.1), and suppose that the linearized system*
231 *in (2.4) is UGAS on \mathcal{S} . Consider a norm $v : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and a scalar $\kappa > 0$ as in*
232 *Lemma 2.2. Then there exists $R > 0$ such that the compact set*

$$233 \quad (3.1) \quad \mathcal{K}_{v,R} := \{x \in \mathbb{R}^n \mid v(x) \leq R\},$$

234 *is forward invariant for (2.1) on \mathcal{S} .*

235 *Proof.* Define $B_{max} := \max_{i \in \{M\}} \{v(b_i)\}$. For any $x \in \mathbb{R}^n$, any $\sigma \in \mathcal{S}$ and any
236 $t \in \mathbb{R}_{\geq 0}$, compute

$$\begin{aligned}
v(\Psi_\sigma(t, x)) &= v \left(\Phi_\sigma(t, 0)x + \int_0^t \Phi_\sigma(t, s)b_{\sigma(s)} ds \right) \\
&\leq v(\Phi_\sigma(t, 0)x) + v \left(\int_0^t \Phi_\sigma(t, s)b_{\sigma(s)} ds \right) \\
&\leq e^{-\kappa t} v(x) + \int_0^t v(\Phi_\sigma(t, s)b_{\sigma(s)}) ds \\
&\leq e^{-\kappa t} v(x) + \int_0^t e^{-\kappa(t-s)} v(b_{\sigma(s)}) ds \\
&\leq e^{-\kappa t} v(x) + B_{max} \int_0^t e^{-\kappa(t-s)} ds \\
&= e^{-\kappa t} v(x) + B_{max} \frac{(1 - e^{-\kappa t})}{\kappa},
\end{aligned}$$

237
238 where we used (2.6) and the fact that, for any $\sigma \in \mathcal{S}$ and any $t, s \in \mathbb{R}_{\geq 0}$ it holds that
239 $\Phi_\sigma(t, s) = \Phi_{\tilde{\sigma}}(t - s, 0)$ with $\tilde{\sigma} \in \mathcal{S}$ defined by $\tilde{\sigma}(t') = \sigma(t' + s)$. It thus holds that
240 if $x \in \mathbb{R}^n$ is such that $v(x) \geq \frac{B_{max}}{\kappa} =: R$, then $v(\Psi_\sigma(t, x)) \leq v(x)$, for all $t \in \mathbb{R}_{\geq 0}$,
241 concluding the proof. \square

242 This statement provides a first outer bound for the minimal forward invariant set
 243 (if it exists). In the following, we show that such a set does exist. To this aim, we
 244 introduce the set-valued map $\mathcal{K} : \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^n$ defined by

$$245 \quad (3.2) \quad \mathcal{K}(t) := \bigcup_{\sigma \in \mathcal{S}} \left\{ \int_0^t \Phi_{\sigma}(t, s) b_{\sigma(s)} ds \right\}.$$

246 Equivalently, for every $t \in \mathbb{R}_{\geq 0}$, the set $\mathcal{K}(t)$ represents the *reachable set* of (2.1) at
 247 time t starting at 0, i.e.

$$248 \quad \mathcal{K}(t) = \{\Psi_{\sigma}(t, 0) \mid \sigma \in \mathcal{S}\}.$$

249 In what follows, studying the properties of the set-valued map $\mathcal{K} : \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^n$, we
 250 prove the existence of the minimal forward invariant set.

251 **THEOREM 3.4.** *Let us consider $\mathcal{F} := \{(A_i, b_i)_{i \in \langle M \rangle} \mid A_i \in \mathbb{R}^{n \times n}, b_i \in \mathbb{R}^n\}$ defining
 252 a switched affine system as in (2.1). If the linearized system (2.4) is UGAS on \mathcal{S} , the
 253 limit*

$$254 \quad (3.3) \quad \mathcal{K}_{\infty} = \lim_{t \rightarrow \infty} \mathcal{K}(t)$$

255 (in the Hausdorff metric on compact sets) of the sequence in (3.2) is well-defined and
 256 equal to the minimal forward invariant set of (2.1) on \mathcal{S} .

257 *Proof.* Without loss of generality, we suppose $b_1 = 0$; indeed, the general case
 258 is reduced to this framework by applying the translation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined as
 259 $T(x) := x + A_1^{-1} b_1$, where A_1^{-1} exists since, by hypothesis, A_1 is a Hurwitz matrix.

260 *Well-definiteness:* Firstly, we prove that, for all $\bar{t} \geq t \geq 0$, we have $\mathcal{K}(t) \subseteq \mathcal{K}(\bar{t})$.
 261 Considering any $x \in \mathcal{K}(t)$, by definition, there exists a $\sigma \in \mathcal{S}$, such that $x = \Psi_{\sigma}(t, 0)$.
 262 Now define $\tilde{\sigma} \in \mathcal{S}$ by

$$263 \quad \tilde{\sigma}(s) := \begin{cases} 1, & \text{if } s < \bar{t} - t, \\ \sigma(s - \bar{t} + t), & \text{if } s \geq \bar{t} - t. \end{cases}$$

264 Since $b_1 = 0$, it is clear that $\Psi_{\tilde{\sigma}}(\bar{t} - t, 0) = 0$, and thus $x = \Psi_{\tilde{\sigma}}(\bar{t}, 0) \in \mathcal{K}(\bar{t})$, proving
 265 that $\mathcal{K}(t) \subseteq \mathcal{K}(\bar{t})$. By Proposition 3.3, for every $t \geq 0$, the reachable sets $\mathcal{K}(t)$ are
 266 uniformly bounded since they are included in the compact set $\mathcal{K}_{v,R}$ defined in (3.1),
 267 recalling that $0 \in \mathcal{K}_{v,R}$ and $\mathcal{K}_{v,R}$ is forward invariant. Thus, the set

$$268 \quad \mathcal{K}_{\infty} = \overline{\bigcup_{t \geq 0} \mathcal{K}(t)}$$

269 is a well-defined compact set, being the limit of an increasing sequence of compact
 270 sets, see [21, Chapter 4.B].

271 *Forward invariance:* Consider first $x \in \bigcup_{t \geq 0} \mathcal{K}(t)$ and consider any $\sigma \in \mathcal{S}$, we prove
 272 that, for any $t_0 \geq 0$, $\Psi_{\sigma}(t_0, x) \in \bigcup_{t \geq 0} \mathcal{K}(t)$. Since $x \in \bigcup_{t \geq 0} \mathcal{K}(t)$ there exists a $T \geq 0$
 273 and a $\sigma_1 \in \mathcal{S}$ such that $x = \Psi_{\sigma_1}(T, 0)$. Defining $\tilde{\sigma} \in \mathcal{S}$ by

$$274 \quad \tilde{\sigma}(s) := \begin{cases} \sigma_1(s) & \text{if } s < T, \\ \sigma(s - T) & \text{if } s \geq T, \end{cases}$$

275 we have $\Psi_{\sigma}(t_0, x) = \Psi_{\tilde{\sigma}}(T + t_0, 0) \in \mathcal{K}(T + t_0) \subseteq \bigcup_{t \geq 0} \mathcal{K}(t)$. Now, for the limit case
 276 where $x \in \mathcal{K}_{\infty} = \overline{\bigcup_{t \geq 0} \mathcal{K}(t)}$, for any $\sigma \in \mathcal{S}$ and any $t_0 \geq 0$, we want to prove that

277 $\Psi_\sigma(t_0, x) \in \mathcal{K}_\infty$. Consider a sequence $(x_k)_{k \in \mathbb{N}}$ such that $x_k \rightarrow x$ as $k \rightarrow \infty$ and
 278 $x_k \in \bigcup_{t \geq 0} \mathcal{K}(t)$ for all $k \in \mathbb{N}$. We have already proven that $\Psi_\sigma(t_0, x_k) \in \bigcup_{t \geq 0} \mathcal{K}(t)$,
 279 for all $k \in \mathbb{N}$. Now by continuity from initial conditions (see [19, Theorem 3.4]) we
 280 have

$$281 \quad \Psi_\sigma(t_0, x) = \lim_{k \rightarrow \infty} \Psi_\sigma(t_0, x_k),$$

282 and thus $\Psi_\sigma(t_0, x) \in \overline{\bigcup_{t \geq 0} \mathcal{K}(t)} = \mathcal{K}_\infty$, concluding the proof.

283 *Minimality:* Consider a compact forward invariant set $C \subset \mathbb{R}^n$, any initial condition
 284 $x \in C$, and the constant switching signal $\sigma(s) = 1$, for all $s \in \mathbb{R}_{\geq 0}$. Consider any
 285 strictly increasing sequence $(t_k)_{k \in \mathbb{N}}$ such that $t_k > 0$ for all $k \in \mathbb{N}$ and $t_k \rightarrow +\infty$ as
 286 $k \rightarrow \infty$. Then $x_k = \Psi_\sigma(t_k, x) \in C$ for all $k \in \mathbb{N}$ by forward invariance of C , and since
 287 $\lim_{k \rightarrow \infty} x_k = \lim_{k \rightarrow \infty} \Psi_\sigma(t_k, x) = \lim_{k \rightarrow \infty} e^{A_1 t_k} x = 0$, by compactness we have $0 \in C$.
 288 Now by definition of forward invariance and since \mathcal{K}_∞ is defined as the reachable set
 289 from 0, we conclude that $\mathcal{K}_\infty \subseteq C$. \square

290 An existence result similar to Theorem 3.4 was proven in [16], in a slightly different
 291 setting and with a different methodology. We presented here our proof based on
 292 Lemma 2.2 since it allows us to not only prove the existence of the minimal forward
 293 invariant set \mathcal{K}_∞ , but also to verify that it is exponentially stable (and thus, in
 294 particular, attractive), as proven in the following proposition.

295 **PROPOSITION 3.5.** *Under the hypothesis of Theorem 3.4, the compact set \mathcal{K}_∞ is*
 296 *exponentially stable for (2.1) on \mathcal{S} . More specifically, there exists a $M > 0$ such that*

$$297 \quad (3.4) \quad |\Psi_\sigma(t, x)|_{\mathcal{K}_\infty} \leq M e^{-\kappa t} |x|_{\mathcal{K}_\infty}, \quad \forall x \in \mathbb{R}^n, \quad \forall \sigma \in \mathcal{S}, \quad \forall t \in \mathbb{R}_{\geq 0},$$

298 *where κ is the exponential decay rate of the linearized system (2.4) given in Lemma 2.2.*
 299

300 *Proof.* Consider a scalar $\kappa > 0$ and a norm $v : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ satisfying the properties
 301 in Lemma 2.2, and define $\mathbb{B}_v := \{x \in \mathbb{R}^n \mid v(x) \leq 1\}$, i.e., the unit ball of the norm v .
 302 We consider $W : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, the *distance from \mathcal{K}_∞ with respect to v* , defined by

$$303 \quad (3.5) \quad W(x) := \min_{y \in \mathcal{K}_\infty} \{v(x - y)\} = \min\{r \in \mathbb{R}_{\geq 0} \mid x \in \mathcal{K}_\infty + r\mathbb{B}_v\}.$$

304 First, for any $x \in \mathcal{K}_\infty$, forward invariance of \mathcal{K}_∞ implies that $W(\Psi_\sigma(t, x)) = 0$, for
 305 all $\sigma \in \mathcal{S}$ and all $t \in \mathbb{R}_{\geq 0}$. Now consider any $x \notin \mathcal{K}_\infty$, any $\sigma \in \mathcal{S}$ and any $t \in \mathbb{R}_{\geq 0}$.
 306 By equation (3.5), we can decompose x as $x = \hat{x} + y$ with $\hat{x} \in \mathcal{K}_\infty$ and $y \in W(x)\mathbb{B}_v$.
 307 Computing

$$308 \quad \begin{aligned} W(\Psi_\sigma(t, x)) &= W\left(\Phi_\sigma(t, 0)(\hat{x} + y) + \int_0^t \Phi_\sigma(t, s)b_{\sigma(s)} ds\right) \\ &= W\left(\Phi_\sigma(t, 0)\hat{x} + \int_0^t \Phi_\sigma(t, s)b_{\sigma(s)} ds + \Phi_\sigma(t, 0)y\right). \end{aligned}$$

309 Recalling again the forward invariance of \mathcal{K}_∞ , we have $\Phi_\sigma(t, 0)\hat{x} + \int_0^t \Phi_\sigma(t, s)b_{\sigma(s)} ds \in$
 310 \mathcal{K}_∞ and, by Lemma 2.2, we have $\Phi_\sigma(t, 0)y \in e^{-\kappa t} W(x)\mathbb{B}_v$, proving that

$$311 \quad W(\Psi_\sigma(t, x)) \leq e^{-\kappa t} W(x).$$

312 Now by equivalence of norms in \mathbb{R}^n , it can be seen that there exist $M_1, M_2 \in \mathbb{R}_{> 0}$
 313 such that

$$314 \quad M_1 |x|_{\mathcal{K}_\infty} \leq W(x) \leq M_2 |x|_{\mathcal{K}_\infty}, \quad \forall x \in \mathbb{R}^n$$

315 where, we recall, $|\cdot|_{\mathcal{K}_\infty}$ denotes the distance from \mathcal{K}_∞ with respect to the Euclidean
 316 distance. Thus we have

$$317 \quad (3.6) \quad |\Psi_\sigma(t, x)|_{\mathcal{K}_\infty} \leq \frac{M_2}{M_1} e^{-\kappa t} |x|_{\mathcal{K}_\infty}, \quad \forall x \in \mathbb{R}^n, \quad \forall \sigma \in \mathcal{S}, \quad \forall t \in \mathbb{R}_{\geq 0},$$

318 proving (3.4) and concluding the proof. \square

319 We note that in Proposition 3.5 we have proven that the decay rate $\kappa > 0$ of the
 320 linearized system (2.4) with respect to the origin is conserved by the switched affine
 321 system (2.1), in the sense that $\kappa > 0$ is also the exponential decay rate of (2.1) with
 322 respect to the minimal forward invariant set \mathcal{K}_∞ , as shown in (3.4).

323 *Remark 3.6* (Properties of \mathcal{K}_∞). Since \mathcal{K}_∞ is formally defined as an infinite union
 324 of sets in (3.3), it can be hard to construct it explicitly, as we will discuss in the next
 325 subsection. However, despite this non-constructive definition of \mathcal{K}_∞ , we are able to
 326 provide some remarkable properties. First of all, given $\mathcal{F} := \{(A_i, b_i)_{i \in \langle M \rangle} \mid A_i \in$
 327 $\mathbb{R}^{n \times n}, b_i \in \mathbb{R}^n\}$, let us consider the set of *Filippov equilibria* defined by

$$328 \quad \text{Fil}_0(\mathcal{F}) := \left\{ y = - \left(\sum_{i \in \langle M \rangle} \bar{\lambda}_i A_i \right)^{-1} \left(\sum_{i \in \langle M \rangle} \bar{\lambda}_i b_i \right) \mid \bar{\lambda} \in \Lambda_M \right\} = \{y \in \mathbb{R}^n \mid 0 \in F(y)\},$$

329 where the set-valued map F is defined in (2.3) and $\Lambda_M := \{\bar{\lambda} \in \mathbb{R}_{\geq 0}^M \mid \sum_{i=1}^M \lambda_i = 1\}$ is
 330 the standard simplex of dimension $M-1$. As discussed in Section 2, solutions to (2.1)
 331 under arbitrary switching rules are dense in the set of solutions to the differential in-
 332 clusion (2.3), and this in particular implies $\text{Fil}_0(\mathcal{F}) \subseteq \mathcal{K}_\infty$. Moreover, we can ensure
 333 that the set \mathcal{K}_∞ is (path) connected: consider $x_1, x_2 \in \mathcal{K}_\infty$, and consider the constant
 334 signal defined by $\hat{\sigma}(t) \equiv 1$. By forward invariance of \mathcal{K}_∞ it is clear that the corre-
 335 sponding trajectories are contained in \mathcal{K}_∞ , more precisely $\{\Psi_{\hat{\sigma}}(t, x_j) \mid t \geq 0\} \subseteq \mathcal{K}_\infty$
 336 for all $j \in \{1, 2\}$. Since \mathcal{K}_∞ is closed and $\lim_{t \rightarrow +\infty} \Psi_{\hat{\sigma}}(t, x_1) = \lim_{t \rightarrow +\infty} \Psi_{\hat{\sigma}}(t, x_2) =$
 337 $-A_1^{-1}b_1$, there is a path in \mathcal{K}_∞ connecting x_1 and x_2 (intuitively, the union of the
 338 two trajectories). We have thus proven that \mathcal{K}_∞ is path-connected. However, in the
 339 subsequent numerical example, we show that \mathcal{K}_∞ is not convex, in general. \triangle

340 *Example 3.7.* Consider a switched affine system as in (2.1) defined by

$$341 \quad A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad b_1 = b_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

342 The set $\text{Fil}_0(\mathcal{F})$ of the Filippov equilibria, defined in (3.7), is depicted in black in
 343 Figure 1 and, as previously discussed, we know that $\text{Fil}_0(\mathcal{F}) \subseteq \mathcal{K}_\infty$. Considering the
 344 convex set \mathcal{Z} depicted in Figure 1, it is possible to show that, for any $x \in \partial\mathcal{Z}$ and for
 345 any $i \in \{1, 2\}$ we have $A_i x + b_i \notin \mathcal{T}_{\mathcal{Z}}(x)$, where $\mathcal{T}_{\mathcal{Z}}(x)$ denotes the tangent cone² to \mathcal{Z}
 346 at x . Applying the Nagumo Theorem [18, Chapter 4, pag. 175], by forward invariance
 347 of \mathcal{K}_∞ , this implies $\text{Int}(\mathcal{Z}) \cap \mathcal{K}_\infty = \emptyset$. Consider the points $x_{e1}, x_{e2} \in \text{Fil}_0(\mathcal{F})$, defined
 348 by $x_{ei} = -A_i^{-1}b_i$ with $i \in \{1, 2\}$, i.e., the equilibria of each affine subsystem. It holds
 349 that $x^* = \frac{1}{2}x_{e1} + \frac{1}{2}x_{e2}$ is both in $\text{Int}(\mathcal{Z})$ and in $\text{co}(\text{Fil}_0(\mathcal{F})) \subseteq \text{co}(\mathcal{K}_\infty)$, which implies
 350 that \mathcal{K}_∞ is non-convex.

²Given a closed convex set $C \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$ the *tangent cone to C at x* is the set $\mathcal{T}_C(x) := \text{cl}(\{z \in \mathbb{R}^n \mid \exists \alpha_0 > 0 \text{ s.t. } x + \alpha z \in C, \forall \alpha \in (0, \alpha_0)\})$, see [22, Appendix A] for further discussions.

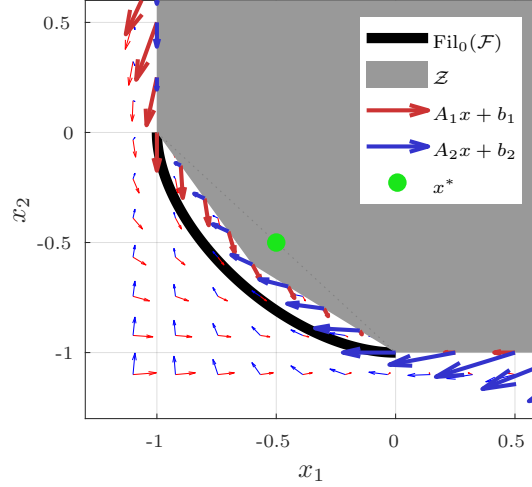


FIGURE 1. Representation of the state-space for Example 3.7 with the set of Filippov equilibria $Fil_0(\mathcal{F})$, the set \mathcal{Z} for which the vector fields $A_i x + b_i$, $i \in \{1, 2\}$ do not point inwards, and the point $x^* = \frac{1}{2}x_{e1} + \frac{1}{2}x_{e2} = [-0.5 \ -0.5]^\top \in co\{Fil_0(\mathcal{F})\}$. This demonstrates that \mathcal{K}_∞ is non-convex since $x^* \in \mathcal{Z}$ implies that $x^* \notin \mathcal{K}_\infty$.

351 This example will be later recalled when we present numerical procedures to over-
 352 approximate \mathcal{K}_∞ , providing further insights in its geometric characterization.

353 Before concluding this subsection, we provide a “negative” result, stating that
 354 (non-trivial) forward invariant sets do not exist if the linearized system (2.4) is un-
 355 stable.

356 **PROPOSITION 3.8.** Consider $\mathcal{F} := \{(A_i, b_i)_{i \in \langle M \rangle} \mid A_i \in \mathbb{R}^{n \times n}, b_i \in \mathbb{R}^n\}$. If the
 357 linearized system (2.4) is unstable on \mathcal{S} (i.e., there exist $x \in \mathbb{R}^n$ and $\sigma \in \mathcal{S}$ such that
 358 $\limsup_{t \rightarrow \infty} |\Phi_\sigma(t, 0)x| = +\infty$) then there does not exist a compact forward invariant
 359 set with non-empty interior for the switched affine system (2.1) on \mathcal{S} . Moreover, if
 360 the set of matrices $\mathcal{A} = \{A_1, \dots, A_M\} \subset \mathbb{R}^{n \times n}$ is also irreducible³ then there exists
 361 no compact non-singleton forward invariant set for (2.1) on \mathcal{S} .

362 *Proof.* Let us consider $x \in \mathbb{R}^n$ and $\sigma \in \mathcal{S}$ which provide an unbounded trajectory
 363 for the linearized system (2.4). Suppose by contradiction that $C \subset \mathbb{R}^n$ is a compact
 364 forward invariant set for (2.1), and $\text{Int}(C) \neq \emptyset$. Consider any $y \in \text{Int}(C)$, and a $\lambda > 0$
 365 small enough such that $z := y + \lambda x \in C$. Recalling the solution (2.5), we obtain

$$366 \quad |\Psi_\sigma(t, z) - \Psi_\sigma(t, y)| = |\Phi_\sigma(t, 0)z - \Phi_\sigma(t, 0)y| = |\Phi_\sigma(t, 0)(z - y)| = \lambda |\Phi_\sigma(t, 0)x|,$$

367 and thus

$$368 \quad \limsup_{t \rightarrow \infty} |\Psi_\sigma(t, z) - \Psi_\sigma(t, y)| = +\infty,$$

369 in contradiction to the fact that $y, z \in C$ and C is a compact forward invariant set.

370 For the second part of the statement, recalling the results contained in [23, Theo-
 371 rem 3] or [24], since \mathcal{A} is irreducible and generates an unstable switched system, there
 372 exist $\rho > 0$ and a norm $v : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that

$$373 \quad (3.8) \quad \forall x \in \mathbb{R}^n, \exists \sigma_x \in \mathcal{S} \text{ such that } v(\Phi_{\sigma_x}(t, 0)x) = e^{\rho t} v(x), \quad \forall t \in \mathbb{R}_{\geq 0}.$$

³A set of matrices $\mathcal{A} \subset \mathbb{R}^{n \times n}$ is said to be *irreducible* if $\{0\}$ and \mathbb{R}^n are the only vector subspaces which are invariant under all matrices in \mathcal{A} .

374 Suppose, by contradiction, that $C \subset \mathbb{R}^n$ is a compact forward invariant set and is not
 375 a singleton. Consider $y, z \in C$ with $y \neq z$ and define $x := z - y$. For $\sigma_x \in \mathcal{S}$ defined
 376 in (3.8), we have

$$377 \quad |\Psi_{\sigma_x}(t, x_2) - \Psi_{\sigma_x}(t, x_1)| = |\Phi_{\sigma_x}(t, 0)z - \Phi_{\sigma_x}(t, 0)y| = |\Phi_{\sigma_x}(t, 0)x|.$$

378 By equivalence of norms, this implies that $\limsup_{t \rightarrow \infty} |\Psi_{\sigma_x}(t, z) - \Psi_{\sigma_x}(t, y)| = +\infty$
 379 and we have again a contradiction, since by hypothesis C is compact and forward
 380 invariant. \square

381 It is easy to see that the singleton-forward invariant case occurs if and only if all the
 382 subsystems share the same equilibrium, or, in other words, the switched affine system
 383 is simply a translation of a linear switched system. We have thus proved that, if
 384 the linearized system (2.4) is unstable and under the mild irreducibility assumption,
 385 non-trivial affine switched systems (2.1) have *no* compact forward invariant sets.

386 **3.2. Outer Approximation of the Minimal Forward Invariant Set.** In
 387 this subsection we propose two numerical methods for computing an outer approx-
 388 imation of \mathcal{K}_∞ , i.e., forward invariant sets $\mathcal{K} \supseteq \mathcal{K}_\infty$ as close as possible to \mathcal{K}_∞ (in
 389 a sense that we will clarify). The first method is a direct consequence of Propo-
 390 sition 3.3 when considering *quadratic norms*, providing forward invariant ellipsoids,
 391 and it is presented in the following statement.

392 **PROPOSITION 3.9.** *If there exists a symmetric matrix $S \in \mathbb{R}^{n \times n}$, a vector $c \in \mathbb{R}^n$*
 393 *and a scalar $\kappa > 0$ satisfying the inequalities*

$$394 \quad (3.9a) \quad SA_i^\top + A_i S \prec -2\kappa S, \quad \forall i \in \langle M \rangle,$$

$$395 \quad (3.9b) \quad \begin{bmatrix} \kappa^2 & (A_i c + b_i)^\top \\ A_i c + b_i & S \end{bmatrix} \succ 0, \quad \forall i \in \langle M \rangle,$$

396 then the ellipsoidal set

$$397 \quad (3.10) \quad \mathcal{K}_Q := \{x \in \mathbb{R}^n \mid (x - c)^\top S^{-1}(x - c) \leq 1\} \supseteq \mathcal{K}_\infty$$

398 is a forward invariant set for system (2.1) on \mathcal{S} .

400 *Proof.* Consider system (2.1) translated to $c \in \mathbb{R}^n$, which is equivalent to replace
 401 $b_i \leftarrow A_i c + b_i$ in (2.1). From the term (2,2) in (3.9b), one has that $S^{-1} \succ 0$. Inequal-
 402 ity (3.9a), in turn, implies that the norm $v(x) = \sqrt{x^\top S^{-1}x}$ satisfies the condition (2.6)
 403 in Lemma 2.2. Moreover, the second inequality, by the Schur Complement Lemma
 404 (see [25, Section A.5.5]), is equivalent to

$$405 \quad (3.11) \quad \kappa^2 > \max_{i \in \langle M \rangle} (A_i c + b_i)^\top S^{-1}(A_i c + b_i) = B_{max}^2$$

407 which ensures that $R = (\frac{B_{max}}{\kappa})^2 < 1$. Therefore, $x \notin \mathcal{K}_Q$ implies $v(x) > 1 > R$ and
 408 therefore, by Proposition 3.3, \mathcal{K}_Q is a forward invariant set. \square

409 An optimization problem to minimize the volume of the ellipsoidal set \mathcal{K}_Q can be
 410 stated considering the objective function $\min_{S, c, \kappa} \ln \det(S)$ subject to (3.9). The vol-
 411 ume of \mathcal{K}_Q is proportional to $\det(S)$ and the \ln function makes the objective function
 412 concave in S . However, for a given $\kappa > 0$, this problem is a concave-minimization
 413 problem, which is generally hard to solve, [26]. Some strategies (see [27]) to handle
 414 this problem are convex-optimization methods for local minimization and branch-
 415 and-bound algorithms for global minimization. A good alternative objective function,

416 leading to a convex optimization problem is $\text{Tr}(S)$, which, instead of minimizing the
 417 volume of the ellipsoid \mathcal{K}_Q , minimizes the sum of the square lengths of the semi-axes,
 418 see [25, Section 2.2.2]. Notice that the point $c \in \mathbb{R}^n$ to which the system is translated
 419 is also a variable of the optimization problem, which allows us not only to estimate
 420 the size of \mathcal{K}_∞ but also to optimize a suitable center in the state space for the ellipsoid
 421 containing it.

422 Nevertheless, it is well known that the existence of a common *quadratic* Lyapunov
 423 function for the linearized part (i.e., a matrix $S > 0$ satisfying (3.9a)) is a restrictive
 424 condition for stability. Moreover, Example 3.7 shows how the set \mathcal{K}_∞ is in general
 425 non-convex; it thus seems natural, in order to improve our estimation of the minimal
 426 forward invariant set \mathcal{K}_∞ , to consider sub-level sets of more general functions. In
 427 what follows, we propose a construction based on sub-level set of non-homogeneous
 428 *sum-of-squares (SOS) polynomials*, see [28] for details.

429 PROPOSITION 3.10 (Outer Approximation via SOS polynomials). *If there exist*
 430 *a non-trivial polynomial $V(x) \in \mathbb{R}[x]$ of degree $d \in \mathbb{N}$ and scalars $r > 0$, $\beta \geq 0$ such*
 431 *that the following SOS constraints are satisfied*

$$432 \quad (3.12a) \quad V(x) - \epsilon \|x\|_d^d \text{ is SOS}$$

$$433 \quad (3.12b) \quad - \left(\frac{\partial V}{\partial x}(x) \right)^\top (A_i x + b_i) - \beta(V(x) - r) \text{ is SOS} \quad \forall i \in \langle M \rangle$$

434
 435 for some $\epsilon > 0$, then the set

$$436 \quad (3.13) \quad \mathcal{K}_{\text{SOS}} := \{x \in \mathbb{R}^n \mid V(x) \leq r\} \supseteq \mathcal{K}_\infty$$

437 is a forward invariant set for system (2.1) on \mathcal{S} .

438 *Proof.* The SOS constraint (3.12a) ensures that the polynomial $V(x) \in \mathbb{R}[x]$ is
 439 positive definite whereas (3.12b) implies

$$440 \quad (3.14) \quad \left(\frac{\partial V}{\partial x}(x) \right)^\top (A_i x + b_i) < 0, \quad \forall i \in \langle M \rangle, \quad \forall x \notin \mathcal{K}_{\text{SOS}},$$

441 recalling the definition of \mathcal{K}_{SOS} in (3.13). This condition implies that $V(x)$ decreases
 442 along solutions of (2.1) that lie outside of \mathcal{K}_{SOS} . Therefore, this set is attractive and
 443 forward invariant for system (2.1) undergoing arbitrary switching. \square

444 *Remark 3.11.* Minimizing, or even computing, the volume of the set \mathcal{K}_{SOS} defined
 445 in (3.13) is, in general, a very intricate task, see [29] for some discussions. Therefore,
 446 a simple way to optimize the over estimation of \mathcal{K}_∞ is to minimize $r > 0$ subject
 447 to (3.12), where $\epsilon > 0$ is chosen to avoid the trivial solution $V(x) = 0$. Also, this
 448 optimization problem can be efficiently solved by a bisection procedure over the scalar
 449 $\beta > 0$, as the problem becomes convex whenever β is given.

450 Another important remark is that any polynomial $V(x)$ satisfying (3.12b) and
 451 $V(x) - \epsilon \|x\|_d^d$ is SOS can be decomposed as $V(x) = V_H(x) + V_R(x)$ where $V_H(x)$ is a
 452 homogeneous polynomial of the same degree $d \in \mathbb{N}$ as $V(x)$ and $V_R(x)$ are the remain-
 453 ing terms. Also, $w(x) = (V_H(x))^{\frac{1}{d}}$ can be shown to be a Lyapunov function for the
 454 linearized system (2.4), satisfying the conditions (2.8) and (2.9) for some $\kappa > 0$. In-
 455 deed, the SOS constraint (3.12b) implies that $-\left(\frac{\partial V}{\partial x}(x)\right)^\top (A_i x + b_i) - \beta r \geq \beta V(x)$ for
 456 all $i \in \langle M \rangle$ which, in turn, implies that $-\left(\frac{\partial V_H}{\partial x}(x)\right)^\top A_i x \geq \beta V_H(x)$ for all $i \in \langle M \rangle$.

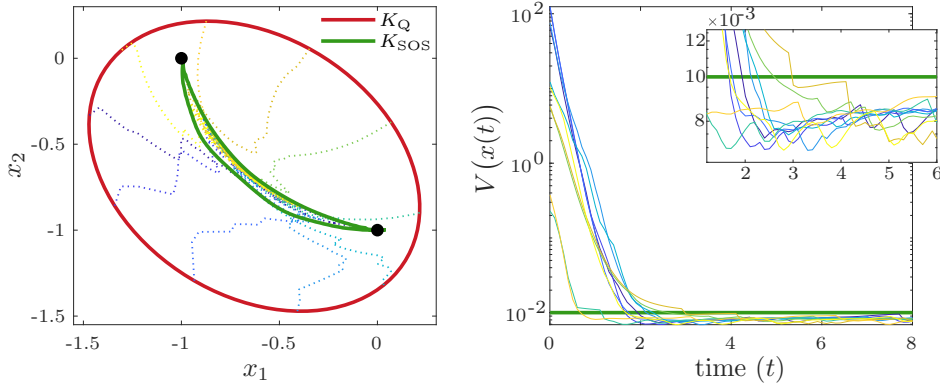


FIGURE 2. Illustration of Example 3.12. On the left, a representation of the state space with the forward invariant sets K_Q and K_{SOS} , the equilibria of each subsystem (black dots), and 10 trajectories undergoing random switching. On the right, the evaluation of the polynomial $V(x)$ associated to K_{SOS} along each of these trajectories and the horizontal line represents the level set used to define K_{SOS} .

457 Therefore, $w(x) = (V_H(x))^{\frac{1}{d}}$ is a common Lyapunov function for the linearized sys-
 458 tem (2.4) homogeneous of degree 1. This in particular implies the existence of a norm
 459 $v : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ satisfying the conditions in Lemma 2.2 for $\kappa = \beta/d$. Thus, recalling
 460 Proposition 3.5, the approximating technique given by Proposition 3.10 provides also
 461 an upper bound on the decay rate of system (2.1). \triangle

462 The following example illustrates both methods presented in this section.

463 *Example 3.12.* Consider the switched affine system introduced in Example 3.7.
 464 We constructed two forward invariant sets that serve as outer approximations of the
 465 minimal one \mathcal{K}_∞ . These sets are respectively based on Proposition 3.9 and Proposi-
 466 tion 3.10, and all related optimization problems were solved using YALMIP [30] and
 467 MOSEK. The first one is the ellipsoidal region \mathcal{K}_Q , which is defined in (3.10) and
 468 relies on the existence of a common quadratic Lyapunov Function for the linearized
 469 system (2.4). For the first outer bound, the obtained matrix $S \succ 0$ and center $c \in \mathbb{R}^n$
 470 were

$$471 \quad S = \begin{bmatrix} 0.7120 & -0.2021 \\ -0.2021 & 0.7120 \end{bmatrix}, \quad c = \begin{bmatrix} -0.6291 \\ -0.6291 \end{bmatrix},$$

472 which solve the problem $\min_{S,c,\kappa} \text{trace}(S)$ subject to (3.9) for $\kappa = 0.4785$. The second
 473 set is \mathcal{K}_{SOS} , which was obtained by solving the SOS optimization problem $\min_{V(x),r} r$
 474 subject to (3.12), where $\epsilon = 10^{-2}$ is chosen to avoid a trivial solution $V(x) \equiv 0$.
 475 Restricting the search for the polynomial $V(x)$ to those with maximum total degree
 476 $d = 12$ and choosing $\beta = 1$, the convex optimization problem yielded the forward
 477 invariant set \mathcal{K}_{SOS} as defined in (3.13). The average elapsed-times for the solution
 478 of these optimization problems were 0.1096 and 0.2128 seconds, respectively, which
 479 were executed on an Intel[®] Core[™] i7-10610U CPU @ 1.80 GHz \times 8 with 16 GB of
 480 memory running Matlab R2020a on Ubuntu 20.04. Both \mathcal{K}_Q and \mathcal{K}_{SOS} are represented
 481 in Figure 2 (left) along with the equilibria of each subsystem (black dots) and 10
 482 trajectories starting on the boundary of \mathcal{K}_Q that undergo random switching. The
 483 value of $V(x)$ evaluated along each of these trajectories is also depicted in Figure 2
 484 (right) as a function of time, where the horizontal line represents the level set defining
 485 \mathcal{K}_{SOS} with $r = 0.0110$. In detail, we can notice that the trajectories were all attracted
 486 to $\mathcal{K}_{SOS} \supseteq \mathcal{K}_\infty$ and never left it once in its interior, as expected.

487 The numerical experiments carried out in the previous example indicate that, al-
 488 though the SOS-based method given in Proposition 3.10 may yield more precise outer
 489 approximations for \mathcal{K}_∞ (possibly non-convex), the LMI-based approach from Proposi-
 490 tion 3.9 can be computationally more attractive since it scales better with the system
 491 dimension. This concludes the stability study of arbitrarily switching signals and we
 492 now move on to present the dwell-time switching results.

493 **4. Dwell-Time Switching Signals.** In many practical situations, stability un-
 494 der *arbitrary* switching signals is a restrictive requirement, and system (2.1) is known
 495 to follow prescribed switching rules $\sigma \in \mathcal{S}$ that satisfy constraints bounding the fre-
 496 quency of the switching events. For that reason, in this section, given any threshold
 497 $\tau > 0$ representing a *dwell-time*, we restrict our analysis to the switching signals
 498 belonging to the *dwell-time signal class* $\mathcal{S}_{\text{dw}}(\tau)$ defined in (2.7).

499 **4.1. Stability Analysis and Ultimate Boundedness.** Stability of switched
 500 linear systems as in (2.4) under dwell-time assumption is a well-studied problem and it
 501 is known that, if all the matrices $A_1, \dots, A_M \in \mathbb{R}^{n \times n}$ are Hurwitz, there exists a (large
 502 enough) dwell-time for which the switched system (2.4) is UGAS [10, Lemma 2], [1,
 503 Chapter 3]. In this subsection, given $\tau > 0$, we analyze the stability/convergence
 504 properties of the switched *affine* system (2.1) on $\mathcal{S}_{\text{dw}}(\tau)$. First of all we observe in
 505 the next result that, if the linearized system is unstable under arbitrary switching
 506 signals, there is no hope to find non-trivial forward invariant sets on $\mathcal{S}_{\text{dw}}(\tau)$, for *any*
 507 dwell time $\tau > 0$.

508 **LEMMA 4.1.** *Consider $\mathcal{F} = \{(A_i, b_i)_{i \in \langle M \rangle} \mid A_i \in \mathbb{R}^{n \times n}, b_i \in \mathbb{R}^n\}$, and suppose that*
 509 *the linearized system (2.4) is unstable on \mathcal{S} . Then, for any $\tau > 0$ there does not exist*
 510 *a compact forward invariant set with non-empty interior on $\mathcal{S}_{\text{dw}}(\tau)$. If moreover the*
 511 *set $\mathcal{A} = \{A_1, \dots, A_M\}$ is irreducible, there exists no compact non-singleton forward*
 512 *invariant set on $\mathcal{S}_{\text{dw}}(\tau)$.*

513 *Sketch of Proof.* The proof follows by Proposition 3.8 and is briefly sketched here.
 514 Given any $\tau > 0$, suppose by contradiction that $C \subset \mathbb{R}^n$ is a compact forward
 515 invariant set with non-empty interior for (2.1) on $\mathcal{S}_{\text{dw}}(\tau)$. Without loss of generality,
 516 by Lemma 3.2, we can suppose that C is convex. By Proposition 3.8, C is not forward
 517 invariant on \mathcal{S} , i.e., there exists an $x \in C$, a $\sigma \in \mathcal{S}$ and a $T > 0$ such that $\Psi_\sigma(T, x) \notin C$.
 518 Consider $\bar{t} := \sup_{t \in [0, T]} \{\Psi_\sigma(t, x) \in C\}$ and denote $y = \Psi_\sigma(\bar{t}, x)$. It can be seen that
 519 $y \in \partial C$ and, by convexity of C and recalling the Nagumo Theorem for differential
 520 inclusions [18, Chapter 4, pag. 175], there exists $i \in \langle M \rangle$ such that $A_i y + b_i \notin \mathcal{T}_C(y)$,
 521 where $\mathcal{T}_C(y)$ denotes the tangent cone to C at y . It is thus clear that considering
 522 the constant signal $\tilde{\sigma} \in \mathcal{S}_{\text{dw}}(\tau)$ defined by $\tilde{\sigma}(s) \equiv i$ for all $s \in \mathbb{R}_{\geq 0}$, we have that
 523 there exists $t_0 > 0$ such that $\Psi_\sigma(t_0, y) \notin C$, contradicting the forward invariance of C
 524 for (2.1) on $\mathcal{S}_{\text{dw}}(\tau)$. The second part, assuming the irreducibility of \mathcal{A} , follows from
 525 the reasoning already presented in the proof of Proposition 3.8. \square

526 Despite this limiting result, UGAS of the linearized system (2.4) on $\mathcal{S}_{\text{dw}}(\tau)$ does imply
 527 some remarkable asymptotic properties of the switched affine system (2.1) on $\mathcal{S}_{\text{dw}}(\tau)$,
 528 and thus we need to recall the following (weak) stability/boundedness notion.

529 **Definition 4.2.** Given any class of switching signals $\tilde{\mathcal{S}} \subset \mathcal{S}$, the switched affine
 530 system (2.1) is said to be *uniformly globally ultimately bounded (UGUB) on $\tilde{\mathcal{S}}$* if there
 531 exists a compact set $\mathcal{V} \subset \mathbb{R}^n$ such that

$$532 \quad \forall x \in \mathbb{R}^n, \forall \sigma \in \tilde{\mathcal{S}}, \exists T(\sigma, x) \geq 0 \text{ such that } \forall t \geq T(\sigma, x), \Psi_\sigma(t, x) \in \mathcal{V}.$$

533 In this case the compact set $\mathcal{V} \subset \mathbb{R}^n$ is said to be a *uniform bounding region*. \triangle

534 Note that the set \mathcal{V} in Definition 4.2, in general, is *not* forward invariant and thus,
 535 not an attractor. In the following we prove that, if for some $\tau > 0$ the linearized
 536 system (2.4) is UGAS on $\mathcal{S}_{\text{dw}}(\tau)$ then the switched affine system (2.1) is UGUB.
 537

538 **THEOREM 4.3.** *For any given $\tau \in \mathbb{R}_{\geq 0}$ and $\mathcal{F} = \{(A_i, b_i)_{i \in \langle M \rangle} \mid A_i \in \mathbb{R}^{n \times n}, b_i \in$
 539 $\mathbb{R}^n\}$, suppose that the linearized system (2.4) is UGAS on $\mathcal{S}_{\text{dw}}(\tau)$. Then the switched
 540 affine system (2.1) is uniformly globally ultimately bounded on $\mathcal{S}_{\text{dw}}(\tau)$.*

541 For any i -th affine subsystem $\dot{x} = A_i x + b_i$, $i \in \langle M \rangle$, we denote by $\Psi_i(\cdot, x) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$
 542 the solution starting at $x(0) = x$ and by $x_{ei} := -A_i^{-1} b_i$ the corresponding equilibrium.
 543 The proof of Theorem 4.3 requires the following preliminary result.

544 **LEMMA 4.4.** *Consider the switched affine system given in (2.1). For some $\tau >$
 545 0 , suppose that the linearized system (2.4) is UGAS on $\mathcal{S}_{\text{dw}}(\tau)$. Then, there exist
 546 translated norms⁴ $\tilde{v}_i : \mathbb{R}^n \rightarrow \mathbb{R}$, a scalar $\tilde{\kappa} > 0$ and a collection of compact sets
 547 $\mathcal{X}_i \subset \mathbb{R}^n$, $i \in \langle M \rangle$, such that*

$$548 \quad (4.1a) \quad x_{ei} \in \text{Int}(\mathcal{X}_i), \quad \forall i \in \langle M \rangle,$$

549

$$550 \quad (4.1b) \quad \tilde{v}_i(\Psi_i(t, x)) \leq \tilde{v}_i(x), \quad \forall x \in \mathbb{R}^n \setminus \text{Int}(\mathcal{X}_i), \quad \forall t \in \mathbb{R}_{\geq 0}, \quad \forall i \in \langle M \rangle,$$

551

$$552 \quad (4.1c) \quad \tilde{v}_i(\Psi_i(\tau, x)) \leq e^{-\tilde{\kappa}\tau} \tilde{v}_j(x), \quad \forall x \in \mathbb{R}^n \setminus \text{Int}(\mathcal{X}_j), \quad \forall (i, j) \in \langle M \rangle^2.$$

553 *Proof.* Since the linearized system (2.4) is UGAS on $\mathcal{S}_{\text{dw}}(\tau)$, we consider the
 554 scalar $\kappa > 0$ and norms $v_1, \dots, v_M : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ verifying the conditions (2.8) and
 555 (2.9) defined in Lemma 2.3. Define $\tilde{v}_i(x) := v_i(x - c_i)$ with arbitrary $c_i \in \mathbb{R}^n$ and take
 556 any positive scalar $\tilde{\kappa} < \kappa$. First, to demonstrate that (2.8) implies (4.1b), consider
 557 any $i \in \langle M \rangle$, any $x \in \mathbb{R}^n$, and any $t \in \mathbb{R}_{\geq 0}$. Thus, computing

$$\begin{aligned} 558 \quad \tilde{v}_i(\Psi_i(t, x)) &= v_i \left(e^{A_i t} x + \int_0^t e^{A_i(t-s)} b_i ds - c_i + \Psi_i(t, c_i) - \Psi_i(t, c_i) \right) \\ 559 &= v_i \left(e^{A_i t} x + \int_0^t e^{A_i(t-s)} b_i ds - e^{A_i t} c_i - \int_0^t e^{A_i(t-s)} b_i ds + \Psi_i(t, c_i) - c_i \right) \\ 560 &\leq v_i \left(e^{A_i t} (x - c_i) \right) + v_i(\Psi_i(t, c_i) - c_i) \\ 561 &\leq e^{-\kappa t} v_i(x - c_i) + v_i(\Psi_i(t, c_i) - c_i) \\ 562 \quad (4.2) &= e^{-\tilde{\kappa} t} \tilde{v}_i(x) + v_i(\Psi_i(t, c_i) - c_i). \end{aligned}$$

⁴A function $w : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a *translated norm* if there exist a norm $v : \mathbb{R}^n \rightarrow \mathbb{R}$ and a vector $c \in \mathbb{R}^n$ (called the *center* of w) such that $w(x) = v(x - c)$, for all $x \in \mathbb{R}^n$.

564 Also, notice that

$$\begin{aligned}
565 \quad \Psi_i(t, c_i) - c_i &= e^{A_i t} c_i + \int_0^t e^{A_i(t-s)} b_i ds - c_i \\
566 &= (e^{A_i t} - I) c_i + \int_0^t e^{A_i(t-s)} b_i ds \\
567 &= \int_0^t e^{A_i(t-s)} A_i c_i ds + \int_0^t e^{A_i(t-s)} b_i ds \\
568 \quad (4.3) &= \int_0^t e^{A_i(t-s)} (A_i c_i + b_i) ds, \\
569 &
\end{aligned}$$

570 which, combined with (4.2) yields

$$\begin{aligned}
571 \quad \tilde{v}_i(\Psi_i(t, x)) &\leq e^{-\kappa t} \tilde{v}_i(x) + v_i \left(\int_0^t e^{A_i(t-s)} (A_i c_i + b_i) ds \right) \\
572 &\leq e^{-\kappa t} \tilde{v}_i(x) + \int_0^t v_i \left(e^{A_i(t-s)} (A_i c_i + b_i) \right) ds \\
573 &\leq e^{-\kappa t} \tilde{v}_i(x) + \int_0^t e^{-\kappa(t-s)} v_i (A_i c_i + b_i) ds \\
574 \quad (4.4) &\leq e^{-\kappa t} \tilde{v}_i(x) + \frac{1 - e^{-\kappa t}}{\kappa} v_i (A_i c_i + b_i), \\
575 &
\end{aligned}$$

576 which ensures (4.1b) for all $x \in \mathbb{R}^n$ such that $\tilde{v}_i(x) \geq \kappa^{-1} v_i (A_i c_i + b_i)$. Moreover we
577 note that (4.4) holds globally if one chooses $c_i = x_{e_i}$ but, for the sake of generality
578 (and for numerical reasons illustrated in what follows), this demonstration is carried
579 out for arbitrary centers c_i , $i \in \langle M \rangle$.

580 To prove (4.1c), consider any $x \in \mathbb{R}^n$ and any $(i, j) \in \langle M \rangle^2$. Reasoning similarly
581 to (4.2), we have

$$\begin{aligned}
582 \quad \tilde{v}_i(\Psi_i(\tau, x)) &\leq v_i (e^{A_i \tau} (x - c_i)) + v_i (\Psi_i(\tau, c_i) - c_i) \\
583 &\leq e^{-\kappa \tau} v_j (x - c_i) + \frac{1 - e^{-\kappa \tau}}{\kappa} v_i (A_i c_i + b_i) \\
584 &\leq e^{-\kappa \tau} (v_j (x - c_j) + v_j (c_j - c_i)) + \frac{1 - e^{-\kappa \tau}}{\kappa} v_i (A_i c_i + b_i) \\
585 \quad (4.5) &= e^{-\kappa \tau} (\tilde{v}_j(x) + \tilde{v}_j(c_i)) + \frac{1 - e^{-\kappa \tau}}{\kappa} v_i (A_i c_i + b_i). \\
586 &
\end{aligned}$$

587 Notice that, for any $0 < \tilde{\kappa} < \kappa$, we have that

$$588 \quad \tilde{v}_i(\Psi_i(\tau, x)) \leq e^{-\tilde{\kappa} \tau} \tilde{v}_j(x),$$

589 if $x \in \mathbb{R}^n$ is such that

$$590 \quad (4.6) \quad \tilde{v}_j(x) \geq \frac{e^{-\kappa \tau} \tilde{v}_j(c_i) + \kappa^{-1} (1 - e^{-\kappa \tau}) v_i (A_i c_i + b_i)}{e^{-\tilde{\kappa} \tau} - e^{-\kappa \tau}} =: R_{ij}.$$

591 Hence, conditions in (4.1c) are verified considering

$$592 \quad (4.7) \quad \mathcal{X}_i := \{x \in \mathbb{R}^n \mid \tilde{v}_i(x) \leq R_{\mathcal{X}}\},$$

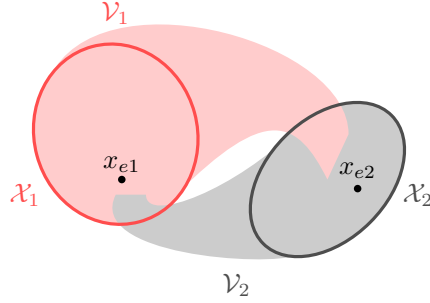


FIGURE 3. For the case of 2-mode switched affine system, a qualitative representation of the sets $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ (solid lines) and $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ (filled area).

593 with $R_{\mathcal{X}} := \max_{(i,j) \in \langle M \rangle^2} R_{ij}$. Notice that (4.1b) also holds for the same \mathcal{X}_i because

$$\begin{aligned}
 594 \quad R_{ij} &= \frac{e^{-\kappa\tau} \tilde{v}_j(c_i) + \kappa^{-1}(1 - e^{-\kappa\tau})v_i(A_i c_i + b_i)}{e^{-\tilde{\kappa}\tau} - e^{-\kappa\tau}} \\
 595 \quad &\geq \frac{\kappa^{-1}(1 - e^{-\kappa\tau})v_i(A_i c_i + b_i)}{e^{-\tilde{\kappa}\tau} - e^{-\kappa\tau}} \\
 596 \quad &> \frac{\kappa^{-1}(1 - e^{-\kappa\tau})v_i(A_i c_i + b_i)}{1 - e^{-\kappa\tau}} \\
 597 \quad (4.8) \quad &= \kappa^{-1}v_i(A_i c_i + b_i).
 \end{aligned}$$

599 Finally, considering $x = x_{ei}$ in (4.4) and any $t > 0$, we note that $\tilde{v}_i(x_{ei}) \leq \kappa^{-1}v_i(A_i c_i +$
 600 $b_i)$. This, by (4.8) implies that $\tilde{v}_i(x_{ei}) < R_{ij}$ for every $(i, j) \in \langle M \rangle^2$, and thus
 601 condition (4.1a) holds, concluding the proof. \square

602 We can now prove Theorem 4.3.

603 *Proof of Theorem 4.3.* Consider translated norms $\tilde{v}_1, \dots, \tilde{v}_M : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, a
 604 scalar $\tilde{\kappa} > 0$ and sets \mathcal{X}_i defined in (4.7) satisfying the conditions of Lemma 4.4.
 605 Define the compact set $\mathcal{X} := \bigcup_{i \in \langle M \rangle} \mathcal{X}_i$ and take

$$\begin{aligned}
 606 \quad (4.9) \quad \mathcal{V} &:= \bigcup_{i \in \langle M \rangle} \mathcal{V}_i, \text{ with} \\
 \mathcal{V}_i &:= \bigcup_{j \in \langle M \rangle} \bigcup_{t \in [0, \tau]} \Psi_j(t, \mathcal{X}_i)
 \end{aligned}$$

607 where, for every $j \in \langle M \rangle$ and $t \in \mathbb{R}_{\geq 0}$, the set-valued map $\Psi_j(t, \cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$
 608 is defined by $\Psi_j(t, S) := \bigcup_{x \in S} \{\Psi_j(t, x)\}$ for any set $S \subseteq \mathbb{R}^n$. An illustration of
 609 the sets \mathcal{V} and \mathcal{X} is shown in Figure 3. Then, we establish uniform global ultimate
 610 boundedness proving the following two claims:

611 Claim 1. For any $x \notin \mathcal{X}$ and any $\sigma \in \mathcal{S}_{\text{dw}}(\tau)$ there exists a finite switching instant
 612 $t_k^\sigma > 0$ such that $\Psi_\sigma(t_k^\sigma, x) \in \mathcal{X}$.

613 Claim 2. For any $x \in \mathcal{X}$ and any $\sigma \in \mathcal{S}_{\text{dw}}(\tau)$, $\Psi_\sigma(t, x) \in \mathcal{V}$, for all $t \in \mathbb{R}_{\geq 0}$.

614 Claims 1 and 2 ensure the UGUB property in Definition 4.2, with the compact set \mathcal{V}
 615 defined in (4.9). Indeed, we note that we can suppose, without loss of generality, that
 616 σ has infinitely many discontinuities: if σ is eventually constant then, considering any
 617 $x \in \mathbb{R}^n$, it holds that $\Psi_\sigma(t, x) \rightarrow x_{ei}$ as $t \rightarrow \infty$ for some $i \in \langle M \rangle$, and $x_{ei} \in \text{Int}(\mathcal{X}_i) \subseteq$
 618 $\mathcal{X} \subseteq \mathcal{V}$ by Lemma 4.4, and thus there exists a $T(\sigma, x) \geq 0$ as in Definition 4.2.

619 To prove Claim 1, consider any $x \notin \mathcal{X}$ and any $\sigma \in \mathcal{S}_{\text{dw}}(\tau)$. Define the instant
 620 $\hat{t}^\sigma := \min\{t_k^\sigma \mid \Psi_\sigma(t_k^\sigma, x) \in \mathcal{X}\}$, i.e., the first switching instant for which $\Psi_\sigma(t, x)$ is
 621 inside \mathcal{X} , and let us show that \hat{t}^σ is always finite. Consider any switching instant
 622 $t_k^\sigma < \hat{t}^\sigma$, and suppose, without loss of generality that $\sigma(t_k^\sigma) = i$. Let us denote, by
 623 simplicity $x(k) := \Psi_\sigma(t_k^\sigma, x)$ and $T_k := t_{k+1}^\sigma - t_k^\sigma \geq \tau$. From the definition of \hat{t}^σ we
 624 have $x(k) \notin \mathcal{X} \supseteq \mathcal{X}_j$ for any $j \in \langle M \rangle$. Therefore, using (4.1b) and (4.1c) we obtain

$$\begin{aligned} \tilde{v}_i(x(k+1)) &= \tilde{v}_i(\Psi_i(T_k, x(k))) \\ (4.10) \quad &\leq \tilde{v}_i(\Psi_i(\tau, x(k))) \\ &\leq e^{-\tilde{\kappa}\tau} \tilde{v}_j(x(k)), \end{aligned}$$

626 for any $j \in \langle M \rangle$. Iterating this property backward yields

$$627 \quad \tilde{v}_i(x(k+1)) \leq e^{-\tilde{\kappa}k\tau} \tilde{v}_j(x),$$

628 for any $j \in \langle M \rangle$. Since this holds for an arbitrary $t_k^\sigma < \hat{t}^\sigma$ and $e^{-\tilde{\kappa}k\tau} \rightarrow 0$ as $k \rightarrow \infty$,
 629 we can conclude that \hat{t}^σ is finite as, otherwise, a contradiction would take place.

630 To prove Claim 2, consider $x \in \mathcal{X}$, which implies that there exists $j \in \langle M \rangle$ such
 631 that $\tilde{v}_j(x) \leq R_{\mathcal{X}}$, from the definition in (4.7). Notice that, given the definition of \mathcal{V} ,
 632 we have $\mathcal{X}_i \subseteq \mathcal{V}_i$ for all $i \in \langle M \rangle$. Suppose $\sigma \in \mathcal{S}_{\text{dw}}(\tau)$ is such that $\sigma(0) = i \in \langle M \rangle$
 633 and take any $t \in [0, \tau]$, which ensures $\sigma(t) = i$ given the dwell-time constraint. From
 634 the definition of \mathcal{V}_i , it is straightforward to verify that $\Psi_i(t, \mathcal{X}_j) \subseteq \mathcal{V}_j$ for all $t \in [0, \tau]$.
 635 Then, we can analyse the worst case with respect to $x \in \mathcal{X}_j$ as

$$\begin{aligned} \tilde{v}_i(\Psi_i(\tau, x)) &\leq \max_{x \in \mathcal{X}_j} \tilde{v}_i(\Psi_i(\tau, x)) \\ 636 \quad &= \max_{x \in \partial \mathcal{X}_j} \tilde{v}_i(\Psi_i(\tau, x)) \\ 637 \quad &\leq \max_{x \in \partial \mathcal{X}_j} e^{-\tilde{\kappa}\tau} \tilde{v}_j(x) \\ 638 \quad &\leq R_{\mathcal{X}}, \end{aligned} \tag{4.11}$$

641 where the equality holds because the maximum of the convex function $\tilde{v}_i(\Psi_i(\tau, \cdot)) : \mathbb{R}^n \rightarrow \mathbb{R}$
 642 inside the convex set \mathcal{X}_j occurs on its boundary $\partial \mathcal{X}_j$ (see [26, Corollary 32.3.2]) and the subsequent inequality follows from (4.1c). This implies that
 643 $\Psi_i(\tau, \mathcal{X}_j) \subseteq \mathcal{X}_i$ for all $(i, j) \in \langle M \rangle^2$ and thus, by (4.1b), $\Psi_i(t, x) \in \mathcal{X}_i$ for all $t \in [\tau, t_1^\sigma]$.
 644 We can iterate this argument for every $k \in \mathbb{N}$ such that $\Psi_\sigma(t_k^\sigma, x) \in \mathcal{X}$, and the proof
 645 is concluded. \square

647 As proven in Lemma 4.1, in general, for any $\tau > 0$, non-trivial forward invariant
 648 sets for (2.1) on the class $\mathcal{S}_{\text{dw}}(\tau)$ do not exist, even if the linearized system (2.4) is
 649 UGAS on $\mathcal{S}_{\text{dw}}(\tau)$. For that reason, in Theorem 4.3 we focused on the weaker *uniform*
 650 *global ultimate boundedness* property, constructing a bounding region \mathcal{V} , starting from
 651 a family of “safety” sets $\mathcal{X}_1, \dots, \mathcal{X}_M$. On the other hand, we can focus on a more re-
 652 laxated notion of forward invariance, which is only concerned by the solutions *evaluated*
 653 *along the discrete sequence of switching points*, as formally defined below.

654 *Definition 4.5.* Given $\tilde{\mathcal{S}} \subset \mathcal{S}$, we say that a compact set $C \subset \mathbb{R}^n$ is *forward*
 655 *invariant* for (2.1) on $\tilde{\mathcal{S}}$ with respect to the switching points if, for all $x \in C$, all $\sigma \in \tilde{\mathcal{S}}$,
 656 we have that

$$657 \quad \Psi_\sigma(t_k^\sigma, x) \in C, \quad \forall x \in C, \quad \forall \sigma \in \tilde{\mathcal{S}}, \quad \forall t_k^\sigma \geq 0,$$

658 where, we recall, $\{t_k^\sigma\}$ denotes the (finite or countable) set of discontinuities of the
 659 signal $\sigma \in \tilde{\mathcal{S}}$.

660 PROPOSITION 4.6 (\mathcal{X} is forward invariant w.r.t. switching instants). *Under the*
 661 *hypothesis of Theorem 4.3, the set $\mathcal{X} = \bigcup_{i \in \langle M \rangle} \mathcal{X}_i$ (where the sublevel sets \mathcal{X}_i are*
 662 *defined in (4.7)) is forward invariant for (2.1) on $\mathcal{S}_{\text{dw}}(\tau)$ w.r.t. the switching points.*

663 *Proof.* First of all, we note that, by (4.1b), for any $i \in \langle M \rangle$, the set \mathcal{X}_i is forward
 664 invariant for the subsystem $\dot{x} = A_i x + b_i$. Let us now consider any $x \in \mathcal{X}$ and
 665 any $\sigma \in \mathcal{S}_{\text{dw}}(\tau)$. By definition of \mathcal{X} there exists a $j \in \langle M \rangle$ such that $x \in \mathcal{X}_j =$
 666 $\{x \in \mathbb{R}^n \mid \tilde{v}_j(x) \leq R_{\mathcal{X}}\}$. Suppose $\sigma(0) = i_0 \in \langle M \rangle$ and consider $t_1^\sigma \geq \tau$ as the
 667 first switching instant of σ . By forward invariance of \mathcal{X}_{i_0} with respect to the i_0 -
 668 subdynamics, if $\Psi_\sigma(\tau, x) \in \mathcal{X}_{i_0}$ then $\Psi_\sigma(t_1^\sigma, x) \in \mathcal{X}_{i_0}$. By (4.11) we have

$$669 \quad \tilde{v}_{i_0}(\Psi_\sigma(\tau, x)) = \tilde{v}_{i_0}(\Psi_{i_0}(\tau, x)) \leq \tilde{v}_j(x) \leq R_{\mathcal{X}}$$

670 proving that $\Psi_{i_0}(t_1^\sigma, x) \in \mathcal{X}_{i_0} \subset \mathcal{X}$. Summarizing, we have proven that, for any $x \in \mathcal{X}$
 671 and any $\sigma \in \mathcal{S}_{\text{dw}}(\tau)$ we have $\Psi_\sigma(t_1^\sigma, x) \in \mathcal{X}$; iterating the argument we conclude that
 672 \mathcal{X} is forward invariant with respect to the switching instants. \square

673 We also note that if the set $\{t_k^\sigma\}$ is finite (corresponding to an eventually constant
 674 switching signal σ), we have that $\lim_{t \rightarrow +\infty} \Psi_\sigma(t, x) = x_{ei} \in \text{Int}(\mathcal{X}_i) \subseteq \text{Int}(\mathcal{X})$, for
 675 some $i \in \langle M \rangle$. This proves that, for any eventually constant signal $\sigma \in \mathcal{S}_{\text{dw}}(\tau)$, there
 676 exists a finite $\bar{T}(\sigma, x) \geq 0$ such that $\Psi_\sigma(t, x) \in \mathcal{X}$ for all $t \in [\bar{T}(\sigma, x), +\infty)$. On the
 677 other hand, the solutions of (2.1) can possibly escape \mathcal{X} between switching times, and
 678 for that reason we introduced the “security” set \mathcal{V} , which characterizes the UGUB
 679 property. For a graphical illustration, see Figure 3.

680 Concluding this subsection, we discuss the relations between the dwell-time pa-
 681 rameter $\tau > 0$, the corresponding decay rate and the sets \mathcal{X}_i and \mathcal{V} .

682 *Remark 4.7* (Dependence of \mathcal{X}_i and \mathcal{V} with respect to τ). Consider the system
 683 $\mathcal{F} = \{(A_i, b_i)_{i \in \langle M \rangle} \mid A_i \in \mathbb{R}^{n \times n}, b_i \in \mathbb{R}^n\}$, and denote $\mathcal{A} = \{A_1, \dots, A_M\}$ the set of
 684 the matrices describing the linear part of the subdynamics. We denote the *minimum*
 685 *dwell-time* by $\tau_{\min}(\mathcal{A})$, formally defined as

$$686 \quad \tau_{\min}(\mathcal{A}) := \inf \{ \tau \geq 0 \mid (2.4) \text{ is GUAS on } \mathcal{S}_{\text{dw}}(\tau) \}.$$

687 By Lemma 2.3, an equivalent definition is

$$688 \quad \tau_{\min}(\mathcal{A}) = \inf_{\tau \geq 0} \tau, \quad \text{s.t.} \\ \exists \kappa > 0 \text{ and norms } v_1, \dots, v_M : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \text{ such that (2.8) and (2.9) hold.}$$

689 We recall, once again, that if A_1, \dots, A_M are Hurwitz, then $\tau_{\min}(\mathcal{A})$ is finite, see [10].
 690 Now, given $\tau > \tau_{\min}(\mathcal{A})$ we define the *decay rate* with respect to $\tau > 0$, i.e.,

$$691 \quad \kappa_{\mathcal{A}}(\tau) := \sup \{ \kappa \geq 0 \mid \exists \text{ norms } v_1, \dots, v_M : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \text{ such that (2.8) and (2.9) hold} \}.$$

692 The computation/approximation of the minimum dwell-time is a well-studied prob-
 693 lem in the literature, see for example [5, 31, 32, 33, 34, 35] and references therein.
 694 As showed in [4] the function $\kappa_{\mathcal{A}} : (\tau_{\min}(\mathcal{A}), \infty) \rightarrow \mathbb{R}_{\geq 0}$ is continuous and non-
 695 decreasing, and it can be seen that $\lim_{\tau \searrow \tau_{\min}(\mathcal{A})} \kappa_{\mathcal{A}}(\tau) = 0$, and thus we extent $\kappa_{\mathcal{A}}$
 696 by continuity defining $\kappa_{\mathcal{A}}(\tau_{\min}(\mathcal{A})) = 0$. Moreover, the function $\kappa_{\mathcal{A}}$ is bounded
 697 from above, by the number $\bar{\kappa} := \min_{i \in \langle M \rangle} \kappa(A_i)$, where, given a matrix $A \in \mathbb{R}^{n \times n}$,
 698 $\kappa(A) := \max_{i \in \{1, \dots, n\}} \{\text{Re}(\lambda_i(A))\}$. As discussed in [4], if the set \mathcal{A} is irreducible, then
 699 the sup in (4.12) can be replaced by a max, i.e., for any $\tau \geq \tau_{\min}(\mathcal{A})$ we can find
 700 norms satisfying (2.8) and (2.9) for $\kappa_{\mathcal{A}}(\tau)$. In what follows we will assume that \mathcal{A} is

701 irreducible, without loss of generality. Moreover, we assume that $\tau_{\min}(\mathcal{A}) > 0$ since
 702 the case $\tau_{\min}(\mathcal{A}) = 0$ correspond to the case of stability under arbitrary switching,
 703 already studied in Section 3.

704 We want now to use these notions to analyze how the sets \mathcal{X}_i and \mathcal{V} are affected
 705 while varying the dwell-time parameter τ . We organize our analysis in two cases,
 706 described in what follows:

- 707 • The case of *slow switching signals*, i.e., when the dwell-time τ becomes arbitrarily
 708 large.
- 709 • The case of *fast switching signals*, i.e., the case $\tau \searrow \tau_{\min}(\mathcal{A})$ in which the dwell-time
 710 threshold approaches the minimum dwell-time $\tau_{\min}(\mathcal{A})$.

711 *Slow Switching:* Consider any $\underline{\tau} > \tau_{\min}(\mathcal{A})$ and consider any $\kappa > 0$ and $v_1, \dots, v_M : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ satisfying the condition in Lemma 2.3. It can be verified that the same κ
 712 and norms v_1, \dots, v_M satisfy the conditions in Lemma 2.3 for any $\tau \geq \underline{\tau}$. For such
 713 $v_1, \dots, v_M : \mathbb{R}^n \rightarrow \mathbb{R}$, we apply the construction in the proof of Lemma 4.4 and we
 714 choose $c_i = x_{ei}$ for any $i \in \langle M \rangle$. Given any $0 < \tilde{\kappa} < \kappa$, from (4.6) it can be seen that,
 715 for all $(i, j) \in \langle M \rangle^2$
 716

$$717 \quad \lim_{\tau \rightarrow +\infty} R_{ij}(\tau) = \lim_{\tau \rightarrow +\infty} \frac{e^{-\kappa\tau} \tilde{v}_j(x_{ei})}{e^{-\tilde{\kappa}\tau} - e^{-\kappa\tau}} = \lim_{\tau \rightarrow +\infty} \frac{\tilde{v}_j(x_{ei})}{e^{(\kappa-\tilde{\kappa})\tau} - 1} = 0.$$

718 This implies that $R_{\mathcal{X}}(\tau) = \max_{(i,j) \in \langle M \rangle^2} R_{ij}(\tau) \rightarrow 0$ as $\tau \rightarrow +\infty$, and thus, consid-
 719 ering the compact sets $\mathcal{X}_i(\tau)$ defined in (4.7) (making explicit the dependence with
 720 respect to τ) we have

$$721 \quad \mathcal{X}_i(\tau) \rightarrow \{x_{ei}\}, \text{ as } \tau \rightarrow +\infty, \quad \forall i \in \langle M \rangle,$$

722 where the convergence is in the Hausdorff sense. Similarly, the set $\mathcal{V}(\tau)$ defined in (4.9)
 723 converges to the union of a finite number of trajectories, i.e., to the set

$$724 \quad \bigcup_{(i,j) \in \langle M \rangle^2} \bigcup_{t \in \mathbb{R}_{\geq 0}} \{\Psi_j(t, x_{ei})\}.$$

725 *Fast Switching:* We study the case $\tau \searrow \tau_{\min}(\mathcal{A})$, showing that the sets \mathcal{X}_i and \mathcal{V}
 726 explode, i.e., are eventually unbounded. This is somewhat expected since we are
 727 approaching the instability limit $\tau_{\min}(\mathcal{A})$ of the linearized system (2.4). To this aim,
 728 consider any decreasing sequence $\tau_k \searrow \tau_{\min}(\mathcal{A})$ and, for any $k \in \mathbb{N}$, consider $\kappa_k :=$
 729 $\kappa_{\mathcal{A}}(\tau_k)$ and $v_1^k, \dots, v_M^k : \mathbb{R}^n \rightarrow \mathbb{R}$ any norms satisfying (2.8) and (2.9) for τ_k and κ_k .
 730 As discussed, it holds that $\kappa_k \searrow 0$ as $k \rightarrow \infty$. In following the construction of proof
 731 of Lemma 4.4 we set $\tilde{\kappa}_k := \kappa_k/2$ and this is without loss of generality, since $\kappa_k \rightarrow 0$.
 732 Similarly, we set $c_i^k = x_{ei}$, for any $k \in \mathbb{N}$; the general case follows the same argument.
 733 Computing, recalling (4.6), for any $(i, j) \in \langle M \rangle^2$ we have

$$734 \quad \lim_{\tau \searrow \tau_{\min}(\mathcal{A})} R_{ij}(\tau) = \lim_{k \rightarrow +\infty} \frac{e^{-\kappa_k \tau_k} \tilde{v}_j^k(x_{ei})}{e^{-\frac{\kappa_k}{2} \tau_k} - e^{-\kappa_k \tau_k}} = \lim_{k \rightarrow +\infty} \frac{\tilde{v}_j^k(x_{ei})}{e^{\frac{\kappa_k}{2} \tau_k} - 1}.$$

735 Since, by hypothesis $\kappa_k \searrow 0$ and $\tau_k \searrow \tau_{\min}(\mathcal{A}) > 0$ as $k \rightarrow \infty$, we have that
 736 $\lim_{\tau \searrow \tau_{\min}(\mathcal{A})} R_{ij}(\tau) = +\infty$. This means that, while approaching the stability margin
 737 given by $\tau_{\min}(\mathcal{A})$, our outer approximations of the safety sets \mathcal{X}_i , $i \in \langle M \rangle$ are growing
 738 unboundedly, providing no information on the asymptotic behavior of system (2.1).
 739 Of course, the same exploding behavior is inherited by the bounding region \mathcal{V} . The
 740 numerical examples in the next section illustrate this phenomenon. \triangle

741 **4.2. Numerical Construction of Bounding Regions.** To devise a numeri-
 742 cal procedure allowing the study of the switched affine system (2.1) under dwell-time
 743 switching we present an LMI-based method relying on sufficient conditions for ob-
 744 taining functions \tilde{v}_i , $i \in \langle M \rangle$ satisfying the conditions (4.1b)-(4.1c) in Lemma (4.4).
 745 For this, we consider norms induced by quadratic forms as

$$746 \quad (4.13) \quad v_i(x) = \sqrt{x^\top P_i x}$$

747 where P_i , $i \in \langle M \rangle$, are positive definite matrices. From these norms, we define the
 748 functions $\tilde{v}_i(x) = v_i(x - c_i)$ for given centers $c_i \in \mathbb{R}^n$, $i \in \langle M \rangle$. The following corollary
 749 is the core of the numerical procedure to construct the bounding region \mathcal{V} .

750 **COROLLARY 4.8.** *For given $\tau \in \mathbb{R}_{\geq 0}$ and $\mathcal{F} = \{(A_i, b_i)_{i \in \langle M \rangle} \mid A_i \in \mathbb{R}^{n \times n}, b_i \in$
 751 $\mathbb{R}^n\}$, the switched affine system (2.1) is uniformly globally ultimately bounded on
 752 $\mathcal{S}_{\text{dw}}(\tau)$ if there exist positive definite matrices $P_i, W_{ij} \in \mathbb{R}^{n \times n}$ and vectors $c_i, d_{ij} \in \mathbb{R}^n$
 753 satisfying the inequalities*

$$754 \quad (4.14a) \quad \mathcal{A}_i^\top P_i + P_i \mathcal{A}_i \prec -\mathcal{E}_{ii} \quad \forall i \in \langle M \rangle$$

$$755 \quad (4.14b) \quad e^{\mathcal{A}_i^\top \tau} P_i e^{\mathcal{A}_i \tau} - P_j \prec -\mathcal{E}_{ij} \quad \forall (i, j) \in \langle M \rangle^2, i \neq j$$

757 with

$$758 \quad (4.15) \quad P_i = \begin{bmatrix} P_i & -P_i c_i \\ -c_i^\top P_i & c_i^\top P_i c_i \end{bmatrix}, \quad \mathcal{A}_i = \begin{bmatrix} A_i & b_i \\ 0 & 0 \end{bmatrix}, \quad \mathcal{E}_{ij} = \begin{bmatrix} W_{ij} & -W_{ij} d_{ij} \\ -d_{ij}^\top W_{ij} & d_{ij}^\top W_{ij} d_{ij} - 1 \end{bmatrix}.$$

759 *Proof.* Let us show that (4.14a)-(4.14b) imply (4.1b)-(4.1c), respectively. First,
 760 notice that, using the augmented state vector $\xi = [x^\top \ 1]^\top$ one can rewrite the sys-
 761 tem (2.1) as $\dot{\xi} = \mathcal{A}_\sigma \xi$ and the functions $\tilde{v}_i(x) = \sqrt{\xi^\top P_i \xi}$. Consequentially, the
 762 solution to this system is simply given by $\Psi_i(t, x) = e^{\mathcal{A}_i t} \xi$. Also, notice that the set
 763 $\mathbb{E}_{ij} := \{x \in \mathbb{R}^n \mid \xi^\top \mathcal{E}_{ij} \xi \leq 0\}$ defines a generic ellipsoid in \mathbb{R}^n , see [36, Section 3.7].
 764 For an arbitrary $x \in \mathbb{R}^n$, multiplying (4.14a) to the left by ξ^\top and to the right by ξ
 765 implies that

$$766 \quad \frac{d}{dt} \tilde{v}_i(\Psi_i(t, x))^2 < -\xi^\top \mathcal{E}_{ii} \xi, \quad \forall t \in \mathbb{R}_{\geq 0},$$

767 which readily implies that (4.1b) holds for some level set \mathcal{X}_i of $\tilde{v}_i(x)$ containing the
 768 ellipsoid \mathbb{E}_{ii} . Analogously, multiplying (4.14b) to the left by ξ^\top and to the right by ξ
 769 implies that

$$770 \quad \tilde{v}_i(\Psi_i(\tau, x))^2 - \tilde{v}_j(x)^2 < -\xi^\top \mathcal{E}_{ij} \xi,$$

771 which, in turn, implies that (4.1b) also holds for some level set \mathcal{X}_i of $\tilde{v}_i(x)$ containing
 772 \mathbb{E}_{ij} with some positive scalar $\tilde{\kappa}$, as strict inequalities are considered. Therefore \mathcal{X}_i can
 773 be defined as a level set of $\tilde{v}_i(x)$ containing \mathbb{E}_{ij} . The proof that $x_{ei} \in \text{Int}(\mathcal{X}_i)$ is done
 774 by multiplying (4.14a) to the left by $\xi_{ei}^\top = [x_{ei}^\top \ 1]$ and to the right by ξ_{ei} and noticing
 775 that $\mathcal{A}_i \xi_{ei} = 0$, which leads to $\xi_{ei}^\top \mathcal{E}_{ii} \xi_{ei} < 0$, concluding the proof. \square

776 *Remark 4.9* (Equivalence with the conditions of [5]). We highlight that the con-
 777 ditions (4.14a)-(4.14b) are equivalent, in terms of conservatism, to the classical dwell-
 778 time stability conditions presented in [5, Theorem 1] for the linearized system (2.4).
 779 In this result, the sufficient condition for stability is given by the existence of matrices
 780 $\tilde{P}_i \in \mathbb{R}^{n \times n}$, $\tilde{P}_i \succ 0$ such that

$$781 \quad (4.16a) \quad A_i^\top \tilde{P}_i + \tilde{P}_i A_i \prec 0 \quad \forall i \in \langle M \rangle$$

$$782 \quad (4.16b) \quad e^{\mathcal{A}_i^\top \tau} \tilde{P}_j e^{\mathcal{A}_i \tau} - \tilde{P}_i \prec 0 \quad \forall (i, j) \in \langle M \rangle^2, i \neq j.$$

784 Indeed, the fact that (4.14a)-(4.14b) imply (4.16a)-(4.16b) follows from the fact
 785 that the latter pair constitutes the matrix blocks (1,1) of the former, taking $\tilde{P}_i =$
 786 $e^{A_i^\top \tau} P_i e^{A_i \tau}$, $i \in \langle M \rangle$, and performing simple algebraic manipulations. Additionally,
 787 whenever (4.16a)-(4.16b) hold for given \tilde{P}_i , $i \in \langle M \rangle$, the inequalities generated by the
 788 matrix blocks (1,1) of (4.14a)-(4.14b) also hold for $P_i = \epsilon e^{-A_i^\top \tau} \tilde{P}_i e^{-A_i \tau}$, $i \in \langle M \rangle$,
 789 with $\epsilon > 0$. Also, for any c_i , $i \in \langle M \rangle$, these conditions can be fully satisfied for ϵ
 790 small enough, some $W_{ij} > 0$ close enough to 0, and d_{ij} such that $W_{ij} d_{ij}$ equal the
 791 off-diagonal terms of the left-hand sides. In the literature concerning dwell-time lin-
 792 ear switched systems, several stability conditions equivalent to (4.16a)-(4.16b) have
 793 been proposed, see for example [31, 34, 37, 38] and references therein. Our proposed
 794 LMI-conditions (4.14a)-(4.14b), while tackling a more challenging problem, have the
 795 same region of feasibility of (4.16a)-(4.16b) with respect to the dwell-time parameter
 796 $\tau > 0$ and the matrices A_i , $i \in \langle M \rangle$. On the other hand, conditions (4.16a)-(4.16b)
 797 are known to be conservative as they restrict the choice of norms in Lemma 2.3 to
 798 *quadratic* norms. To asymptotically reach tight conditions for stability analysis of
 799 dwell-time linear switched systems one could consider SOS or polyhedral Lyapunov
 800 functions (see [32] and [33], respectively) increasing the computational complexity.
 801 Of course these approaches can be adapted in our switched affine systems framework,
 802 but for simplicity we do not proceed in this direction. \triangle

803 In contrast with (4.16a)-(4.16b), the direct verification of (4.14a)-(4.14b) is dif-
 804 ficult to be performed numerically due to the product of many decision variables.
 805 However, selecting the center of the functions $\tilde{v}_i : \mathbb{R}^n \rightarrow \mathbb{R}$ at the equilibrium of the
 806 i -th subsystem (i.e., choosing $c_i = x_{ei}$) allows us to restate it as an LMI problem
 807 given as:

$$808 \quad (4.17a) \quad \min_{P_i, M_{ij}, d_{ij}} \sum_{(i,j) \in \langle M \rangle^2} \text{Tr } M_{ij}, \text{ s.t.}$$

$$809 \quad (4.17b) \quad \begin{bmatrix} -Q_{ij}(P_i, P_j) + D & T(d_{ij}) \\ T(d_{ij})^\top & M_{ij} \end{bmatrix} \succ 0, \quad \forall (i, j) \in \langle M \rangle^2$$

811 where $T(d_{ij})^\top = [I \ d_{ij}]$, $D = \text{diag}(0, \dots, 0, 1)$ and

$$812 \quad Q_{ij}(P_i, P_j) = \begin{cases} A_i^\top P_i + P_i A_i, & \text{if } i = j \\ e^{A_i^\top \tau} P_i e^{A_i \tau} - P_j, & \text{if } i \neq j \end{cases}.$$

813 By applying the Schur Complement Lemma (see [25, Section A.5.5]) with respect to
 814 M_{ij} in the LMI (4.17b), one can easily verify that these conditions are equivalent
 815 to (4.14a)-(4.14b) by taking $W_{ij} = M_{ij}^{-1}$. The objective function (4.17a) seeks to
 816 indirectly reduce the size of \mathcal{V} by minimizing the sum of the squared lengths of the
 817 semi-axes of the ellipsoids \mathcal{X}_i , $i \in \langle M \rangle$, which provides a tight estimation of the mini-
 818 mum bounding region. Indeed, exactly minimizing the volume of \mathcal{V} is overwhelmingly
 819 difficult since it is a non-convex set constituted by the union of uncountable many
 820 ellipsoids, recall the definition in (4.9).

821 Once a solution P_i , M_{ij} , d_{ij} , $(i, j) \in \langle M \rangle^2$ has been found, the regions \mathcal{X}_i , $i \in$
 822 $\langle M \rangle$ can be obtained by solving

$$823 \quad (4.18a) \quad \min_{\beta_{ij} \geq 0, \gamma \geq 0} \gamma, \text{ s.t.}$$

$$824 \quad (4.18b) \quad \beta_{ij} \mathcal{E}_{ij} - P_j + \gamma D \succ 0, \quad \forall (i, j) \in \langle M \rangle^2,$$

826 with \mathcal{E}_{ij} defined in (4.15) and $W_{ij} = M_{ij}^{-1}$. According to the discussions presented
 827 in [36, Section 3.7.1], taking $R_{\mathcal{X}} = \sqrt{\gamma}$ implies that $\mathcal{X}_i \supseteq \mathbb{E}_{ij} = \{x \in \mathbb{R}^n \mid \xi^\top \mathcal{E}_{ij} \xi \leq$
 828 $0\}$, $\forall (i, j) \in \langle M \rangle^2$. Finally, once the region \mathcal{X}_i is characterized, for an arbitrary
 829 point $x \in \mathbb{R}^n$ the test whether $x \in \mathcal{V}$ can be performed without major difficulties, as
 830 described in the next proposition.

831 **PROPOSITION 4.10.** *A point $x \in \mathbb{R}^n$ is contained in the bounding region \mathcal{V} defined*
 832 *in (4.9) with norms v_1, \dots, v_M given in (4.13), if and only if there exist $t \in [0, \tau]$ and*
 833 *a pair $(i, j) \in \langle M \rangle^2$ such that*

$$834 \quad (4.19) \quad (x - e^{A_j t}(c_i - x_{ej}) - x_{ej})^\top e^{-A_j^\top t} P_i e^{-A_j t} (x - e^{A_j t}(c_i - x_{ej}) - x_{ej}) \leq R_{\mathcal{X}}^2.$$

835 *Proof.* By definition, $x \in \mathcal{V}$ if for some pair $(i, j) \in \langle M \rangle$ there exists $x_0 \in \mathcal{X}_i$ and
 836 $t \in [0, \tau]$ such that $x = \Psi_j(t, x_0)$. Notice that, for all $x_0 \in \mathcal{X}_i$ we have

$$837 \quad R_{\mathcal{X}}^2 \geq (x_0 - c_i)^\top P_i (x_0 - c_i)$$

$$838 \quad (4.20) \quad = (x_0 - c_i)^\top e^{A_j^\top t} e^{-A_j^\top t} P_i e^{-A_j t} e^{A_j t} (x_0 - c_i)$$

840 and also that

$$841 \quad e^{A_j t}(x_0 - c_i) = e^{A_j t}(x_0 - x_{ej} - c_i + x_{ej})$$

$$842 \quad (4.21) \quad = \Psi_j(t, x_0) - x_{ej} - e^{A_j t}(c_i - x_{ej})$$

844 because one can rewrite $\Psi_j(t, x_0) = e^{A_j t}(x_0 - x_{ej}) + x_{ej}$. Replacing this last equality
 845 in (4.20) yields (4.19) for $x = \Psi_j(t, x_0)$, concluding the proof. \square

846 It is noteworthy to mention that (4.19) can be easily verified through a line search
 847 for $t \in [0, \tau]$ during which the inequality is evaluated for each pair $(i, j) \in \langle M \rangle^2$. The
 848 next example illustrates the overall procedure presented in this section.

849 *Example 4.11.* Let us adapt Example 1 in [5] by introducing affine terms and
 850 consider a switched affine system (2.1) defined by

$$851 \quad (4.22) \quad A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.5 \end{bmatrix}, b_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, b_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

852 Solving the optimization problem (4.17a)-(4.17b) for a minimum dwell-time of $\tau =$
 853 2.76 yields the matrices

$$854 \quad P_1 = \begin{bmatrix} 6.4537 & 0.1019 \\ 0.1019 & 0.8028 \end{bmatrix} \times 10^{-3}, P_2 = \begin{bmatrix} 0.4290 & 0.0320 \\ 0.0320 & 3.0485 \end{bmatrix} \times 10^{-3}$$

855 as solution. This is the least dwell-time value for which we have obtained a feasible
 856 solution and it matches the one found in [5] for the linearized system (2.4), as proved
 857 in Remark 4.9. To obtain the regions \mathcal{X}_i , $i \in \langle M \rangle$ and subsequently the bounding
 858 region \mathcal{V} , we solved the optimization problem (4.18a)-(4.18b) and obtained $R_{\mathcal{X}} =$
 859 $\sqrt{\gamma} = 22.3274$ which allows us to draw \mathcal{X}_1 , \mathcal{X}_2 , \mathcal{V}_1 and \mathcal{V}_2 as given in Figure 4. These
 860 same regions are also represented in Figure 4 for $\tau = 5$ and $\tau = 10$. The bounding
 861 region $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ ensures by Theorem 4.3 that, for any initial condition $x \in \mathbb{R}^n$, we
 862 have the solution $\Psi_\sigma(t, x) \in \mathcal{V}$ for all t after a finite amount of time and all $\sigma \in \mathcal{S}_{\text{dw}}(\tau)$.
 863 This figure also illustrates that the area of these regions decreases as τ increases, as
 864 discussed in Remark 4.7.

865 The above example showed how the methodology developed in this paper allows the
 866 estimation of bounding regions for switched affine systems under dwell-time switching.
 867 In the following section, we present a path-following algorithm based on [39] that is
 868 shown to provide more accurate bounding regions.

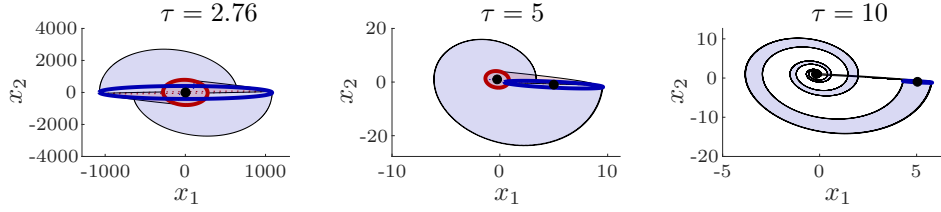


FIGURE 4. For three different values of dwell-time τ , the regions X_1 (red line), X_2 (black line), \mathcal{V}_1 (red area) and \mathcal{V}_2 (black area) are represented. The bounding region $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ is to where all solutions converge under any dwell-time switching signal $\sigma \in \mathcal{S}_{\text{dw}}(\tau)$. The black points represent the equilibria of each subsystem and the dotted regions represent $\Phi_2(\tau, \mathcal{X}_1)$ and $\Phi_1(\tau, \mathcal{X}_2)$ keeping the same color pattern.

869 **4.2.1. Path-following method to enhance the ellipsoid positioning.** In-
 870 spired by the methodology presented in [39], we derive a local optimization procedure
 871 for selecting the centers c_i of the functions $\tilde{v}_i(x) = \sqrt{(x - c_i)^\top P_i (x - c_i)}$, $i \in \langle M \rangle$.
 872 Particularly, the present problem is well adapted to be solved by this path-following
 873 method given that the choice of $c_i \in \mathbb{R}^n$, $i \in \langle M \rangle$ does not interfere in the feasibility
 874 of the optimization problem in (4.17a)-(4.17b). Also, though this method only guar-
 875 antees convergence to local optima, the fact that suitable points $c_i \in \mathbb{R}^n$, $i \in \langle M \rangle$,
 876 should lie close to x_{ei} (tending to it when $\tau \rightarrow \infty$) allows us to efficiently warm-
 877 start this method. As described in [39], the idea behind the path-following algorithm
 878 is to linearize the non-linear constraints using a first-order approximation around a
 879 given feasible solution and iteratively compute a direction in the decision space that
 880 slightly improves the objective function. Before presenting the algorithm adapted to
 881 our context, let us use (4.17a) and (4.17b) to define the following objective function

$$882 \quad (4.23) \quad f(M_{ij}) = \sum_{(i,j) \in \langle M \rangle^2} \text{Tr } M_{ij}$$

883 and the matrix-valued function

$$884 \quad (4.24) \quad h_{ij}(P_i, P_j, c_i, c_j, M_{ij}, d_{ij}) = \begin{bmatrix} -Q_{ij}(P_i, P_j) + D & T(d_{ij}) \\ T(d_{ij})^\top & M_{ij} \end{bmatrix}$$

885 where the dependence on c_i and c_j happens through the the matrices P_i and P_j in
 886 the definition of Q_{ij} .

887 Algorithm 4.1 performs the optimization with respect to the centers c_i , $i \in \langle M \rangle$.
 888 With some abuse of notation, we refer to the solution tuple $(P_i, c_i, M_{ij}, d_{ij})_{(i,j) \in \langle M \rangle^2}$
 889 simply as $(P_i, c_i, M_{ij}, d_{ij})$. In Lines 1 and 2 the algorithm initializes each c_i with
 890 the associated equilibrium x_{ei} and the variables P_i, M_{ij}, d_{ij} with the corresponding
 891 solution to (4.17a)-(4.17b). Then, in Line 4, we solve the first-order approximation
 892 of (4.17a)-(4.17b) taking into account c_i as variables. The first constraint charac-
 893 terizes the linear approximation of the nonlinear constraint (4.17b) around the point
 894 defined by the current variables $(P_i, c_i, M_{ij}, d_{ij})$ and the second constraint should
 895 be interpreted as an element-wise bound on the difference between (P_i, c_i) and the
 896 ‘‘perturbed’’ variables (\hat{P}_i, \hat{c}_i) . This constraint is not applied to \hat{M}_{ij} and \hat{d}_{ij} as the
 897 original problem is convex with respect to them. The control parameter $\delta = 10^{-1}$
 898 was chosen in our numerical experiments, which was sufficiently small such that the
 899 first-order approximation is valid and large enough such that convergence is met in
 900 a few iterations. Afterward, another feasible point $(P_i, c_i, M_{ij}, d_{ij})$ is determined by
 901 solving (4.17a)-(4.17b) for new centers c_i and this procedure is repeated until a con-

Algorithm 4.1 Path-following method for optimizing the centers c_i , $i \in \langle M \rangle$

Require: System matrices (A_i, b_i) , $i \in \langle M \rangle$, a dwell-time τ , a precision $\varepsilon > 0$ and a step-bound $\delta > 0$

1: $c_i \leftarrow x_{ei}$, $\forall i \in \langle M \rangle$

2: $(P_i, M_{ij}, d_{ij}) \leftarrow \arg \min_{P_i, M_{ij}, d_{ij}} f(M_{ij})$ s.t. (4.17b)

3: **repeat**

4: $(\hat{P}_i, \hat{c}_i, \hat{M}_{ij}, \hat{d}_{ij}) \leftarrow \arg \min_{\hat{P}_i, \hat{c}_i, \hat{M}_{ij}, \hat{d}_{ij}} f(\hat{M}_{ij})$ s.t.

$$h_{ij}(P_i, P_j, c_i, c_j, M_{ij}, d_{ij}) + \sum_{\substack{X \in \{P_i, P_j, c_i, \\ c_j, M_{ij}, d_{ij}\}}} \frac{\partial h_{ij}(P_i, P_j, c_i, c_j, M_{ij}, d_{ij})}{\partial X} (\hat{X} - X) > 0,$$

for $(i, j) \in \langle M \rangle^2$,

$$-\delta < \hat{X} - X < \delta,$$

for $X \in (P_i, c_i)$,

5: $c_i \leftarrow \hat{c}_i$

6: $(P_i, M_{ij}, d_{ij}) \leftarrow \arg \min_{P_i, M_{ij}, d_{ij}} f(M_{ij})$ s.t. (4.17b)

7: **until** $|f(M_{ij}) - f(\hat{M}_{ij})| > \varepsilon f(M_{ij})$

8: **return** $(P_i, c_i, M_{ij}, d_{ij})$

902 convergence criterion is satisfied. The next example illustrates cases where the use of
903 Algorithm 4.1 allows the determination of tighter bounding regions.

904 *Example 4.12.* Consider a switched affine system (2.1) defined by

$$905 \quad A_1 = \begin{bmatrix} -5 & 1 \\ -1 & -4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -5 & -1 \\ 1 & -4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -2 & 8 \\ -5 & -5 \end{bmatrix}, \quad b_1 = \begin{bmatrix} -50 \\ -10 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -10 \\ -40 \end{bmatrix}, \quad b_3 = 0.$$

906 The equilibrium points of each subsystem are $x_{e1} = [-10 \ 0]^\top$, $x_{e2} = [0 \ -10]^\top$ and
907 $x_{e3} = 0$. For two different values of dwell-time $\tau \in \{0.1, 0.5\}$, we applied Algorithm 4.1
908 to determine suitable centers c_i , $i \in \langle M \rangle$, for the functions $\tilde{v}_i(x)$ used to define the
909 sets \mathcal{X}_i , $i \in \langle M \rangle$, and the bounding region \mathcal{V} , given in (4.9). Considering $\varepsilon = 10^{-3}$
910 For $\tau = 0.1$, after 48 iterations the algorithm converged to the centers

$$911 \quad c_1 = \begin{bmatrix} -6.9714 \\ -1.9481 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -0.2497 \\ -7.2263 \end{bmatrix}, \quad c_3 = \begin{bmatrix} -0.9154 \\ -1.5790 \end{bmatrix}$$

912 and for $\tau = 0.5$, after 8 iterations the algorithm converged to

$$913 \quad c_1 = \begin{bmatrix} -9.7879 \\ -0.5505 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -0.2918 \\ -9.5003 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0.3777 \\ 0.3399 \end{bmatrix}$$

914 The value of the objective function $f(W_{ij})$ over each iteration for each case is depicted
915 in Figure 5. Notice that for the smaller dwell-time value, optimizing the centers
916 allowed us to reduce more significantly the value of the objective function, which
917 indicates that, for the larger dwell-time, the equilibria x_{ei} were closer to the locally
918 best points c_i , $i \in \langle M \rangle$. The corresponding regions \mathcal{X}_i and \mathcal{V}_i are given in Figure 6.

919 **5. Conclusion.** Stability properties of continuous-time switched affine systems
920 under arbitrary and dwell-time switching were discussed in this paper. We demon-
921 strated that, in both cases, the boundedness of the state trajectories is closely related
922 to the global asymptotic stability of the associated linearized system, i.e., when the
923 affine terms are ignored. However, differently from switched linear systems, switched

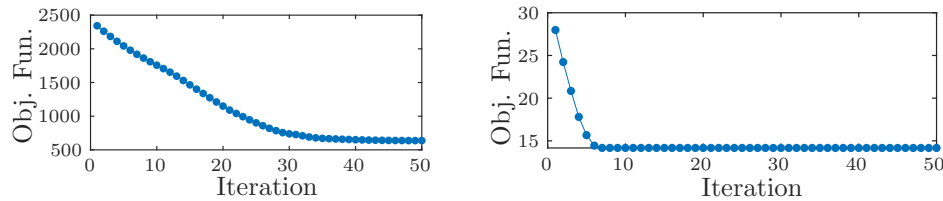


FIGURE 5. In the course of several iterations of Algorithm 4.1, the value of the objective function $f(M_{ij})$ is depicted for $\tau = 0.1$ (left) and $\tau = 0.5$ (right).

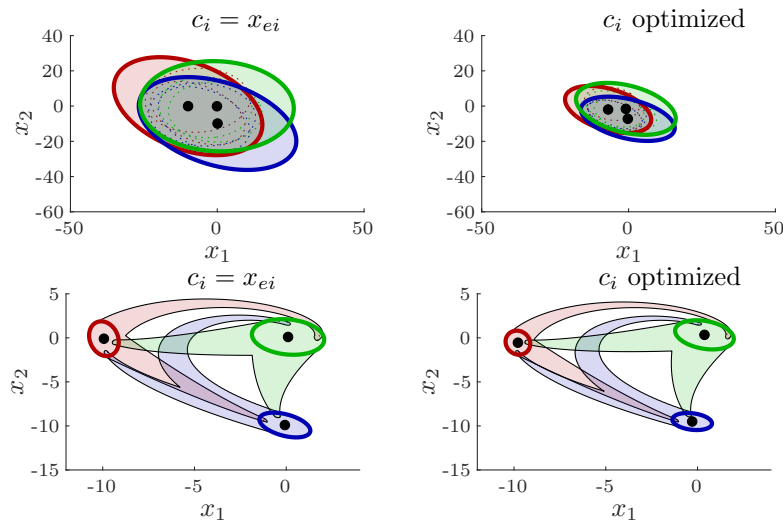


FIGURE 6. Regions \mathcal{X}_1 , \mathcal{X}_2 and \mathcal{X}_3 (red, black and green lines, resp.) and regions \mathcal{V}_1 , \mathcal{V}_2 and \mathcal{V}_3 (red, black and green areas, resp.) for $\tau = 0.1$ (top) and $\tau = 0.5$ (bottom). The figures on the left take $c_i = x_{ei}$ for all $i \in \langle M \rangle$ whereas the ones on the right have c_i obtained from Algorithm 4.1.

924 affine systems are not stable with respect to a point but rather to sets. In the arbitrary
 925 switching case, we characterized the existence of forward invariant sets. On the
 926 other hand, when dwell-time switching is considered, forward invariant sets need not
 927 exist and bounding regions are considered. Theoretical results ensuring the existence
 928 and non-existence of such sets were given and numerical methods based on convex
 929 optimization were devised to outer approximate them. These results were illustrated
 930 by numerical examples for each case.

931 In the future, we plan to study the stability of switched affine systems under
 932 other classes of switching signals such as periodic, path-constrained, and Markov-jump
 933 switching. The case of state-dependent switching systems will also be considered,
 934 since it naturally arises considering affine dynamics as representation of a local first-
 935 order approximation of general smooth vector-fields. Moreover, this shall provide
 936 novel insights into the analysis of symbolic-abstraction based systems, for which local
 937 approximations techniques represent a central tool.

938

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