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Continuous-Time Switched Systems with Switching Frequency Constraints: Path-Complete Stability Criteria

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Abstract

We propose a novel Lyapunov construction for continuous-time switched systems relying on a graph theoretical Lyapunov construction. Starting with a finite family of continuously differentiable functions, suitable inequalities involving these functions and the vector fields defining the switched system are encoded in a direct and labeled graph. We then provide sufficient conditions for (asymptotic) stability subject to constrained switching times, by relying on the path-completeness of the chosen graph. The analysis is first carried out under the hypothesis of constant switching frequency. Then, the results are generalized to dwell time setting. In the case of linear dynamics, the graph formalism allows us to interpret the existing results on dwell time stability in a unified language. Some numerical examples illustrate the usefulness of the conditions.

Keywords: Switching systems, Path-Complete Graphs, Dwell Time Switching, Multiple Lyapunov Functions.

1. Introduction

This paper is devoted to the study of stability certificates for continuous-time switched systems. Due to the importance, both in a theoretical context and in engineering applications, of this class of dynamical systems, the stability problem has been extensively studied in recent years, see for example (Liberzon, 2003), (Shorten et al., 2007). To properly model the evolution of trajectories resulting from time-dependent switching among a finite number of dynamical subsystems, we consider a finite set $\mathscr{I} \subset \mathbb{N}$, the *index set*, and for each $j \in \mathscr{I}$, we associate a continuous-time dynamical subsystem $\dot{x} = f_j(x)$, with $f_j : \mathbb{R}^n \to \mathbb{R}^n$. We thus consider the switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \tag{1}$$

where $\sigma : [0, \infty) \to \mathscr{I}$ denotes a time-dependent switching signal. The stability of (1) under *arbitrary switching* signals is a challenging problem and is related with the existence of *common* Lyapunov functions, see (Shorten et al., 2007).

On the other hand, in many situations, further information is available, and system (1) is known to follow prescribed switching rules σ that satisfy various kind of constraints which bound the frequency of the switching events. In this constrained setting, the existence of a common Lyapunov function is a very restrictive (and not necessary) condition, and, in general more "convoluted" constructions are needed. A frequent assumption on signals $\sigma : [0, \infty) \to \mathscr{I}$ is obtained imposing a *dwell time*, i.e. a minimum time-threshold during which no switching occurs. Stability of (1) under dwell time assumption is a well-studied problem. When considering linear sub dynamics, (i.e. $f_j(x) \equiv A_j x$, with $A_j \in \mathbb{R}^{n\times n}$ for all $j \in \mathscr{I}$), it is well known that, if all the matrices $A_1, \dots, A_K$ are Hurwitz, there exists a (large enough) dwell time for which the switched system (1) is asymptotically stable (Morse, 1996, Lemma 2). Various numerical approximations of the minimal dwell time $\tau_{\text{dw}} \in \mathbb{R}_+$ for which system (1) is stable have been proposed, most of them relying on multiple Lyapunov-functions approaches, see (Geromel and Colaneri, 2006; Allerhand and Shaked, 2011; Briat, 2015; Chesi et al., 2012; Blanchini and Colaneri, 2010; Yuan et al., 2021) and references therein. Regarding converse Lyapunov constructions in this setting, see the results presented in (Wirth, 2005; Chitour et al., 2021). Dwell time certificates for stability of *non-linear* switched systems have been proposed in (Hespanha and Morse, 1999), (De Persis et al., 2003), (Vu et al., 2007), (Zhang and Tanwani, 2019) and references therein.

The main inspiration of this work is given, in the context of *discrete-time switched systems*, in (Lee and Dullerud, 2006; Ahmadi et al., 2014; Philippe et al., 2019). In these works, the concept of *path-complete Lyapunov functions* is introduced and studied: starting from a finite family of functions $\mathscr{V} := \{V_s\}_{s\in S} \subset \mathscr{C}^1(\mathbb{R}^n, \mathbb{R})$, the inequalities involving these functions and the system-data are encoded in a direct and labeled graph $\mathscr{G} = (S, E)$, with $E \subset S \times S \times \mathscr{I}$. In this article, we adapt this formalism to the continuous-time set-

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ting. Graph-based stability conditions have been recently proposed in (Chitour et al., 2021), where the system (1) is modeled as a system constrained on a graph. In our work, instead, the chosen graph affects the *structure* of the Lyapunov inequalities which have to be satisfied. In this sense, the graph template is a parameter which the user has to properly choose, in order to reduce the conservatism of the underlying sufficient conditions. In the discrete-time setting, several "reasonable" hierarchies of graphs based on *lifts* have been proposed in (Philippe et al., 2016, Section 3) in order to reduce the conservatism. These techniques can be adapted, mutatis mutandis, in the continuous time setting, providing *collections* of sufficient conditions. Roughly speaking, to have less conservative stability criteria, one needs to consider large graphs (i.e. with a large number of edges), thus increasing the number of inequalities to be checked, and inevitably increasing the computational complexity. This is consistent with the existing converse Lyapunov results in the linear subsystems case (Wirth, 2005; Chitour et al., 2021; Chesi et al., 2012), which established the existence of (polyhedral) Lyapunov norms and SOS-Lyapunov functions (respectively) and thus in general asking for an unbounded number of faces/polynomial degree (respectively). In this paper we revise these results and we show how reasonable choices of the graph-structure can improve the estimation of the minimal dwell time, proposing novel LMI-based conditions for the linear case.

After the required preliminaries arising from graph theory and having formally introduced the system's setting, we focus, as starting point, to system (1) constrained to a *fixed time* switching policy. This simple framework allows us to develop in an intuitive way the theoretical tools that are then used to propose stability certificates in the dwell time setting. We propose a novel Lyapunov construction, together with a discussion about the computational complexity of the proposed conditions. We show that most of the results concerning the dwell time stability in the linear case (notably (Geromel and Colaneri, 2006) and the converse Lyapunov results (Wirth, 2005; Chitour et al., 2021)) can somehow be re-stated in a path-complete graph framework. This will allow us to clarify the relations and the links between our approach and the existing ones. The efficacy of the theoretical developments is then illustrated by virtue of numerical examples.

Notation: The symbol $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of n-dimensional real-vectors and $\mathbb{R}_+ := \{s \in \mathbb{R} \,|\, s \geq 0\}$, the set of non-negative numbers. The symbol $x \cdot y$, (or $x^\top y$) denotes the *Euclidean scalar product* of $x, y \in \mathbb{R}^n$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is *positive definite* ($\alpha \in \mathcal{PD}$) if it is continuous, $\alpha(0) = 0$, and $\alpha(s) > 0$ if $s \neq 0$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is of *class* $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, $\alpha(0) = 0$, and strictly increasing; it is of *class* $\mathcal{K}_\infty$ if it is also unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, s)$ is of class $\mathcal{K}$ for all s, and $\beta(r, \cdot)$ is decreasing and $\beta(r,s) \to 0$ as $s \to \infty$, for all r.

2. Preliminaries

We recall some basic notions and notation which relate to stability of switching systems and to graph theory.

2.1. Setting and Definitions

Given $N \in \mathbb{N}$, and a finite family of vector fields $\mathcal{F} := \{f_1, \dots, f_N\} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, we study the switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)). \tag{2}$$

Defining $\mathcal{I} := \{1, \dots N\}$, the *switching signals* σ are selected, in general, among the set $\mathcal{S}$ defined by

$$\mathcal{S} := \{\sigma : \mathbb{R}_+ \to \mathcal{I} \mid \sigma \text{ piecewise constant}\}. \tag{3}$$

Without loss of generality we suppose that signals $\sigma \in \mathcal{S}$ are right-continuous. We recall that piecewise constancy in $\mathbb{R}_+$ implies that $\sigma \in \mathcal{S}$ has a finite number of discontinuities in any bounded subinterval of $\mathbb{R}_+$. Given a $\sigma \in \mathcal{S}$, we denote the sequence of switching instants, that is, the points at which σ is discontinuous, by $\{t_i^\sigma\}$. The set $\{t_i^\sigma\}$ may be infinite or finite, possibly reduced to the initial instant $t_0 := 0$; if it is infinite, then it is unbounded. Given a point $x_0 \in \mathbb{R}^n$, and a signal $\sigma \in \mathcal{S}$, we denote with $\phi(t, x_0, \sigma)$ the solution of (2) starting at x_0, evaluated at some instant $t \in \mathbb{R}$.

Definition 1. *Consider* $\mathcal{F} := \{f_1, \dots, f_N\} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ *and a set* $\widehat{\mathcal{S}} \subset \mathcal{S}$. *The switched system* (2) *is said to be* uniformly globally stable (UGS) on $\widehat{\mathcal{S}}$ *if there exists an* $\alpha \in \mathcal{K}_\infty$ *such that, for all* $\sigma \in \widehat{\mathcal{S}}$, *for all* $x_0 \in \mathbb{R}^n$ *and for all* $t \geq 0$,

$$|\phi(t, x_0, \sigma)| \leq \alpha(|x_0|).$$

System (2) *is said to be* uniformly globally asymptotically stable (UGAS) on $\widehat{\mathcal{S}}$, *if there exists an* $\beta \in \mathcal{KL}$ *such that, for all* $\sigma \in \widehat{\mathcal{S}}$, *for all* $x_0 \in \mathbb{R}^n$ *and for all* $t \geq 0$,

$$|\phi(t, x_0, \sigma)| \leq \beta(|x_0|, t). \qquad \triangle$$

We will consider two proper subclasses of $\mathcal{S}$: given $\tau > 0$, we define the class of *fixed time switching signals*, as the set

$$\mathcal{S}_{\text{fix}}(\tau) := \left\{\sigma \in \mathcal{S} \mid \frac{t_i^\sigma - t_{i-1}^\sigma}{\tau} \in \mathbb{N},\ \forall t_i^\sigma > 0\right\}, \tag{4}$$

and the class of *dwell time switching signals*, given by

$$\mathcal{S}_{\text{dw}}(\tau) := \left\{\sigma \in \mathcal{S} \mid t_i^\sigma - t_{i-1}^\sigma \geq \tau,\ \forall\, t_i^\sigma > 0\right\}. \tag{5}$$

It is clear that $\mathcal{S}_{\text{fix}}(\tau) \subset \mathcal{S}_{\text{dw}}(\tau)$, for any $\tau > 0$.

Assumption 1. *Given* $\mathcal{F} = \{f_j\}_{j \in \mathcal{I}} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ *we suppose that* $f_j(0) = 0$ *and* $f_j(x) \neq 0$ *for any* $x \in \mathbb{R}^n \setminus \{0\}$, *for any* $j \in \mathcal{I}$. *Moreover, considering any* $f_j \in \mathcal{F}$ *and the system*

$$\dot{x}(t) = f_j(x(t)), \tag{6}$$

we suppose that maximal solutions of (6) *are complete (forward and backward).*

Given $j \in \mathscr{I}$, we define the *flow map* $\phi_j : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\phi_j(t,x) := \text{ solution of (6) starting at } x \text{ evaluated at } t.$$

Moreover, in what follows, we use the following properties.

Property 1. *For any $j \in \mathscr{I}$, the map $\phi_j : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ satisfies*

1. $\phi_j(t,0) = 0$, *for any* $t \in \mathbb{R}$,
2. $\phi_j(0,x) = x$, *for all* $x \in \mathbb{R}^n$,
3. $\phi_j(s,\phi_j(t,x)) = \phi_j(s+t,x)$, *for all* $x \in \mathbb{R}^n$, *for all* $s,t \in \mathbb{R}$,
4. $\phi_j \in \mathscr{C}^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$,
5. $\frac{\partial}{\partial t}\phi_j(t,x) = f_j(\phi_j(t,x))$, *for all* $(t,x) \in \mathbb{R}\times\mathbb{R}^n$,
6. $\frac{\partial}{\partial x}\phi_j(t,x)\cdot f_j(x) = f_j(\phi_j(t,x))$, *for all* $(t,x) \in \mathbb{R}\times\mathbb{R}^n$.

Items *1., 2., 3., 5.* of Property 1 follow from the definition of solution and from Assumption 1. Item *4.* is proved in (Khalil, 2002, Section 3.3), Item *6.* is a consequence of the so-called variational equation associated to the dynamical system $\dot{x} = f_j(x)$, see for example (Sontag, 1998, Theorem 1, pag. 58).

2.2. Graphs and Path-Completeness

Our main goal is the study of stability of (2), using a multiple-Lyapunov functions approach. We thus introduce a formalism that allows us to represent inequalities involving scalar functions by graphs. Given a discrete alphabet $\mathscr{I} \subset \mathbb{N}$, a *direct and labeled graph* $\mathscr{G} = (S,E)$ is defined by a finite set S (the set of nodes) and $E \subset S \times S \times \mathscr{I}$ (the set of edges).

Definition 2 (Path-Completeness). *A graph $\mathscr{G} = (S,E)$ is* path-complete *for $\mathscr{I}$ if, for any $K \geq 1$ and any "word" $j_1 \ldots j_K$, with $j_k \in \mathscr{I}$, there exists a* path $\{(s_k,s_{k+1},j_k)\}_{1\leq k\leq K}$ *such that $(s_k,s_{k+1},j_k) \in E$, for each $1 \leq k \leq K$.* △

This formalism was introduced in (Ahmadi et al., 2014) and (Philippe et al., 2019) in the context of *discrete-time* switched systems. For a thorough discussion we refer to (Ahmadi et al., 2014), see Figures 1 and 2 for graphical representations of specific path-complete graphs. Given a graph $\mathscr{G} = (S,E)$, its *dual graph* $\mathscr{G}' = (S',E')$ is defined by $S' = S$ and $(a,b,j) \in E \iff (b,a,j) \in E'$, that is, the graph obtained reversing the direction of each edge. A graph $\mathscr{G} = (S,E)$ is path-complete if and only if its dual graph $\mathscr{G}' = (S,E')$ is path-complete, see (Ahmadi et al., 2014, Theorem 3.2).

3. An Intermediate Step: Fixed Time Switching

In this section, given $\mathscr{I} = \{1,\ldots,N\}$, $\mathscr{F} := \{f_j\}_{j\in\mathscr{I}} \subset \mathscr{C}^1(\mathbb{R}^n,\mathbb{R}^n)$ and $\tau > 0$, we study the stability of system (2) under fixed time switching signals $\mathscr{S}_{\text{fix}}(\tau)$ defined in (4). We want to point out that this problem can be "translated" and then approached in a discrete-time setting (as in (Jungers, 2009; Ahmadi et al., 2014; Philippe et al., 2019)), considering stability of the difference inclusion $x^+ \in \text{co}\left\{\phi_j(\tau,x) \mid j \in \mathscr{I}\right\}$. On the other hand, the developments provided in what follows can be considered as necessary intermediate steps in generalizing the path-complete Lyapunov construction from the discrete-time setting to continuous-time switched systems under the dwell-time assumption.

3.1. Stability via Path-Complete Lyapunov Functions

We give a result of stability for (2) under fixed time-switching rule, relying on graphs associated to a candidate vector-valued Lyapunov function, as defined in what follows.

Definition 3. *Given a finite set S, a* candidate vector-valued Lyapunov function *is a map $V : \mathbb{R}^n \to \mathbb{R}^{|S|}$, such that $V_\ell \in \mathscr{C}^1(\mathbb{R}^n,\mathbb{R})$ is positive definite and radially unbounded for each $\ell \in S$. More explicitly, for all $\ell \in S$, there exists $\underline{\alpha}_\ell, \overline{\alpha}_\ell \in \mathscr{K}_\infty$ such that $\underline{\alpha}_\ell(|x|) \leq V_\ell(x) \leq \overline{\alpha}_\ell(|x|)$, $\forall\, x \in \mathbb{R}^n$.* △

Given system (2), let us fix a threshold $\tau > 0$ and consider a candidate vector-valued Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}^{|S|}$. Consider a continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}$; given $a,b \in S$ and $j \in \mathscr{I}$, we define a set of labeled edges E between nodes in S according to the following rule: an *edge* $(a,b,j)_\tau \in E$ if

$$V_b(\phi_j(\tau,x)) - V_a(x) \leq -\rho(|x|), \quad \forall x \in \mathbb{R}^n. \tag{7}$$

The choice of the function $\rho : \mathbb{R}_+ \to \mathbb{R}$ depends on the problem we are interested in: if we are interested in UGS, we take $\rho \equiv 0$, while in studying UGAS we choose $\rho \in \mathscr{PD}$. With this definition, given any $\mathscr{F} = \{f_j\}_{j\in\mathscr{I}} \subset \mathscr{C}^1(\mathbb{R}^n,\mathbb{R}^n)$, any $\rho \in \mathscr{C}(\mathbb{R}_+,\mathbb{R})$ and any candidate vector-valued Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}^{|S|}$, we associate to V a direct and labeled graph $\mathscr{G} = (S,E)$ over the alphabet $\mathscr{I}$. More precisely, given $V : \mathbb{R}^n \to \mathbb{R}^{|S|}$, the associated graph $\mathscr{G} = (S,E)$ is defined by: $(a,b,j)_\tau \in E$ if and only if inequality (7) is satisfied by V_a, V_b, and f_j. We now give our first stability result.

Proposition 1. *Consider a $\tau > 0$, a function $\rho \in \mathscr{PD}$, a finite set S, and $V : \mathbb{R}^n \to \mathbb{R}^{|S|}$ a candidate vector-valued Lyapunov function. If the associated graph $\mathscr{G} = (S,E)$ is path-complete for $\mathscr{I}$ then system (2) is UGAS on $\mathscr{S}_{\text{fix}}(\tau)$. If the same holds considering $\rho \equiv 0$, then system (2) is UGS on $\mathscr{S}_{\text{fix}}(\tau)$.*

The proof of this statement can be trivially obtained from its discrete-time counterpart presented in (Ahmadi et al., 2014). In what follows, instead, we propose a novel proof, which allows us to illustrate a multiple-Lyapunov construction which will be used to study the general dwell-time case. The following lemma is used in the proof of Proposition 1.

Lemma 1. *Consider $f_j \in \mathscr{F}$ and V_a, V_b with $a,b \in S$, and suppose that condition (7) holds for some $\rho \in \mathscr{PD}$. Then, there exists a function $U_{a,b,j} : [0,\tau] \times \mathbb{R}^n \to \mathbb{R}$ continuously differentiable in $(0,\tau)\times\mathbb{R}^n$ and $\underline{\alpha}_{a,b,j}$, $\overline{\alpha}_{a,b,j} \in \mathscr{K}_\infty$ such that*

$$U_{a,b,j}(0,x) = V_a(x),\ U_{a,b,j}(\tau,x) = V_b(x),\ \forall\, x \in \mathbb{R}^n, \tag{8a}$$

$$\underline{\alpha}_{a,b,j}(|x|) \leq U_{a,b,j}(t,x) \leq \overline{\alpha}_{a,b,j}(|x|),\ \forall\,(t,x) \in [0,\tau]\times\mathbb{R}^n, \tag{8b}$$

$$\frac{d}{dt}U_{a,b,j}(t,\phi_j(t,x)) \leq -\frac{1}{\tau}\rho(|x|),\ \forall\,(t,x) \in (0,\tau)\times\mathbb{R}^n. \tag{8c}$$

If condition (7) *is satisfied with* $\rho \equiv 0$, (8c) *holds with* $\rho \equiv 0$.

Proof. Define $U_1, U_2 : [0,\tau] \times \mathbb{R}^n \to \mathbb{R}$ as,

$$\begin{aligned} U_1(t,x) &:= V_a(\phi_j(-t,x)), \\ U_2(t,x) &:= V_b(\phi_j(\tau - t,x)). \end{aligned} \tag{9}$$

Since $V_a, V_b \in \mathscr{C}^1(\mathbb{R}^n,\mathbb{R})$ and recalling Item *4.* of Property 1, U_1, U_2 are continuously differentiable in (t,x). Moreover, applying (Khalil, 2002, Lemma 4.3), it can be seen that there exist $\underline{\alpha}_1, \underline{\alpha}_2, \overline{\alpha}_1, \overline{\alpha}_2 \in \mathscr{K}_\infty$ such that

$$\underline{\alpha}_i(|x|) \le U_i(t,x) \le \overline{\alpha}_i(|x|),\ \forall (t,x) \in [0,\tau] \times \mathbb{R}^n,\ \forall\, i \in \{1,2\}.$$

For any $i = 1,2$ and any $(t,x) \in (0,\tau) \times \mathbb{R}^n$, we introduce the notation $\dot{U}_i(t,x) := \frac{\partial U_i}{\partial t}(t,x) + \frac{\partial U_i}{\partial x}(t,x) \cdot f_j(x)$. Given any $(t,x) \in (0,\tau) \times \mathbb{R}^n$, using Property 1 and (9), we have $\dot{U}_1(t,x) = \frac{d}{ds} U_1(t+s, \phi_j(s,x)))|_{s=0} = \frac{d}{ds} V_a(\phi_j(-t,x))|_{s=0} = 0$. Similarly, it can be seen that $\dot{U}_2(t,x) = 0$. Define $U_{a,b,j} : [0,\tau] \times \mathbb{R}^n \to \mathbb{R}$ by

$$U_{a,b,j}(t,x) := \frac{\tau - t}{\tau} U_1(t,x) + \frac{t}{\tau} U_2(t,x), \tag{10}$$

by (9), it is clear that $U_{a,b,j}(0,x) = V_a(x)$ and $U_{a,b,j}(\tau,x) = V_b(x)$, $\forall\, x \in \mathbb{R}^n$. Computing the derivative of U along solutions of (6), using (7) we obtain

$$\begin{aligned} \frac{d}{dt} U_{a,b,j}(t,\phi_j(t,x)) &= \frac{U_2(t,\phi_j(t,x))}{\tau} - \frac{U_1(t,\phi_j(t,x))}{\tau} \\ &\quad + \frac{\tau - t}{\tau} \dot{U}_1(t,x) + \frac{t}{\tau} \dot{U}_2(t,x) \\ &= \frac{V_b(\phi_j(\tau,x)) - V_a(x)}{\tau} \le -\frac{\rho(|x|)}{\tau}, \end{aligned}$$

$\forall\, t \in (0,\tau)$ and $\forall\, x \in \mathbb{R}^n$. Defining $\underline{\alpha}_{a,b,j}(s) := \min\{\underline{\alpha}_1(s), \underline{\alpha}_2(s)\}$ and $\overline{\alpha}_{a,b,j}(s) := \max\{\overline{\alpha}_1(s), \overline{\alpha}_2(s)\}$ for any $s \in \mathbb{R}_+$, we conclude the proof. □

The possibility of concluding stability for continuous-time systems from "sampled" or fixed-time Lyapunov conditions as in (7) was also noted, using different techniques, in (Aeyels and Peuteman, 1998). In Lemma 1 we have shown that (7) implies the existence of a $\mathscr{C}^1$ function decreasing along solutions, and it is the main tool of the following argument.

Proof of Proposition 1. Without loss of generality, we consider $\rho \in \mathscr{P}\mathscr{D}$, proving UGAS; the case with $\rho \equiv 0$ ensuring UGS is similar. For any $\sigma \in \mathscr{S}_{\text{fix}}(\tau)$, in what follows we construct a positive definite and continuous function $U_\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ decreasing along solutions of (2), for any initial point $x \in \mathbb{R}^n$. Moreover we show that the decay rate of U_σ along solutions of (2) does not depend on $\sigma \in \mathscr{S}_{\text{fix}}(\tau)$. Consider any $\sigma \in \mathscr{S}_{\text{fix}}(\tau)$, we construct recursively the associated "word" as follows: for each $t_i^\sigma > 0$ consider the number $n(i) = \frac{t_i^\sigma - t_{i-1}^\sigma}{\tau}$; by definition of $\mathscr{S}_{\text{fix}}(\tau)$, $n(i) \in \mathbb{N}$. If $\sigma(t_{i-1}^\sigma) = j$, then add a string of j's of length $n(i)$. If the sequence $\{t_i^\sigma\}$ is finite, add an infinite sequence of j_M's , where $j_M = \sigma(\max\{t_i^\sigma\})$. By path-completeness of $\mathscr{G}$, we can consider a path in $\mathscr{G} = (V,E)$ corresponding to this sequence. Suppose that the first edge of the selected path is $(a,b,j)_\tau \in E$, for some $a,b \in S$ and $j \in \mathscr{I}$. We start defining $U_\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ posing $U_\sigma|_{[0,\tau]\times\mathbb{R}^n} \equiv U_{a,b,j}$ as defined in Lemma 1. If the second edge is (b,c,i) we will prolongate the function U to the interval $[\tau, 2\tau]$ accordingly, since the reasoning of Lemma 1 can be straightforwardly adapted for any interval $[k\tau, (k+1)\tau]$, for any $k \in \mathbb{N}$. Iterating this procedure, we construct $U_\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$. The continuity of this function at times $t = k\tau$ for any $k \in \mathbb{N}$ is assured by condition (8a). Moreover there exist $\underline{\alpha}', \overline{\alpha}' \in \mathscr{K}_\infty$ (independent of σ) such that $\underline{\alpha}'(|x|) \le U_\sigma(t,x) \le \overline{\alpha}'(|x|)$, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$: it suffices to define

$$\underline{\alpha}'(s) := \min_{(\ell,\ell',j)\in E} \underline{\alpha}_{\ell,\ell',j}(s), \quad \overline{\alpha}'(s) := \max_{(\ell,\ell',j)\in E} \overline{\alpha}_{\ell,\ell',j}(s),$$

where $\underline{\alpha}_{\ell,\ell',j}$, $\overline{\alpha}_{\ell,\ell',j}(s)$ are given by Lemma 1. Moreover, considering any $x \in \mathbb{R}^n$ and the solution $\phi(\cdot,x,\sigma)$ of (2) under the switching signal σ, again by Lemma 1 we have

$$\frac{d}{dt} U_\sigma(t,\phi(t,x,\sigma)) \le -\frac{1}{\tau} \rho(|\phi(k\tau,x,\sigma)|),$$

$\forall\, t \in (k\tau, (k+1)\tau)$ and $\forall\, k \in \mathbb{N}$ (and thus, almost everywhere in $\mathbb{R}_+$). By arbitrariness of $\sigma \in \mathscr{S}_{\text{fix}}(\tau)$ and since $\underline{\alpha}', \overline{\alpha}'$ and ρ do not depend on σ, using a standard comparison argument (see for example (Khalil, 2002, Theorem 4.8), we conclude that (2) is (UGAS) on $\mathscr{S}_{\text{fix}}(\tau)$. □

3.2. Relaxed Conditions: "Splitting Edges"

Actually, the main strength of any Lyapunov direct result for continuous-time systems lies in the fact that stability can be ensured *without* computing solutions. On the other hand inequality (7) depends on the solutions of the system $\dot{x} = f_j(x)$ at time τ. In what follows, we propose conditions implying (7), without the necessity of computing solutions.

Lemma 2. *Consider* $f_j \in \mathscr{F}$, $\tau > 0$ *and a* $K \in \mathbb{N}\setminus\{0\}$. *Suppose there exist* $V_0, \dots, V_K \in \mathscr{C}^1(\mathbb{R}^n,\mathbb{R})$ *positive definite and* $\widetilde{\rho} \in \mathscr{P}\mathscr{D}$ *such that*

$$\begin{cases} \nabla V_k(x) \cdot f_j(x) \quad + \frac{K(V_k(x) - V_{k-1}(x))}{\tau} \le -\widetilde{\rho}(|x|),\ \forall\, x \in \mathbb{R}^n, \\ \nabla V_{k-1}(x) \cdot f_j(x) + \frac{K(V_k(x) - V_{k-1}(x))}{\tau} \le -\widetilde{\rho}(|x|),\ \forall\, x \in \mathbb{R}^n. \end{cases} \tag{11}$$

for all $k \in \{1,\dots,K\}$. *This implies that* $\exists\, \rho \in \mathscr{P}\mathscr{D}$ *such that*

$$V_K(\phi_j(\tau,x)) - V_0(x) \le -\rho(|x|),\quad \forall\, x \in \mathbb{R}^n. \tag{12}$$

If conditions (11) *hold with* $\widetilde{\rho} \equiv 0$, (12) *holds with* $\rho \equiv 0$.

Sketch of the Proof. Define $t_k = \frac{k}{K}\tau$ for $k \in \{0,\dots,K\}$ and consider the function $U : [0,\tau] \times \mathbb{R}^n \to \mathbb{R}$ as

$$U(t,x) = \frac{K(t - t_{k-1})}{\tau} V_k(x) + \frac{K(t_k - t)}{\tau} V_{k-1}(x)$$

for $t \in [t_{k-1}, t_k]$ and for all $k \in \{1,\dots,K\}$. Using (11), it can be seen that

$$V_K(\phi_j(\tau,x))-V_0(x)=U(\tau,x(\tau))-U(0,x(0))$$
$$=\int_0^\tau \dot U(t,x(t))dt\le -\int_0^\tau \widetilde\rho(|x(t)|)dt.$$

To conclude, it suffices to consider any function $\rho\in\mathscr{PD}$ such that $\rho(|x|)\le\int_0^\tau \widetilde\rho(|\phi(t,x)|)dt,\ \forall\ x\in\mathbb{R}^n$, which is possible applying (Khalil, 2002, Lemma 4.3). The case with $\widetilde\rho\equiv\rho\equiv 0$ follows the same steps. □

Lemma 2 provides a procedure to check if a function $W\in\mathscr{C}^1(\mathbb{R}^n,\mathbb{R})$ evaluated at solutions of (6) after a time $\tau>0$ is always smaller than another function $V\in\mathscr{C}^1(\mathbb{R}^n,\mathbb{R})$ evaluated at the initial conditions. An important question now arises: are these two conditions equivalent? For the case $\rho\in\mathscr{PD}$, we have the following "semiglobal-practical" (i.e. for generic compact sets away from the origin) result.

Theorem 1. *Consider $V,W\in\mathscr{C}^2(\mathbb{R}^n,\mathbb{R})$ positive definite, $f_j\in\mathscr{F}$ and $\tau>0$. If there exists a $\rho\in\mathscr{PD}$ such that*

$$W(\phi_j(\tau,x))-V(x)\le-\rho(|x|),\ \forall\,x\in\mathbb{R}^n, \tag{13}$$

then, for every compact set $\mathscr{K}\subset\mathbb{R}^n\setminus\{0\}$, there exist a (large enough) $K\in\mathbb{N}$, $V_0,\dots,V_K\in\mathscr{C}^2(\mathbb{R}^n,\mathbb{R})$ positive definite with $V_0\equiv V$, $V_K\equiv W$ and a $\widetilde\rho\in\mathscr{PD}$ such that

$$\begin{cases}\nabla V_k(x)\cdot f_j(x) \quad+\frac{K(V_k(x)-V_{k-1}(x))}{\tau}\le-\widetilde\rho(|x|),\ \forall\,x\in\mathscr{K},\\ \nabla V_{k-1}(x)\cdot f_j(x)+\frac{K(V_k(x)-V_{k-1}(x))}{\tau}\le-\widetilde\rho(|x|),\ \forall\,x\in\mathscr{K},\end{cases} \tag{14}$$

for all $k\in\{1,\dots,K\}$.

Proof. Without loss of generality, we suppose $\tau=1$. We use again the construction already proposed in the proof of Lemma 1: define $U_1,U_2:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ as in (9), i.e.

$$\begin{aligned}U_1(t,x)&:=V(\phi_j(-t,x)),\\ U_2(t,x)&:=W(\phi_j(1-t,x)),\end{aligned} \tag{15}$$

and we define $U:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ by $U(t,x):=(1-t)U_1(t,x)+tU_2(t,x)$. Again, it is clear that $U(0,x)=V(x)$ and $U(1,x)=W(x)$, for any $x\in\mathbb{R}^n$. Let us now fix $K\in\mathbb{N}$ and split the interval $[0,1]$ considering $t_0=0$, $t_1=\frac{1}{K}$, $t_2=\frac{2}{K},\dots,t_K=1$. Define, for any $k\in\{0,\dots K\}$, $V_k:\mathbb{R}^n\to\mathbb{R}$ as

$$V_k(x):=U(t_k,x),\ \ \forall\,x\in\mathbb{R}^n. \tag{16}$$

Given $k\in\{0,\dots,K\}$ we compute

$$\begin{aligned}\nabla V_k(x)&=(1-t_k)\frac{\partial}{\partial x}U_1(t_k,x)+t_k\frac{\partial}{\partial x}U_2(t_k,x)\\ &=(1-t_k)\nabla V(\phi_j(-t_k,x))\frac{\partial}{\partial x}\phi_j(-t_k,x)\\ &\quad+t_k\nabla W(\phi_j(1-t_k,x))\frac{\partial}{\partial x}\phi_j(1-t_k,x).\end{aligned}$$

And thus, using Item 6. of Property 1,

$$\begin{aligned}\nabla V_k(x)\cdot f_j(x)&=(1-t_k)\nabla V(\phi_j(-t_k,x))\cdot f_j(\phi_j(-t_k,x))\\ &\quad+t_k\nabla W(\phi_j(1-t_k,x))\cdot f_j(\phi_j(1-t_k,x)),\end{aligned} \tag{17}$$

for any $x\in\mathbb{R}^n$ and any $k\in\mathbb{N}$. Let us call $\delta=\frac{1}{K}$, we have

$$\begin{aligned}&\frac{1}{\delta}(V_k(x)-V_{k-1}(x))\\ &=\frac{1}{\delta}\Big[(1-t_k)U_1(t_k,x)+t_kU_2(t_k,x)\\ &\quad-(1-(t_k-\delta))U_1(t_k-\delta,x)-(t_k-\delta)U_2(t_k-\delta,x)\Big]\\ &=\frac{1}{\delta}\left[(1-t_k)(U_1(t_k,x)-U_1(t_k-\delta,x))\right]\\ &\quad+\frac{1}{\delta}\left[t_k(U_2(t_k,x)-U_2(t_k-\delta,x))\right]\\ &\quad+U_2(t_k-\delta,x)-U_1(t_k-\delta,x).\end{aligned} \tag{18}$$

Inequalities in (14) can now be obtained from equations (17), (18) using the Taylor-Lagrange Theorem. Considering a compact set *away from the origin* is necessary for the existence of a $\widetilde\rho\in\mathscr{PD}$ as in (14), in order to have an uniform bound on the remainder of the Taylor-Lagrange approximation. □

It can be seen that the implication in Theorem 1 holds in a global sense (and not only on a compact set $\mathscr{K}\subset\mathbb{R}^n\setminus\{0\}$) when considering homogeneous of degree 1 subsystems (i.e. $f_j(\lambda x)=\lambda f_j(x)$, for all $\lambda\ge 0$, for all $x\in\mathbb{R}^n$) and scalar functions W,V homogeneous (of any degree). This, together with Lemma 2, proves an equivalence between conditions in (13) and (14) in this particular case. On the other hand this theoretical equivalence do not provide any insight of the choice of the "splitting parameter" $K\in\mathbb{N}$, which, in general, can be arbitrarily large. The same equivalence is proven, with a different technique, in (Xiang, 2015), in the case of linear subsystems and quadratic real-valued functions.

4. Dwell Time Switching

In this section we study the stability of system (2) under dwell time switching signals $\mathscr{S}_{\text{dw}}(\tau)$ defined in (5), adapting the preliminary ideas presented in Section 3.

4.1. Main Stability Result

In order to provide a dwell time counterpart of Proposition 1, we need to reinforce the conditions encoded in a generic edge. Given the system (2), let us fix $\tau>0$ and consider a candidate vector-valued Lyapunov function $V:\mathbb{R}^n\to\mathbb{R}^{|S|}$. Consider a continuous function $\rho:\mathbb{R}_+\to\mathbb{R}$; given $a,b\in S$ and $j\in\mathscr{I}$, we say that there is a *"dwell time" edge* $(a,b,j)^{\text{dw}}_\tau\in E^{\text{dw}}$ if

$$V_b(\phi_j(t,x))-V_a(x)\le-\rho(|x|),\ \forall x\in\mathbb{R}^n,\ \forall\,t\in[\tau,2\tau). \tag{19}$$

Again, the choice of $\rho:\mathbb{R}_+\to\mathbb{R}$ depends on the problem we are interested in: $\rho\equiv 0$ for UGS, $\rho\in\mathscr{PD}$ for UGAS.

Theorem 2 (Dwell-Time Lyapunov Direct Method)**.** *Consider a $\tau > 0$, a function $\rho \in \mathcal{PD}$, a finite set S, and $V : \mathbb{R}^n \to \mathbb{R}^{|S|}$ a candidate vector-valued Lyapunov function. Suppose the associated graph $\mathcal{G} = (S, E^{dw})$ is path-complete for $\mathcal{I}$. Then system* (2) *is UGAS on $\mathcal{S}_{\rm dw}(\tau)$. If the same hold considering $\rho \equiv 0$, then system* (2) *UGS on $\mathcal{S}_{\rm dw}(\tau)$.*

Proof. Consider any $\sigma \in \mathcal{S}_{\rm dw}(\tau)$, for any $t_i^\sigma > 0$ define $\underline{n}(i) = \lfloor \frac{t_i^\sigma - t_{i-1}^\sigma}{\tau} \rfloor$, where, given a $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r. By definition of $\mathcal{S}_{\rm dw}(\tau)$ in (5), $\underline{n}(i) \geq 1$, for all $t_i^\sigma > 0$. Similarly to the proof of Proposition 1, we construct the "word" associated to σ as follows: for each $t_i^\sigma > 0$, if $\sigma(t_{i-1}^\sigma) = j$, then add a string of j's of length $\underline{n}(i)$. If the sequence $\{t_i^\sigma\}$ is finite, add an infinite sequence of j_M's, where $j_M = \sigma(\max\{t_i^\sigma\})$. It suffices now to consider the path in $\mathcal{G} = (S, E^{\rm dw})$ corresponding to this word. We define the function $U : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ as in proof of Proposition 1, with the peculiarity that here we split each interval $[t_{i-1}^\sigma, t_i^\sigma]$ in $\underline{n}(i) - 1$ sub-intervals of length τ,

$$\left[t_{i-1}^\sigma, t_{i-1}^\sigma + \tau\right], \left[t_{i-1}^\sigma + \tau,\, t_{i-1}^\sigma + 2\tau\right], \ldots$$

and one last sub-interval, $\left[t_{i-1}^\sigma + \left(\underline{n}(i) - 1\right)\tau,\, t_i^\sigma\right]$. By definition of $\mathcal{S}_{\rm dw}(\tau)$ and $\underline{n}(i)$, this last sub-interval has length equal to $t_i^\sigma - t_{i-1}^\sigma - \left(\underline{n}(i) - 1\right)\tau \in [\tau, 2\tau)$. Following the idea of proof of Proposition 1, we construct a continuous function $U_\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, positive definite and decreasing along solutions of (2), with decay rate independent of $\sigma \in \mathcal{S}_{dw}(\tau)$. ☐

We now see how we can verify the condition $(a, b, j)_\tau^{\rm dw} \in E^{\rm dw}$, without explicitly computing solutions, following the same idea introduced in Lemma 2.

Lemma 3. *Consider $f_j \in \mathcal{F}$, any $\tau > 0$ and a $K \in \mathbb{N} \setminus \{0\}$. Suppose there exist $V_0, \ldots, V_K \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ and $\widetilde{\rho} \in \mathcal{PD}$ such that*

$$\begin{cases} \nabla V_k(x) \cdot f_j(x) \quad + \frac{K(V_k(x) - V_{k-1}(x))}{s} \leq -\widetilde{\rho}(|x|),\ \forall\, x \in \mathbb{R}^n, \\ \nabla V_{k-1}(x) \cdot f_j(x) + \frac{K(V_k(x) - V_{k-1}(x))}{s} \leq -\widetilde{\rho}(|x|),\ \forall\, x \in \mathbb{R}^n, \end{cases} \tag{20}$$

for both $s = \tau$ *and* $s = 2\tau$*, and for all $k \in \{1, \ldots, K\}$. This implies that there exists a $\rho \in \mathcal{PD}$ such that*

$$V_K(\phi_j(t, x)) - V_0(x) \leq -\rho(|x|), \quad \forall\, x \in \mathbb{R}^n,\ \forall\, t \in [\tau, 2\tau). \tag{21}$$

If (20) *hold with $\widetilde{\rho} \equiv 0$, the inequality* (21) *holds with $\rho \equiv 0$.*

Proof. Consider any $k \in \{1, \ldots, K\}$ and $V_{k-1}, V_k \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ such that (20) hold for both $s = \tau$ and $s = 2\tau$. Consider the first inequality in (20); since it holds for both $s = \tau$ and $s = 2\tau$, consider any $\lambda \in [0, 1]$, we multiply the two obtained inequalities by λ and $(1 - \lambda)$ respectively. Adding, we obtain

$$\begin{aligned} &\nabla V_k(x) \cdot f_j(x) + \tfrac{\lambda K}{\tau}(V_k(x) - V_{k-1}(x)) \\ &\quad + \tfrac{(1-\lambda)K}{2\tau}(V_k(x) - V_{k-1}(x)) \\ &= \nabla V_k(x) \cdot f_j(x) + \tfrac{(1+\lambda)K}{2\tau}(V_a(x) - V_b(x)) \leq -\widetilde{\rho}(|x|). \end{aligned}$$

Since the function $\phi : [0, 1] \to [1, 2]$ defined by $\phi(\lambda) = \frac{2}{1+\lambda}$ is bijective, we conclude that

$$\nabla V_k(x) \cdot f_j(x) - \frac{K}{\widetilde{\tau}}(V_k(x) - V_{k-1}(x)) \leq -\widetilde{\rho}(|x|),$$

for all $\widetilde{\tau} \in \left[\frac{\tau}{K}, 2\frac{\tau}{K}\right]$. Similarly, considering the second inequalities (for $s = \tau$ and $s = 2\tau$) in (20), we also have

$$\nabla V_{k-1}(x) \cdot f_j(x) + \frac{K}{\widetilde{\tau}}(V_k(x) - V_{k-1}(x)) \leq -\widetilde{\rho}(|x|),$$

for all $\widetilde{\tau} \in \left[\frac{\tau}{K}, 2\frac{\tau}{K}\right]$. By arbitrariness of $k \in \{1 \ldots, K\}$, we can conclude: it suffices, for any $t \in [\tau, 2\tau)$, to split the interval $[0, t]$ in the K sub-intervals of length $\frac{t}{K} \in \left[\frac{\tau}{K}, 2\frac{\tau}{K}\right]$, and then follow the construction given in proof of Lemma 2. ☐

In Lemma 3 we presented one possible way to ensure the conditions encoded in (19) without explicitly compute solutions. We require to verify $4K$ inequalities involving gradients of some auxiliary functions ($V_0, \ldots V_K \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$). One drawback of this technique is that the number of inequalities increases rapidly as we increase the number of nodes and K, as required by Theorem 1 to reduce the level of conservatism.

4.2. Particular Cases and Comparison with existing results

In general, our construction relying on path-complete graphs (Theorem 2) do not require to construct sub-Lyapunov functions for each subsystems, as is often the case in multiple-Lyapunov functions results (as in (Liberzon, 2003, Sections 3.1. and 3.2)). We show here how the proposed path-complete conditions simplify if some of the nodes represents a Lyapunov function for a subsystem. Consider $\mathcal{I} = \{1, \ldots, N\}$, $\mathcal{F} := \{f_j\}_{j \in \mathcal{I}} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, $\tau > 0$, a discrete set S and a vector-valued candidate Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}^{|S|}$. In what follows, without loss of generality, we always suppose that $\rho \in \mathcal{PD}$. Let us analyze the conditions associated to *self-loops*: suppose that $(\ell, \ell, j)_\tau \in E$ for some $\ell \in S$ and $j \in \mathcal{I}$, that means

$$V_\ell(\phi_j(\tau, x)) - V_\ell(x) \leq -\rho(|x|), \quad \forall\, x \in \mathbb{R}^n,$$

while $(\ell, \ell, j)_\tau^{\rm dw} \in E^{\rm dw}$ means

$$V_\ell(\phi_j(t, x)) - V_\ell(x) \leq -\rho(|x|), \quad \forall\, x \in \mathbb{R}^n,\ \forall\, t \in [\tau, 2\tau].$$

In Lemmas 2 and 3 we presented how to verify these two conditions without computing the solutions explicitly. We note that these two conditions are satisfied if V_ℓ is a Lyapunov function for the subsystem $\dot{x} = f_j(x)$, i.e. the inequality

$$V_\ell(\phi_j(t, x)) \leq \beta(V_\ell(x), t), \quad \forall\, t \geq 0,\ \forall x \in \mathbb{R}^n \tag{22}$$

holds for some $\beta \in \mathcal{KL}$ such that $\beta(s, 0) = s$ for any $s \in \mathbb{R}_+$. If V_ℓ is locally Lipschitz, this is ensured if the "usual" Lyapunov inequality

$$\nabla V_\ell(x) \cdot f_j(x) \leq -\widetilde{\rho}(|x|), \quad \text{for almost all } x \in \mathbb{R}^n \tag{23}$$

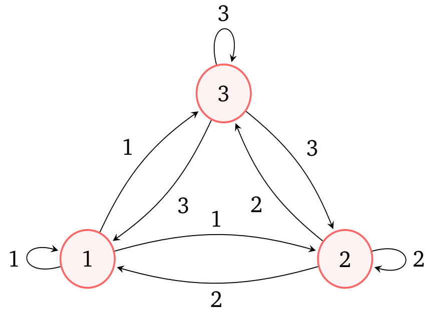

Figure 1: The graph $\mathcal{G}_1$, in the case $\mathcal{I} = \{1,2,3\}$. For simplicity, for each edge we denoted only the label.

is satisfied for some $\widetilde{\rho} \in \mathcal{PD}$ see (Kellet, 2014, Lemma 20) (see also (Teel and Praly, 2000) for the possibility of having "almost everywhere" conditions, since the $f_1,\dots f_N$ are continuous). If (22) (or, equivalently, (23)) is satisfied, we say that the self-loop (ℓ,ℓ,j) is in the *strong form* and we denote it by $(\ell,\ell,j)_{\rm st} \in E$, and, clearly, this implies both $(\ell,\ell,j)_t \in E$ and $(\ell,\ell,j)_t^{\rm dw} \in E^{\rm dw}$, for any $t \geq 0$. Self loops in strong form are in particular useful in the dwell-time context, since the following result holds true.

Lemma 4. *Consider $\mathcal{I} = \{1,\dots,N\}$, $\mathcal{F} := \{f_j\}_{j\in\mathcal{I}} \subset \mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^n)$, $\tau > 0$, any $\rho \in \mathcal{PD}$, a discrete set S and a vector-valued candidate Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}^{|S|}$. For any $j \in \mathcal{I}$ and any $a,b \in S$ we have the following implications*

$$(a,b,j)_\tau \in E \wedge (b,b,j)_{st} \in E \Rightarrow (a,b,j)_\tau^{dw} \in E^{dw} \tag{24a}$$

$$(a,b,j)_\tau \in E \wedge (a,a,j)_{st} \in E \Rightarrow (a,b,j)_\tau^{dw} \in E^{dw} \tag{24b}$$

Proof. For proving (24a), consider any $t \in [\tau, 2\tau]$, and any $x \in \mathbb{R}^n$ we have

$$\begin{aligned} &V_b(\phi_j(t,x)) - V_a(x) \\ &= V_b(\phi_j(t,x)) - V_b(\phi_j(\tau,x)) + V_b(\phi_j(\tau,x)) - V_a(x) \\ &\leq 0 - \rho(|x|) = -\rho(|x|), \end{aligned}$$

since $t \geq \tau$ and $(a,b,j)_\tau \in E$ and $(b,b,j)_{\rm st} \in E$. Similarly, for (24b), for any $t \in [\tau, 2\tau]$, and any $x \in \mathbb{R}^n$ we have $V_b(\phi_j(t,x)) - V_a(x) = V_b(\phi_j(t,x)) - V_a(\phi_j(t-\tau,x)) + V_a(\phi_j(t-\tau,x)) - V_a(x) \leq -\rho(|x|) + 0 = -\rho(|x|)$, since $t \geq \tau$ and $(a,b,j)_\tau \in E$ and $(a,a,j)_{\rm st} \in E$. □

Intuitively, Lemma 4 states that if we ensure that $(a,b,j)_\tau \in E$ and at the departing or arrival node there is a strong self loop with the same label $j \in \mathcal{I}$, we know that also the condition encoded in $(a,b,j)_\tau^{\rm dw}$ is satisfied. This lemma allows us to see how several results proposed in the past could be recovered in a path-complete graph framework.

Remark 1 (Literature Review). *In the linear case (i.e. $f_j(x) \equiv A_j x$, with $A_j \in \mathbb{R}^{n\times n}$ for all $j \in \mathcal{I} := \{1,\dots,N\}$), it is well known that, if all the matrices $A_1,\dots,A_N$ are Hurwitz, there exists a (large enough) dwell time $\tau > 0$ for which the switched system* (2) *is asymptotically stable on $\mathcal{S}_{dw}(\tau)$, see (Morse, 1996, Lemma 2). A widely studied problem is the numerical approximation of the minimal dwell time $\tau_{dw} \in \mathbb{R}_+$ for which this holds. In (Geromel and Colaneri, 2006) a Lyapunov construction is proposed, starting from a family of quadratics, in order to compute upper-bounds on τ_{dw}. The proposed conditions require the existence of $P_1,\dots,P_N > 0$ such that*

$$\begin{aligned} &P_j A_j + A_j^\top P_j < 0, \;\; \forall\, j \in \mathcal{I} \\ &e^{A_j^\top \tau} P_{j'} e^{A_j \tau} - P_j < 0, \;\; \forall\, j \neq j' \in \mathcal{I}. \end{aligned}$$

It easy to see that these conditions exactly correspond to the ones obtained considering entrywise quadratic *candidate vector-valued functions $V : \mathbb{R}^n \to \mathbb{R}^{|S_1|}$ and a graph $\mathcal{G}_1 = (S_1, E_1)$, defined by $S_1 \equiv \mathcal{I}$ and*

$$\begin{aligned} &(j,j,j)_{st} \in E_1 \quad \forall\, j \in \mathcal{I}, \\ &(j,j',j)_\tau \in E_1 \quad \forall\, j \neq j' \in \mathcal{I}, \end{aligned} \tag{25}$$

Roughly speaking, E contains all the possible N strong self loops, and each node $j \in S_1 = \mathcal{I}$ has $N-1$ outgoing edges of the form $(j,j',j)_\tau \in E$, for any $j' \neq j \in \mathcal{I}$, see Figure 1 for a graphical representation. It is clear that $\mathcal{G}_1$ is path-complete; the graph $\mathcal{G}_1$ is referred to as the De Bruijn graph of dimension 1, see (Ahmadi et al., 2014, Section 6) and references therein. Recalling Lemma 4, this result can be seen as a specification of Theorem 2 for a particular graph-structure. The dual version of these conditions (i.e. considering again entrywise quadratic vector valued function, but imposing the conditions arising considering $\mathcal{G}_1' = (S_1, E_1')$) have been proposed in (Allerhand and Shaked, 2011) and (Xiang, 2015). Converse Lyapunov results for dwell time stability in the linear context are presented in (Wirth, 2005; Chitour et al., 2021); interestingly, also these results could be "translated" in a path-complete formalism. In fact (Wirth, 2005, Corollary 6.5) states that, if the switching system (2) *with linear sub-dynamics defined by $\mathcal{A} = \{A_1,\dots,A_N\}$ is asymptotically stable on $\mathcal{S}_{dw}(\tau)$, then there exist N* norms, *$v_1,\dots v_N : \mathbb{R}^n \to \mathbb{R}$ for which the condition encoded in the* dual graph *$\mathcal{G}_1' = (S_1, E_1')$ are satisfied, when the strong self-loops condition encoded* (22) *hold. A similar converse result (with the same graph-structure) is proved in (Blanchini and Colaneri, 2010), considering polyhedral norms (norms with polyhedral sub-level sets), while in (Chesi et al., 2012), the idea is to reach non-conservative conditions, approaching these polyhedral norms using homogeneous polynomial functions, via LMI-based conditions. In (Chitour et al., 2021), a graph-structure is used to model a class of dwell-time switching systems, and then a converse Lyapunov result is proposed, together with an algorithm to construct polyhedral norms satisfying the conditions encoded in $\mathcal{G}_1'$. In (Allerhand and Shaked, 2011), the authors proposed conditions equivalent to the ones obtained in (Geromel and Colaneri, 2006), using a splitting technique similar to the one presented in Lemma 2. For a comparison of the numerical complexity of these approaches, we refer to (Briat, 2015). Rephrasing, we have seen that a large class of existing results can be seen as particular instances of Theorem 2 for a fixed graph ($\mathcal{G}_1$, or its dual $\mathcal{G}_1'$) and for various kind of candidate functions templates (quadratics functions, SOS polynomials, polyhedral norms, etc.). In our work,* *instead, the graph structure is a parameter that the user has to choose/fix a-priori, leading to several sets of stability criteria whose conservatism depends on the chosen graph.*

As we said in Remark 1, several results ensuring dwell-time stability in the literature are presented using the graph $\mathcal{G}_1 = (S_1, E_1)$ (depicted in Fig. 1), or its dual $\mathcal{G}_1' = (S_1, E_1')$. In many of these works (for example (Xiang, 2015), (Chesi et al., 2012), (Allerhand and Shaked, 2011)), it is claimed, without formal proof, that conditions relying on $\mathcal{G}_1$ are equivalent, in term of conservatism, to the ones relying on $\mathcal{G}_1'$, when the chosen family of functions is the same (in the cited cases quadratic functions and (polyhedral) norms). In the following result we formalize this idea in a general setting.

Lemma 5. *Consider $\mathcal{F} = \{f_1, \dots, f_N\} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ such that each $f_j \in \mathcal{F}$ is homogeneous of degree 1. The conditions encoded in $\mathcal{G}_1 = (S_1, E_1)$ (with strong self-loops) are satisfied for an $\rho \in \mathcal{PD}$ by an homogeneous of degree $q \geq 1$ vector-valued function $V : \mathbb{R}^n \to \mathbb{R}^{|S_1|}$ if and only if the conditions encoded in $\mathcal{G}_1' = (S_1, E_1')$ (with strong self-loops) are satisfied for a $\rho' \in \mathcal{PD}$ by an homogeneous of degree q vector-valued function $V' : \mathbb{R}^n \to \mathbb{R}^{|S_1|}$.*

Sketch of the Proof. Suppose that conditions encoded in $\mathcal{G}_1$ (with strong self-loops) are satisfied by a homogeneous of degree $q \geq 1$ vector-valued function $V : \mathbb{R}^n \to \mathbb{R}^{|S_1|}$. Given any $j \in \mathcal{I}$, let us define $V_j'(x) := V_j(\phi_j(-\tau, x))$. The functions V_j' are homogeneous of degree q, since $V_j'(\lambda x) = V_j(\phi_j(-\tau, \lambda x)) = V_j(\lambda \phi_j(-\tau, x)) = \lambda^q V_j(\phi_j(-\tau, x)) = \lambda^q V_j'(x)$ for any $x \in \mathbb{R}^n$, $\lambda \geq 0$, and they satisfy the conditions encoded in $\mathcal{G}_1'$ (with strong self-loops). □

A corollary of this lemma can be obtained considering entrywise quadratic functions and linear sub-vector fields ($f_j(x) \equiv A_j x$ for some $A_j \in \mathbb{R}^{n \times n}$, for any $j \in \mathcal{I}$), since the V_j' defined in the proof of Lemma 5 would be, again, quadratics, for any $j \in \mathcal{I}$. We want to underline how the "equivalence via duality" proved in Lemma 5 for the graph $\mathcal{G}_1$, does not hold for a generic path-complete graph.

5. Linear Case and Numerical Results

Let us consider $\mathcal{A} = \{A_1, \dots, A_K\} \subset \mathbb{R}^{n \times n}$, we define the *linear* switched system, as

$$\dot{x}(t) = A_{\sigma(t)} x(t), \tag{26}$$

where the signals σ are again selected in (a subclass of) $\mathcal{S}$, see (3). In this section we focus on entrywise quadratics vector-valued functions, that is, maps $V : \mathbb{R}^n \to \mathbb{R}^{|S|}$, for some finite set S, such that, for any $\ell \in S$, we have that $V_\ell(x) = x^\top P_\ell x$, for some $P_\ell \in \mathbb{R}^{n \times n}$, $P_\ell > 0$. In this setting, the conditions encoded in edges are translated into LMIs. Considering entrywise quadratics vector-valued functions and the system (26), conditions $(a, b, j)_\tau \in E$ as defined in (7), become

$$e^{A_j^\top \tau} P_b e^{A_j \tau} - P_a < 0, \tag{27}$$

which, applying Theorem 1, is equivalent to the existence a large enough $K \in N$ and K positive definite matrices $P_0, \dots, P_K$, with $P_0 = P_a$, $P_K = P_b$ such that

$$\begin{cases} P_k A_j + A_j^\top P_k - \frac{K}{\tau}(P_k - P_{k-1}) < 0, \\ P_{k-1} A_j + A_j^\top P_{k-1} - \frac{K}{\tau}(P_k - P_{k-1}) < 0, \end{cases} \tag{28}$$

for all $k \in \{1, \dots, K\}$. Then, considering the conditions encoded in the dwell time edge $(a, b, j)_\tau^{\mathrm{dw}} \in E^{\mathrm{dw}}$ as defined in (19), is translated into condition

$$e^{A_j^\top t} P_b e^{A_j t} - P_a < 0, \quad \forall\, t \in [\tau, 2\tau], \tag{29}$$

which is implied, as proved in Lemma 3, by the existence of $P_0, \dots, P_K > 0$, with $P_0 = P_a$, $P_K = P_b$ such that

$$\begin{cases} P_k A_j + A_j^\top P_k & -\frac{K}{s}(P_k - P_{k-1}) < 0, \\ P_{k-1} A_j + A_j^\top P_{k-1} & -\frac{K}{s}(P_k - P_{k-1}) < 0, \end{cases} \tag{30}$$

for any $s \in \{\tau, 2\tau\}$, for all $k \in \{1, \dots, K\}$. Finally, the strong self-loop condition $(a, a, j)_{\mathrm{st}}$ as defined in (23) can be ensured, simply verifying the "classical" LMI: $P_a A_j + A_j^\top P_a < 0$.

Remark 2 (Choice of the Graph and Numerical Complexity)**.** *The sufficient conditions proposed in Theorem 2 depend on the choice of the path-complete graph $\mathcal{G}$, whose structure induces the Lyapunov inequalities to be checked. In related discrete-time switching systems literature, several hierarchies of path-complete graphs have been proposed in order to reduce the conservatism of path-complete stability certificates, as for example the De Bruijn graphs in (Ahmadi et al., 2014, Remark 6.1), or several abstract* lifts *in (Philippe et al., 2016, Section 3). These techniques can be straightforwardly adapted in the continuous time setting. Of course, these methods require to increase the "size" of the graph $\mathcal{G}$ (i.e. the number of edges) in order to have less restrictive conditions, and thus increasing the number of inequalities. This is consistent with the fact that existing converse Lyapunov results (Wirth (2005); Chesi et al. (2012); Chitour et al. (2021)) state the existence of polyhedral Lyapunov norms/SOS-Lyapunov functions, and thus requiring an unbounded number of vertices/ unbounded degree (respectively). In the linear case, our method, once fixed a graph $\mathcal{G} = (S, E)$, requires to solve $2|E|$ LMIs (as in* (29)*) with $|S|$ decision variables. As observed in Lemma 4, if the self-loops are imposed to hold in the strong sense* (23)*, the number of inequalities can be reduced. Moreover, to facilitate the use in uncertain setting, we adopt convex conditions in the matrices A_j, without resorting to matrix exponentials $e^{A_j \tau}$, fixing an integer $K \in \mathbb{N}$ (a "relaxation degree") and then check the K-refinement conditions in* (30)*, i.e. checking the feasibility $2K|E|$ LMIs (in general). To provide upper bounds on the minimal dwell-time, the parameter τ can be minimized via a bisection technique.*

Concluding this section, we underline how, for a large class of non-linear systems, the path-dependent Lyapunov conditions can be algorithmically checked adapting the SOS-techniques proposed in (Prajna et al., 2002) and references therein. For the sake of readability we do not analyze further this case, presenting, in what follows, a benchmark-example in the linear setting, in order to compare our results with existing ones.

5.1. Numerical Example

We study a linear switched system as in (26), which was already introduced in the literature Allerhand and Shaked (2011). We aim to estimate (as precisely as possible) $\tau_{dw} \in \mathbb{R}_+$, the *minimal dwell time* for which the system is UGAS on the classes $\mathcal{S}_{\mathrm{dw}}(\tau)$, for all $\tau \geq \tau_{dw}$, using the ideas presented in the previous sections.

Example 1. Consider the following example, introduced in (Allerhand and Shaked, 2011, Example 1), of system (26) with $\mathcal{A} = \{A_1, A_2\} \subset \mathbb{R}^{3\times 3}$ defined by

$$A_1 = \begin{bmatrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 0 & 6 \\ -1 & -1 & -5 \\ 0 & 1 & -1 \end{bmatrix}$$

In (Allerhand and Shaked, 2011) the authors establish that the system is stable under the class of switching $\mathcal{S}_{dw}(\tau)$ with $\tau = 0.4$, firstly using the ideas proposed in (Geromel and Colaneri, 2006), computing the exponential matrices $e^{A_i t}$ for $i = 1, 2$ and then checking the inequalities encoded in the graph $\mathcal{G}_1$ (on the alphabet $\{1, 2\}$), as presented in Remark 1. The same value is then obtained without computing the exponential matrices, relying on matrix inequalities as in (28). In our preliminary work (Della Rossa et al., 2020), we consider the graph $\mathcal{G}_1$, obtaining the same upper bound $\tau = 0.4$ (the numerical details are reported in Table 1).
Here, instead, we consider a De Bruijn graph of dimension 2 (see (Ahmadi et al., 2014, Remark 6.1) for the definition of this family of graphs), denoted by $\mathcal{G}_2$ and depicted in Figure 2, with which, at the cost of an increased complexity, we aim to reduce the conservativeness of the conditions, i.e. to provide a smaller upper bound for the minimal dwell-time τ. We used the solver SDPT3 (Toh et al., 1999), which allowed us to determine an upper bound on the dwell-time $\tau = 0.21$, thus improving the previously known bound of $\tau = 0.4$. Tables 1 and 2 show the results obtained by applying Theorem 2, using different splitting parameters K, when either the De Bruijn graph of dimension 1 or 2 are considered. To better detail the computational complexity of the method, the number of LMIs and semidefinite matrix decision variables are shown in each case. It is noted that, while at the beginning a slightly increase in the splitting coefficient K leads to a notably better estimation of τ, at some point increasing K – even significantly – does not lead to a marked improvement of the dwell time upper bound estimation (the condition with $\tau = 0.20$, for $\mathcal{G}_2$, is still infeasible with $K = 500$). To conclude, Table 3 compares the best results obtained with the different methods and graphs, both using our method or the ones already presented in literature, and recalled above.

τ	K	LMIs	Vars
1	3	18	8
0.6	5	26	12
0.5	8	38	18
0.45	20	86	42
0.4	**125**	**506**	**252**

Table 1: Dwell-time upper-bounds, obtained applying Theorem 2 and Lemma 3, using a De Bruijn graph of dimension 1.

τ	K	LMIs	Vars
1.45	3	82	22
0.9	5	122	34
0.25	8	182	52
0.22	20	422	124
0.21	**125**	**2522**	**754**

Table 2: Dwell-time upper-bounds, obtained applying Theorem 2 and Lemma 3, using a De Bruijn graph of dimension 2.

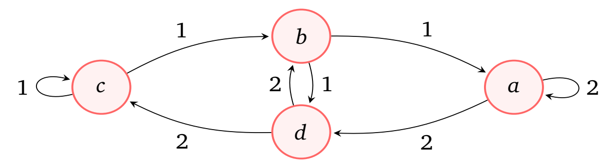

Figure 2: The graph $\mathcal{G}_2$, the De Bruijn graph of dimension 2 on $\mathcal{I} = \{1, 2\}$ used in Example 1.

6. Conclusion

In this paper we investigated how the *path-complete* graphs formalism, originally proposed for discrete-time switched systems, can be adapted and used in a continuous-time setting. We studied the relations of our approach and the existing literature, with particular attention to the problem of estimation of the minimal dwell time, when considering linear subdynamics. Possible open questions for further research are the case of uncertain subsystems and the investigation of a hierarchy or a “quantitative comparison" between conditions arising from different path-complete graphs.

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	Thm. 2 and Lem. 3 – $\mathscr{G}_1$	**Thm. 2 and Lem. 3 – $\mathscr{G}_2$**	(Allerhand and Shaked, 2011)	(Geromel and Colaneri, 2006)
τ	0.4	**0.21**	0.4	0.4
LMIs	506	**2522**	384	4
Vars	252	**754**	192	2
E.M.	No	**No**	No	Yes

Table 3: Comparison of different dwell-time upper-bounds obtained through our method, using a De Bruijn graph of dimension 1 or 2, the method from Allerhand and Shaked (2011), the one from (Geromel and Colaneri, 2006). The last line denotes the necessity of computing matrix exponentials.

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