The statistical mechanics of innovation diffusion

Annals of Operations Research
https://doi.org/10.1007/s10479-025-06894-0

ORIGINAL RESEARCH

The statistical mechanics of innovation diffusion

Giacomo Masali[1] · Maria Elena Bruni[2] · Guido Perboli[1,3]

Received: 31 December 2024 / Accepted: 6 October 2025

Abstract

This paper proposes a novel agent-based model for new-product diffusion, grounded in the kinetic theory of statistical mechanics. The model describes a population of utility-driven agents whose adoption decisions emerge from belief dynamics shaped by peer influence, advertising, and stochastic factors. By adapting the Boltzmann equation, a closed-form expression for the adoption curve is derived as an emergent property of decentralized, compromise-based interactions. Unlike aggregate models such as the Generalized Bass Model or the Bass Logit Diffusion Model, the proposed Kinetic Innovation Diffusion (KID) model links micro-level behavioral rules to macro-level adoption dynamics without relying on top-down assumptions. Empirical validation on benchmark products—including color televisions, air conditioners, clothes dryers, and freezers—demonstrates that the KID model outperforms existing methods in both fit and early-stage forecast accuracy. Thanks to its tractable structure, the model enables estimation of key strategic quantities from minimal data, making it a valuable tool for pre-launch planning and early diffusion monitoring. Its flexibility also supports extensions to incorporate heterogeneity, abandonment, or strategic firm behavior—offering a unified, analytically grounded framework for innovation diffusion.

Keywords Agent-based model · Innovation diffusion · Statistical mechanics · New-product

✉ Giacomo Masali
giacomo.masali@polito.it

Maria Elena Bruni
mariaelena.bruni@unical.it

Guido Perboli
guido.perboli@polito.it

[1] DIGEP, Politecnico di Torino, Turin, Italy

[2] DIMEG, Università della Calabria, Rende, Italy

[3] Arisk srl, Milan, Italy

Published online: 27 October 2025

1 Introduction

How innovations diffuse in the population is object of investigation in a broad literature, spanning from sociology to economics and management science (see Young, 2009, for a discussion of the structural differences between these approaches). In the operations research literature, to which the authors primarily refer, the Bass model (Bass, 1969) remains the most widely adopted framework for modeling new product diffusion. It counts hundreds of successful applications ranging from satellite TV to smartphones (Bass, 2004), from medical technologies (Simon & Sebastian, 1987) to commerce platforms (Li et al., 2020). At its core, the Bass Model (BM, henceforth) divides buyers into two categories: *innovators* and *imitators*. Innovators purchase the product independently of others, typically driven by intrinsic interest or exposure to advertising. These people are pivotal in the early stages of the diffusion process, since their purchase does not depend on previous sales of the product. Imitators buy the product under social pressure—whether framed as word-of-mouth, contagion, or internal influence. This pressure grows with the number of buyers, becoming more and more dominant as time goes by. Being the population limited, the process then slows down when approaching the saturation point.

The BM is an aggregate model: it starts from the observed macro-pattern (typically an S-shaped adoption curve) and models the phenomenon accordingly. Aggregate models are favored by managers and decision-makers for their simplicity and interpretability. They provide a compact, analytical summary of sales dynamics and can be estimated from readily available aggregate data. However, they rely on strong assumptions on the underlying behavioral processes, and they are less equipped to capture heterogeneity or psychological nuance at the individual level. Despite the descriptive power of the BM and its many variants, it remains an aggregate-level approach, often disconnected from individual-level behaviors. This work aims to fill this gap by deriving an aggregate adoption curve from micro-level rules, grounded in agent interaction dynamics.

Agent-based (AB) models are based on a bottom-up approach that can overcome some of the aforementioned shortcomings. In AB modeling, no assumptions are made on the whole system: individual-level rules are set for the agents and the population-level behavior emerges from the interactions of those agents. This makes it possible to model differences in behavior, preferences, and psychological traits. However, most AB models in the literature are simulation-based, which limits their analytical tractability. They typically simulate a population of agents over time and track adoption at each step, which can require significant computational resources. Moreover, calibration can be difficult because the available data is often aggregate, while the model parameters operate at the micro-level. This paper proposes an agent-based new-product diffusion model that combines the flexibility of agent-based models with the analytical clarity of aggregate models. This approach is based on a version of the Boltzmann equation, originally developed in kinetic gas theory, but adapted to describe social interactions. The idea is to draw a parallelism between the molecules interacting in a gas and the people interacting in society, in order to be able to use the same mathematical formalism. When molecules collide, they exchange kinetic energy following the laws of mechanics and this modifies their position and velocity, i.e. their *microscopic state*. In this model, the microscopic state of the agents is set to be the perceived value of the new product. When people interact with each other (or with the advertisement campaign), they exchange information and change their perceived value. This work will assume people follow a *compromise dynamics*: when they interact, they move partially toward each other's beliefs.

This modeling approach offers several advantages:

- It adopts a bottom-up perspective, linking aggregate outcomes to micro-level behavioral rules.
- It performs competitively against state-of-the-art extensions of the Bass model (Cosguner & Seetharaman, 2022).
- Its parameters are directly interpretable in terms of marketing levers (e.g., price, advertising intensity), making it useful for managerial decision-making.
- It provides a flexible framework: by adjusting the agent-level rules, one can model repeat purchases, abandonment, segmentation, and more—without needing to assume a specific functional form for the aggregate adoption curve.

The paper is organized as follows. The next section contains a review of the literature on on Bass-type aggregate models, agent-based models, and the use of kinetic theory in social systems. Section 3 describes the Boltzmann equation and the kinetic framework that are the foundations of this model. Later, Sect. 4 presents the kinetic diffusion model and Sect. 5 evaluates its performance. Section 6 concludes the paper.

2 Literature review

This section provides a focused review of the innovation diffusion literature in operations research. For broader overviews in sociology and economics, see Rogers (2003) and Hall (2004), respectively. The success and wide range of applications of the BM are clear if one looks at the number of extensions developed to address a range of challenges marketers face when launching new products. These include, but are not limited to, product substitution, pricing, and advertisement calibration. For a complete review of the literature on new-product diffusion models up to 2010 see Peres et al. (2010).

The first adaptation of the BM to product substitution appeared in Norton and Bass (1987) and was later refined in several improved versions (Jiang & Jain, 2012; Singh et al., 2012). The problem of dynamic pricing for new products has been studied both within and outside the BM framework– see, for instance, Bass and Bultez (1982), Kalish and Lilien (1983), Dockner and Jørgensen (1988). Because the original BM lacks explicit marketing levers, it was extended by Bass et al. (1994) into the Generalized Bass Model (GBM), which incorporates price and advertising as part of the marketing mix. Since then, many versions of the GBM have appeared to address the various decision-making needs, particularly around pricing and advertising. Important contributions in this area include Robinson and Lakhani (1975), Krishnan et al. (1999) and Krishnan and Jain (2006). Other aspects of the diffusion process that have been studied inside the BM framework are its dependence on opinion distribution in a population (Fan et al., 2017; Han et al., 2022) and supply restrictions (Ho et al., 2002; Kumar & Swaminathan, 2003; Shen et al., 2011). For the sake of completeness, it should be stressed that both these topics have been extensively studied outside the BM framework as well, see as examples Assenova (2018) and Iyengar et al. (2011) for opinion-based models and Keith et al. (2017) for supply constraints. Some authors have criticized the GBM as a prescriptive model for marketing strategy (Bass et al., 2000; Fruchter & Van den Bulte, 2011). The recent work by Cosguner and Seetharaman (2022) overcomes some of these limitations using a utility-based approach.

Because the BM is an aggregate model, it requires strong assumptions about system behavior. This has motivated efforts to disaggregate the diffusion process, by modeling individual-level adoption and deriving the resulting aggregate patterns. These efforts went through different paths: Sinha and Chandrashekaran (1992) used hazard models to incorporate different timing in adoption for different buyers; a spatial dimension is taken into account in the works by Bronnenberg and Mela (2004) and Bell and Song (2007) while the work by Li et al. (2014) is based on network theory. This approach falls under the broader class of agent-based (AB) models, which trace back to Goldenberg et al. (2000). Aggregate models and AB models represent two opposite approaches to the same problem and can be used in parallel to satisfy different needs (Rahmandad & Sterman, 2008). The vast majority of AB models for new-product diffusion are simulation-based, built to explore the diffusion of specific products or behaviors in particular market contexts. Notable applications include Palmer et al. (2015) on private photovoltaic in Italy; Stummer et al. (2015) on repeat purchase modeling; Bastani et al. (2016) on energy-savings behaviors, Zhang et al. (2022) on photovoltaic adoption in Singapore; Rotaris and Scorrano (2023) on car-sharing diffusion in an Italian region. Important exceptions to simulation-based AB modeling exist. Fibich and Gibori (2010), for example, aggregate agent dynamics on a network to derive analytical expressions for product diffusion. Another promising direction is the mean-field game (MFG) approach,which leverages game theory to model strategic multi-agent behavior. In this stream of literature, Chenavaz et al. (2021) study price optimization in a multi-firm new-product diffusion process; while Chaab et al. (2022) imagine a Stackelberg game between a firm and a large number of strategic consumers and draw the diffusion dynamics from the mean-field approximated Nash equilibrium.

In this paper, an alternative kinetic approach is adopted, grounded in the mathematical formalism of statistical mechanics. Inspired by the kinetic theory of gases, this approach models the population as a system of interacting agents whose beliefs and behaviors evolve over time–analogous to molecules exchanging energy upon collision. By treating interactions as belief-updating events, a closed-form differential equations for aggregate quantities is derived from individual-level rules. For a comprehensive introduction to this methodology, see Pareschi and Toscani (2013). In recent years, the kinetic framework has been successfully employed to describe a wide variety of social and biological phenomena. Of particular interest for this paper are the applications regarding opinion formation (Düring & Wright, 2022; Toscani, 2006) and epidemiology (Della Marca et al., 2021; Loy & Tosin, 2021).

3 Kinetic framework

In this section, the classical mathematical framework of the Boltzmann equation is reinterpreted from the perspective of operations research and economics. The goal is to model how individual beliefs about the value of a new product evolve due to interactions and information flow. Inspired by kinetic theory, which studies how microscopic interactions in gases lead to macroscopic behaviors, the present study borrows the structure of those models and applies it to agent-based settings where individuals update their beliefs through social and media interactions.

Rather than modeling particles in a gas, agents are viewed as consumers or market participants who revise their evaluation of a product over time. This revision is influenced by peer interactions (e.g., word-of-mouth), background signals (e.g., advertising), and stochastic fluctuations that represent individual differences or unmodeled factors.

3.1 The Boltzmann equation

Let $f(t, v) \geq 0$ denote the distribution of agents over perceived product values v at time t. The function f can be interpreted as a probability density over a continuum of belief states in the market, in the sense that the quantity $f(t, v)$ indicates the number of agents with perceived value $v \in [v, v + dv]$ at time t. Its evolution is modeled through an interaction-based dynamic equation inspired by models of collective behavior, namely the Boltzmann equation (Boltzmann, 1872), which, in this context, captures the effect of interactions on agents' beliefs:

$$\frac{\partial f(t, v)}{\partial t} = \mu Q(f, f)(t, v). \tag{1}$$

Where $Q(f, f)(t, v)$ represents the effect of pairwise interactions on the belief distribution, and μ is the frequency of the interactions. The evolution of f is driven by two components:

- Peer influence: when two agents interact, they influence each other's perception of the product.
- External signals: agents may also adjust their perception based on external campaigns or real-world experience.

An important feature of the interaction mechanism is that it preserves the total number of agents. This implies:

$$\int_V Q(f, f)(t, v) dv = 0 \quad \forall t. \tag{2}$$

This property allows to give the density $f(t, v)$ a probabilistic sense. Starting with a density $f(0, v)$ such that $\int_V f(0, v) dv = 1$, this will remain valid for all subsequent times. Hence

$$\int_V f(t, v) dv = 1 \quad \forall t > 0. \tag{3}$$

Using this, it is possible to compute expectations of relevant market quantities. Let $\varphi(v)$ denote any measurable characteristic of an agent as a function of their valuation. Then the average value of $\varphi(v)$ at time t is given by:

$$\langle \varphi(v) \rangle = \int_{\mathbb{R}} \varphi(v) f(t, v) dv, \tag{4}$$

where the dependency on time t is omitted to lighten the notation. Two specific statistics will be central in this analysis:

- $\rho(t) = \int f(t, v)dv$: the total number of agents in the population,
- $m(t) = \int v \cdot f(t, v)dv$: the average perceived value of the product across the population.

Note that it is not trivial to compute the number of agents because a label switch mechanism will be employed: while the whole population remains constant, the population associated with the different labels will change. For a comprehensive discussion on the Boltzmann equation see Cercignani (1988).

3.2 Binary interactions

The interactions correspond to every moment in which value v may change due to external inputs, namely word-of-mouth, advertising or actual usage. When two agents interact, their perceived values are assumed to shift toward a compromise. This mechanism reflects bounded rationality: agents cannot compute optimal decisions in isolation, so they rely on peer comparisons. Mathematically, if v and v_* represent the pre-interaction valuations, the post-interaction value is:

$$\begin{aligned} v^{post} &= I(v, v_*) \\ v_*^{post} &= I(v_*, v), \end{aligned} \tag{5}$$

where $I(\cdot, \cdot)$ is the interaction function encoding how beliefs are exchanged or adjusted during interaction. It will be symmetric and stochastic. To capture the dynamics of this belief-updating process at the population level, the effect of a small time interval Δt, during which interactions may or may not occur, is considered. A binary interaction is assumed to take place with probability $\mu\Delta t$, where μ represents the interaction rate between the agents. This is modeled using a Bernoulli random variable, independent of the specific content of the interaction. Under this setup, an agent's valuation is updated with probability $\mu\Delta t$ and left unchanged otherwise:

$$v' = I(v, v_*) \cdot \mu\Delta t + v \cdot (1 - \mu\Delta t), \tag{6}$$

and hence

$$\langle\varphi(v')\rangle = \langle\varphi(I(v, v_*))\rangle \cdot \mu\Delta t + \langle\varphi(v)\rangle(1 - \mu\Delta t). \tag{7}$$

Re-organizing the members and computing the limit for small time intervals it is easy to prove the following equality (Pareschi & Toscani, 2013):

$$\frac{d}{dt}\langle\varphi(v)\rangle = \mu(\langle\varphi(v')\rangle - \langle\varphi(v)\rangle). \tag{8}$$

By making the average explicit through Eq. (4) one obtains

$$\frac{d}{dt}\int_V \varphi(v)f(t,v)dv = \mu\left\langle \iint_{V\times V_*} (\varphi(v') - \varphi(v))f(t,v)f(t,v_*)dvdv_* \right\rangle, \tag{9}$$

where statistical independence between the interacting agents is assumed–i.e., the joint distribution $f^{(2)}(t,v,v_*)$ factorizes as $f(t,v)f(t,v_*)$.

To reflect heterogeneity in information processing and judgment, a stochastic noise term is introduced in the interaction function $I(\cdot,\cdot)$. This captures private signals, unobservable traits, and random deviations from strict imitation or compromise. With a change in the notation, $\langle\cdot\rangle$ indicates now the expected value over the stochastic interaction function.

This represents the weak form of the Boltzmann equation. The meaning of this formula is that the variation in time of the average of an observable quantity in an interacting system is equal to the mean variation of that quantity in the interactions.

3.2.1 Interaction with a background

Agents are not influenced only by peers, they are also exposed to marketing signals. This is modeled using a background distribution $G(v)$, representing the message conveyed by an advertising campaign. The strength and content of the campaign are encoded by the weight and position of peaks in $G(v)$.

$$G(v) = \sum_{i=0}^{n} w_i \delta(v - v_i), \tag{10}$$

where w_i (such that $\sum_{i=0}^{n} w_i = 1$) are the weights of information and $\delta(\cdot)$ is the Dirac delta function (Hassani & Hassani, 2009). This background influence can be incorporated in the same interaction formalism, treating it as a one-sided interaction with a fixed signal.:

$$\frac{d}{dt}\int_V \varphi(v)f(t,v)dv = \mu\left\langle \iint_{V\times V_*} (\varphi(v') - \varphi(v))f(t,v)G(v_*)dvdv_* \right\rangle. \tag{11}$$

Although $G(v)$ is assumed to be fixed, it can be generalized to a time-dependent function to reflect dynamic advertising strategies.

3.3 Label switch process

To model the decision to adopt or not adopt the innovation, the population is divided in two groups: *users* and *non-users*. Agents transition from non-user to user status when their perceived value exceeds the product price. This is formalized through a Markovian label-switching process similar to Loy and Tosin (2020), where the transition probability depends on individual valuation. Each agent is equipped with a label $x \in \mathcal{I} = 1, 2$ and a transition event is defined as an event where a purchase may occur. This will happen according to frequency λ. If such an event occurs, the transition probability from one label to the other will be:

$$T(v; x|y) \in [0,1] \quad \forall v \in V, x, y \in \mathcal{I}. \tag{12}$$

A Markov process is defined where the transition probability is belief-dependent. To ensure the process is well-defined and the total population remains constant, it is required that:

$$\sum_x T(v; x|y) = 1 \quad \forall v \in \ V, y \in \mathcal{I}. \tag{13}$$

By plugging this into the kinetic framework for a distribution $f(t, x, v)$ of agents with label x, one obtains:

$$\sum_{i=1}^{n} \varphi(i) \frac{d}{dt} f_i(t) = \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} (\varphi(i) - \varphi(j)) T(i|j) f_j(t), \tag{14}$$

where the subscript indicates the distribution restricted to some label. By choosing $\varphi(i) = 1$ for $i \in \mathcal{I}$ and $\varphi(j) = 0 \ \ \forall j \in \mathcal{I} \setminus i$ one is counting agents that belong to group i. Thus, it is possible compute the evolution of density for group i as

$$\frac{d}{dt} f_i(t) = \lambda \left(\sum_{j=1}^{n} T(i|j) f_j(t) - f_i(t) \right) \quad i = 1, \ldots, n. \tag{15}$$

Combining this label-switching mechanism with the belief-updating interactions described earlier leads to a joint evolution of agents' beliefs and their adoption status. Taking distribution function $f(t, x, v)$ yields, from mass conservation:

$$\sum_{i=1}^{n} \int_V f_i(t, v) dv = 1 \quad \forall t > 0. \tag{16}$$

The addends of the sum are the masses of the agents with label i:

$$\rho_i(t) := \int_V f_i(t, v) dv. \tag{17}$$

By considering the label switch process and the interaction process (both with other agents and with the background) happening in parallel with frequencies λ and μ, respectively, and with a time interval Δt small enough to neglect the second order contributions one obtains a non-conservative weak Boltzmann equation for a single group i of the form:

$$\begin{aligned} \frac{d}{dt} \int_V \varphi(v) f_i(t, v) dv =& \lambda \int_V \varphi(v) \left(\sum_{j=1}^{n} T(v; i|j) f_j(t, v) - f_i(t, v) \right) dv \\ &+ \mu \sum_{j=1}^{n} \iiint_{V^3} \langle \varphi(v') - \varphi(v) \rangle f_i(t, v) f_j(t, v_*) G(\hat{v}) dv dv_* d\hat{v} \end{aligned} \tag{18}$$

In this case, the interaction rule is the same for every agent, regardless the group it belongs to. From here, it is easy to describe a situation where the interaction rule depends on the label of the interacting agents by differentiating the cases in the second sum.

Of course, the mass conservation property holds only for the whole system and not for each group. Taking $\varphi(v) = 1$ in (18) allows to compute the evolution of the mass of agents with label *i* as

$$\frac{d}{dt}\rho_i(t) = \lambda \sum_{j=1}^{n} \int_V T(v;i|j) f_j(t,v)dv - \lambda\rho_i(t). \tag{19}$$

This approach connects the micro-level belief dynamics to the macro-level adoption curve: as more agents update their beliefs and find the product worthwhile, the adoption rate increases.

3.4 Time-scale separation

The description of the evolution of a system by means of the Boltzmann equation is tied to the microscopic quantities that form this equation. On the one hand, this means that it may not be best suited to understand the long-term behavior of the analyzed system. On the other hand, if two events occur on very different time scales, separated dynamics may better capture the phenomenon's essence. In this model, agents revise their beliefs about the product's value relatively frequently, whereas actual adoption decisions—once beliefs cross a threshold—occur less often. Capturing this time-scale separation is essential for modeling innovation diffusion in a realistic and tractable way.

This is addressed by distinguishing between two types of dynamics:

- Belief dynamics: how agents update their perceived value of the product through interaction and information flow.
- Adoption dynamics: how agents decide to switch from non-user to user status once their belief crosses a threshold.

Formally, a system with two types of interactions is considered: frequent updates to valuation (e.g., social influence, advertising), and infrequent decisions to adopt. Each of these processes contributes differently to the overall evolution of the system. Let $f(t, v)$ be the distribution of agents over valuations at time *t*. The full model includes two terms: one governing fast (frequent) interactions and another governing slow (infrequent) transitions:

$$\frac{d}{dt} f(t,v) = \lambda Q(f,f)(t,v) + \mu P(f,f)(t,v), \tag{20}$$

where $Q(f,f)$ describes belief updates, $P(f,f)$ describes label transitions such as adoption and $\lambda \sim 1$ and $\mu \sim \delta^{-1}$, $\delta \to 0$ are the respective frequencies, representing time-scale separation. Furthermore, a new time scale $\tau := \delta^{-1}t$ is defined and referred to as the fast time. The slow time will be *t*. The distribution function is now re-scaled as

$$\tilde{f}(\tau, v) = f(t, v). \tag{21}$$

Using this definition, Eq. (20) can be split into the two scales, given that the contribution of *P*(*f*, *f*) is of the same order as δ, as

$$\begin{aligned} D_\tau f(\tau, v) &= P(f, f)(\tau, v) \\ D_t f(t, v) &= Q(f, f)(t, v)\ . \end{aligned} \tag{22}$$

The interpretation of this is the following. The system evolves by the action of the fast process *P*(*f*, *f*) on the fast time τ toward its possible equilibrium configuration, undisturbed by the slow process *Q*(*f*, *f*). On the slow time scale *t* this configuration evolves by action of the slow process.

3.5 Quasi-invariant limit

It is difficult to use the Boltzmann equation to analytically evaluate the equilibrium distribution of a quantity. The quasi-invariant interactions limit allows to do so. By assuming that each binary interaction produces a small shift in the perceived value *v* and by re-scaling the time in such a way that one unit contains many of those interactions, one can study how the system behaves asymptotically. Mathematically, the binary interaction can be expressed as

$$v' = v + \varepsilon I(v, v_*) + \eta, \tag{23}$$

where η is a random variable of null mean and variance σ^2 and it is assumed that $\varepsilon, \sigma^2 \to 0^+$. A new time scale $\tau := \varepsilon t$ and a re-scaled distribution function $g(\tau, v) := f(\frac{\tau}{\varepsilon}, v)$ are then defined. Boltzmann equation now reads:

$$\frac{d}{d\tau} \int_V \varphi(v) g(\tau, v) dv = \frac{1}{\varepsilon} \left\langle \iint_{V \times V} (\varphi(v') - \varphi(v)) g(\tau, v) \cdot g(\tau, v_*) dv dv_* \right\rangle. \tag{24}$$

Quantity σ^2 is related to the contribution to the shift in the microscopic value of the stochastic fluctuations due to self-reasoning, personal preferences of the agents and other unaddressed influences. Quantity ε is related to the contribution to the shift in the microscopic value of the interactions with the outside world. The limit value is defined as $\frac{\sigma^2}{\varepsilon} \to \gamma > 0$ when $\varepsilon, \sigma^2 \to 0^+$. By taking the series expansion and performing the limit for $\varepsilon, \sigma^2 \to 0^+$, it is possible to write the so-called Fokker-Planck equation (Risken, 1996):

$$\partial_\tau g(\tau, v) = -\partial_v \Big(g(\tau, v) \int_V I(v, v_*) g(\tau, v_*) dv_* \Big) + \frac{\gamma}{2} \partial_v^2 g(\tau, v). \tag{25}$$

This equation will be used to investigate the asymptotic behavior of the distribution of the perceived value of the new product in the population.

4 Multi-agent bass model

It is assumed that the market consists of N agents. Each agent is either a *non-user* or a *user*. This status is encoded in a label $x \in \{1, 2\}$ where $x = 1$ corresponds to *non-user* and $x = 2$ to *user*. Every agent is also equipped with a microscopic state $v \in V \subseteq \mathbb{R}_+$ that represents its perceived value of the product. The weak Boltzmann equation (18) describes the evolution of the mass and mean perceived value in the population for the label-restricted distribution function $f_i(t, v)$. The equation is reported here, for clarity:

$$\begin{aligned} \frac{d}{dt}\int_V \varphi(v) f_i(t,v)dv =& \lambda \int_V \varphi(v)\left(\sum_{j=1}^{n} T(t,v;i|j) f_j(t,v) - f_i(t,v)\right) dv \\ &+ \mu \sum_{j=1}^{n} \iiint_{V^3} \langle \varphi(v_{i'}) - \varphi(v)\rangle f_i(t,v) f_j(t,v_*) G_i(\hat{v}) dv dv_* d\hat{v}. \end{aligned} \tag{26}$$

The first term in the right-hand side of the equation accounts for the changes in evaluation in group *i* due to purchases; the second term accounts for changes in the evaluation due to interactions between agents, i.e. word-of-mouth, or with a background $G_i(\hat{v})$, i.e. advertisement for *non-users* and actual usage of the product for *users*. First, focus on the evolution of group sizes. Equation (19) yields:

$$\frac{d}{dt}\rho_i(t) = \lambda \sum_{j=1}^{N} \int_V T(t,v;i|j) f_j(t,v)dv - \lambda \rho_i(t) \quad i = 1, 2. \tag{27}$$

In this work, agents are not allowed to abandon the innovation. The transition probability matrix thus becomes:

$$\begin{matrix} T(v;1|1) = 1 - H(v-P) & T(v;2|1) = H(v-P) \\ T(v;1|2) = 0 & T(v;2|2) = 1. \end{matrix} \tag{28}$$

Where P is the product price and $H(\cdot)$ is the Heaviside step function (Weisstein, 2002). The presence of the step function as a transition probability means that the agents behave as rational entities and as soon as their perceived value matches the real price of the product and a transition event occurs, they become *user*. The evolution of the mass in the two groups is therefore:

$$\frac{d}{dt}\rho_2(t) = \lambda \int_P^{+\infty} f_1(t,v)dv = -\frac{d}{dt}\rho_1(t). \tag{29}$$

This functional form of the differential equation governing the diffusion is different from those appearing in Young (2009), where contagion, social influence and social learning are considered. Given the presence of the integral over the perceived values, one can think of this model as an opinion-dynamics based model for innovation diffusion.

Next, the interaction rules used in the second term of the right-hand side of equation (26) are defined. These rules differ for *users* and *non-users*:

- *Non-users* update their valuation based on word-of-mouth and external information (e.g., advertising):

$$v'_{1,i} = v - a_i(v - v_*) - b(v - \hat{v}) + v\eta. \tag{30}$$

- a_i captures the strength of peer influence (word-of-mouth), and b captures the effect of external signals (advertising). Parameters a_i and b can be stochastic but the mean-field approach of Eq. (9) will average out any distributional effect. The structure of the interaction allows for different word-of-mouth parameters depending on the group of the interacting agent. This models, for instance, the situation where agents value more the opinion of a direct user rather than the opinion of a *non-user*. The stochastic term η is a random variable with $\langle \eta \rangle = 0$ and variance σ^2. It accounts for unobserved mechanisms that influence the perceived value, such as personal preferences, internal reasoning or external events. Here, v_* represents the perceived value of the interacting agent and $\hat{v}$ the advertisement and is sampled from $G_1(\hat{v})$.
- *Users* revise their valuation based solely on usage experience and product quality:

$$v'_2 = v - c(v - \hat{v}) + v\xi. \tag{31}$$

- c is the experience-driven adjustment parameter, and ξ is the stochastic term (random variable with $\langle \xi \rangle = 0$ and variance σ^2). Here, $\hat{v}$ represents the actual value of the product and is sampled from $G_2(\hat{v})$.

The background $G_1(\hat{v})$ in this model reflects both the content of the advertising campaign and individual skepticism about it. For simplicity, it is modeled as a distribution with two point masses:

$$G_1(\hat{v}) = (1 - w)\delta(v) + w\delta(v - P), \tag{32}$$

where $\delta(\cdot)$ is the Dirac delta function, P is the product price, and $w \in [0, 1]$ represents the effectiveness or credibility of the campaign. Conversely, background $G_2(\hat{v})$ represents the actual value of the technology. It reads:

$$G_2(\hat{v}) = \delta(v - rP), \tag{33}$$

where $r > 0$ is a coefficient capturing the perceived quality or real value of the product relative to price. It is important to ensure that the interactions do not bring the perceived value outside its domain, i.e. $v'_i \geq 0, i = 1, 2$. This holds for

$$\eta \geq a_i + b - 1 \tag{34}$$

$$\xi \geq c - 1. \tag{35}$$

This paper focuses on regimes where a_i, b, c and σ^2 are sufficiently small to guarantee non-negativity. While this interaction rule may appear mechanical, it captures a compromise-based updating process similar to bounded rationality models in behavioral economics. Each agent updates its valuation by averaging with others, reflecting limited cognitive processing and social influence.

Consider the *non-users* and assume that decision-making occurs on a slower timescale than information acquisition: agents make up their minds before buying the product and impulse purchase is ruled out. This time lag is realistic and reflects the cognitive and logistical delays between forming an opinion and acting on it. Under this assumption, the time-scale separation described in Sect. 3.4 can be applied to the label switch process and the interaction process. Taking $\lambda = 1$ and $\mu = \delta^{-1}, \delta \to 0$, the weak form of the split Boltzmann equation (22) for the first group reads:

$$\frac{d}{d\tau}\int_V \varphi(v)\tilde{f}_1(\tau,v)dv = \sum_{i=1,2}\iiint_{V^3} \langle \varphi(v_{1,i'}) - \varphi(v)\rangle \tilde{f}_1(\tau,v)\tilde{f}_i(\tau,v_*)G_1(\hat{v})dvdv_*d\hat{v} \tag{36}$$

$$\frac{d}{dt}\int_V \varphi(v)f_1(t,v)dv = -\int_V \varphi(v)H(v-P)f_1(t,v)dv \tag{37}$$

In the same way, they can be written them for group 2. The interpretation is as follows: the distribution $\tilde{f}_i(\tau, v)$ will quickly evolve and reach equilibrium following the interaction dynamics. Once this equilibrium is reached, the slow adoption process updates the group composition, prompting a new belief adjustment cycle, and so on. The evolution of mass and mean perceived value on the fast scale will read:

$$\frac{d}{d\tau}\rho_1(\tau) = \frac{d}{d\tau}\rho_2(\tau) = 0 \tag{38}$$

$$\frac{d}{d\tau}m_1(\tau) = -a_2\rho_2(m_1 - m_2) - b(m_1 - wP) \tag{39}$$

$$\frac{d}{d\tau}m_2(\tau) = -c(m_2 - rP). \tag{40}$$

It is to be noted here that the dynamics of the mean perceived value for the *non-users* m_1 does not depend on the intra-group word-of-mouth parameter a_1. On the slow scale *t*, the evolution of the mass for the two groups is the same as in (29):

$$\frac{d}{dt}\rho_2(t) = \int_P^{+\infty} f_1(t,v)dv = -\frac{d}{dt}\rho_1(t), \tag{41}$$

where the distribution $f_1(t, v)$ is the equilibrium distribution reached by the fast process.

In order to evaluate the equilibrium distribution $f_1(t, v)$, the Fokker-Planck equation introduced in Sect. 3.5 is employed. The quasi-invariant limit is satisfied for a_i, b and $\sigma^2 \to 0^+$. Let $\gamma = \frac{\sigma^2}{b}$ and $\alpha_i = \frac{a_i}{b}$. The interpretation of this is that γ represents the ratio between the importance of the self-reasoning process and the interactions and α the ratio between the word-of-mouth effect and the advertisement effect. In this case, Fokker-Planck equation (25) reads:

$$\begin{aligned}\partial_\tau f_1(\tau,v) = -\partial_v\Bigg[& f_1(\tau,v)\Big(\sum_{i=1,2}\iint_{V^2}[\alpha_i(v_*-v)+(\hat{v}-v)]\cdot \\ & \cdot f_i(\tau,v_*)G(\hat{v})dv_*d\hat{v}\Big)\Bigg]+\frac{\gamma}{2}\partial_v^2(v^2\cdot f).\end{aligned} \tag{42}$$

To find stationary solutions, impose time derivative to 0 and integrate:

$$0=\partial_v\left[-f_1^\infty(v)\cdot(\alpha_1\rho_1 m_1+\alpha_2\rho_2 m_2+wP-(1+\alpha_1\rho_1+\alpha_2\rho_2)v)+\frac{\gamma}{2}\partial_v\big(v^2\cdot f_1^\infty(v)\big)\right]. \tag{43}$$

This holds if the left-hand side of the equation under the derivative is constant. Assume it to be null to allow $f_1^\infty(v)=0$ to be a solution. A stationary solution on the fast time scale is obtained to be:

$$f_1^\infty(v)=C\cdot\frac{e^{-\frac{2}{\gamma}\frac{\alpha_1\rho_1 m_1+\alpha_2\rho_2 m_2+wP}{v}}}{v^{2+\frac{2(1+\alpha_1\rho_1+\alpha_2\rho_2)}{\gamma}}}. \tag{44}$$

Where $C=K\rho_1$ is the normalizing constant over ρ_1. If, on the fast time, one solves Eq. (39), the mean perceived value for group 1 reads:

$$m_1(\tau)=C_0\cdot e^{-(a_2\rho_2+b)t}+\frac{a_2\rho_2 rP+bwP}{a_2\rho_2+b}. \tag{45}$$

This means that the value of m_1 will go to $\frac{a_2\rho_2 r+bw}{a_2\rho_2+b}\cdot P$ as τ goes to infinity. Similarly, m_2 will go to *rP*.

Equation (44) is now fully defined: the steady state is reached with respect to the fast time τ and thus the curve is parameterized by macroscopic quantities $\rho_1(t)$, $\rho_2(t)$ and $m_2(t)$, that evolve on the slow time *t*. Quantity m_1 is expressed as a function of ρ_2 and m_2. Quantity m_2 goes to *rP*. In conclusion and by re-integrating the market size *N*, the equation for the slow time can be written as:

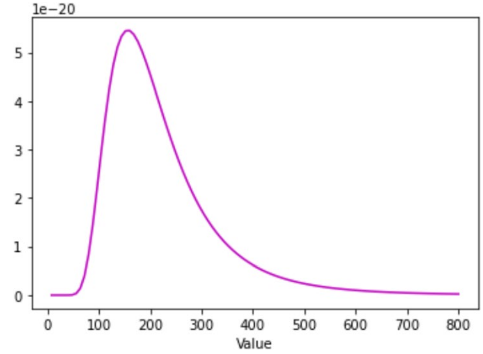
$$\frac{d}{dt}\rho_2(t)=N(1-\rho_2)K\int_P^{+\infty}\frac{e^{-\frac{2}{\gamma}\frac{P}{v}\left[\alpha_1\rho_1\frac{a_2\rho_2 r+bw}{a_2\rho_2+b}+\alpha_2\rho_2 r+w\right]}}{v^{2+\frac{2(1+\alpha_1\rho_1+\alpha_2\rho_2)}{\gamma}}}dv. \tag{46}$$

The expression under the integral sign represents the equilibrium profile of the statistical distribution of the perceived value in the population. Figure 1 shows an example of this distribution. This profile exhibits a fat tail for high values of *v*.

For clarity, here is a recap of the quantities forming this equation:

- *N* is the size of the market.
- γ is linked to the variance of the distribution of perceived value among the *non-users*. It is a measure of the divisiveness of the product.
- *P* is the price of the product. It can vary over time to model different pricing strategies of the product.

Fig. 1 Example of perceived value distribution in the population



- a_2 is the word-of-mouth parameter between groups. It affects how much each interaction with a *user* has the power to change the perceived value of *non-users*.
- *b* is the advertisement parameter. It affects how much each interaction with the advertisement campaign has the power to change the perceived value of agents.
- *r* is the real value of the product, as the mean value perceived by the *users*.
- *w* is the strength of the advertisement campaign. By spending more money or better designing the campaign, managers can modify this number over time. However, it is outside the scope of this paper to design a ready-to-use mapping tool to convert actual advertising effort into parameter *w*.
- α_i is the ratio $\frac{a_i}{b}$, with $i = 1, 2$.

Equation (46) describes the evolution of the *non-users* population in time. It states that the number of agents that become *user* in a given instant is equal to the number of agents whose perceived value *in equilibrium* is higher than the sale price of the product in that instant.

The kinetic approach proposed in this paper shares some structural similarities with the mean-field game model introduced by Chaab et al. (2022), but also presents important conceptual and methodological differences. Both frameworks operate at the mean-field level, where each agent is influenced by the aggregate state of the population. However, while Chaab et al. adopt a mean-field game approach–modeling a Stackelberg game between a forward-looking firm and strategic consumers–this model follows a kinetic perspective rooted in statistical mechanics. In their framework, the firm (leader) commits to a dynamic pricing and advertising policy, while consumers (followers) are segmented into individualists, influenced mainly by advertising, and conformists, who respond to social forces. Each consumer solves an optimization problem to determine the optimal time of adoption within a finite horizon. In contrast, the agents in this kinetic model are behaviorally simpler: they adopt the product as soon as their perceived value exceeds the price. This assumption reflects bounded rationality and informational frictions more typical of early-stage market behavior, where optimization over a full time horizon may be unrealistic. Heterogeneity, such as the distinction between individualists and conformists, can be incorporated in the kinetic model by assigning different interaction parameters (e.g., different coefficients a_i and *b*) to different agent classes. However, doing so may complicate closed-form solutions. One of the main advantages of the kinetic approach lies in its ability to yield an explicit adoption curve derived from micro-level behavioral rules. Unlike MFG-based models, which require numerical solutions, the kinetic model provides a continuous-time diffusion law in closed form. This makes it particularly useful for forecasting and estimation in data-sparse environments.

By solving the integral in equation (46), one obtains an operative equation for the evolution of the diffusion process:

$$\frac{d}{dt}\rho_2(t) = N(1-\rho_2)\tilde{\gamma}\left(1 + \frac{2(1+\alpha_1\rho_1+\alpha_2\rho_2)}{\gamma}, \frac{2}{\gamma}\left[\alpha_1\rho_1\frac{a_2\rho_2 r + bw}{a_2\rho_2 + b} + \alpha_2\rho_2 r + w\right]\right). \quad (47)$$

Here, $\tilde{\gamma}(\cdot)$ is the regularized lower incomplete gamma function. Note here that the explicit price parameter *P* cancels out explicitly, but the adoption dynamics still depend on marketing levers through the real value parameter *r* and the advertisement parameter *w*. Equation (47) is computationally tractable and produces an S-shaped adoption curve, a hallmark of innovation diffusion. While Eq. (47) is the most general form of the model, simpler variants can be used in practice. For example, Sect. 5.1 considers the case where *users* and *non-users* exert equal influence on others ($a_1 = a_2$). Another tractable simplification assumes that only *user* interactions are effective ($a_1 = 0$). The next section evaluates the empirical performance of the Kinetic Innovation Diffusion (KID) model using sales data from four real-world products.

5 Results

The performance of the Kinetic Innovation Diffusion (KID) model has been evaluated on real sales data for four benchmark consumer products: color televisions, air conditioners, clothes dryers, and freezers. These products are commonly used in the diffusion modeling literature and have appeared in several foundational studies (Bass et al., 1994; Cosguner & Seetharaman, 2022; Jiang et al., 2006). The data for these products are contained in Cosguner and Seetharaman (2022), and the fittings obtained in the same paper, the most recent work on this topic, are reported here for reference. Model fitting was conducted using a non-linear least squares method applied to the general form of the KID model, implemented in Python. The emphasis of this section is on the out-of-sample forecasting performance of the model, rather than the quality of the in-sample fit. Given the KID model's larger parameter space, a lower sum of squared errors (SSE) is expected compared to the benchmark models. This expectation is confirmed in Table 1, which reports the in-sample SSE from the best-fitting KID model alongside the SSE values reported by Cosguner and Seetharaman (2022). Table 2 shows the parameters that yield these best fits, for completeness.

To obtain optimal parameters, a Monte Carlo parameter space exploration was used, repeatedly fitting the model from randomly sampled initial conditions. This stochastic approach reduces the risk of converging to local minima and ensures a more robust fit.

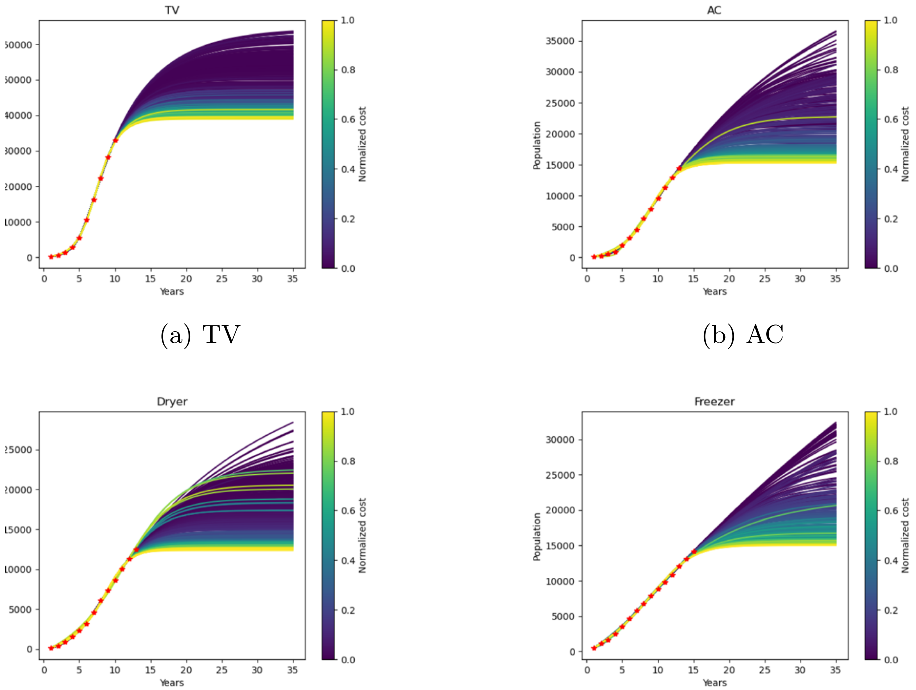
It is important to stress that the parameter estimates reported here should not be interpreted as accurate representations of the underlying behavioral parameters. Instead, they

Table 1 The SSE for the four products of the compared models (smallest error for each product is in bold)

	KID	BM	GBM	BGDM	BLDM
TV	**130,740**	996,421	402,514	457,051	457,387
AC	**53,635**	341,468	294,659	90,727	89,550
Dryer	**76,886**	212,777	210,770	129,485	129,227
Freezer	**123,521**	242,672	209,251	136,767	136,728

Table 2 The parameters of the kinetic diffusion model for the four products

	a_1	a_2	b	γ	w	r	N
TV	3.49 1e-3	9.58 1e-2	7.49 1e-3	0.165	0.533	0.989	51775
AC	4.04 1e-2	1.46 1e-3	4.30 1e-5	1.296	0.929	0.966	21694
Dryer	7.38 1e-3	3.83 1e-2	4.11 1e-3	0.069	0.7636	0.931	28182
Freezer	4.32 1e-3	3.56 1e-3	1.60 1e-5	1.691	0.834	0.893	63746



(a) TV (b) AC

(c) Dryer (d) Freezer

Fig. 2 Possible diffusion S-curves obtained by the model on the available data (stars in red). The scale from dark to light color indicates the SSE and hence is inversely correlated with the likelihood of the realization

serve to demonstrate the model's fitting and forecasting capability. For a more accurate estimation of real-world mechanisms, ensemble methods should be preferred over point estimates. Figure 2 plots the results of this Monte Carlo procedure, the color scale of the plots indicates the likelihood of the realization.

In the next section, focus shifts from in-sample fitting to the managerial relevance of the KID model, particularly in terms of forecasting future adoption trends—arguably the most critical use case for practitioners.

5.1 Early stage forecasting method

Forecasts on future sales, to have managerial value, need to be reliable as early as possible in the diffusion process. Even a very accurate prediction of the peak in the sales or the market potential is useless for managers and decision makers, if it becomes available when it is too late to properly program production, stocking and distribution of the product. To forecast the final 20% of a diffusion curve, as done in some prior studies (Cosguner and Seetharaman 2022, e.g.) provides little actionable insight for decision-makers. In practice, early-stage parameter estimation is often conducted through expert judgment or qualitative techniques (Lawrence & Lawton, 1981). The structure of the KID model supports this early estimation task. In particular, its theoretical formulation offers a direct expression, namely equation (46), for the distribution of perceived value in the population–even before the product has launched. At a pre-launch phase, where the number of *users* is zero and assuming $a_1 = a_2 = a$ and hence $\alpha_1 = \alpha_2 = \alpha$, that reads:

$$f_1^{\infty}(v) = C \cdot \frac{e^{-\frac{2}{\gamma}\frac{P\cdot w}{v}(1+\alpha)}}{v^{2+\frac{2(1+\alpha)}{\gamma}}}. \tag{48}$$

This insight opens the door to parameter inference from pre-launch surveys: one can administer a survey to a target population exposed to the advertising campaign and fit equation (48) to the resulting distribution of perceived values. This procedure allows estimation of the advertising strength w and the ratio $\frac{1+\alpha}{\gamma}$. The market potential N is typically known or can be reasonably estimated by experienced marketing analysts. The real value r can be approximated using focus groups exposed to product samples. Even if only one of the parameters a and b is known, the model remains usable by fixing one at a plausible level (e.g. $b = 0.001$), and computing the other via the relationship $a = b \cdot \alpha$. This framework allows practitioners to simulate alternative adoption trajectories under different strategic assumptions–making it a powerful tool for early-stage decision-making. This hypothesis are tested via a Monte Carlo implementation of this method. While actual belief data was unavailable for the benchmark products, the early-stage estimation process was simulated by perturbing the parameters obtained from the full-sample fit with random noise of $\pm 20\%$ of amplitude. These perturbed values were then used as initial estimates for fitting the model to only 3 or 5 early data points. This process was repeated 1000 times and mean and standard deviation of some significant quantities, namely market potential, time of the sales peak, sales peak and market saturation (calculated as the moment when 95% of the market potential is reached), were computed. These are key strategic quantities to plan production, stocking and distribution.

Tables 3 and 4 show the results for the considered products. Predictions made after the first 5 years are generally accurate within an acceptable statistical error (the only exceptions are freezer's peak sales and AC's saturation point). Predictions made after only 3 years are accurate for TV and dryer, while the standard deviation for AC and freezer is too large to derive accurate information. These findings demonstrate that the KID model is well suited for early-stage forecasting and can support managerial decisions at a time when such insights are most valuable. It offers quantitatively grounded predictions of the most critical planning parameters–without requiring the entire adoption trajectory to unfold.

Table 3 Mean and standard deviation of forecast values of market potential and market saturation with 5 and 3 years of data compared with the values evaluated with the full sample, for the four products

Pts	Prod.	Market Potential			Saturation		
		Mean	STD	Full S	Mean	STD	Full S
5	TV	63560	25946	53562	21	2	20
	AC	32820	11679	25319	25	4	30
	Dryer	16353	4945	15598	19	3	18
	Freezer	21240	7182	17059	22	3	20
3	TV	58610	13481	53562	22	4	20
	AC	38697	24940	25319	38	9	30
	Dryer	15205	2657	15598	16	4	18
	Freezer	32627	34652	17059	38	18	20

Table 4 Mean and standard deviation of forecast values of sales peak and flex point with 5 and 3 years of data compared with the actual values, for the four products

Pts	Prod.	Peak			Flex		
		Mean	STD	Act	Mean	STD	Act
5	TV	7244	4316	5982	8	1	8
	AC	2925	1250	1828	9	1	8
	Dryer	1536	760	1523	10	2	8
	Freezer	1598	828	1205	12	3	13
3	TV	6493	2775	5982	8	1	8
	AC	2386	2446	1828	12	4	8
	Dryer	2308	1037	1523	8	2	8
	Freezer	680	225	1205	4	7	13

6 Conclusions

This paper has introduced a novel methodology for modeling new-product diffusion, grounded in the tools of statistical mechanics. The proposed Kinetic Innovation Diffusion (KID) model is a utility-driven, agent-based framework in which agents decide to adopt based on their individual perceptions of product value. These perceptions evolve through both peer interactions and exposure to advertising, with updates governed by a compromise-based rule. Unobserved individual variability is captured through stochastic noise. The model provides a closed-form analytical expression for the adoption curve, derived directly from micro-level behavioral assumptions. Importantly, it bridges the gap between aggregate diffusion models and simulation-based agent-based models. Unlike aggregate models, which rely on top-down assumptions about market dynamics, the KID model builds diffusion from the bottom up, treating it as an emergent phenomenon of individual belief dynamics. The results shown in Sect. 5 confirm the validity of the model. The KID proved to explain better than the compared methods, namely BM, GBM, BLDM and BGDM, the sales data of four key products: color TV, air conditioning, clothes dryer and freezer. Moreover, the structure of the KID model allows a new method for pre-launch or early-stage sales forecasts. This enables managers to use this model for strategic planning.

Conceptually, the KID model offers a synthesis between system dynamics and agent-based modeling. It retains the analytical tractability of aggregate models while preserving the flexibility of micro-founded simulations. Features such as consumer segmentation, product abandonment, leader follower dynamics, or nonlinear interaction rules can be naturally incorporated within this framework without requiring assumptions about the aggregate adoption curve.

Nonetheless, the model's analytical richness comes at the cost of mathematical complexity. As additional behavioral features are introduced, the model may become less tractable and the notation more cumbersome—an inherent trade-off relative to purely simulation-based AB models.

Future research could expand this framework in several directions. First, the model could be extended to capture more complex adoption phenomena, such as technology hype cycles, which remain underexplored in the diffusion literature. Second, it offers a foundation for studying optimal pricing and advertising strategies, as these variables enter explicitly into the marketing mix of the model's equations. Finally, a systematic investigation of parameter uncertainty—particularly when estimates are derived from noisy pre-launch survey data—could improve the robustness of the model for decision-making under uncertainty.

Funding Open access funding provided by Politecnico di Torino within the CRUI-CARE Agreement.

Declarations

Conflict of interest All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

References

Assenova, V. A. (2018). Modeling the diffusion of complex innovations as a process of opinion formation through social networks. *PloS one, 13*(5), 0196699.

Bass, F.M., Jain, D., Krishnan, T., and others. (2000). Modeling the marketing-mix influence in new-product diffusion. New-product diffusion models, 99–122.

Bass, F. M. (1969). A new product growth for model consumer durables. *Management science, 15*(5), 215–227.

Bass, F. M. (2004). The bass model. *Management Science, 50*(12 Supplement), 1833–1840.

Bass, F. M., & Bultez, A. V. (1982). A note on optimal strategic pricing of technological innovations. *Marketing Science, 1*(4), 371–378.

Bass, F. M., Krishnan, T. V., & Jain, D. C. (1994). Why the bass model fits without decision variables. *Marketing science, 13*(3), 203–223.

Bastani, M. S., Asadi, S., & Anumba, C. J. (2016). Application of bass diffusion theory to simulate the impact of feedback and word of mouth on occupants' behavior in commercial buildings: An agent-based approach. *Journal of Architectural Engineering, 22*(4), 04016013.

Bell, D. R., & Song, S. (2007). Neighborhood effects and trial on the internet: Evidence from online grocery retailing. *Quantitative Marketing and Economics, 5*, 361–400.

Boltzmann, L. (1872). Weitere studien über das wärmegleichgewicht unter gasmolekülen sitzungberichte der kaiserlichen akademie der wissenschaften mathematisch-naturwissenschaftliche classe 66, 275–370. The kinetic theory of gases. *History of Modern Physical Sciences, 1*, 262–349.
Bronnenberg, B. J., & Mela, C. F. (2004). Market roll-out and retailer adoption for new brands. *Marketing Science, 23*(4), 500–518.
Cercignani, C. (1988). The boltzmann equation. In: The Boltzmann Equation and Its Applications, pp. 40–103. Springer, New York.
Chaab, J., Salhab, R., & Zaccour, G. (2022). Dynamic pricing and advertising in the presence of strategic consumers and social contagion: A mean-field game approach. *Omega, 109*, 102606.
Chenavaz, R., Paraschiv, C., & Turinici, G. (2021). Dynamic pricing of new products in competitive markets: A mean-field game approach. *Dynamic Games and Applications, 11*(3), 463–490.
Cosguner, K., & Seetharaman, P. (2022). Dynamic pricing for new products using a utility-based generalization of the bass diffusion model. *Management Science, 68*(3), 1904–1922.
Dockner, E., & Jørgensen, S. (1988). Optimal pricing strategies for new products in dynamic oligopolies. *Marketing Science, 7*(4), 315–334.
Düring, B., & Wright, O. (2022). On a kinetic opinion formation model for pre-election polling. *Philosophical Transactions of the Royal Society A, 380*(2224), 20210154.
Fan, Z.-P., Che, Y.-J., & Chen, Z.-Y. (2017). Product sales forecasting using online reviews and historical sales data: A method combining the bass model and sentiment analysis. *Journal of Business Research, 74*, 90–100.
Fibich, G., & Gibori, R. (2010). Aggregate diffusion dynamics in agent-based models with a spatial structure. *Operations Research, 58*(5), 1450–1468.
Fruchter, G. E., & Bulte, C. (2011). Why the generalized bass model leads to odd optimal advertising policies. *International Journal of Research in Marketing, 28*(3), 218–230.
Goldenberg, J., Libai, B., Solomon, S., Jan, N., & Stauffer, D. (2000). Marketing percolation. *Physica A: statistical mechanics and its applications, 284*(1–4), 335–347.
Hall, B. H. (2004). *Innovation and diffusion*. National Bureau of Economic Research Cambridge: Mass., USA.
Han, Z., Tang, Z., & He, B. (2022). Improved bass model for predicting the popularity of product information posted on microblogs. *Technological Forecasting and Social Change, 176*, 121458.
Hassani, S., & Hassani, S. (2009). Dirac delta function. Mathematical Methods: For Students of Physics and Related Fields, 139–170.
Ho, T.-H., Savin, S., & Terwiesch, C. (2002). Managing demand and sales dynamics in new product diffusion under supply constraint. *Management Science, 48*(2), 187–206.
Iyengar, R., Bulte, C., & Valente, T. W. (2011). Opinion leadership and social contagion in new product diffusion. *Marketing Science, 30*(2), 195–212.
Jiang, Z., Bass, F. M., & Bass, P. I. (2006). Virtual bass model and the left-hand data-truncation bias in diffusion of innovation studies. *International Journal of Research in Marketing, 23*(1), 93–106.
Jiang, Z., & Jain, D. C. (2012). A generalized norton-bass model for multigeneration diffusion. *Management Science, 58*(10), 1887–1897.
Kalish, S., & Lilien, G. L. (1983). Optimal price subsidy policy for accelerating the diffusion of innovation. *Marketing Science, 2*(4), 407–420.
Keith, D. R., Sterman, J. D., & Struben, J. (2017). Supply constraints and waitlists in new product diffusion. *System Dynamics Review, 33*(3–4), 254–279.
Krishnan, T. V., Bass, F. M., & Jain, D. C. (1999). Optimal pricing strategy for new products. *Management Science, 45*(12), 1650–1663.
Krishnan, T. V., & Jain, D. C. (2006). Optimal dynamic advertising policy for new products. *Management Science, 52*(12), 1957–1969.
Kumar, S., & Swaminathan, J. M. (2003). Diffusion of innovations under supply constraints. *Operations Research, 51*(6), 866–879.
Lawrence, K.D., & Lawton, W.H. (1981). Applications of diffusion models: some empirical results. New product forecasting, 529–541.
Li, S., Jin, Z., et al. (2014). Modeling and analysis of new products diffusion on heterogeneous networks. *Journal of Applied Mathematics, 2014*, 940623.
Li, X., Yuan, J., Shi, Y., Wang, T., Hu, X., Chan, F. T. S., & Ruan, J. (2020). An extended bass model on consumer quantity of b2c commerce platforms. *Electronic Commerce Research, 20*, 609–628.
Loy, N., & Tosin, A. (2020). Boltzmann-type equations for multi-agent systems with label switching. arXiv preprint arXiv:2006.15550.
Loy, N., & Tosin, A. (2021). A viral load-based model for epidemic spread on spatial networks. arXiv preprint arXiv:2104.12107.

Marca, R.D., Loy, N., & Tosin, A. (2021). An sir-like kinetic model tracking individuals' viral load. arXiv preprint arXiv:2106.14480

Norton, J. A., & Bass, F. M. (1987). A diffusion theory model of adoption and substitution for successive generations of high-technology products. *Management Science, 33*(9), 1069–1086.

Palmer, J., Sorda, G., & Madlener, R. (2015). Modeling the diffusion of residential photovoltaic systems in Italy: An agent-based simulation. *Technological Forecasting and Social Change, 99*, 106–131.

Pareschi, L., & Toscani, G. (2013). *Interacting Multiagent Systems: Kinetic Equations and Monte Carlo Methods*. OUP.

Peres, R., Muller, E., & Mahajan, V. (2010). Innovation diffusion and new product growth models: A critical review and research directions. *International Journal of Research in Marketing, 27*(2), 91–106.

Rahmandad, H., & Sterman, J. (2008). Heterogeneity and network structure in the dynamics of diffusion: Comparing agent-based and differential equation models. *Management Science, 54*(5), 998–1014.

Risken, H. (1996). *Fokker-planck Equation*. Springer.

Robinson, B., & Lakhani, C. (1975). Dynamic price models for new-product planning. *Management Science, 21*(10), 1113–1122.

Rogers, E. (2003). Diffusion of innovations. revised. New York: Simon & Schuster.

Rotaris, L., & Scorrano, M. (2023). Insights into peer-to-peer carsharing: Modelling and scenario analysis via a bass diffusion agent-based model. *Research in Transportation Economics, 97*, 101261.

Shen, W., Duenyas, I., & Kapuscinski, R. (2011). New product diffusion decisions under supply constraints. *Management Science, 57*(10), 1802–1810.

Simon, H., & Sebastian, K.-H. (1987). Diffusion and advertising: The German telephone campaign. *Management Science, 33*(4), 451–466.

Singh, O., Anand, A., Kapur, P., & Aggrawal, D. (2012). Consumer behaviour-based innovation diffusion modelling using stochastic differential equation incorporating change in adoption rate. *International Journal of Technology Marketing, 7*(4), 346–360.

Sinha, R. K., & Chandrashekaran, M. (1992). A split hazard model for analyzing the diffusion of innovations. *Journal of Marketing Research, 29*(1), 116–127.

Stummer, C., Kiesling, E., Günther, M., & Vetschera, R. (2015). Innovation diffusion of repeat purchase products in a competitive market: An agent-based simulation approach. *European Journal of Operational Research, 245*(1), 157–167.

Toscani, G. (2006). Kinetic models of opinion formation. *Communications in Mathematical Sciences, 4*(3), 481–496.

Weisstein, E.W. (2002). Heaviside step function. https://mathworld.wolfram.com/

Young, H. P. (2009). Innovation diffusion in heterogeneous populations: Contagion, social influence, and social learning. *American Economic Review, 99*(5), 1899–1924.

Zhang, N., Lu, Y., Chen, J., & Hwang, B.-G. (2022). An agent-based diffusion model for residential photovoltaic deployment in Singapore: Perspective of consumers' behaviour. *Journal of Cleaner Production, 367*, 132793.