Mini-Workshop: Hardy Inequalities in Discrete and Continuum Settings

Mathematisches Forschungsinstitut Oberwolfach

Report No. 14/2025

DOI: 10.4171/OWR/2025/14

Mini-Workshop: Hardy Inequalities in Discrete and Continuum Settings

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2 March – 7 March 2025

Abstract. Hardy inequalities play a fundamental role in various fields, such as mathematical physics, operator theory, the analysis of partial differential equations, potential theory, and probability theory. The goal of this workshop was to bring together leading experts to foster cross-interactions between them, with special focus on the following three topics, which naturally overlap: optimal Hardy weights on manifolds and graphs; fractional Laplacians; Sub-Riemannian analogues, magnetic Laplacians and Dirac operators.

Mathematics Subject Classification (2020): Primary: 35J10, 35R02, 43A80, 53C23, 47A10. Secondary: 26D15, 47D07, 53C21, 39A12, 31C25.

Introduction by the Organizers

The *Mini-Workshop: Hardy Inequalities in Discrete and Continuum Settings* (02 March – 07 March 2025) was organized by E. Berchio (Torino), M. Keller (Potsdam), Y. Pinchover (Haifa) and L. Roncal (Bilbao). It took place in a hybrid format, with a total of 14 invited researchers (including 3 who participated online), 1 master's student and 4 organizers, with a broad geographic and academic age representation.

On the technical side, there were 12 onsite talks and 3 online talks broadcast via Zoom. Onsite talks were approximately 75 minutes long, while online talks lasted about 50 minutes, with ample time allotted for discussion in both formats. The participants came from various mathematical backgrounds, spanning the theory of partial differential equations, mathematical physics, graph theory, harmonic analysis and spectral theory. Despite their diverse expertise, they shared a common interest: *Hardy inequalities.* Generally, the talks reported on recent advancements in the field, by the researchers and their collaborators. In order to address the different mathematical backgrounds, the participants made special effort to present talks containing the latest results in the respective fields, in a manner accessible to all participants. The talks were scheduled from Monday to Friday, with additional discussion sessions held on Tuesday afternoon and Friday morning, following the final presentations.

The topics discussed in the workshop can be summarized as follows.

- *Optimal Hardy weights on manifolds and graphs.* In the last decade, a new focus came into play, which was set on optimality of the Hardy weight itself going back to a question posed by Agmon. Specifically, optimality has both sharpness of the constant and non-existence of a minimizer as a consequence. Revisiting the original discrete roots of the problem has led to the surprising insight that, while the constant is optimal, the original weight can be improved. In $\mathbb{Z}^d$, $d \geq 3$, a precise form of an optimal Hardy weight can be proven; however, the higher order terms seem to be non-definite. Beyond $p = 2$, the optimality theory for graphs and general p exists by now, but a specific inequality for $\mathbb{Z}^d$ is still not known. Moreover, generalizations to Riemannian manifolds have been intensively pursued and analogous questions arise. During the workshop, these issues were addressed. Asymptotic behaviour of the optimal constant as the dimension $d \to \infty$ was discussed, via interesting connections between discrete Hardy inequalities on lattices with continuous Hardy-type inequalities on the torus. Additionally, new approaches to obtain optimal Poincaré–Hardy inequalities on the hyperbolic spaces were presented, and generalizations to certain classes of Riemannian manifolds were discussed, including transference methods to the discrete setting. Finally, it was shown how recent improvements in optimality have contributed to progress on Landis' conjecture for graphs.

- *Fractional Laplacians.* Hardy inequalities for the fractional Laplacian on $\mathbb{R}^d$ have been investigated by several authors, and interest was later regained with the seminal work using a non-local ground state representation approach. Despite recent advances in more general settings, there are several contexts in which fractional versions of the Hardy inequality are yet to be fully understood. In particular, there is a probabilistic interpretation of powers of the Laplacian in terms of an anomalous diffusion. From this perspective, the powers $(-\Delta)^\alpha$ in $L^2((0,\infty))$ with $\alpha \in \mathbb{N}$ are subcritical and there is no criticality transition in powers. It was discussed in the workshop the surprising fact that the situation is very different in the discrete setting, since it was shown that the integer powers of the discrete Laplacian $(-\Delta)^\alpha$ on $\ell^2(\mathbb{N})$ are subcritical if and only if $\alpha = 1$.

- *Sub-Riemannian analogues, magnetic Laplacians and Dirac operators.* Beyond the Euclidean space, lattices, and Riemannian manifolds, the study of Hardy and related inequalities has been extended to the framework of sub-Riemannian geometry, particularly in the Heisenberg group. Another vibrant branch of research concerns the study of Schrödinger-type operators in the presence of magnetic fields. Introducing non-trivial magnetic perturbations of Hamiltonian operators induces repulsive effects in quantum mechanics which have been quantified by Hardy-type inequalities. Some of the talks in these directions addressed Hardy, Hardy–Rellich, and Rellich identities and inequalities with sharp constants for Grushin vector fields, quantitative Hardy-type inequalities for the magnetic Laplacian by proving a similar inequality first for the Pauli operator.

The interaction between the different communities turned out to be very lively, featuring broad discussions on the latest open problems in the field and an exchange of state-of-the-art tools. Among the topics that sparked significant activity and exchange of knowledge were: extended Dirichlet spaces and criticality theory for nonlinear Dirichlet forms; affirmative answers to the Landis conjecture on $\mathbb{R}^d$ under a positivity assumption on the operator involved; connections of superharmonic functions for Hardy-type inequalities for Sobolev-Bregman forms and trade-offs in fractional Hardy inequalities. The mini-workshop was overall highly productive and fostered stimulating discussions among researchers from diverse backgrounds, which will, hopefully, lead to future scientific collaborations.