

Node-dependent kinematics approach for progressive damage analysis of pultruded composite structures based on 3D Tsai-Wu damage model

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1 Node-dependent kinematics approach for progressive  
2 damage analysis of pultruded composite structures based  
3 on 3D Tsai-Wu damage model

4 J. Shen <sup>a</sup>, M.R.T Arruda <sup>b\*</sup>, A. Pagani <sup>a</sup>, M. Petrolo <sup>a</sup>,  
E. Carrera <sup>a</sup>, R. Augello <sup>a</sup>, E. Zappino <sup>a</sup>,

<sup>a</sup> *Mul2* Lab, Department of Mechanical and Aerospace Engineering,  
Politecnico di Torino, Italy

<sup>b</sup> LNEC, National Laboratory of Civil Engineering,  
Department of Structures - Structural Behaviour Unit, Portugal

5 **Abstract:** *This work investigates the progressive damage analysis of pultruded composite*  
6 *structures using the higher-order beam models with Node-Dependent Kinematics (NDK) based*  
7 *on Carrera Unified Formulation (CUF). A three-dimensional (3D) Tsai-Wu orthotropic dam-*  
8 *age model is implemented, in which the fracture energy and viscous regularization techniques*  
9 *are coupled. Subsequently, three experimental benchmarks involving compact tension, three-*  
10 *point bending, and four-point bending tests on pultruded composite structures are selected to*  
11 *validate the proposed numerical framework. The comparison between numerical and experi-*  
12 *mental results shows that the numerical framework is able to provide accurate and objective*  
13 *structural responses. Moreover, the use of NDK enhances the computational efficiency of the*  
14 *progressive damage analysis without affecting accuracy.*

15  
16 **Keywords:** Damage mechanics; Carrera Unified Formulation; Fracture energy regulariza-  
17 tion; Viscous regularization; Pultruded composites

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18  
19 **1 Introduction**

20 Pultruded composites are Fiber-Reinforced Polymer (FRP) composite material manufactured  
21 through the pultrusion process. Due to its high-strength and lightweight nature, this type of  
22 composites has been used across a wide range of industries, from aerospace to construction.  
23 Composite structures are often prone to complex damage mechanisms such as fiber breakage,  
24 matrix cracking, and delamination. These failure modes often develop progressively under  
25 quasi-static loading. Therefore, correct design and efficient use of these composite materials  
26 require the progressive damage analysis, including the failure onset and propagation.

27 To overcome the problems related to demanding experimental tests, various numerical  
28 models have been widely proposed to analyze the progressive damage of composite structures.  
29 The first challenge is to select the suitable failure criterion. There are many failure theories in

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\*Corresponding author. E-mail: mario.rui.arruda@tecnico.ulisboa.pt

30 the literature, such as Tsai-Wu [1], Hashin [2], and Puck [3]. Despite their success in predicting  
31 the failure of composite structures, these models have been observed to underestimate the  
32 experimental strength when applied to pultruded FRP elements [4, 5], as they neglect the  
33 progressive damage. Therefore, it is necessary to take damage propagation into account in  
34 the failure analysis.

35 In Continuum Damage Mechanics (CDM), the damage propagation indicates the degrada-  
36 tion of elastic moduli due to the growth and coalesce of voids or micro-cracks. An exponential  
37 damage evolution model coupled with a failure initiation criterion was first proposed in [6],  
38 and is commonly referred to as the Matzenmiller–Lubliner–Taylor (MLT) model. The MLT  
39 model produces reasonably accurate results [7], but requires the parameterization of damage  
40 coefficients that control the ‘smoothness’ of damage evolution. A decohesion Finite Element  
41 (FE) was developed by [8] as an alternative method for simulating delamination between  
42 solid elements using a mixed-mode progressive damage model. Their formulation incorpo-  
43 rated a strain-softening behavior governed by fracture energy, which was previously applied  
44 to concrete [9]. Compared to MLT, this method offers the advantage of employing physically  
45 meaningful parameters, such as fracture energies. However, the experimental determination  
46 of fracture energies for each failure mode is difficult. Moreover, the fracture energies measured  
47 may exhibit a wide range for similar materials [10, 11, 12].

48 Nevertheless, some of the aforementioned models were included in commercial software,  
49 and were widely utilized in numerical analysis. A damage initiation and progression model,  
50 presented in [13] and based on the works of Hashin [2], MLT [6], and decohesion [8], is  
51 included in ABAQUS. Recently, a two-dimensional (2D) orthotropic damage model based  
52 on the Tsai–Wu failure criterion was developed in [14, 15], in which failure modes can be  
53 identified once the criterion is met.

54 The previous 2D orthotropic damage models are only limited to shell elements due to  
55 their transverse isotropic assumption [16]. Moreover, classical lower-order structural theories  
56 have been reported to be insufficient for progressive failure analysis in composite laminates  
57 [17]. Thus, high-fidelity simulations necessitate the incorporation of three-dimensional (3D)  
58 orthotropic damage models based on appropriate failure criteria into solid elements. For  
59 instance, the Hashin failure criterion has been extended to 3D applications, as presented  
60 in [18], though interlaminar damage was not considered. Practical implementation of the  
61 LaRC05 failure criterion and its variations for 3D analysis was explored in [19, 20]. In [21],  
62 a fully 3D formulation of the Hashin criterion was demonstrated coupled with simplified  
63 degradation models, tailored for bolted connections.

64 The development of 3D composite constitutive relations is also associated with the need  
65 for simulating connections, mainly with bolts [20, 22]. Certain damage models are capa-  
66 ble of predicting not only the failure of FRP laminates, but also their post-failure behavior  
67 [23]. However, these models often require high-density meshes and significant computational  
68 resources, primarily due to the need to resolve individual plies and/or implement cohesive  
69 interfaces [24], making them impractical for the analysis of large-scale engineering problems.  
70 Some authors have chosen novel formulations [25] using standard concepts that couple the  
71 damage initiation between the fiber, matrix and interlaminar to successfully simulate compos-  
72 ites beam-column connections [26]. Recently, a 3D orthotropic damage model based on 3D  
73 Hashin failure including interlaminar failure was proposed in [27]. Moreover, the 2D Tsai-Wu  
74 damage model [14] was extended to 3D in which the interlaminar failure is included in [28].  
75 In [27, 28], fracture energy regularization was employed to mitigate the mesh dependency and  
76 viscous regularization was utilized to alleviate the convergence issues.

77 Although the aforementioned 3D models can obtain accurate results, the computational

78 costs can increase significantly. Carrera Unified Formulation (CUF) [29] provide an alternative  
79 solution for 3D solid elements. In the CUF framework, 3D displacement fields of an one-  
80 dimensional (1D) beam model can be obtained through cross-sectional expansion functions.  
81 Higher-order beam models allow for accurate failure analyses since accurate stresses and  
82 strains can be recovered [29]. CUF has been successfully applied for the damage analysis of  
83 concrete and reinforced concrete structures [30, 31]. Moreover, it has also been exploited for  
84 composite laminates [32, 33], in which the CODAM2 model [34] was integrated into 2D CUF-  
85 based layer-wise models. Recently, the 3D orthotropic damage model based on 3D Hashin  
86 failure criterion was implemented in the CUF-based beam models for composite structures  
87 [35].

88 In CUF, the Node-Dependent Kinematics (NDK) approach [36] allows the employment  
89 of different cross-sectional kinematics within the same beam model. With the help of this  
90 approach, the structure can be divided into critical zones with refined models and non-critical  
91 zones with lower-order models so that the computational costs can be further reduced without  
92 sacrificing accuracy. This approach was investigated for the analysis of composite structures  
93 [37, 38, 39], most of which are limited to the linear static analysis [40].

94 This work aims to apply the CUF-based higher-order beam models to the progressive  
95 damage analysis of pultruded composite structures. In addition to the novelties stemmed  
96 from previous works [32, 35], this work employs a new orthotropic damage model based on  
97 3D Tsai-Wu failure criterion in which only one single scalar failure surface is needed and  
98 different failure modes can be distinguished at the onset of failure. Moreover, this work aims  
99 to mitigate the mesh dependency more effectively through a new characteristic element length  
100 from [41]. Additionally, the viscous regularization, which was ignored in the previous works, is  
101 implemented in this work to mitigate the convergence difficulties, especially for brittle failure  
102 case. This work employs the NDK approach to enhance the computational efficiency of CUF-  
103 based beam models in the progressive damage analysis of pultruded composite structures for  
104 the first time.

105 The paper is organized as follows: Section 2 introduces the fundamentals of CUF-based  
106 models with NDK briefly; Section 3 presents a novel orthotropic damage model based on  
107 3D Tsai-Wu failure criterion, in which the fracture energy regularization and viscous regu-  
108 larization techniques are introduced; Subsequently, Section 4 illustrates and discusses three  
109 numerical examples. Finally, Section 5 draws the conclusions.

## 110 2 Unified higher-order finite beam elements

111 In this section, higher-order beam theories based on CUF are briefly introduced. As shown  
112 in Fig. 1, this theory overcomes the assumption of rigid cross-sections in a 1D domain by  
113 approximating the 3D displacement field through the following equation:

$$\mathbf{u}(x, y, z) = F_\tau(x, z)\mathbf{u}_\tau(y), \quad \tau = 1, 2, \dots, M \quad (1)$$

114 where  $y$  is the axial direction of the beam, and  $x$  and  $z$  are the cross-sectional coordinates;  
115  $\mathbf{u}_\tau(y)$  denotes the generalized displacements along the axial direction;  $F_\tau(x, z)$  is the cross-  
116 sectional expansion function;  $\tau$  represents a summation over terms, and  $M$  denotes the number  
117 of terms in cross-sectional expansion functions.

118 The displacement of the beam model in the 1D case along the axial direction can be ap-  
119 proximated using the finite element shape functions  $N_i(y)$ , as shown in Fig. 1, which includes  
120 different beam elements of linear (B2), quadratic (B3), and cubic (B4) orders. Therefore, Eq.

121 (1) can be expressed as:

$$\mathbf{u}(x, y, z) = F_\tau(x, z)N_i(y)\mathbf{u}_{\tau i}, \quad i = 1, \dots, N_{NE} \quad (2)$$

122 where  $N_{NE}$  denotes the number of nodes per beam element;  $\mathbf{u}_{\tau i}$  is the nodal displacement  
 123 vector.

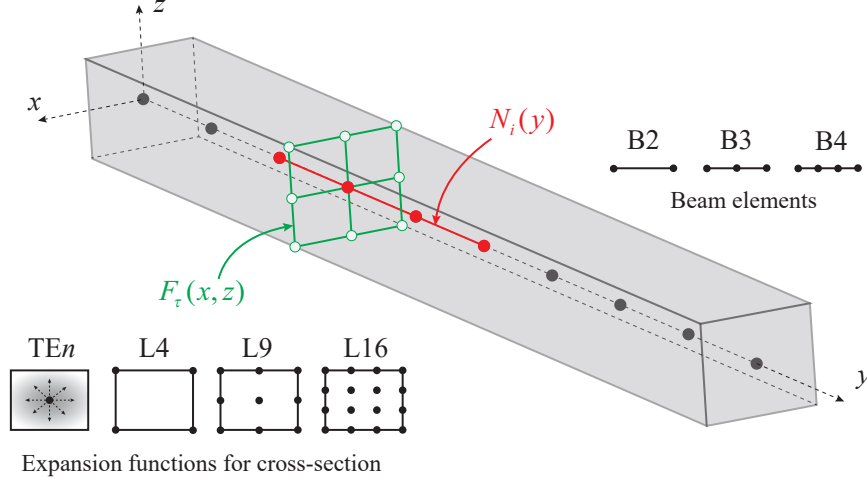


Figure 1: Reference framework, beam elements, and cross-section expansion

## 124 2.1 Cross-sectional expansions

125 Various types of  $F_\tau$  such as polynomials, harmonics, and exponentials can be implemented  
 126 through CUF. The choice of a particular expansion characterizes the capabilities of a struc-  
 127 tural theory: the richer the expansion, the more accurate the results [29]. The most commonly  
 128 implemented Taylor-like and Lagrange polynomial expansions, as shown in Fig. 1, are briefly  
 129 introduced below.

130 Taylor-like polynomials consists of the 2D base  $x^i z^j$ , where  $i$  and  $j$  are positive integers.  
 131 The 1D beam model based on such polynomials is referred to as TE $n$  model, where  $n$  denotes  
 132 the expansion order. For instance, the complete quadratic (i.e.,  $n = 2$ ) Taylor-like polynomial  
 133 basis takes the form:  $F_1 = 1$ ,  $F_2 = x$ ,  $F_3 = z$ ,  $F_4 = x^2$ ,  $F_5 = xz$ ,  $F_6 = z^2$ . Then, the  
 134 displacement field can be expressed as follows:

$$\begin{aligned} u_x &= u_{x1} + xu_{x2} + zu_{x3} + x^2u_{x4} + xzu_{x5} + z^2u_{x6} \\ u_y &= u_{y1} + xu_{y2} + zu_{y3} + x^2u_{y4} + xzu_{y5} + z^2u_{y6} \\ u_z &= u_{z1} + xu_{z2} + zu_{z3} + x^2u_{z4} + xzu_{z5} + z^2u_{z6} \end{aligned} \quad (3)$$

135 where  $u_{x1} \cdots u_{z3}$  are unknown variables. It is important to note that classical beam theories,  
 136 such as Euler-Bernoulli and Timoshenko, are special cases of TE1-based 1D CUF models.

137 Similarly, 1D CUF models based on 2D Lagrange polynomial expansions can be referred  
 138 to as LE models. Based on the order of Lagrange polynomials, the options of quadrilateral  
 139 elements include bilinear four-point (L4), quadratic nine-point (L9), and cubic sixteen-point

140 (L16). The expression for one L9 element is shown here as an example:

$$\begin{aligned}
 F_\tau &= \frac{1}{4} (r^2 + rr_\tau) (s^2 + ss_\tau), \quad \tau = 1, 3, 5, 7 \\
 F_\tau &= \frac{1}{2} s_\tau^2 (s^2 + ss_\tau) (1 - r^2) + \frac{1}{2} r_\tau^2 (r^2 + rr_\tau) (1 - s^2), \quad \tau = 2, 4, 6, 8 \\
 F_\tau &= (1 - r^2) (1 - s^2), \quad \tau = 9
 \end{aligned} \tag{4}$$

141 where  $(r, s)$  are the normalized coordinates, while  $(r_\tau, s_\tau)$  represent the normalized coordinates  
 142 of each node  $\tau$ .

143 Compared to TE models, unknown variables of LE models, which are pure displacement  
 144 components, can be located above the physical surfaces of the structure. Moreover, locally  
 145 refined models can be easily built using LE.

## 146 2.2 Node-dependent kinematics

147 The Node-dependent kinematics approach was first proposed in [37] to mix different structural  
 148 theories in the CUF framework. The key idea is allowing for different kinematics at each  
 149 node of a 1D beam element, which works as a transition to connect elements with different  
 150 kinematics without any displacement discontinuities. As seen in Fig. 2, the whole domain  
 151 is divided into two zones where two different cross-sectional kinematics are employed. A  
 152 transition element with node-dependent kinematics is used to connect these two zones. The  
 153 kinematics description of each point in this transition element is independent, and they are  
 154 smeared by the shape functions that ensure a smooth transition between the displacement of  
 155 the two nodes. Therefore, the displacement continuity can be guaranteed between two zones  
 156 without any need for special coupling techniques.

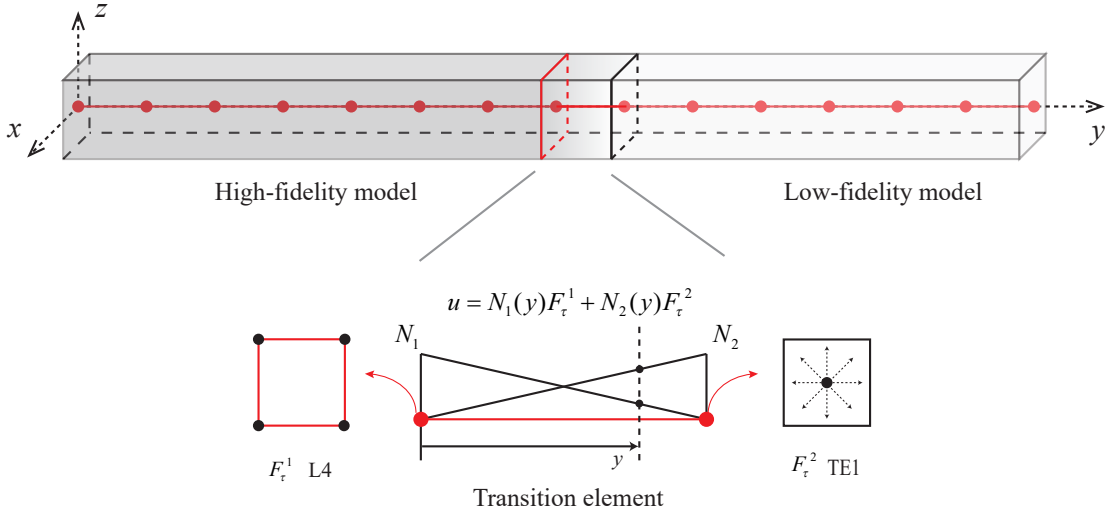


Figure 2: NDK approach for beams

157 This idea can be easily extended to any order beam models in CUF. In general, the  
 158 displacement field of 1D element with node-dependent kinematics can be rewritten as:

$$\mathbf{u}(x, y, z) = F_\tau^i(x, z) N_i(y) \mathbf{u}_{\tau i}, \quad \tau = 1, \dots, M^i \quad i = 1, \dots, N_{NE} \tag{5}$$

159 where  $F_\tau^i$  represents the expansion function at node point  $i$  and  $M^i$  is the corresponding  
 160 number of expansion terms.

161 With NDK, the numerical accuracy can be improved using a refinement in the kinematics  
 162 assumptions without mesh refinement of FE model. Furthermore, a balance between accuracy  
 163 and computational costs can be achieved where refined models can be employed for the  
 164 selected zones of interest such as critical zones vulnerable to damage and coarse models can  
 165 be adopted for the non-critical regions of the structure. In the present work, the critical  
 166 regions are modelled using LE, while the remaining structure is modelled using lower-order  
 167 TE.

## 168 2.3 Governing equations

169 The governing equations can be derived using the Principle of Virtual Displacement (PVD)  
 170 which assumes that the external virtual work ( $\delta L_{\text{ext}}$ ) is equal to the internal virtual work  
 171 ( $\delta L_{\text{int}}$ ) in the static case.

172 The internal and external virtual work can be written as:

$$\delta L_{\text{int}} = \int_V \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV \quad (6)$$

$$\delta L_{\text{ext}} = \int_V \delta \mathbf{u}^T \mathbf{P} dV \quad (7)$$

173 where  $\boldsymbol{\varepsilon}$  and  $\boldsymbol{\sigma}$  are stress and strain vectors, respectively;  $\mathbf{P}$  is the external load.

174 The relationship between displacement and strain fields can be obtained from the geomet-  
 175 rical relation:

$$\boldsymbol{\varepsilon} = \mathbf{D} \mathbf{u} \quad (8)$$

176 where  $\mathbf{D}$  is a differential operator. Moreover, the constitutive law permits to derive the  
 177 relation between stress and strain fields which is expressed as:

$$\boldsymbol{\sigma} = \mathbf{C}^d \boldsymbol{\varepsilon} \quad (9)$$

178 where  $\mathbf{C}^d$  is the  $6 \times 6$  material stiffness matrix including damage.

179 Substituting Eq. (8) and (9) and the displacement field given in Eq. (5) into the Eq. (6),  
 180 the internal virtual work can be rewritten as:

$$\begin{aligned} \delta L_{\text{int}} &= \delta \mathbf{u}_{sj}^T \int_V [N_j(y) F_s^j(x, z) \mathbf{D}^T \mathbf{C}^d \mathbf{D} F_\tau^i(x, z) N_i(y)] dV \mathbf{u}_{\tau i} \\ &= \delta \mathbf{u}_{sj}^T \mathbf{K}^{ij\tau s} \mathbf{u}_{\tau i} \end{aligned} \quad (10)$$

181 where  $\mathbf{K}^{ij\tau s}$  is a  $3 \times 3$  matrix denoted as Fundamental Nucleus (FN), which enables the  
 182 assembly of the element stiffness matrix independently of the structural theory order. The  
 183 explicit expression of  $\mathbf{K}^{ij\tau s}$  can be found in [37]. For the sake of clarity, the first component  
 184  $k_{xx}^{ij\tau s}$  is reported:

$$\begin{aligned} k_{xx}^{ij\tau s} &= \int_V C_{11}^d \frac{\partial}{\partial x} (N_j F_s^j) \frac{\partial}{\partial x} (N_i F_\tau^i) dV + \int_V C_{16}^d \frac{\partial}{\partial x} (N_j F_s^j) \frac{\partial}{\partial y} (N_i F_\tau^i) dV \\ &+ \int_V C_{44}^d \frac{\partial}{\partial z} (N_j F_s^j) \frac{\partial}{\partial z} (N_i F_\tau^i) dV + \int_V C_{16}^d \frac{\partial}{\partial y} (N_j F_s^j) \frac{\partial}{\partial x} (N_i F_\tau^i) dV \\ &+ \int_V C_{66}^d \frac{\partial}{\partial y} (N_j F_s^j) \frac{\partial}{\partial y} (N_i F_\tau^i) dV \end{aligned} \quad (11)$$

185 Similarly, the external virtual work can be rewritten as Eq. (12) with the introduction of  
 186 Eq. (5).

$$\delta L_{\text{ext}} = \delta \mathbf{u}_{sj}^T \int_V \delta N_j(y) F_s^j(x, z) \mathbf{P} dV = \delta \mathbf{u}_{sj}^T \mathbf{P}_{sj} \quad (12)$$

187 where  $\mathbf{P}_{sj}$  is the external load acting on beam node  $j$  and cross-sectional node  $s$ .

188 Then, the governing equation for static problems is derived by applying the PVD and  
 189 integrating Eq. (10) with Eq. (12), yielding:

$$\mathbf{K}^{\tau sij} \mathbf{u}_{\tau i} = \mathbf{P}_{sj} \quad (13)$$

## 190 2.4 Implicit solution for physical nonlinearity

191 This work involves physical nonlinearity due to damage. The nonlinear equilibrium equation  
 192 derived from the CUF-based beam formulation is solved using an implicit displacement-based  
 193 iterative scheme. At each pseudo-time step  $t + \Delta t$ , the residual force is expressed as:

$$\mathbf{R}(t + \Delta t) = \mathbf{F}_{\text{ext}}(t + \Delta t) - \mathbf{F}_{\text{int}}(t) \quad (14)$$

194 where  $\mathbf{F}_{\text{int}}(t)$  is the internal force evaluated using current damage state at time  $t$ ;  $\mathbf{F}_{\text{ext}}(t + \Delta t)$   
 195 is the applied external load at time  $t + \Delta t$ ;  $\mathbf{R}(t + \Delta t)$  is the unbalanced force at the beginning  
 196 of the increment.

197 At each time step, the displacement increment  $\Delta \mathbf{u}$  is computed by solving the following  
 198 linear system:

$$\mathbf{K}_{\text{sec}}(t + \Delta t) \cdot \Delta \mathbf{u} = \mathbf{R}(t + \Delta t) \quad (15)$$

199 in which  $\mathbf{K}_{\text{sec}}(t + \Delta t)$  is the secant stiffness matrix constructed based on the damage at time  
 200  $t + \Delta t$ . In the CUF framework, Eq. (15) reads:

$$\mathbf{K}_{\text{sec}}^{\tau sij} \Delta \mathbf{u}_{\tau i} = \mathbf{r}_{sj} \quad (16)$$

201 where  $\mathbf{K}_{\text{sec}}^{\tau sij}$  is the FN of the secant stiffness matrix involving damage, and  $\mathbf{r}_{sj}$  is the FN of  
 202 the residual force vector.

203 After solving the Eq. (15) or (16), the displacement is updated as  $\mathbf{u}(t + \Delta t) = \mathbf{u}(t) + \Delta \mathbf{u}$ .  
 204 The internal force vector and damage variables are then recomputed accordingly. Due to  
 205 the nonlinearity introduced by damage evolution, the solution generally does not converge  
 206 within a single step. Therefore, multiple iterations are performed within each time increment.  
 207 Convergence is checked based on the relative norm of the iterative displacement correction  
 208 with respect to the accumulated increment:

$$\frac{\|\Delta \mathbf{u}^k\|}{\|\mathbf{u}^k - \mathbf{u}^0\|} < \varepsilon \quad (17)$$

209 where  $\Delta \mathbf{u}^k = \mathbf{u}^k - \mathbf{u}^{k-1}$  is the displacement correction at iteration  $k$ , and  $\mathbf{u}^0$  is the displacement  
 210 at the beginning of the current time step  $t$ .

211 In this modified Newton-Raphson method, the secant stiffness matrix is adopted instead of  
 212 tangent stiffness due to its straightforward implementation. Both the secant stiffness matrix  
 213 and the internal force vector at each iteration are dependent on the damage state, which is  
 214 evaluated using the damage model introduced in the following section.

### 3 3D Tsai-Wu damage model

This section introduces the local orthotropic damage model for composite materials, in which fiber, matrix, interlamina, and shear damage states are taken into account. The corresponding constitutive equations, 3D Tsai-Wu failure criteria, and its related damage evolution law are presented. Fracture energy regularization for higher-order beam elements is employed to mitigate the mesh dependency.

#### 3.1 Damaged material response

Damage caused by a series of microcracks or microvoids is often physically considered as the loss of stiffness in the CDM theory [42]. The tensile damage and compressive damage are activated using the effective stress  $\hat{\sigma}$ . Based on the idea of the model of [6], the relation between the effective stress,  $\hat{\sigma}$ , and the nominal stress,  $\sigma$ , is proposed as:

$$\hat{\sigma} = \mathbf{M}\sigma \quad (18)$$

where  $\mathbf{M}$  is the damage operator, which is expressed considering the 3D stress state:

$$\mathbf{M} = \begin{bmatrix} \frac{1}{1-d_f} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1-d_m} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-d_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1-d_{s12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1-d_{s13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1-d_{s23}} \end{bmatrix} \quad (19)$$

where  $d_f$ ,  $d_m$ , and  $d_i$  represent damage variables for fiber, matrix, and interlamina failure modes, repetitively.  $d_{s12}$ ,  $d_{s13}$ , and  $d_{s23}$  denote three shear damage variables.

Based on the Tsai-Wu criterion,  $d_f$ ,  $d_m$ , and  $d_i$  are independent and regulated by the corresponding tension and compression values which will be demoted by subscripts  $t$  and  $s$ . In addition, shear damage terms are assumed to be dependent on the fiber, matrix, and interlaminar damages. The explicit expressions of damage variables are:

$$\begin{cases} d_f = 1 - (1 - d_{ft})(1 - d_{fc}) \\ d_m = 1 - (1 - d_{mt})(1 - d_{mc}) \\ d_i = 1 - (1 - d_{it})(1 - d_{ic}) \end{cases} \quad \begin{cases} d_{s12} = 1 - (1 - d_f)(1 - d_m) \\ d_{s13} = 1 - (1 - d_f)(1 - d_i) \\ d_{s23} = 1 - (1 - d_m)(1 - d_i) \end{cases} \quad (20)$$

Considering the damage operator matrix Eq. (19), the damaged compliance matrix can be obtained as:

$$\mathbf{H}^d = \begin{bmatrix} \frac{1}{(1-d_f)E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{(1-d_m)E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{(1-d_i)E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{(1-d_{s12})G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{(1-d_{s13})G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{(1-d_{s23})G_{23}} \end{bmatrix} \quad (21)$$

235 and the corresponding damaged stiffness matrix is obtained by inverting the compliance ma-  
 236 trix:

$$\mathbf{C}^d = \frac{1}{\Delta^d} \begin{bmatrix} C_{11}^d & C_{12}^d & C_{13}^d & 0 & 0 & 0 \\ C_{21}^d & C_{22}^d & C_{23}^d & 0 & 0 & 0 \\ C_{31}^d & C_{32}^d & C_{33}^d & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^d & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55}^d & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66}^d \end{bmatrix} \quad (22)$$

237 with

$$\Delta^d = 1 - (1 - d_f)(1 - d_m)v_{12}v_{21} - (1 - d_f)(1 - d_i)v_{13}v_{31} - (1 - d_m)(1 - d_i)v_{23}v_{32} \\ - 2 \times (1 - d_f)(1 - d_m)(1 - d_i)v_{12}v_{23}v_{31}$$

238

$$\begin{cases} C_{11}^d = (1 - d_f)E_1 [1 - (1 - d_m)(1 - d_i)v_{23}v_{32}] \\ C_{12}^d = (1 - d_f)(1 - d_m)E_1 [v_{21} + (1 - d_i)v_{23}v_{31}] \\ C_{13}^d = (1 - d_f)(1 - d_i)E_1 [v_{31} + (1 - d_m)v_{21}v_{32}] \\ C_{21}^d = (1 - d_f)(1 - d_m)E_2 [v_{12} + (1 - d_i)v_{13}v_{32}] \\ C_{22}^d = (1 - d_m)E_2 [1 - (1 - d_f)(1 - d_i)v_{13}v_{31}] \\ C_{23}^d = (1 - d_m)(1 - d_i)E_2 [v_{32} + (1 - d_f)v_{12}v_{31}] \\ C_{31}^d = (1 - d_f)(1 - d_i)E_3 [v_{13} + (1 - d_m)v_{12}v_{23}] \\ C_{32}^d = (1 - d_m)(1 - d_i)E_3 [v_{23} + (1 - d_f)v_{13}v_{21}] \\ C_{33}^d = (1 - d_i)E_3 [1 - (1 - d_f)(1 - d_m)v_{12}v_{21}] \\ C_{44}^d = \Delta(1 - d_{s12})G_{12} \\ C_{55}^d = \Delta(1 - d_{s13})G_{13} \\ C_{66}^d = \Delta(1 - d_{s23})G_{23} \end{cases}$$

239 where  $E_1$ ,  $E_2$ ,  $E_3$ ,  $G_{12}$ ,  $G_{13}$ , and  $G_{23}$  are undamaged material moduli;  $v_{12}$ ,  $v_{21}$ ,  $v_{13}$ ,  $v_{31}$ ,  $v_{23}$ ,  
 240 and  $v_{32}$  are undamaged Poisson's ratios.

## 241 3.2 Failure initiation criteria

242 Damage initiation indicates the onset of stiffness degradation at a material point, which  
 243 depends on the current stress field. The most widely used form of the Tsai-Wu criterion  
 244 employs a scalar failure surface,  $\text{TW}(\hat{\boldsymbol{\sigma}})$ , defined in the effective stress space for orthotropic  
 245 materials:

$$\text{TW}(\hat{\boldsymbol{\sigma}}) = F_{11}\hat{\sigma}_1^2 + F_{22}\hat{\sigma}_2^2 + F_{33}\hat{\sigma}_3^2 + 2F_{23}\hat{\sigma}_2\hat{\sigma}_3 + 2F_{13}\hat{\sigma}_1\hat{\sigma}_3 + 2F_{12}\hat{\sigma}_1\hat{\sigma}_2 + \\ F_1\hat{\sigma}_1 + F_2\hat{\sigma}_2 + F_3\hat{\sigma}_3 + F_{44}\hat{\tau}_{23}^2 + F_{55}\hat{\tau}_{12}^2 + F_{66}\hat{\tau}_{12}^2 \quad (23)$$

246 where  $F_i$  and  $F_{ij}$  ( $i, j = 1, \dots, 6$ ) are strength tensors of the second and fourth rank, respec-  
 247 tively. Due to the symmetry of orthotropic materials, certain components such as  $F_4$  and  
 248  $F_{45}$  are expected to vanish. The failure criterion claims that the structure is safe if  $\text{TW} < 1$ ,  
 249 whereas the critical condition for failure occurs when  $\text{TW} = 1$ .

250  $F_i$  and  $F_{ii}$  are related to the one-directional engineering strengths, which can be measured  
 251 experimentally in the laboratory. These components are defined as:

$$F_{11} = \frac{1}{X_t X_c}; \quad F_{22} = \frac{1}{Y_t Y_c}; \quad F_{33} = \frac{1}{Z_t Z_c}; \quad (24)$$

$$F_{44} = \frac{1}{S_I^2}; \quad F_{55} = \frac{1}{S_T^2}; \quad F_{66} = \frac{1}{S_L^2}$$

$$F_1 = \frac{1}{X_t} - \frac{1}{X_c}; \quad F_2 = \frac{1}{Y_t} - \frac{1}{Y_c}; \quad F_3 = \frac{1}{Z_t} - \frac{1}{Z_c} \quad (25)$$

253 where  $X_t$  and  $X_c$  represent the uniaxial tensile and compressive fiber strengths;  $Y_t$  and  $Y_c$   
 254 correspond to the tensile and compressive matrix strengths;  $Z_t$  and  $Z_c$  denote the tensile  
 255 and compressive interlaminar strengths;  $S_I$ ,  $S_T$ , and  $S_L$  are the interlaminar shear strength,  
 256 transverse shear strength, and longitudinal shear strength, respectively.

257 The determination of the off-diagonal components,  $F_{ij}$ , in the Tsai-Wu criterion is chal-  
 258 lenging. In the original formulation of the Tsai-Wu criterion [1], the failure envelope is  
 259 constrained to be an ellipsoid, ensuring that it remains closed. This constraint imposes a  
 260 condition on  $F_{ij}$ :

$$F_{ij}^2 < F_{ii} F_{jj} \quad (26)$$

261 The constraint of Eq. (26) provides a range for  $F_{ij}$  rather than a fixed value. Additionally,  
 262 [1] noted that  $F_{ij}$  can be determined under combined stress states, which can be represented  
 263 as:

$$\begin{aligned} F_{12} &= \frac{1}{2B_{12}^2} (1 - B_{12}(F_1 + F_2) - B_{12}^2(F_{11} + F_{22})) \\ F_{13} &= \frac{1}{2B_{13}^2} (1 - B_{13}(F_1 + F_3) - B_{13}^2(F_{11} + F_{33})) \\ F_{23} &= \frac{1}{2B_{23}^2} (1 - B_{23}(F_2 + F_3) - B_{23}^2(F_{22} + F_{33})) \end{aligned} \quad (27)$$

264 where  $B_{ij}$  corresponds to the tensile bi-stress strength, which requires difficult biaxial stress  
 265 tests. This term can be estimated using several outputs according to [43]. Eq. (28) from  
 266 [43], where a lower bound estimate can be obtained using the maximum failure envelope, is  
 267 employed in this work.

$$\begin{aligned} B_{12} &= \min \left\{ \frac{Y_t}{1 - \nu_{21}}, \frac{X_t}{1 - \nu_{12}} \right\} \\ B_{13} &= \min \left\{ \frac{Z_t}{1 - \nu_{31}}, \frac{X_t}{1 - \nu_{13}} \right\} \\ B_{23} &= \min \left\{ \frac{Z_t}{1 - \nu_{32}}, \frac{Y_t}{1 - \nu_{23}} \right\} \end{aligned} \quad (28)$$

268 An alternative empirical approach based on the generalization of the von Mises criterion  
 269 [44, 45] was also proposed, which satisfies the constraint in Eq. (26). This empirical expression  
 270 is given by:

$$F_{ij} = -\frac{1}{2} \sqrt{F_{ii} F_{jj}} \quad (29)$$

271 As demonstrated in [46], the assumption of a closed failure envelope is not universally valid  
 272 for all composites, and the constraint in Eq. (26) may be relaxed under certain circumstances.  
 273 Nevertheless, for the composites considered in this work,  $F_{ij}$  satisfies Eq. (26). Accordingly,  
 274  $F_{ij}$  is first calculated using Eq. (27). If the computed values violate the condition in Eq. (26),  
 275 the empirical expression in Eq. (29) is applied instead.

### 276 3.3 Damage evolution

277 Once  $TW(\hat{\boldsymbol{\sigma}}) = 1$ , further loading will cause the degradation of material stiffness coefficients.  
 278 The reduction in stiffness coefficients is governed by damage variables that range from zero  
 279 for intact material to one for completely damaged material.

280 In this framework, the evolution laws for each damage variable can be formulated in the  
 281 strain space, as described in [6]. New variables are introduced based on the definitions of  
 282 equivalent stresses and strains. These equivalent measures are derived using the generalized  
 283 equivalent principal virtual work, expressed as:

$$\sigma_{eq}\varepsilon_{eq} = \sum \sigma_{ij}\varepsilon_{ij}, \quad \text{and} \quad \varepsilon_{eq} = \sqrt{\sum \varepsilon_{ij}^2}, \quad i, j = 1, 2, \dots, 6 \quad (30)$$

284 For the specific cases of fiber, matrix, and interlaminar failure, the corresponding equiva-  
 285 lent strains and stresses are reformulated:

$$\left\{ \begin{array}{l} \varepsilon_{eq,ft} = \sqrt{\langle \varepsilon_1 \rangle^2 + \gamma_{12}^2 + \gamma_{13}^2} \\ \varepsilon_{eq,fc} = \langle -\varepsilon_1 \rangle \\ \varepsilon_{eq,mt} = \sqrt{\langle \varepsilon_2 \rangle^2 + \langle \varepsilon_3 \rangle^2 + \gamma_{21}^2 + \gamma_{23}^2 + \gamma_{31}^2} \\ \varepsilon_{eq,mc} = \sqrt{\langle -\varepsilon_2 \rangle^2 + \langle -\varepsilon_3 \rangle^2 + \gamma_{21}^2 + \gamma_{23}^2 + \gamma_{31}^2} \\ \varepsilon_{eq,it} = \sqrt{\langle \varepsilon_3 \rangle^2 + \gamma_{31}^2 + \gamma_{32}^2} \\ \varepsilon_{eq,ic} = \langle -\varepsilon_3 \rangle \end{array} \right. \quad (31)$$

$$\left\{ \begin{array}{l} \sigma_{eq,ft} = \frac{\langle \sigma_1 \rangle \langle \varepsilon_1 \rangle + \tau_{12}\gamma_{12} + \tau_{13}\gamma_{13}}{\sqrt{\langle \varepsilon_1 \rangle^2 + \gamma_{12}^2 + \gamma_{13}^2}} \\ \sigma_{eq,fc} = \langle -\sigma_1 \rangle \\ \sigma_{eq,mt} = \frac{\langle \sigma_2 \rangle \langle \varepsilon_2 \rangle + \langle \sigma_3 \rangle \langle \varepsilon_3 \rangle + \tau_{21}\gamma_{21} + \tau_{23}\gamma_{23} + \tau_{31}\gamma_{31}}{\sqrt{\langle \varepsilon_2 \rangle^2 + \langle \varepsilon_3 \rangle^2 + \gamma_{21}^2 + \gamma_{23}^2 + \gamma_{31}^2}} \\ \sigma_{eq,mc} = \frac{\langle -\sigma_2 \rangle \langle -\varepsilon_2 \rangle + \langle -\sigma_3 \rangle \langle -\varepsilon_3 \rangle + \tau_{21}\gamma_{21} + \tau_{23}\gamma_{23} + \tau_{31}\gamma_{31}}{\sqrt{\langle -\varepsilon_2 \rangle^2 + \langle -\varepsilon_3 \rangle^2 + \gamma_{21}^2 + \gamma_{23}^2 + \gamma_{31}^2}} \\ \sigma_{eq,it} = \frac{\langle \sigma_3 \rangle \langle \varepsilon_3 \rangle + \tau_{31}\gamma_{31} + \tau_{32}\gamma_{32}}{\sqrt{\langle \varepsilon_3 \rangle^2 + \gamma_{31}^2 + \gamma_{32}^2}} \\ \sigma_{eq,ic} = \langle -\sigma_3 \rangle \end{array} \right. \quad (32)$$

286 where  $\langle \cdot \rangle$  is the Macaulay bracket to have non negative values.

287 To define the damage evolution laws, it is necessary to obtain the equivalent stress,  $\sigma_{eq,I}^0$ ,  
 288 and equivalent strain,  $\sigma_{eq,I}^0$ , at the onset of damage initiation. In the effective stress space,  
 289 the effective stress varies linearly with strain. Thus, the following relation is obtained:

$$\hat{\sigma}_{ij}^{\text{onset}} = \beta \hat{\sigma}_{ij}(d_0), \quad \hat{\sigma}_{ij}(d_0) \propto \epsilon_{ij}(d_0) \implies \epsilon_{ij}^{\text{onset}} = \beta \epsilon_{ij}(d_0) \quad (33)$$

290 where  $\hat{\sigma}_{ij}^{\text{onset}}$  denotes the effective stress at the onset of damage, which can be determined  
 291 by scaling the effective stress  $\hat{\sigma}_{ij}(d_0)$  with damage  $d_0$ . Similarly, the strain at the onset of  
 292 damage,  $\epsilon_{ij}^{\text{onset}}$ , is determined by scaling the strain  $\epsilon_{ij}(d_0)$  with damage  $d_0$ .

293

The determination of the scaling factor  $\beta$  involves solving the Tsai–Wu criterion:

$$\text{TW}(\beta\hat{\boldsymbol{\sigma}}) = 1 \quad (34)$$

294

where TW is defined in Eq. (23). Substituting Eq. (23) into Eq. (34) leads to a quadratic equation for  $\beta$ , as shown below:

295

$$\beta^2 \underbrace{(2F_{12}\hat{\sigma}_1\hat{\sigma}_2 + 2F_{13}\hat{\sigma}_1\hat{\sigma}_3 + 2F_{23}\hat{\sigma}_2\hat{\sigma}_3 + F_{11}\hat{\sigma}_1^2 + F_{22}\hat{\sigma}_2^2 + F_{33}\hat{\sigma}_3^2)}_a + \underbrace{(F_{66}\hat{\tau}_{12}^2 + F_{55}\hat{\tau}_{13}^2 + F_{44}\hat{\tau}_{23}^2)}_a + \beta \underbrace{(F_1\hat{\sigma}_1 + F_2\hat{\sigma}_2 + F_3\hat{\sigma}_3)}_b + \underbrace{(-1)}_c = 0 \quad (35)$$

296

in which  $a$ ,  $b$ , and  $c$  are coefficients of the quadratic equation. Therefore,  $\beta$  is given by:

$$\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (36)$$

297

In many orthotropic damage models, it is assumed that damage evolution primarily depends on the stresses and strains associated with the corresponding damage plane. Therefore, the Tsai–Wu failure criterion must be independently solved for fiber, matrix, and interlaminar behavior under both tension and compression. To ensure consistency, Eq. (35), used to calculate the  $\beta$  variable, must include only the stress components corresponding to the equivalent stresses defined in Eq. (32) for each failure mode  $I$ , i.e., fiber, matrix, and interlaminar, in tension and compression.

304

In this work, the damage evolution laws for each failure mode  $I$  are defined based on a linear softening path. To mitigate the mesh dependency, an equivalent displacement-stress relationship, as shown in Fig. 3, is obtained by introducing the characteristic element length  $l_c$ . Then, the damage evolution function of each damage variable is expressed as:

305

306

307

$$d_I = \frac{\delta_{eq,I}^u (\delta_{eq,I} - \delta_{eq,I}^0)}{\delta_{eq,I} (\delta_{eq,I}^u - \delta_{eq,I}^0)}, \quad \text{if } \delta_{eq,I}^0 \leq \delta_{eq,I} \leq \delta_{eq,I}^u \quad (37)$$

308 with

$$\delta_{eq,I}^0 = l_c \varepsilon_{eq,I}^0 \quad \text{and} \quad \delta_{eq,I}^u = \frac{2G_{f,I}}{\sigma_{eq,I}^0} \quad (38)$$

309

where  $\delta_{eq,I}^0$  and  $\sigma_{eq,I}^0$  are the equivalent displacements and stresses, respectively, at the onset of damage.  $\delta_{eq,I}^u$  is the ultimate equivalent displacement when the material is fully damaged.

311

It is evident that  $l_c$  directly influences the softening curve by controlling the ultimate equivalent displacement  $\delta_{eq,I}^u$ . Smaller values of  $l_c$  lead to gradual softening curves, while larger values result in steeper softening. This relationship ensures that the total energy dissipation remains consistent with the experimental fracture energy.

314

315

### 3.4 Characteristic element length

316

Determining the characteristic element length is critical in fracture energy regularization techniques to maintain the objectivity of energy dissipation. This length should reflect the evolving width of the fracture zone throughout the fracture process [47]. Traditional methods that rely solely on constant calculations of element areas or volumes are often insufficient [48]. As highlighted by [49], the determination of  $l_c$  depends on several factors, including the

320

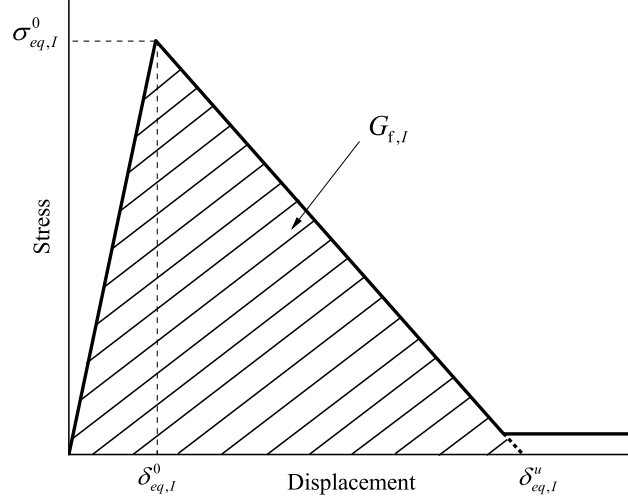


Figure 3: Equivalent displacement-stress for fiber, matrix, and interlamina

321 shape and size of element, the interpolation function employed, the integration scheme, and  
 322 the mesh line orientation. This work employs the projection method proposed by [41].

323 Figure 4 employs a B3 beam element with one L9 expansion element to explain the idea of  
 324 [41]. First, a volume is constructed using the beam element through Lagrange expansion. Two  
 325 parameters  $\alpha_1$  and  $\alpha_2$  are introduced to scale the beam element length and cross-sectional  
 326 dimensions, leading to a scaled sub-volumes. For second-order elements shown in Fig. 4,  
 327  $\alpha_1$  and  $\alpha_2$  are set to 0.5 according to [41]. Subsequently, a projection method is applied  
 328 to the scaled sub-volumes containing the target point, i.e., Gauss point (GP). This projection  
 329 method determines the  $l_c$  along the unit vector of the major principal strain at each GP,  
 330 indicating each GP from the same element has an individual  $l_c$ . Therefore, the formulation  
 331 for  $l_c$  of each GP,  $\mathbf{x}$ , can be generalized as:

$$l_c(\mathbf{x}) = |\max[\mathbf{x}_n \cdot \mathbf{n}(\mathbf{x})] - \min[\mathbf{x}_n \cdot \mathbf{n}(\mathbf{x})]| \quad (39)$$

332 where  $\mathbf{x}_n$  is the middle point of the edge in a scaled volume that contains the targeted Gaussian  
 333 point  $\mathbf{x}$ , as illustrated in Fig. 4.  $\mathbf{n}(\mathbf{x})$  is the unit vector in the major principal strain direction.

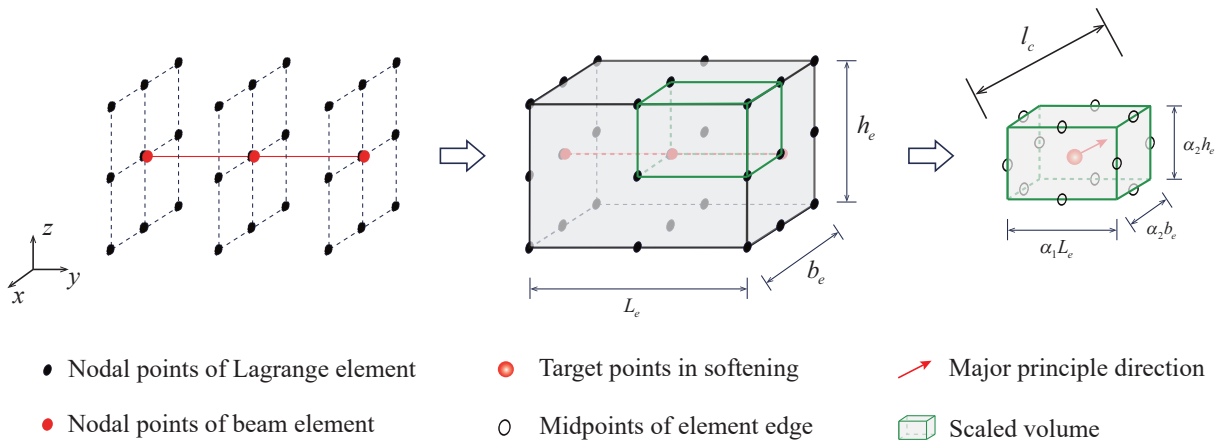


Figure 4: Illustration of calculation for  $l_c$

### 3.5 Viscous regularization

Although the fracture energy regularization ensures consistent fracture energy dissipation, the governing equations remain ill-posed, highlighting that fracture energy regularization is only a partial regularization technique. This issue can result in the loss of positive definiteness of the tangent stiffness matrix and numerical convergence difficulties.

Therefore, a viscous regularization scheme can be implemented directly to mitigate these numerical issues. This approach relies on the artificial Duvaut-Lions viscosity model, which ensures that the tangent stiffness matrix of softening materials remains positive definite for sufficiently small time increments.

In damage modeling, the viscous damage variable in the regularization scheme is defined as follows:

$$\delta d_I^v = \frac{1}{\eta_I} (d_I - d_I^v) \times \delta t \quad (40)$$

where  $\eta_I$  represents a viscosity coefficient indicating the relaxation time of the viscous system,  $d_I$  means the non-viscous damage variable and  $d_I^v$  denotes viscous damage variable corresponding to mode  $I$  (including various concrete and composite damage modes). The damaged material response is then evaluated using the viscous damage variables.

A numerical algorithm is required to solve Eq. (40). A straightforward approach is to update the viscous damage variable using the backward Euler method, expressed as:

$$d_I^v(t_0 + \Delta t) = \frac{\Delta t}{\eta + \Delta t} d_I^v(t_0 + \Delta t) + \frac{\eta}{\eta + \Delta t} d_I^v(t_0) \quad (41)$$

where  $\Delta t$  is size of time step.

Eq. (41) indicates the current viscous damage variable  $d_I^v(t_0 + \Delta t)$  is determined as a weighted combination of the current non-viscous damage variable  $d_I^v(t_0 + \Delta t)$  and the previous viscous damage variable  $d_I^v(t_0)$ .

## 4 Numerical examples

This section investigates the progressive damage behavior of three pultruded composite structures using the proposed numerical models. For all numerical tests, the displacement control method was employed. The load-displacement responses and damage patterns were obtained and compared to the experimental data.

### 4.1 Compact tension test

The first example is based on an orthotropic Glass Fiber Reinforced Polymer (GFRP) specimen which was conducted experimentally in the work of [50] to assess the fracture energy through a Compact Tension (CT) test. The full details of the experimental layout is shown in Fig. 5(a). The loading and geometrical conditions of the CT specimen are illustrated in Fig. 5(b). The specimen has a thickness of 10 mm, with the lower pin clamped and a vertical displacement applied to the upper pin until a maximum value of 1.2 mm is reached. The material properties relevant to the strength and fracture energies of the GFRP material are summarized in Table 1.

Beam elements can be assigned along the thickness of the specimen which is shown in Fig. 6(a). The cross-sectional discretization is illustrated in Fig. 6(b), where Lagrange elements are employed. In this cross-section, the fiber direction is oriented along the z-axis, and the

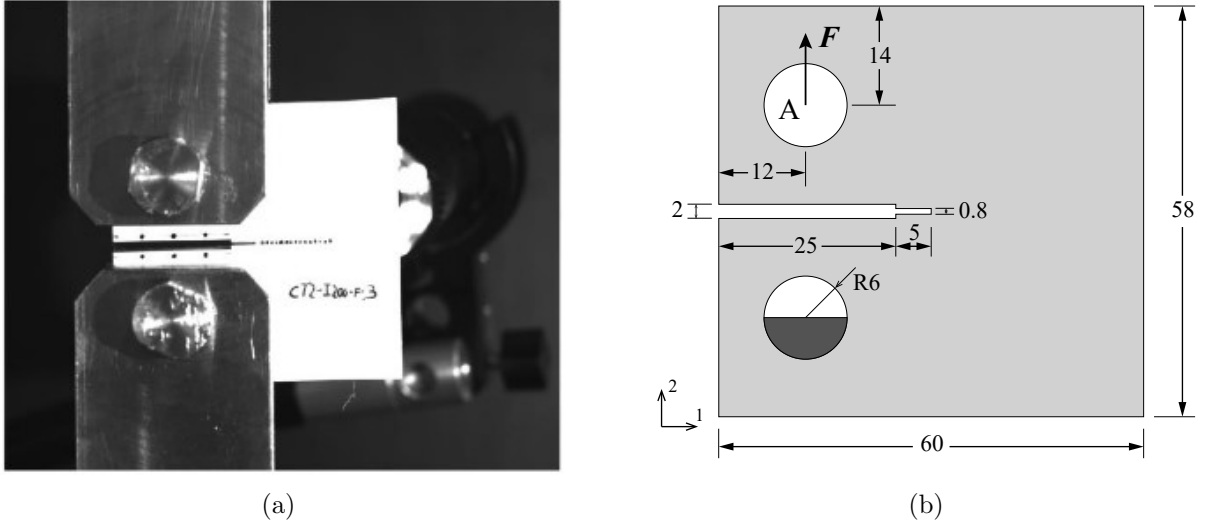


Figure 5: (a) Experimental layout [50] and (b) geometry for CT test (mm)

Table 1: Material properties of the notched GFRP specimen

Properties	Units	Symbols	Values	Symbols	Values
Elastic moduli	GPa	$E_{11}$	30	$G_{12}$	3.0
		$E_{22}$	12	$G_{13}$	3.0
		$E_{33}$	12	$G_{23}$	3.0
Poisson's ratios	-	$\nu_{12}$	0.24		
		$\nu_{13}$	0.24		
		$\nu_{23}$	0.3		
Strengths	MPa	$X_t$	323	$X_c$	426
		$Y_t$	71	$Y_c$	71
		$Z_t$	71	$Z_c$	71
		$S_L$	67	$S_T$	64
		$S_I$	64		
Fracture energies	MPa·mm	$G_{ft}$	100	$G_{fc}$	100
		$G_{mt}$	20	$G_{mc}$	20
		$G_{it}$	20	$G_{ic}$	20

372 matrix is aligned along the x-axis. The gray circular supports in Fig. 6(b) are modeled with  
 373 a linear isotropic steel material.

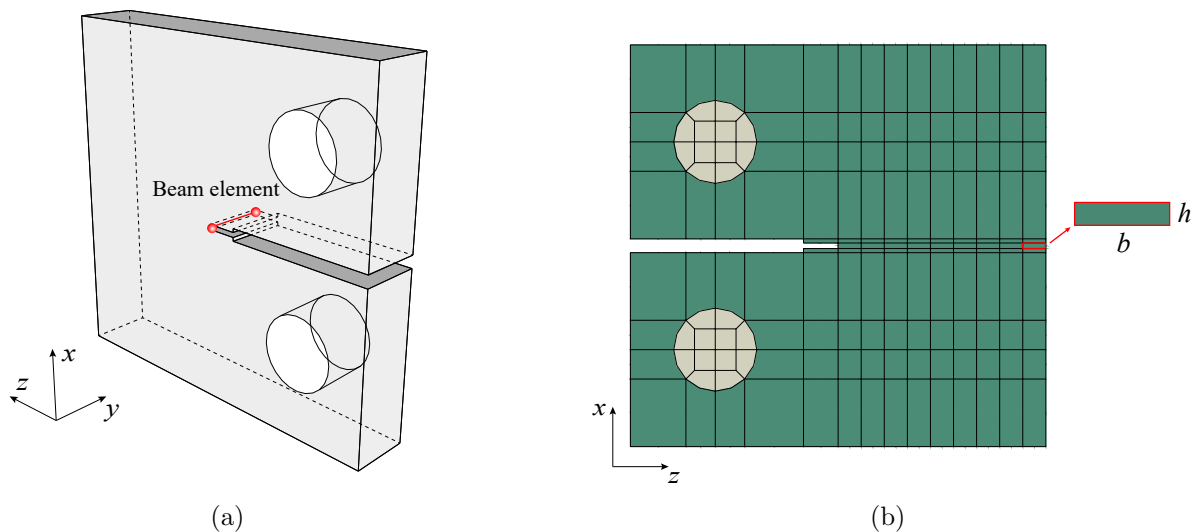


Figure 6: (a) Beam elements and (b) cross-sectional discretizations of the CT specimen

374 To fully validate the proposed fracture energy regularization technique for mitigating mesh  
 375 dependency, Table 2 lists ten CUF-based beam models. These models account for variations  
 376 in both the size and order of beam elements as well as the cross-sectional discretization, i.e.,  
 377 structural theory. The meaning of 'b' and 'h' in Table 2 can be seen in Fig. 6(b).

Table 2: Model information for CT test

Model No.	Cross-sectional discretization		Beam discretization		DOF
	Number	$b \times h$ (mm $\times$ mm) <sup>1</sup>	Number	Length (mm) <sup>2</sup>	
Model 1	157 L4	3.33 $\times$ 0.8	1 B2	10	1,332
Model 2	157 L4	3.33 $\times$ 0.8	1 B3	10	1,896
Model 3	157 L4	3.33 $\times$ 0.8	1 B4	10	2,460
Model 4	157 L9	3.33 $\times$ 0.8	1 B4	10	8,925
Model 5	157 L16	3.33 $\times$ 0.8	1 B4	10	19,500
Model 6	175 L4	3.33 $\times$ 0.267	1 B4	10	2,700
Model 7	388 L4	1.0 $\times$ 0.8	1 B4	10	5,484
Model 8	388 L16	1.0 $\times$ 0.8	1 B4	10	44,916
Model 9	388 L16	1.0 $\times$ 0.8	2 B4	5	77,730
Model 10	388 L16	1.0 $\times$ 0.8	3 B4	3.33	110,544

<sup>1</sup> The individual discretized cross-sectional mesh size in the notch.

<sup>2</sup> The individual beam element length.

378 The load-displacement diagrams of all numerical models are plotted in Fig. 7, including  
 379 experimental results for comparison. In Fig. 7(a), Models 1 to 3, which vary in beam element  
 380 order, show nearly identical curves, suggesting that the beam element order has minimal  
 381 impact on numerical results in this case. However, with higher-order LE, as seen in Models 4  
 382 and 5, the initial stiffness and post-peak softening responses are slightly reduced compared to  
 383 the first three models. This minor variation can be attributed to the improved capability of  
 384 higher-order LE to capture cross-sectional deformation. Overall, the numerical results closely

align with the experimental data in Fig. 7(a), confirming that the proposed model effectively mitigates the influence of beam element and cross-sectional LE order.

In Fig. 7(b), Models 3, 6, and 7 explore the influence of LE size by varying the number of LE, while Models 8 to 10 examine the influence of beam element size by adjusting the number of beam elements along the thickness. The curves in Fig. 7(b) can be separated into two distinct groups: one group with L4 models and the other with L16 models. Within each group, the curves are nearly coincident and within the experimental ranges, indicating the proposed method effectively achieves mesh size independence. However, L4 models yield slightly higher curves than L16 models, likely due to the linear approximation in L4 models compared to the higher-order approximation in L16 models. This distinction highlights the enhanced accuracy of L16 models in capturing the structural response.

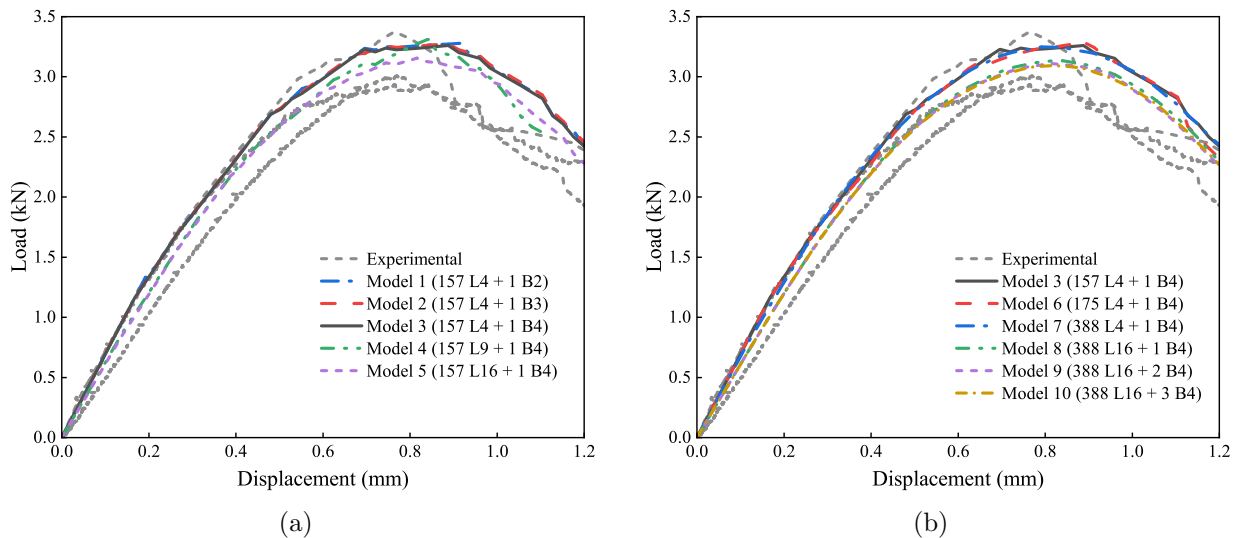


Figure 7: Reaction load-displacement curves of the CT test from various CUF-based models: (a) Influence of order and (b) influence of mesh size

Since external loads are applied along the x-axis, aligned with the direction of the matrix, Fig. 8 shows the progression of the tensile and compressive damage within the matrix in the CT test of Model 5. The results indicate that the damage begins at the corner of the notch and propagates into the specimen as a narrow band. Compressive matrix damage occurs on the right side of the notch, indicating that it can also be captured in the proposed damage model framework. The applied displacement of 0.816 mm corresponds to the peak load in the load-displacement diagram. Compared to the damage distribution at 1.2 mm displacement, the matrix tensile damage at the peak load appears to have only propagated halfway. Meanwhile, the matrix compressive damage seems to be in its early stages, with only minimal areas showing signs of damage. This phenomenon suggests that satisfying the Tsai-Wu criterion does not necessarily imply the complete failure of the structure. Furthermore, even when the structure reaches its maximum load-carrying capacity, it does not entirely fail and retains some residual loading capacity.

The tensile and compressive matrix damage distributions from other models are similar as those presented in Fig. 8. Although six damage parameters, which are fiber, matrix, and interlaminar damage under compression and tension, are considered in the proposed orthotropic damage models, some of them are almost negligible in this case and will not be presented here.

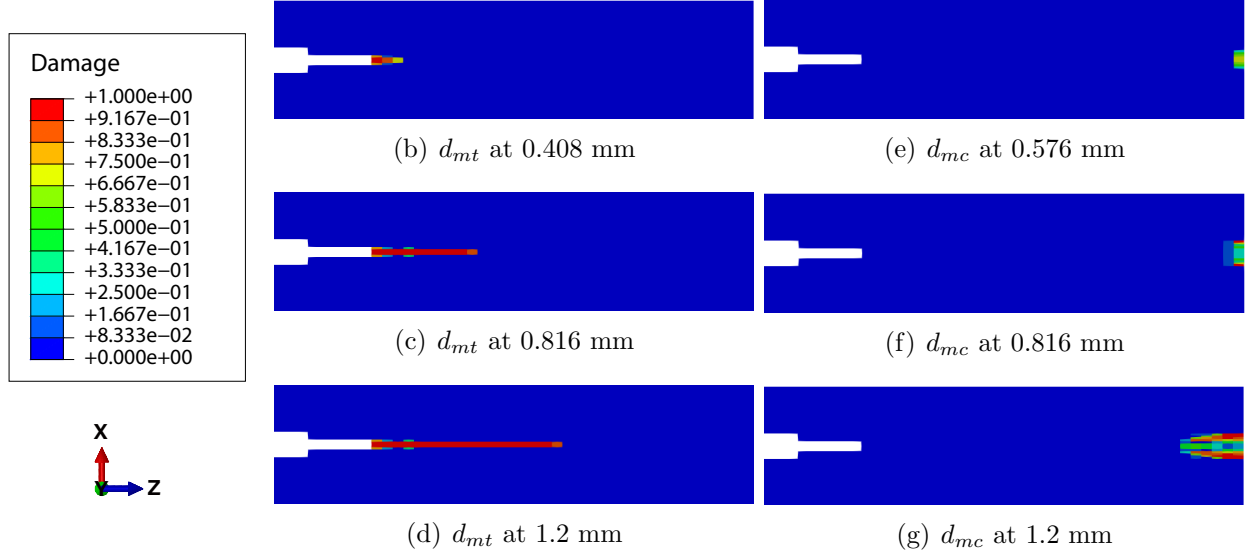


Figure 8: Tensile and compressive matrix damage propagations of the CT test from Model 5

## 4.2 Three-point bending beam

The pultruded GFRP beam with a notch at mid-span in a Three-Point Bending (TPB) test was investigated originally by [51], in which the mode I fracture properties of pultruded GFRP composites along the matrix direction were extracted. The dimension and loading layout can be seen in Fig. 9. In this case, the loading is controlled by displacement with the maximum value of 1.2 mm and the Crack Mouth Opening Displacements (CMOD) at the notch bottom are obtained for plotting the load-CMOD curves. Based on the work of [27], material properties are summarized in Table 3. The measured fracture energies from [51] and [27] are different. In this work, the average values are employed.

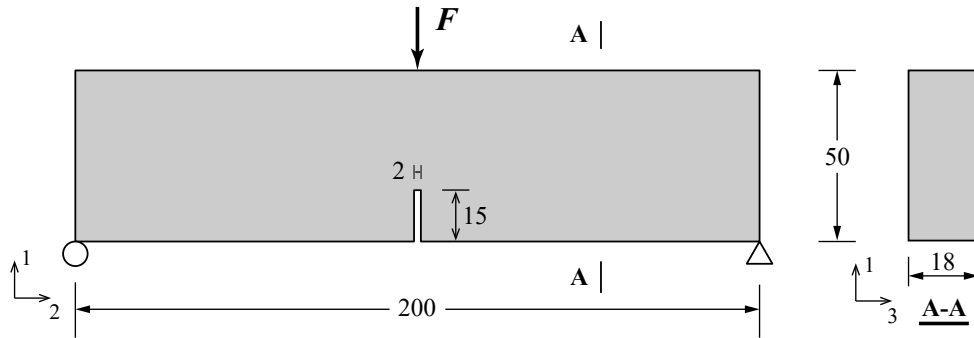


Figure 9: Load and geometry of the TPB beam (mm)

To reduce computational costs, a quarter structure is employed for numerical analysis, which can be seen in Fig. 10. The beam elements are assigned as shown in Fig. 10(a), where only half length of the beam is employed. Six B4 elements are adopted for the un-notched part. Different order and number of beam elements are only employed for the middle notch to investigate the mesh dependency. Five beam element assignments in the notch are shown in Fig. 10(a). The cross-section is halved and discretized as show in Fig. 10(b). Different cross-sectional discretizations are taken into account. Overall, seven CUF models are considered, as detailed in Table 4, in which beam element size and order as well as cross-sectional discretizations are considered.

Table 3: Material properties of the TPB specimen

Properties	Units	Symbols	Values	Symbols	Values
Elastic moduli	GPa	$E_{11}$	30	$G_{12}$	2.0
		$E_{22}$	10.8	$G_{13}$	2.0
		$E_{33}$	10.8	$G_{23}$	2.0
Poisson's ratios	-	$\nu_{12}$	0.24		
		$\nu_{13}$	0.24		
		$\nu_{23}$	0.4		
Strengths	MPa	$X_t$	323	$X_c$	423
		$Y_t$	37	$Y_c$	80
		$Z_t$	37	$Z_c$	80
		$S_L$	33	$S_T$	33
		$S_I$	33		
Fracture energies	MPa·mm	$G_{ft}$	130	$G_{fc}$	130
		$G_{mt}$	5.12	$G_{mc}$	5.12
		$G_{it}$	5.12	$G_{ic}$	5.12

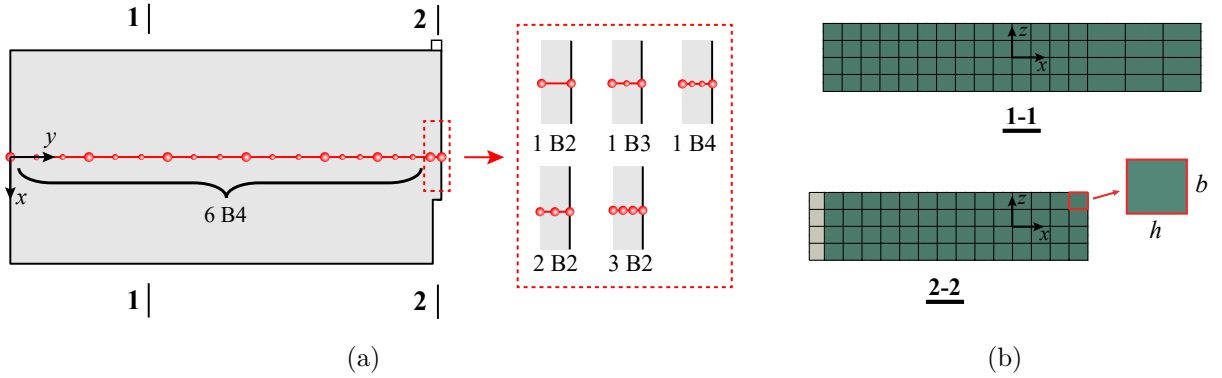


Figure 10: (a) Beam element and (b) cross-sectional discretizations of the TPB beam

Table 4: Model information for TPB beam

Model No.	Cross-sectional discretization		Beam discretization		DOF
	Number <sup>1</sup>	b×h (mm×mm)	Number	Length (mm) <sup>2</sup>	
Model 1	45 L9	3.00×2.92	6 B4 + 1 B2	1.00	12,978
Model 2	45 L9	3.00×2.92	6 B4 + 1 B3	1.00	13,545
Model 3	45 L9	3.00×2.92	6 B4 + 1 B4	1.00	14,112
Model 4	68 L9	2.25×2.50	6 B4 + 1 B2	1.00	18,846
Model 5	80 L9	2.25×2.06	6 B4 + 1 B2	1.00	22,086
Model 6	80 L9	2.25×2.06	6 B4 + 2 B2	0.50	23,085
Model 7	80 L9	2.25×2.06	6 B4 + 3 B2	0.33	24,084

<sup>1</sup> The number of L9 in Section 1-1 shown in Fig. 10(b).

<sup>2</sup> The individual beam element length in the notch.

432 Figure 11 shows the vertical reaction load-CMOD curves of seven CUF models, including  
433 experimental results from [51] for comparison. For the linear part or before the crack load,  
434 all numerical results are close to the experimental results. From Models 1 to 3, it is evident  
435 that the use of higher-order beam elements in the notch causes the decrease of peak load,  
436 which can be attributed to mesh sensitivity to the element order. The curves of Models 1,  
437 4 and 5 are close with each other, indicating the cross-sectional discretization influence the  
438 results significantly. When increasing the number of B2 elements in the notch, Models 6 and 7  
439 present slightly lower curves than Model 5, which can be attributed to the better convergence.  
440 However, Models 6 and 7 are close to each other, indicating no mesh dependency.

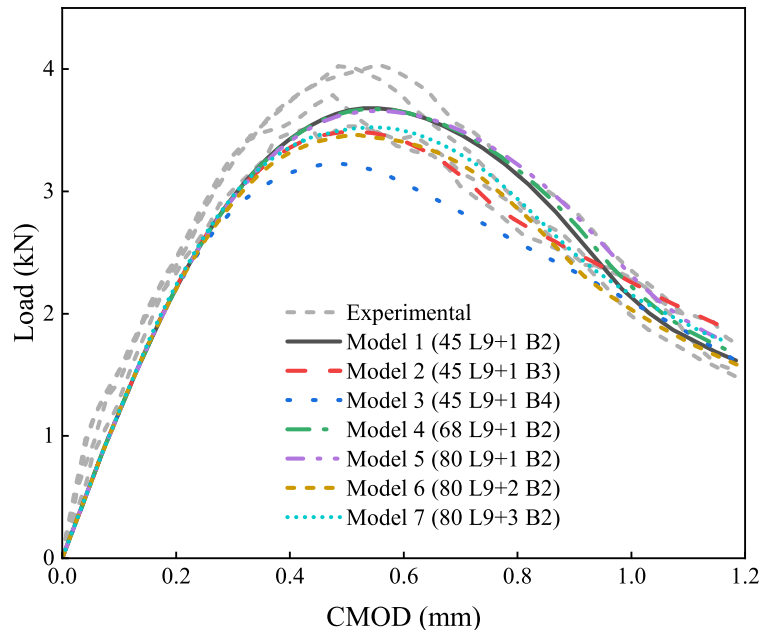


Figure 11: Load-CMOD curves of the TPB beam across different CUF models

441 Under TPB loading, the matrix, aligned along the y-axis, exhibits significant damage  
442 compared to other damage variables. Figure 12 shows that all models exhibit a distinct and  
443 narrow tensile damage band in the midspan notched section. Meanwhile, Figure 13 displays  
444 that all models exhibit a slightly compressive damage near the loading plate. These con-  
445 sistent matrix damage distributions across all models demonstrate the accuracy of proposed  
446 numerical models.

447 The most efficient model in Table 4 is Model 1 which requires 12,978 DOF. Since the  
448 critical zone of this structure is the notched part in the middle which is small compared to  
449 the whole structure, it is possible to apply the NDK approach to enhance the efficiency of  
450 the CUF model. Based on the Model 1, three NDK models are shown in Fig. 14. In NDK 1,  
451 beam nodes in two B4 elements close to the notch employ LE, while the rest employ TE. In  
452 Fig. 14(b), the zone using LE is further reduced to save more computational costs. The zone  
453 using LE in NDK 3 is the same as that in NDK 2. However, some beam nodes using TE3 are  
454 adopted between LE and TE1, which can work as a transition.

455 Based on the three NDK configurations, Table 5 lists five NDK models used in this study.  
456 These NDK models are based on Model 1 which adopts seven beam elements and twenty  
457 beam nodes. Additionally, the cross-sectional discretization of Model 1 is retained in all NDK  
458 models. Models 8 and 9 utilize the NDK 1. However, Models 8 and 9 employ the TE1  
459 and TE3 in non-critical zones, respectively, to evaluate the impact of higher-order TE on  
460 structural accuracy. Similarly, Models 10 and 11 are based on NDK 2 and the effects of TE

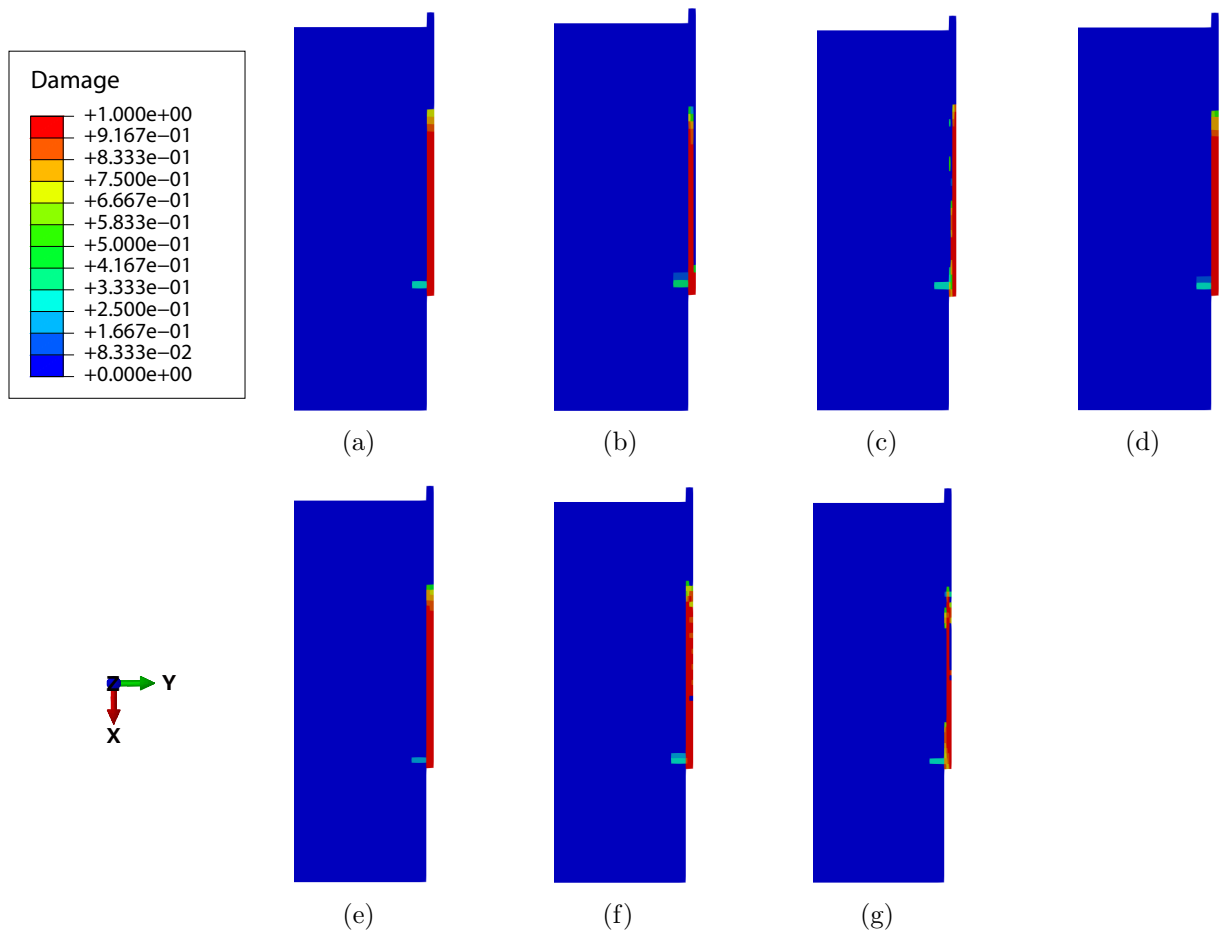


Figure 12: Matrix tensile damage distribution of the TPB beam from (a) Model 1; (b) Model 2; (c) Model 3; (d) Model 4; (e) Model 5; (f) Model 6; (g) Model 7.

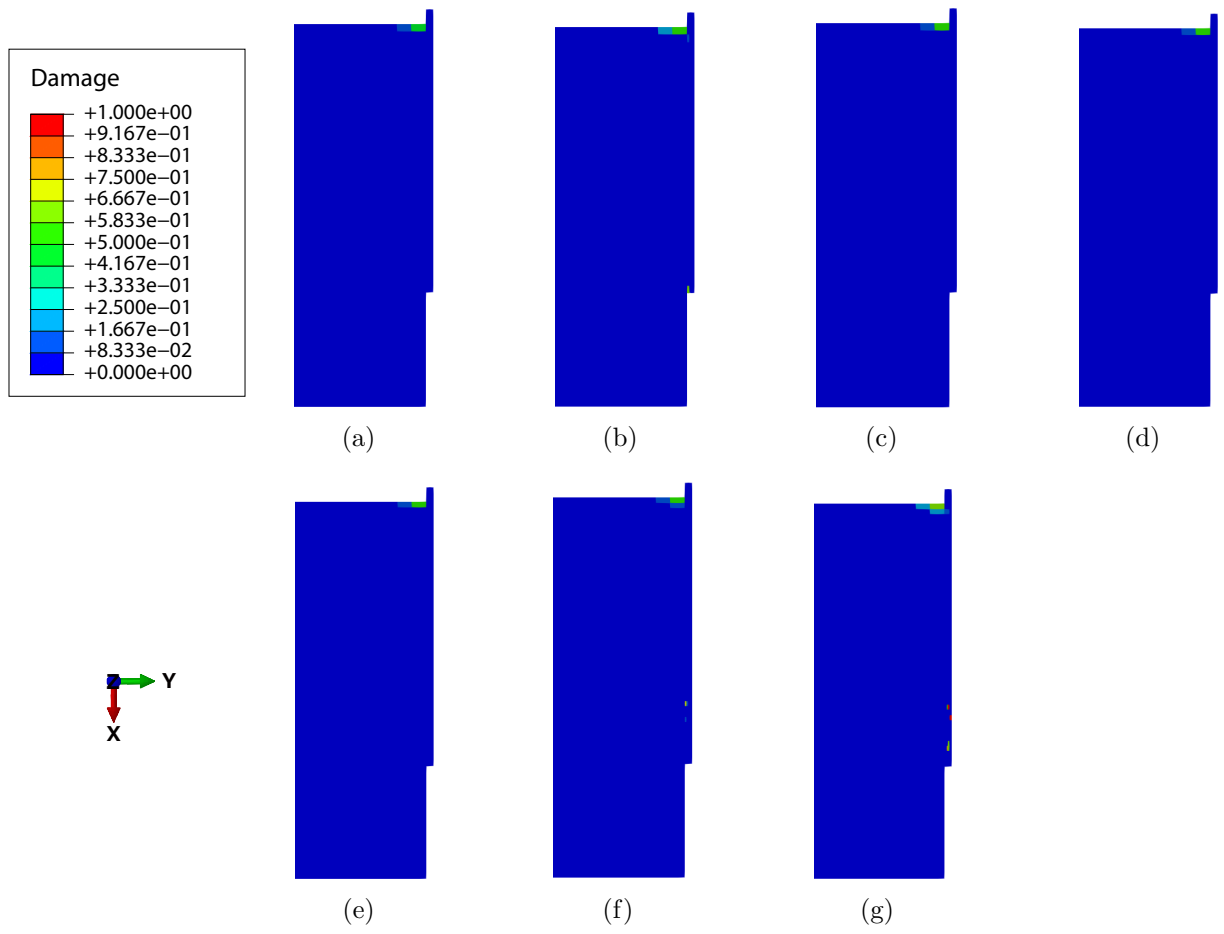


Figure 13: Matrix compressive damage distribution of the TPB beam from (a) Model 1; (b) Model 2; (c) Model 3; (d) Model 4; (e) Model 5; (f) Model 6; (g) Model 7.

461 orders are investigated. Moreover, Model 12 adopts the NDK 3, where TE3 elements form a  
 462 transition zone between LE and TE1.

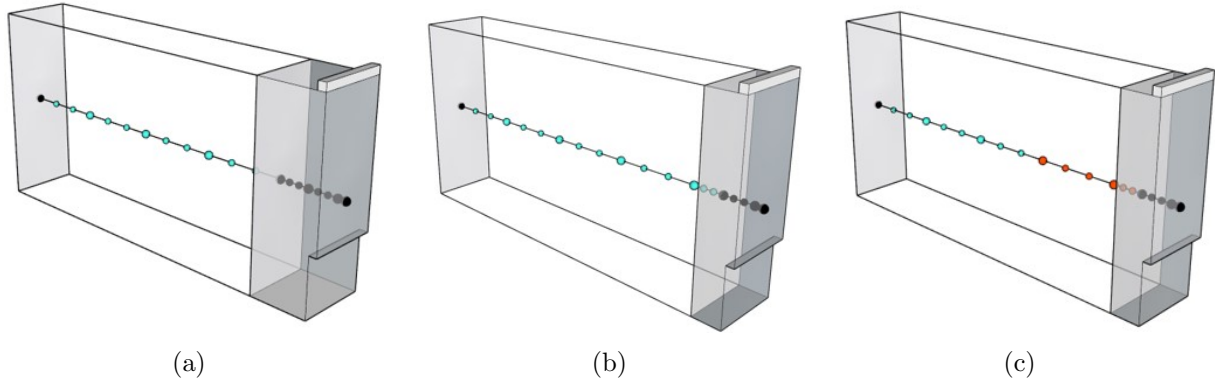


Figure 14: (a) NDK 1 model, (b) NDK 2 model, and (c) NDK 3 model based on Model 1 for the analysis of the TPB beam-Cyan represents TE1 and red denotes TE3

Table 5: NDK models based on Model 1 for the notched TPB beam

Model No.	NDK type <sup>1</sup>	Expansion of beam nodes <sup>2</sup>	DOF
Model 8	NDK 1	9 LE + 11 TE1	5,916
Model 9	NDK 1	9 LE + 11 TE3	6,147
Model 10	NDK 2	6 LE + 14 TE1	3,390
Model 11	NDK 2	6 LE + 14 TE3	4,284
Model 12	NDK 3	6 LE + 8 TE1 + 6 TE3	4,116

<sup>1</sup> See Fig. 14.

<sup>2</sup> Each model includes 20 beam nodes in total.

463 The load-CMOD curves of five NDK models are shown in Fig. 15, in which results from the  
 464 experimental campaign and Model 1 are included for comparison. It is evident that all NDK  
 465 models cause a higher initial stiffness than Model 1, especially for Model 10 whose curve is  
 466 out of the experimental range. This phenomenon is expected, as using lower-order kinematics  
 467 in non-critical zones leads to stiffer numerical models compared to those using higher-order  
 468 kinematics. Among the NDK models, Model 9 presents the closest curve to the Model 1,  
 469 as it retains a relatively larger region using LE and the use of TE3 in non-critical zones. In  
 470 contrast, Model 8, also using the NDK1 but with TE1 in non-critical zones, displays a higher  
 471 initial stiffness and peak load. For models employing NDK 2, Model 11 closely matches  
 472 experimental results, while Model 10 provides an incorrect curve outside the experimental  
 473 range. The accuracy of Model 11 is due to the use of TE3 in non-critical zones. This outcome  
 474 indicates that higher-order TE in non-critical zones can enhance numerical accuracy when  
 475 the critical zone is too small. In addition, Model 12 performs similarly to Model 11, but it  
 476 requires slightly less DOF. These results highlight that the introduction of transition zones  
 477 using higher-order TE is beneficial for minimizing the required DOF. Except Model 10, all  
 478 NDK models present structural curves that fall within the experimental range. Moreover, it  
 479 is worth noting that all NDK models present similar softening curves to Model 1, indicating  
 480 the the importance of accurately defining the critical zones.

481 The matrix damage patterns from various NDK models are presented in Figs. 16 and  
 482 17. Model 10 exhibits notable deviations, with additional matrix tensile damage appearing

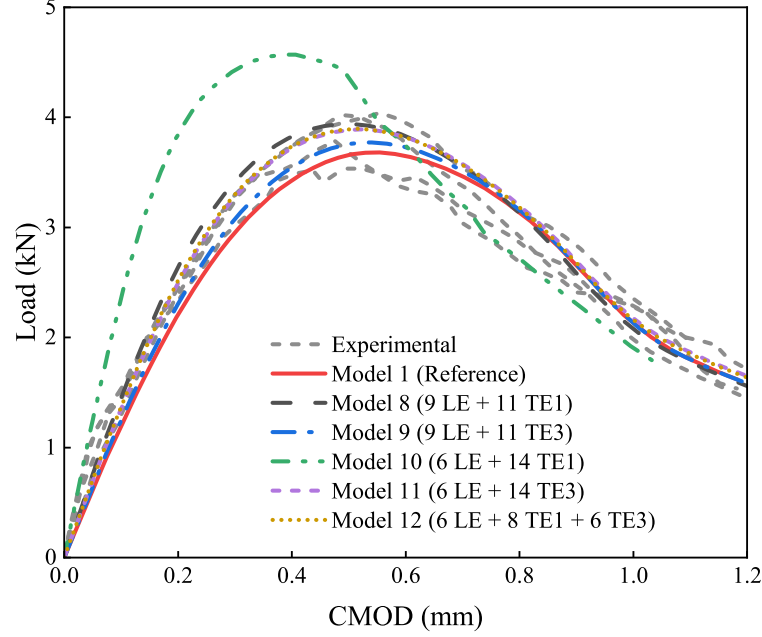


Figure 15: Load-CMOD curves of different NDK models for the TPB beam

483 outside the notch and more pronounced matrix compressive damage near the loading plate.  
 484 In contrast, the remaining models display consistent matrix damage patterns, closely aligning  
 485 with those of Model 1. This consistency suggests that, except for Model 10, all other NDK  
 486 models effectively replicate the matrix damage behavior observed in the refined model.

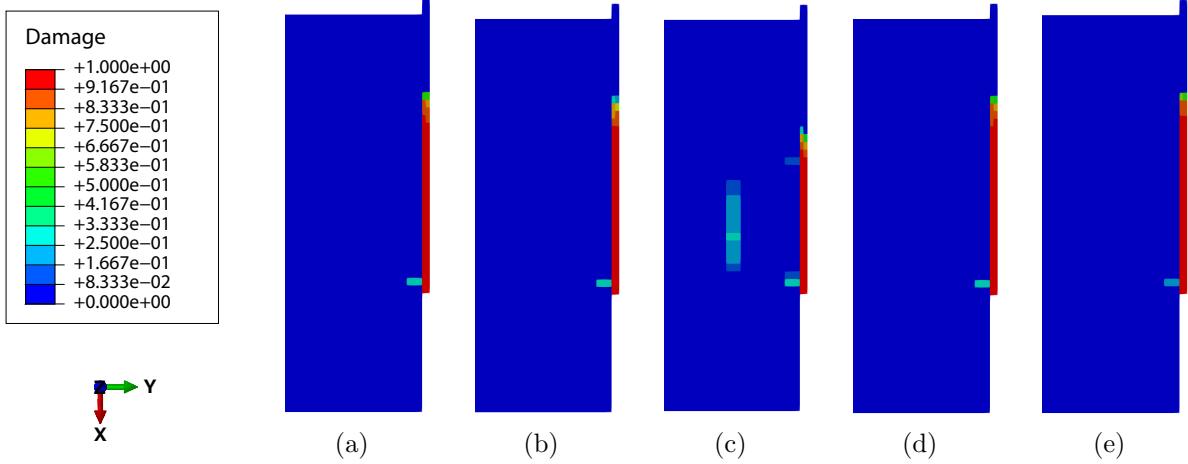


Figure 16: Matrix tensile damage distribution of the TPB beam from (a) Model 8; (b) Model 9; (c) Model 10; (d) Model 11; (e) Model 12.

### 487 4.3 Four-point bending test

488 The final test involves simulating the collapse of a pultruded GFRP beam under Four-Point  
 489 Bending (FPB) conditions. The experimental campaign of this GFRP beam under FPB test  
 490 is a part of a large R&D project named CLICKHOUSE [52] and was studied numerically in  
 491 [53]. The beam is simply supported over a span of approximately 3000 mm and subjected  
 492 to two-point loading in a classic FPB configuration, as illustrated in Fig. 18. The loads are

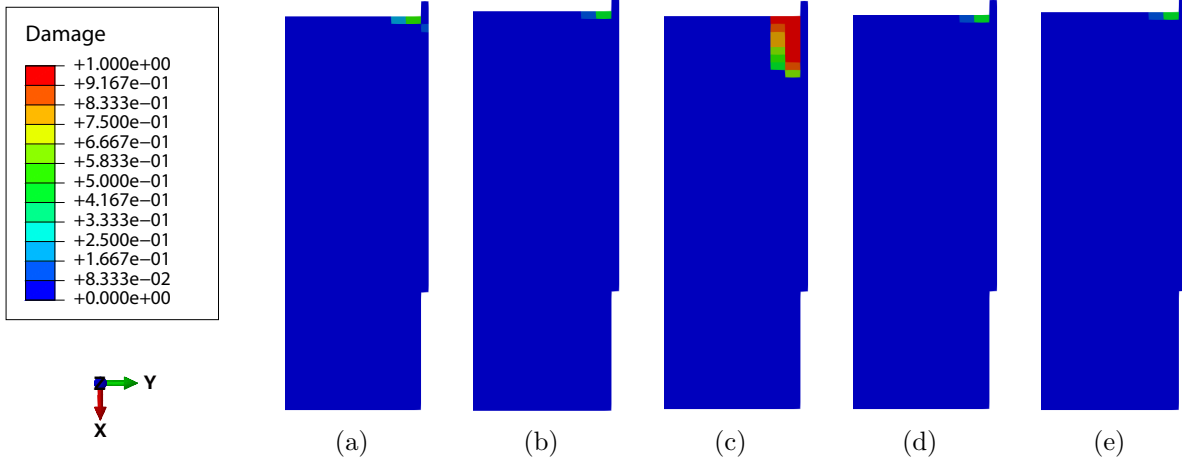


Figure 17: Matrix compressive damage distribution of the TPB beam from (a) Model 8; (b) Model 9; (c) Model 10; (d) Model 11; (e) Model 12.

493 symmetrically applied at approximately one-third of the span length. The cross-section of  
 494 the beam is a hollow square profile with an external side length of 120 mm and a thickness  
 495 of 10 mm. Material properties used for the GFRP composite are obtained from [53] and  
 496 summarized in Table 6.

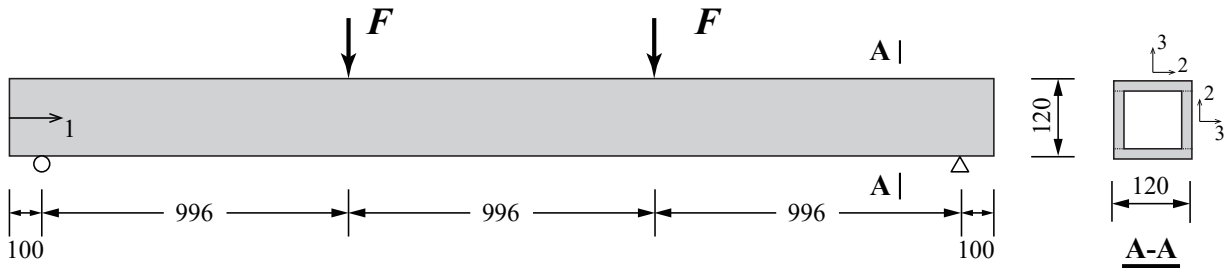


Figure 18: Loads and geometry of the FPB beam (mm)

497 A half-model of the structure is utilized for numerical analysis to reduce computational  
 498 costs. Figure 19(a) illustrates the uniform assignment of beam elements. The length of each  
 499 beam element is approximately 50 mm, which results in a total of 36 beam elements. If the  
 500 beam element length is reduced to around 30 mm, 56 beam elements will be needed. Three  
 501 types of cross-sectional discretizations are shown in Fig. 19(b) in which L9 elements are  
 502 employed with a mesh size of 10 mm.

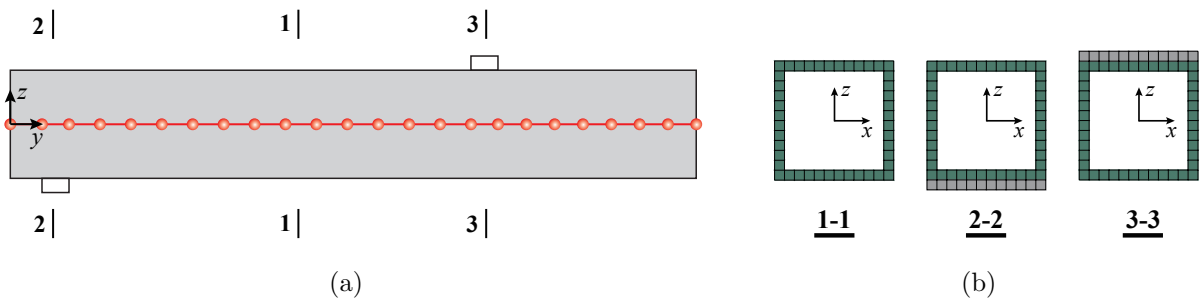


Figure 19: (a) Beam element and (b) cross-sectional discretizations of the FPB beam

Table 6: Material properties of the FPB beam

Properties	Units	Symbols	Values	Symbols	Values
Elastic moduli	GPa	$E_{11}$	32.4	$G_{12}$	3.2
		$E_{22}$	4.8	$G_{13}$	3.2
		$E_{33}$	4.8	$G_{23}$	3.2
Poisson's ratios	-	$\nu_{12}$	0.3		
		$\nu_{13}$	0.3		
		$\nu_{23}$	0.3		
Strengths	MPa	$X_t$	326.2	$X_c$	326.2
		$Y_t$	88.9	$Y_c$	88.9
		$Z_t$	88.9	$Z_c$	88.9
		$S_L$	41.4	$S_T$	58.7
		$S_I$	41.4		
Fracture energies	MPa·mm	$G_{ft}$	55	$G_{fc}$	95
		$G_{mt}$	11.5	$G_{mc}$	11.5
		$G_{it}$	11.5	$G_{ic}$	11.5

503 Eight CUF-based models, detailed in Table 7, are investigated, each employing the identical  
504 cal cross-sectional discretizations. The first four models utilize LE across the entire structure.  
505 The key differences among these models are the order and number of beam elements used,  
506 aimed at evaluating the regularization technique. Models 5 and 6, based on Model 1, aim  
507 to reduce computational costs through the NDK approach. Figure 20 illustrates two NDK  
508 configurations. The NDK 1 model adopts a conservative strategy, with a relatively small non-  
509 critical zone employing TE1 models. In contrast, the NDK 2 model introduces a transition  
510 zone employing TE3 models to achieve at least 50 % reduction in DOF. Similarly, Models 7  
511 and 8 apply the NDK strategy based on Model 2 to assess the efficiency of NDK in higher-  
512 order beam models. For comparison, a 3D model based on ABAQUS, referred to as the ABQ  
513 3D model, is also included. This model uses C3D8 solid elements with an approximate ele-  
514 ment size of 10 mm. The same damage model is implemented in the ABQ 3D model using  
515 the UMAT subroutine.

Table 7: Model information the GFRP beam for the FPB test

Model No.	Model type	Beam elements	Size (mm) <sup>1</sup>	DOF
Model 1	Full LE	36 B2	50	30,204
Model 2	Full LE	36 B3	50	59,316
Model 3	Full LE	36 B4	50	88,428
Model 4	Full LE	56 B2	30	46,044
Model 5	NDK 1	36 B2	50	19,725
Model 6	NDK 2	36 B2	50	15,153
Model 7	NDK 1	36 B3	50	37,375
Model 8	NDK 2	36 B3	50	29,955
ABQ 3D	-	-	-	42,972

<sup>1</sup> Individual beam element length.

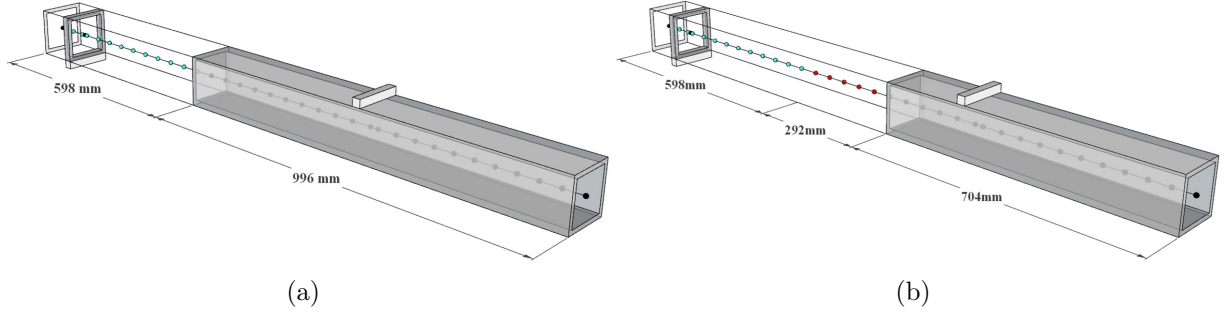


Figure 20: (a) NDK 1 model and (b) NDK 2 model for the FPB beam-Black represents LE, cyan stands for TE1, and red denotes TE3

516 The load-displacement curves in Fig. 21 provide a detailed comparison of different nu-  
 517 merical models applied to the FPB beam under bending. The experimental data from [53]  
 518 reported a peak load of approximately 81 kN, serving as a reference for validation. Figure  
 519 21(a) highlights the results from various refined beam models with different beam element  
 520 configurations. Figure 21(b) compares the refined models with their corresponding NDK  
 521 models to evaluate the accuracy of the NDK approach. Overall, all numerical models, in-  
 522 cluding the CUF-based models and the ABQ 3D model, effectively capture the brittle failure  
 523 behavior of the FPB beam, which is characterized by a linear elastic phase followed by a rapid  
 524 loss of load-bearing capacity after reaching the peak load. In the elastic phase, all models  
 525 exhibit consistent load-displacement curves with minimal variations in stiffness. Although  
 526 the NDK models show slightly higher stiffness compared to the refined models, this difference  
 527 is not pronounced in the plotted curves due to the conservative selection of non-critical zones.  
 528 The alignment between the CUF-based models and the ABQ 3D results further proves the  
 529 accuracy of the CUF framework in capturing the elastic response of the FPB beam. At the  
 530 peak load, all models predict slightly higher values than the experimental peak load of 81  
 531 kN. This consistent results across different CUF-based models demonstrates their ability to  
 532 capture the ultimate load capacity. Additionally, their close agreement with the ABQ 3D  
 533 model confirms the accuracy of the CUF-based framework.

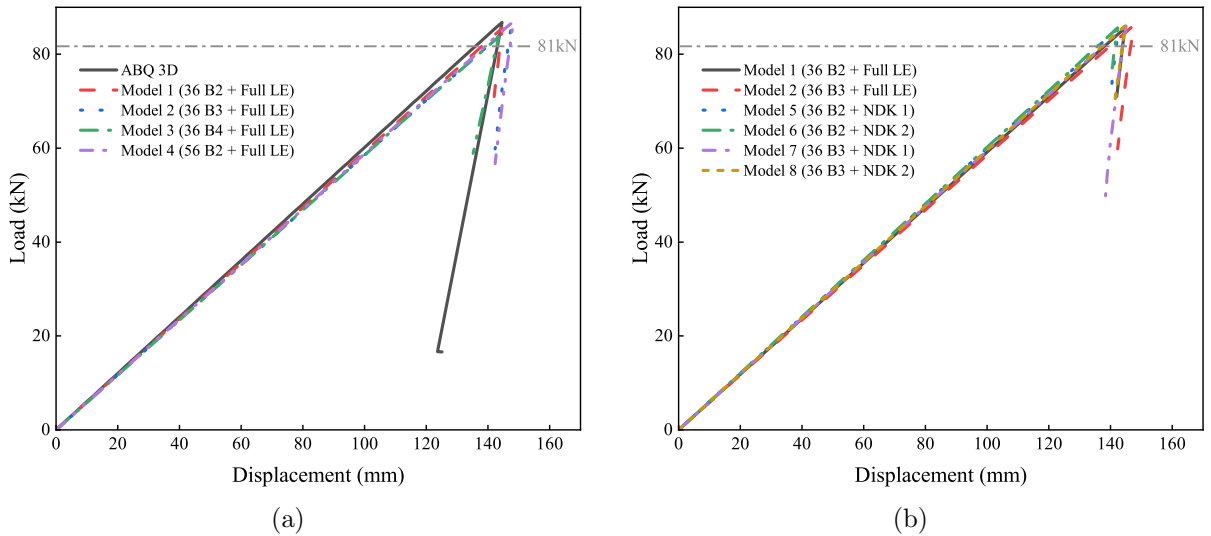


Figure 21: Load-displacement curves of the FPB beam at mid-span from various numerical models

534 To further assess model performance, Table 8 summarizes the peak loads and correspond-  
535 ing displacements for each model. Excluding Model 3, the other refined models estimate a  
536 peak load of approximately 86 kN, representing a minor overestimation of about 6% compared  
537 to the experimental data. The results from the refined models closely align with those of the  
538 ABQ 3D model, indicating that mesh dependency has been mitigated to a significant extent.  
539 Moreover, all NDK models predict peak loads around 86 kN, which closely match their refined  
540 counterparts. However, Models 5 and 6 show slightly lower displacements at peak loads than  
541 Model 1. This small difference is likely due to the stiffer response of the NDK models caused  
542 by using lower-order kinematics in non-critical regions. A similar pattern is seen in Models  
543 2, 7, and 8. Nevertheless, these differences are minor, confirming the accuracy of the NDK  
544 models.

Table 8: Peak results of the FPB GFRP beam from various models and experiment

Model No.	Model type	Peak load (kN)	Displacement (mm) <sup>1</sup>
Exp [53]	-	81.0	-
Model 1	36 B2 + Full LE	85.57	144.65
Model 2	36 B3 + Full LE	86.02	147.34
Model 3	36 B4 + Full LE	83.66	143.50
Model 4	56 B2 + Full LE	86.85	148.29
Model 5	36 B2 + NDK 1	85.58	142.53
Model 6	36 B2 + NDK 2	85.59	142.24
Model 7	36 B3 + NDK 1	85.93	145.07
Model 8	36 B3 + NDK 2	86.08	144.81
ABQ 3D	-	85.61	146.84

<sup>1</sup> It indicates the displacement corresponding to the peak load.

545 As shown in Fig. 22, experimental failure occurred at the intersection of the top flange  
546 with the webs, starting near the load application points, from which the fractures developed  
547 and propagated. Additionally, crack is also evident at the junction of the web and flange at the  
548 bottom of the beam beneath the right loading point. This kind of failure is attributed to the  
549 weaker material properties of web-flange junctions in the pultruded GFRP beams. Therefore,  
550 shear damage has been identified as the primary failure mechanism in this pultruded GFRP  
551 beam [53].

552 Figure 23 presents the shear damage patterns predicted by all numerical models. Sig-  
553 nificant damage localization near the right-side loading plate is observed across all models,  
554 which is similar to the experimental cracks. Model 3 additionally shows pronounced damage  
555 localization near the left-side loading plate, which explains its slightly lower peak load. While  
556 Models 2 through 4 do not capture damage at the bottom junction of the flange and web, as  
557 seen in Model 1, they accurately predict similar damage patterns at the top junction of the  
558 flange and web near the loading plate, which is the primary damage characteristic for this  
559 beam. Models 5 and 6, using the NDK approach based on Model 1, do not show damage at  
560 the bottom junction of the flange and web. However, they capture the key damage patterns  
561 at the top flange-web junction near the loading plate. Similar observations are noted for  
562 Models 2, 7, and 8, which utilize B3 beam elements. Model 7 shows some damage at the  
563 bottom flange-web junction, while Model 8 does not. Despite these differences, both mod-  
564 els effectively capture the critical damage patterns at the top flange-web junction near the  
565 loading plate. The ABQ 3D model provides a clear depiction of damage, showing localization

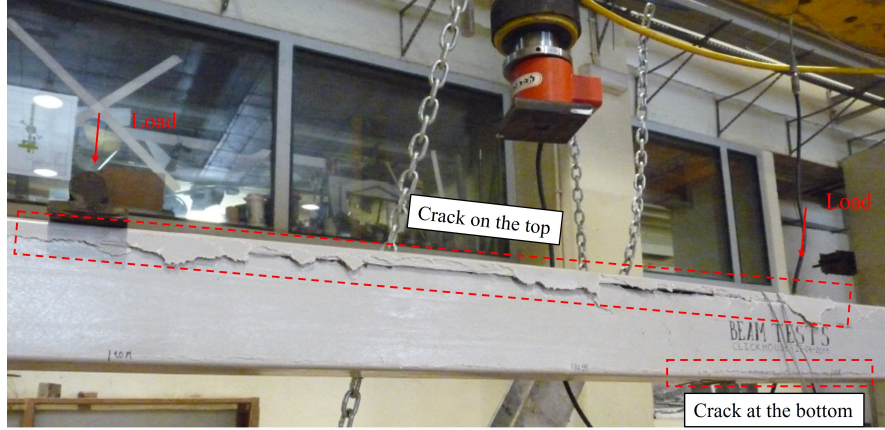


Figure 22: Experimental failure mode of the FPB beam [53]

566 below the loading plate and horizontal propagation along both the top and bottom flange-web  
 567 junctions. The occurrence of damage at the bottom junction in some models, including the  
 568 ABQ 3D model, is likely due to their convergence at lower post-peak residual loads, whereas  
 569 certain CUF-based models experience convergence challenges at higher residual loads.

570 In summary, both the CUF-based models and the ABQ 3D model successfully reproduce  
 571 the essential damage patterns observed in the experimental benchmark, which are similar to  
 572 numerical results from [53] using the classical Hashin failure criteria.

## 573 5 Conclusions

574 This work has presented a comprehensive numerical investigation of the progressive damage  
 575 analysis of pultruded composite structures. A novel 3D orthotropic damage model was inte-  
 576 grated into one-dimensional CUF-based beam models, where regularization techniques were  
 577 applied to mitigate mesh dependency. The NDK approach was also employed to improve the  
 578 computational efficiency of the CUF-based beam models. The proposed framework was val-  
 579 idated through three experimental benchmark tests: compact tension, three-point bending,  
 580 and four-point bending. Based on the comparison of numerical and experimental results, the  
 581 following conclusions are drawn:

- 582 1) CUF-based beam models with the 3D Tsai-Wu orthotropic damage model can effec-  
 583 tively provided numerical progressive damage results, closely matching with experimen-  
 584 tal data. Notably, Model 1 in the CT test required only 1,332 DOF to achieve 3D-level  
 585 accuracy, highlighting the computational efficiency of CUF-based beam models.
- 586 2) Despite minor mesh order sensitivity, most refined CUF-based models consistently pro-  
 587 duced accurate results. These consistencies confirmed the effectiveness of proposed  
 588 method for calculating characteristic element lengths in mitigating mesh dependency.
- 589 3) The proposed framework successfully simulated quasi-brittle failure, as evidenced in the  
 590 CT and TPB tests. It also demonstrated its applicability to brittle failure scenarios, as  
 591 the FPB test shows.
- 592 4) The application of the NDK approach in the TPB and FPB tests demonstrated that  
 593 carefully selecting non-critical zones can achieve results comparable to those of the cor-

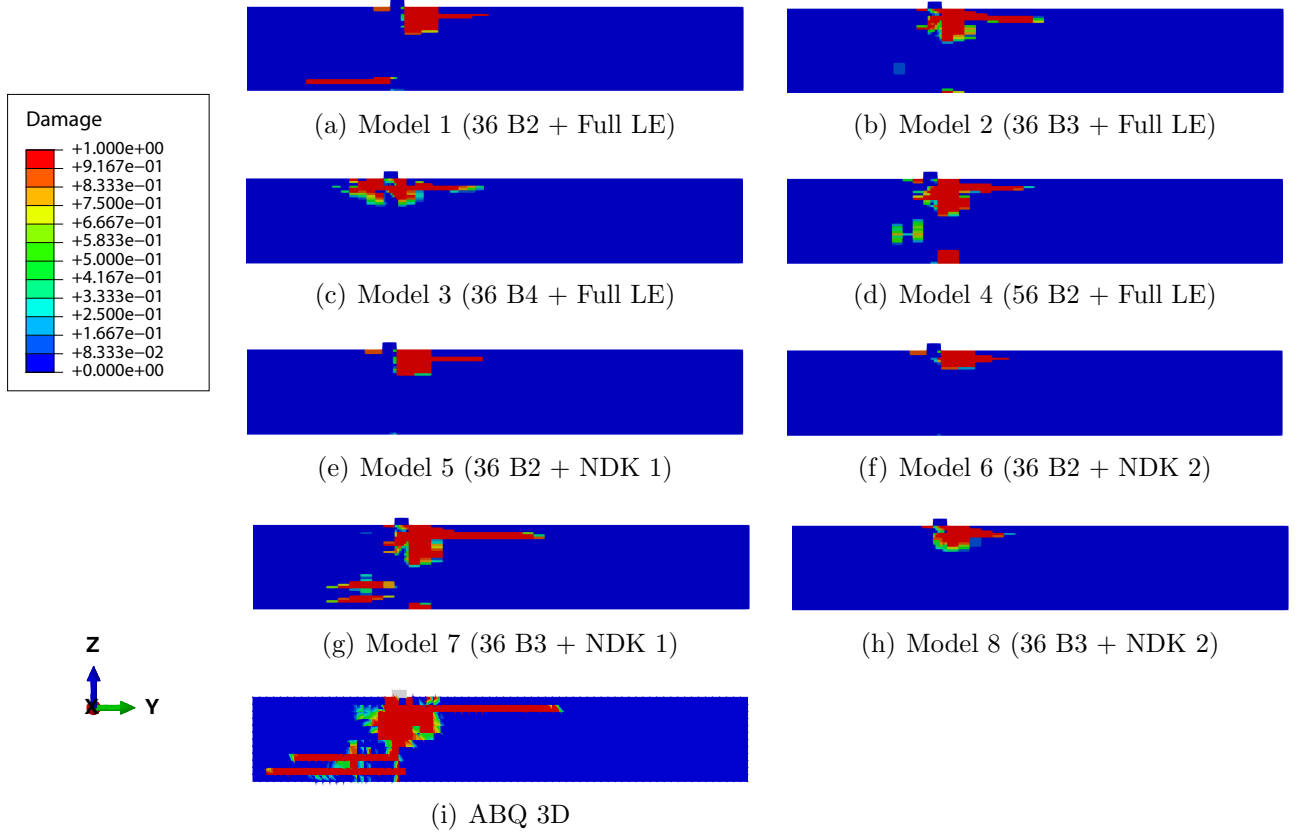


Figure 23: Shear damage distribution of the FPB beam from various models

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responding refined models. Additionally, incorporating a transition region using higher-order TE contributed to the balance between computational efficiency and accuracy.

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5) Across the TPB and FPB tests, NDK models achieved at least a 50 % reduction in DOF without sacrificing too much accuracy compared to their refined counterparts. Moreover, there is still a potential for further reductions in the FPB test because a large number of beam elements were assigned.

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6) The scalability of the NDK approach should be highlighted, allowing for straightforward enhancements to beam models by adjusting the cross-sectional kinematics at individual beam nodes without requiring remeshing. This feature demonstrates the adaptability of the proposed framework.

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While the current NDK models demonstrate satisfactory performance, their accuracy still depends on manually defined non-critical, critical, and transition zones based on prior knowledge and trial-and-error. Future research may focus on incorporating automated optimization strategies, such as the best theory diagrams [54], for zone definition to maximize the potential of the proposed framework.

## 609 Acknowledgement

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611

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<sup>614</sup> SEPINOV - Sistemas Estruturais e Produtos Inovadores from LNEC.

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