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Invariant Monge–Ampère equations on contactified para–Kähler manifolds

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Abstract
We develop a method for describing invariant PDEs of Monge–Ampère type in the sense of Lychagin and Morimoto (MAE) on a homogeneous contact manifold N of a semisimple Lie group G, which is the *contactification* of the homogeneous symplectic manifold $M = G/H = \mathrm{Ad}_G Z \subset \mathfrak{g}$, where M is the adjoint orbit of a splittable closed element Z of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. The method is then applied to a ten-dimensional semisimple orbit M of the exceptional Lie group G_2 and a complete list of mutually non-equivalent MAEs on N is obtained.

Keywords Para–Kähler structures · Homogeneous contact manifolds · Jet spaces · G-invariant PDEs

Mathematics Subject Classification 53C30 · 53D10 · 57S20 · 58A20 · 58J70.

Introduction

Monge–Ampère equations (MAEs) form a distinguished class of nonlinear second–order PDEs. They were introduced by Monge in 1784 in his pioneering study of optimal transportation problem and continued by Ampère in 1820. The classical MAE has the form

$$\det \mathrm{Hess}\, u(x) = f(x, u(x))\,, \quad x \in \mathbb{R}^n\,,$$

where $\mathrm{Hess}\, u(x) = D^2 u$ is the Hessian and $f(x, u)$ is a given function.

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Numerous applications in differential geometry, meteorology, cosmology, hydrodynamics, economics, optimal mass transportation problem, etc., lead to consideration of a more general class of MAEs, given by

$$\det[\operatorname{Hess} u - A(x, u, Du)] = f(x, u, Du),$$

where A is a symmetric matrix.

MAEs are intensively studied: see, e.g., [12, 41, 44]. Many deep results about existence and uniqueness of solutions are obtained in the case when $\operatorname{Hess} u(x)$ is positively defined, and as such it may be considered as a Riemannian metric (the so called Hessian metric), see [33, 37]. A short history of MAEs and their complex and quaternionic versions can be found in the notes [43] by Verbitsky.

Lychagin [30, 31] and Morimoto [34] proposed a geometric construction of general MAEs in terms of contact and symplectic geometry. More precisely, let (N, D) be a $2n + 1$ dimensional manifold with contact structure D, i.e., a codimension–one distribution locally defined by 1-form θ with $d\theta^n \wedge \theta \neq 0$: then there are particular n-forms Ω that define MAEs $\mathcal{E}_\Omega$ in the sense that a solution of an equation $\mathcal{E}_\Omega$ is a Legendrian submanifold of N which annihilates Ω. Among the aforementioned forms there are those that can be considered as n-forms on the cotangent space of an n–dimensional manifold, in which case the corresponding MAEs $\mathcal{E}_\Omega$ are called symplectic, and the solutions are Lagrangian submanifolds annihilating Ω.

Starting from the paper by Monge, the most important application of MAEs remains that to the optimal transportation problem and related matters. New geometric approach to this subject had been developed in a series of papers [19, 23, 24, 45, 46] by Kim, McCann, Warren, Harvey and Lawson. They established a closed relationship between the classical Monge–Kantorovich mass transportation problem and para–Kähler geometry. More precisely, they proved that, under some assumptions, the solution of the Monge–Kantorovich problem reduces to the construction of special Lagrangian submanifolds in some $2n$-dimensional para–Kähler manifold.

In this paper, we develop an approach for describing invariant MAEs, in the sense of Lychagin–Morimoto, on homogeneous contact manifolds N of a semisimple Lie group G. More precisely, we consider manifolds $N = G/L$ that are contactifications of homogeneous para–Kähler manifolds $M = G/K$, described in [3, 4]. The method is applied to classify all invariant MAEs on the 11-dimensional contact manifold $N^{11}_3 := \mathsf{G}_2/\,\mathsf{SL}(2, \mathbb{R})$ of the exceptional non-compact Lie group G_2, associated to the 10-dimensional para–Kähler real flag manifold $M^{10}_3 := \mathsf{G}_2/\,\mathsf{GL}(2, \mathbb{R})$. We stress that the 10-dimensional homogeneous manifold M^{10}_3 refers to the non-compact real form of G_2, though it has the same complexification as that of another 10-dimensional homogeneous manifold, this time of the compact real form of G_2, namely the Grassmannian of oriented planes in $\mathbb{R}^7$, see [14, Section 3.5].

Note that invariant second–order PDEs with G_2 symmetries had been studied by K. Yamaguchi [47] and a description of invariant PDEs with G_2 symmetry, different from MAEs, was obtained in the remarkable paper [40] by D. The. Classifications of different classes of MAEs were given in the papers [22, 25–27, 32] and the problem of equivalence and the conditions of linearizability of MAEs was treated in [28, 29].

Structure of the paper

In Sect. 1 we remind that the notion of a bi–Lagrangian manifold is equivalent to the notion of a para–Kähler manifold; then we pass to the G–homogeneous case and recall some useful results: the adjoint orbit $\mathrm{Ad}_G(Z)$ of Z possesses a bi–Lagrangian structure if and only if Z

is splittable, in which case all the bi–Lagrangian structures on $\mathrm{Ad}_G(Z)$ are in one–to–one correspondence with distinguished $\mathbb{Z}$–gradings of $\mathfrak{g} = \mathrm{Lie}(G)$, which is called *fundamental*. Next, we introduce the *contactification* of a homogeneous para–Kähler manifold and, finally, by employing the generalized Gauss decomposition, we prove a theorem that allows to locally identify a para–Kähler manifold with the cotangent bundle to a suitable flag manifold.

In Sect. 2 we recall the construction of a general MAE in terms of contact and symplectic geometry given by Lychagin–Morimoto. The local expression of general MAEs in Darboux coordinates is obtained later, together with the natural interpretation of the fibers of a MAE as hyperplane sections of a Lagrangian Grassmannian.

In Sect. 3 we work out some examples of fundamental gradings: in particular, we describe all fundamental gradings of the algebras $\mathfrak{sl}_n$ and $\mathrm{Lie}(\mathsf{G}_2)$.

In Sect. 4 we focus on the 10-dimensional G_2–homogeneous manifold $M^{10}_3 = \mathsf{G}_2/\mathsf{GL}_2(\mathbb{R})$ and we construct a basis of the space of invariant effective 5-forms on the contactification N^{11}_3 of M^{10}_3. This allows to provide a list of all G_2–invariant MAEs on N^{11}_3, that are second–order (nonlinear) PDEs in 5 independent variables; we finally establish which of them are contact–equivalent.

Notations and conventions

The index 3 in the symbols M^{10}_3 and N^{11}_3 refers to the classification of para–Kähler manifolds of the non-compact real form of G_2 given in Sect. 3.2.2.

1 Homogeneous para–Kähler manifolds and their contactification

1.1 Bi–Lagrangian and para–Kähler structures

Below we introduce two equivalent categories made of the objects we will be working with.

Definition 1.1 An *almost para–complex structure* on a $2n$-dimensional manifold M is a field of endomorphisms $I \in \mathrm{End}\,(TM)$ such that $I^2 = \mathrm{Id}$, the eigenvalues of I are $+1$ and -1 and the corresponding eigendistributions $T^\pm M$ are n-dimensional:

$$TM = T^+M \oplus T^-M\,. \tag{1}$$

An almost para–complex structure is called a *para–complex structure* if both the distributions $T^\pm M$ are integrable.

Definition 1.2 The decomposition (1) on a symplectic manifold (M, ω) is called *almost bi–Lagrangian* if $\omega|_{T^\pm M}$ vanishes identically. If, moreover, the distributions $T^\pm M$ are integrable, it is called a *bi–Lagrangian structure* and (M, ω, I) is called a *bi–Lagrangian manifold*.

The integrable submanifolds (of maximal dimension) of the distributions $T^\pm M$ of a bi–Lagrangian structure on a $2n$-dimensional symplectic manifold (M, ω) are Lagrangian submanifolds of M, i.e., they are n-dimensional and ω vanishes identically on them; see also [11].

Definition 1.3 An *almost para–Hermitian manifold* is a pseudo–Riemannian manifold (M, g) equipped with an almost para–complex structure I, such that the distributions $T^\pm M$ are (absolutely) isotropic, i.e., $g|_{T^\pm M}$ vanishes identically.

Given an almost para–Hermitian manifold (M, g, I), we define the two–form

$$\omega := g(\cdot\, , I(\cdot))\,. \tag{2}$$

Definition 1.4 An almost para–Hermitian manifold (M, g, I), such that I is a para–complex structure, is called a *para–Hermitian manifold*. A para–Hermitian manifold (M, g, I) is called a *para–Kähler manifold* if the para–complex structure I is parallel with respect to the Levi–Civita connection ∇^g of g, i.e., if $\nabla^g I = 0$.

The next well–known results show that the notion of a bi–Lagrangian manifold (M, ω, I) is equivalent to the notion of a para-Kähler manifold (M, g, I).

Lemma 1.1 *Let (M, g, I) be a para–Hermitian manifold: then $\nabla^g I = 0$ if and only if the two-form ω given by (2) is closed, i.e., if $d\omega = 0$.*

Proof It suffices to show that the identities

$$d\omega(v, w, u) = g(\nabla^g_w(I)(v), u) + g(\nabla^g_u(I)(w), v) + g(\nabla^g_v(I)(u), w), \tag{3}$$

$$d\omega(v, w, u) + d\omega(v, I(w), I(u)) = -2g(\nabla^g_v(I)(w), u), \tag{4}$$

hold for all commuting vector fields $v, w, u, I(w), I(u)$ on M, see [9, Proposition 4.16].

In view of such a commutativity and of (2), we obtain

$$\begin{aligned} d\omega(v, w, u) &= v(\omega(w, u)) + w(\omega(u, v)) + u(\omega(v, w)) \\ &= v(g(w, I(u))) + w(g(u, I(v))) + u(g(v, I(w))), \\ &= g(\nabla^g_v(w), I(u)) + g(w, \nabla^g_v(I(u))) + \dots \\ &= g(\nabla^g_v(w), I(u)) + g(w, \nabla^g_v(I)(u)) + g(w, I(\nabla^g_v(u))) + \dots \\ &= g(w, \nabla^g_v(I)(u)) + \omega(\nabla^g_v(w), u) + \omega(w, \nabla^g_v(u)) + \dots \\ &= g(\nabla^g_w(I)(v), u) + g(\nabla^g_u(I)(w), v) + g(\nabla^g_v(I)(u), w), \end{aligned}$$

where the dots denote cyclic permutations of (v, w, u), that is formula (3).

Let us observe now that, in view of the Leibniz rule for the covariant derivative and the fundamental identity $\nabla^g(g) = 0$, it holds

$$\begin{aligned} 2g(\nabla^g_v(I)(w), u) &= 2g(\nabla^g_v(I(w)), u) - 2g(I(\nabla^g_v(w)), u) \\ &= 2g(\nabla^g_v(I(w)), u) + 2g(\nabla^g_v(w), I(u)). \end{aligned} \tag{5}$$

The Koszul formula, applied to both the addends of (5), yields

$$\begin{aligned} 2g(\nabla^g_v(I(w)), u) &= v(g(I(w), u)) + I(w)(g(v, u)) - u(g(v, I(w))) \\ &= v(\omega(u, w)) + I(w)(\omega(v, I(u))) + u(\omega(w, v)) \end{aligned} \tag{6}$$

and

$$\begin{aligned} 2g(\nabla^g_v(w), I(u)) &= v(g(w, I(u))) + w(g(v, I(u))) - I(u)(g(v, w)) \\ &= v(\omega(w, u)) + w(\omega(v, u)) + I(u)(\omega(I(w), v)), \end{aligned} \tag{7}$$

respectively. By taking the sum of (6) and (7) we obtain

$$\begin{aligned} 2g(\nabla^g_v(I)(w), u) = {} & v(\omega(u, w)) + I(w)(\omega(v, I(u))) + u(\omega(w, v)) + v(\omega(w, u)) \\ & + w(\omega(v, u)) + I(u)(\omega(I(w), v)) = I(w)(\omega(v, I(u))) + u(\omega(w, v)) \\ & + w(\omega(v, u)) + I(u)(\omega(I(w), v)). \end{aligned} \tag{8}$$

It is then easy to see that

$$\begin{aligned} & d\omega(v,w,u) + d\omega(v, I(w), I(u)) \\ & = v(\omega(w,u)) + w(\omega(u,v)) + u(\omega(v,w)) + \\ & + v(\omega(I(w), I(u))) + I(w)(\omega(I(u), v)) + I(u)(\omega(v, I(w))) \\ & = v(\omega(w,u)) + w(\omega(u,v)) + u(\omega(v,w)) + \\ & - v(\omega(w,u)) + I(w)(\omega(I(u), v)) + I(u)(\omega(v, I(w))), \end{aligned}$$

after switching the arguments in all the instances of ω, coincides with (8): this proves (4).

Proposition 1.1 *If (M, g, I) is a para–Kähler manifold, then (M, ω, I) is a bi–Lagrangian manifold with ω given by (2); conversely, if (M, ω, I) is a bi–Lagrangian manifold, then (M, g, I) is a para–Kähler manifold, with g given by*

$$g := \omega(\,\cdot\,, I(\,\cdot\,))\,. \tag{9}$$

Proof If a para–Kähler manifold (M, g, I) is given, then the form ω defined by (2) is symplectic thanks to Lemma 1.1; it only remains to show that $T^{\pm}M$ are Lagrangian, but this is an immediate consequence of $T^{\pm}M$ being g–isotropic:

$$v, w \in T^{\pm}M \Rightarrow \omega(v, w) = g(v, I(w)) = g(v, \pm w) = 0\,. \tag{10}$$

If a bi–Lagrangian manifold (M, ω, I) is given, then one has to show that (9) defines indeed a pseudo–Riemannian metric, such that $T^{\pm}M$ are g–isotropic and I is g–parallel.

The first claim, recalling that $T^{\pm}M$ are Lagrangian, is easy to see and it formally mirrors (10):

$$\begin{aligned} v, w \in T^{\pm}M &\Rightarrow g(v,w) = \omega(v, I(w)) = \omega(v, \pm w) = 0, \\ v \in T^{\pm}M, w \in T^{\mp}M &\Rightarrow g(v,w) = \omega(v, I(w)) = \omega(v, \mp w) \\ &\qquad = \pm\omega(w, v) = \omega(w, I(v)) = g(w, v). \end{aligned}$$

The second claim follows again from Lemma 1.1.

1.2 Classification of para-Kähler homogeneous manifolds of a semisimple Lie group

In what follows G will denote a real Lie group and Z an element of its Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$; the Killing form on $\mathfrak{g}$ will be denoted by the symbol B.

Definition 1.5 An element $Z \in \mathfrak{g}$ is called *semisimple* if such is its adjoint operator ad_Z. A semisimple element $Z \in \mathfrak{g}$ is called *splittable* if ad_Z has real eigenvalues; it is called *closed*, if it generates a closed 1-parameter subgroup $\{\exp tZ \mid t \in \mathbb{R}\} \simeq \mathbb{R}$ of G.

It is a classical result by Kirillov, Kostant and Souriau that, up to a central extension, any homogeneous symplectic manifold $M = G/H$ is a coadjoint orbit in the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G. If G is semisimple, then the Killing form B is not degenerate and therefore the coadjoint orbit $\mathrm{Ad}^*_G\xi \subset \mathfrak{g}^*$ of an element $\xi \in \mathfrak{g}^*$ can be identified with the adjoint orbit $\mathrm{Ad}_G Z \subset \mathfrak{g}$, with Z given by $Z = B^{-1} \circ \xi$.

Theorem 1.1 *[Kirillov–Kostant–Souriau] Up to a covering, any homogeneous symplectic manifold $(M = G/H, \omega)$ of a semisimple Lie group G is isomorphic to an adjoint orbit $\mathrm{Ad}_G Z \subset \mathfrak{g}$ equipped with the G–invariant symplectic form ω given by*

$$\omega_x(X, Y) := B(x, [\mathrm{Ad}_{x^{-1}*}(X), \mathrm{Ad}_{x^{-1}*}(Y)])\,,$$

for any $x \in M \subset \mathfrak{g}$ and any $X, Y \in T_xM \subset T_xG$, where $\mathrm{Ad}_{x^{-1}*} : T_xG \to T_eG = \mathfrak{g}$ is the tangent map of $\mathrm{Ad}_{x^{-1}}$ at the point x.

Corollary 1.1 *The G–invariant symplectic forms on the adjoint orbit $M = \mathrm{Ad}_G Z = G/H$ are parametrized by central elements $Z \in \mathfrak{g}$, such that $C_{\mathfrak{g}}(Z) = \mathfrak{h}$. The symplectic form ω^Z at the point $o = eH$ is given by*

$$\omega_o^Z(X, Y) = B(Z, [X, Y]) ,$$

$X, Y \in T_oM = \mathfrak{m}$, where $\mathfrak{m}$ is Ad_G–invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$, identified with the tangent space T_oM.

Definition 1.6 Let G be a semisimple Lie group and let $\mathfrak{m}$ be a B–orthogonal complement of $C_{\mathfrak{g}}(Z)$: then the unique G–invariant symplectic structure ω^Z on $M = \mathrm{Ad}_G Z$, such that

$$\omega^Z(X, Y) = B(Z, [X, Y]) , \quad \forall X, Y \in \mathfrak{m} = T_Z M , \tag{11}$$

is called *the symplectic structure on M associated with $Z \in \mathfrak{g}$.*

Theorem 1.2 *[Hou-Deng-Kaneyuki-Nishiyama [4, 20]] Let G be a semisimple real Lie group and $(M = \mathrm{Ad}_G Z, \omega)$ an adjoint orbit of an element $Z \in g = \mathrm{Lie}(G)$, equipped with the invariant symplectic structure ω associated with Z. Then the manifold M admits a G–invariant integrable bi–Lagrangian (or, equivalently, para–Kähler) structure if and only if Z is a splittable element.*

Indeed, if Z is splittable, then all the eigenvalues of the operator ad_Z are real, so that

$$\mathfrak{g} = \sum_{j \in \mathcal{A}} \mathfrak{g}_j ,$$

where $\mathfrak{g}_j$ denotes the eigenspace of ad_Z that corresponds to the eigenvalue j and $\mathcal{A} := \{j \in \mathbb{R} \mid \det(\mathrm{ad}_Z - j\,\mathrm{id}_{\mathfrak{g}}) = 0\}$ is the set of real eigenvalues of ad_Z. In turn, this allows to refine the B–orthogonal reductive decomposition $\mathfrak{g} = C_{\mathfrak{g}}(Z) + \mathfrak{m}$ as follows:

$$\mathfrak{g} = \mathfrak{g}^- + \mathfrak{g}^0 + \mathfrak{g}^+ = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+ , \tag{12}$$

where $\mathfrak{g}^0 = \mathfrak{h} = C_{\mathfrak{g}}(Z)$, and

$$\mathfrak{g}^\pm = \mathfrak{n}^\pm := \sum_{j \in \mathcal{A} \cap \mathbb{R}^\pm} \mathfrak{g}_j . \tag{13}$$

The $\mathrm{ad}_{\mathfrak{h}}$–invariant decomposition (12) extends to a G–invariant decomposition $TM = T^-M \oplus T^+M$ of TM, which turns out to be bi–Lagrangian, because $\mathfrak{n}^\pm$ is a sub–algebra (which ensures integrability) and from the definition (11) of ω^Z it follows that the symplectic form ω^Z vanishes on $\mathfrak{n}^\pm$.

The $\mathrm{ad}_{\mathfrak{h}}$–invariant decomposition (12) is called *a generalized Gauss decomposition*, see Definition 1.9 below.

Corollary 1.2 *Let $N^\pm$ be the nilpotent connected subgroup generated by the Lie algebra $\mathfrak{n}^\pm$: for any $g \in G$ the submanifold $gN^\pm Z \subseteq M$ is an integral submanifold of the distribution $T^\pm M$ passing through the point $x = gZ \in M$.*

A classification of homogeneous bi–Lagrangian manifolds of a (complex or real) semisimple Lie group G can be obtained by employing certain gradings of the corresponding Lie algebra $\mathfrak{g}$. We recall that a $\mathbb{Z}$–grading of a Lie algebra $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \mathfrak{g}^{-k} + \ldots + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \ldots + \mathfrak{g}^k , \tag{14}$$

satisfying $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ for all $i, j \in \mathbb{Z}$, see, e.g., [15, 36]. A special kind of $\mathbb{Z}$–grading, that appeared in the works of N. Tanaka as a natural Lie–algebraic counterpart of a bracket–generating differential system, has been called *fundamental* [39, Definition 1.3]: the same concept has been employed in [3] in the formulation of the classification theorem below.

Definition 1.7 A $\mathbb{Z}$–grading (14) is called *fundamental* if the subalgebra

$$\mathfrak{g}^- := \mathfrak{g}^{-k} + \ldots + \mathfrak{g}^{-1}$$

is generated by $\mathfrak{g}^{-1}$.

Theorem 1.3 below follows from the remark that the operator D given, for any j, by $D|_{\mathfrak{g}^j} := j\,\mathrm{id}_{\mathfrak{g}^j}$, is a derivation of the Lie algebra (14), and any derivation of a semisimple Lie algebra is inner, i.e., $D = \mathrm{ad}_d$, with $d \in \mathfrak{g}$.

Theorem 1.3 *[Alekseevsky–Medori [3]] Let* $(M = \mathrm{Ad}_G(Z) = G/H, \omega)$ *be as in Theorem 1.2 and let* $\mathfrak{h} = C_{\mathfrak{g}}(Z) = \mathrm{Lie}(H)$. *Then there exists a natural one–to–one correspondence between*

i) invariant bi-Lagrangian structures $TM = T^+M \oplus T^-M$*;*
ii) H*–invariant decompositions (called* bi–isotropic*) of the Lie algebra*

$$\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+ , \tag{15}$$

where $\mathfrak{n}^\pm$ *given by* (13) *are subalgebras such that* $B|_{\mathfrak{n}^\pm} = 0$*;*
iii) fundamental H*–invariant* $\mathbb{Z}$*–gradings* (14) *with* $\mathfrak{g}^0 = \mathfrak{h}$.

More precisely, the bi–isotropic decomposition (15), which corresponds to the fundamental grading (14), is given by

$$\mathfrak{n}^\pm = \sum_{\pm i > 0} \mathfrak{g}^i$$

and $\mathfrak{h} = \mathfrak{g}^0$, while the bi–Lagrangian decomposition $TM = T^+M \oplus T^-M$ associated with (15) is the natural G–invariant extension of the H–invariant decomposition $T_oM = \mathfrak{n}^+ + \mathfrak{n}^-$ of the tangent space of $M = G/H$ at $o = eH = [e]_H$, under the standard identification $T_oM = \mathfrak{g}/\mathfrak{h} = \mathfrak{n}^+ + \mathfrak{n}^-$.

1.3 Contactification of a homogeneous para–Kähler manifold

Let G be a real semisimple Lie group and $M := \mathrm{Ad}_G Z = G/H$ the adjoint orbit of a closed and splittable semisimple element $Z \in \mathfrak{g}$: consider the B-orthogonal decomposition $\mathfrak{h} = \mathfrak{l} + \mathbb{R}Z$ of the stability subalgebra $\mathfrak{h} = \mathrm{Lie}(H)$, where $\mathfrak{l} = (\mathbb{R}Z)^\perp$.

Theorem 1.4 *The Lie algebra* $\mathfrak{l}$ *generates a closed subgroup* $L \subset G$ *and the homogeneous manifold* $N := G/L$ *has the reductive decomposition*

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{n} ,$$

with $\mathfrak{n} = \mathbb{R}Z + \mathfrak{m}$, *where* $\mathfrak{m} = T_Z M$.

The Ad_L*-invariant 1-form* $B \circ Z$ *defines an invariant contact structure* θ *on* N*, where the invariant contact 1-form* θ *is the* G*–invariant extension of* $B \circ Z$.

Proof See [6, Proposition 3.5] and [1].

The natural projection

$$\pi : N = G/L \longrightarrow M = G/H \tag{16}$$

is a G-equivariant principal bundle with structure group $\mathbb{R} = \{\exp tZ \mid t \in \mathbb{R}\}$, and H identifies with the product $H = L \cdot \{\exp tZ \mid t \in \mathbb{R}\}$; moreover, the contact form $\theta : TN \longrightarrow \mathbb{R}$ turns out to be the connection form of a principal connection on π, whose curvature form is the symplectic form $\omega = d\theta$ on M, see [10, Theorem 4].

Definition 1.8 The G–homogeneous contact manifold (N, θ) is called the *contactification* of the G–homogeneous para–Kähler manifold (M, ω).

1.4 Homogeneous para–Kähler manifolds as a completion of the cotangent bundle of a real flag manifold

Let (M, ω) be as in Sect. 1.3 above; recall also the bi–isotropic decomposition (15) from Theorem 1.3 and the subgroups $N^\pm$ from Corollary 1.2. Letting $P^\pm := H \cdot N^\pm$ be the closed real parabolic subgroup of G with nilradical $N^\pm$, which is a semidirect product $P^\pm = H \ltimes N^\pm$, we have the following well–known result.

Proposition 1.2 *The map*

$$\begin{aligned} N^- \times H \times N^+ &\longrightarrow G, \\ (n^-, h, n^+) &\longmapsto g = n^- h n^+, \end{aligned}$$

is a diffeomorphism onto an open dense submanifold $\widetilde{G} \subset G$.

Definition 1.9 The decomposition

$$\widetilde{G} = N^- H N^+$$

is called the *generalized Gauss decomposition*.

We shall need the real flag manifold $F := G/P^+$ corresponding to the parabolic subgroup $P^+ = H \cdot N^+$, together with the natural projection

$$\pi : M = G/H \longrightarrow F = G/P^+ . \tag{17}$$

Let ω_{std} denote the standard symplectic form on the cotangent space to a smooth manifold.

Theorem 1.5 *Up to a zero–measure subset, the homogeneous symplectic manifold (M, ω) is symplectomorphic to the symplectic manifold T^*F, equipped with the symplectic form ω_{T^*F} unambiguously defined by*

$$\omega_{T^*F}|_{\mathfrak{g}^j + \mathfrak{g}^{-j}} = j\omega_{\mathrm{std}} , \quad \forall j \in \mathbb{N} . \tag{18}$$

Proof Let $o = [e]_H = eH$ be the origin of M and let us take $\pi(o) = eP^+$ as the origin of F: this means that $o \in M = G/H$ corresponds to $Z \in M = \mathrm{Ad}_G(Z)$. The following n-dimensional manifolds

$$\begin{aligned} N_M &:= N^- o = \mathrm{Ad}_{N^-}(Z) \subset M, \\ N_F &:= N^- \pi(o) = \widetilde{G}/P^+ \subset F, \end{aligned}$$

are then related to each other by an N^-–equivariant diffeomorphism. Indeed, $\pi : M \longrightarrow F$ restricts to an N^-–equivariant diffeomorphism $\pi : N_M \longrightarrow N_F$; moreover, by Proposition 1.2, the orbit N_F is an open dense submanifold of F.

Next, we prove that the N^-–homogeneous vector bundles T^*N_M and $T^+M|_{N_M}$ can be identified by means of the (G–invariant) isomorphism between T^*M and TM given by the symplectic form ø: to this end it suffices to consider the fibers at the origin, that are

$$\begin{aligned} T_o N_M &= \mathfrak{n}^- \subset T_o M = \mathfrak{n}^- + \mathfrak{n}^+, \\ T_o^+ N_M &= \mathfrak{n}^+, \end{aligned}$$

respectively, and to observe that

$$\begin{aligned} \mathfrak{n}^+ &\longrightarrow (\mathfrak{n}^-)^*, \\ X^+ &\longrightarrow \omega_o(X^+, \cdot), \end{aligned}$$

is an isomorphism. Therefore, the flag manifold F can be replaced by the open dense subset $N_F \subset F$: taking into account the N^-–equivariant isomorphism $T^*N_F \simeq T^+M|_{N_M}$, it remains to construct a diffeomorphism $\Phi : T^+M|_{N_M} \longrightarrow M$. This will be given by the following exponential map:

$$\begin{aligned} \Phi : T^+M|_{N_M} &\longrightarrow M, \\ n_*^- X^+ &\longmapsto n^-(\exp X^+)o, \end{aligned} \tag{19}$$

where $n^- \in N^-$, $X^+ \in T_o^+M = \mathfrak{n}^+$, $\exp : \mathfrak{n}^+ \to N^+$ is the exponential map of the Lie algebra $\mathfrak{n}^+$ into the group N^+, and $n_*^- : T_o^+M \to T_{n^-o}^+M$ is the differential at o of the action of n^-.

By Corollary 1.2, the restriction $\Phi|_{T_{n^-o}^+M}$ of Φ to the fiber $T_{n^-o}^+M$ of the rank–n bundle $T^+M|_{N_M}$ at the point n^-o of the n-dimensional manifold N_M is a diffeomorphism onto its image, which is the n-dimensional maximal integral submanifold of T^+M that passes through n^-o; since the set of all points of the form n^-o, that is the orbit $N^-o = N_M$, is transversal to the aforementioned integral manifolds, the latter make up a (smooth) bundle over N_M: since (19) is manifestly N^-–equivariant, the whole map Φ, regarded as the N^-–equivariant extension of the diffeomorphism $\Phi|_{T_{n^-o}^+M}$, is a diffeomorphism itself.

It remains to show that the diffeomorphism (19), regarded as a map

$$\Phi : T^*N_F \longrightarrow M,$$

pulls back the symplectic form ω^Z given by (11) to the “deformation” ω_{T^*F} of the standard symplectic form ω_{std} on T^*N_F given by (18). To begin with, we compare ω_{T^*F} with $\Phi^*(\omega^Z)$ at the zero $0 \in T_{\pi(o)}^*N_F$ of the fiber of T^*N_F at the origin $\pi(o)$ of N_F. Let us recall that

$$\omega_{\mathrm{std}}|_0 \in \Lambda^2(T_0^*(T^*N_F)) = \Lambda^2((T_{\pi(o)}N_F \oplus T_0(T_{\pi(o)}^*N_F))^*) = \Lambda^2(T_{\pi(o)}^*N_F \oplus T_{\pi(o)}N_F)$$

is given by

$$\omega_{\mathrm{std}}|_0 = e_i \wedge \varepsilon^i \ ,$$

where $\{e_i\}$ is a basis of $T_{\pi(o)}N_F$ and $\{\varepsilon^i\}$ is its dual basis in $T_{\pi(o)}^*N_F$.

We shall need a root system R of $\mathfrak{g}$, where $R = R^+ \cup R^-$ is its splitting into positive and negative roots, and $\mathfrak{g}_\alpha$ denotes the eigenspace of each $\alpha \in R$.

Then it is possible to choose as basis of $T_{\pi(o)}N_F \equiv \mathfrak{n}^-$ a system $\{E_\alpha \mid \alpha \in R^+\}$, where each E_α is a generator of $\mathfrak{g}_\alpha$ for all $\alpha \in R$, and

$$B(E_\alpha, E_{-\beta}) = \delta_{\alpha,\beta} \ , \quad \forall \alpha, \beta \in R^+ \ ,$$

Since Φ maps $0 \in T^*N_F$ to $o = Z \in M$, one needs to calculate

$$\omega(E_\alpha, E_\beta) = \omega^Z(E_\alpha, E_\beta)$$

for all $\alpha, \beta \in R$: by definition,

$$\omega^Z(E_\alpha, E_\beta) = B(Z, [E_\alpha, E_\beta]), \tag{20}$$

and $[E_\alpha, E_\beta] = 0$ unless $\alpha + \beta = 0$, in which case

$$\omega^Z(E_\alpha, E_{-\alpha}) = B(Z, H_\alpha) = j, \tag{21}$$

where j is the degree of E_α and H_α makes, together with $E_{\pm\alpha}$, a standard $\mathfrak{sl}_2$–triple, which matches with the definition (18) of ω_{T^*F}.

The general claim follows from invariance arguments.

2 Monge–Ampère equations and their generalization

In this section we review some basic facts concerning the contact geometry of Monge–Ampère equations (MAEs); more details can be found in [5, 27].

2.1 MAEs via *n*-forms on jet spaces

Let

$$J^k(n, 1) := \{[f]^k_x \mid f : \mathcal{U} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}\}$$

be the k-order jet space of smooth functions. Let $j^k(f) : \mathcal{U} \subseteq \mathbb{R}^n \to J^k$ be the k-jet prolongation of $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$, given by $j^k(f)(x) := [f]^k_x$, and Ω a differential form on J^k. Then the differential form on $\mathcal{U}$

$$j^k(f)^*(\Omega)$$

is called the *horizontalization* of Ω.

Definition 2.1 For any n–form Ω on J^1, we call $\mathcal{E}_\Omega := \{[f]^2_x \in J^2 \mid j^1(f)^*(\Omega) = 0\}$ the *Monge–Ampère equation (MAE)* associated with Ω (see also Sect. 2.2 below). A function $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$, such that $[f]^2_x \in \mathcal{E}_\Omega$ $\forall x \in \mathcal{U}$, is a *solution* of $\mathcal{E}_\Omega$.

2.2 MAEs on contact and symplectic manifolds

Given a contact manifold $(N, \mathcal{C})$, that is a $(2n+1)$-dimensional manifold equipped with a completely non-integrable $2n$-dimensional distribution $\mathcal{C}$, the 2-form $\omega := d\theta|_{\mathcal{C}}$ defines a conformal symplectic structure on $\mathcal{C}$, where θ is any 1–form, such that $\ker(\theta) = \mathcal{C}$. In turn, this conformal symplectic structure allows to introduce the *Lagrangian Grassmannian* of $(\mathcal{C}_p, \omega_p)$, that is the set

$$\begin{aligned}\mathcal{L}(\mathcal{C}_p) &\overset{\text{def.}}{=} \{L_p \mid L_p \text{ is a Lagrangian } n\text{-dimensional subspace of } \mathcal{C}_p\} \\ &= \{L_p \in \mathrm{Gr}(n, \mathcal{C}_p) \mid \omega_p|_{L_p} \equiv 0\}\end{aligned}$$

of all Lagrangian n-dimensional subspaces (henceforth called simply “planes” by analogy with the case $n = 2$) at a given point $p \in N$.

Definition 2.2 The *prolongation* of a contact manifold $(N, \mathcal{C})$ is the fiber bundle $\pi : N^{(1)} \longrightarrow N$, where

$$N^{(1)} := \bigcup_{p \in N} \mathcal{L}(\mathcal{C}_p),$$

and π is the natural projection.

Points $p^{(1)}$ of $N^{(1)}$ can be then understood as Lagrangian planes $L_{p^{(1)}}$ of $(\mathcal{C}_p, \omega_p)$, where $p = \pi(p^{(1)})$: there is then a natural correspondence

$$p^{(1)} \in N^{(1)} \Longleftrightarrow L_{p^{(1)}} \in \mathcal{L}(\mathcal{C}_p)\,, \quad p = \pi(p^{(1)})\,.$$

According to V. V. Lychagin [27] and M. T. Morimoto [34], Definition 2.1 can be generalized by Definition 2.3 below. Given a contact manifold $(N, \mathcal{C})$, with $\mathcal{C} = \ker(\theta)$, let $\mathcal{I}_\mathcal{C}$ be the (graded) ideal of the algebra $\Lambda^\bullet(N)$ made of differential forms that vanish on all integral submanifolds of $\mathcal{C}$: the following forms

$$\theta \wedge \alpha + d\theta \wedge \beta\,, \quad \alpha, \beta \in \Lambda^\bullet(N)\,,$$

generate $\mathcal{I}_\mathcal{C}$, see [27]. Recall also that $L_{p^{(1)}} \subset T_{\pi(p^{(1)})}N$ is the Lagrangian plane associated with $p^{(1)} \in N^{(1)}$.

Definition 2.3 The hypersurface $\mathcal{E}_\Omega$ of $N^{(1)}$ given by

$$\mathcal{E}_\Omega := \left\{ p^{(1)} \in N^{(1)} \mid \Omega|_{L_{p^{(1)}}} \equiv 0 \right\}\,, \tag{22}$$

where $\Omega \in \Lambda^n(N)$, is called a *(general) MAE*.

MAEs can be defined also on symplectic manifolds rather than on contact ones: it is enough to replace the above contact manifold N with a symplectic manifold M: details of such a construction are omitted and we refer the reader to, e.g., [17, 38].

Remark 2.1 Since, in the definition (22) of a general MAE, the restriction $\Omega|_{L_{p^{(1)}}}$ is identically zero if the form $\Omega \in \Lambda^n(N)$ belongs to the contact ideal $\mathcal{I}_\mathcal{C}$, instead of the form Ω one can use the equivalence class

$$[\Omega] \in \frac{\Lambda^n(N)}{\mathcal{I}_\mathcal{C}}\,.$$

Elements of the quotient

$$\Lambda^\bullet(N) \longrightarrow \frac{\Lambda^\bullet(N)}{\mathcal{I}_\mathcal{C}} \tag{23}$$

are called *effective* forms; accordingly, some speak of MAE associated with an *effective* differential form.

2.3 MAEs in Darboux coordinates

If F is an n-dimensional smooth manifold, equipped with coordinates $\{x^1, \ldots, x^n\}$, then the standard symplectic form $\omega_{\rm std}$ on T^*F reads

$$\omega_{\rm std} = dx^i \wedge du_i\,, \tag{24}$$

where u_i is the *momentum* conjugate with x^i; together with the coordinate u that corresponds to the value of a function, the x^i's and the momenta u_i's form the so–called Darboux coordinate system of the contact manifold

$$J^1(F, \mathbb{R}) = T^*F \times \mathbb{R}\,.$$

In such a coordinate system, the contact form θ reads

$$\theta = du - u_i dx^i\,.$$

The second–order jet space $J^2(F,\mathbb{R})$ is an affine bundle over $J^1(F,\mathbb{R})$, its fiber at $p^1 \in J^1(F,\mathbb{R})$ being modeled by the symmetric power $S^2(T^*_xF)$, where x is the natural projection of p^1 on F: this allows to extend the Darboux coordinate system by adding the coordinates u_{ij} that correspond to second–order derivatives. A point p^2 of the fiber of $J^2(F,\mathbb{R})$ over $p^1 \in J^1(F,\mathbb{R})$ has coordinates u_{ij}, if

$$u_{ij}dx^i dx^j \in S^2(T^*_xF)$$

is the symmetric form that corresponds to p^2; the same point can be regarded as a Lagrangian plane of $\mathcal{C}_{p^1}$ by means of the embedding

$$\begin{aligned} S^2(T^*_xF) &\longrightarrow \mathrm{LGr}(\mathcal{C}_{p^1}),\\ p^2 = u_{ij}dx^i dx^j &\longmapsto L_{p^2} = \mathrm{span}\left(D_{x^i} + u_{ij}\partial_{u_j} \mid i = 1,\ldots,n\right), \end{aligned} \tag{25}$$

where

$$D_{x^i} = \partial_{x^i} + u_i\partial_u.$$

A coordinate expression of the general MAE $\mathcal{E}_\Omega$ defined by an n-form Ω on $J^1(F,\mathbb{R})$ can be then obtained by employing the extended Darboux coordinates introduced above; to this end we shall need the Plücker embedding

$$\begin{aligned} \mathrm{LGr}(\mathcal{C}_{p^1}) &\longrightarrow \mathbb{P}(\Lambda^n\mathcal{C}_{p^1}),\\ L = \mathrm{span}\,(\ell_1\ldots,\ell_n) &\longmapsto \mathrm{vol}\,(L) := [\ell_1\wedge\cdots\wedge\ell_n], \end{aligned} \tag{26}$$

as well as a coordinate expression

$$\Omega = A dx^1\wedge\cdots\wedge dx^n + B^j_i dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n\wedge du_j + \cdots + C du_1\wedge\cdots\wedge du_n \tag{27}$$

of Ω, where $\widehat{\ }$ denotes the removal of a factor. By combining (25) with (26), we see that the point p^2 is mapped to the projective class of

$$D_{x^1}\wedge\cdots\wedge D_{x^n} + u_{ij}D_{x^1}\wedge\cdots\wedge\widehat{D_{x^i}}\wedge\cdots\wedge D_{x^n}\wedge\partial_{u_j} + \cdots + \det(u_{ij})\partial_{u_1}\wedge\cdots\wedge\partial_{u_n}.$$

Then, by applying the formula (22), we find out that

$$\mathcal{E}_\Omega = \{F_\Omega(u_{ij}) = 0\},$$

where $F_\Omega(u_{ij})$ is a linear combination of the minors of the matrix (u_{ij}), unambiguously defined by

$$\Omega|_{L_{p^2}} = F_\Omega(u_{ij})dx^1\wedge\cdots\wedge dx^n. \tag{28}$$

Example 2.1 Let $n = 2$ and consider the 2-form $\Omega = du_1\wedge du_2$. The restriction of Ω to the generic Lagrangian plane

$$\mathrm{span}\left(D_{x^1} + u_{11}\partial_{u_1} + u_{12}\partial_{u_2},\ D_{x^2} + u_{12}\partial_{u_1} + u_{22}\partial_{u_2}\right)$$

is given by

$$(u_{11}dx^1 + u_{12}dx^2)\wedge(u_{12}dx^1 + u_{22}dx^2) = (u_{11}u_{22} - u_{12}^2)dx^1\wedge dx^2$$

so that we obtain the MAE

$$\mathcal{E}_\Omega = \{u_{11}u_{22} - u_{12}^2 = 0\}.$$

Note that $u_{11}dx^1 + u_{12}dx^2$ and $u_{12}dx^1 + u_{22}dx^2$ are the horizontalization of the forms du_1 and du_2, respectively (cf. Section 2.1). For arbitrary n, if all coefficients in (27) are zero, except C, then $\mathcal{E}_\Omega$ is the Monge–Ampère equation

$$\mathcal{E}_\Omega = \{\det(u_{ij}) = 0\}\,.$$

We refer the reader to [16, 18, 27] and references therein for more details.

2.4 Fibers of MAE as hyperplane sections

At each point p of the $(2n+1)$-dimensional contact manifold $(N, \mathcal{C})$, let us consider the Plücker embedding (26): then the fiber

$$(\mathcal{E}_\Omega)_p := \mathcal{E}_\Omega \cap N_p^{(1)} = \mathcal{E}_\Omega \cap \mathrm{LGr}(\mathcal{C}_p)$$

of the MAE $\mathcal{E}_\Omega$ defined by (22) turns out to be a *hyperplane section* of $\mathrm{LGr}(\mathcal{C}_p)$. Indeed, the evaluation Ω_p of Ω at $p \in N$ is an element of the exterior algebra

$$\Lambda^n(\mathcal{C}_p^*) = (\Lambda^n(\mathcal{C}_p))^*$$

of the dual vector space $\mathcal{C}_p^*$ and, as such, the linear equation

$$\Omega_p = 0 \tag{29}$$

defines a projective hyperplane in $\mathbb{P}(\Lambda^n\mathcal{C}_p)$, whose intersection with $\mathrm{LGr}(\mathcal{C}_p)$ gives precisely $(\mathcal{E}_\Omega)_p$. It is worth stressing that (29) depends only on the effective form determined by Ω via the projection (23), see Remark 2.1.

In view of the natural projection

$$T_p^*N \longrightarrow \mathcal{C}_p^*$$

the basis

$$(dx^1)_p\,, \ldots\,, (dx^n)_p\,, (du)_p\,, (du_1)_p\,, \ldots\,, (du_n)_p\,,$$

of T_p^*N gives rises to the basis

$$(dx^1)_p\,, \ldots\,, (dx^n)_p\,, (du_1)_p\,, \ldots\,, (du_n)_p\,, \tag{30}$$

of $\mathcal{C}_p^*$. In the present paper, we will use (30) as a standard basis of $\mathcal{C}_p^*$; moreover, since homogeneity allows to restrict ourselves to the fiber of N at the origin, the index p will be omitted in the symbols above.

3 Examples of para–Kähler homogeneous manifolds and their contactifications

3.1 Fundamental gradings of a (complex or real) semisimple Lie algebra $\mathfrak{g}$

Let

$$\mathfrak{g} = \mathfrak{a} + \sum_{\alpha\in R} \mathfrak{g}_\alpha$$

be a root space decomposition of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{a}$. We fix a system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subset R$, that is a basis of $\mathfrak{a}^*$, such that any root $\alpha \in R$ has integer coefficients with respect to Π of the same sign (non-negative or non-positive).

Any disjoint decomposition

$$\Pi = \Pi^0 \cup \Pi^1 \tag{31}$$

of Π defines a fundamental grading of $\mathfrak{g}$ as follows (see Definition 1.7). First, define the function $d : R \longrightarrow \mathbb{Z}$ by

$$\begin{aligned} d|_{\Pi^0} &:= 0, \\ d|_{\Pi^1} &:= 1, \\ d(\alpha) &:= \sum k_i d(\alpha_i) \quad \forall \alpha = \sum k_i \alpha_i \in R. \end{aligned}$$

Let

$$R^i := \{\alpha \in R \mid d(\alpha) = i\} \tag{32}$$

be the level set of d: then the fundamental grading associated with (31) is given by

$$\mathfrak{g}^0 = \mathfrak{a} + \sum_{\alpha \in R^0} \mathfrak{g}_\alpha = \mathfrak{a} + \sum_{\alpha \in \Pi^0} \mathfrak{g}_\alpha ,$$

and

$$\mathfrak{g}^i = \sum_{\alpha \in R^i} \mathfrak{g}_\alpha , \quad \forall i \neq 0 .$$

Notice that any fundamental grading of $\mathfrak{g}$ is conjugated to a unique grading of such a form: see [35, Section 3.5].

Any real semisimple Lie algebra $\widehat{\mathfrak{g}}$ is a real form of a complex semisimple Lie algebra $\mathfrak{g}$, that is $\widehat{\mathfrak{g}} = \mathfrak{g}^\sigma$ is the fixed point set of some antilinear involution σ of $\mathfrak{g}$. We can always assume that σ preserves a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ and induces an automorphism of the root system R. A root $\alpha \in R$ is called *compact* (or *black*) if $\sigma(\alpha) = -\alpha$. It is always possible to choose a system of simple roots $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ such that, for any non-compact root $\alpha_i \in \Pi$, the corresponding root $\sigma(\alpha_i)$ is a sum of one non-compact root $\alpha_j \in \Pi$ and a linear combination of compact roots from Π: the roots α_i and α_j are called *equivalent*.

Proposition 3.1 *[Alekseevsky–Medori [3]] Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ an antilinear involution, and $\mathfrak{g}^\sigma$ the corresponding real form. The grading of $\mathfrak{g}$, associated with a decomposition $\Pi = \Pi^0 \cup \Pi^1$, defines a grading $\mathfrak{g}^\sigma = \Sigma(\mathfrak{g}^i)^\sigma$ of $\mathfrak{g}^\sigma$ if and only if Π^1 consists of non-compact roots and any two equivalent roots are either both in Π^0 or both in Π^1.*

3.2 Examples of fundamental gradings

3.2.1 Fundamental gradings of $\mathfrak{sl}(V)$

Let V be a (complex or real) vector space and $V = V^1 \oplus \dots \oplus V^k$ a decomposition of V into a direct sum of subspaces: this defines a fundamental grading

$$\mathfrak{sl}(V) = \sum_{i=-k}^{k} \mathfrak{g}_i$$

of the Lie algebra $\mathfrak{sl}(V)$, where

$$\mathfrak{g}^i = \{A \in \mathfrak{sl}(V) \mid A(V^j) \subset V^{i+j} \quad \forall j = 1, \dots, k\} . \tag{33}$$

Proposition 3.2 *Any fundamental grading of* $\mathfrak{sl}(V)$ *is of the form* (33).

Proof Formally analogous to the proof in the case of $\mathfrak{su}(n)$, see: [2, Section 5].

3.2.2 Fundamental gradings of $\mathfrak{g}_2$

The root system of the complex exceptional Lie algebra $\mathfrak{g}_2$ has the form

$$R = \{\pm\epsilon_i, \pm(\epsilon_i - \epsilon_j) \mid i, j = 1, 2, 3\}, \tag{34}$$

where the vectors ϵ_i satisfy

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = 0, \tag{35}$$

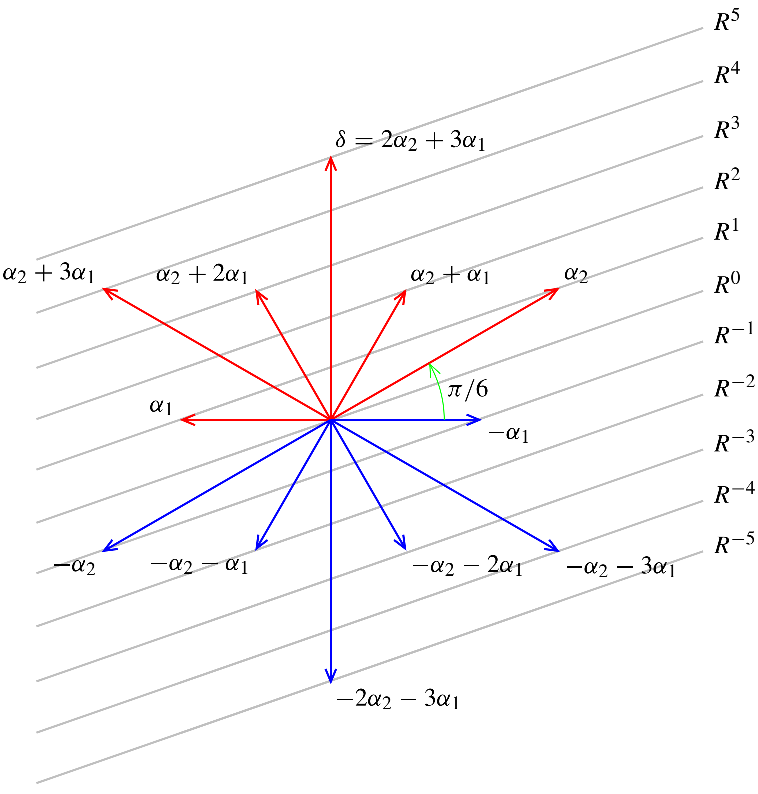
$$(\epsilon_i, \epsilon_i) = \frac{2}{3}, \tag{36}$$

$$(\epsilon_i, \epsilon_j) = -\frac{1}{3}, \quad i \neq j. \tag{37}$$

Consider the system of simple roots $\Pi = \{\alpha_1 := -\epsilon_2, \alpha_2 := \epsilon_2 - \epsilon_3\}$. The corresponding system of positive roots is

$$R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\},$$

and it is represented by the upper six (red) arrows in the figure below:



There are three fundamental gradings for the complex Lie algebra $\mathfrak{g}_2$. For any of such gradings, we give below the subset $\Pi^1 \subset \Pi$ and the level sets R^i of the grading function $d : R \longrightarrow \mathbb{Z}$, cf. (32).

(1) $\Pi^1 = \Pi$:

$$\begin{aligned} R^0 &= \emptyset, \\ R^1 &= \{\alpha_1, \alpha_2\}, \\ R^2 &= \{\alpha_1 + \alpha_2\}, \\ R^3 &= \{2\alpha_1 + \alpha_2\}, \\ R^4 &= \{3\alpha_1 + \alpha_2\}, \\ R^5 &= \{3\alpha_1 + 2\alpha_2\}. \end{aligned}$$

All the level sets R^i, with $i = -5, \dots, +5$ are represented by the gray parallel lines in the picture above.

(2) $\Pi^1 = \{\alpha_1\}$:

$$\begin{aligned} R^0 &= \{\alpha_2\}, \\ R^1 &= \{\alpha_1, \alpha_1 + \alpha_2\}, \\ R^2 &= \{2\alpha_1 + \alpha_2\}, \\ R^3 &= \{3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}. \end{aligned}$$

(3) $\Pi^1 = \{\alpha_2\}$:

$$\begin{aligned} R^0 &= \{\alpha_1\}, \\ R^1 &= \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\}, \\ R^2 &= \{3\alpha_1 + 2\alpha_2\}. \end{aligned}$$

There are just two real forms of the complex Lie algebra $\mathfrak{g}_2$: the compact form, which has no non-trivial $\mathbb{Z}$–gradings,[1] and the normal form $\mathfrak{g}_2^\sigma$, which has a diagonalizable Cartan subalgebra and no compact roots. The above–listed gradings of the complex Lie algebra $\mathfrak{g}_2$ define three gradings of the real Lie algebra $\mathfrak{g}_2^\sigma$; they also correspond to the gradings labeled (3), (2) and (1), respectively, in [13, Theorem 2], see also [8].

In the sequel we will be working with the non-compact real form of the Lie group G_2 and with its fundamental grading corresponding to the third choice of Π^1 in the above list.

4 Classification of G_2–invariant MAEs on the contactification of $\mathsf{G}_2/\,\mathsf{GL}_2(\mathbb{R})$

4.1 Invariant effective *n*-forms on the contact manifold *N*

We recall below the general definition of an invariant differential form on a homogeneous manifold.

[1] There are nonetheless non-trivial gradings over other groups, e.g., over $\mathbb{Z}_2$, $\mathbb{Z}_2^2$, and $\mathbb{Z}_2^3$.

4.1.1 Invariant forms on a homogenous manifold

A differential form ξ on $N = G/L$ is called a relative G–invariant differential form if

$$g^*(\xi) = \lambda\xi \quad \forall g \in G\,,$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$, where $g^* : \Lambda^\bullet(N) \to \Lambda^\bullet(N)$ is the pull–back of the diffeomorphism g associated to the group element g; if $\lambda = 1$, then we omit "relative": it follows from the properties of the pullback that the subset

$$\Lambda^\bullet(N)^G \subseteq \Lambda^\bullet(N)$$

of all G–invariant differential forms on N is a graded differential subalgebra. Moreover, since $\mathfrak{n}$ is the tangent space of N at the origin, it is possible to identify

$$\Lambda^\bullet(N)^G \simeq \Lambda^\bullet(\mathfrak{n}^*)^L := \{\xi \in \Lambda^\bullet(\mathfrak{n}^*) \mid \mathrm{Ad}^*_g(\xi) = 0 \quad \forall g \in L\} \tag{38}$$

or, in the case of connected L,

$$\Lambda^\bullet(N)^G \simeq \Lambda^\bullet(\mathfrak{n}^*)^{\mathfrak{l}} := \{\xi \in \Lambda^\bullet(\mathfrak{n}) \mid \ \mathrm{ad}^*_X(\xi) = 0 \quad \forall X \in \mathfrak{l}\}\,. \tag{39}$$

Remark 4.1 If $V_1 \subseteq \Lambda^\bullet(\mathfrak{n})$ is a one-dimensional $\mathfrak{l}$–submodule, then any element $\xi \in V_1 \setminus \{0\}$ is a relative invariant form.

4.1.2 Invariant effective forms on the contactification *N* of a homogenous para–Kähler manifold

We consider now the contactification $N = G/L$ of the G–homogeneous para–Kähler manifold $M = G/H$, and recall the reductive decompositions $\mathfrak{g} = \mathfrak{h}+\mathfrak{m}$ and $\mathfrak{g} = \mathfrak{l}+\mathfrak{n}$, associated with M and N, respectively, where $\mathfrak{n} = \mathfrak{m} + \mathbb{R}Z$, $\mathfrak{m} = \mathfrak{n}_+ + \mathfrak{n}_-$, and $\mathfrak{h} = \mathfrak{l} + \mathbb{R}Z$.

The hyperplane $\mathfrak{m} \subset \mathfrak{n}$ can be then interpreted both as the contact plane $\mathcal{C}_{eL}$ at the origin eL of N and as the tangent plane T_oM at the origin $o = eH$ of M: the former is mapped isomorphically to the latter by the differential at eL of the G–equivariant natural projection π in (16). Accordingly, the algebra epimorphism

$$\Lambda^\bullet(\mathfrak{n}^*)^{\mathfrak{l}} \longmapsto \Lambda^\bullet(\mathfrak{m}^*)^{\mathfrak{l}}\,, \tag{40}$$

that is dual to the inclusion $\mathfrak{m} \subset \mathfrak{n}$, maps G–invariant forms on N to effective G–invariant forms on N. The projection (40) can be regarded as the G–invariant counterpart of the natural projection (23).

Remark 4.2 Since $\mathfrak{h} = \mathfrak{l} + \mathbb{R}Z$, the algebra $\Lambda^\bullet(\mathfrak{m}^*)^{\mathfrak{h}}$ of G–invariant differential forms on the homogeneous para–Kähler manifold G/H is a proper subalgebra of $\Lambda^\bullet(\mathfrak{m}^*)^{\mathfrak{l}}$:

$$\Lambda^\bullet(\mathfrak{m}^*)^{\mathfrak{h}} = \{\xi \in \Lambda^\bullet(\mathfrak{m}^*)^{\mathfrak{l}} \mid \mathrm{ad}^*_Z(\xi) = 0\}\,.$$

Elements of $\Lambda^\bullet(\mathfrak{m}^*)^{\mathfrak{l}}$ that are eigenvectors of ad^*_Z correspond to *relative* G–invariant differential forms on M.

As pointed out in Sect. 2.4, the fiber $(\mathcal{E}_\Omega)_{eL}$ at the origin $eL \in N$ of a MAE $\mathcal{E}_\Omega$ is given by the equation $\Omega_{eL} = 0$, where Ω_{eL} is an element of $\Lambda^n(\mathcal{C}^*_{eL}) = \Lambda^n(\mathfrak{m}^*)$. In particular, if $\mathcal{E}_\Omega$ is G–invariant, then it is unambiguously defined by its fiber $(\mathcal{E}_\Omega)_{eL}$: it follows that

G–invariant MAEs on N are in one–to–one correspondence with hypersurfaces $\{\xi = 0\}$, where $\xi \in \Lambda^n(\mathfrak{m}^*)^{\mathfrak{l}}$ is a G–invariant *effective* n-form on N.

A key step towards a classification of invariant MAEs on N is then the description of G-invariant effective n-forms on N, i.e., the description of the space $\Lambda^n(\mathfrak{m}^*)^{\mathfrak{l}} \simeq \Lambda^n(\mathfrak{m})^{\mathfrak{l}}$.

Since $\mathfrak{n}^{\pm}$ are $\mathfrak{l}$-submoduli, one can decompose further

$$\Lambda^n(\mathfrak{m})^{\mathfrak{l}} = \sum_{p+q=n} (\Lambda^{\mathfrak{l}})^{p,q} ,$$

where

$$(\Lambda^{\mathfrak{l}})^{p,q} := \Lambda^p(\mathfrak{n}^+)^{\mathfrak{l}} \wedge \Lambda^q(\mathfrak{n}^-)^{\mathfrak{l}}.$$

4.2 Properties of the roots of the exceptional Lie algebra $\mathfrak{g}_2$

Let $\mathfrak{g}_2$ be the non-compact real Lie algebra of type G_2 with a Cartan subalgebra $\mathfrak{a}$; following the notation of Gorbatsevich, Onishchik and Vinberg [35], we take the root system R given by (34), where ϵ_1, ϵ_2 and ϵ_3 satisfy (35)–(36)–(37), and a set of simple roots is $\Pi = \{\alpha_1 = -\epsilon_2,\ \alpha_2 = \epsilon_2 - \epsilon_3\}$, with

$$(\alpha_1, \alpha_1) = 2/3\,, \quad (\alpha_2, \alpha_2) = 2\,, \quad (\alpha_1, \alpha_2) = -1\,,$$

see Sect. 3.2.2. The maximal root is

$$\delta := 3\alpha_1 + 2\alpha_2 = \epsilon_1 - \epsilon_3\,,$$

in particular,

$$\begin{aligned} < \alpha|\delta > := \frac{2(\alpha, \delta)}{(\delta, \delta)} &= (\delta, \alpha) \quad \forall \alpha \in R, \\ < \alpha_1|\delta > &= 0, \\ < \alpha_2 + k\alpha_1|\delta > &= (\alpha_2, \delta) = 1 \quad \forall k, \end{aligned}$$

see the picture at page 10. It will be convenient to choose the generators $E_{\pm\alpha}$ of $\mathfrak{g}_{\pm\alpha}$ in such a way that

$$\begin{aligned} B(E_\alpha, E_\alpha) &= \frac{2}{(\alpha, \alpha)} \quad \forall \alpha \in R, \\ [H_{\alpha_i}, E_\beta] &= \frac{2(\alpha_i, \beta)}{(\alpha_i, \alpha_i)} E_\beta \quad \forall \alpha_i \in \Pi \quad \forall \beta \in R, \end{aligned}$$

where $H_\alpha = [E_\alpha, E_{-\alpha}]$: this can be accomplished in a rather standard way (see the aforementioned book).

The operator ad_{H_δ} acts on the vector $E_{\alpha_2+k\alpha_1}$ as the identity Id and on the dual covector $E^*_{\alpha_2+k\alpha_1} = (E_{\alpha_2+k\alpha_1}, \cdot)$ as $-$Id .

4.3 Invariant five-forms on the ten-dimensional manifold $M^{10}_3 = \mathsf{G}_2/\,\mathsf{GL}_2(\mathbb{R})$

This case corresponds to the fundamental grading (3) introduced earlier in Sect. 3.2.2: the subalgebra $\mathfrak{h}$ of maximal rank associated with the root α_1 is given by

$$\mathfrak{h} = \mathfrak{gl}_2(\mathbb{R})^{\alpha_1} := \mathbb{R}H_\delta + \mathfrak{sl}_2(\mathbb{R})^{\alpha_1} = \mathrm{span}\left(H_\delta, H_{\alpha_1}, E_{\pm\alpha_1}\right)\,, \tag{41}$$

where

$$\mathfrak{sl}_2(\mathbb{R})^{\alpha_1} := \mathrm{span}\left(H_{\alpha_1}, E_{\pm\alpha_1}\right) \simeq \mathfrak{sl}_2$$

is the standard $\mathfrak{sl}_2$–triple determined by $\pm\alpha_1$. In the sequel we will denote either by $\mathsf{GL}_2(\mathbb{R})^{\alpha_1}$, or simply by H, the subgroup of G whose Lie algebra is given by (41): such a subgroup is clearly isomorphic to $\mathsf{GL}_2(\mathbb{R})$. Since $(\alpha_1, \delta) = 0$, the subgroup H turns out to be the centralizer of the element $Z = H_\delta$, which acts on root vectors as

$$\begin{aligned} \mathrm{ad}_Z E_{\alpha_2+k\alpha_1} &= < \alpha_2 + k\alpha_1 | \delta > E_{\alpha_2+k\alpha_1} = E_{\alpha_2+k\alpha_1}, \\ \mathrm{ad}_Z E_\delta &= 2E_\delta, \end{aligned}$$

and similarly for root vectors corresponding to negative roots. Therefore, the reductive decomposition (12) reads

$$\mathfrak{g}_2 = \mathfrak{h} + \mathfrak{m} = \mathfrak{h} + \mathfrak{n}_+ + \mathfrak{n}_- ,$$

whereas the decomposition (13) reduces to

$$\mathfrak{n}_\pm = \mathfrak{n}_{\pm 1} + \mathfrak{n}_{\pm 2} . \tag{42}$$

A basis of the tangent space $\mathfrak{m} = T_o M$ of the manifold $M = M^{10}_3 = \mathsf{G}_2/H = \mathsf{G}_2/\,\mathsf{GL}_2^{\alpha_1}(\mathbb{R})$ is $\{E_{\pm\gamma_i}, E_{\pm\delta} \mid i = 0, 1, 2, 3\}$, where

$$\gamma_i := \alpha_2 + i\alpha_1 , \quad i = 0, 1, 2, 3 , \quad \delta = 2\alpha_2 + 3\alpha_1 ,$$

are the positive roots of $\mathfrak{g}_2$ different from α_1, whereas $\{H_\delta, H_{\alpha_1}, E_{\pm\alpha_1}\}$ is a basis of the stability subalgebra $\mathfrak{h}$, and $\{H_{\alpha_1}, E_{\pm\alpha_1}\}$ is a basis of the corresponding derived subalgebra $\mathfrak{h}' = \mathfrak{sl}_2(\mathbb{R})$.

Our choice of the coroots leads to a Chevalley basis, i.e., a basis whose corresponding structure constants N are integers: in particular, the action of the operator $\mathrm{ad}_{E_{\pm\alpha_1}}$ on the root vectors from $\mathfrak{m}^+$ is given by

$$\mathrm{ad}_{E_{\pm\alpha_1}} E_{\pm\delta} = 0, \tag{43}$$

$$\mathrm{ad}_{E_{\pm\alpha_1}} E_{\gamma_i} = N_{\pm\alpha_1,\gamma_i} E_{\gamma_{i\pm 1}}, \tag{44}$$

where the only nonzero N_{α_1,γ_i}'s are:

$$\begin{aligned} N_{\alpha_1,\gamma_0} = N_{\alpha_1,\alpha_2} &= 1, \\ N_{\alpha_1,\gamma_1} &= 2, \\ N_{\alpha_1,\gamma_2} &= 3, \\ N_{\alpha_1,-\gamma_1} &= -3, \\ N_{\alpha_1,-\gamma_2} &= -2, \\ N_{\alpha_1,-\gamma_3} &= -1, \end{aligned}$$

whereas the only nonzero $N_{-\alpha_1,\gamma_i}$'s are:

$$\begin{aligned} N_{-\alpha_1,\gamma_1} &= 3, \\ N_{-\alpha_1,\gamma_2} &= 2, \\ N_{-\alpha_1,\gamma_3} &= 1, \\ N_{-\alpha_1,-\gamma_0} = N_{-\alpha_1,-\alpha_2} &= -1, \\ N_{-\alpha_1,-\gamma_1} &= -2, \\ N_{-\alpha_1,-\gamma_2} &= -3, \end{aligned}$$

see, eg., [21, Section 33.5]. In the next propositions we will identify $\mathfrak{m}$ with its dual $\mathfrak{m}^*$: in particular, this means that the dual of the adjoint operator in (38)–(39) can be safely replaced by the adjoint operator itself.

Proposition 4.1 *The space $\Lambda^1(\mathfrak{m}^*)^{\mathfrak{h}'} \cong \Lambda^1(\mathfrak{m})^{\mathfrak{h}'}$ of $\mathfrak{h}'$–invariant one-forms on $\mathfrak{m}$ is generated by $E_{\pm\delta}$.*

Proof If

$$\omega^1 = \mu_+ E_\delta + \sum_{i=0}^{3} \lambda_+^i E_{\gamma_i} + \sum_{i=0}^{3} \lambda_-^i E_{-\gamma_i} + \mu_- E_{-\delta}\,,$$

then (43)–(44) show that

$$\mathrm{ad}_{E_{\alpha_1}}(\omega^1) = \sum_{i=0}^{2} \lambda_+^i N_{\alpha_1,\gamma_i} E_{\gamma_i+1} + \sum_{i=1}^{3} \lambda_-^i N_{\alpha_1,-\gamma_i} E_{-\gamma_i-1}$$

vanishes if and only if $\lambda_+^0 = \lambda_+^1 = \lambda_+^2 = \lambda_-^1 = \lambda_-^2 = \lambda_-^3 = 0$, i.e., $\omega^1 = \mu_+ E_\delta + \lambda_+^3 E_{\gamma_3} + \lambda_-^0 E_{-\gamma_0} + \mu_- E_{-\delta}$. We can now apply $\mathrm{ad}_{E_{-\alpha_1}}$ and find out that

$$\mathrm{ad}_{E_{-\alpha_1}}(\omega^1) = \lambda_+^3 N_{-\alpha_1,\gamma_3} E_{\gamma_2} + \lambda_-^0 N_{-\alpha_1,-\gamma_0} E_{-\gamma_1}$$

vanishes if and only if $\lambda_+^3 = \lambda_-^0 = 0$, whence $\omega^1 = \mu_+ E_\delta + \mu_- E_{-\delta}$.

Proposition 4.2 *The space $\Lambda^2(\mathfrak{m}^*)^{\mathfrak{h}'} \cong \Lambda^2(\mathfrak{m})^{\mathfrak{h}'}$ of $\mathfrak{h}'$–invariant two-forms on $\mathfrak{m}$ is generated by $E_\delta \wedge E_{-\delta}$, together with*

$$\begin{aligned} \omega_\pm^2 &:= E_{\pm\gamma_1} \wedge E_{\pm\gamma_2} - 3E_{\pm\gamma_0} \wedge E_{\pm\gamma_3}\,, \\ \omega^2 &:= 3E_{\gamma_0} \wedge E_{-\gamma_0} + E_{\gamma_1} \wedge E_{-\gamma_1} + E_{\gamma_2} \wedge E_{-\gamma_2} + 3E_{\gamma_3} \wedge E_{-\gamma_3}\,. \end{aligned} \tag{45}$$

Proof From the splitting (42) it follows the splitting

$$\begin{aligned} \Lambda^2(\mathfrak{m}) = {} & \Lambda^2(\mathfrak{n}_{+1}) \oplus (\mathfrak{n}_{+1} \otimes \mathfrak{n}_{-1}) \oplus \Lambda^2(\mathfrak{n}_{-1}) \oplus (\mathfrak{n}_{+1} \otimes \mathfrak{n}_{+2}) \oplus (\mathfrak{n}_{-1} \otimes \mathfrak{n}_{+2}) \\ & \oplus (\mathfrak{n}_{+1} \otimes \mathfrak{n}_{-2}) \oplus (\mathfrak{n}_{-1} \otimes \mathfrak{n}_{-2}) \oplus (\mathfrak{n}_{+2} \otimes \mathfrak{n}_{-2}) \end{aligned}$$

of the 45-dimensional space $\Lambda^2(\mathfrak{m})$. Since $\mathrm{ad}_{E_{\pm\alpha_1}}(\mathfrak{n}_{\pm 1}) \subseteq \mathfrak{n}_{\pm 1}$ and $\mathrm{ad}_{E_{\pm\alpha_1}}(\mathfrak{n}_{\pm 2}) = 0$, all the constituents of the above decomposition are $\mathfrak{h}'$–invariant, so that we can analyze each one separately.

We begin by observing that the two-form $E_\delta \wedge E_{-\delta}$, i.e., the generator of $\mathfrak{n}_{+2} \otimes \mathfrak{n}_{-2}$, is $\mathfrak{h}'$–invariant in view of (43). Then, for any one-form $\omega^1 \in \mathfrak{n}_{\pm 1}$, the two-form $\omega^1 \wedge E_{\pm\delta}$ is $\mathfrak{h}'$–invariant, i.e.,

$$\mathrm{ad}_{E_{\pm\alpha_1}}(\omega^1 \wedge E_{\pm\delta}) = \mathrm{ad}_{E_{\pm\alpha_1}}(\omega^1) \wedge E_{\pm\delta} = 0\,,$$

if and only if $\mathrm{ad}_{E_{\pm\alpha_1}}(\omega^1) = 0$, i.e., ω^1 is $\mathfrak{h}'$–invariant, because $\mathfrak{n}_{\pm 1}$ is an $\mathfrak{h}'$–module: by Proposition 4.1, ω^1 must then be a linear combination of $E_{\pm\delta}$ and then equal to zero, because $\omega^1 \in \mathfrak{n}_{\pm 1}$. This proves that among the last five constituents of $\Lambda^2(\mathfrak{m})$ there is only one nonzero $\mathfrak{h}'$–invariant one, namely $\mathfrak{n}_{+2} \otimes \mathfrak{n}_{-2}$. We pass now to the first three constituents.

In order to deal with $\Lambda^2(\mathfrak{n}_{+1})^{\mathfrak{h}'}$, it is convenient to introduce the basis

$$\omega_{ij} := E_{\gamma_i} \wedge E_{\gamma_j} \tag{46}$$

of the six-dimensional space $\Lambda^2(\mathfrak{n}_{+1})$: if

$$\omega^2 = \lambda^{01}\omega_{01} + \lambda^{02}\omega_{02} + \lambda^{03}\omega_{03} + \lambda^{12}\omega_{12} + \lambda^{13}\omega_{13} + \lambda^{23}\omega_{23}\,, \tag{47}$$

then

$$\begin{aligned}\mathrm{ad}_{E_{\alpha_1}}(\omega^2) &= \lambda^{01}\mathrm{ad}_{E_{\alpha_1}}(\omega_{01}) + \lambda^{02}\mathrm{ad}_{E_{\alpha_1}}(\omega_{02}) + \lambda^{03}\mathrm{ad}_{E_{\alpha_1}}(\omega_{03}) + \lambda^{12}\mathrm{ad}_{E_{\alpha_1}}(\omega_{12}) \\ &\quad + \lambda^{13}\mathrm{ad}_{E_{\alpha_1}}(\omega_{13}) + \lambda^{23}\mathrm{ad}_{E_{\alpha_1}}(\omega_{23}) \\ &= \lambda^{01}N_{\alpha_1,\gamma_1}\omega_{02} + \lambda^{02}(N_{\alpha_1,\gamma_0}\omega_{12} + N_{\alpha_1,\gamma_2}\omega_{03}) \\ &\quad + \lambda^{03}N_{\alpha_1,\gamma_0}\omega_{13} + \lambda^{12}N_{\alpha_1,\gamma_2}\omega_{13} + \lambda^{13}N_{\alpha_1,\gamma_1}\omega_{23}\end{aligned}$$

vanishes if and only if

$$\lambda^{01} = \lambda^{02} = 0\,, \quad \lambda^{03}N_{\alpha_1,\gamma_0} + \lambda^{12}N_{\alpha_1,\gamma_2} = \lambda^{03} + 3\lambda^{12} = 0\,, \quad \lambda^{13} = 0\,,$$

i.e., ω^2 belongs to the linear span of $\omega_{12} - 3\omega_{03}$ and ω_{23}: such a two-form will be $\mathrm{ad}_{E_{-\alpha_1}}$–invariant if and only if

$$\begin{aligned}\mathrm{ad}_{E_{-\alpha_1}}(\lambda^{12}(\omega_{12}-3\omega_{03})+\lambda^{23}\omega_{23}) &= \lambda^{12}(N_{-\alpha_1,\gamma_1}\omega_{02}-3N_{-\alpha_1,\gamma_3}\omega_{02})+\lambda^{23}\mathrm{ad}_{E_{-\alpha_1}}(\omega_{23}) \\ &= \lambda^{12}(3-3\cdot 1)\omega_{02} + \lambda^{23}\mathrm{ad}_{E_{-\alpha_1}}(\omega_{23}) = \lambda^{23}\mathrm{ad}_{E_{-\alpha_1}}(\omega_{23}) = 0\,,\end{aligned}$$

that is $\lambda^{23} = 0$, which proves that $\Lambda^2(\mathfrak{n}_{+1})^{\mathfrak{h}'}$ is the one-dimensional subspace spanned by $\omega^2_+ = \omega_{12} - 3\omega_{03}$.

The case of $\Lambda^2(\mathfrak{n}_{-1})^{\mathfrak{h}'}$ is formally analogous: instead of (46) we shall have

$$\omega'_{ij} := E_{-\gamma_i} \wedge E_{-\gamma_j}\,, \tag{48}$$

so that, having defined ω^2 as in (47) with ω'_{ij} in place of ω_{ij}, it turns out that

$$\begin{aligned}\mathrm{ad}_{E_{-\alpha_1}}(\omega^2) &= \lambda^{01}N_{-\alpha_1,-\gamma_1}\omega'_{02} + \lambda^{02}(N_{-\alpha_1,-\gamma_0}\omega'_{12} + N_{-\alpha_1,-\gamma_2}\omega'_{03}) + \lambda^{03}N_{-\alpha_1,-\gamma_0}\omega'_{13} \\ &\quad + \lambda^{12}N_{-\alpha_1,-\gamma_2}\omega'_{13} + \lambda^{13}N_{-\alpha_1,-\gamma_1}\omega'_{23}\end{aligned}$$

vanishes if and only if

$$\lambda^{01} = \lambda^{02} = 0\,, \quad \lambda^{03}N_{-\alpha_1,-\gamma_0} + \lambda^{12}N_{-\alpha_1,-\gamma_2} = -\lambda^{03} - 3\lambda^{12} = 0\,, \quad \lambda^{13} = 0\,,$$

i.e., ω^2 belongs to the linear span of $\omega'_{12} - 3\omega'_{03}$ and ω'_{23}: such a two-form will be $\mathrm{ad}_{E_{\alpha_1}}$–invariant if and only if

$$\begin{aligned}\mathrm{ad}_{E_{\alpha_1}}(\lambda^{12}(\omega'_{12}-3\omega'_{03})+\lambda^{23}\omega'_{23}) &= \lambda^{12}(N_{\alpha_1,-\gamma_1}\omega'_{02}-3N_{\alpha_1,-\gamma_3}\omega'_{02})+\lambda^{23}\mathrm{ad}_{E_{-\alpha_1}}(\omega'_{23}) \\ &= (-3-3\cdot(-1))\omega'_{02} + \lambda^{23}\mathrm{ad}_{E_{-\alpha_1}}(\omega'_{23}) = \lambda^{23}\mathrm{ad}_{E_{-\alpha_1}}(\omega'_{23}) = 0\,,\end{aligned}$$

that is $\lambda^{23} = 0$, which proves that $\Lambda^2(\mathfrak{n}_{-1})^{\mathfrak{h}'}$ is the one-dimensional subspace spanned by $\omega^2_- = \omega'_{12} - 3\omega'_{03}$.

By an abuse of notation, we denote now by

$$\omega_{ij} := E_{\gamma_i} \wedge E_{-\gamma_j}$$

the basis elements of the sixteen-dimensional space $\mathfrak{n}_{+1} \otimes \mathfrak{n}_{-1}$: if

$$\omega^2 = \sum_{i,j=0}^{3} \lambda^{ij}\omega_{ij}\,,$$

then

$$\begin{aligned}
\mathrm{ad}_{E_{\alpha_1}}(\omega^2) &= \sum_{i,j=0}^{3} \lambda^{ij}\mathrm{ad}_{E_{\alpha_1}}(\omega_{ij}) \\
&= \sum_{i,j=0}^{3} \lambda^{ij}(N_{\alpha_1,\gamma_i}\omega_{i+1,j} + N_{\alpha_1,-\gamma_j}\omega_{i,j-1}) \\
&= \sum_{i,j=0}^{3} \lambda^{ij}N_{\alpha_1,\gamma_i}\omega_{i+1,j} + \sum_{i,j=0}^{3} \lambda^{ij}N_{\alpha_1,-\gamma_j}\omega_{i,j-1},
\end{aligned}$$

where the ω_{ij}'s with an index beyond the range $\{0, 1, 2, 3\}$ must be considered zero. Therefore

$$\begin{aligned}
\mathrm{ad}_{E_{\alpha_1}}(\omega^2) &= \sum_{j=0}^{3}\sum_{i=0}^{2} \lambda^{ij}N_{\alpha_1,\gamma_i}\omega_{i+1,j} + \sum_{i=0}^{3}\sum_{j=1}^{3} \lambda^{ij}N_{\alpha_1,-\gamma_j}\omega_{i,j-1} \\
&= \sum_{j=0}^{3}(\lambda^{0j}\omega_{1j} + 2\lambda^{1j}\omega_{2j} + 3\lambda^{2j}\omega_{3j}) + \sum_{i=0}^{3}(-3\lambda^{i1}\omega_{i0} - 2\lambda^{i2}\omega_{i1} - \lambda^{i3}\omega_{i2}) \\
&= \lambda^{00}\omega_{10} + 2\lambda^{10}\omega_{20} + 3\lambda^{20}\omega_{30} + \lambda^{01}\omega_{11} + 2\lambda^{11}\omega_{21} + 3\lambda^{21}\omega_{31} \\
&\quad +\lambda^{02}\omega_{12} + 2\lambda^{12}\omega_{22} + 3\lambda^{22}\omega_{32} + \lambda^{03}\omega_{13} + 2\lambda^{13}\omega_{23} + 3\lambda^{23}\omega_{33} \\
&\quad -3\lambda^{01}\omega_{00} - 2\lambda^{02}\omega_{01} - \lambda^{03}\omega_{02} - 3\lambda^{11}\omega_{10} - 2\lambda^{12}\omega_{11} - \lambda^{13}\omega_{12} \\
&\quad -3\lambda^{21}\omega_{20} - 2\lambda^{22}\omega_{21} - \lambda^{23}\omega_{22} - 3\lambda^{31}\omega_{30} - 2\lambda^{32}\omega_{31} - \lambda^{33}\omega_{32} \\
&= (\lambda^{00} - 3\lambda^{11})\omega_{10} + (2\lambda^{10} - 3\lambda^{21})\omega_{20} + 3(\lambda^{20} - \lambda^{31})\omega_{30} \\
&\quad +(\lambda^{01} - 2\lambda^{12})\omega_{11} + 2(\lambda^{11} - \lambda^{22})\omega_{21} + (3\lambda^{21} - 2\lambda^{32})\omega_{31} \\
&\quad +(\lambda^{02} - \lambda^{13})\omega_{12} + (2\lambda^{12} - \lambda^{23})\omega_{22} + (3\lambda^{22} - \lambda^{33})\omega_{32} \\
&\quad +\lambda^{03}\omega_{13} + 2\lambda^{13}\omega_{23} + 3\lambda^{23}\omega_{33} - 3\lambda^{01}\omega_{00} - 2\lambda^{02}\omega_{01} - \lambda^{03}\omega_{02}.
\end{aligned}$$

Analogously,

$$\begin{aligned}
\mathrm{ad}_{E_{-\alpha_1}}(\omega^2) &= 3\lambda^{10}\omega_{00} + 2\lambda^{20}\omega_{10} + \lambda^{30}\omega_{20} + (3\lambda^{11} - \lambda^{00})\omega_{01} + (2\lambda^{21} - \lambda^{10})\omega_{11} + (\lambda^{31} - \lambda^{20})\omega_{21} \\
&\quad +(3\lambda^{12} - 2\lambda^{01})\omega_{02} + 2(\lambda^{22} - \lambda^{11})\omega_{12} + (\lambda^{32} - 2\lambda^{21})\omega_{22} + 3(\lambda^{13} - \lambda^{02})\omega_{03} \\
&\quad +(2\lambda^{23} - 3\lambda^{12})\omega_{13} + (\lambda^{33} - 3\lambda^{22})\omega_{23} - \lambda^{30}\omega_{31} - \lambda^{31}\omega_{32} - \lambda^{32}\omega_{33}\,.
\end{aligned}$$

It is not hard to see that the two equations $\mathrm{ad}_{E_{\pm\alpha_1}}(\omega^2) = 0$ are satisfied if and only if $\lambda^{ij} = 0$ for all $i \neq j$ and the following three conditions hold:

$$\lambda^{11} = \lambda^{22}, \quad \lambda^{00} = 3\lambda^{11}, \quad \lambda^{33} = 3\lambda^{22},$$

thus finishing the proof.

In the next Proposition 4.3 we construct *some* independent generators of the space $\Lambda^4(\mathfrak{m}^*)^{\mathfrak{h}'}$: it will be clear later in Corollary 4.1 why they suffice for our purposes.

Proposition 4.3 *The space $\Lambda^4(\mathfrak{m}^*)^{\mathfrak{h}'} \cong \Lambda^4(\mathfrak{m})^{\mathfrak{h}'}$ contains the following (linearly independent) three four-forms:*

$$\begin{aligned}
\omega^4_{\pm} &:= E_{\pm\gamma_0} \wedge E_{\pm\gamma_1} \wedge E_{\pm\gamma_2} \wedge E_{\pm\gamma_3}, \\
\omega^4 &:= E_{\gamma_0} \wedge E_{\gamma_1} \wedge E_{-\gamma_0} \wedge E_{-\gamma_1} + E_{\gamma_0} \wedge E_{\gamma_2} \wedge E_{-\gamma_0} \wedge E_{-\gamma_2} + \\
&\quad +E_{\gamma_0} \wedge E_{\gamma_3} \wedge E_{-\gamma_1} \wedge E_{-\gamma_2} + E_{\gamma_1} \wedge E_{\gamma_2} \wedge E_{-\gamma_0} \wedge E_{-\gamma_3} + \\
&\quad +E_{\gamma_1} \wedge E_{\gamma_3} \wedge E_{-\gamma_1} \wedge E_{-\gamma_3} + E_{\gamma_2} \wedge E_{\gamma_3} \wedge E_{-\gamma_2} \wedge E_{-\gamma_3}.
\end{aligned} \tag{49}$$

Proof The six-dimensional $\mathfrak{sl}_2$–representations $\Lambda^2(\mathfrak{n}_{\pm 1})$ can be decomposed into sums of irreducible $\mathfrak{sl}_2$–representations, that are

$$\Lambda^2(\mathfrak{n}_{\pm 1}) = \mathbb{R}\omega^2_\pm + V_4^\pm\,,$$

where $\omega^2_\pm$ are given by (45), and $V_4^\pm$ are two copies of the irreducible $\mathfrak{sl}_2$–representation V_4 of highest weight 4, that are dual to each other: $V_4^+ := V_4$, $V_4^- := V_4^*$. More precisely, by using the six basis elements ω_{ij} of $\Lambda^2(\mathfrak{n}_{+1})$ defined in (46), we let

$$v_0 := \omega_{23}\,,\quad v_1 := 2\omega_{13}\,,\quad v_2 := \omega_{12} + 3\omega_{03}\,,\quad v_3 := 2\omega_{02}\,,\quad v_4 := \omega_{01}\,. \tag{50}$$

Then the five elements (50) span the irreducible five-dimensional $\mathfrak{sl}_2$–representation V_4 contained in $\Lambda^2(\mathfrak{n}_{+1})$, such that v_4 is its lowest weight vector (of weight -4).

The decomposition of the tensor product $\Lambda^2(\mathfrak{n}_{+1}) \otimes \Lambda^2(\mathfrak{n}_{-1})$ into a sum of irreducible $\mathfrak{sl}_2$–representations contains exactly two one-dimensional constituents:

$$\Lambda^2(\mathfrak{n}_{+1}) \otimes \Lambda^2(\mathfrak{n}_{-1}) = \mathbb{R}\omega^2_+ \wedge \omega^2_- + V_0 + \text{irreducible modules of } \dim > 1\,,$$

having employed the natural embedding

$$\Lambda^2(\mathfrak{n}_{+1}) \otimes \Lambda^2(\mathfrak{n}_{-1}) \subset \Lambda^4(\mathfrak{n}_{+1} + \mathfrak{n}_{-1})\,.$$

In order to construct a generator $\widehat{\omega}^4$ of the one-dimensional representation V_0, that is, the unique one-dimensional constituent of

$$V_4^+ \otimes V_4^- \simeq V_4 \otimes V_4^* \cong V_8 \oplus V_6 \oplus V_4 \oplus V_2 \oplus V_0\,,$$

we use now the basis ω'_{ij} of $\Lambda^2(\mathfrak{n}_{-1})$ defined in (48): first, we define a basis

$$v'_0 := \omega'_{01}\,,\quad v'_1 := -2\omega'_{02}\,,\quad v'_2 := \omega'_{12} + 3\omega'_{03}\,,\quad v'_3 := -2\omega'_{13}\,,\quad v'_4 := \omega'_{23}\,, \tag{51}$$

of $V'_4 \subset \Lambda^2(\mathfrak{n}_{-1})$; then we observe that, since v_i and v'_i have both weight $4-2i$, the subspace in which $\mathrm{ad}_{H_{\alpha_1}}$ acts with zero weight is the 5-dimensional subspace generated by

$$v_0 \otimes v'_4\,,\quad v_1 \otimes v'_3\,,\quad v_2 \otimes v'_2\,,\quad v_3 \otimes v'_1\,,\quad v_4 \otimes v'_0\,.$$

It can be easily showed that, up to a nonzero scalar, the only linear combination of the above elements on which $\mathrm{ad}_{E_{\alpha_1}}$ acts also with zero weight is given by

$$12v_0 \otimes v'_4 - 3v_1 \otimes v'_3 + 2v_2 \otimes v'_2 - 3v_3 \otimes v'_1 + 12v_4 \otimes v'_0\,. \tag{52}$$

Finally, by substituting (50) and (51) into (52), we obtain

$$\widehat{\omega}^4 = 12\omega_{01}\otimes\omega'_{01} + 12\omega_{02}\otimes\omega'_{02} + 2(\omega_{12}+3\omega_{03})\otimes(\omega'_{12}+3\omega'_{03}) + 12\omega_{13}\otimes\omega'_{13} + 12\omega_{23}\otimes\omega'_{23}\,,$$

which is by construction $\mathfrak{h}'$–invariant; since $(\omega_{12} + 3\omega_{03}) \otimes (\omega'_{12} + 3\omega'_{03}) = \omega^2_+ \wedge \omega^2_-$ is invariant as well, so it will be of the form

$$\omega^4 = \frac{1}{12}(\widehat{\omega}^4 - 2\omega^2_+ \wedge \omega^2_-)$$

defined in (49).

The forms $\omega^4_\pm$ are the obvious generators of the one-dimensional modules $\Lambda^4(\mathfrak{n}_{\pm 1})$.

Corollary 4.1 below combines the results of Proposition 4.1, Proposition 4.2 and Proposition 4.3, to obtain a system of independent generators for the algebra $\Lambda^\bullet(\mathfrak{m}^*)^{\mathfrak{h}'} \cong \Lambda^\bullet(\mathfrak{m})^{\mathfrak{h}'}$ of $\mathfrak{h}'$–invariant forms on $\mathfrak{m}$, up to degree five: its proof relies on a simple result of multilinear algebra.

Lemma 4.1 *Let $V = A \oplus B$ be a vector space, $\{\kappa_i\}_{i\in I} \subset \Lambda^k(A)$ an independent system of k-forms on A, and $\{\chi_j\}_{j\in J} \subset \Lambda^h(B)$ an independent system of h-forms on B: then the system $\{\kappa_i \wedge \chi_j\}_{(i,j)\in I\times J} \subset \Lambda^{k+h}(V)$ of $(k+h)$–forms on V is linearly independent as well.*

Proof Enough to observe that $\Lambda^{k+h}(V)$ contains the direct summand $\Lambda^k(A) \otimes \Lambda^h(B)$ and to recall that the tensor products of basis elements of $\Lambda^k(A)$ by basis elements of $\Lambda^h(B)$ define a basis of $\Lambda^k(A) \otimes \Lambda^h(B)$. □

Corollary 4.1 *The algebra $\Lambda^\bullet(\mathfrak{m}^*)^{\mathfrak{h}'} \cong \Lambda^\bullet(\mathfrak{m})^{\mathfrak{h}'}$ of $\mathfrak{h}'$–invariant forms is generated, up to degree 5, by the eight elements*

$$E_\delta, E_{-\delta}, \omega_+^2, \omega_-^2, \omega^2, \omega_+^4, \omega_-^4, \omega^4\,, \tag{53}$$

and the dimensions of $\Lambda^i(\mathfrak{m})^{\mathfrak{h}'}$ are 2, 4, 6, 9 and 12 for $i = 1, 2, 3, 4, 5$, respectively.

Proof A quick application of the LiE program [42] allows to populate the table:

$i =$	1	2	3	4	5
number of one-dimensional submodules of $\Lambda^i(\mathfrak{m})$:	2	4	6	9	12

and then the dimensions $\dim \Lambda^i(\mathfrak{m})^{\mathfrak{h}'}$ with $i = 1, 2, 3, 4, 5$ cannot exceed $2, 4, 6, 9, 12$, respectively: see Remark 4.1.

Proposition 4.1 showed that $E_{\pm\delta}$ is a basis of $\Lambda^1(\mathfrak{m})^{\mathfrak{h}'}$ and, among the four independent generators

$$\omega_\pm^2, \omega^2, E_\delta \wedge E_{-\delta}$$

of $\Lambda^2(\mathfrak{m})^{\mathfrak{h}'}$ constructed in Proposition 4.2, the last one needs not to be added to the list (53) because it is the product of two already existing elements: E_δ and $E_{-\delta}$.

To deal with $\Lambda^3(\mathfrak{m})^{\mathfrak{h}'}$, let us employ the two independent generators of $\Lambda^1(\mathfrak{m})^{\mathfrak{h}'}$ and the three generators of $\Lambda^2(\mathfrak{m})^{\mathfrak{h}'}$ different to $E_\delta \wedge E_{-\delta}$ from the aforementioned propositions: their wedge products lead to six three-forms

$$E_\delta \wedge \omega_\pm^2\,, \quad E_{-\delta} \wedge \omega_\pm^2\,, \quad E_{\pm\delta} \wedge \omega^2\,,$$

that are linearly independent in view of Lemma 4.1. Therefore, by dimensional reasons, they must generate $\Lambda^3(\mathfrak{m})^{\mathfrak{h}'}$.

In the case of $\Lambda^4(\mathfrak{m})^{\mathfrak{h}'}$, by taking all possible (nonzero) wedge products of the three two-forms constructed in Proposition 4.2, we obtain six four-forms that are linearly independent by Lemma 4.1; the three four-forms $\omega_\pm^4$ and ω^4 obtained in Proposition 4.3 are linearly independent and belong to a direct summand different from the first six: taken together, the nine four-forms

$$
\begin{aligned}
E_\delta \wedge E_{-\delta} \wedge \omega_\pm^2 &= E_\delta \wedge E_{-\delta} \wedge (E_{\pm\gamma_1} \wedge E_{\pm\gamma_2} - 3E_{\pm\gamma_0} \wedge E_{\pm\gamma_3}),\\
E_\delta \wedge E_{-\delta} \wedge \omega^2 &= E_\delta \wedge E_{-\delta} \wedge (3E_{\gamma_0} \wedge E_{-\gamma_0} + E_{\gamma_1} \wedge E_{-\gamma_1}\\
&\quad + E_{\gamma_2} \wedge E_{-\gamma_2} + 3E_{\gamma_3} \wedge E_{-\gamma_3}),\\
\omega_+^2 \wedge \omega_-^2 &= E_{\gamma_1} \wedge E_{\gamma_2} \wedge E_{-\gamma_1} \wedge E_{-\gamma_2} - 3(E_{\gamma_0} \wedge E_{\gamma_3} \wedge E_{-\gamma_1} \wedge E_{-\gamma_2}\\
&\quad + E_{\gamma_1} \wedge E_{\gamma_2} \wedge E_{-\gamma_0} \wedge E_{-\gamma_3})\\
&\quad + 9E_{\gamma_0} \wedge E_{\gamma_3} \wedge E_{-\gamma_0} \wedge E_{-\gamma_3},\\
\omega_+^2 \wedge \omega^2 &= 3(E_{\gamma_1} \wedge E_{\gamma_2} \wedge E_{\gamma_0} \wedge E_{-\gamma_0} + E_{\gamma_1} \wedge E_{\gamma_2} \wedge E_{\gamma_3} \wedge E_{-\gamma_3}\\
&\quad - E_{\gamma_0} \wedge E_{\gamma_3} \wedge E_{\gamma_1} \wedge E_{-\gamma_1}\\
&\quad - E_{\gamma_0} \wedge E_{\gamma_3} \wedge E_{\gamma_2} \wedge E_{-\gamma_2}),\\
\omega_-^2 \wedge \omega^2 &= 3(E_{-\gamma_1} \wedge E_{-\gamma_2} \wedge E_{\gamma_0} \wedge E_{-\gamma_0} + E_{-\gamma_1} \wedge E_{-\gamma_2} \wedge E_{\gamma_3} \wedge E_{-\gamma_3}\\
&\quad - E_{-\gamma_0} \wedge E_{-\gamma_3} \wedge E_{\gamma_1} \wedge E_{-\gamma_1} - E_{-\gamma_0} \wedge E_{-\gamma_3} \wedge E_{\gamma_2} \wedge E_{-\gamma_2}),\\
\omega_\pm^4 &= E_{\pm\gamma_0} \wedge E_{\pm\gamma_1} \wedge E_{\pm\gamma_2} \wedge E_{\pm\gamma_3},\\
\omega^4 &= E_{\gamma_0} \wedge E_{\gamma_1} \wedge E_{-\gamma_0} \wedge E_{-\gamma_1} + E_{\gamma_0} \wedge E_{\gamma_2} \wedge E_{-\gamma_0} \wedge E_{-\gamma_2} +\\
&\quad + E_{\gamma_0} \wedge E_{\gamma_3} \wedge E_{-\gamma_1} \wedge E_{-\gamma_2} + E_{\gamma_1} \wedge E_{\gamma_2} \wedge E_{-\gamma_0} \wedge E_{-\gamma_3} +\\
&\quad + E_{\gamma_1} \wedge E_{\gamma_3} \wedge E_{-\gamma_1} \wedge E_{-\gamma_3} + E_{\gamma_2} \wedge E_{\gamma_3} \wedge E_{-\gamma_2} \wedge E_{-\gamma_3},
\end{aligned}
$$

fill up a basis of $\Lambda^4(\mathfrak{m})^{\mathfrak{h}'}$.

Finally, $\Lambda^5(\mathfrak{m})^{\mathfrak{h}'}$ is generated by the twelve elements

$$
\omega_+^2 \wedge \omega_-^2 \wedge E_{\pm\delta}, \quad \omega_+^2 \wedge \omega^2 \wedge E_{\pm\delta}, \quad \omega_-^2 \wedge \omega^2 \wedge E_{\pm\delta}, \quad \omega_+^4 \wedge E_{\pm\delta}, \quad \omega_-^4 \wedge E_{\pm\delta}, \quad \omega^4 \wedge E_{\pm\delta}, \tag{54}
$$

whose linear independence is ensured by Lemma 4.1.

Theorem 4.1 *Elements (54) generate the space of* G_2*–invariant effective 5-forms on the contact manifold* $N_3^{11} = \mathsf{G}_2/\,\mathsf{SL}_2(\mathbb{R})^{\alpha_1}$. *They define also relative* G_2*–invariant 5-forms on the bi-Lagrangian manifold* $M_3^{10} = \mathsf{G}_2/\,\mathsf{GL}_2(\mathbb{R})^{\alpha_1}$.

Proof The first statement follows immediately from the definition of a G–invariant effective n–form on the contactification $N = G/L$ of $M = G/H$, with $n = 5$, $G = \mathsf{G}_2$, $\mathfrak{h} = \mathfrak{gl}_2(\mathbb{R})^{\alpha_1}$ and $\mathfrak{l} = \mathfrak{h}' = \mathfrak{sl}_2(\mathbb{R})^{\alpha_1}$ (see Sect. 4.1.2). Straightforward calculations show that all elements (54) are eigenvectors for ad_Z, which proves the second statement (see Remark 4.2).

4.4 Coordinate expressions of the G_2–invariant MAEs

We employ now the results of Sect. 1.4 to obtain a coordinate expression of the G_2–invariant MAEs associated with the G_2–invariant effective 5-forms (54). Recall that, up to a negligible subset, the homogeneous symplectic manifold M is symplectomorphic to the symplectic manifold T^*F, equipped with the symplectic form ω_{T^*F} given by (18): see Theorem 1.5. In particular, the 5-dimensional homogeneous manifold $F = \mathsf{G}_2/P^+$ (cf. (17)) can be equipped, locally, with the coordinates

$$
\{x^0, x^1, x^2, x^3, x^4\} \tag{55}
$$

via the exponential map:

$$\begin{aligned} T_oF \simeq \mathfrak{n}_- &\longrightarrow N^-, \\ X^- = x^0E_{-\gamma_0} + \cdots + x^3E_{-\gamma_3} + x^4E_{-\delta} &\longmapsto \exp(X^-). \end{aligned}$$

Remark 4.3 By switching the roles of N^- and N^+ we obtain another chart: these two charts are enough to cover the whole M.

To the above local coordinate system on F we associate the Darboux coordinate system

$$\{x^0, x^1, x^2, x^3, x^4, u_0, u_1, u_2, u_3, u_4\}, \tag{56}$$

on T^*F, in which the standard symplectic form ω_{std} reads as in (24): to keep the notation light, we shall denote the differentials $(dx^i)_o$, $(du_i)_o$, $i = 0, 1, 2, 3, 4$, at the origin o of the above coordinate functions, simply by

$$dx^i, \; du_i, \quad i = 0, 1, 2, 3, 4.$$

Accordingly, we obtain from (18) the evaluation at o of the symplectic form ω_{T^*F}, more precisely

$$\omega_{T^*F}|_o = \sum_{i=0}^{3} dx^i \wedge du_i + 2dx^4 \wedge du_4,$$

which is an element of $\Lambda^2(\mathcal{C}_o^*) = \Lambda^2(\mathfrak{n}^*)$, that we usually identify with $\Lambda^2(\mathfrak{n})$. On the other hand, (20)–(21) imply that

$$\omega_{T^*F}|_o = \sum_{i=0}^{3} E_{\gamma_i} \wedge E_{-\gamma_i} + 2E_\delta \wedge E_{-\delta},$$

which means that the linear isomorphism given by

$$\begin{aligned} dx^i &\longleftrightarrow E_{\gamma_i}, \quad i = 0, 1, 2, 3, \\ dx^4 &\longleftrightarrow E_\delta, \\ du_i &\longleftrightarrow E_{-\gamma_i}, \quad i = 0, 1, 2, 3, \\ du_4 &\longleftrightarrow E_{-\delta}, \end{aligned} \tag{57}$$

preserves the symplectic form ω_{T^*F} at the origin.

Now we wish to apply the results recalled in Sect. 2.3 to find a coordinate expression of the MAEs associated with the twelve invariant 5-forms (54).

By employing the symplectomorphism (57), the forms (54), that are covariant tensors on the vector space $\mathcal{C}_o$, can be written down as degree–five (skew–symmetric) polynomials of the basis elements of $\mathcal{C}_o^*$, see also the remark at the end of Sect. 2.4. These are, in that order:

$$\begin{aligned}
\omega_+^2 \wedge \omega_-^2 \wedge E_\delta &= (dx^1 \wedge dx^2 \wedge du_1 \wedge du_2 - 3(dx^0 \wedge dx^3 \wedge du_1 \wedge du_2 \\
&\quad + dx^1 \wedge dx^2 \wedge du_0 \wedge du_3) + 9dx^0 \wedge dx^3 \wedge du_0 \wedge du_3) \wedge dx^4 \\
\omega_+^2 \wedge \omega_-^2 \wedge E_{-\delta} &= (dx^1 \wedge dx^2 \wedge du_1 \wedge du_2 - 3(dx^0 \wedge dx^3 \wedge du_1 \wedge du_2 \\
&\quad + dx^1 \wedge dx^2 \wedge du_0 \wedge du_3) + 9dx^0 \wedge dx^3 \wedge du_0 \wedge du_3) \wedge du_4 \\
\omega_+^2 \wedge \omega^2 \wedge E_\delta &= 3(dx^1 \wedge dx^2 \wedge dx^0 \wedge du_0 + dx^1 \wedge dx^2 \wedge dx^3 \wedge du_3 \\
&\quad - dx^0 \wedge dx^3 \wedge dx^1 \wedge du_1 - dx^0 \wedge dx^3 \wedge dx^2 \wedge du_2) \wedge dx^4 \\
\omega_+^2 \wedge \omega^2 \wedge E_{-\delta} &= 3(dx^1 \wedge dx^2 \wedge dx^0 \wedge du_0 + dx^1 \wedge dx^2 \wedge dx^3 \wedge du_3 \\
&\quad - dx^0 \wedge dx^3 \wedge dx^1 \wedge du_1 - dx^0 \wedge dx^3 \wedge dx^2 \wedge du_2) \wedge du_4 \\
\omega_-^2 \wedge \omega^2 \wedge E_\delta &= 3(du_1 \wedge du_2 \wedge dx^0 \wedge du_0 + du_1 \wedge du_2 \wedge dx^3 \wedge du_3 \\
&\quad - du_0 \wedge du_3 \wedge dx^1 \wedge du_1 - du_0 \wedge du_3 \wedge dx^2 \wedge du_2) \wedge dx^4, \\
\omega_-^2 \wedge \omega^2 \wedge E_{-\delta} &= 3(du_1 \wedge du_2 \wedge dx^0 \wedge du_0 + du_1 \wedge du_2 \wedge dx^3 \wedge du_3 \\
&\quad - du_0 \wedge du_3 \wedge dx^1 \wedge du_1 - du_0 \wedge du_3 \wedge dx^2 \wedge du_2) \wedge du_4, \\
\omega_+^4 \wedge E_\delta &= dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4, \\
\omega_+^4 \wedge E_{-\delta} &= dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge du_4, \\
\omega_-^4 \wedge E_\delta &= du_0 \wedge du_1 \wedge du_2 \wedge du_3 \wedge dx^4, \\
\omega_-^4 \wedge E_{-\delta} &= du_0 \wedge du_1 \wedge du_2 \wedge du_3 \wedge du_4, \\
\omega^4 \wedge E_\delta &= (dx^0 \wedge dx^1 \wedge du_0 \wedge du_1 + dx^0 \wedge dx^2 \wedge du_0 \wedge du_2 + \\
&\quad + dx^0 \wedge dx^3 \wedge du_1 \wedge du_2 + dx^1 \wedge dx^2 \wedge du_0 \wedge du_3 + \\
&\quad + dx^1 \wedge dx^3 \wedge du_1 \wedge du_3 + dx^2 \wedge dx^3 \wedge du_2 \wedge du_3) \wedge dx^4, \\
\omega^4 \wedge E_{-\delta} &= (dx^0 \wedge dx^1 \wedge du_0 \wedge du_1 + dx^0 \wedge dx^2 \wedge du_0 \wedge du_2 + \\
&\quad + dx^0 \wedge dx^3 \wedge du_1 \wedge du_2 + dx^1 \wedge dx^2 \wedge du_0 \wedge du_3 + \\
&\quad + dx^1 \wedge dx^3 \wedge du_1 \wedge du_3 + dx^2 \wedge dx^3 \wedge du_2 \wedge du_3) \wedge du_4.
\end{aligned}$$

Let us denote by ρ any of the twelve $(0, 5)$–tensors on $\mathcal{C}_o$ above: then there is a unique G_2–invariant five–form Ω^ρ on M, such that $\Omega_o^\rho = \rho$. Then, formula (22) allows to define twelve MAEs $\mathcal{E}_{\Omega^\rho}$, which can be now locally expressed in Darboux coordinates by means of the function $F_{\Omega^\rho}(u_{ij})$, cf. (28).

A general method for constructing G-invariant PDEs on homogeneous manifolds has been proposed in [7], based on the fact that a G-invariant equation $\mathcal{E}$ coincides with the *G–equivariant extension* of its fiber $\mathcal{E}_p$ at an arbitrary point p, i.e.:

$$\mathcal{E} = \bigcup_{g \in G} g(\mathcal{E}_p)\,. \tag{58}$$

Definition 4.1 The right–hand side of (58) is called the *G–equivariant extension* of the fiber $\mathcal{E}_p$.

The global expression of $F_{\Omega^\rho}(u_{ij})$ can be obtained from its restriction

$$F_\rho := F_{\Omega^\rho}|_{\mathrm{LGr}(\mathcal{C}_o,5)}$$

to the fiber $\mathrm{LGr}(\mathcal{C}_o, 5)$ of N^1 at $o \in N$: this corresponds to the G_2–equivariant extension

$$\mathcal{E}_{\Omega^\rho} = \mathsf{G}_2 \cdot \mathcal{E}_\rho := \bigcup_{g \in \mathsf{G}_2} g(\mathcal{E}_\rho), \quad \mathcal{E}_\rho := \{F_\rho = 0\},$$

of the hypersurface $\mathcal{E}_\rho$ of $\mathrm{LGr}(\mathcal{C}_o, 5)$. Below we compute the functions F_ρ for all the twelve $(0, 5)$–tensors above; to this end, we shall need the following symbols:

$$\begin{aligned}
M_a^l &\overset{\text{def.}}{=} \textit{rank} - 4 \textit{ minor of } u_{ij} \textit{ obtained by removing } a^{\text{th}} \textit{ row and } l^{\text{th}} \textit{ column},\\
M_{ab}^{lm} &\overset{\text{def.}}{=} \textit{rank} - 3 \textit{ minor of } u_{ij} \textit{ obtained by removing rows } a,\ b, \textit{ and columns } l,\ m,\\
M_{abc}^{lmn} &\overset{\text{def.}}{=} \textit{rank} - 2 \textit{ minor of } u_{ij} \textit{ obtained by removing rows } a,\ b,\ c, \textit{ and columns } l,\ m,\ n.
\end{aligned}$$

Then, the functions F_ρ read as follows:

$$\begin{aligned}
\mathcal{E}_{\omega_+^2 \wedge \omega_-^2 \wedge E_\delta} &\longleftrightarrow M_{124}^{034} - 3\left(M_{034}^{034} + M_{124}^{124}\right) + 9M_{034}^{124} = 10M_{124}^{034} - 3\left(M_{034}^{034} + M_{124}^{124}\right),\\
\mathcal{E}_{\omega_+^2 \wedge \omega_-^2 \wedge E_{-\delta}} &\longleftrightarrow M_{12}^{03} - 3\left(M_{03}^{03} + M_{12}^{12}\right) + 9M_{03}^{12} = 10M_{12}^{03} - 3\left(M_{03}^{03} + M_{12}^{12}\right),\\
\mathcal{E}_{\omega_+^2 \wedge \omega^2 \wedge E_\delta} &\longleftrightarrow 6(u_{03} + u_{12}),\\
\mathcal{E}_{\omega_+^2 \wedge \omega^2 \wedge E_{-\delta}} &\longleftrightarrow 6\left(M_{012}^{123} + M_{013}^{023}\right),\\
\mathcal{E}_{\omega_-^2 \wedge \omega^2 \wedge E_\delta} &\longleftrightarrow -3\left(M_{04}^{34} + M_{34}^{04} + M_{14}^{24} + M_{24}^{14}\right) = -6\left(M_{04}^{34} + M_{14}^{24}\right),\\
\mathcal{E}_{\omega_-^2 \wedge \omega^2 \wedge E_{-\delta}} &\longleftrightarrow 3\left(M_0^3 + M_3^0 + M_1^2 + M_2^1\right) = 3\left(M_0^3 + M_1^2\right),\\
\mathcal{E}_{\omega_+^4 \wedge E_\delta} &\longleftrightarrow \det(u_{ij}),\\
\mathcal{E}_{\omega_+^4 \wedge E_{-\delta}} &\longleftrightarrow u_{44},\\
\mathcal{E}_{\omega_-^4 \wedge E_\delta} &\longleftrightarrow M_4^4,\\
\mathcal{E}_{\omega_-^4 \wedge E_{-\delta}} &\longleftrightarrow \emptyset,\\
\mathcal{E}_{\omega^4 \wedge E_\delta} &\longleftrightarrow M_{014}^{234} + M_{024}^{134} + M_{034}^{034} + M_{124}^{124} + M_{134}^{024} + M_{234}^{014}\\
&\qquad = 2M_{014}^{234} + 2M_{024}^{134} + M_{034}^{034} + M_{124}^{124},\\
\mathcal{E}_{\omega^4 \wedge E_{-\delta}} &\longleftrightarrow M_{01}^{23} + M_{02}^{13} + M_{03}^{03} + M_{12}^{12} + M_{13}^{02} + M_{23}^{01}\\
&\qquad = 2M_{01}^{23} + 2M_{02}^{13} + M_{03}^{03} + M_{12}^{12}.
\end{aligned}$$

4.5 Classification up to contactomorphisms

We will need a particular element τ of the linear symplectic group $\mathrm{Sp}(\mathcal{C}_o) = \mathrm{Sp}(\mathfrak{m}) = \mathrm{Sp}(\mathfrak{n}_+ + \mathfrak{n}_-)$: it is given by the 10×10 matrix

$$\tau := \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}.$$

Let us now observe that any element $g \in \mathsf{G}_2$ is, by construction, a contactomorphism of the manifold N, which preserves the integrable Lagrangian distributions T^+M and T^-M corresponding to the subalgebras $\mathfrak{n}_+$ and $\mathfrak{n}_- \simeq \mathfrak{n}_+^*$. This means that the differential

$$(dg)_o \in \mathrm{Hom}(\mathcal{C}_o, \mathcal{C}_{go}) \simeq \mathrm{Hom}(\mathfrak{n}_+ \oplus \mathfrak{n}_+^*, \mathfrak{n}_+ \oplus \mathfrak{n}_+^*)$$

preserves the symplectic structure of $\mathfrak{n}_+ \oplus \mathfrak{n}_+^*$, preserves $\mathfrak{n}_+$ and $\mathfrak{n}_+^*$ separately and, moreover, the action on $\mathfrak{n}_+^*$ is dual to the action on $\mathfrak{n}_+$: this means that $(dg)_o$ can be regarded as an element of $\mathrm{Sp}(\mathfrak{n}_+ \oplus \mathfrak{n}_+^*)$, that is

$$(dg)_o = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \tag{59}$$

for some 5×5 matrix A.

By employing the Darboux coordinates (56) induced by the coordinates (55) of F, we define the (total) Legendre transform

$$\begin{aligned} &\Phi : N \longrightarrow N, \\ &\quad (x^0, x^1, x^2, x^3, x^4, u, u_0, u_1, u_2, u_3, u_4) \\ &\quad \longmapsto (u_0, u_1, u_2, u_3, u_4, u - x^i u_i, -x^0, -x^1, -x^2, -x^3, -x^4), \end{aligned}$$

which is a (local) contactomorphism of N, such that:

- Φ sends the leaf $\mathcal{L}_p^{\pm}$ at $p \in N$ of the Lagrangian distribution $T^{\pm}M$ to the leaf $\mathcal{L}_{\Phi(p)}^{\mp}$ at $\Phi(p) \in N$;
- the differential $(d\Phi)_p$, regarded as a linear symplectomorphism of $\mathfrak{n}_+ \oplus \mathfrak{n}_+^*$, coincides with τ.

Theorem 4.2 *The MAE $\mathcal{E}_{\Omega^\rho}$ is contact–equivalent to the MAE $\mathcal{E}_{\Omega^{\tau(\rho)}}$.*

Proof It is enough to prove the identity

$$\mathsf{G}_2 \cdot (\tau(\mathcal{E}_\rho)) = \Phi(\mathsf{G}_2 \cdot \mathcal{E}_\rho)\,. \tag{60}$$

Since any point $p^1 \in \mathcal{E}_\rho$ is interpreted as a Lagrangian plane L_{p^1}, the identity (60) reads

$$\left\{\big((dg)_o \circ \tau\big)(L_{p^1}) \mid p^1 \in \mathcal{E}_\rho\,, g \in \mathsf{G}_2\right\} = \left\{\big((d\Phi)_{\overline{g}o} \circ (d\overline{g})_o\big)(L_{p^1}) \mid p^1 \in \mathcal{E}_\rho\,, \overline{g} \in \mathsf{G}_2\right\}\,. \tag{61}$$

Let us observe that, for any $g \in \mathsf{G}_2$, there exists another $\overline{g} \in \mathsf{G}_2$, such that

$$(d\overline{g})_o = \begin{pmatrix} (A^{-1})^t & 0 \\ 0 & A \end{pmatrix}\,, \tag{62}$$

Then, by using (59) and (62), it is easy to show that

$$(dg)_o \circ \tau = \tau \circ (d\overline{g})_o\,. \tag{63}$$

In light of the properties of Φ, formula (63) shows that (61) holds.

It only should be stressed that all the identifications we made in this proof are well defined up to an element of the stabilizer SL_2 of o: this does not affect the final result, because the hypersurface $\mathcal{E}_\rho$ is SL_2–invariant.

Corollary 4.2 *A G_2–invariant MAE that is obtained as the G_2–equivariant extension of any of the twelve hypersurfaces $\{F_\rho = 0\}$ above is contactomorphic to the G_2–equivariant extension of one of the following six G_2–invariant hypersurfaces:*

$$\begin{aligned} \textit{quadratic}\ (\mathrm{Q1}){:}\quad & 10M^{034}_{124} - 3\left(M^{034}_{034} + M^{124}_{124}\right) = 0, \\ \textit{linear}\ (\mathrm{L1}){:}\quad & u_{03} + u_{12} = 0, \\ \textit{quadratic}\ (\mathrm{Q2}){:}\quad & M^{123}_{012} + M^{023}_{013} = 0, \\ \textit{determinant}\ (\mathrm{D}){:}\quad & \det(u_{ij}) = 0, \\ \textit{linear}\ (\mathrm{L2}){:}\quad & u_{44} = 0, \\ \textit{quadratic}\ (\mathrm{Q3}){:}\quad & 2M^{234}_{014} + 2M^{134}_{024} + M^{034}_{034} + M^{124}_{124} = 0. \end{aligned}$$

Proof Follows from the following identities, easily checked by direct computations:

$$
\begin{aligned}
\tau(\mathcal{E}_{\omega_+^2\wedge\omega_-^2\wedge E_\delta}) &= \mathcal{E}_{\omega_+^2\wedge\omega_-^2\wedge E_{-\delta}},\\
\tau(\mathcal{E}_{\omega_+^2\wedge\omega^2\wedge E_\delta}) &= -\mathcal{E}_{\omega_-^2\wedge\omega^2\wedge E_{-\delta}},\\
\tau(\mathcal{E}_{\omega_+^2\wedge\omega^2\wedge E_{-\delta}}) &= -\mathcal{E}_{\omega_-^2\wedge\omega^2\wedge E_\delta},\\
\tau(\mathcal{E}_{\omega_+^4\wedge E_\delta}) &= \mathcal{E}_{\omega_-^4\wedge E_{-\delta}},\\
\tau(\mathcal{E}_{\omega_+^4\wedge E_{-\delta}}) &= \mathcal{E}_{\omega_-^4\wedge E_\delta},\\
\tau(\mathcal{E}_{\omega^4\wedge E_\delta}) &= \mathcal{E}_{\omega^4\wedge E_{-\delta}}.
\end{aligned}
$$

Theorem 4.3 *The following results hold:*

(1) MAEs that are the G_2*–equivariant extension of the hypersurfaces labeled above (L1) and (Q2) are contact–equivalent;*
(2) MAEs that are the G_2*–equivariant extension of the hypersurfaces labeled above (D) and (L2) are contact–equivalent;*
(3) MAEs that are the G_2*–equivariant extension of the hypersurfaces labeled above (Q1) and (L1) are not contact–equivalent.*

Proof Claims (1) and (2) can be proved analogously to the proof of Theorem 4.2, if τ is replaced by

$$
\xi := \begin{pmatrix} \mathrm{id}_4 & 0 & 0 & 0\\ 0 & 0 & 0_4 & 1\\ 0_4 & 0 & \mathrm{id}_4 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix},
$$

and Φ is replaced by

$$
\begin{aligned}
&\Xi : N \longrightarrow N,\\
&\quad (x^0, x^1, x^2, x^3, x^4, u, u_0, u_1, u_2, u_3, u_4)\\
&\quad \longmapsto (x^0, x^1, x^2, x^3, u_4, u - x^4u_4, u_0, u_1, u_2, u_3, -x^4).
\end{aligned}
$$

Claim (3) can be proved by an analysis of the symbol. Since (L1) is linear, its symbol is constant and, as such, its rank never drops: it is constant to 4. On the other hand, the symbol of (Q1) is given, up to proportionality, by the 5×5 matrix

$$
\begin{aligned}
&\mathrm{Smbl(Q1)}\\
&= \begin{pmatrix}
-3a_{12}^2 & 5a_{10}a_{12} & -5a_7a_{12} & 3a_3a_{12} & 0\\
5a_{10}a_{12} & -3a_9a_{12} & 3a_6a_{12} & -5a_2a_{12} & 0\\
-5a_7a_{12} & 3a_6a_{12} & -3a_5a_{12} & 5a_1a_{12} & 0\\
3a_3a_{12} & -5a_2a_{12} & 5a_1a_{12} & -3a_3^2-3a_6^2+10a_2a_7+3a_5a_9-10a_1a_{10} & 0\\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\end{aligned}
$$

where we have solved

$$
u_{00} = \frac{3u_{03}^2+3u_{12}^2-10u_{02}u_{13}-3u_{11}u_{22}+10u_{01}u_{23}}{3u_{33}}
$$

and we have set

$$
a_1 := u_{01}, \quad a_2 := u_{02}, \ldots, \quad a_{14} := u_{44}.
$$

The rank of the above matrix is generically 4, but there are nonempty sets of points, where it drops.

In terms of u_{ij} variables, the list of G_2–invariant MAEs, up to contact equivalence, reduces to the G_2–equivariant extensions of the following hypersurfaces:

$$\begin{aligned}
\text{quadratic Q1:}\quad & 3u_{03}^2 + 3u_{12}^2 - 10u_{02}u_{13} - 3u_{11}u_{22} + 10u_{01}u_{23} - 3u_{00}u_{33} = 0,\\
\text{linear L1:}\quad & u_{03} + u_{12} = 0,\\
\text{linear L2:}\quad & u_{44} = 0,\\
\text{quadratic Q3:}\quad & 2u_{02}u_{13} + u_{11}u_{22} + 2u_{01}u_{23} + u_{00}u_{33} - u_{03}^2 - 4u_{12}u_{03} - u_{12}^2 = 0.
\end{aligned}$$

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Conflict of interest The authors have no Conflict of interest to declare that are relevant to this article

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