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ON COUNT DATA MODELS BASED ON BERNSTEIN FUNCTIONS OR THEIR INVERSES

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Abstract. We present a class of positive discrete random variables extending the Conway–Maxwell-Poisson distribution. This class emerges in a natural way from an application in queueing theory and contains distributions exhibiting quite different features. Some of these distributions are characterized by the presence of Bernstein and inverse Bernstein functions. As a byproduct, we describe these inverses for which the existing literature is limited. Moreover, we investigate dispersion properties for these count data models, giving necessary and/or sufficient conditions to obtain both over and underdispersion. We also provide neat expressions for the factorial moments of any order. This furnishes us with a compact form also in the case of the Conway–Maxwell-Poisson.

1. Introduction

Starting from the seminal paper by Conway and Maxwell [13], the interest in the COM-Poisson distribution (also known as CMP distribution) as a model of count data, has grown continuously in different fields such as statistics [48, 56], social and natural sciences [7, 43], economics [47, 49] (see also the recent monograph by Sellers [46]). By looking at the probability mass function of the COM-Poisson random variable,

$$f_n(\zeta) = \frac{\zeta^n}{n!^\delta}\frac{1}{C_\delta(\zeta)}, \qquad C_\delta(\zeta) = \sum_{n=0}^{\infty}\frac{\zeta^n}{n!^\delta}. \qquad n \in \mathbb{N} = \{0,1,\dots\},\ \delta,\zeta>0, \tag{1.1}$$

we note that its structure strongly resembles that of its celebrated special case, the Poisson random variable ($\delta = 1$). Despite this similarity, the two distributions differ on a fundamental aspect: the COM-Poisson can exhibit a variance to mean ratio smaller, equal, or larger than unity. A long list of variants of the Poisson and COM-Poisson distributions have been proposed during the last decades to address specific modeling needs. The related literature is extensive and therefore we refer to the following recent articles: [6, 9, 10, 16, 18, 23, 36, 37, 42]. However, by inspecting the very form of the probability mass function of the COM-Poisson random variable a natural extension emerges. The power function which distinguishes the COM-Poisson by the Poisson is, in fact, the most prominent example of a Bernstein function or of an inverse Bernstein function, depending on whether the exponent is respectively smaller or larger than unity. Led by this intuition, we consider a class of random variables, in which the role of the power function is played by ϕ, chosen in the space of Bernstein functions or that of their inverses:

$$P_0^\phi(\zeta) = \frac{1}{Z(\zeta,\phi)}, \quad P_n^\phi(\zeta) = \frac{\zeta^n}{\prod_{k=1}^n \phi(k)}\frac{1}{Z(\zeta,\phi)}, \quad n \in \mathbb{N}^* = \{1,2,\dots\},\ \zeta>0, \tag{1.2}$$

where

$$Z(\zeta,\phi) = 1 + \sum_{n=1}^{\infty}\frac{\zeta^n}{\prod_{k=1}^n \phi(k)}. \tag{1.3}$$

This class of random variables turns out to be very rich and includes the COM-Poisson and the Poisson random variables as well as several other variants of the COM-Poisson that appeared in the literature. Our intuition is confirmed by a model of interest in queueing theory with

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state-dependent service times. We outline it in Section 2 in the spirit of [13], in which (1.2) arises as a stationary distribution.

Bernstein functions are characterized by the regular behavior of their derivatives, have a useful integral representation and are often studied thanks to their connection to Lévy processes (for this reason they also go under the name of Laplace exponents), see [45] for a full discussion. To investigate the properties of (1.2) as a function of ϕ, we make use of the Bernstein-gamma function [40]. The latter is the solution W_ϕ to the functional equation $W_\phi(z+1) = \phi(z)W_\phi(z)$, $W_\phi(1) = 1$, and it generalizes the classical gamma function. The Bernstein-gamma function has been used, for example, to generalize the Mittag–Leffler function [41] and to study the properties of the Wright function [4]. Here, we define and outline the properties of the compositional inverse of Bernstein functions. These inverses appear, for instance, in [5]. However, to the best of our knowledge, an extended treatment on inverses constitutes a novelty in the literature. Moreover, we obtain an analog of the Bernstein-gamma function for eventually log-convex functions in the spirit of [55]. Among the strengths of these functions, in relation to count data models, we mention their convexity that is ultimately related to the underdispersion property of (1.2).

For a random variable, the property of underdispersion refers to the variance to mean ratio being less than unity. If, otherwise, it is greater than unity, we speak of overdispersion. In the literature on count data models these properties are always put in contrast to the undesired mean-equal-variance property of the Poisson model. Real-world count data quite often exhibit overdispersion (e.g. heavy-tails), but underdispersion is also possible (e.g. zero-inflation). Overdispersed models arises with relatively simple constructions (see [24] for a discussion), while models capable of dealing with underdispersed data are less frequent in the literature (see e.g. [12, 15, 26, 29, 30, 43, 44, 48, 57]). We provide sufficient conditions for the underdispersion (overdispersion) according to the properties of ϕ. In particular, the class of models (1.2) can be either overdispersed or underdispersed, depending on whether ϕ is chosen in the set of Bernstein functions or of their inverses.

The paper is organized as follows: Section 2 describes the queueing model in which (1.2) arises as a stationary distribution. In Section 3, Bernstein functions are recalled, together with some of their properties. The set of inverse Bernstein functions is presented in the same section, while some of their properties are analyzed in the following Section 4. Section 5 deals with fundamental properties of the introduced model, while Section 6 is specifically devoted to the analysis of its dispersion properties. Lastly, in Section 6, an extension of the model is explored, obtaining, among other results, a compact expression for the factorial moments.

2. The model

Following [13] we consider a model of the service in a queueing system whose rate depends on the state of the system. In particular, we consider the mean service rate $\mu_n = \phi(n)\mu$, where n is the number of units in the system and $\phi : (0, \Lambda) \to (0, \infty)$ where Λ can be infinity. Further, $1/(\mu\phi(1))$ is the service time if only one unit is in the system. Let now define $N(t)$, $t \geq 0$, as the random number of units in the system at time t and $P_n(t) = \mathbb{P}(N(t) = n)$, $n \in \mathbb{N}_\Lambda = \mathbb{N} \cap (0, \Lambda)$. We furthermore assume that the inter-arrival times are independent and exponentially distributed with mean λ. The function ϕ is chosen such that N admits a stationary distribution, that is equation (4) of [1] is satisfied:

$$\sum_{i=1}^{\lfloor \Lambda \rfloor} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{\prod_{j=1}^{i} \phi(j)} < \infty. \tag{2.1}$$

By using the notation of [13] the system of differential difference equations is, for every $n \in \mathbb{N}_\Lambda$:

$$\begin{aligned} P_0(t+\Delta) &= (1-\lambda\Delta)P_0(t) + \phi(1)\mu\Delta P_1(t), \\ P_n(t+\Delta) &= (1-\lambda\Delta-\phi(n)\mu\Delta)P_n(t) + \lambda\Delta P_{n-1}(t) + \phi(n+1)\mu\Delta P_{n+1}(t). \end{aligned}$$

Letting $\Delta \to 0$

$$\begin{aligned} P_0'(t) &= -\lambda P_0(t) + \phi(1)\mu P_1(t), \\ P_n'(t) &= -(\lambda + \phi(n)\mu)P_n(t) + \lambda P_{n-1}(t) + \phi(n+1)\mu P_{n+1}(t). \end{aligned}$$

Since $N(t)$ admits a stationary distribution, letting $\rho = \lambda/\mu$, we get

$$\begin{aligned} \frac{\rho}{\phi(1)} P_0 &= P_1, \\ (\rho + \phi(n))P_n &= \rho P_{n-1} + \phi(n+1)P_{n+1}, \qquad n \in \mathbb{N}_\Lambda^* = \mathbb{N}_\Lambda \backslash \{0\}, \end{aligned}$$

and hence

$$P_n = \frac{\rho^n}{\prod_{k=1}^n \phi(k)} P_0. \tag{2.2}$$

Then, from (2.2), using the normalization condition, we obtain

$$P_0^\phi(\rho) = \frac{1}{Z(\rho,\phi)}, \qquad P_n^\phi(\rho) = \frac{\rho^n}{\prod_{k=1}^n \phi(k)} \frac{1}{Z(\rho,\phi)}, \qquad n \in \mathbb{N}_\Lambda^*, \tag{2.3}$$

where

$$Z(\rho,\phi) = 1 + \sum_{n=1}^{\lfloor \Lambda \rfloor} \frac{\rho^n}{\prod_{k=1}^n \phi(k)}. \tag{2.4}$$

Note that (2.4) converges thanks to (2.1). In the following, we will refer to distribution (2.2) as the extended COM-Poisson distribution (eCOM-Poisson) with characteristic couple (ρ, ϕ), due to its analogy to the COM-Poisson that appears as a special case for $\phi(n) = n^\delta$, $\delta > 0$.

In the next sections 3 and 4 we will analyze some possible choices for the function ϕ.

3. Bernstein functions and their compositional inverses

We recall that a function $f : (0,\infty) \to \mathbb{R}$ is a Bernstein function if $f \in \mathcal{C}^\infty$, $f(\lambda) \geq 0$, $\forall\, \lambda > 0$, and

$$(-1)^{n-1} f^{(n)}(\lambda) \geq 0, \quad \forall\, n \in \mathbb{N}^* = \{1,2,\dots\}, \quad \forall \lambda > 0, \tag{3.1}$$

where $f^{(n)}$ denotes the n-th derivative of f. The space of Bernstein functions is usually denoted by $\mathcal{BF}$. Correspondingly, $\mathcal{BF}_b$ denotes the space of bounded Bernstein functions. Furthermore we denote by $\mathcal{BF}^0$ the space of Bernstein functions f such that $\lim_{\lambda\to 0+} f(\lambda) = 0$. We will only recall in this section the results that will be directly used in the following. For an extensive discussion on the properties of this class of functions see [45].

A further characterization of a Bernstein function $f : (0,\infty) \to \mathbb{R}$ is given by the following representation:

$$f(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(\mathrm{d}t) \tag{3.2}$$

where $a, b \geq 0$ and μ is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)} \min(1,t)\mu(\mathrm{d}t) < \infty$. In particular, f is said to be a complete Bernstein function ($f \in \mathcal{CBF}$) if μ has a completely monotone density with respect to the Lebesgue measure.

Let us now recall some known properties of the functions mentioned above.

Proposition 3.1. *i) A function $f \in \mathcal{BF}_b$ if, and only if, in (3.2) $b = 0$ and $\mu(0,\infty) < \infty$; ii) If $f \in \mathcal{CBF}$, $f \neq 0$ then the functions $\frac{\lambda}{f(\lambda)}$, $1/f\left(\frac{1}{\lambda}\right)$ and $\lambda f\left(\frac{1}{\lambda}\right)$ belong to $\mathcal{CBF}$.*

A function $f \in \mathcal{BF}$ is said to be a special Bernstein function, and we write $f \in \mathcal{SBF}$, if the function $\lambda/f(\lambda) \in \mathcal{BF}$. It follows that if $f \in \mathcal{SBF}$, then $\lambda/f(\lambda) \in \mathcal{SBF}$.

Let f be a non-constant Bernstein function such that $\lim_{\lambda\to 0+} f(\lambda) = 0$. Then we define its (compositional) inverse h as

$$h(s) := \inf\{x > 0\colon f(x) > s\}, \qquad 0 < s < \lim_{\lambda\to\infty} f(\lambda) = \Lambda. \tag{3.3}$$

Plainly, $h\colon (0,\Lambda)\to\mathbb{R}$ are positive, convex, $\mathcal{C}^\infty$ functions. We will write $\mathcal{IBF}$ for the space of such inverse functions. Correspondingly, we have,

$$f(\lambda) := \inf\{x\in(0,\Lambda)\colon h(x)>\lambda\}. \tag{3.4}$$

Notice that $\Lambda=\lim_{\lambda\to\infty} f(\lambda)$ is also the point at which $\lim_{s\to\Lambda-} h(s)=\infty$.

The n-th derivative of h can be derived by recurring at the formula for higher order derivatives of inverse functions (see e.g [28]):

$$\begin{aligned} h^{(n)}(s) &= \frac{(-1)^{n-1}}{[f^{(1)}(h(s))]^{2n-1}} \sum_{\substack{s_1+s_2+\dots s_n=n-1\\ s_1+2\cdot s_2+\dots+n\cdot s_n=2n-2}} \frac{(-1)^{s_1}(2n-s_1-2)![f^{(1)}(h(s))]^{s_1}\dots[f^{(n)}(h(s))]^{s_n}}{(2!)^{s_2}s_2!\dots(n!)^{s_n}s_n!} \\ &= \frac{1}{f^{(1)}(h(s))}\mathcal{C}_{2n-2,n-1}\left(\frac{1}{f^{(1)}(h(s))}, -\frac{f^{(2)}(h(s))}{f^{(1)}(h(s))},\dots,-\frac{f^{(n)}(h(s))}{f^{(1)}(h(s))}\right), \end{aligned}$$

where

$$\begin{aligned} &\mathcal{C}_{h,k}(x_1,\dots,x_{h-k+1}) \\ &\quad= \sum_{\substack{j_1+j_2+\dots+j_{h-k+1}=k\\ j_1+2\cdot j_2+\dots+(h-k+1)j_{h-k+1}=h}} \frac{1}{\binom{h}{j_1}}\frac{h!}{j_1!j_2!\dots j_{h-k+1}!}\left[\frac{x_1}{1!}\right]^{j_1}\left[\frac{x_2}{2!}\right]^{j_2}\dots\left[\frac{x_{h-k+1}}{(h-k+1)!}\right]^{j_{h-k+1}} \end{aligned}$$

are weighted exponential Bell polynomials with weights $\binom{h}{j_1}^{-1}$.

Remark 3.1. *Note that a characterization of h through the signs of its derivatives cannot be given as it is done in the case of Bernstein functions. As an example take the function $h(s)=s^\beta$, $\beta>1$.*

Example 3.1 (Pairs of Bernstein and corresponding inverse Bernstein functions)**.**

$$f(\lambda)=\lambda^\alpha; \qquad h(s)=s^{1/\alpha}, \quad \alpha\in(0,1); \tag{3.5}$$

$$f(\lambda)=\frac{a\lambda}{\lambda+1}, \quad a\in(0,\infty); \qquad h(s)=\frac{s}{a-s}, \quad s\in(0,a); \tag{3.6}$$

$$f(\lambda)=(\lambda+1)^\alpha-1; \qquad h(s)=(s+1)^{1/\alpha}-1, \quad \alpha\in(0,1); \tag{3.7}$$

$$f(\lambda)=\sqrt{\frac{a\lambda}{\lambda+1}}, \quad a\in(0,\infty); \qquad h(s)=\frac{s^2}{a-s^2}, \quad s\in(0,\sqrt{a}); \tag{3.8}$$

$$f(\lambda)=\mathcal{W}(\lambda); \qquad h(s)=se^s, \tag{3.9}$$

where $\mathcal{W}$ is the Lambert function on the positive real line;

$$f(\lambda)=\log(1+\lambda^\alpha); \qquad h(s)=(e^s-1)^{1/\alpha}, \quad \alpha\in(0,1); \tag{3.10}$$

$$f(\lambda)=\log(\cosh(\sqrt{2\lambda})); \qquad h(s)=\frac{1}{2}\log^2(e^s+\sqrt{e^{2s}-1}); \tag{3.11}$$

$$f(\lambda)=\log\left(1+\frac{\lambda}{a}\right); \qquad h(s)=a(e^s-1), \quad a>0. \tag{3.12}$$

Example 3.2 (Construction based on Lévy–Laplace exponents). *A large set of inverses of Bernstein functions emerges as follows. The class of Lévy–Laplace exponents has been considered in* [5, 8] *and it is defined as the set*

$$\mathcal{LE} = \left\{ \Psi : \Psi(s) = a + bs + cs^2 + \int_{(0,\infty)} (e^{-sx} - 1 + sx)\nu(\mathrm{d}x), \quad a, b, c \geq 0 \right\}, \tag{3.13}$$

where ν is a positive measure on $(0,\infty)$ such that $\int_{(0,\infty)}(x \wedge x^2)\nu(\mathrm{d}x) < \infty$ and (a,b,c,ν) is the quadruple characteristics of Ψ.

If $\Psi \in \mathcal{LE}$ and $a = \Psi(0) = 0$, then Ψ is the Laplace exponent of a spectrally negative Lévy process $Z = (Z_t)_{t\geq 0}$, that is a Lévy process with no positive jumps that do not drift to $-\infty$ (see [32]*). Thus, it holds*

$$e^{t\Psi(\lambda)} = \mathbb{E}[e^{\lambda Z_t}], \qquad t, \lambda \geq 0. \tag{3.14}$$

Moreover, $\Psi(\lambda) = \lambda f(\lambda)$ where f is a Bernstein function [11].

The class of functions in $\mathcal{LE}$ with $a = 0$ is denoted by B_3 in [8] *and is also known as the class of branching mechanisms for (sub)critical continuous state branching processes* [33].

From Proposition 9 in [8]*, every function in B_3 is such that its inverse is a Bernstein function, implying that every function Ψ belonging to $\mathcal{LE}$ with $a = 0$ can be considered as the inverse function of a Bernstein function $\widetilde{f}$. We stress that $\widetilde{f}$ is not usually the Bernstein function associated to the process Z.*

One can give the following interpretation: if $\widetilde{f}$ is the the inverse of $\Psi \in B_3$, the Bernstein function $[\frac{\mathrm{d}}{\mathrm{d}\lambda}\widetilde{f}(\lambda)]^{-1}$ is the exponent of the subordinator defined as the inverse of the local time at zero of Z.

4. Some properties of inverse Bernstein functions

4.1. **Log-concave and exp-convex functions.** We noted in the previous section that the inverse functions defined as in (3.3) are convex. We discuss here further properties. To this end, we recall that a positive, twice-differentiable function $g\colon I \subset \mathbb{R} \to \mathbb{R}^+$, is log-convex (-concave) if $\log g$ is a convex (concave) function. Similarly, we define exponentially-convex (-concave) functions. A characterizing property of log-convex (-concave) functions is that $g''(x)g(x) - (g'(x))^2$ is non-negative (non-positive) [55]. Finally, we recall that a function g is eventually log-convex if there exists $m \in \mathbb{R}$ such that $\log g$ is convex in (m,∞).

We have the following results.

Lemma 4.1. *Let $g(x)$, $x \in \mathbb{R}^+$, be an exponentially-concave (-convex) function. Then,*

$$g''(x) + \left(g'(x)\right)^2 \leq (\geq)\ 0, \qquad \forall\, x \in \mathbb{R}^+. \tag{4.1}$$

Proof. Let g be an exponentially-concave function. Clearly, $\frac{d^2}{dx^2}e^{g(x)} \leq 0$ and the statement follows after straightforward calculations. The exponentially-convex case is proved analogously. □

Lemma 4.2. *Let f be an exponentially-concave Bernstein function such that $\lim_{\lambda\to 0+} f(\lambda) = 0$. Then, its compositional inverse h is eventually log-convex.*

Proof. We want to prove that $h''(s)h(s) - (h'(s))^2 \geq 0$, that is a characterizing property of log-convex functions.

Since $h(f(\lambda)) = \lambda$, we have that

$$\begin{aligned} f'(\lambda) &= \frac{1}{h'(f(\lambda))} \\ f''(\lambda) &= -\frac{h''(f(\lambda))}{[h'(f(\lambda))]^3}. \end{aligned}$$

From Lemma 4.1 we have that $f''(\lambda)+(f'(\lambda))^2\le 0$, $\forall\lambda$, that, written in terms of the derivatives of h, gives

$$-\frac{h''(f(\lambda))-[h'(f(\lambda))]^2}{[h'(f(\lambda))]^3}\le 0. \tag{4.2}$$

Since h is a non-decreasing function, we conclude that

$$h''(f(\lambda))-\left[h'(f\lambda))\right]^2\ge 0 \tag{4.3}$$

and thus

$$h''(f(\lambda))h(f(\lambda))-\left[h'(f(\lambda))\right]^2=h''(f(\lambda))\lambda-\left[h'(f(\lambda))\right]^2\ge 0,\quad \forall\lambda\ge 1 \tag{4.4}$$

proving that h is eventually log-convex with $m=f(1)$. □

4.2. **Bernstein-Gamma function and its analogue for inverse Bernstein functions.** In the following we write, for a function ϕ as in Section 2,

$$W_\phi(1)=1,\qquad W_\phi(n)=\prod_{k=1}^{n-1}\phi(k),\quad n\in\{2,3,\dots\}. \tag{4.5}$$

In particular, if $\phi\in\mathcal{BF}$, then W_ϕ stands for the unique solution, in the space of positive definite functions, to the functional equation

$$W_\phi(n+1)=\phi(n)W_\phi(n) \tag{4.6}$$

with $W_\phi(1)=1$, and we refer to [40] for a thorough account on this set of functions that generalizes the gamma function, which appears as a special case when $\phi(n)=n$. Further, Theorem 4.1 of [55] extends W_ϕ to the real line.

If ϕ is an eventually log-convex function such that [25]

$$\lim_{n\to\infty}\frac{\phi(n)}{\phi(n+1)}=1, \tag{4.7}$$

then, the following theorem enables us to extend W_ϕ to the real line as well.

Theorem 4.1. *Let ϕ be an eventually log-convex function on $\mathbb{R}^+$ satisfying condition* (4.7). *Then, there exists a unique function $\widetilde{W}_\phi$ satisfying the functional equation $\widetilde{W}_\phi(x+1)=\phi(x)\widetilde{W}_\phi(x)$, $x\in\mathbb{R}_+$, with the initial condition $\widetilde{W}_\phi(1)=1$. Moreover*

$$\widetilde{W}_\phi(x)=\lim_{n\to\infty}\frac{\phi(n)\cdots\phi(1)\phi^x(n)}{\phi(n+x)\cdots\phi(x)},\qquad x>0. \tag{4.8}$$

Proof. The proof follows the lines of Theorem 4.1 in [55], properly adapted to the case of log-convex functions. □

Remark 4.1. *Recalling Lemma 4.2, we have that inverses of exponentially-concave $\mathcal{BF}^0$ functions satisfy Theorem 4.1.*

5. Some properties of the eCOM-Poisson distribution

Let us consider a discrete random variable X with eCOM-Poisson distribution with characteristic couple (ρ,ϕ). We first observe that the ratio of successive probabilities

$$\frac{\mathbb{P}(X=n-1)}{\mathbb{P}(X=n)}=\frac{\phi(n)}{\rho},\qquad n\in\mathbb{N}^*, \tag{5.1}$$

gives us the rate of decay of the tail of the distribution. Moreover, we also have

$$\frac{\mathbb{P}(X=n-s)}{\mathbb{P}(X=n)}=\frac{\phi(n)\phi(n-1)\dots\phi(n-s+1)}{\rho^s},\qquad n\ge s. \tag{5.2}$$

To evaluate the moments of X, following [35] (see also [27]) where a power series distribution is considered, we make use of the probability generating function of X, i.e.

$$\mathbb{E}u^X = \frac{Z(u\rho,\phi)}{Z(\rho,\phi)}, \qquad |u| \le 1. \tag{5.3}$$

Remark 5.1. *If $\phi \in \mathcal{BF}$ expression* (5.3) *can be written as*

$$\mathbb{E}u^X = \frac{{}_1F_1(1;\phi;\rho u)}{{}_1F_1(1;\phi;\rho)}, \tag{5.4}$$

where ${}_1F_1(1;\phi;\cdot)$ is the extension of the hypergeometric function ${}_1F_1$ given in [17]*. From this expression one can calculate summary statistics of X using properties of this function. Note the analogy with the case of the COM-Poisson probability generating function (see* [39]*).*

The factorial moments of order $s \in \mathbb{N}^*$ of X are given by

$$m_s = \frac{\mathrm{d}^s}{\mathrm{d}u^s}\mathbb{E}u^X\Big|_{u=1} = \mathbb{E}\left[X(X-1)(X-2)\cdot\ldots\cdot(X-s+1)\right] = (-1)^s\mathbb{E}(-X)_s, \tag{5.5}$$

where $(y)_s = y(y+1)\cdot\cdots\cdot(y+s-1)$ is the rising factorial (also known as the Pochhammer symbol). Now, by virtue of the Viète–Girard formulae, that relate the coefficients of a polynomial to sums and products of its roots, we expand $X(X-1)(X-2)\cdots(X-s+1)$ as

$$m_s = \sum_{r=1}^{s}(-1)^{s-r}e_r\,\mathbb{E}X^r, \tag{5.6}$$

where

$$e_r = e_r(l_1,l_2,\ldots,l_r) = \sum_{1\le l_1\le l_2\le\cdots\le l_r\le s} l_1 l_2\cdot\ldots\cdot l_r. \tag{5.7}$$

By using the relation between moments and factorial moments we have

$$\mathbb{E}X^s = \sum_{j=0}^{s}(-1)^j\left\{{s \atop j}\right\}\mathbb{E}(-X)_j \tag{5.8}$$

where $\left\{{s \atop j}\right\} = \frac{1}{j}\sum_{m=0}^{j}(-1)^{j-m}\binom{j}{m}m^s$ are the Stirling numbers of the second kind. Considering that

$$\mathbb{E}(-X)_j = \frac{1}{Z(\rho,\phi)}\sum_{k=1}^{\infty}(-k)_j\frac{\rho^k}{\prod_{i=1}^{k}\phi(i)}, \tag{5.9}$$

in which we used that $(0)_j = 0$, it follows

$$\mathbb{E}X^s = \frac{1}{Z(\rho,\phi)}\sum_{j=0}^{s}(-1)^j\left\{{s \atop j}\right\}\sum_{k=1}^{\infty}(-k)_j\frac{\rho^k}{\prod_{i=1}^{k}\phi(i)}, \qquad s\in\mathbb{N}^*. \tag{5.10}$$

Writing $(-k)_j = (-1)^j k!/(k-j)!$, we identify in (5.10) the operator (ρD) characterized by the well known formula [22]

$$(\rho D)^s f(\rho) = \sum_{j=0}^{s}\left\{{s \atop j}\right\}\rho^j D^j f(\rho), \tag{5.11}$$

where $D = \mathrm{d}/\mathrm{d}\rho$, and f is a suitable function. The previous steps prove the following proposition.

Proposition 5.1. *Let $X \sim$ eCOM-Poisson(ρ,ϕ). Then,*

$$\mathbb{E}X^s = \frac{1}{Z(\rho,\phi)}(\rho D)^s Z(\rho,\phi), \quad s\in\mathbb{N}^*, \tag{5.12}$$

$$m_s = \frac{1}{Z(\rho,\phi)}\rho^s D^s Z(\rho,\phi), \quad s\in\mathbb{N}^*. \tag{5.13}$$

In general, the normalizing function $Z(\rho,\phi)$ does not permit neat closed-form expressions for the statistical quantities related to this distribution. However, asymptotic results on the product $\prod_{i=1}^{k}\phi(i)$ are available in the case of Bernstein functions, see Theorem 4.2 in [40] (see also [38]). A closed form for a sort of a generalization of the factorial moments is presented in the following remark.

Remark 5.2. *For $X \sim$ eCOM-Poisson(ρ,ϕ), by direct calculation, we have*

$$ {}_{\phi}m_s = \mathbb{E}[\phi(X)\phi(X-1)\dots\phi(X-s+1)] = \rho^s, \qquad s\in\mathbb{N}^*, \tag{5.14}$$

which remarkably coincide with the factorial moments of a Poisson random variable of parameter ρ. Further, (5.14) *specialize to the factorial moments if ϕ is the identity function (Poisson case). Further, if $s=1$ we have*

$$\mathbb{E}\phi(X) = \rho. \tag{5.15}$$

Remark 5.3. *By the extended Markov's inequality for arbitrary non-negative and non-decreasing functions, we have, for every $a>0$,*

$$\mathbb{P}(X\geq a) \leq \frac{\mathbb{E}\phi(X)}{\phi(a)} = \frac{\rho}{\phi(a)}, \tag{5.16}$$

where we have used (5.15)*. For $a\in\mathbb{N}^*$, considering* (5.1)*, we have*

$$\mathbb{P}(X\geq a) \leq \frac{\mathbb{P}(X=a)}{\mathbb{P}(X=a-1)}. \tag{5.17}$$

In the following we present a few examples of eCOM-Poisson distributions for different choices of the function ϕ belonging to either the class of Bernstein functions or that of their inverses.

Example 5.1. *Let $X\sim$ eCOM-Poisson(ρ,ϕ) such that $\rho\in(0,1)$ and ϕ is the Bernstein function $\phi(r)=\frac{r}{r+1}$. Condition $\rho\in(0,1)$ guarantees that* (2.1) *is satisfied. Then $\prod_{r=1}^{n}\phi(r) = \frac{1}{n+1}$ and $Z(\rho,\phi)=\sum_{n=0}^{\infty}(n+1)\rho^n = (1-\rho)^{-2}$. From Proposition 5.1, we obtain*

$$\begin{aligned}\mathbb{E}X^s &= (1-\rho)^2(\rho D)^s(1-\rho)^{-2} = (1-\rho)^2(\rho D)^s\sum_{j=1}^{\infty} j\rho^{j-1}\\ &= (1-\rho)^2\sum_{j=0}^{\infty}(j+1)j^s\rho^j = (1-\rho)^2[Li_{-s-1}(\rho)+Li_{-s}(\rho)],\end{aligned}$$

where Li_{-s} is the polylogarithm function. In particular, the first moment of X and its variance read

$$\mathbb{E}X = \frac{2\rho}{1-\rho}, \qquad \mathbb{V}arX = \frac{2\rho}{(1-\rho)^2}. \tag{5.18}$$

Example 5.2. *Consider the inverse Bernstein function $\phi(r)=(r+1)^2-1=r(r+2)$. Then $\prod_{r=1}^{n}\phi(r) = \frac{n!(n+2)!}{2}$ and Proposition 5.1 can be applied for $Z(\rho,\phi)=\sum_{n=0}^{\infty}\frac{2\rho^n}{(n+1)(n+2)n!^2}$. Clearly, condition* (2.1) *holds true. This function satisfies the Euler differential equation of the second order:*

$$\rho^2 S''(\rho) + 4\rho S'(\rho) + 2S(\rho) = 2C_0(\rho), \tag{5.19}$$

where $C_0(\rho)=\sum_{n=0}^{\infty}\frac{\rho^n}{n!^2}$ is the Tricomi function.

Example 5.3 (COM-Poisson)**.** *Consider the function $\phi(r)=r^\delta$, for $\delta>0$. Then $\prod_{r=1}^{n}\phi(r) = n!^\delta$, $Z(\rho,\phi)=\sum_{n=0}^{\infty}\frac{\rho^n}{n!^\delta}=C_\delta(\rho)$ is the Le Roy function, introduced in* [34] *(see also* [19, 21, 51] *for a more general Le Roy-type function). Note that if $\delta\in(0,1)$, then $\phi(r)\in\mathcal{BF}$, if $\delta>1$,* *then* $\phi(r) \in \mathcal{IBF}$. *There does not exist closed form formulae for the moments of the COM-Poisson distribution, although recurrent relations* [14], [50] *or asymptotic expansions* [20] *are available. An alternative formula can be obtained from Proposition 5.1:*

$$\mathbb{E}X^s = \frac{(\rho D)^s C_\delta(\rho)}{C_\delta(\rho)}. \tag{5.20}$$

Example 5.4. *Consider the Bernstein function* $\phi(r) = \frac{r+a}{r+b}$, $a, b > 0$. *Then,* $\prod_{r=0}^{n} \phi(r) = \frac{(a)_{n+1}}{(b)_{n+1}}$ *and*

$$Z(\rho, \phi) = \frac{b}{a} \sum_{n=0}^{\infty} \frac{(1)_n (b+1)_n}{(a+1)_n} \frac{\rho^n}{n!} = \frac{b}{a}\, {}_2F_1(1, b+1; a+1; \rho), \tag{5.21}$$

where ${}_2F_1$ *is the Gauss hypergeometric function. From Proposition 5.1, we obtain*

$$\mathbb{E}X^s = \frac{(\rho D)^s\ {}_2F_1(1, b+1; a+1; \rho)}{{}_2F_1(1, b+1; a+1; \rho)}, \tag{5.22}$$

and using the well-known differentiation formula (see 15.2.2 in [2]*):*

$$D^j\, {}_2F_1(\mu, \nu; \lambda; \rho) = \frac{(\mu)_j (\nu)_j}{(\lambda)_j} {}_2F_1(\mu + j, \nu + j; \lambda + j; \rho), \tag{5.23}$$

we get the expression for the factorial moments

$$m_s = \rho^s \frac{(1)_s (b+1)_s}{(a+1)_s} \frac{{}_2F_1(1+s, b+1+s; a+1+s; \rho)}{{}_2F_1(\mu + j, \nu + j; \lambda + j; \rho)}. \tag{5.24}$$

Finally, we consider and example in which ϕ is neither a Bernstein nor an inverse Bernstein function.

Example 5.5. *Consider the function* $\phi(r) = \frac{r^2+r+1}{r^2-r+1}$. *For* $\rho \in (0, 1)$ *condition* (2.1) *is satisfied. Then,* $\prod_{r=1}^{n} \phi(r) = n^2 + n + 1$ *and*

$$Z(\rho, \phi) = \sum_{n=0}^{\infty} \frac{\rho^n}{n^2 + n + 1} = \sum_{n=0}^{\infty} \frac{\rho^n}{(n + 1/2)^2 + 3/4}. \tag{5.25}$$

The latter quantity is a Mathieu series [52]. *Consider formula (5) in (*[54]*, page 386), due to Gegenbauer,*

$$\frac{1}{(p^2 + a^2)^\mu} = \frac{\sqrt{\pi}}{(2a)^{\mu - 1/2}\Gamma(\mu)} \int_0^\infty e^{-px} x^{\mu - 1/2} J_{\mu - 1/2}(ax) \mathrm{d}x \qquad a \in \mathbb{R}, \mu, p > 0, \tag{5.26}$$

where $J_{\mu-1/2}(ax)$ *is a Bessel function of the first kind. Taking* $p = n + 1/2, \mu = 1, a = \frac{\sqrt{3}}{2}$, *we get*

$$\frac{1}{(n + 1/2)^2 + \frac{3}{4}} = 3^{-1/4} \sqrt{\pi} \int_0^\infty e^{-(n+1/2)x} \sqrt{x}\, J_{1/2}\left(\frac{\sqrt{3}}{2} x\right) \mathrm{d}x, \qquad n \in \mathbb{N}. \tag{5.27}$$

From here, we get the following integral representation

$$Z(\rho, \phi) = \frac{2}{\sqrt{3}} \int_0^\infty \frac{e^{\frac{x}{2}}}{e^x - \rho} \sin\left(\frac{\sqrt{3}}{2} x\right) \mathrm{d}x. \tag{5.28}$$

6. Dispersion properties

In this section we consider $\Lambda = \infty$. Recall the Poisson probability mass function $P_n(\rho)$, $\rho > 0$, reads

$$P_n^{\mathrm{id}}(\rho) = P_n(\rho) = \frac{\rho^n}{n!} e^{-\rho}, \qquad n \in \mathbb{N}, \tag{6.1}$$

where id is the identity function. Clearly,

$$P_n^\phi(\rho) = \frac{\frac{P_n(\rho)}{\prod_{j=1}^n \phi(j)}}{\sum_{k=0}^\infty \frac{P_k(\rho)}{\prod_{j=1}^k \phi(j)}}, \qquad n \in \mathbb{N}, \tag{6.2}$$

and hence it can be viewed as a weighted Poisson distribution,

$$P_n^\phi(\rho) = \frac{\rho^n w(n) e^{-\rho}}{n!\, \mathbb{E} w(Y)}, \qquad n \in \mathbb{N}, \tag{6.3}$$

where $Y \sim \text{Poisson}(\rho)$ and $w(n) = n!/\prod_{j=1}^n \phi(j)$ is the weight function. Note that $w(n)$ does not depend on ρ.

In order to study overdispersion and underdispersion properties of the model we consider Theorem 3 of [31] together with the corollary following Theorem 4 of the same paper, and thus we study log-convexity of the weight function $w(n)$, that is convexity of

$$\log w(n) = \log \frac{n!}{\prod_{k=1}^n \phi(k)}. \tag{6.4}$$

It is well-known that a sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ is convex if it satisfies the inequality

$$2a_n \le a_{n-1} + a_{n+1}, \qquad \text{for every } n \in \mathbb{N}^*. \tag{6.5}$$

Hence, $w(n)$ is log-convex, and hence the model exhibits overdispersion, if

$$2\log \frac{n!}{\prod_{k=1}^n \phi(k)} - \log \frac{(n-1)!}{\prod_{k=1}^{n-1} \phi(k)} - \log \frac{(n+1)!}{\prod_{k=1}^{n+1} \phi(k)} \le 0, \qquad \text{for every } n \in \mathbb{N}^*. \tag{6.6}$$

This is equivalent to

$$\log\left(\frac{n}{n+1}\frac{\phi(n+1)}{\phi(n)}\right) \le 0, \qquad \text{for every } n \in \mathbb{N}^*, \tag{6.7}$$

leading, in turn, to

$$\frac{n+1}{n} \ge \frac{\phi(n+1)}{\phi(n)}, \qquad \text{for every } n \in \mathbb{N}^*. \tag{6.8}$$

Equivalently, for the dispersion function $d(\lambda) = \lambda/\phi(\lambda)$, $\lambda \in (0,\infty)$,

$$d(n+1) \ge d(n), \qquad \text{for every } n \in \mathbb{N}^*, \tag{6.9}$$

i.e. $d(n)$ is non-decreasing. This leads to the following theorem.

Theorem 6.1. *Let X be the random variable with probability mass function* (2.3)*. Then X is overdispersed (underdispersed) if $d(n) = n/\phi(n)$ is non-decreasing (non-increasing).*

Corollary 6.1. *If $\phi \in \mathcal{C}^1$ is non-decreasing, then X is overdispersed (underdispersed) if for every $\lambda \in (0,\infty)$, $\phi'(\lambda) \le (\ge)\, \phi(\lambda)/\lambda$.*

Corollary 6.2. *If $\phi \in \mathcal{SBF}$, the random variable X is overdispersed.*

Proof. Since $\phi \in \mathcal{SBF}$, then also $\lambda/\phi(\lambda) = d(\lambda) \in \mathcal{SBF}$ and then $d(\lambda)$ is non-decreasing. We conclude by means of Theorem 6.1. □

Consider $\mathcal{ISBF}$, the space of inverse functions of special Bernstein functions. We have, in this case, the following corollary.

Corollary 6.3. *If $\phi \in \mathcal{ISBF}$, the random variable X is underdispersed.*

Proof. Since $\phi \in \mathcal{ISBF}$, then there exists a function $g \in \mathcal{SBF}$ such that $g(\phi(\lambda)) = \lambda$. Hence,

$$d(\lambda) = \frac{\lambda}{\phi(\lambda)} = \frac{g(\phi(\lambda))}{\phi(\lambda)}. \tag{6.10}$$

Since $g \in \mathcal{SBF}$, then there exists $h \in \mathcal{SBF}$ such that $h(\phi(\lambda)) = \phi(\lambda)/g(\phi(\lambda))$. Then,

$$d(\lambda) = 1/h(\phi(\lambda)) \tag{6.11}$$

and hence, being $\phi \in \mathcal{ISBF}$, the dispersion function d is non-increasing. Finally, by Theorem 6.1 we get underdispersion. □

Remark 6.1. *Note that, if $\phi \in \mathcal{SBF}$, so is d, and vice versa. Further, consider that $w(n) = \prod_{k=1}^n d(k)$. This last comment leads us to the duality property in which $w(n)$ and $W_\phi(n) = \prod_{k=1}^n \phi(k)$ can be exchanged.*

7. A further extension

Consider a non–negative random variable X having probability mass function

$$P_n^\phi(\rho) = \frac{\Gamma(n+\gamma)\rho^n}{n!V_\phi(\alpha n+\beta)} \frac{1}{Z_{\alpha,\beta,\gamma}(\rho,\phi)}, \qquad \alpha,\beta,\gamma > 0,\ n \in \mathbb{N}, \tag{7.1}$$

where

$$Z_{\alpha,\beta,\gamma}(\rho,\phi) = \sum_{k=0}^\infty \frac{\Gamma(k+\gamma)\rho^k}{k!V_\phi(\alpha k+\beta)} \tag{7.2}$$

and V_ϕ is either the Bernstein-gamma function W_ϕ or the the function $\widetilde{W}_\phi$ in Theorem 4.1. If $\alpha = \beta = \gamma = 1$ we will write $Z_{1,1,1}(\rho,\phi) = Z(\rho,\phi)$.

Remark 7.1. *From* (7.1) *we retrieve the following known special cases.*

- *If $\gamma = \beta = \alpha = 1$, X is eCOM-Poisson(ρ,ϕ).*
- *If $\gamma = \beta = \alpha = 1$, $\phi \equiv \mathrm{id}$, X is Poisson with parameter ρ.*
- *If $\gamma = \beta = \alpha = 1$, $\phi(x) = x^\delta$, with $\delta \in \mathbb{R}_+$, X is COM-Poisson with parameters ρ and δ* [13].
- *If $\gamma = \alpha = 1$, $\phi \equiv \mathrm{id}$, X has the hyper-Poisson distribution* [3] *with parameters ρ and β.*
- *If $\gamma = 1$, $\phi \equiv \mathrm{id}$, X has the alternative Mittag–Leffler distribution (see* [6, 23]*).*
- *If $\phi \equiv \mathrm{id}$ we obtain the alternative generalized Mittag-Leffler distribution* [42] *(see also* [53] *for properties of the generalized Mittag–Leffler function).*

Furthermore, note that distribution (7.1) *for $\phi(x) = x^\delta$, $\delta \in \mathbb{R}_+$, does not coincide with that in Section 3.1 of* [9] *although both based on the principle of generalizing the factorial by a gamma-type function, and choosing the power function (see also* [18] *if in addition $\gamma = 1$).*

Proposition 7.1. *The factorial moments of X write*

$$m_s = \rho^s \frac{Z_{\alpha,s\alpha+\beta,\gamma+s}(\rho,\phi)}{Z_{\alpha,\beta,\gamma}(\rho,\phi)}, \qquad s \in \mathbb{N}. \tag{7.3}$$

Proof. The result follows directly from the application of Proposition 5.1. □

Corollary 7.1. *The moments read*

$$\mathbb{E}X^s = \frac{1}{Z_{\alpha,\beta,\gamma}(\rho,\phi)} \sum_{r=1}^s \rho^r \left\{ \begin{matrix} s \\ r \end{matrix} \right\} Z_{\alpha,r\alpha+\beta,\gamma+r}(\rho,\phi), \qquad s \in \mathbb{N}. \tag{7.4}$$

In particular, the first two moments of X read

$$\mathbb{E}X = \rho \frac{Z_{\alpha,\alpha+\beta,\gamma+1}(\rho,\phi)}{Z_{\alpha,\beta,\gamma}(\rho,\phi)}, \tag{7.5}$$

$$\mathbb{E}X^2 = \rho^2 \frac{Z_{\alpha,2\alpha+\beta,\gamma+2}(\rho,\phi)}{Z_{\alpha,\beta,\gamma}(\rho,\phi)} + \rho \frac{Z_{\alpha,\alpha+\beta,\gamma+1}(\rho,\phi)}{Z_{\alpha,\beta,\gamma}(\rho,\phi)}. \tag{7.6}$$

Corollary 7.2. *If $\gamma = \beta = \alpha = 1$, (eCOM-Poisson case) the factorial moments* (7.3) *simplify to*

$$m_s = \rho^s \frac{Z_s(\rho,\phi)}{Z(\rho,\phi)}, \qquad s \in \mathbb{N}, \tag{7.7}$$

where $Z_s(\rho,\phi) = Z_{1,s+1,s+1}(\rho,\phi)$. If, in addition, $\phi(x) = x^\delta$, $\delta \in \mathbb{R}_+$, formula (7.7)*, in agreement with (52) of* [9]*, gives the factorial moments of the COM-Poisson distribution:*

$$m_s = \frac{\rho^s}{C_\delta(\rho)} \sum_{k=0}^{\infty} \frac{\rho^k}{k!}(k+s)!^{1-\delta}, \qquad n \in \mathbb{N}, \tag{7.8}$$

where $C_\delta(\rho)$ is the Le Roy function.

Remark 7.2. *The following Turán-type inequality for the function Z holds:*

$$Z_{\alpha,\alpha+\beta,\gamma+1}(\rho,\phi) + \rho Z_{2\alpha,\alpha+\beta,\gamma+2}(\rho,\phi) \geq \rho \frac{Z^2_{\alpha,\alpha+\beta,\gamma+1}(\rho,\phi)}{Z_{\alpha,\beta,\gamma}(\rho,\phi)}. \tag{7.9}$$

In the two following propositions we provide conditions for over and underdispersion of X. Note that, in this case, V_ϕ cannot be expressed in terms of products of functions ϕ.

Proposition 7.2. *The random variable X is overdispersed (underdispersed) if and only if*

$$Z_{\alpha,\beta,\gamma}(\rho,\phi) Z_{\alpha,2\alpha+\beta,\gamma+2}(\rho,\phi) > (<)\ Z^2_{\alpha,\alpha+\beta,\gamma+1}(\rho,\phi). \tag{7.10}$$

Proof. We prove the characterization for the overdispersion case. The same reasoning applies to the case of underdispersion. The variable X is overdispersed if and only if $m_2 > m_1^2$ which in this case, recalling Proposition 7.1, corresponds to (7.10). □

Remark 7.3. *Note that, although this distribution is somehow more general than the eCOM-Poisson, Proposition 7.2 is stronger than Theorem 6.1.*

Next, we provide a sufficient condition for overdispersion (underdispersion) that involves the first derivative of the function

$$\psi_\phi(y) := \frac{\mathrm{d}}{\mathrm{dy}} \log V_\phi(y), \qquad y > 0, \tag{7.11}$$

where, in particular, ψ_{id} is the classical digamma function.

Proposition 7.3. *If*

$$\psi\prime_{\mathrm{id}}(y+\gamma) \geq (\leq)\ \alpha \psi\prime_\phi(\alpha y + \beta) \tag{7.12}$$

where $\psi\prime_\phi$ is the first derivative of the function defined in (7.11)*, then X is overdispersed (underdispersed).*

Proof. Using Theorem 4 of [31], we consider log-convexity of the weight function $w(y) = \Gamma(y+\gamma)/V_\phi(\alpha y + \beta)$, that is

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} \log \frac{\Gamma(y+\gamma)}{V_\phi(\alpha y+\beta)} = \frac{\mathrm{d}}{\mathrm{d}y} \left[\psi_{\mathbf{id}}(y+\gamma) - \alpha\psi_\phi(\alpha y + \beta)\right] \geq (\leq)\, 0 \tag{7.13}$$

from which the statement immediately follows. □

Remark 7.4. *Note that for $\alpha = 1$,* (7.12) *reduces to*

$$\frac{\psi\prime_{\mathrm{id}}(y+\gamma)}{\psi\prime_\phi(y+\beta)} \geq (\leq)\ 1. \tag{7.14}$$

If, in addition, $\phi \equiv \mathrm{id}$, then (7.12) *coincides with condition* (58) *of* [9] *for the alternative generalized Mittag-Leffler distribution* [42].

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