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A pro-p version of Sela's accessibility and Poincaré duality pro-p groups

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Abstract. We prove a pro-p version of Sela's theorem (1997) stating that a finitely generated group is k-acylindrically accessible. This result is then used to prove that PD^n pro-p groups admit a unique k-acylindrical JSJ-decomposition.

Dedicated to the 65th birthday of Peter H. Kropholler

1. Introduction

Since 1970, the Bass–Serre theory of groups acting on trees stood out as one of the major advances in the classical combinatorial group theory. The main notion of the Bass–Serre theory is the notion of graph of groups. The fundamental group of a graph of groups acts naturally on a standard (universal) tree that allows to describe subgroups of these constructions. This theory raised naturally the question of accessibility, namely, whether we can continue to split G into an amalgamated free product or an HNN-extension forever, or do we reach the situation, after finitely many steps, where we cannot split it any more. In other words, accessibility is the question whether splittings of G as the fundamental group of a graph of groups have natural bound. Accessibility of splittings over finite groups (i.e., as a graph of groups with finite edge-groups) was studied by Dunwoody [3,4], who proved that finitely presented groups are accessible but found an example of an inaccessible finitely generated group. This initiated naturally a search for a kind of accessibility that holds for finitely generated groups. The breakthrough in this direction is due to Sela [18], who proved k-acylindrical accessibility for any finitely generated group: accessibility provided the stabilizer of any segment of length k of the group acting on its standard tree is trivial for some k.

The profinite version of Bass–Serre theory was developed by Luis Ribes, Oleg Melnikov and the second author. However, the pro-p version of Bass–Serre theory does not give subgroup structure theorems the way it does in the classical Bass–Serre theory: even in the pro-p case, if G acts on a pro-p tree T, then a maximal subtree of the

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quotient graph $G \setminus T$ does not always exist, and even if it exists, it does not always lift to T. Nevertheless, the pro-p version of the subgroups structure theorem works for pro-p groups acting on a pro-p trees that are accessible with respect to splitting over edge stabilizers; see [2, Theorem 6.3]. This shows additional importance of studying accessibility of pro-p groups. In general, finitely generated pro-p groups are not accessible, as shown by Wilkes [24], and it is an open question whether finitely presented are. Our main result in this direction is the pro-p version of the celebrated Sela's result [18] (cf. Theorem 3.13).

Theorem 1.1. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental group of a finite reduced k-acylindrical graph of pro-p groups. Then $|E(\Gamma)| \leq d(G)(4k+1) - 1$, $|V(\Gamma)| \leq 4kd(G)$.*

We use our accessibility theorem to establish the Kropholler type [12, Theorem A2] JSJ-decomposition for Poincaré duality pro-p groups. First JSJ-decompositions appeared in 3-dimensional topology with the theory of the characteristic submanifold by Jaco–Shalen and Johannson. These topological ideas were carried over to the group theory first by Kropholler [12] for some Poincaré duality groups. Later constructions of JSJ-decompositions were given in various settings by Sela for torsion-free hyperbolic groups [19], and in various settings by Rips–Sela [17], Bowditch [1], Dunwoody–Sageev [5], Fujiwara–Papasoglu [7], Dunwoody–Swenson [6], etc. This has had a vast influence and a range of applications in the geometric and combinatorial group theory.

The result below can be considered as the first step towards this theory in the category of pro-p groups. We establish a canonical JSJ-decomposition of Poincaré duality pro-p groups of dimension n (i.e., PD^n pro-p groups) which is a pro-p version of the Kropholler [12, Theorem A2]. It also can be viewed as a pro-p version of the torus decomposition theorem for 3-manifolds (cf. Theorem 4.5).

Theorem 1.2. *For every PD^n pro-p group G, $n > 2$, there exists a (possibly trivial) k-acylindrical pro-p G-tree $\mathcal{T}$ satisfying the following properties:*

(i) *every edge stabilizer is a maximal polycyclic subgroup of G of Hirsch length $n-1$;*

(ii) *every polycyclic subgroup of G of Hirsch length > 1 stabilizes a vertex;*

(iii) *the underlying graph of groups does not split further k-acylindrically over polycyclic subgroups of G of Hirsch length $n-1$.*

Moreover, every two pro-p G-trees satisfying the properties above are G-isomorphic.

Examples of JSJ-decompositions of PD^3 pro-p groups can be obtained by the pro-p completion of an abstract JSJ-decomposition of some 3-manifolds (see [23]). The pro-p completion of PD^n groups in general were studied in [8, 9, 11, 12, 22].

The structure of the paper is as follows. Section 2 recalls the notions of a pro-p tree, a pro-p fundamental group and a graph of pro-p groups with a special focus on finite graphs of pro-p groups. Throughout the paper, finite graphs of pro-p groups will be often required to be reduced and proper (see Definitions 2.12 and 2.17) but Remarks 2.13 and 2.18 show that such an assumption is not restrictive. Section 3 is devoted to the proof of the pro-p version of Sela's accessibility which states that every finitely generated pro-p group is k-acylindrically accessible. Recall that a profinite graph of pro-p groups $(\mathcal{G}, \Gamma)$ is *k-acylindrical* if the action of the pro-p fundamental group on its standard pro-p tree is k-acylindrical (cf. Section 2.2). In this section, we also prove the pro-p version of Karras–Solitar result describing 2-generated subgroups of free products with malnormal amalgamation (see Theorem 3.19). Finally, Section 4 deals with splittings of PD^n pro-p groups and culminates with a JSJ-decomposition for PD^n pro-p groups (see Theorem 1.2) which is a pro-p version of the Kropholler theorem [12, Theorem A2]. Note that the Kropholler theorem gives also information on vertex-groups of a JSJ-splitting that is based on the Kropholler–Roller decomposition theorem [13, Theorem B] that states that a PD^n group G having a PD^{n-1} subgroup H virtually splits as a free product with amalgamation or HNN-extension over a subgroup commensurable with it if $\mathrm{cd}(H \cap H^g) \neq n-2$ for each $g \in G$. In fact, by [13, Theorem C], G virtually splits over H if H is polycyclic.

Unfortunately, Kropholler–Roller theorems do not hold in the prop-p case as shown by the following example, which has been constructed in communication with Peter Kropholler during the visit of the second author to the University of Southampton.

Example 1.3. Let G be an open pro-p subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ and H be the intersection of the Borel subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ with G. Then H is a malnormal metacyclic subgroup of G and therefore is a PD^2 pro-p group. The group G is an analytic pro-p group of dimension 3 and so is a PD^3 pro-p group. However, G does not split as an amalgamated free pro-p product or HNN-extension at all.

In Section 5, we provide the details of the statement written in the example above. Here we just remark that the absence of the Kroholler–Roller splitting result is an obstacle for obtaining information on vertex-groups of a JSJ-splitting from Theorem 1.2.

2. Notation, definitions and basic results

2.1. Notation. We shall denote by $d(G)$ the number of a minimal set of generators of a pro-p group G and by $\Phi(G)$ its Frattini subgroup. If a pro-p group G continuously acts on a profinite space X, we denote by G_x the stabilizer of x in G. If $x \in X$ and $g \in G$, then $G_{gx} = gG_xg^{-1}$. We shall use the notation $h^g = g^{-1}hg$ for conjugation. For a subgroup H of G, H^G will stand for the (topological) normal closure of H in G. If G is an abstract group, $\widehat{G}$ will mean the pro-p completion of G.

2.2. Conventions. Throughout the paper, unless otherwise stated, groups are pro-p, subgroups will be closed and morphisms will be continuous. Finite graphs of groups will be proper and reduced (see Definitions 2.12 and 2.17). Actions of a pro-p group G on a profinite graph Γ will a priori be supposed to be faithful (i.e., the action has no kernel), unless we consider actions on subgraphs of Γ.

Next we collect basic definitions, following [15].

2.1. Profinite graphs

Definition 2.3. A *profinite graph* is a triple (Γ, d_0, d_1), where Γ is a profinite (i.e., boolean) space, and $d_0, d_1\colon \Gamma \to \Gamma$ are continuous maps such that $d_i d_j = d_j$ for $i, j \in \{0, 1\}$. The elements of $V(\Gamma) := d_0(G) \cup d_1(G)$ are called the *vertices* of Γ, and the elements of $E(\Gamma) := \Gamma \setminus V(\Gamma)$ are called the *edges* of Γ. If $e \in E(\Gamma)$, then $d_0(e)$ and $d_1(e)$ are called the *initial* and *terminal vertices* of e. A vertex with only one incident edge is called *pending*. If there is no confusion, one can just write Γ instead of (Γ, d_0, d_1).

Definition 2.4. A *morphism* $f\colon \Gamma \to \Delta$ of graphs is a map f which commutes with the d_i's. Thus it will send vertices to vertices, but might send an edge to a vertex.[1]

2.5. Collapsing edges. We do not require for a morphism to send edges to edges. If Γ is a graph and e is an edge which is not a loop, we can *collapse* the edge e by removing $\{e\}$ from the edge set of Γ, and identify $d_0(e)$ and $d_1(e)$ with a new vertex y. That is, Γ' is the graph given by $V(\Gamma') = V(\Gamma) \setminus \{d_0(e), d_1(e)\} \cup \{y\}$ (where y is the new vertex), and $E(\Gamma') = E(\Gamma) \setminus \{e\}$. We define $\pi\colon \Gamma \to \Gamma'$ by setting $\pi(m) = m$ if $m \notin \{e, d_0(e), d_1(e)\}$, $\pi(e) = \pi(d_0(e)) = \pi(d_1(e)) = y$. The maps $d_i'\colon \Gamma' \to \Gamma'$ are defined so that π is a morphism of graphs. Another way of describing Γ' is that $\Gamma' = \Gamma/\Delta$, where Δ is the subgraph $\{e, d_0(e), d_1(e)\}$ collapsed into the vertex y.

Definition 2.6. Every profinite graph Γ can be represented as an inverse limit $\Gamma = \varprojlim \Gamma_i$ of its finite quotient graphs [15, Proposition 1.5].

A profinite graph Γ is said to be *connected* if all its finite quotient graphs are connected. Every profinite graph is an abstract graph, but a connected profinite graph is not necessarily connected as an abstract graph.

A connected finite graph without circuits is called a *tree*. In the next subsection, we shall explain how this notion extends to the pro-p context. The *valency* of a vertex is the number of edges connected to it. Hence, a vertex is pending if it has valency 1. A tree with two pending vertices will be called a *line*.

2.2. Pro-p trees

2.7. The fundamental group of a profinite graph. Let Γ be a connected profinite graph. If $\Gamma = \varprojlim \Gamma_i$ is the inverse limit of the finite graphs Γ_i, then it induces the inverse system $\{\pi_1(\Gamma_i) = \hat{\pi}_1^{abs}(\Gamma_i)\}$ of the pro-p completions of the abstract (usual) fundamental groups $\pi_1^{abs}(\Gamma_i)$. So the pro-p fundamental group $\pi_1(\Gamma)$ can be defined as $\pi_1(\Gamma) = \varprojlim_i \pi_1(\Gamma_i)$. If $\pi_1(\Gamma) = 1$, then Γ is called a *pro-p tree*.

If T is a pro-p tree, then we say that a pro-p group G *acts on* T if it acts continuously on T and the action commutes with d_0 and d_1.

[1] It is called a *quasimorphism* in [14].

If $t \in V(T) \cup E(T)$, we denote by G_t the stabilizer of t in G. For a pro-p group G acting on a pro-p tree T, we let $\widetilde{G}$ denote the subgroup generated by all vertex stabilizers. Moreover, for any two vertices v and w of T, we let $[v,w]$ denote the geodesic connecting v to w in T, i.e., the (unique) smallest pro-p subtree of T that contains v and w. The geodesic connecting two vertices might not be a finite segment. By length of a geodesic we will mean the number of edges in the geodesic, which in general may be infinite. The fundamental group $\pi_1(\Gamma)$ acts freely on a pro-p tree $\widetilde{\Gamma}$ (universal cover) such that $\pi_1(\Gamma) \setminus \widetilde{\Gamma} = \Gamma$ (see [26, Section 3] or [14, Chapter 3] for details).

An action of a pro-p group on a pro-p tree T is called *k-acylindrical* if the stabilizer of any geodesic in T of length greater than k is trivial. For instance, 0-acylindrical refers to an action with trivial edge stabilizers, and 1-acylindrical implies that edge stabilizers are malnormal in vertex-groups.

Lemma 2.8. *Let G be a pro-p group acting k-acylindrically on a pro-p tree T. Then every polycyclic subgroup A of G of Hirsch length > 1 fixes a vertex.*

Proof. Let $A \leq G$ be a polycyclic group of Hirsch length > 1. By contradiction, assume that A does not fix any vertex of T. By [15, Theorem 3.18], there exists a normal subgroup N of A stabilizing some vertex $v \in V(T)$. Since $A \neq A_v$, the minimal subtree T_A containing Av is fixed by N (see [15, Theorem 3.7]). Since T is k-acylindrical, T_A has diameter at most k, so A stabilizes a vertex. ∎

2.3. Finite graphs of pro-p groups

In this subsection, we recall the definition of a finite graph of pro-p groups $(\mathcal{G}, \Gamma)$ and its fundamental pro-p group $\Pi_1(\mathcal{G}, \Gamma)$. When we say that $\mathcal{G}$ is a finite graph of pro-p groups, we mean that it contains the data of the underlying finite graph, the edge pro-p groups, the vertex pro-p groups and the attaching continuous maps. More precisely, one gives the following definitions.

Definition 2.9. Let Γ be a connected finite graph. A graph of pro-p groups $(\mathcal{G}, \Gamma)$ over Γ consists of specifying a pro-p group $\mathcal{G}(m)$ for each $m \in \Gamma$ (i.e., $\mathcal{G} = \dot{\bigcup}_{m\in\Gamma} \mathcal{G}(m)$), and continuous monomorphisms $\partial_i : \mathcal{G}(e) \to \mathcal{G}(d_i(e))$ for each edge $e \in E(\Gamma)$, $i = 1, 2$.

Definition 2.10. (1) A *morphism* of graphs of pro-p groups $(\mathcal{G}, \Gamma) \to (\mathcal{H}, \Delta)$ is a pair $(\alpha, \overline{\alpha})$ of maps, with $\alpha : \mathcal{G} \to \mathcal{H}$ a continuous map, and $\overline{\alpha} : \Gamma \to \Delta$ a morphism of graphs, and such that $\alpha_{\mathcal{G}(m)} : \mathcal{G}(m) \to \mathcal{H}(\overline{\alpha}(m))$ is a homomorphism for each $m \in \Gamma$ and which commutes with the appropriate ∂_i. Thus the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\alpha"] \arrow[d, "\partial_i"] & \mathcal{H} \arrow[d, "\partial_i"] \\ \mathcal{G} \arrow[r, "\alpha"] & \mathcal{H} \end{tikzcd}$$

is commutative.

(2) We say that $(\alpha, \overline{\alpha})$ is a *monomorphism* if both α, $\overline{\alpha}$ are injective. In this case, its image will be called a *subgraph of groups* of $(\mathcal{H}, \Delta)$. In other words, a subgraph of groups of a graph of pro-p-groups $(\mathcal{G}, \Gamma)$ is a graph of groups $(\mathcal{H}, \Delta)$, where Δ is a subgraph of Γ (i.e., $E(\Delta) \subseteq E(\Gamma)$ and $V(\Delta) \subseteq V(\Gamma)$, the maps d_i on Δ are the restrictions of the maps d_i on Γ), and for each $m \in \Delta$, $\mathcal{H}(m) \leq \mathcal{G}(m)$.

2.11. Definition of the fundamental pro-*p* group. In [27, §3.3], the fundamental group G is defined explicitly in terms of generators and relations associated to a chosen subtree D. Namely,

$$\begin{aligned} G = \langle \mathcal{G}(v), t_e \mid v \in V(\Gamma), nE(\Gamma), t_e = 1 \text{ for } e \in D, \\ \partial_0(g) = t_e \partial_1(g) t_e^{-1} \text{ for } g \in \mathcal{G}(e) \rangle. \end{aligned} \tag{1}$$

That is, if one takes the abstract fundamental group $G_0 = \pi_1(\mathcal{G}, \Gamma)$, then $\Pi_1(\mathcal{G}, \Gamma) = \varprojlim_N G_0/N$, where N ranges over all normal subgroups of G_0 of index being a power of p and with $N \cap \mathcal{G}(v)$ open in $\mathcal{G}(v)$ for all $v \in V(\Gamma)$. Note that this last condition is automatic if $\mathcal{G}(v)$ is finitely generated (as a pro-p group) by [16, Theorem 4.2.8]. It is also proved in [27] that the definition given above is independent on the choice of the maximal subtree D.

The main examples of $\Pi_1(\mathcal{G}, \Gamma)$ are an amalgamated free pro-p product $G_1 \amalg_H G_2$ and an HNN-extension $\mathrm{HNN}(G, H, t)$ that correspond to the cases of Γ having one edge and either two vertices or only one vertex, respectively.

Definition 2.12. We call the graph of groups $(\mathcal{G}, \Gamma)$ *proper* (injective in the terminology of [14]) if the natural map $\mathcal{G}(v) \to \Pi_1(\mathcal{G}, \Gamma)$ is an embedding for all $v \in V(\Gamma)$.

Remark 2.13. In the pro-p case, a graph of groups $(\mathcal{G}, \Gamma)$ is not always proper. However, the vertex- and edge-groups can always be replaced by their images in $\Pi_1(\mathcal{G}, \Gamma)$, so that $(\mathcal{G}, \Gamma)$ becomes proper and $\Pi_1(\mathcal{G}, \Gamma)$ does not change. Thus throughout the paper, we shall only consider proper graphs of pro-p groups. In particular, all our free amalgamated pro-p products are proper. Thus we shall always identify vertex- and edge-groups of $(\mathcal{G}, \Gamma)$ with their images in $\Pi_1(\mathcal{G}, \Gamma)$.

If $(\mathcal{G}, \Gamma)$ is a finite graph of finitely generated pro-p groups, then by a theorem of Serre (stating that every finite index subgroup of a finitely generated pro-p group is open, cf. [16, Theorem 4.2.8]), the fundamental pro-p group $G = \Pi_1(\mathcal{G}, \Gamma)$ of $(\mathcal{G}, \Gamma)$ is the pro-p completion of the usual fundamental group $\pi_1(\mathcal{G}, \Gamma)$ (cf. [20, §5.1]). Note that $(\mathcal{G}, \Gamma)$ is proper if and only if $\pi_1(\mathcal{G}, \Gamma)$ is residually p. In particular, edge- and vertex-groups will be subgroups of $\Pi_1(\mathcal{G}, \Gamma)$.

Proposition 2.14. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental pro-p group of a finite proper graph of pro-p groups and U a normal subgroup of G. Put $\widetilde{U} = \langle \mathcal{G}(v)^g \cap U \mid g \in G, v \in V(\Gamma) \rangle$. Then $\widetilde{U}$ is normal in G and $G/\widetilde{U} = \Pi_1(\mathcal{G}_U, \Gamma)$, where $\mathcal{G}_U(m) = \mathcal{G}(m)U/U$ for each $m \in \Gamma$ with ∂_0, ∂_1 being natural inclusions in G/U.*

Proof. The fundamental group $\Pi_1(\mathcal{G}_U, \Gamma)$ has a presentation

$$\langle \mathcal{G}_U(v),\ t_e \mid v \in V(\Gamma),\ e \in E(\Gamma),\ t_e = 1 \text{ for } e \in D,\\ \partial_0(g) = t_e \partial_1(g) t_e^{-1},\ \text{for } g \in \mathcal{G}_U(e)\rangle. \tag{2}$$

Therefore, the kernel of the epimorphism $\Pi_1(\mathcal{G}, \Gamma) \to \Pi_1(\mathcal{G}_U, \Gamma)$ induced by the natural morphism $(\mathcal{G}, \Gamma) \to (\mathcal{G}_U, \Gamma)$ is generated as a normal subgroup by $\mathcal{G}(v) \cap U$, $v \in V(\Gamma)$ as needed. ∎

Let $(\mathcal{G}, \Gamma)$ be a profinite graph of pro-p groups and Δ a subgraph of Γ. Then by $(\mathcal{G}, \Delta)$ we shall denote the graph of groups restricted to Δ. We shall often use the following assertion.

Lemma 2.15 ([21, Lemma 2.4]). *Let $(\mathcal{G}, \Gamma)$ be a proper finite graph of pro-p groups, and let Δ be a connected subgraph of Γ. Then the natural homomorphism $\Pi_1(\mathcal{G}, \Delta) \to \Pi_1(\mathcal{G}, \Gamma)$ is a monomorphism.*

Proposition 2.16. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental group of a proper finite graph of pro-p groups. Suppose there exists an edge e such that $G(e) = 1$ and $G(d_i(e)) \neq 1$ for $i = 0, 1$. Then G splits as a free pro-p product.*

Proof. Suppose $\Gamma \setminus \{e\}$ is not connected. Then $\Pi_1(\mathcal{G}, \Gamma) = G_1 \amalg G_2$, where G_1 and G_2 are the fundamental groups of the graphs of groups restricted to the connected components C_1, C_2 of $\Gamma \setminus \{e\}$ (cf. Lemma 2.15). So the result holds in this case.

Otherwise, let D be a maximal subtree of Γ not containing an edge e. Then $G = \mathrm{HNN}(G_1, G(e), t)$, where G_1 is the fundamental group of the graph of groups restricted to $\Gamma \setminus \{e\}$ (cf. Lemma 2.15). But since $G(e) = 1$, we have $G = G_1 \amalg \langle t \rangle$. ∎

Definition 2.17. A finite graph of pro-p groups $(\mathcal{G}, \Gamma)$ is said to be *reduced*, if for every edge e which is not a loop, neither $\partial_1(e)\colon \mathcal{G}(e) \to \mathcal{G}(d_1(e))$ nor $\partial_0(e)\colon \mathcal{G}(e) \to \mathcal{G}(d_0(e))$ is an isomorphism.

Remark 2.18. Any finite graph of pro-p groups can be transformed into a reduced finite graph of pro-p groups by the following procedure: If $\{e\}$ is an edge which is not a loop and for which one of ∂_0, ∂_1 is an isomorphism, we can collapse $\{e\}$ to a vertex y (as explained in §2.5). Let Γ' be the finite graph given by $V(\Gamma') = \{y\} \cup V(\Gamma) \setminus \{d_0(e), d_1(e)\}$ and $E(\Gamma') = E(\Gamma) \setminus \{e\}$, and let $(\mathcal{G}', \Gamma')$ denote the finite graph of groups based on Γ' given by $\mathcal{G}'(y) = \mathcal{G}(d_1(e))$ if $\partial_0(e)$ is an isomorphism, and $\mathcal{G}'(y) = \mathcal{G}(d_0(e))$ if $\partial_0(e)$ is not an isomorphism.

This procedure can be continued until $\partial_0(e), \partial_1(e)$ are not surjective for all edges not defining loops. Note that the reduction process does not change the fundamental pro-p group, i.e., one has a canonical isomorphism $\Pi_1(\mathcal{G}, \Gamma) \simeq \Pi_1(\mathcal{G}_{\mathrm{red}}, \Gamma_{\mathrm{red}})$. So, if the pro-$p$ group G is the fundamental group of a finite graph of pro-p groups, we may assume that the finite graph of pro-p groups is reduced.

Remark 2.19. The procedure of collapsing in the graph of pro-p groups $(\mathcal{G}, \Gamma)$ can be generalized using Lemma 2.15. If Δ is a connected subgraph, then we can collapse Δ to a vertex v and put $G(v) = \Pi_1(\mathcal{G}, \Delta)$ leaving the rest of edge- and vertex-groups unchanged. The fundamental group $\Pi_1(\mathcal{G}_\Delta, \Gamma/\Delta) = \Pi_1(\mathcal{G}, \Gamma)$. The graph of groups $(\mathcal{G}_\Delta, \Gamma/\Delta)$ will be called *collapsed*.

Lemma 2.20. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental pro-p group of a finite reduced tree of pro-p groups $(\mathcal{G}, \Gamma)$, and let $d(G)$ be the minimal number of generators of G. Then a minimal subset V of $V(\Gamma)$ with $G = \langle \mathcal{G}(v) \mid v \in V \rangle$ contains all pending vertices of Γ and has no more than $d(G)$ elements.*

Proof. For every pending vertex v of Γ and the (unique) edge $e \in \Gamma$ connected to it, $\bar{\mathcal{G}}(v) = \mathcal{G}(v)/\mathcal{G}(e)^{\mathcal{G}(v)}$ is non-trivial, because the tree of groups $(\mathcal{G}, \Gamma)$ is reduced, and the groups are pro-p. Define the quotient tree of groups $(\bar{\mathcal{G}}, \Gamma)$ by putting $\bar{\mathcal{G}}(m) = 1$ if $m \in \Gamma$ is not a pending vertex, and $\bar{\mathcal{G}}(v) = \mathcal{G}(v)/\mathcal{G}(e)^{\mathcal{G}(v)} \neq 1$ if v is a pending vertex. Then from presentation (1) for $\Pi_1(\bar{\mathcal{G}}, \Gamma)$, it follows that

$$\Pi_1(\bar{\mathcal{G}}, \Gamma) = \coprod_{v \in V(\Gamma)} \bar{\mathcal{G}}(v) = \coprod_{v \in P_\Gamma} \bar{\mathcal{G}}(v),$$

where P_Γ is the set of pending vertices of Γ. The natural morphism $(\mathcal{G}, \Gamma) \to (\bar{\mathcal{G}}, \Gamma)$ induces then the epimorphism $G = \Pi_1(\mathcal{G}, \Gamma) \to \bar{G} = \Pi_1(\bar{\mathcal{G}}, \Gamma)$. This shows that $P_\Gamma \subseteq V$.

To show that $|V| \leq d(G)$, consider the Frattini quotient $\bar{G} = G/\Phi(G)$ and use overline for the images of subgroups of G in $\bar{G}$. Since $d(G) = d(\bar{G})$ and $\bar{G}$ is finite elementary abelian, one can choose finite $V_i \subseteq V$ with $|V_1| = 1$ and $|V_{i+1}| = |V_i| + 1$ such that $\langle \bar{\mathcal{G}}(v) \mid v \in V_i \rangle$ is strictly increasing sequence of subgroups of $\bar{G}$. Then the number of terms in this sequence is $\leq d(G)$. Hence the number of vertices of V is at most $d(\bar{G}) \leq d(G)$. ∎

Proposition 2.21. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be a finite graph of pro-p groups, and let D be a maximal subtree of Γ. Suppose G is finitely generated. If $\mathcal{G}(e)$ is finitely generated for every $e \in \Gamma \setminus D$, then $\Pi_1(\mathcal{G}, D)$ is finitely generated with $d(\Pi_1(\mathcal{G}, D)) \leq d(G) + \sum_{e \in \Gamma \setminus D}(d(\mathcal{G}(e)) - 1)$.*

Proof. Since Γ is finite, we can think of G as $G = \mathrm{HNN}(\Pi_1(\mathcal{G}, D), \mathcal{G}(e), t_e)$, $e \in \Gamma \setminus D$.

Let $A = \Pi_1(\mathcal{G}, D)/\Phi(\Pi_1(\mathcal{G}, D))$ and let B be a subgroup generated by the images of $\mathcal{G}(e)$ in A for $e \in \Gamma \setminus D$. Since $\mathcal{G}(e)$ is finitely generated for each $e \in \Gamma \setminus D$, the group B is finite. Then there exists an epimorphism $G = \mathrm{HNN}(\Pi_1(\mathcal{G}, D), \mathcal{G}(e), t_e) \to A/B \oplus \mathbb{F}_p[\Gamma \setminus D]$ that sends $\Pi_1(\mathcal{G}, D)$ to A/B and t_e to e in the vector space $\mathbb{F}_p[\Gamma \setminus D]$. Since $A/B \oplus \mathbb{F}_p[\Gamma \setminus D]$ is finite, A/B is finite, and so A is finite implying that $\Pi_1(\mathcal{G}, D)$ is finitely generated. Since $d(\Pi_1(\mathcal{G}, D)) = d(A) = d(A/B) + d(B)$, $d(G) \geq d(A/B) + |\Gamma \setminus D| = d(A) - d(B) + |\Gamma \setminus D|$, one deduces

$$d(\Pi_1(\mathcal{G}, D)) = d(A) \leq d(G) - |\Gamma \setminus D| + d(B) \leq d(G) + \sum_{e \in \Gamma \setminus D} (d(\mathcal{G}(e)) - 1). \quad \blacksquare$$

2.22. Standard (universal) pro-p tree. Associated with the finite graph of pro-p groups $(\mathcal{G}, \Gamma)$, there is a corresponding *standard pro-p tree* (or universal covering graph) $T = T(G) = \bigcup_{m\in\Gamma} G/\mathcal{G}(m)$ (cf. [27, Proposition 3.8]). The vertices of T are those cosets of the form $g\mathcal{G}(v)$, with $v \in V(\Gamma)$ and $g \in G$; its edges are the cosets of the form $g\mathcal{G}(e)$, with $e \in E(\Gamma)$; and the incidence maps of T are given by the formulas

$$d_0(g\mathcal{G}(e)) = g\mathcal{G}(d_0(e)), \quad d_1(g\mathcal{G}(e)) = gt_e\mathcal{G}(d_1(e)) \quad e \in E(\Gamma),\ t_e = 1 \text{ if } e \in D.$$

There is a natural continuous action of G on T, and clearly $G \setminus T = \Gamma$. Remark also that since Γ is finite, $E(T)$ is compact.

3. Acylindrical accessibility

In this section, we shall prove a pro-p version of Sela's accessibility. Note that Sela used $\mathbb{R}$-trees for the proof; later Weidmann [22, Theorem 4] found another proof using Nielsen method and established a bound. Both methods are not available in the pro-p case.

We shall start with two auxiliary results on free amalgamated product and its generalization for abstract groups.

Lemma 3.1. *Let $G = G_1 *_H G_2$ be a splitting of a group as an amalgamated free product, and $H_1 \leq G_1$, $H_2 \leq G_2$. Then $\langle H_1, H_2\rangle = L_1 *_K L_2$, where $L_1 = \langle H_1, H_2 \cap H\rangle$ and $L_2 = \langle H_2, H_1 \cap H\rangle$ and $K = \langle H_1 \cap H, H_2 \cap H\rangle$. In particular, if $H_1 \cap H \leq U \geq H \cap H_2$ for some normal subgroup U of G, then $L_1 \leq H_1(U \cap G_1)$, $L_2 \leq H_2(U \cap G_2)$, $K \leq H \cap U$.*

Proof. First note that it follows from the Bass–Serre theory [20] that $\langle H_1, H_2\rangle$ is a free amalgamated product whose factors are contained in G_1 and G_2, respectively. To see this, it suffices to consider the Bass–Serre tree T associated to G and denote by e the edge whose vertices have stabilizers G_1 and G_2, respectively. Now one notices that the $\langle H_1, H_2\rangle$-orbit of e in T is connected, and it provides a tree acted on by $\langle H_1, H_2\rangle$ with a single edge orbit.

Therefore, we need to prove that the factors of the splitting are L_1 and L_2, and the amalgamated subgroup is K. To this end, we claim that an element $x \in \langle H_1, H_2\rangle$ has a reduced form $h = x_1x_2\cdots x_n$ with $x_i \in L_1 \cup L_2$. Suppose not, and $x = a_1a_2\cdots a_m$ is an expression as a product of the minimal length of alternating elements from H_1 or H_2 (i.e., if $a_i \in H_1$, then $a_{i+1} \in H_2$) such that a reduced word of it is not of the desired form. Then a reduced word for $a_2\cdots a_n$ has a reduced form $a_2\cdots a_m = l_1\cdots l_k$ with $l_i \in L_1 \cup L_2$.

Recall that $a_1 \in H_i \leq L_i$ for $i = 1$ or 2. Since the word $a_1l_1\cdots l_k$ is not reduced and $l_1\cdots l_k$ is, the reduction happens in a_1l_1 that can occur in the free amalgamated product $G = G_1 *_H G_2$ only if $a_1, l_1 \in H$. In particular, either $a_1 \in H_1 \cap H$ or $a_1 \in H_2 \cap H$, and so $a_1l_1\cdots l_k$ is a reduced word of needed form if a_1 and l_1 belong to different L_i's;

if a_1 and l_1 belong to the same L_i's, then the consolidated word $(a_1 l_1) \cdots l_k$ has entries from $L_1 \cup L_2$ and is reduced. This gives a contradiction.

It remains to prove that $K = \langle H_1 \cap H, H_2 \cap H \rangle$. For $k \in K$, write minimal expressions $k = x_1 \cdots x_n$ and $k = y_1 \cdots y_m$ as alternating products of elements of H_1, $H_2 \cap H$ and H_2, $H_1 \cap H$, respectively. Thus $x_1 \cdots x_n = y_1 \cdots y_m$. If $k \notin \langle H_1 \cap H, H_2 \cap H \rangle$, then there are $x_i, y_j \notin H$ for some i, j, and we can choose i maximal and j minimal with this property. But then the product $y_m^{-1} \cdots y_1^{-1} x_1 \cdots x_m$ cannot be reduced to 1, since y_j^{-1} and x_i cannot be cancelled. ∎

Proposition 3.2. *Let $G = \pi_1(\mathcal{G}, \Gamma)$ be the fundamental group of a finite tree of groups, and let $H_v \leq G(v)$ be a subgroup of $G(v)$ for each $v \in V(\Gamma)$. Then $H = \langle H_v \mid v \in V(\Gamma) \rangle = \pi_1(\mathcal{L}, \Gamma)$ such that $L(v) = \langle H_v, G(e) \cap H_w \rangle$ and $L(e) = \langle H_v \cap G(e), H_w \cap G(e) \rangle$, where e ranges over the edges incident to v, and w is the other vertex of e. In particular, if U is a normal subgroup of G and, for each edge e and its vertex v, one has $H_v \cap G(e) \leq U$, then $L(v) \leq H_v U$ and $L(e) \leq U \cap G(e)$.*

Proof. We use induction on $|\Gamma|$. If Γ has one edge only, the result follows from Lemma 3.1. Let e be an edge of Γ having w as a pending vertex. Then $G = G_1 *_{G_e} G_w$. Let v be the other vertex of e and put $H'_v = \langle H_v, H_w \cap G(e) \rangle$. Let $H_1 = \langle H_u, H'_v \mid u \in V(\Gamma) \setminus \{w\} \rangle$. By the induction hypothesis, $H_1 = \pi_1(\mathcal{L}_1, \Delta)$, with $\Delta = \Gamma \setminus \{e, w\}$ and vertex- and edge-groups satisfying the statement of the proposition. Applying Lemma 3.1, we get $\langle H_1, H_w \rangle = L_1 *_K L(w)$, where $L_1 = \langle H_1, G(w) \cap G(e) \rangle$, $L(w) = \langle H_w, H_1 \cap G(e) \rangle$ and $K = \langle H_1 \cap G(e), H_w \cap G(e) \rangle$. It follows that $H = \langle H_v \mid v \in V(\Gamma) \rangle = \langle H_1, H_w \rangle = \langle H_1, \langle H_u, H_w \cap G(e) \rangle \rangle = \pi_1(\mathcal{L}, \Delta) \amalg_K L(w) = \pi_1(\mathcal{L}, \Gamma)$ with the desired properties. ∎

Lemma 3.3. *Let $G = G_1 \amalg_H G_2$ be a splitting of the pro-p group G as an amalgamated free pro-p product of pro-p groups G_1, G_2 and $H_1 \leq G_1$, $H_2 \leq G_2$ be subgroups such that $H_1 \cap H \leq U \geq H \cap H_2$ for some open normal subgroup U of G. Then $\langle H_1, H_2 \rangle = L_1 \amalg_K L_2$ with $L_1 \leq H_1 U$, $L_2 \leq H_2 U$, $K \leq HU$.*

Proof. By [28, Proposition 4.4], $\langle H_1, H_2 \rangle = L_1 \amalg_K L_2$ with $L_1 \leq G_1$, $L_2 \leq G_2$, $K \leq H$. By Lemma 3.1 combined with §2.11, $L_1 \leq H_1 U$, $L_2 \leq H_2 U$, $K \leq HU$. ∎

Corollary 3.4. *Let $G = G_1 \amalg_H G_2$ be a splitting of the pro-p group G as an amalgamated free pro-p product of pro-p groups G_1, G_2 and $H_1 \leq G_1$, $H_2 \leq G_2$ be subgroups such that $H_1 \cap H = 1 = H_2 \cap H$. Then $\langle H_1, H_2 \rangle = H_1 \amalg H_2$.*

Proof. Since U in the preceding lemma is arbitrary, the result follows. ∎

Proposition 3.5. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental pro-p group of a finite tree of pro-p groups and $H_v \leq G(v)$ for $v \in V(\Gamma)$. Let U be an open normal subgroup of G, and suppose that for each edge e, one has $H_v \cap G(e) \leq U \geq H_w \cap G(e)$. Then $H = \langle H_v \mid v \in V \rangle = \Pi_1(\mathcal{H}, \Gamma)$ such that $H(v) \leq H_v U$ for all $v \in V(\Gamma)$ and $H(e) \leq U \cap \mathcal{G}(e)$ for all $e \in E(\Gamma)$.*

Proof. By [28, Proposition 4.4], $\langle H_v \mid v \in V(\Gamma)\rangle = \Pi_1(\mathcal{H}, \Gamma)$ with $H(m) \leq G(m)$. By Proposition 3.2 combined with §2.11, $H(v) \leq H_v U$ and $H(e) \leq U \cap G(e)$. ■

Corollary 3.6. *Suppose $H_v \cap G(e) = 1$ for all $v \in V(\Gamma)$ and each $e \in E(\Gamma)$. Then $H = \coprod_{v \in V(\Gamma)} H_v$.*

Proof. Since U in Proposition 3.5 is an arbitrary open normal subgroup, $H(v) = H_v$ and $\mathcal{H}(e) = 1$ for each $e \in D$. Hence $H = \coprod_{v \in V(\Gamma)} H_v$ by [14, Example 6.2.3]. ■

Definition 3.7. We say that a profinite graph of pro-p groups $(\mathcal{G}, \Gamma)$ is *k-acylindrical* if the action of the fundamental group $\Pi_1(\mathcal{G}, \Gamma)$ on its standard pro-p tree is k-acylindrical.

Proposition 3.8. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental pro-p group of a k-acylindrical finite graph of pro-p groups. Let v, w be vertices at distance $\geq 2k + 1$. Then*

$$\langle G(v), G(w)\rangle = G(v) \amalg G(w).$$

Proof. Let $[v, w]$ be a shortest path between v and w. Let $G(v, w)$ be the fundamental group of the graph of pro-p groups restricted to $[v, w]$. By Lemma 2.15, $G(v, w)$ is a subgroup of G generated by vertex stabilizers of $[v, w]$. Let e be an edge of $[v, w]$ at distance $> k$ from w and v. Then $G(v) \cap G(e) = 1 = G(w) \cap G(e)$. Note that $G(v, w)$ splits over $G(e)$ as a free amalgamated pro-p product $G(v, w) = G_1 \amalg_{G(e)} G_2$, where G_1, G_2 are pro-p groups generated by vertex-groups of the connected components of $[v, w] \setminus e$ (see Lemma 2.15), so that $G(v) \leq G_1$ and $G(w) \leq G_2$. By Corollary 3.4, $\langle G(v), G(w)\rangle = G(v) \amalg G(w)$ as required. ■

Proposition 3.9. *Suppose $\Gamma = [v, w]$ is a line of pro-p groups such that $G = \Pi_1(\mathcal{G}, \Gamma) = G(v) \amalg G(w)$. Let $(\mathcal{G}, \Gamma_{\mathrm{red}})$ be a reduced graph of pro-p groups obtained from $(\mathcal{G}, \Gamma)$ by the procedure described in Remark 2.18. If Γ_{red} is not a vertex, then one of the following holds:*

(i) *Γ_{red} has only two edges e_1, e_2 with pending vertices v, w and one middle vertex u such that $G(u) = G(e_1) \amalg G(e_2)$;*

(ii) *Γ_{red} has only one edge, a trivial edge-group and $G(v)$, $G(w)$ as vertex-groups.*

Proof. Let U be an open normal subgroup of G and $G_U(v) = G(v)U/U$, $G_U(w) = G(w)U/U$. Let $U(v, w) = (\langle U \cap G(v), U \cap G(w)\rangle)^G$ and $G_U = G/U(v, w) = G_U(v) \amalg G_U(w)$ (cf. Proposition 2.14). Then $G = \varprojlim_U G_U$, where $G_U = \Pi_1(\mathcal{G}_U, \Gamma_{\mathrm{red}})$ and $\mathcal{G}_U = \cup G(m)U(v, w)/U(v, w)$ for every $m \in \Gamma$. Starting with some U, the graph of groups $(\mathcal{G}_U, \Gamma_{\mathrm{red}})$ is reduced, and without loss of generality, we may assume that it is reduced for every U.

Suppose that Γ_{red} has one edge. Then $G_U = G_U(v) \amalg_{G_U(e)} G_U(w)$. It follows that $G_U(e) = 1$ for each U and therefore so is $G(e)$.

Suppose now that Γ_{red} has more than one edge; we shall use induction on the sum $|G_U(v)| + |G_U(w)|$ of the orders of the free factors of $G_U = G_U(v) \amalg G_U(w)$ to show that Γ_{red} satisfies (i) or (ii). Let e_1, e_2 be edges of Γ incident to v and w respectively, and v_1, w_1 the other vertices of e_1 and e_2. Since we have only finitely many vertices in $[v_1, w_1]$, we can apply the pro-p version of the Kurosh subgroup theorem [14, Theorem 9.6.1 (a)], and so

$$\begin{aligned}\Pi_1(\mathcal{G}_U, [v_1, w_1]) &= (G_U(v) \cap \Pi_1(\mathcal{G}_U, [v_1, w_1])) \amalg (\Pi_1(\mathcal{G}_U, [v_1, w_1]) \cap G_U(w)) \amalg L \\ &= G_U(e_1) \amalg G_U(e_2) \amalg L.\end{aligned}$$

Thus, $G_U = G_U(v) \amalg L \amalg G_U(w)$. Hence $L = 1$ and $\Pi_1(\mathcal{G}_U, [v_1, w_1]) = G_U(e_1) \amalg G_U(e_2)$. If $v_1 = w_1$, then we are in case (i). Suppose $v_1 \neq w_1$. By induction hypothesis, $(\mathcal{G}_U, [v_1, w_1])$ satisfy (i) or (ii), and so in either case $G_U(v_1) = G_U(e_1)$ and $G_U(w_1) = G_U(e_2)$. Hence edges e_1 and e_2 are fictitious, a contradiction. Therefore, $v_1 = w_1$. Thus putting $u = v_1 = w_1$, we have $G_U(u) = G_U(e_1) \amalg G_U(e_2)$, and so $G(u) = G(e_1) \amalg G(e_2)$. ∎

Proposition 3.10. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental pro-p group of a k-acylindrical finite tree of pro-p groups. Suppose there exists a subset $V \subset V(\Gamma)$ such that*

(i) $[v, w]$ *has at least $2k + 1$ edges whenever $v \neq w \in V$;*

(ii) $G = \langle G(v) \mid v \in V \rangle$.

Then $G = \coprod_{v \in V} G(v)$.

Proof. For every $v \in V$, we collapse the ball of radius k centered at v to the vertex v itself and consider the collapsed graph of groups Γ' obtained in this way from Remark 2.19. Setting $H_v = G_v$ for $v \in V$ and $H_v = 1$ for $v \notin V$, we achieve premises of Corollary 3.6, since the action is k-acylindrical, deducing from it the result. ∎

Corollary 3.11. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental pro-p group of a reduced k-acylindrical finite line of pro-p groups ($k > 0$). Let V be the minimal subset of $V(\Gamma)$ such that $G = \langle G(v) \mid v \in V \rangle$. If G is finitely generated, then $|E(\Gamma)| \leq 2k|V|$.*

Proof. We just need to show that the distance between two vertices of V is at most $2k$. Suppose on the contrary v, w are vertices of V such that $[v, w]$ has at least $2k + 1$ edges. Let $]v, w[= [v, w] \setminus \{v, w\}$. Collapsing connected components C_v and C_w of $\Gamma \setminus]v, w[$ and considering the collapsed graph of pro-p groups (see Remark 2.19) instead of $(\mathcal{G}, \Gamma)$, we may assume that $\Gamma = [v, w]$. By Proposition 3.8, $G = G(v) \amalg G(w)$. But then Proposition 3.9 forces $[v, w]$ to have at most two edges, a contradiction. ∎

Corollary 3.12. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental group of a proper finite k-acylindrical tree of pro-p groups. Let V be the minimal subset of $V(\Gamma)$ such that $G = \langle G(v) \mid v \in V \rangle$. Suppose there exists a vertex $v \in V$ such that the distance $l(v, w)$ is at least $2k + 1$ for every $w \in V$. Then G splits as a free pro-p product.*

Proof. Divide V as the disjoint union $\{v\} \cup \bigcup_{i=1}^{l} V_i$, where the sets V_i are defined as follows: $V_i = V \cap C_i$, where C_i is a connected component of $\Gamma \setminus B(v, 2k)$ and $B(v, 2k)$ is the ball of radius $2k$ with the centre in v. Denote by Δ_i the span of V_i, and let $G_i = \Pi_1(G, \Delta_i)$ be the fundamental group of a graph of groups restricted to Δ_i. Using Remark 2.19, we can collapse all Δ_i. The obtained graph of groups satisfies premises of Proposition 3.10 and by hypothesis possesses more then one vertex. Hence, by Proposition 3.10, it is a non-trivial free pro-p product. ■

Theorem 3.13. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental group of a finite reduced k-acylindrical graph of pro-p groups. Then $|E(\Gamma)| \leq d(G)(4k+1) - 1$ and $|V(\Gamma)| \leq 4kd(G)$.*

Proof. Suppose G is a non-trivial free pro-p product, i.e., $G = G_1 \coprod G_2$. In this case, one proceeds by induction on $d(G)$. Indeed, $d(G) = d(G_1) + d(G_2)$ and one computes

$$\begin{aligned} E(\Gamma) &= |E(\Gamma_1)| + |E(\Gamma_2)| + 1 \leq d(G_1)(4k+1) - 1 + d(G_2)(4k+1) - 1 + 1 \\ &\leq (d(G_1) + d(G_2))(4k-1) - 1 = d(G)(4k-1) - 1, \\ V(\Gamma) &= |V(\Gamma_1)| + |V(\Gamma_2)| \leq 4kd(G_1) + 4kd(G_2) = 4kd(G). \end{aligned}$$

Thus, we can assume that G does not split as non-trivial free pro-p product. Let D be a maximal subtree of Γ. By [2, Lemma 3.6], there are at most $d(G)$ edges in $\Gamma \setminus D$. Let V be a minimal subset of $V(\Gamma)$ such that $G = \langle G(v), t_e \mid v \in V, e \in \Gamma \setminus D \rangle$. Looking at $G/\Phi(G)$, one easily deduces that $|V| \leq d(G)$. Let V' be the set of vertices connected to vertices of V by an edge $e \in \Gamma \setminus D$. Then $|V \cup V'| \leq 2d(G)$ as follows from presentation of $\Pi_1(\mathcal{G}, D) = \langle G(v) \mid v \in V \cup V' \rangle$; indeed if not, then we can factor out all these $G(v)$'s and get a non-trivial free product $\pi_1(\Gamma) \amalg L$ for some L that contradicts $G = \langle G(v), t_e \mid v \in V, e \in \Gamma \setminus D \rangle$. For every $v \in V \cup V'$, we collapse the ball of radius k centred at v to the vertex v itself, and we consider the tree of groups Γ' obtained in this way from Remark 2.19. Now we are left with a minimal set of vertices of Γ', say $\overline{V}$ (which is the image in Γ' of the set $V \cup V'$). But $\overline{V}$ consists of a single vertex because otherwise, by Corollary 3.12, G is a non-trivial free pro-p product. This means that, for every $v, w \in V \cup V'$, the geodesic $[v, w] \subseteq D$ has length $\leq 2k$ if it does not contain a middle vertex from $V \cup V'$; hence the number of vertices in D is at most $4kd(G)$, and therefore the number of edges of Γ is at most $4kd(G) - 1 + d(G) = d(G)(4k+1) - 1$. ■

Corollary 3.14. *Let G be a free amalgamated pro-p product $G = G_1 \amalg_H G_2$ of coherent pro-p groups over an analytic pro-p group H. If H is malnormal in G_1, then G is coherent.*

Proof. Let K be a finitely generated subgroup of G. Then K acts at most 2-acylindrically on the standard pro-p tree $T(G)$. By Theorem 3.13, K is 2-acylindrically accessible. By [2, Theorem 3.6], $K = \Pi_1(\mathcal{H}, \Gamma)$ is the fundamental group of a finite graph of finite pro-p groups with edge-groups being conjugate to subgroups of H. Hence, for each edge

$e \in \Gamma$, one has $d(\mathcal{H}(e)) \le \mathrm{rank}(H)$, where $\mathrm{rank}(H)$ means the Prüfer rank. Therefore, K is finitely presented (cf. (1)). ∎

Theorem 3.15. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental group of a finite reduced k-acylindrical graph of pro-p groups. If H is a finitely generated pro-p subgroup of G, then H is the fundamental group of a finite graph $(\mathcal{H}, \Lambda)$ where all edge- and vertex-groups of $\mathcal{H}$ are conjugate into edge- and vertex-groups of $\mathcal{G}$, respectively.*

Proof. Let T be the standard pro-p tree associated to the finite graph $(\mathcal{G}, \Lambda)$. Given a finitely generated subgroup H of G, denote by $\mathcal{F}_H$ the family consisting of all subgroups of H which are conjugate to subgroups of edge-groups in $\mathcal{G}$. Since H acts on T, by [2, Corollary 4.4], H splits as the pro-p fundamental group of a reduced finite graph $(\mathcal{H}, \Delta)$ of pro-p groups with edge-groups in $\mathcal{F}_H$. Hence, by Theorem 3.13, H is $\mathcal{F}_H$-accessible and the size of Δ is bounded. ∎

Theorem 3.16. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental group of a finite reduced k-acylindrical graph of pro-p groups with $d(G(e)) \le n$ for each $e \in E(\Gamma)$. Suppose G is finitely generated. Then $|E(\Gamma)| \le (2kn + 1)d(G)$.*

Proof. Let D be a maximal subtree of Γ. By [2, Lemma 3.6], there are at most $d(G)$ edges in $\Gamma \setminus D$. By Proposition 2.21, $d(\Pi_1(\mathcal{G}, D)) \le d(G) + (n-1)d(G) = nd(G)$. By Lemma 2.20, D has at most $nd(G)$ pending vertices in D. Let V be a minimal set of vertices such that $\Pi_1(\mathcal{G}, D) = \langle G(v) \mid v \in V\rangle$. Then $|V| \le d(\Pi_1(\mathcal{G}, D))$ and so, by Corollary 3.11, $|E(D)| \le 2knd(G)$. So $|E(\Gamma)| \le (2kn + 1)d(G)$. ∎

Corollary 3.17. *Suppose all edge-groups are 2-generated and $k = 1$. Then $|E(\Gamma)| \le 5d(G)$.*

We finish the section with a pro-p version of the Karras–Solitar theorem [10, Theorem 6] but we start with the lemma below where generation symbols $\langle\,\rangle$ mean abstract generation, unlike in the rest of the paper where $\langle\,\rangle$ means topological generation.

Lemma 3.18. *Let $G = G_1 \amalg_H G_2$ be a non-fictitious free pro-p product with malnormal amalgamation. Suppose G is 2-generated. Then H is trivial and G_1, G_2 are cyclic.*

Proof. Let $x \in G_1$, $y \in G_2$ such that G is generated by x and y. Consider the abstract subgroup $\langle x, y\rangle$ of the abstract free amalgamated product $G_1 *_H G_2$. By [10, Theorem 6], $\langle x, y\rangle$ is a free product $\langle x\rangle * \langle y\rangle$. By Lemma 3.1, $\langle x\rangle \cap H = 1 = H \cap \langle y\rangle$. Hence $\overline{\langle x\rangle} \cap H = 1 = \overline{\langle y\rangle} \cap H$ and, by Corollary 3.4, $G = \overline{\langle x\rangle} \amalg \overline{\langle y\rangle}$. Thus the result follows from Proposition 3.10 (ii). ∎

Theorem 3.19. *Let $G = G_1 \amalg_H G_2$ be a free pro-p product with malnormal amalgamation, and let K be a 2-generated subgroup of G. If K is not conjugate to a subgroup of G_1 or G_2, then K is a free pro-p product of two cyclic groups.*

Proof. Consider the action of K on the standard pro-p tree $T(G)$. Then the action is acylindrical. We assume that K does not stabilize a vertex (if it does it is conjugate into G_1 or G_2). Suppose first that K is generated by vertex stabilizers. By [2, Theorem 4.2, Case 1 of the proof], there exists a non-trivial splitting $K = K_1 \amalg_{K_e} K_2$ as a free pro-p product with amalgamation over an edge stabilizer. Then the result follows from Corollary 3.18.

Suppose now K is not generated by vertex stabilizers. By [2, Theorem 4.2, Case 2 of the proof], there exists a non-trivial splitting $K = \mathrm{HNN}(L, K_e, t)$ as a pro-p HNN-extension over an edge stabilizer. Note that $K = \langle x, t\rangle$, where $x \in G_v$, for some $v \in V(T)$, and t is a stable letter that is not conjugate into $G_1 \cup G_2$. Then from the acylindricity of the action, we deduce that x and x^t cannot stabilize the same vertex of T. So, by [2, Theorem 4.2, Case 1 of the proof], $R = \langle x, x^t\rangle = R_1 \amalg_{R_e} R_2$, and every vertex-group of R belongs either to R_1 or R_2 up to conjugation. It follows that x and x^t belong to different factors. Then, by Corollary 3.18, $R = \langle x\rangle \amalg \langle x^t\rangle$ is a free pro-p product. It follows that $K = \langle x\rangle \amalg \langle t\rangle$ as needed. ∎

4. Decomposing PD^n pro-p groups

4.1. Pro-p PD^n-pairs

In [25], Wilkes defined the profinite version of group pairs but we shall need only a simple version of it. A pro-p group pair $(G, \mathcal{S})$ consists of a pro-p group G and of a finite family $\mathcal{S}$ of closed subgroups S_x of G indexed over a set (we allow repetitions in this family). Given a closed subgroup H of G, let $\mathcal{S}^H$ denote the family of subgroups

$$\{H \cap \sigma(y) S_x \sigma(y)^{-1} \mid x \in X,\ y \in H \backslash G / S_x\}, \tag{3}$$

indexed over

$$H \backslash G/\mathcal{S} := \bigsqcup_{x \in X} H \backslash G/S_x,$$

where $\sigma\colon G/H \to G$ is a section[2] of the quotient map $G \to G/H$.

In [25], the author develops the theory of the cohomology of a profinite group relative to a collection of closed subgroups and defines profinite Poincaré duality pairs (or PD^n-pairs for short), and the reader is referred to [25, Section 5] for rigorous definitions and basic results. A pro-p group pair $(G, \mathcal{S})$ is a pro-p PD^n-pair, for some $n \in \mathbb{N}$, if the double of G over the groups in $\mathcal{S}$ is a pro-p PD^n-group. Here the double of G over $\mathcal{S}$ refers to the fundamental group of a graph of groups with two vertices and $|\mathcal{S}|$ edges, where a copy of G is over each vertex and groups of $\mathcal{S}$ are over the edges, with natural boundary maps.

[2] A different section only affects the family $\mathcal{S}^H$ by changing its members by conjugacy in H.

Example 4.1. Let G be a PD^n pro-p group isomorphic to the fundamental group of a reduced proper finite graph of pro-p groups $(\mathcal{G}, \Gamma)$ whose edge-groups are PD^{n-1} subgroups of G. For each vertex $v \in V(\Gamma)$, denote by $\mathcal{E}_v$ the collection of all the subgroups of $G(v)$ which are images $\partial_i(G(e))$ of those edge-groups such that $d_i(e) = v$. Then $(G(v), \mathcal{E}_v)$ is a pro-p PD^n-pair by [25, Theorem 5.18 (2)] for $\mathcal{S} = \emptyset$.

We say that a pro-p PD^n-pair $(G, \mathcal{S})$ splits as an amalgamated free pro-p product $G = G_1 \amalg_H G_2$ (resp. as HNN-extension HNN(G_1, H, t)) if each S_i is conjugate to either G_1 or G_2 (resp. G_1).

The following proposition was communicated to us by Gareth Wilkes.

Proposition 4.2 (Wilkes). *Let $(G, \mathcal{S})$ be a pro-p PD^n-pair with $\mathcal{S} = \{S_1, \dots, S_n\}$. Then, for every $i = 1, \dots, n$, $(G, \mathcal{S})$ does not split over S_i as a pair.*

The proof relies on the following.

Lemma 4.3. *Let G be a pro-p group such that $(G, \mathcal{S})$ is a PD^n-pair. Suppose $S_1 = S_2$. Then $m = 2$ and $S_1 = S_2 = G$.*

Proof. By [25, Theorem 5.17 (1)], the pro-p HNN-extension $\widetilde{G} = \mathrm{HNN}(G, S_1 = S_2, t)$ with $s^t = s$ for $s \in S_1$ is a PD^n-pair relative to the collection $\{S_3, \dots, S_n\}$. Since $\widetilde{G}$ contains the pro-p PD^n-group $S_1 \times \langle t \rangle$ (cf. [25, Proposition 5.9]), one has $\mathrm{cd}_p(\widetilde{G}) = n$. By [25, Corollary 5.8], $\{S_3, \dots, S_n\}$ is empty and $m = 2$.

If $G \neq S_1$, take an open subgroup U containing S_1. If $\mathcal{S}^U$ is the collection defined in (3), then $(U, \mathcal{S}^U)$ is a PD^n-pair (see the proof of [25, Proposition 5.11]). But $|\mathcal{S}^U| = 2|U \setminus G/S_1| > 2$ and $\{S_1, S_2\} \subset \mathcal{S}^U$, contradicting the first part. ■

Proof of Proposition 4.2. Suppose by contradiction that $(G, \mathcal{S})$ does split over some S_i. Assume without loss of generality $i = 1$. Up to changing $\mathcal{S}$ by conjugacy, G is either isomorphic to HNN(G_1, S_1, t), with $S_k \leq G_1$ for every $k = 1, \dots, n$, or isomorphic to $G_1 \amalg_H G_2$ with $S_k \leq G_1$ or $S_k \leq G_2$ for every $k = 1, \dots, n$. In the latter case, $\mathcal{S}$ can be decomposed as $\mathcal{S}_1 \sqcup \mathcal{S}_2$, where each $\mathcal{S}_j$, $j = 1, 2$, contains only elements from $\mathcal{S}$ which are also subgroups of G_j. Assume without loss of generality that $S_1 \in \mathcal{S}_1$. Then by [25, Theorem 5.16 (2)] for $G \cong G_1 \amalg_H G_2$ and by [25, Theorem 5.17 (2)] if $G \cong$ HNN(G_1, H, t) the pair $(G_1, \mathcal{S} \sqcup \{H\})$ is a PD^n-pair which contradicts Lemma 4.3. ■

4.2. Splitting over polycyclic subgroups

Here we collect some results that will be used later in the proof of the main theorem.

We say that a pro-p group G admits a *k-acylindrical splitting* if G is isomorphic to the fundamental pro-p group $\Pi_1(\mathcal{G}, \Gamma)$ of a k-acylindrical proper reduced finite graph of pro-p groups.

Proposition 4.4. *Let G be a pro-p PD^n-group which is the fundamental group $\Pi_1(\mathcal{G}, \Gamma)$ of a finite reduced graph of pro-p groups with PD^{n-1} edge-subgroup of G. Then the stabilizers of two adjacent edges of T are not commensurable.*

Proof. We just need to show that two adjacent edge-groups $G(e_1)$, $G(e_2)$ do not intersect by a subgroup of finite index. Suppose they do. Then there exists an open subgroup U of G such that $U \cap G(e_1) = G(e_1) \cap G(e_2) = U \cap G(e_2)$. So replacing G by U, we may assume that $G(e_1) = G(e_2)$. Let v be their common vertex. By Example 4.1, $(G(v), \mathcal{E}_v)$ is a pro-p PD^n pair with $G(e_1), G(e_2) \in \mathcal{E}_v$ contradicting Lemma 4.3. ∎

The following theorem establishes a JSJ-decomposition for PD^n pro-p groups analogous to one from [12, Theorem A2].

Theorem 4.5. *For every* PD^n *pro-p group G, $n > 2$, there exists a (possibly trivial) k-acylindrical pro-p G-tree $\mathcal{T}$ satisfying the following properties:*

(i) *every edge stabilizer is a maximal polycyclic subgroup of G of Hirsch length $n-1$;*

(ii) *a polycyclic subgroup of G of Hirsch length* > 1 *stabilizes a vertex;*

(iii) *the underlying graph of groups does not split further k-acylindrically over a polycyclic subgroup of G of Hirsch length $n-1$.*

Moreover, every two pro-p G-trees satisfying the properties above are G-isomorphic.

Proof. By Theorem 3.16, a k-acylindrical decomposition as fundamental group of a reduced finite graph of pro-p groups $(\mathcal{G}, \Lambda)$ with polycyclic subgroup of G of Hirsch length $n-1$ has a bound, so we can choose one with a maximal number of edge-groups. In particular, the edge-groups satisfy property (i).

By Lemma 2.8, the standard pro-p tree $\mathcal{T}$ also satisfies property (ii). We shall show now property (iii).

First notice that the vertex-stabilizers of $\mathcal{T}$ cannot decompose k-acylindrically over polycyclic subgroups of Hirsch length > 1 at all. Indeed, suppose on the contrary that some vertex-group $G(v)$ of $(\mathcal{G}, \Lambda)$ splits k-acylindrically either as $G_0 \amalg_A G_1$ or $\mathrm{HNN}(G_0, A, t)$, where A is polycyclic of Hirsch length > 1. Then, by Lemma 2.8, the edge-groups of all adjacent edges to $G(v)$ are conjugate into either G_0 or G_1. Denote by E_i the set of edges in $\mathrm{star}_\Lambda(v)$, whose edge-group is conjugate into G_i with $i = 0, 1$. Thus we can replace the vertex v by an edge e with two vertices v_1 and v_2, connecting the edges $e_i \in E_i$ to v_i, together with boundary maps $\partial_i \colon G(e_i) \to G(v_i)$ given by correspondent conjugation for every $e_i \in E_i$. Note that the construction of this map is continuous because $\mathrm{star}_\Lambda(v)$ is finite. This contradicts the maximality of the decomposition.

Given any two pro-p trees $\mathcal{T}$ and $\overline{\mathcal{T}}$ satisfying the properties (i)–(iii), we claim that there exists a G-equivariant morphism $\phi \colon \mathcal{T} \to \overline{\mathcal{T}}$. Let us prove the claim. In order to construct ϕ, we need to map G-equivariantly each edge e of $\mathcal{T}$ to an edge $\overline{e}$ of $\overline{\mathcal{T}}$. Let $v = d_0(e)$ and $w = d_1(e)$. Therefore, there exist vertices $\overline{v}$ and $\overline{w}$ such that $G_v \subseteq G_{\overline{v}}$ and $G_w \subseteq G_{\overline{w}}$. Hence it suffices to prove that $\overline{v}$ and $\overline{w}$ are at distance 1 in the tree $\overline{\mathcal{T}}$ and set $\phi(e) = \overline{e}$, where $\overline{e}$ denotes the edge connecting $\overline{v}$ to $\overline{w}$. By Proposition 4.4 , edge-groups of distinct edges in Λ are not commensurable. Therefore, one sees that $G(v_1) \cap G(v_2)$ is a polycyclic subgroup of G of Hirsch length $n-1$ that implies adjacency. ∎

The uniqueness of $\mathcal{T}$ in Theorem 1.2 induces an action on it by the automorphism group $\mathrm{Aut}(G)$. This gives a splitting structure on $\mathrm{Aut}(G)$ if $\mathcal{T}$ is non-trivial. We state this as the following assertion.

Corollary 4.6. *The automorphism group* $\mathrm{Aut}(G)$ *acts on* $\mathcal{T}$. *Moreover, if* $\mathcal{T}$ *is not a vertex, then* $\mathrm{Aut}(G)$ *splits as non-trivial amalgamated free pro-*p *product or pro-*p HNN-*extension.*

5. Example

Theorem 5.1. *Let* G *be an abstract* PD^n *group and* H *its* PD^{n-1} *subgroup.*

(i) ([13, Theorem B]) *Suppose that* $\mathrm{cd}(H \cap H^g) \neq n-2$ *for each* $g \in G$. *Then* G *splits as an amalgamated free product or* HNN-*extension over a group commensurable with* H.

(ii) ([13, Theorem C]) *Suppose* H *is polycyclic. Then some finite index subgroup of* G *splits as an amalgamated free product or* HNN-*extension over a group commensurable with* H.

The following example shows that both Kropholler–Roller theorems do not hold in the pro-p case, i.e., neither of the statements of the theorems above.

Example 5.2. Let G be an open pro-p subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$, say the first principal congruence pro-p subgroup if $p > 2$, and the second principal congruence pro-p subgroup if $p = 2$. Let $H = B \cap G$ be the intersection of the Borel subgroup B of $\mathrm{SL}_2(\mathbb{Z}_p)$ with G. Then H is a maximal metacyclic subgroup of G and therefore is a PD^2 pro-p group. Moreover, H is malnormal. Indeed, the group of upper unipotent matrices is a normal subgroup of H which is isomorphic to $\mathbb{Z}_p$ on which a subgroup of diagonal matrices $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ acts as multiplication by t^2; recalling that the group of units of $\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_p \times C_{p-1}$ (if $p > 2$) and to $\mathbb{Z}_2 \times C_2$ (if $p = 2$), we see that H is metacyclic, say $H = U \rtimes T$, where U consists of unipotent elements and T consists of diagonal elements.

To see that H is malnormal in G, consider $A = H \cap H^g$ for some $g \in G \setminus B$. First observe that a straightforward calculation shows that for the unipotent upper triangle group U, one has $U \cap U^g = 1$ for $g \notin B$. Now if $B \cap B^g$ intersects U non-trivially, then this intersection is cyclic, since otherwise it is open in B contradicting the preceding sentence. Therefore, it is normal in $\langle B, g \rangle$ since it is normal in both B and B^g (cf. [14, Lemma 15.2.1 (a)]); so $B \cap B^g \cap U \leq U^g \cap U = 1$. It follows that $B \cap B^g$ is generated by a semisimple element s and, as it is not scalar (the scalars have order 2), its eigenvalues are disjoint t, t^{-1}. This matrix s has two 1-dimensional eigen submodules of $\mathbb{Z}_p \oplus \mathbb{Z}_p$: V_t associated with eigenvalue t and $V_{t^{-1}}$ associated with eigenvalue t^{-1}. Hence $V_t \cap V_{t^{-1}} = 0$. Note that if $v \in V_t$, then $v/p \in \mathbb{Z}_p \oplus \mathbb{Z}_p$ implies $v/p \in V_t$. This means that $V_t/p \neq V_{t^{-1}}/p$ in $\mathbb{F}_p \oplus \mathbb{F}_p$. But g is trivial modulo p and so $gV_t/p = V_{t^{-1}}/p$, a contradiction.

The group G is an analytic pro-p group of dimension 3 and so is a PD^3 pro-p group. It has no non-abelian pro-p subgroups, and it is not soluble. So by [15, Theorems 4.7 and 4.8], it does not split as an amalgamated free pro-p product or HNN-extension.

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