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(Article begins on next page)

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SUBGROUPS OF PRO-p PD^3-GROUPS

I. CASTELLANO AND P. ZALESSKII

Abstract. We study 3-dimensional Poincaré duality pro-p groups in the spirit of the work by Robert Bieri and Jonathan Hillmann, and show that if such a pro-p group G has a nontrivial finitely presented subnormal subgroup of infinite index, then either the subgroup is cyclic and normal, or the subgroup is cyclic and the group is polycyclic, or the subgroup is Demushkin and normal in an open subgroup of G.

Also, we describe the centralizers of finitely generated subgroups of 3-dimensional Poincaré duality pro-p groups.

In algebraic topology, Poincaré duality theorem expresses the symmetry between the homology and cohomology of closed orientable manifolds. The notion of a Poincaré duality group of dimension n (or PD^n-group, for short) originates as a purely algebraic analogue of the notion of an n-manifold group, that is, the fundamental group of an aspherical n-manifold. For dimension $n = 1$ or 2, the modelling of n-manifold groups by PD^n-groups is precise: the only such Poincaré duality groups are $\mathbb{Z}$ and surface groups. In low dimension, the critical case is $n = 3$; whether every PD^3-group is a 3-manifold group is still open and represents the main problem in this area.

The notion of a duality group carries over to the realm of profinite groups but the literature has independently developed two definitions of a profinite PD^n-group G at a prime p whose equivalence is uncertain. The definitions differ in that one requires the profinite group G to be of type FP$_\infty$ over $\mathbb{Z}_p$ and the other does not. For pro-p groups the two definitions coincide and the theory becomes more amenable. The cyclic group $\mathbb{Z}_p$ is the only pro-p PD^1-group, whereas the pro-p PD^2-groups are the Demushkin groups.

In this article we focus on pro-p PD^3-groups and we start an investigation of their subgroups in the spirit of the work by Robert Bieri and Jonathan Hillmann [2, 6], who addressed subgroup structure questions on PD^3-groups motivated by considerations from 3-manifold topology (see also [7] and references there). Typically, the existence of a subgroup satisfying some prescribed properties gives rather strong consequences for the structure of the

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PD^3-group itself. For instance, for pro-p PD^3-groups we prove the following result (see Theorem 4.2):

Theorem. *Let G be a pro-p PD^3-group which has a nontrivial finitely presented subnormal subgroup H of infinite index. Then one of the following holds:*

1. *H is Demushkin and it is normal in an open subgroup U of G such that $U/H \cong \mathbb{Z}_p$;*
2. *H is cyclic and G is virtually polycyclic;*
3. *H is cyclic and normal in G with G/H virtually Demushkin.*

The result is the pro-p analogue of Bieri and Hillman's Theorem proved in [2]. The structure of the proof we provide has much in common with the classical result: we discuss all possible cases determined by the cohomological dimension of the subnormal subgroup H. But each case needs arguments specific to pro-p groups. For example, some of those arguments relies on well-known results on profinite PD^n-groups of [4, 17].

Following the approach of [6], we also study centralizers of subgroups of pro-p PD^3-groups. We produce the description below which must account additional cases (especially for pro-2 groups) that do not occur in the discrete case (see Theorem 3.1):

Theorem. *Let G be a pro-p PD^3-group and $H \neq 1$ is a finitely generated subgroup of G with $C_G(H) \neq 1$. Then one of the following holds:*

1. *$H \cong \mathbb{Z}_p$ and $C_G(H)/H$ is virtually Demushkin;*
2. *H is a non-abelian Demushkin group and $C_G(H) \cong \mathbb{Z}_p$,;*
3. *$H \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $C_G(H) \cong \mathbb{Z}_p^3$ and G is virtually $\mathbb{Z}_p^3$;*
4. *$H \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_2$ is generalized dihedral pro-2 group, $C_G(H) \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_2$ and $G \cong \mathbb{Z}_2 \times (\mathbb{Z}_2 \rtimes \mathbb{Z}_2)$;*
5. *H and $C_G(H)$ are free pro-p groups;*
6. *H and $C_G(H)$ are polycyclic;*
7. *$H \cong \mathbb{Z}_p$ and $C_G(H)/H$ is virtually free pro-p.*
8. *H is cyclic by virtually free and $C_G(H) \cong \mathbb{Z}_p$.*

Notations. Morphisms of topological groups are assumed to be continuous and subgroups are closed. Therefore, we simply write $H \leq G$ for a closed subgroup H of G, whereas $K \leq_o G$ denote an open subgroup K of G. Similarly, we use $H \trianglelefteq G$ and $K \trianglelefteq_o G$ for normal H and K.

In presence of either a topological module or a topological group, the term finitely generated (resp. presented) indicates the property of being topologically finitely generated (resp. presented).

1. Preliminaries

1.1. **Cohomological dimension of pro-p groups.** For an arbitrary prime p and a pro-p group G, the *cohomological dimension of G* can be defined as

$$\mathrm{cd}(G) = \sup\{n \in \mathbb{N} \mid H^n(G, \mathbb{F}_p) \neq 0\}.$$

One has the following well-known result.

Proposition 1.1 ([11, Section 7])**.** *For a pro-p group G one has*

(1) $\mathrm{cd}(G) \leq n \in \mathbb{N}$ *if and only if* $H^{n+1}(G, \mathbb{F}_p) = 0$.
(2) $\mathrm{cd}(G) = n$ *implies* $H^n(G, M) \neq 0$ *for every finite G-module M.*
(3) $\mathrm{cd}(G) = 1$ *if and only if G is free of rank at least 1.*

The *virtual cohomological dimension of G* is defined by

$$\mathrm{vcd}(G) = \inf\{\mathrm{cd}(H) \mid H \leq_o G\}.$$

Proposition 1.2 ([14])**.** *For a torsion-free pro-p group G,* $\mathrm{vcd}(G) = \mathrm{cd}(G)$. *In particular, every torsion free virtually free pro-p group is free pro-p.*

Next we state known results on pro-p groups of finite cohomological dimension frequently used in the paper.

Proposition 1.3 ([17, Theorem 1.1])**.** *Let G be a pro-p group of finite cohomological dimension n and let N be a closed normal subgroup of G of cohomological dimension k such that the order of $H^k(N, \mathbb{F}_p)$ is finite.Then* $\mathrm{vcd}(G/N) = n - k$.

We shall often use the following well-known Schur's theorem (see, for example, [9, Theorem 8.19]).

Lemma 1.4. *Let G be a group having center of finite index. Then the commutator $[G, G]$ is finite.*

The following is the pro-p analogue of [1, Theorem 8.8].

Corollary 1.5. *Let G be a pro-p group of finite cohomological dimension n. If the center $\mathcal{Z}(G)$ has cohomological dimension n, then G is abelian.*

Proof. Since $\mathcal{Z}(G)$ is a torsion-free abelian pro-p group, $\mathcal{Z}(G)$ is a free abelian pro-p group of rank $= n$ (see [11, Theorem 4.3.4]). By Proposition 1.3, $\mathcal{Z}(G)$ has finite index in G and so the commutator $\overline{[G, G]}$ is finite by Lemma 1.4. Since G is torsion-free, $\overline{[G, G]} = 1$ and G is abelian. □

1.2. **Pro-p PD^n-groups.** The pro-p group G is called a *pro-p Poincaré duality group of dimension n* (or pro-p PD^n-group, for short) if the following properties are satisfied:

(D1) $\mathrm{cd}(G) = n$,
(D2) $|H^k(G, \mathbb{F}_p)|$ is finite for all $k \leq n$,
(D3) $H^k(G, \mathbb{F}_p[[G]]) = 0$ for $1 \leq k \leq n-1$, and
(D4) $H^n(G, \mathbb{F}_p[[G]]) \cong \mathbb{F}_p$.

All pro-p PD^1-groups are infinite cyclic and all pro-p PD^2-groups are Demushkin. The Demushkin groups D are one-relator pro-p groups of cohomological dimension 2 such that $H^2(D, \mathbb{F}_p) \cong \mathbb{F}_p$ and the cup-product

$$\cup\colon H^1(D, \mathbb{F}_p) \times H^1(D, \mathbb{F}_p) \to H^2(D, \mathbb{F}_p)$$

is a non-singular bilinear form.

We end the preliminaries by collecting several well-known results on pro-p PD^n-groups that will be used further on.

Proposition 1.6 ([16, Proposition 4.4.1])**.** *Let G be a pro-p group of finite cohomological dimension n and U an open subgroup. Then G is a pro-p PD^n-group if and only if U is a pro-p PD^n-group.*

Note that the previous result is stated for pro-p groups but it holds in general for profinite groups.

Proposition 1.7 ([14, I.§4, Exercise 5(b)])**.** *A subgroup of infinite index in a pro-p PD^n-group has cohomological dimension at most $n-1$.*

Proposition 1.8 ([4, Corollary 1.5])**.** *Let G be a pro-p PD^3-group and N a non-trivial finitely generated normal subgroup of G of infinite index. Then either N is infinite cyclic and G/N is virtually Demushkin or N is Demushkin and G/N is virtually infinite cyclic.*

We shall need also the following

Proposition 1.9. *Let $H \neq 1$ be a finitely presented subgroup of a pro-p PD^3-group G. If $\mathrm{cd}(H) = \mathrm{cd}(N_G(H))$, then H is open in $N_G(H)$.*

Proof. For $\mathrm{cd}(H) = 3$ the result follows by Proposition 1.7. If $\mathrm{cd}(H) < 3$, then H is open in $N_G(H)$ by Proposition 1.3. □

2. Polycyclic pro-p groups

A pro-p group G is called *polycyclic* if there is a finite series of closed normal subgroups of G

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G \tag{1}$$

such that each factor group G_i/G_{i-1} is cyclic for $i = 1, \ldots, n$.

Let G be a polycyclic pro-p group. The *Hirsch length* of G, denoted by $h(G)$, is the number of factors G_i/G_{i-1} in the series (1) that are isomorphic to $\mathbb{Z}_p$. Clearly, this number is independent on the choice of the series.

Many results on abstract polycyclic groups find their analogue in the pro-p world and the proofs often carry over up to minimal adjustments (see [13] for the abstract case).

Lemma 2.1. *Let G be a pro-p group, H a finitely generated normal subgroup of G and $K \trianglelefteq_o H \trianglelefteq G$. Then the normal core K^G is open in K.*

Proof. If $g \in G$, then $K^g \trianglelefteq_o H$ and $H/K^g \cong H/K$. There are finitely many morphisms of H onto H/K, since there are at most $|H/K|$ possible images for each of the finitely many generators of H. Therefore, there are finitely many distinct groups among the K^g as g runs through G; call them $K_1, \ldots, K_n$. Clearly, $K^G = K_1 \cap \ldots \cap K_n$ is open in K and normal in G. □

Remark 2.2. The proof above also shows that the normalizer $N_G(K)$ has finite index in G whenever H is finitely generated and $K \trianglelefteq_o H \trianglelefteq G$ for some pro-p group G.

Proposition 2.3. *Every polycyclic pro-p group G with $h(G) \geq 1$ is poly-$\mathbb{Z}_p$ by finite.*

Proof. Suppose that G has a finite series of subgroups $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ with cyclic factors G_i/G_{i-1}. We need to prove that G has a poly-$\mathbb{Z}_p$ subgroup that is normal and open. If $n = 1$ there is nothing to prove. For $n > 1$ we proceed by induction. Suppose that G_{n-1} has a poly-$\mathbb{Z}_p$ subgroup N which is normal and open. If G/G_{n-1} is finite, then the core N^G is a poly-$\mathbb{Z}_p$ subgroup of G that is normal and open. If G/G_{n-1} is infinite, we apply Lemma 2.1 with N for K and G_{n-1} for H. There exists an open subgroup $U \trianglelefteq_o N \trianglelefteq_o G_{n-1}$ such that $U \trianglelefteq G$. In particular, G/U is virtually $\mathbb{Z}_p$, i.e., there is a finite index subgroup K/U of G/U such that $K/U \cong \mathbb{Z}_p$. Thus, K is poly-$\mathbb{Z}_p$ because U is poly-$\mathbb{Z}_p$ and $K/U \cong \mathbb{Z}_p$. □

By a combination of the last paragraph of [3] and [18, Theorem 1.3], one obtains the following list of isomorphism types for torsion-free virtually abelian pro-p groups.

Proposition 2.4. *Every virtually abelian pro-p group of cohomological dimension 3 has one of the following isomorphism types:*

(i) for $p > 3$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$;
(ii) for $p = 3$, in addition to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, one has a torsion-free extension of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ by C_3;
(iii) for $p = 2$, in addition to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, one has a torsion-free extension of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by one of the following finite 2-groups: $C_2, C_4, C_8, D_2, D_4, D_8, Q_{16}$.

Proposition 2.5. *Let G be a pro-p PD^3-group such that $\mathcal{Z}(G)$ is not cyclic. If $p > 2$, then G is abelian. If $p = 2$, then $G \cong \mathbb{Z}_2 \times (\mathbb{Z}_2 \rtimes \mathbb{Z}_2)$.*

Proof. Finitely generated torsion-free abelian pro-p groups are free abelian pro-p of finite rank, then the rank of $\mathcal{Z}(G)$ equals either 2 or 3.

If $\mathcal{Z}(G) \cong \mathbb{Z}_p^3$, Proposition 1.7 implies that $\mathcal{Z}(G)$ has finite index in G and the commutator is finite by Lemma 1.4. For G is torsion-free, G is abelian.

Suppose $\mathcal{Z}(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Since G is torsion free, it is easy to check that $Z(G/Z(G))$ is torsion free. By Proposition 1.8, the quotient $G/\mathcal{Z}(G)$ is virtually $\mathbb{Z}_p$, i.e. $G/\mathcal{Z}(G) \cong \mathbb{Z}_p$ for $p > 2$, and either $\mathbb{Z}_2$ or $\mathbb{Z}_2 \rtimes C_2$ for $p = 2$ (since $\mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_p \times C_{p-1}$ or $\mathbb{Z}_2 \times C_2$). But center by cyclic group is abelian so $p = 2$ and $G/\mathcal{Z}(G) \cong \mathbb{Z}_2 \rtimes C_2$ is infinite dihedral. Thus G is an extension of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}_2 \rtimes C_2$ and easy computations yield $G \cong \mathbb{Z}_2 \times (\mathbb{Z}_2 \rtimes \mathbb{Z}_2)$. □

We finish the section with the description of torsion-free polycyclic subgroups of Hirsch length 3. They are exactly polycyclic pro-p PD^3-groups.

Theorem 2.6. *Let G be a torsion-free polycyclic pro-p group of Hirsch length 3. Then one of the following holds:*

(i) G is virtually abelian and so described in Proposition 2.4;

(ii) $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ *for* $p > 2$ *and has a subgroup of index 2 isomorphic to* $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ *if* $p = 2$.

Proof. Let A be a maximal normal abelian subgroup of G. Then $C_G(A)$ is the kernel of the homomorphism $G \longrightarrow \mathrm{Aut}(A)$. If A is cyclic, then $\mathrm{Aut}(A)$ is virtually cyclic; therefore $C_G(A)$ has Hirsch length ≥ 2 and so is virtually abelian of rank 2, a contradiction.

If $rank(A) = 3$, we are in case (i). Suppose $rank(A) = 2$. Then G/A is virtually cyclic and maximality of A together with Proposition 2.4 implies that $\mathbb{Z}(G/\mathbb{Z}(G))$ is torsion free. Therefore, for $p > 2$, $G/A \cong \mathbb{Z}_p$ and $G \cong A \rtimes \mathbb{Z}_p$. For $p = 2$, the quotient group G/A can also be infinite dihedral and contain an infinite cyclic subgroup of index 2 whose inverse image in G has the required structure. This finishes the proof. □

Corollary 2.7. *Let G be a torsion-free polycyclic pro-p group of Hirsch length 3. If $p > 3$ then $G = \mathbb{Z}_p \times \mathbb{Z}_p \rtimes \mathbb{Z}_p$.*

3. Centralizers in pro-p PD^3 groups

Theorem 3.1. *Let G be a pro-p PD^3-group and $H \neq 1$ is a finitely generated subgroup of G with $C_G(H) \neq 1$. Then one of the following holds:*

(1) $H \cong \mathbb{Z}_p$ and $C_G(H)/H$ is virtually Demushkin;
(2) H is a non-abelian Demushkin group and $C_G(H) \cong \mathbb{Z}_p$,;
(3) $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $C_G(H) \cong \mathbb{Z}_p^3$ and G is virtually $\mathbb{Z}_p^3$;
(4) $H \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_2$ is generalized dihedral pro-2 group, $C_G(H) \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_2$ and $G \cong \mathbb{Z}_2 \times (\mathbb{Z}_2 \rtimes \mathbb{Z}_2)$;
(5) H and $C_G(H)$ are non-abelian free pro-p groups;
(6) H and $C_G(H)$ are polycyclic;
(7) $H \cong \mathbb{Z}_p$ and $C_G(H)/H$ is virtually free pro-p.
(8) H is cyclic by virtually free and $C_G(H) \cong \mathbb{Z}_p$.
(9) $C_G(H)$ is cyclic, H is open and G is cyclic by virtually Demushkin.

Proof. We distinguish three cases each of which has several subcases.

Case 1: Suppose $HC_G(H) \trianglelefteq_o G$. By Proposition 1.6, $HC_G(H)$ is a pro-p PD^3-group satisfying the conclusions of Proposition 1.8 for H finitely generated and normal. Then either $H \cong \mathbb{Z}_p$ or H is Demushkin or H is open PD^3. For $H \cong \mathbb{Z}_p$, $C_G(H) = HC_G(H)$ and $C_G(H)/H$ is virtually Demushkin by Proposition 1.8 again (i.e. (1) holds). If H is Demushkin with trivial center, then $HC_G(H) \cong H \times C_G(H)$ and $C_G(H) \cong \mathbb{Z}_p$ (see Proposition 1.2) and so (2) holds. Demushkin groups with non-trivial center are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ or to the Klein bottle pro-2 group $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$. For $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $C_G(H) = HC_G(H)$ and for $p > 2$ is central extension of H by $\mathbb{Z}_p$ and so $C_G(H) \cong \mathbb{Z}_p^3$, i.e. (3) holds. If $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $H \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_2$, then $C_G(H)$ contains $\mathbb{Z}_2 \times \mathbb{Z}_2$ and so, by Theorem 2.6, G is virtually $\mathbb{Z}_2 \times (\mathbb{Z}_2 \rtimes \mathbb{Z}_2)$. Thus (4) holds.

Finally, if H is open in G then the center $\mathcal{Z}(H^G)$ of the normal core of H is non-trivial and normal in G. Then we have either (6) or (9) by Proposition 1.8.

Case 2: Suppose $\mathrm{cd}(HC_G(H)) = 2$. If $\mathcal{Z}(H) = 1$, then $HC_G(H) \cong H \times C_G(H)$ and both H and $C_G(H)$ are free pro-p so (5) holds.

Assume $\mathcal{Z}(H)$ is non-trivial. Then by Corollary 1.5 either $HC_G(H)$ is abelian (hence $HC_G(H) = C_G(H)$ and we have case (6)) or $\mathcal{Z}(H) \cong \mathbb{Z}_p$. In the latter case, by Proposition 1.3, the quotient $HC_G(H)/\mathcal{Z}(H)$ is virtually free and $H/\mathcal{Z}(H)$ is either open in $HC_G(H)/\mathcal{Z}(H)$ or finite. So either $\mathcal{Z}(H) \trianglelefteq_o H$ or $H \trianglelefteq_o HC_G(H)$.

In the first case $H = \mathcal{Z}(H) \cong \mathbb{Z}_p$. So $HC_G(H) = C_G(H)$ which is $\mathbb{Z}_p$ by virtually free by Proposition 1.3, i.e. (7) holds.

In the second case ($H \trianglelefteq_o HC_G(H)$) we have $\mathcal{Z}(H) \trianglelefteq_o C_G(H)$ and so $C_G(H) \cong \mathbb{Z}_p \cong \mathcal{Z}(H)$. Then either H is polycyclic and so is $HC_G(H)$ (i.e. (6) holds) or $H/\mathcal{Z}(H)$ is virtually free non-abelian and so (8) holds.

Case 3: Suppose $HC_G(H)$ is free. Then H and $C_G(H)$ are free pro-p. But if H is free pro-p non-abelian then $C_G(H) = 1$ contradicting the hypothesis. So $H = C_G(H) \cong \mathbb{Z}_p$, i.e. case (6) holds.

□

Remark 3.2. It is well-known that discrete PD^3-groups do not contain direct products of nonabelian free groups [10]. At this stage, we do not know whether case (5) can occur in the pro-p context.

In cases (1), (7) and (8) the finite subgroups of the mentioned virtually Demushkin and virtually free pro-p group are actually cyclic, since the inverse images of them in G are torsion-free virtually cyclic and therefore are cyclic. Moreover, $C_G(H)$ in case (7) and H in case (8) are the fundamental groups of finite graphs of infinite cyclic groups by [5]. It follows that, in this cases, $C_G(H)$ and H are the pro-p completions of abstract fundamental groups of finite graph of cyclic groups.

A natural example of $C_G(H)$ in case (1) is the pro-p completion of a residually-p fundamental group of a Seifert 3-manifold.

4. Subnormal subgroups of pro-p PD^3-groups

Lemma 4.1. *Let G be a pro-p group and N a maximal infinite cyclic normal subgroup of G. If G/N does not have nontrivial normal cyclic subgroups, then N is characteristic.*

Proof. Let $K \cong \mathbb{Z}_p$ be a normal subgroup of G. The projection of K to G/N is normal and cyclic, and then trivial. Hence all normal cyclic subgroups of G are contained in N. In particular, $\phi(N) \leq N$ for every $\phi \in \mathrm{Aut}(G)$. □

Theorem 4.2. *Let G be a pro-p PD^3-group which has a nontrivial finitely presented subgroup H which is subnormal and of infinite index in G. Then one of the following holds:*

(1) *H is Demushkin and it is normal in an open subgroup U of G such that $U/H \cong \mathbb{Z}_p$;*
(2) *H is cyclic and G is polycyclic;*
(3) *H is cyclic and normal in G with G/H virtually Demushkin.*

Proof. Let $\{J_i \mid 0 \leq i \leq n\}$ be a chain of minimal length n among subnormal chains in G with $H = J_0$ and let $k = \min\{i = 1, \dots, n \mid [J_i\colon H] = \infty\}$. Since H has infinite index in G, it has cohomological dimension at most 2 by Proposition 1.7. Therefore, H is either infinite cyclic, a nonabelian free pro-p group or it has cohomological dimension 2. We therefore distinguish three cases some of which have several subcases.

Case 1: Assume $\mathrm{cd}(H) = 2$. Since H is finitely presented and H is open in J_{k-1}, J_{k-1} is finitely presented and so the order of $H^2(J_{k-1}, \mathbb{F}_p)$ is finite. Proposition 1.3 yields $\mathrm{cd}(J_k) = 3$ because J_{k-1} has infinite index in J_k. By Proposition 1.7, J_k is open in G and J_k is a pro-p PD^3-group (see Proposition 1.6). Proposition 1.8 implies that J_{k-1} is Demushkin and J_k/J_{k-1} is virtually cyclic. Finally, since H is open in J_{k-1}, H is Demushkin and, by Remark 2.2, $N_{J_k}(H)$ is open in J_k. Hence (1) holds in this case.

For the cases with $\mathrm{cd}(H) = 1$ we may assume $k = 1$ by Proposition 1.2. Moreover, Proposition 1.9 yields $\mathrm{cd}(J_1) > 1$ and so J_1 is not free pro-p. By Schreier index formula, there exist a maximal subgroup among finitely generated normal free pro-p subgroups of J_1 containing H as an open subgroup, so from now on, we assume H is maximal.

Case 2: Suppose H is non-abelian. First we note that $\mathrm{cd}(J_1) \neq 3$ because H is not cyclic (see Propositions 1.6, 1.7 and 1.8). Thus, $\mathrm{cd}(J_1) = 2$ and J_1/H is virtually free pro-p by Proposition 1.3.

Let $g \in G$ be an element that normalizes J_1. Then $HH^g/H \cong H^g/H \cap H^g$ is a finitely generated normal subgroup of the virtually free pro-p group J_1/H and so either HH^g/H is finite or it has finite index in J_1/H.

Subcase 2a: HH^g/H is finite for every $g \in J_2$. Then H is open in HH^g and by the maximality of H, $HH^g = H$ for every $g \in J_2$. Then, H is normal in J_2 contradicting our assumption that the length n was minimal. Thus this subcase does not occur.

Subcase 2b: HH^g/H has finite index in J_1/H for some $g \in J_2$. It follows that HH^g has finite index in J_1 and J_1 is finitely generated. Then J_1/H is finitely generated, and so J_1 is finitely presented since it is virtually the semidirect product of finitely generated free pro-p groups. By arguing as in Case 1 with J_1 for H, we deduce that J_1 is a pro-p PD^2, i.e., J_1 is Demushkin. The group J_1 can not be polycyclic since it contains the non-abelian free pro-p subgroup H. If J_1 is Demushkin but not polycyclic this contradicts the fact that a non-soluble Demushkin group does not possess finitely generated normal free subgroups (see [8, Theorem 3] for example). Thus this subcase can not occur either, i.e. Case 2 does not occur.

Case 3: Assume $H \cong \mathbb{Z}_p$.

Claim. If H is not normal in G, then G is polycyclic.

Proof. If there is $g \in J_2$ such that HH^g is 2-generated, then $cd(HH^g) = 2$ and so $J_1/(HH^g)$ is virtually free. If $J_1/(HH^g)$ is infinite, it follows that J_1 contains a subgroup $HH^g \rtimes C$, with C infinite cyclic, whose cohomological dimension is 3 and, hence, it is open in G as needed. Otherwise, HH^g is open in J_1 and so $cd(J_1) = 2$. As in Subcase 2b, J_1 is finitely presented. Then J_2/J_1 is virtually free and, as above, J_2 contains $J_1 \rtimes C$ that has cohomological dimension 3. So J_2 is open in G and therefore, by Proposition 1.8, J_2/J_1 is virtually cyclic. Since HH^g is polycyclic, we deduce that G is polycyclic.

Assume now that HH^g is cyclic for every $g \in J_2$. Then $[HH^g : H]$ is finite and let $S = \bigcup S_j$ be an ascending union of finite subsets of J_2 such that S is dense in J_2. Then

$$A = \overline{\bigcup_j \prod_{g \in S_j} H^g},$$

(where $\prod$ means the internal product of normal subgroups in J_1) is an abelian normal subgroup of J_2 that has to be of rank at most 3. If A is not cyclic then as before $A \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and J_2/A is virtually free; hence J_2 contains $A \rtimes C$ with C infinite cyclic, so $A \rtimes C$ and therefore J_2 is open in G and hence G is polycyclic. If A is cyclic then H is open in A and so is normal in J_2, a contradiction. This proves the claim.

If H is normal in G, then Proposition 1.8 implies that G/H is virtually Demushkin and (3) holds.

□

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