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A quick journey on stable sheaves and vector bundles

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Abstract
In this article, we collect main results obtained from our collaboration with Edoardo Ballico. The topics cover various aspects on the theory of vector bundles and stable sheaves related to Castelnuovo–Mumford regularity, cohomological splitting criteria, globally generated vector bundles, arithmetically Cohen–Macaulay sheaves, logarithmic vector bundles, stable sheaves of rank zero and co-Higgs sheaves.

Introduction

In his 1987 popular essay *Pour l'honneur de l'esprit humain* Jean Dieudonné (see [34]) states: "Let us instead consider one of the most fertile theories of contemporary mathematics, sheaf cohomology; elaborated in 1946, it is more or less contemporary with the double helix of molecular biology and has made possible advances of similar magnitude." Indeed, due to their general nature and versatility, sheaves have several applications in topology and especially in algebraic and differential geometry. Sheaf theory is one of the fundamental tools for studying the geometric properties of objects. A sheaf allows us to express the relationships between small regions of a topological space and the total space. Sheaf cohomology is a generalization of singular cohomology, allowing more general "coefficients" than a simple Abelian group. Since the 1950s, sheaf cohomology has become a central part of algebraic geometry and complex analysis, partly because of the importance of the sheaf of regular functions and the sheaf of holomorphic functions.

In algebraic geometry, two types of sheaves are particularly important, namely coherent sheaves and locally free sheaves, or vector bundles. There are and have been many open problems concerning coherent sheaves and vector bundles on various algebraic varieties. Even the existence of low-rank bundles on projective spaces is very difficult to determine. A famous conjecture from 1974 by Robin Hartshorne, which is still open, states: there are no rank two non-split vector bundles on $\mathbb{P}^n$ with $n \geq 6$. Even in $\mathbb{P}^5$ there are no known indecomposable rank two bundles except in the case of positive characteristics. Then there

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are problems concerning the characterization, stability of vector bundles and the geometry of moduli spaces (see [60] and the references therein).

Edoardo Ballico has been working very fruitfully on these issues since the 1980s. This article is not intended to be an overview of Ballico's vast research activity on the subject, but our aim is to recount some significant stages of our collaboration with him. We began our collaboration with him when we were at the early stages of our research activity and we hope to be able to express in these pages our gratitude for everything we have learned from him. In particular, we hope that his tireless enthusiasm for research and his ability to train young mathematicians will shine through.

Here we summarize the structure of this article. In Sect. 1 we describe some notions of Castelnuovo–Mumford regularity in particular on quadric hypersurfaces, Grassmannian lines and multiprojective spaces, inspired by the derived categories but using only the natural exact sequences in the various contexts. We then use these notions of regularity to have splitting criteria and cohomological characterizations. This was the context in which the collaboration between the second author and Edoardo Ballico began. Section 2 is devoted to the classification of globally generated vector bundles with small first Chern class in particular on quadric surfaces and threefolds, complete intersection Calabi–Yau threefolds and Segre threefolds. In this context the collaboration with the first author has also started. In Sect. 3 we consider the study of arithmetically Cohen–Macaulay sheaves. In Sect. 4 we deal with logarithmic vector bundles, in Sect. 5 with stable sheaves of rank zero and in Sect. 6 with co-Higgs sheaves.

1 Castelnuovo–Mumford regularity for coherent sheaves on projective varieties

Let be $\mathbb{K}$ an algebraically closed field of characteristic zero. We want to start with the following well known splitting criterion for vector bundles on projective spaces (see [46]):

Theorem 1.1 (Horrocks [46]) *Let $\mathscr{E}$ be a vector bundle on $\mathbb{P}^n$, $n \geq 1$ then $\mathscr{E}$ splits as a direct sum of line bundles if and only if $\forall i = 1, \dots, n-1$, $\forall k \in \mathbb{Z}$,*

$$\mathrm{h}^i(\mathscr{E}(k)) = 0.$$

An important improvement of Horrocks' criterion is given in [38]:

Theorem 1.2 (Evans–Griffiths [38]) *Let $\mathscr{E}$ be a vector bundle of rank r on $\mathbb{P}^n$, $n \geq 2$, then $\mathscr{E}$ splits as a direct sum of line bundles if and only if $\forall i = 1, \dots, r-1$ $\forall k \in \mathbb{Z}$,*

$$\mathrm{h}^i(\mathscr{E}(k)) = 0.$$

In particular for rank 2 bundles $\mathscr{E}$ on $\mathbb{P}^n$ we have that

$$\mathscr{E} \text{ splits}$$
$$\Updownarrow$$
$$\mathrm{H}^1_*(\mathscr{E}) = 0.$$

In chapter 14 of [52] Mumford introduced the following concept of regularity for a coherent sheaf on a projective space:

Definition 1.3 (Mumford [52]) A coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n$ $(n \geq 2)$ is said to be m-regular if $\mathrm{H}^i(\mathscr{F}(m-i)) = 0$ for $i = 1, \dots, n$. We will say regular instead of 0-regular.

In [52] Mumford proved the following fundamental properties:

$\mathscr{F}$ regular $\Rightarrow \mathscr{F}(1)$ regular and $\mathscr{F}$ globally generated

It soon became clear that the Castelnuovo–Mumford definition of regularity was a key notion and a fundamental tool in many areas of algebraic geometry such as the construction of moduli spaces and commutative algebra. Here we see how it can be useful for proving splitting criteria.

We may use this notion in order to give a simple proof of Horrocks criterion:

We may assume that $\mathscr{E}$ is regular, while $\mathscr{E}(-1)$ is not.

Since, by hypothesis, all intermediate cohomology vanishes, $\mathscr{E}(-1)$ is not regular if and only if $\mathrm{H}^n(\mathscr{E}(-1-n)) \neq 0$, i.e. $\mathrm{H}^0(\mathscr{E}^\vee) \neq 0$ (Serre duality). Thus there is a non-zero map $f : \mathscr{E} \to \mathscr{O}$. Since $\mathscr{E}$ is globally generated, there is a map $g : \mathscr{O} \to \mathscr{E}$ such that $f \circ g \neq 0$. Hence $f \circ g$ is a non-zero multiple of the identity map $\mathscr{O} \to \mathscr{O}$. Hence $\mathscr{O}$ is a direct summand of $\mathscr{E}$. By induction on the rank we obtain the splitting criterion.

The above proof can be found in [49, (7.3.10.)]. It is also given a simple proof of the splitting criterion by Evans–Griffiths. The ingredients are Castelnuovo–Mumford regularity + Le Potier vanishing theorem (see [49, (7.3.5.)]).

Remark 1.4 In order to prove splitting criteria in other varieties we need notions of regularity on a n-dimensional X with the condition $\mathrm{H}^n(\mathscr{E}(1) \otimes \omega_X) = 0$. We will see, for instance, that on quadric hypersurfaces the top cohomology condition is $\mathrm{H}^n(\mathscr{E}(-n+1)) = 0$ since $\omega_X = \mathscr{O}_X(-n)$.

Let $(X, \mathscr{O}_X)$ be a nonsingular projective variety over an algebraically closed field K of characteristic zero with a fixed very ample invertible sheaf $\mathscr{O}_X(1)$ and $\mathrm{Pic}(X) = \mathbb{Z}$. If X is not $\mathbb{P}^n$, Horrocks' criterion does not apply and we have the following definition:

Definition 1.5 A vector bundle $\mathscr{E}$ on a nonsingular projective variety X of dimension n is called aCM (arithmetically Cohen–Macaulay) if $\forall i = 1, \dots, n-1$ $\forall k \in \mathbb{Z}$,

$$\mathrm{h}^i(\mathscr{E}(k)) = 0$$

but $\mathscr{E}$ is not split as a direct sum of line bundles.

1.1 The case of quadric hypersurfaces

Let $\mathscr{Q} = \mathscr{Q}_n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface. We briefly recall Ottaviani's construction of the spinor bundles [58]. Set $k := \lfloor n/2 \rfloor$. For all integers $m > r > 0$, let $G(r, m)$ denote the Grassmannian of all $(m-r)$-dimensional linear subspaces of $\mathbb{K}^m$. Let $U_{r,m}$ be the universal rank $(m-r)$ subbundle of $G(r, m)$. We first assume that n is odd. Ottaviani used the geometry of the variety of all k-dimensional linear subspaces of $\mathscr{Q}_n$ to construct a morphism $s_n : \mathscr{Q}_n \to G(2^k, 2^{k+1})$. Set $\Sigma_n(-1) := s_n^*(U_{2^k,2^{k+1}})$. Now assume that n is even. In this case we have two morphisms $s_{1,n} : \mathscr{Q}_n \to G(2^{k-1}, 2^k)$ and $s_{2,n} : \mathscr{Q}_n \to G(2^{k-1}, 2^k)$. Set $\Sigma_{1,n}(-1) := s_{1,n}^*(U_{2^{k-1},2^k})$ and $\Sigma_{2,n}(-1) := s_{2,n}^*(U_{2^{k-1},2^k})$. For instance on $\mathscr{Q}_4$ the two spinor bundles $\Sigma_{1,4}$ and $\Sigma_{2,4}$ correspond to the not isomorphic rank two universal bundles on $G(2, 4)$. If n is odd and we see $\mathscr{Q}_{n-1}$ as a smooth hyperplane section of $\mathscr{Q}_n$, then

$$\Sigma_n | Q_{n-1} \cong \Sigma_{1,n-1} \oplus \Sigma_{2,n-1}.$$

If n is even and we see $\mathscr{Q}_{n-1}$ as a smooth hyperplane section of $\mathscr{Q}_n$, then

$$\Sigma_{1,n} | \mathscr{Q}_{n-1} \cong \Sigma_{2,n} | \mathscr{Q}_{n-1} \cong \Sigma_{n-1}.$$

Since here we fix the integer n, we write Σ (resp. Σ_1 and Σ_2) instead of Σ_n (resp. $\Sigma_{1,n}$ and $\Sigma_{2,n}$) if n is odd (resp. even).

We use the unified notation Σ_* meaning that for even n both the spinor bundles Σ_1 and Σ_2 are considered, while $\Sigma_* = \Sigma$ if n is odd.

On quadric hypersurfaces aCM bundles have been completely classified in [48]:

Theorem 1.6 (Knörrer [48]) *Every aCM bundle $\mathscr{E}$ on $\mathscr{Q}_n$ is direct sum of line bundles and spinor bundles with some twist.*

In [25] has been given the following definition of regularity for quadric hypersurfaces:

Definition 1.7 A coherent sheaf $\mathscr{F}$ on $\mathscr{Q}_n$ ($n \geq 2$) is said to be m-Qregular if $\mathrm{H}^i(\mathscr{F}(m-i)) = 0$ for $i = 1, \dots, n-1$, $\mathrm{H}^{n-1}(\mathscr{F}(m) \otimes \Sigma_*(-n+1)) = 0$, and $\mathrm{H}^n(\mathscr{F}(m-n+1)) = 0$.

We will say Qregular instead of 0-Qregular.

This notion of regularity satisfies the fundamental properties of Castelnuovo–Mumford regularity (see [25]):

$\mathscr{F}$ Qregular $\Rightarrow$ $\mathscr{F}(1)$ Qregular and $\mathscr{F}$ globally generated.

Moreover the condition $\mathrm{H}^n(\mathscr{E}(1) \otimes \omega_{\mathscr{Q}}) = 0$ is includes so it is possible to prove splitting criteria:

Theorem 1.8 *Let $\mathscr{E}$ be a rank r vector bundle on $\mathscr{Q}_n$, $n \geq 2$. Then the following conditions are equivalent*:

(a) $\mathrm{H}^i_*(\mathscr{E}) = 0$ *for every* $i = 1, \dots, \min\{r-1, n-2\}$, *and* $\mathrm{H}^{n-1}_*(\mathscr{E}) = 0$.
(b) *$\mathscr{E}$ is a direct sum of line bundles and twists of spinor bundles.*

The complete proof can be found in [25]. Here we just give an idea.

Idea of the proof: We may assume (up to twist) that $\mathscr{E}$ is Qregular, while $\mathscr{E}(-1)$ is not.

Since we assume the cohomological vanishing conditions (a), $\mathscr{E}(-1)$ is not Qregular if and only if $\mathrm{H}^n(\mathscr{E}(-1) \otimes \mathscr{O}(-n+1)) \neq 0$ or $\mathrm{H}^{n-1}(\mathscr{E}(-1) \otimes \Sigma_*(-n+1)) \neq 0$.

- By arguing as in Theorem 1.1, $\mathrm{H}^n(\mathscr{E}(-n)) \neq 0 \Rightarrow \mathscr{O}$ is a direct summand of $\mathscr{E}$.
- Let $\mathrm{H}^{n-1}(\mathscr{E}(-1) \otimes \Sigma_1(-n+1)) \neq 0$ (the argument for $\mathrm{H}^{n-1}(\mathscr{E}(-1) \otimes \Sigma_2(-n+1)) \neq 0$ is very similar). Thanks to the vanishing conditions (a) we get the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}^{n-1}(\mathscr{E}(-1) \otimes \Sigma_1(-n+1)) \otimes \mathrm{H}^1(\mathscr{E}^\vee(1) \otimes \Sigma_1^\vee(-1)) & \xrightarrow{\sigma} & \mathrm{H}^n(\Sigma_1^\vee(-1) \otimes \Sigma_1(-n+1)) \\
\downarrow & & \downarrow \\
\mathrm{H}^0(\mathscr{E}(-1) \otimes \Sigma_2) \otimes \mathrm{H}^1(\mathscr{E}^\vee(1) \otimes \Sigma_1^\vee(-1)) & \xrightarrow{\mu} & \mathrm{H}^1(\Sigma_1^\vee(-1) \otimes \Sigma_2) \\
\downarrow & & \downarrow \\
\mathrm{H}^0(\mathscr{E}(-1) \otimes \Sigma_2) \otimes \mathrm{H}^0(\mathscr{E}^\vee(1) \otimes \Sigma_2^\vee) & \xrightarrow{\tau} & \mathrm{H}^0(\Sigma_2^\vee \otimes \Sigma_2) \cong \mathbb{C} \\
\uparrow \cong & & \uparrow \cong \\
Hom(\Sigma_2^\vee, \mathscr{E}(-1)) \otimes Hom(\mathscr{E}(-1), \Sigma_2^\vee) & \xrightarrow{\gamma} & Hom(\Sigma_2^\vee, \Sigma_2^\vee)
\end{array}$$

The map σ comes from Serre duality and it is not zero. The right vertical maps are isomorphisms. The left vertical maps are surjective, and $\tau \neq 0$.

Thus $\eta \circ \beta \neq 0$. Hence $\eta \circ \beta$ is a non-zero multiple of the identity. Hence $\Sigma_2^\vee$ is a direct summand of $\mathscr{E}(-1)$.

Remark 1.9 The above Theorem gives an improvement of Theorem 1.6 with an easier proof and an analogous of Theorem 1.2 for quadrics. The hypothesis $\mathrm{H}^{n-1}_*(\mathscr{E}) = 0$ does not appear in the Evans–Griffiths criterion on $\mathbb{P}^n$. On $\mathscr{Q}_n$ it is necessary, because on $\mathscr{Q}_n$ there are many indecomposable bundles with $\mathrm{H}^1_*(\mathscr{E}) = \cdots = \mathrm{H}^{n-2}_*(\mathscr{E}) = 0$ but $\mathrm{H}^{n-1}_*(\mathscr{E}) \neq 0$ (see [25, Remark 4.5]).

For instance on $\mathscr{Q}_4$ there is the rank 3 bundle P_4 arising from the following exact sequence:

$$0 \to \mathscr{O} \to \Sigma_1 \oplus \Sigma_2 \to P_4 \to 0.$$

On $\mathscr{Q}_5$ there is the rank 3 bundle P_5 arising from the following exact sequence:

$$0 \to \mathscr{O} \to \Sigma \to P_5 \to 0.$$

If $\mathrm{rank}(\mathscr{E}) = 2$, then $\mathscr{E}^\vee \cong \mathscr{E}(c_1(\mathscr{E}))$. Hence the assumption $\mathrm{H}^{n-1}_*(\mathscr{E}) = 0$ may be omitted if $\mathrm{rank}(\mathscr{E}) = 2$. Moreover we have the following proposition [25]:

Proposition 1.10 *Let $\mathscr{E}$ be a rank 2 bundle on $\mathscr{Q}_n$ with $Qreg(\mathscr{E}) = 0$ and $\mathrm{H}^1(\mathscr{E}(-2)) = \mathrm{H}^1(\mathscr{E}(c_1)) = 0$, where $c_1 := c_1(\mathscr{E})$. Then $\mathscr{E}$ is a direct sum of line bundles and twists of spinor bundles.*

If $n > 4$, then $\mathscr{E} \cong \mathscr{O} \oplus \mathscr{O}(c_1)$.

1.2 The case of multiprojective spaces

Notice that the product of two projective lines $\mathbb{P}^1 \times \mathbb{P}^1$ is, in the same time, a quadric surface and a Grassmannian. Here the notion of Qregularity and Gregularity given on Grassmannians of lines in [3] coincide. In [24] has been introduced the following definition (different from the one given by Hoffman and Wang in [45]) for regularity in the product of two projective spaces:

Definition 1.11 A coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n \times \mathbb{P}^m$ is said to be (p, p')-*regular* if, for all $i > 0$,

$$\mathrm{H}^i(\mathscr{F}(p, p') \otimes \mathscr{O}(j, k)) = 0$$

whenever $j + k = -i$, $-n \leq j \leq 0$ and $-m \leq k \leq 0$.

We often say "*regular*" instead of "(0, 0)-regular", and "*p-regular*" instead of "(p, p)-regular". We define the *regularity* of $\mathscr{F}$, $Reg(\mathscr{F})$, as the least integer p such that $\mathscr{F}$ is p-regular.

This notion of regularity satisfies the fundamental properties of Castelnuovo–Mumford regularity (see [24]):

$\mathscr{F}$ regular $\Rightarrow \mathscr{F}(p, p')$ regular for any $p, p' \geq 0$ and $\mathscr{F}$ globally generated.

The above definition is suitable (as in the case of Qregularly) to have splitting criteria:

Theorem 1.12 *Let $\mathscr{E}$ be a rank r vector bundle on $\mathbb{P}^n \times \mathbb{P}^m$. Then the following conditions are equivalent*:

(i) *for any $i = 1, \ldots, m+n-1$ and for any integer t, $\mathrm{H}^i(\mathscr{E}(t, t) \otimes \mathscr{O}(j, k)) = 0$ whenever $j + k = -i$, $-n \leq j \leq 0$ and $-m \leq k \leq 0$.*
(ii) *There are r integers $t_1, \ldots, t_r$ such that $\mathscr{E} \cong \bigoplus_{i=1}^r \mathscr{O}(t_i, t_i)$.*

Moreover, it is possible to prove the following result:

Theorem 1.13 *Let $\mathscr{E}$ be a vector bundle on $\mathbb{P}^n \times \mathbb{P}^m$. Then the following conditions are equivalent*:

(a) *for any* $i = 1, \dots, m+n-1$ *and for any integer* t, $\mathrm{H}^i(\mathscr{E}(t,t) \otimes \mathscr{O}(j,k)) = 0$ *whenever* $-i \leq j+k \leq 0$, $-n \leq j \leq 0$ *and* $-m \leq k \leq 0$ *but* $(j,k) \neq (-n,0), (0,-m)$.
(b) $\mathscr{E}$ *is a direct sum of the line bundles* $\mathscr{O}$, $\mathscr{O}(0,1)$ *and* $\mathscr{O}(1,0)$ *with some twist* (t,t).

It is possible to generalize the notion of regularity on $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ (see [24])

Definition 1.14 A coherent sheaf $\mathscr{F}$ on $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ is said to be $(p_1, \dots, p_s)$-*regular* if, for all $i > 0$,

$$\mathrm{H}^i(\mathscr{F}(p_1, \dots, p_s) \otimes \mathscr{O}(k_1, \dots, k_s)) = 0$$

whenever $k_1 + \dots, +k_s = -i$ and $-n_j \leq k_j \leq 0$ for any $j = 1, \dots, s$.

This notion of regularity satisfies the fundamental properties of Castelnuovo–Mumford regularity (see [24]):
$\mathscr{F}$ regular $\Rightarrow \mathscr{F}(p_1, \dots, p_s)$ regular for any $p_1, \dots, p_s \geq 0$ and $\mathscr{F}$ globally generated.
The following splitting criteria are the generalizations of Theorems 1.12 and 1.13:

Theorem 1.15 *Let* $\mathscr{E}$ *be a rank* r *vector bundle on* $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$. *Set* $d = n_1 + \cdots + n_s$. *Then the following conditions are equivalent*:

(1) *for any* $i = 1, \dots, d-1$ *and for any integer* t, $\mathrm{H}^i(\mathscr{E}(t, \dots, t) \otimes \mathscr{O}(k_1, \dots, k_s)) = 0$ *whenever* $k_1 + \dots, +k_s = -i$ *and* $-n_j \leq k_j \leq 0$ *for any* $j = 1, \dots, s$.
(2) *There are* r *integer* $t_1, \dots, t_r$ *such that* $\mathscr{E} \cong \bigoplus_{i=1}^r \mathscr{O}(t_i, \dots, t_i)$.

Theorem 1.16 *Let* $\mathscr{E}$ *be a rank* r *vector bundle on* $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$. *Set* $d = n_1 + \cdots + n_s$. *Then the following conditions are equivalent*:

(1) *for any* $i = 1, \dots, d-1$ *and for any integer* t, $\mathrm{H}^i(\mathscr{E}(t, \dots, t) \otimes \mathscr{O}(k_1, \dots, k_s)) = 0$ *whenever* $k_1 + \dots, +k_s \geq -i$ *and* $-n_j - 1 \leq k_j \leq 1$ *for any* $j = 1, \dots, s$, *but with an index* j *such that* $k_j \neq 0, -n_j$.
(2) $\mathscr{E}$ *is a direct sum of line bundles* $\mathscr{O}(l_1, \dots, l_s)$ *(where* $l_j = 1$ *or* $l_j = 0$ *for any* $j = 1, \dots, s$*) with some twist* $(t, \dots, t)$.

These results have been improved in [55] and very recently in [51]. Similar splitting criteria have been proved with similar methods and analogues notions of regularity in weighted projective spaces in [53] and in rational normal scroll surfaces in [33].

Recently in [64], Schreyer proved a splitting criterion for torsion free sheaves on Segre-Veronese varieties $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \to \mathbb{P}^N$ embedded by the complete linear system of a very ample line bundle $\mathscr{O}(H) = \mathscr{O}(d_1, \dots, d_t)$, so $N = \left(\sum_{j=1}^t \binom{n_j + d_j}{d_j}\right) - 1$.

Theorem 1.17 (Schreyer) *Let* $\mathscr{O}(H) = \mathscr{O}(d_1, \dots, d_s)$ *be a very ample line bundle on a product of projective spaces* $P = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ *of dimension* $d = n_1 + \cdots + n_s$ *with* $s \geq 2$ *factors. A torsion-free sheaf* $\mathscr{F}$ *on* P *splits into a direct sum* $\mathscr{F} \cong \bigoplus \mathscr{O}(t_j H)$ *if and only if*

$$\forall i \in \{1, \dots, d-1\}, H^i(P, \mathscr{F}(a_1, \dots, a_s)) = 0$$

for all twists with $\mathscr{O}(a_1, \dots, a_s)$ *such that the cohomology groups* $H^i(P, \mathscr{O}(tH) \otimes \mathscr{O}(a_1, \dots, a_s))$ *vanish for all* $i \in \{1, \dots, d-1\}$ *and all* $t \in \mathbb{Z}$.

The proof is based on the Tate resolution for a coherent sheaf. Here we give a different and shorter proof using the notion of regularity of Definition 1.14:

Proof Let t_1 be an integer such that $\mathscr{F}(t_1H) \otimes \mathscr{O}(-1, \dots, -1)$ is not regular but $\mathscr{F}((t_1 + 1)H) \otimes \mathscr{O}(-1, \dots, -1)$ is regular. Notice that for any integer t and for $i = 1, \dots d - 1$,

$$\mathrm{H}^i(\mathscr{O}(tH) \otimes \mathscr{O}(k_1 - 1, \dots, k_s - 1)) = \mathrm{H}^i(\mathscr{O}(td_1 + k_1 - 1, \dots, td_sk_s - 1)) = 0$$

whenever $-n_j \le k_j \le 0$ for any $j = 1, \dots, s$.

In fact, since $d_1, \dots, d_s \ge 1$, when $t > 0$

$$td_1 + k_1 - 1 \ge -n_1, \dots, td_s + k_s - 1 \ge -n_s,$$

and when $t \le 0$

$$td_1 + k_1 - 1 \le 0, \dots, td_s + k_s - 1 \le 0.$$

So, by hypothesis, for any $i = 1, \dots d - 1$,

$$\mathrm{H}^i(\mathscr{F}(t_1H) \otimes \mathscr{O}(k_1 - 1, \dots, k_s - 1)) = 0$$

whenever $-n_j \le k_j \le 0$ for any $j = 1, \dots, s$.

In particular $\mathscr{F}(t_1H) \otimes \mathscr{O}(-1, \dots, -1)$ is not regular if and only if

$$\mathrm{H}^d(\mathscr{F}(t_1H) \otimes \mathscr{O}(-n_1 - 1, \dots, -n_s - 1)) \neq 0.$$

Claim: $\mathscr{F}(t_1H)$ is regular.

Notice that for any integer t and for $i = 1, \dots, d - 1$,

$$\mathrm{H}^i(\mathscr{O}(tH) \otimes \mathscr{O}(k_1, \dots, k_s)) = \mathrm{H}^i(\mathscr{O}(td_1 + k_1, \dots, td_sk_s)) = 0$$

whenever $-n_j \le k_j \le 0$ for any $j = 1, \dots, s$.

In fact, since $d_1, \dots, d_s \ge 1$, when $t > 0$

$$td_1 + k_1 \ge -n_1, \dots, td_s + k_s \ge -n_s,$$

and when $t \le 0$

$$td_1 + k_1 \le 0, \dots, td_s + k_s \le 0.$$

So, by hypothesis, for any $i = 1, \dots m - 1$,

$$\mathrm{H}^i(\mathscr{F}(t_1H) \otimes \mathscr{O}(k_1, \dots, k_s)) = 0$$

whenever $-n_j \le k_j \le 0$ for any $j = 1, \dots, s$.

So it remains to prove that

$$\mathrm{H}^d(\mathscr{F}(t_1H) \otimes \mathscr{O}(-n_1, \dots, -n_s)) = 0.$$

Since $\mathscr{F}((t_1 + 1)H) \otimes \mathscr{O}(-1, \dots, -1)$ is regular, we know that

$$\mathrm{H}^d(\mathscr{F}(t_1H) \otimes \mathscr{O}(d_1 - 1 - n_1, \dots, d_s - 1 - n_s)) = 0.$$

If $d_1 = \cdots = d_s = 1$ the Claim is proved. Now, let assume $d_j > 1$ for some $j = 1, \dots s$. For simplicity, unless we rearrange the variables, we can consider $j = 1$. Let us consider the exact sequence

$$0 \to \mathscr{O}(-n_1 - 1, \dots, -n_s - 1) \to \mathscr{O}(-n_1, \dots, -n_s - 1)^{\binom{n_1+1}{n_1}} \to \cdots \to \mathscr{O}(0, \dots, -n_s - 1) \to 0, \tag{1}$$

tensored by $\mathscr{F}(t_1H) \otimes \mathscr{O}(d_1 - 1, d_2 \dots, d_s)$. Since

$$\mathrm{H}^d(\mathscr{F}(t_1H) \otimes \mathscr{O}(d_1 - 1 - n_1, \dots, d_s - 1 - n_s)) = 0$$

and also

$$\begin{aligned} & \mathrm{H}^{m-1}(\mathscr{F}(t_1H)\otimes\mathscr{O}(d_1-n_1,\ldots,d_s-1-n_s))=\cdots \\ & \quad =\mathrm{H}^{m-n_1}(\mathscr{F}(t_1H)\otimes\mathscr{O}(d_1-1,\ldots,d_s-1-n_s))=0, \end{aligned}$$

we obtain

$$\mathrm{H}^d(\mathscr{F}(t_1H)\otimes\mathscr{O}(d_1-2-n_1,\ldots,d_s-1-n_s))=0.$$

Now, if $d_1 > 2$, we consider he exact sequence (1) tensored by $\mathscr{F}(t_1H)\otimes\mathscr{O}(d_1-2,d_2\ldots,d_s)$. Since

$$\mathrm{H}^d(\mathscr{F}(t_1H)\otimes\mathscr{O}(d_1-2-n_1,\ldots,d_s-1-n_s))=0$$

and also

$$\begin{aligned} & \mathrm{H}^{m-1}(\mathscr{F}(t_1H)\otimes\mathscr{O}(d_1-n_1,\ldots,d_s-1-n_s))=\cdots \\ & \quad =\mathrm{H}^{m-n_1}(\mathscr{F}(t_1H)\otimes\mathscr{O}(d_1-2,\ldots,d_s-1-n_s))=0, \end{aligned}$$

we obtain

$$\mathrm{H}^d(\mathscr{F}(t_1H)\otimes\mathscr{O}(d_1-2-n_1,\ldots,d_s-1-n_s))=0.$$

We repeat this argument d_1-1 times and we obtain

$$\mathrm{H}^d(\mathscr{F}(t_1H)\otimes\mathscr{O}(-n_1,\ldots,d_s-1-n_s))=0.$$

For the same argument for any $d_j > 1$ we get

$$\mathrm{H}^d(\mathscr{F}(t_1H)\otimes\mathscr{O}(-n_1,\ldots,-n_s))=0$$

and the Claim is proved.

Then, by Serre duality, since

$$\mathrm{H}^d(\mathscr{F}(t_1H)\otimes\mathscr{O}(-n_1-1,\ldots,-n_s-1))\neq 0,$$

$\mathrm{H}^0(\mathscr{F}(t_1H)^\vee)\neq 0$. Moreover, by the Claim, $\mathscr{F}(t_1H)$ is globally generated and we get a summand

$$\mathscr{F}\cong\mathscr{O}(t_1H)\oplus\mathscr{F}'.$$

If rank $\mathscr{F}=1$, we are done: $\mathscr{F}'=0$ since $\mathscr{F}$ is torsion free. Otherwise, we can argue by induction on the rank since $\mathscr{F}'$ satisfies the assumption of the Theorem again. □

2 Globally generated vector bundles

Globally generated vector bundles on projective varieties play an important role in classical algebraic geometry. In fact, every globally generated bundles $\mathscr{E}$ of rank k on an algebraic variety X and every $(N+1)$-dimensional linear subspace $V\subset\mathrm{H}^0(\mathscr{E})$ such that $V\otimes\mathscr{O}_X\to\mathscr{E}$ is an epimorphism, determine a regular morphism

$$\varphi_V : X\longrightarrow \mathrm{Gr}(k,N+1)$$

to the Grassmannian variety parametrizing k-dimensional subspaces in the dual space of V. Conversely, every such a regular morphism corresponds to a globally generated vector bundle $\mathscr{E}$ of rank k on X and an $(N+1)$-dimensional linear subspace $V\subset\mathrm{H}^0(\mathscr{E})$.

If globally generated vector bundles are nontrivial, then the restriction of the bundles to a rational curve $C \subset X$ must have positive first Chern class. We have classified globally generated vector bundles for which this positive first Chern class for the lines are small, on various varieties:

- smooth quadric hypersurfaces $\mathscr{Q}_2$ and $\mathscr{Q}_3$;
- smooth complete intersection Calabi–Yau threefold;
- Segre threefolds with Picard rank two and three.

One of the main ingredients is to use the Hartshorne–Serre correspondence; see [2], from which one can get the following exact sequence in case $\dim X = 3$,

$$0 \to \mathscr{O}_X^{\oplus(r-1)} \xrightarrow{\sigma} \mathscr{E} \to \mathscr{I}_{C,X}(c_1) \to 0,$$

for a globally generated vector bundle $\mathscr{E}$ of rank r on X with the first Chern class c_1 and a smooth curve $C \subset X$. Here, the map σ is induced by an $(r-1)$-dimensional vector subspace of

$$\mathrm{Ext}^1_X(\mathscr{I}_{C,X}(c_1), \mathscr{O}_X) \cong \mathrm{H}^2(\mathscr{I}_{C,X}(c_1) \otimes \omega_X)^\vee.$$

From now on, we will denote the Chern class $c_i = c_i(\mathscr{E})$ of a coherent sheaf $\mathscr{E}$ by a k-tuple of integers if there is no confusion, e.g. the first Chern class of a vector bundle on $\mathscr{Q}_3$ will be denoted by an integer $c_1 \in \mathbb{Z}$. In particular, the dimension of the space $\mathrm{Ext}^1_X(\mathscr{I}_{C,X}(c_1), \mathscr{O}_X)$ gives an upper bound for the rank of indecomposable globally generated vector bundles with prescribed Chern classes. First, we obtain the following classification on a smooth quadric threefold $\mathscr{Q} = \mathscr{Q}_3$ for rank two case; see [14].

Theorem 2.1 *There exists an indecomposable and globally generated vector bundle $\mathscr{E}$ of rank 2 on $\mathscr{Q}$ with the Chern classes (c_1, c_2), $c_1 \le 3$ in the following cases:*

(1) $(c_1 = 1, c_2 = 1)$, *$\mathscr{E}$ is the spinor bundle Σ and C is a line.*
(2) $(c_1 = 2, c_2 = 4)$, *$\mathscr{E}$ is a pull-back of a null-correlation bundle on $\mathbb{P}^3$ twisted by 1 and C is the disjoint union of two conics.*
(3) $(c_1 = 3, c_2 = 5)$ *and $\mathscr{E} \cong \Sigma(1)$, where Σ is the spinor bundle of $\mathscr{Q}$.*
(4) $(c_1 = 3, 6 \le c_2 \le 9)$, *$\mathscr{E}$ is the homology of a monad*

$$0 \to \mathscr{O}_{\mathscr{Q}}^{\oplus(c_2-5)}(1) \to \Sigma^{\oplus(c_2-4)}(1) \to \mathscr{O}_{\mathscr{Q}}(2)^{\oplus(c_2-5)} \to 0,$$

and C is a smooth elliptic curve of degree c_2.

The same methodology using the Hartshorne–Serre correspondence, used in the proof of Theorem 2.1, is applied to classify the globally generated vector bundles of higher rank on $\mathscr{Q}$ with the small first Chern classes; see [16].

Theorem 2.2 *Let $\mathscr{E}$ be a non-split vector bundle of rank 3 on $\mathscr{Q}$ with $c_1 \le 2$. Then $\mathscr{E}$ is globally generated if and only if $\mathscr{E}$ admits an exact sequence,*

$$0 \to \mathscr{O}_{\mathscr{Q}}^{\oplus 2} \to \mathscr{E} \to \mathscr{I}_{C,\mathscr{Q}}(c_1) \to 0,$$

where C is a smooth irreducible curve of degree d and genus g:

$$\begin{aligned} c_1 = 1 \Rightarrow\ & (d,g) = (1,0)\ ;\ \mathscr{E} \simeq \Sigma \oplus \mathscr{O}_{\mathscr{Q}} \\ & (d,g) = (2,0)\ ;\ \mathscr{E} \simeq \mathscr{A}_p \\ c_1 = 2 \Rightarrow\ & (d,g) = (3,0)\ ;\ \mathscr{E} \simeq \Sigma \oplus \mathscr{O}_{\mathscr{Q}}(1) \end{aligned}$$

$$(d, g) \in \{(4,0), (4,1), (5,1), (6,2), (8,5)\}, \textit{ or}$$

$\mathscr{E}$ is isomorphic to $\mathscr{A}_p^\vee(1)$ or a pull-back of $\mathscr{N}_{\mathbb{P}^3}(1) \oplus \mathscr{O}_{\mathbb{P}^3}$ whose associated curve C is a disjoint union of two conics.

Here, $\mathscr{A}_p$ is the pull-back of $T_{\mathbb{P}^3}(-1)$ along the linear projection $\mathscr{Q} \to \mathbb{P}^3$ with a center $p \in \mathbb{P}^4 \backslash \mathscr{Q}$ and $\mathscr{N}_{\mathbb{P}^3}$ is a null-correlation bundle on $\mathbb{P}^3$. The vector bundle $\mathscr{E}$ has the Chern classes (c_2, c_3) with $c_2 = \deg(C)$ and $c_3 = 2g - 2 + d(3 - c_1)$.

Theorem 2.3 *There exists a globally generated and indecomposable vector bundle of rank $r \geq 3$ on $\mathscr{Q}$ with the Chern classes (c_1, c_2, c_3), $c_1 \leq 2$, if and only if the numeric data $(c_1, c_2, c_3; r)$ is one of the following*:

$$\begin{aligned}
&(1, 2, 2; 3 \leq r \leq 4),\\
&(2, 4, 0; 3),\ (2, 4, 2; 3),\ (2, 4, 4; 3 \leq r \leq 4),\\
&(2, 5, 5; 3 \leq r \leq 5),\ (2, 6, 8; 3 \leq r \leq 7),\ (2, 8, 16; 3 \leq r \leq 13).
\end{aligned}$$

To show the existence of indecomposable vector bundles on $\mathscr{Q}$ with $c_1 = 2$, we consider a family of indecomposable vector bundles with $c_1 = 1$ and construct self-extensions of elements in the family, for which he checks the indecomposability case by case. As an automatic consequence, every globally generated vector bundle of rank $r \geq 14$ on $\mathscr{Q}$ with $c_1 \leq 2$ is decomposable. With the same spirit, we deal with the case of a smooth quadric surface to obtain the following result; see [17].

Theorem 2.4 *There exists an indecomposable and globally generated vector bundle of rank $r \geq 2$ on a smooth quadric surface $\mathscr{Q}_2$ with the Chern classes (c_1, c_2) such that $c_1 = (a, b) \leq (2, 2)$, $a \leq b$ if and only if $(c_1, c_2; r)$ is one of the following*:

$$\begin{aligned}
&(1, 1, 2; r = 2, 3),\\
&(1, 2, 2; 2), (1, 2, 3; r = 2, 3), (1, 2, 4; r = 2, 3, 4, 5),\\
&(2, 2, 3; 2), (2, 2, 4; r = 2, 3), (2, 2, 5; r = 2, 3), (2, 2, 6; r = 2, 3, 4, 5),\\
&(2, 2, 8; r = 2, 3, 4, 5, 6, 7, 8).
\end{aligned}$$

Then, we turn our attention to the smooth complete intersection Calabi–Yau (CICY) threefolds; see [18]. There are only five types of such:

(i) the quintic $X_5 \subset \mathbb{P}^4$,
(ii) the intersection $X_{2,4} \subset \mathbb{P}^5$ of a quadric and a quartic,
(iii) the intersection $X_{3,3} \subset \mathbb{P}^5$ of two cubics,
(iv) the intersection $X_{2,2,3} \subset \mathbb{P}^6$ of two quadrics and a cubic, and
(v) the intersection $X_{2,2,2,2} \subset \mathbb{P}^7$ of four quadrics.

Theorem 2.5 *Let $\mathscr{E}$ be a globally generated vector bundle of rank $r \geq 2$ without trivial factors on a smooth quintic hypersurface $X = X_5$ in $\mathbb{P}^4$. If $c_1(\mathscr{E}) \leq 2$, then one of the following holds*:

(1) $\mathscr{E} \cong T_{\mathbb{P}^4}(-1)_{|X}$ *or* $\mathscr{E} \cong \pi_p^*(T_{\mathbb{P}^3}(-1))$.
(2) $\mathscr{E} \cong \mathscr{O}_X(1)^{\oplus 2}$, *or* $\mathscr{E} \cong \pi_p^*(\mathscr{N}_{\mathbb{P}^3}(1))$ *where $\mathscr{N}_{\mathbb{P}^3}$ is a null correlation bundle on $\mathbb{P}^3$.*
(3) $\mathscr{E} \cong \pi_p^*(\Omega^1_{\mathbb{P}^3}(2))$.
(4) $0 \to \mathscr{O}_X(-2) \to \mathscr{O}_X^{\oplus(r+1)} \to \mathscr{E} \to 0$, *with* $3 \leq r \leq 14$.
(5) $0 \to \mathscr{O}_X(-1)^{\oplus 2} \to \mathscr{O}_X^{\oplus(r+2)} \to \mathscr{E} \to 0$, *with* $3 \leq r \leq 8$.

(6) $0 \to \mathscr{O}_X(-1) \to \mathscr{O}_X^{\oplus r} \oplus \mathscr{O}_X(1) \to \mathscr{E} \to 0$, *with* $3 \le r \le 5$.

Here, $\pi_p : X \to \mathbb{P}^3$ *is a linear projection from a point* $p \in \mathbb{P}^4 \setminus X$.

Secondly, we focus on the case of CICY threefolds of codimension 2.

Theorem 2.6 *Let* $\mathscr{E}$ *be a globally generated vector bundle of rank* 2 *on a CICY threefold* X *of codimension* 2*. If* $c_1(\mathscr{E}) \le 2$*, then the possible* $c_2(\mathscr{E})$ *is as follows*:

(1) *on* $X = X_{2,4}$*, we have* $c_2(\mathscr{E}) \in \{0, 4, 8, 11, 16\}$;
(2) *on* $X = X_{3,3}$*, we have* $c_2(\mathscr{E}) \in \{0, 9, 12, 15, 16, 18\}$.

And for each Chern classes there exist corresponding globally generated vector bundles on X.

Note that there are three types of Segre varieties of dimension 3: $\mathbb{P}^3$, $\mathbb{P}^2 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Since the classification over $\mathbb{P}^3$ is sufficiently done by others, e.g. [1, 54, 63, 65, 66], we consider the other types of Segre threefolds. The following result is on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in [20], and the similar result on $\mathbb{P}^2 \times \mathbb{P}^1$ can be found in [19].

Theorem 2.7 *Let* $\mathscr{E}$ *be an indecomposable and globally generated vector bundle of rank* $r \ge 2$ *on* $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *with the Chern classes* $c_1 = (a_1, a_2, a_3)$ *and* $c_2 = (e_1, e_2, e_3)$*. Let* s *be the number of connected components of associated curve to* $\mathscr{E}$ *via the Hartshorne–Serre correspondence. If* $c_1 \in \{(1, 1, 1), (2, 1, 1), (2, 2, 1)\}$*, the quadruple* $(s; e_1, e_2, e_3)$ *and the possible rank* r *are as follows:*

$(\mathbf{r} = \mathbf{2})$ (i) $c_1(\mathscr{E}) = (2, 1, 1)$: *up to permutations on* (e_2, e_3)

$$\{(3; 0, 3, 3), (2; 0, 2, 2), (1; 2, 1, 1), (1; 1, 1, 1), (1; 1, 2, 0)\};$$

(ii) $c_1(\mathscr{E}) = (2, 2, 1)$: *up to permutations on* (e_1, e_2)

$$\{(1; 2, 1, 2), (1; 3, 1, 2), (1; 4, 1, 2), (2; 2, 0, 4), (3; 3, 0, 6)\}.$$

$(\mathbf{r} \ge \mathbf{3})$ (iii) $c_1(\mathscr{E}) = (1, 1, 1)$: $\{(1; 1, 1, 1 ; 3 \le r \le 7)\}$;
(iv) $c_1(\mathscr{E}) = (2, 1, 1)$: *up to permutations on* (e_2, e_3)

$$\begin{aligned}&\{(3; 0, 3, 3 ; 3 \le r \le 4), (1; 2, 2, 2 ; 3 \le r \le 5),\\ &\ (1; 2, 3, 3 ; 3 \le r \le 8), (1; 2, 4, 4 ; 3 \le r \le 11)\\ &\ (1; 1, a, b ; 3 \le r \le a + b) \bigm| 3 \le a + b\}.\end{aligned}$$

Moreover, there exist globally generated vector bundles in each case.

More recently globally generated vector bundles with low c_1 has been classified on a projective space blown up at finitely many points (see [5]), on a projective space blown up along a line (see [57]) on the del Pezzo threefold of degree 6 with Picard number 2 (see [56]).

3 Arithmetically Cohen–Macaulay sheaves

For a projective scheme $X \subset \mathbb{P}^N$, a coherent sheaf $\mathscr{E}$ on X is called *arithmetically Cohen–Macaulay* (for short, aCM) if the following hold:

(i) the dimension of the support of $\mathscr{E}$ is equal to $\dim(X)$,
(ii) the stalk $\mathscr{E}_x$ has positive depth for any point x on X, and

(iii) $\mathrm{H}^i(\mathscr{E}(t)) = 0$ for all $t \in \mathbb{Z}$ and $i = 1, \ldots, \dim(X) - 1$.

The aCM sheaves attract much attention, mainly because it is a common belief that the category generated by aCM sheaves on X measures the complexity of X. Indeed, a classification of aCM varieties was proposed as *finite, tame or wild* representation type according to the complexity of this category in [36]. The classification of aCM vector bundles has been done for several projective varieties such as smooth quadric hypersurfaces in [47, 48], cubic surfaces in [29, 39], prime Fano threefolds in [50], Grassmannian varieties in [31] and others. In fact, in [37] a complete list of varieties supporting a finite number of aCM sheaves is provided. Varieties that only support one dimensional families of aCM vector bundles, *tame varieties*, are known by the classical work of Atiyah in [4] for elliptic curves, and much more recently by work of Faenzi and Malaspina in [40] for rational scrolls of degree four. In [32] it is shown that all the Segre varieties have a *wild* behaviour, namely they support families of arbitrary dimension, consisting of indecomposable and pairwise non-isomorphic aCM sheaves. Finally, it has been finally shown that the rest of aCM integral projective varieties are wild; see [41].

Our contribution to the study on such sheaves is over the varieties that are either aCM or not reduced. One direction is to classify the aCM sheaves with low rank on such varieties, and the other is to show the wildness of a certain family of algebraic surfaces by constructing an irreducible family of indecomposable and pairwise non-isomorphic aCM sheaves with dimension as high as possible. The former is done over singular quadric surface with corank at least two; see [21, 22]. This line of work goes back to the seminal work in [28]. One of the main result in [22] is the classification of aCM kernel sheaf of simple type with pure rank two on $X_2 = H_1 \cup H_2$ a quadric surface of corank two with $L = H_1 \cap H_2$ the intersecting line.

Theorem 3.1 *Let $\mathscr{E}$ be an aCM kernel sheaf of simple type with pure rank two on X_2. Up to twist it is one of the following*:

- *a direct sum of two line bundles*;
- *an extension of a twisted ideal sheaf $\mathscr{I}_{p,X_2}(1)$ of a point $p \notin L$ by $\mathscr{O}_{X_2}$, which is locally free*;
- *a sheaf whose restriction to each component of X_2 satisfies*

  (i) $\mathscr{E}_{|H_i} \cong \mathscr{O}_{H_i}(c) \oplus \mathscr{O}_{H_i}$ *and*
  (ii) $0 \to \mathscr{O}_{H_j}(k) \to \mathscr{E}_{|H_j} \to \mathscr{I}_{Z,H_j}(c-k) \to 0$

  for an integer $c \geq 2$ and $\{i, j\} = \{1, 2\}$, where $Z \subset L$ is a zero-dimensional subscheme with $k = |Z|$ such that $0 \leq k < c \leq 2k + 2$.

Then, concerning the trichotomy classification of aCM varieties, one can use the classification of aCM sheaves of rank one and two on X_2 to obtain:

Corollary 3.2 *The quadric surface X_2 is of wild type in a very strong sense, that is, there are arbitrarily large dimensional families of pairwise non-isomorphic aCM sheaves of rank one and two on X_2.*

Then we step into the case $X = 2H$ the double plane. Recalling that in general the *rank* of $\mathscr{E}$ is defined to be $\mathrm{rank}(\mathscr{E}) := \frac{\mu(\mathscr{E})}{\mu(\mathscr{O}_X)}$, one get that $\mathrm{rank}(\mathscr{E}) \in (1/2)\mathbb{N}$ on $X = 2H$. We classify all the aCM sheaves of rank at most 3/2, up to twist; see [21].

Theorem 3.3 *Any aCM sheaf of rank one on X is either $\mathscr{O}_X$, $\mathscr{O}_H(a) \oplus \mathscr{O}_H$ with $a \in \mathbb{Z}$, or $\mathscr{I}_{C,X}$ for a plane curve $C \subset H$, up to twists.*

As an automatic consequence, we could spot a new kind of wildness that does not occur in the previous works.

Corollary 3.4 *The double plane X is of wild representation type in a very strong sense, that is, there exist arbitrarily large dimensional families of pairwise non-isomorphic aCM sheaves of fixed rank one on X.*

To classify the aCM sheaves of rank $3/2$, we used a special type of sheaf, called the *layered* sheaf; see [29, Definition 6.5] and [21]. A coherent sheaf $\mathscr{E}$ of rank $r \in (1/2)\mathbb{N}$ on X is said to be *layered* if there exists a filtration $0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_{2r-1} \subset \mathscr{E}_{2r} = \mathscr{E}$ of $\mathscr{E}$ with $\mathscr{E}_i/\mathscr{E}_{i-1} \cong \mathscr{O}_H(t_i)$ with $t_i \in \mathbb{Z}$ for all $i = 1, \ldots, 2r$.

Theorem 3.5 *Let $\mathscr{E}$ be an aCM sheaf of rank $3/2$ on X. Then up to a twist we have the following:*

(1) *If $\mathscr{E}$ is non-layered, then it is unique and it is the only simple one. Moreover, it is an Ulrich sheaf.*
(2) *If $\mathscr{E}$ is layered and indecomposable, then either*
   (i) *$\mathscr{E}$ admits a filtration $0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}_3 = \mathscr{E}$ such that $\mathscr{E}_i \big/ \mathscr{E}_{i-1} \cong \mathscr{O}_H$ for each $i = 1, 2, 3$,*
   (ii) $0 \to \mathscr{O}_H \to \mathscr{E} \to \mathscr{I}_{L,X} \to 0$ *for a line $L \subset H$, or*
   (iii) *it fits into the following sequence;*
$$0 \to \mathscr{I}_{C,X}(a) \to \mathscr{E} \to \mathscr{O}_H \to 0$$
   for a plane curve $C \subset H$ and an integer $a \geq \deg(C)$.

Moreover, we observed that any aCM vector bundle of rank two on a quadric surface of corank at least 2 is a direct sum of two line bundles; see [21, 22]. Then we could apply the induction on the dimension of the quadric hypersurface $\mathscr{Q}$ and the criterion in [59] for a vector bundle on a smooth quadric hypersurface to split, to obtain the following. Note that the lower bound $N \geq 8$ is used to obtain a smooth linear section of $\mathscr{Q}$, but it is still unknown whether this bound is sharp.

Corollary 3.6 *Let $\mathscr{Q} \subset \mathbb{P}^N$ with $N \geq 8$ be any quadric hypersurface. If $\mathscr{E}$ is an aCM vector bundle of rank two on $\mathscr{Q}$, then it splits.*

Now in the latter direction of study, we tried to construct an irreducible family of indecomposable and pairwise non-isomorphic aCM sheaves on two different types of surfaces: (i) smooth surfaces in $\mathbb{P}^3$, and (ii) smooth surfaces with non-negative Kodaira dimension, to determine their representation type; see [12, 23]. The main result in [12] is the following.

Theorem 3.7 *Let $X = X_d \subset \mathbb{P}^3$ be a surface of degree $d \geq 4$ with $X_{\mathrm{reg}} \neq \emptyset$. Assume further that either $\mathrm{Pic}(X) = \mathbb{Z}\langle \mathscr{O}_X(1)\rangle$ or that X is integral. For every $r \in 2\mathbb{N}$, there exists a family $\{\mathscr{E}_\lambda\}_{\lambda\in\Lambda}$ of indecomposable aCM vector bundles of rank r such that Λ is an integral quasi-projective variety with $\dim \Lambda = r$ and $\mathscr{E}_\lambda \not\cong \mathscr{E}_{\lambda'}$ for all $\lambda \neq \lambda'$ in Λ. In particular, X is of wild representation type.*

For the construction of aCM vector bundles, the Cayley–Bacharach condition is considered in a delicate way, while a tricky argument using monodromy is used to show that two bundles in the constructed family are non-isomorphic. In Theorem 3.7 we constructed aCM vector bundles of even rank, basically by applying the iteration of Cayley–Bacharach construction.

In the proof, we constructed a much bigger dimensional family of aCM vector bundles, while we only considered its subvariety of dimension equal to the rank of the bundles so that any two distinct points in this subvariety represent non-isomorphic aCM bundles.

Then we run through the same mission on surfaces with non-negative Kodaira dimension, obtaining the following result in [23].

Theorem 3.8 *Let X be an integral projective surface with a fixed ample line bundle $\mathscr{O}_X(1)$ listed below. Then for each integer $r \geq 2$ there exists an $b_X(r)$-dimensional irreducible family $\{\mathscr{E}_\alpha\}_{\alpha\in\Gamma}$ of indecomposable aCM vector bundles of rank r on X such that for each $\alpha \in \Gamma$ there are only finitely many $\beta \in \Gamma$ with $\mathscr{E}_\alpha \cong \mathscr{E}_\beta$.*

No.	X	$b_X(r)$
1	$\pi : X \to Y$ a birational morphism with $\omega_Y \cong \mathscr{O}_Y$ and $q(Y) = 0$ such that $\pi^{-1}(Y_{\text{sing}}) \cong Y_{\text{sing}}$	$2r$
2	$\omega_X \not\cong \mathscr{O}_X$ locally free with $\mathrm{h}^0(\omega_X) = 0$ and $\mathrm{h}^0(\omega_X^{\otimes 2}) = 1$, and $q(X) = 0$	$2\lceil \frac{r}{2} \rceil$
3	smooth and $q(X) = 1$ with $\omega_X^\vee \otimes \mathscr{O}_X(1)$ trivial or ample	1
4	$\pi : X \to Y$ a birational morphism with an abelian surface Y and $\omega_X^\vee \otimes \mathscr{O}_X(1)$ trivial or ample	$r + 1$
5	$\pi : X \to Y$ a birational morphism with a hyperelliptic surface Y	1
6	$\omega_X \cong \mathscr{O}_X(1)$ with $\mathrm{h}^1(\omega_X^{\otimes n}) = 0$ for all $n \in \mathbb{Z}$ and $p_g \geq 3$	r

Theorem 3.8 shows that the projective surfaces of Kodaira dimension zero, possibly with singularities, are of wild representation type, except the case of hyperelliptic surfaces. Mostly we consider sets of points on the surface, with which one can construct indecomposable aCM sheaves of any rank, using the iterated extension. Then we apply the Hartshorne–Serre correspondence type method to construct aCM vector bundles, given by extending the forementioned sheaves by trivial bundles. The result is important in a sense that it can give the lower bound of the function

$$a_{X,\mathscr{O}_X(1)}(r) := \sup_\Gamma \left\{ \dim \Gamma \;\middle|\; \Gamma \text{ runs over the parameter spaces of indecomposable aCM vector bundles of rank} r \text{on} X \right\}$$

for the polarized variety in Theorem 3.8. One of the natural questions is to find this function in a closed form in r.

4 Logarithmic vector bundles

Let X be a smooth projective variety and $D = D_1 + \cdots + D_m$ be a reduced and effective divisor with each D_i irreducible. Then one can define the *logarithmic sheaf* $\Omega_X^1(\log D)$, the sheaf of differential 1-forms with logarithmic poles along D. If D has simple normal crossings, its logarithmic sheaf is known to be locally free and so it can be called to be the logarithmic bundle. It admits the residue exact sequence

$$0 \to \Omega_X^1 \to \Omega_X^1(\log D) \xrightarrow{\text{res}} \bigoplus \varepsilon_{i*}\mathscr{O}_{D_i} \to 0 \tag{2}$$

where $\varepsilon_i : D_i \to X$ is the embedding and the map res is the Poincaré residue morphism. Its dual $T_X(-\log D) := \Omega_X^1(\log D)^\vee$ is the sheaf the logarithmic vector fields tangent along D, fitting into the exact sequence

$$0 \to T_X(-\log D) \to T_X \to \oplus\varepsilon_{i*}\mathscr{O}_{D_i}(D_i) \to 0,$$

where T_X is the tangent bundle of X. One of our main contribution on the logarithmic vector bundle is on the Torelli problem, i.e. when can we tell $\Omega_X^1(\log D) \not\cong \Omega_X^1(\log D')$ for two distinct divisors D and D'. We have several results on the case when X is a smooth quadric hypersurface and a multiprojective space; see [15].

Proposition 4.1 *Let $D = \mathscr{Q} \cap H$ be a smooth hyperplane section of a smooth quadric hypersurface $\mathscr{Q} \subset \mathbb{P}^{n+1}$ with $n \geq 1$ and H a hyperplane. Setting*

$$E(D) := \left\{f \in \operatorname{Aut}(\mathbb{P}^{n+1}) \,\middle|\, f(\mathscr{Q}) = \mathscr{Q} \text{ and } f(D) = D\right\};$$
$$S(D) := \left\{o \in \mathbb{P}^{n+1} \setminus \mathscr{Q} \,\middle|\, \forall f \in E(D),\, f(o) = o\right\},$$

we have $S(D) = \{p\}$, where p is the point apolar to H with respect to $\mathscr{Q}$.

Here again, the map $\pi_p : \mathscr{Q} \to \mathbb{P}^n$ is the linear projection from the point $p \in \mathbb{P}^{n+1} \setminus \mathscr{Q}$. For the notion of apolarity, we refer to [35, Chapter 1]. One can see clearly that $p \in S(D)$ and the statement is true for $n = 1$. Then one can apply the induction by considering a general hyperplane containing p. As an automatic consequence of Proposition 4.1 together with the stability of $\Omega_{\mathscr{Q}}^1(\log D)$ and the splitting of $\pi_p^*(T_{\mathbb{P}^n}(-2))$, we get the following; see [15].

Corollary 4.2 *Let $D = \mathscr{Q} \cap H$ be a smooth hyperplane section of $\mathscr{Q} \subset \mathbb{P}^{n+1}$. Then we have*

$$\Omega_{\mathscr{Q}}^1(\log D) \cong \pi_p^*(T_{\mathbb{P}^n}(-2)),$$

where p is the point apolar to H with respect to $\mathscr{Q}$. In particular, the map $\Phi : (\mathbb{P}^{n+1})^\vee \dashrightarrow \mathbf{M}$ sending $[H] \in (\mathbb{P}^{n+1})^\vee$ to $\Omega_{\mathscr{Q}}^1(\log \mathscr{Q} \cap H)$ is generically injective. Here, $\mathbf{M}$ is the moduli space of stable sheaves of rank two on $\mathscr{Q}$ with Chern classes $c_1 = \mathscr{O}_{\mathscr{Q}}(-1)$ and $c_2 = 2$.

In general, for a smooth projective variety $X \subset \mathbb{P}^N$ and a divisor $D = D_1 + \cdots + D_m$ with each $D \in |\mathscr{O}_X(1)|$, we have Torelli property, if $m \geq N + 3$; see [15, Corollary 3.7]. In other words, one can expect to have a positive answer to the Torelli question, if the number of irreducible components in a divisor is big enough. On the other hand, one can have a negative answer if the number is small, as in below.

Proposition 4.3 *Let $\mathscr{Q} = \mathscr{Q}_2 \subset \mathbb{P}^3$ be a smooth quadric surface. For a divisor $D = D_1 + D_2$ with each D_i a smooth conic, the zeros of the unique section in $\mathrm{H}^0(\Omega_{\mathscr{Q}}^1(\log \mathrm{D}))$ are the singular points of the two singular conics in the pencil spanned by D_1 and D_2. Moreover, the map Ψ from the family of such divisors D to the corresponding moduli space is not generically injective.*

Non-Torelli property can be easily deduced by dimension counting, while the first part of Proposition 4.3 can be obtained by considering a free resolution of $\Omega^1_{\mathcal{Q}}(\log D)$, from which we can see that the zero locus is the zeros of the equations $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$ on $\mathcal{Q}$, where F is the defining equation of $\mathcal{Q}$ and x, y are residual coordinates of $\mathbb{P}^3$ to the hyperplanes associated to each D_i.

The result in [67] asserts that the Torelli property holds for the logarithmic tangent bundles associated to smooth hypersurfaces in the projective space of degree at least three if and only if its defining equation is not of Sebastiani–Thom type. Using the induction, we generalize this result to the multiprojective space to obtain the following.

Theorem 4.4 *For $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ with $s \geq 2$, the map defined by $D \mapsto T_X(-\log D)$, is generically injective for $D \in |\mathcal{O}_X(a_1, \ldots, a_s)|$ with $a_i \geq 3$ for all i.*

Another direction of our contribution on logarithmic vector bundle is on vanishing of cohomology of the bundle, i.e. whether the logarithmic bundle is aCM or not. One can say that a divisor D is of *aCM type* if its corresponding logarithmic vector bundle $\Omega^1_X(\log D)$ is aCM; see [13].

Theorem 4.5 *Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of dimension $n \geq 2$; in case $n = 2$ assume further that X is very general. If D is a reduced and effective divisor of aCM type on X with respect to $\mathcal{O}_X(1)$, then one of the following holds.*

(i) *$X = \mathbb{P}^n$ and $D = D_1 + \cdots + D_m$ with $1 \leq m \leq n+1$ and each D_i a hyperplane;*
(ii) *$X = \mathcal{Q}_2$ a smooth quadric surface and $D = A_1 + \ldots + A_a + B_1 + \cdots + B_b$ the union of $a + b$ distinct lines with $1 \leq a, b \leq 3$ such that $A_i \in |\mathcal{O}_Q(1,0)|$ and $B_j \in |\mathcal{O}_Q(0,1)|$.*

In case when $n = 2$, i.e. X is a smooth projective surface, the vanishing condition $\mathrm{h}^0(T_X) = 0$ would provide a strong restriction on the divisors D_i. We also suggest several results on the (non-)existence of aCM divisors over surfaces with $\mathrm{h}^0(T_X) = 0$, which include the following:

(i) X is of general type;
(ii) the minimal model of X is a K3 surface or an Enriques surface;
(iii) most surface with $\kappa(X) = 1$ and $\kappa(X) = -\infty$;
(iv) X is obtained by blowing up a Del Pezzo surface X of degree four at finitely many points.

Indeed, the vanishing $\mathrm{h}^0(T_X) = 0$ enforces the vanishing of $\mathrm{h}^0(\mathcal{O}_{D_i}(D_i))$ for the aCM type divisor $D = D_1 + \cdots + D_m$ by the Poincaré residue morphism. Note that the last class contains the smooth cubic surfaces in $\mathbb{P}^3$ and the smooth complete intersection $X \subset \mathbb{P}^4$ of two quadric hypersurfaces, which are dealt in the proof of Theorem 4.5.

We also defined the *deficiency module* of degree i associated to a divisor D

$$\mathrm{H}^i_*(D) := \oplus_{t \in \mathbb{Z}} \mathrm{H}^i(\Omega^1_X(\log D) \otimes \mathcal{O}_X(t))$$

for each $i = 1, \ldots, n-1$, which is a module over the ring $S = S_X := \oplus_{t \geq 0} \mathrm{H}^0(\mathcal{O}_X(t))$. Then we asked the Torelli question in terms of the deficiency module to obtain a positive answer in some interesting cases, i.e. these modules uniquely determine D. Similarly we also define *T-deficiency module* to be $\mathrm{H}^i_*(D^T) := \oplus_{t \in \mathbb{Z}} \mathrm{H}^i(T_X(-\log D) \otimes \mathcal{O}_X(t))$ of degree i associated to D to ask the same Torelli-type question. Here, the superscript "T" in D^T indicates that we consider the logarithmic tangent bundle $T_X(-\log D)$ instead of $\Omega^1_X(\log D)$. We can show that a certain arrangement over a projective variety is determined by the deficiency module, e.g. over an Enriques surface as in the following result in [13].

Proposition 4.6 *Let X be an Enriques surface with a fixed ample line bundle $\mathscr{O}_X(1)$. Fix a divisor $D = D_1 + \cdots + D_m$ with each $D_i \in |\mathscr{O}_X(a) \otimes \omega_X|$ for some positive integer a with* $\mathrm{h}^1(\Omega_X^1(a) \otimes \omega_X) = 0$. *Then the multiplication map*

$$\gamma : \mathrm{H}^0(\omega_X(a)) \otimes \mathrm{H}^1(\Omega_X^1(\log D) \otimes \omega_X) \to \mathrm{H}^1(\Omega_X^1(\log D)(a))$$

determines D.

Indeed, for each nonzero $f \in \mathrm{H}^0(\omega_X(a))$, the induced map

$$\gamma_f : \mathrm{H}^1(\Omega_X^1(\log D) \otimes \omega_X) \to \mathrm{H}^1(\Omega_X^1(\log D)(a))$$

is checked to be surjective if and only if f is not a scalar multiple of a defining equation f_i of D_i for some i, while the map has corank one for $f = f_i$.

5 Stable sheaves of rank zero

One of the main mathematical topic that E. Ballico devoted his life to, is the moduli space of curves in a smooth projective variety X. Depending on the direction of study, there are many different moduli spaces parametrizing the same curves, such as Hilbert scheme of curves, moduli space of stable sheaves of rank zero supporting curves, moduli space of stable maps, moduli space of stable pairs. His contribution on this topic is to give explicit description of the moduli space together with its topological property, and see the birational relationship with other moduli spaces.

Definition 5.1 Let $\mathscr{F}$ be a pure sheaf of dimension 1 on a smooth projective variety X with the Hilbert polynomial $\chi_{\mathscr{F}}(t) = \mu t + \chi$ with respect to $\mathscr{O}_X(1)$. The *p-slope* of $\mathscr{F}$ is defined to be $p(\mathscr{F}) = \chi/\mu$. The sheaf $\mathscr{F}$ is called *semistable (stable)* if

(1) $\mathscr{F}$ does not have any 0-dimensional torsion, and
(2) for any proper subsheaf $\mathscr{F}'$, we have

$$p(\mathscr{F}') = \frac{\chi'}{\mu'} \leq (<) \frac{\chi}{\mu} = p(\mathscr{F})$$

where $\chi_{\mathscr{F}'}(t) = \mu' + \chi'$.

Then one can define the moduli space of semistable sheaves on X with linear Hilbert polynomial $\chi(t) = \mu t + \chi$, denoted by $\mathbf{M}_X(\mu, \chi)$. The Hilbert scheme $\mathbf{Hilb}_X(\mu, \chi)$ of curves on X is the space of curves in X with the Hilbert polynomial $\mu t + \chi$ with respect to $\mathscr{O}_X(1)$. Similarly, one can define α-(semi)stability for the pair $(s, \mathscr{F})$ and a positive rational number $\alpha \in \mathscr{Q}_{>0}$, where $\mathscr{F}$ is a purely 1-dimensional sheaf on X and $s : \mathscr{O}_X \to \mathscr{F}$ is a nonzero section, using α-slope

$$\mu_\alpha(s, \mathscr{F}) := \frac{\chi + \alpha}{\mu}.$$

One of our contributions is to relate these moduli spaces birationally to obtain their geometric properties. The following is some of his contribution over a smooth quadric threefold $\mathscr{Q} = \mathscr{Q}_3$; see [8].

Theorem 5.2 *We have* $\mathbf{Hilb}_{\mathscr{Q}}(2, 1) \cong \mathbf{M}_{\mathscr{Q}}(2, 1) \cong \mathrm{Gr}(3, 5)$, *the Grassmannian variety parametrizing projective planes in* $\mathbb{P}^4$.

Theorem 5.3 *For the Hilbert polynomial $\chi(t) = 2t + 2$, we have the following description.*

(1) $\mathbf{Hilb}_{\mathscr{Q}}(2, 2)$ *consists of two rational irreducible components*, $\mathbf{H}_1$ *and* $\mathbf{H}_2$, *of dimension* 9 *and* 6 *respectively*, *and it is smooth outside* $\mathbf{H}_1 \cap \mathbf{H}_2$.
(2) $\mathbf{M}_{\mathscr{Q}}(2, 2)$ *consists of two irreducible components*, $\mathbf{M}_1$ *and* $\mathbf{M}_2$, *of dimension* 6 *both*, *and* $\mathbf{M}_1$ *is rational and smooth outside* $\mathbf{M}_1 \cap \mathbf{M}_2$.
(3) $\mathbf{M}^{\alpha}_{\mathscr{Q}}(2, 2)$ *consists of two rational irreducible components*, $\mathbf{N}_1$ *and* $\mathbf{N}_2$, *of dimension* 7 *and* 6 *respectively*, *and it is smooth outside* $\mathbf{N}_1 \cap \mathbf{N}_2$.

Note that the statement (3) is independent on the choice of α; there is no wall-crossing among the family of moduli spaces $\mathbf{M}^{\alpha}_{\mathscr{Q}}(2, 2)$. Note also that there is no geometric description on $\mathbf{M}_2$ because it consists only of strictly semistable sheaves. Moreover, we suggest a full description of elements in each irreducible components and intersections.

- $\mathbf{H}_1$ consists of non-locally Cohen–Macaulay curves and $\mathbf{H}_2$ is the closure of locally Cohen–Macaulay curves. Their intersection consists of singular conics D with an extra point $p \in D_{\text{sing}}$ such that the hyperplane section containing the curve is singular at p.
- $\mathbf{M}_{1,\text{red}}$ is parametrized by the space of conics in $\mathscr{Q}$ and $(\mathbf{M}_1 \cap \mathbf{M}_2)_{\text{red}}$ is parametrized by the space of singular conics in $\mathscr{Q}$. $\mathbf{M}_2$ has a one-to-one correspondence to $\text{Sym}^2(\mathbb{P}^3)$, the set of pairs of two lines in $\mathscr{Q}$.
- $\mathbf{N}_1$ is birational to the incidence variety of the space of conics in $\mathscr{Q}$ and $\mathbf{N}_2$ is birational to $\text{Sym}^2(\mathbb{P}^3)$.

One of the main ingredient in the work is a deep understanding on multiple curves on X; in this case the double lines; see [26, 27]. Our other contribution is on the topological property of the moduli space, specifically connectedness. In [43] the connectedness of the Hilbert scheme of curves is proven for the fixed degree and genus of curves, although it is classically known that the locus of smooth curves may not be connected. In [6] we investigate the connectedness of the Hilbert scheme of locally Cohen–Macaulay curves on the Segre threefold $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 5.4 *Let* $\mathbf{H}(e_1, e_2, e_3, \chi)_{+,\text{red}}$ *be the reduced Hilbert scheme of locally Cohen–Macaulay curves* C *in* X *with tridegree* (e_1, e_2, e_3) *and* $\chi(\mathscr{O}_C) = \chi$.

(i) $\mathbf{H}(2, 0, 0, a)_{+,\text{red}}$ *is irreducible and rational for* $a \geq 2$;
(ii) $\mathbf{H}(2, 1, 0, a)_{+,\text{red}}$ *has the two irreducible components for* $a \geq 3$;
(iii) $\mathbf{H}(1, 1, 1, a)_{+,\text{red}}$ *is irreducible and rational for* $a \in \{1, 3\}$, *while* $\mathbf{H}(1, 1, 1, 2)_{+,\text{red}}$ *has the three connected components that are rational*;
(iv) $\mathbf{H}(2, 1, 1, 1)_{+,\text{red}}$ *is irreducible and rational.*

The main ingredient in the study of Hilbert schemes of locally Cohen–Macaulay curves with low degree is a rational ribbon and the Ferrand construction, together with the applications of deformation/obstruction theory of moduli space of corresponding stable maps. We delicately use the intersecting property of the double lines with other lines in X to investigate irreducible and connected components of the Hilbert schemes respectively. We applied the same methodology to the case when $X = \mathbb{P}^2 \times \mathbb{P}^1$ the Segre threefold with Picard rank two to obtain several results in the same context.

6 Co-Higgs sheaves

A *co-Higgs* sheaf on a smooth projective variety X is a pair $(\mathscr{E}, \Phi)$ where $\mathscr{E}$ is a torsion-free coherent sheaf on X and $\Phi \in \mathrm{H}^0(\mathscr{E}nd(\mathscr{E}) \otimes T_X)$ for which $\Phi \wedge \Phi = 0$ as an element of $\mathrm{H}^0(\mathscr{E}nd(\mathscr{E}) \otimes \bigwedge^2 T_X)$. Here, Φ is called the *co-Higgs field* of $(\mathscr{E}, \Phi)$ and the condition $\Phi \wedge \Phi = 0$ is an integrability condition. Over the field of complex numbers, it is a generalized holomorphic bundle over a smooth complex projective variety X, considered as a generalized complex manifold [42, 44]. Motivated by the works by Rayan in [61, 62], we investigated the (non-)existence of co-Higgs sheaves on various polarized variety and extended it to the logarithmic setting. First we discovered the relationship between the semistability of a co-Higgs sheaf and the semistability of its ground sheaf in [7].

Theorem 6.1 *For a co-Higgs bundle $(\mathscr{E}, \Phi)$ on a polarized smooth variety $(X, \mathscr{O}_X(1))$ with non-negative Kodaira dimension, the semistability of $(\mathscr{E}, \Phi)$ implies the semistability of $\mathscr{E}$.*

Basically we consider the Harder–Narasimhan filtration of $\mathscr{E}$ with length at least two, by assuming that $\mathscr{E}$ is not semistable. Then we restrict the co-Higgs field to a curve with high degree to obtain the non-semistability of the field. In fact, we disregard the condition on Kodaira dimension for the rank two case, to obtain a similar statement on stability under the assumption that the strict order of instability is low. Here, the strict order of instability of $\mathscr{E}$ is defined to be the maximal integer k such that $\mathrm{h}^0(\det(\mathscr{E} \otimes \mathscr{R})(-k)) > 0$ for all $\mathscr{R} \in \mathrm{Pic}(X)$ with $\mathrm{h}^0(\mathscr{E} \otimes \mathscr{R}) > 0$; see [7, Proposition 2.8]. The results give a way to study moduli of semistable co-Higgs bundles, with a base on the study of moduli of semistable bundles. Indeed, we also assert that any semistable co-Higgs bundle of rank two over a surface of general type has a trivial co-Higgs field, and so the semistability of co-Higgs bundles of rank two is equivalent to the semistability of bundles.

Then we pay our attention to the case when X is a surface and the rank of co-Higgs bundles is two. In [7] we observed that, under the condition of vanishing $\mathrm{H}^0(T_X) = 0$, the existence of unstable co-Higgs bundle of rank two implies that the co-Higgs field is nilpotent, while any non-trivial global tangent vector field suggests an example of strictly semistable co-Higgs bundle of arbitrary rank with injective co-Higgs field. It motivates us to seek for additional conditions to assure that a semistable co-Higgs bundle has a nilpotent co-Higgs field. We prove the following assertion, implying that any surface can achieve this vanishing after a finite number of blow-ups.

Theorem 6.2 *For a smooth projective surface X, there exists another surface X' together with a birational morphism $\sigma : X' \to X$, satisfying the following property: if $\upsilon : X'' \to X'$ is any birational morphism, then every rank two co-Higgs field on X'' is nilpotent.*

We investigated the existence and non-existence of a co-Higgs sheaf with a nilpotent co-Higgs field, especially over the projective space, in [9]. The Hartshorne–Serre correspondence states that the construction of vector bundles of rank at least two is closely related with the structure of two-codimensional locally complete intersection subschemes. Using this correspondence we suggest a way of constructing a nilpotent co-Higgs structure on bundles satisfying a certain condition. Indeed, for a vector bundle admitting an exact sequence

$$0 \to \mathscr{O}_X^{\oplus(r-1)} \xrightarrow{u} \mathscr{E} \xrightarrow{v} \mathscr{I}_{Z,X} \otimes \mathscr{A} \to 0, \tag{3}$$

for a line bundle $\mathscr{A}$ with $\mathrm{h}^0(T_X \otimes \mathscr{A}^\vee) > 0$, one can consider the following composition

$$\mathscr{E} \xrightarrow{v} \mathscr{I}_{Z,X} \otimes \mathscr{A} \xrightarrow{h} T_X \xrightarrow{g} \mathscr{O}_X^{\oplus(r-1)} \otimes T_X \xrightarrow{u \otimes \mathrm{id}} \mathscr{E} \otimes T_X,$$

where the map h is given by a nonzero section in $\mathrm{H}^0(T_X \otimes \mathscr{A}^\vee)$ and the map g is induced by an inclusion $\mathscr{O}_X \to \mathscr{O}_X^{\oplus(r-1)}$. Then we obtain a number of statements, including the following two assertions.

Theorem 6.3 *The set of nilpotent co-Higgs fields on a fixed stable reflexive sheaf $\mathscr{E}$ of rank two on $\mathbb{P}^n$ is identified with the total space of* $\mathrm{H}^0(\mathscr{E}(1)) \otimes \mathscr{O}_{\mathbb{P}^n}(-1)$ *with the zero section blown down to a point corresponding to the trivial field, only if* $c_1(\mathscr{E}) + 2x_{\mathscr{E}} = -3$. *Here, $x_{\mathscr{E}}$ is the maximal integer x such that* $\mathrm{H}^0(\mathscr{E}(-x)) \neq 0$. *In the other cases the set is trivial.*

Theorem 6.4 *For each positive integer c_2, there exist both strictly semistable indecomposable bundle and stable bundle of rank three on $\mathbb{P}^3$ with trivial first Chern class, on which there are nilpotent co-Higgs structures Φ with* $\ker(\Phi) = \mathscr{O}_{\mathbb{P}^3}^{\oplus 2}$.

As a natural generalization, we replace the tangent bundle T_X by the logarithmic tangent bundle $T_X(-\log D)$ in the definition of co-Higgs field, to investigate the existence of nilpotent co-Higgs sheaves with a co-Higgs field vanishing in the normal direction to a given divisor D of X, *i.e.*, a pair $(\mathscr{E}, \Phi)$ of a torsion-free coherent sheaf $\mathscr{E}$ and a morphism $\Phi : \mathscr{E} \to \mathscr{E} \otimes T_X(-\log D)$ with the integrability condition satisfied. The pair is called a *D-logarithmic co-Higgs sheaf* and it is called *2-nilpotent* if $\Phi \circ \Phi$ is trivial. The first result in this direction is on the existence of nilpotent D-logarithmic co-Higgs sheaves of rank at least two in [10] and we use the same trick as in [9].

Theorem 6.5 *Let X be a projective manifold with $\dim(X) \geq 2$ and $D \subset X$ be a simple normal crossing divisor. For a fixed line bundle $\mathscr{L} \in \mathrm{Pic}(X)$ and an integer $r \geq 2$, there exists a 2-nilpotent D-logarithmic co-Higgs sheaf $(\mathscr{E}, \Phi)$, where $\Phi \neq 0$ and $\mathscr{E}$ is reflexive and indecomposable with $c_1(\mathscr{E}) \cong \mathscr{L}$ and* $\mathrm{rank}(\mathscr{E}) = r$.

Indeed, one can strengthen the statement of Theorem 6.5 by requiring $\mathscr{E}$ to be locally free, in cases $\dim(X) = 2$ or $r \geq \dim(X)$, due to the statement of the Hartshorne–Serre correspondence and the dimension of non-locally free locus. One can also notice that the logarithmic co-Higgs sheaves constructed in Theorem 6.5 are highly unstable, which is consistent with the general philosophy on the existence of stable co-Higgs bundles; see [30, Theorem 1.1] for example. We also produce several examples of nilpotent semistable logarithmic co-Higgs sheaves on projective spaces and a smooth quadric surface, using a simple way of construction in [9, 10]. Since the logarithmic co-Higgs sheaves are co-Higgs sheaves in the usual sense with an additional vanishing condition in the normal direction of divisors, their moduli space is a closed subvariety of the moduli of the usual co-Higgs sheaves.

Theorem 6.6 *For $D \in |\mathscr{O}_{\mathbb{P}^2}(1)|$, let $\mathbf{M}_{\mathbb{P}^2}(D; c_1, c_2)$ be the moduli of semistable trace-free D-logarithmic co-Higgs bundles of rank two on $\mathbb{P}^2$ with the Chern classes (c_1, c_2).*

(i) $\mathbf{M}_{\mathbb{P}^2}(D; -1, 0)$ *is isomorphic to the total space of* $\mathscr{O}_D(-2)^{\oplus 6}$.
(ii) $\mathbf{M}_{\mathbb{P}^2}(D; 1, 0)$ *contains the total space of $\mathscr{O}_{\mathbb{P}^5}(-2)$ with the zero section contracted to a point, as an open dense subset.*

Now the additional condition for a co-Higgs field to vanish in the normal direction to D with higher degree, forces the associated bundle to be unstable. Thus one can put more interest on the logarithmic co-Higgs sheaves associated to divisors with high degree and assume that the length of Harder–Narasimhan filtration of a sheaf $\mathscr{E}$

$$\{0\} = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_s = \mathscr{E} \tag{4}$$

is at least two. We fixed the length s at least two of the filtration together with rank r_i and degree d_i of $\mathscr{F}_i/\mathscr{F}_{i-1}$. Setting $\gamma := \deg T_X(-\log D)$ and $\mu_i := d_i/r_i$, one can assume that $\mu_s - \mu_1 \leq \gamma < 0$ as the least requirement for the existence of the non-trivial co-Higgs field. Indeed, if $\mu_s - \mu_1 > \gamma$, then we get $\mathrm{Hom}(\mathscr{E}, \mathscr{E} \otimes T_X(-\log D)) = 0$; see [11, Remark 3.9]. Then we investigate the numeric criterion for the sheaf to admit a non-trivial co-Higgs field as follows in [11].

Theorem 6.7 *Fix the numeric data for the Harder–Narasimhan filtration and denote by* $\mathbb{U}$ *the set of the torsion-free sheaves on an algebraic curve* X *with these data. Then the following hold*:

(i) *there exists an unstable sheaf in* $\mathbb{U}$ *with non-trivial co-Higgs field*;
(ii) *the inequality* $\mu_s - \mu_1 \geq \gamma + 1 - g$ *implies the existence of an unstable sheaf in* $\mathbb{U}$ *with no non-trivial co-Higgs field*;
(iii) *the inequality* $\mu_s - \mu_1 < \gamma + 1 - g$ *implies that every sheaf in* $\mathbb{U}$ *admits a non-trivial co-Higgs field.*

Furthermore we extend the notion of *Segre invariant* to the setting of logarithmic co-Higgs sheaves and show that it is well-defined over curves under the assumption that $\gamma < 0$ and that this invariant is same as the usual Segre invariant under a certain condition.

Theorem 6.8 *For a logarithmic co-Higgs sheaf* $(\mathscr{E}, \Phi)$ *on an algebraic curve* X *with* $\gamma < 0$*, the* k^{th}*-Segre invariant* $s_k(\mathscr{E}, \Phi)$ *is well-defined. It is also equal to the Segre invariant* $s_k(\mathscr{E})$ *in the usual sense, if* $\mathscr{E}$ *admits the complete Harder–Narasimhan filtration, i.e.* $r_i = 1$ *for all* i*.*

Over algebraic curves the bundle $T_X(-\log D)$ is automatically semistable, because the sheaf has rank one. So, as the counterpart to the case of algebraic curves, we deal with the case when the dimension of X is at least 2 and $T_X(-\log D)$ is not semistable. Under the assumption that the biggest slope in the Harder–Narasimhan filtration of $T_X(-\log D)$ is negative, we give a recipe to construct all the pairs $(\mathscr{E}, \Phi)$ with $\mathscr{E}$ reflexive of $\mathrm{rk}(\mathscr{E}) = r \in \{2, 3\}$ and non-trivial co-Higgs field $\Phi : \mathscr{E} \to \mathscr{E} \otimes T_X(-\log D)$. When $r = 2$ and in most cases with $r = 3$, the map Φ is always 2-nilpotent and so it is integrable.

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