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Analytical expressions for the effective coefficients of fibre-reinforced composite materials under the influence of inelastic distortions

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We aim to deduce analytic expressions for the homogenised coefficients that describe the mechanical behaviour of a uniaxially fibre-reinforced composite material consisting of two solid constituents undergoing inelastic distortions, one representing the extracellular matrix, and the other representing the inclusions that model the fibres. While our work is mathematical in nature, our underlying goal is to explore questions related to biology, as biological systems, such as soft and hard tissues, can change their properties in response to various internal and external factors. One of our key motivations is to tackle the computational complexities involved in determining the effective, macroscopic properties of such biological systems. This requires addressing the interactions across different scales of the so-called cell and homogenised problems. To achieve this, we begin by formulating the governing equations that describe the dynamics of the composite's constituents, provided by the balance of linear momentum and the law for the evolution of the inelastic distortions. We then employ the asymptotic homogenisation technique to derive the individual cell-level problems and the homogenised macroscopic equations. This process also yields expressions for effective coefficients that capture the overall behaviour of the composite material. In a first step towards our investigations and to illustrate the capabilities of our approach, we consider the case in which the composite under study possesses a fibre-reinforced structure. Together with additional hypotheses, we concentrate on the calculation of the effective properties using complex variable methods. Finally, after obtaining general formulae, we focus on providing numerical results.

1 | INTRODUCTION

The mathematical modelling of composite materials that can change their mechanical properties in response to internal and external stimuli presents a challenging endeavour. Of special interest is the case of biological composites and the need for an infrastructure based on mathematical models, numerical simulations and analytical techniques capable of mimicking and predicting the evolution of their intrinsic properties.

In this work, we seek to provide insights into the complex interactions occurring in heterogeneous media with multi-scale organisation and complex internal geometry. We are particularly interested in exploring how structural transformations impact the overall properties of heterogeneous media. Such phenomena are prevalent in various biological systems and are associated with incompatible deformations or anelastic distortions of the medium in which they occur [1]. Among the different classes of internal transformations, here, we focus on remodelling [2], interpreted with the production of anelastic distortions. These involve the start and progression of transformations of an irreversible nature that are not resolved in terms of the visible deformation of the medium and induce variations of mechanical properties [3, 4]. Examples of biological systems undergoing internal transformations of the type just described include bones [5, 6], brain tissue [7], and focal adhesions [8], among others.

Our modelling approach builds up from some of our previous works (see, e.g., refs. [4, 9]) by contextualising the Bilby-Kröner-Lee (BKL) decomposition of the deformation gradient tensor (see refs. [1, 10–12] and references therein) in a multi-scale framework. Namely, for each constituent of the composite, the decomposition involves multiplicatively accounting for a purely elastic contribution and a tensor quantity associated with the presence of anelastic distortions and referred to as *remodelling tensor* [13–15]. In line with this description, within a purely mechanical framework, the governing equations of our model rely on the balance of linear momentum and the evolution law for the remodelling tensor. These are recast through the Principle of Virtual Work (PVW) [16], which has been widely utilised in the analysis of inelastic processes (see, e.g., refs. [14, 17–19] and references therein). In particular, the evolution law for the remodelling tensor stems from a balance of generalised, tensor-valued forces work-conjugate to the virtual variations of the remodelling tensor and the study of the dissipation inequality [14, 19–21].

The existing literature on the study of composite materials develops according to different approaches including mixture theory [22–27], which models heterogeneous materials as continuous mixtures of their constituent phases, average field theory [28–34], including the representative volume element (RVE) and Eshelby-based techniques, and asymptotic homogenisation (AH) [35–39] (refer to ref. [40] for a review on these). In this work, we employ the two-scale AH approach, which is a mathematically rigorous technique effectively utilised to calculate the overall properties of various types of heterogeneous media involving coupled phenomena (see, for instance, refs. [41–49]). As far as biomechanics is concerned, the usage of AH is frequent in nanomedicine [50], poroelasticity [51, 52], and biomaterial modelling such as bone [34, 49, 53] or cardiac tissue [54, 55]. A critical aspect of the AH involves elucidating sets of partial differential equations, commonly referred to as *local* or *cell problems*, which are typically defined within a periodic cell. The solutions to these local problems are highly significant, as they produce the *averaged* or *effective coefficients* characterising the homogeneous medium that replaces the original one where the heterogeneities are explicitly accounted for. The finite element method (FEM) and analytical techniques have been employed to solve the cell problems, providing different means to determine the effective coefficients. While analytical solutions to local problems are relatively uncommon and limited in handling geometric complexities, they are more computationally cost-effective than numerical methods and provide valuable benchmark data. In the case of layered composites, analytical expressions for the effective coefficients are well established (see, e.g., refs. [4, 56–59]). For uniaxially fibre-reinforced composites with simple geometrical features, solutions to the local problems are also available (see, for instance, refs. [57, 60–66]). In this specific case of uniaxially fibre-reinforced composites, the local problems describing the unit cell are decoupled into plane and antiplane formulations for which the solutions can be derived using potential methods involving complex variables and the properties of the quasi-periodic Weierstrass zeta function [67–69]. On the other hand, the solution of the cell problems has also been considered in refs. [43, 70] based on the multipole method [71, 72]. A discussion on the numerical solutions to cell problems arising from the AH approach is available in refs. [46, 73].

As in refs. [4, 9], our goal is to formulate a multi-scale description of remodelling. However, a key difference from refs. [4, 9] lies in our objective to provide analytical expressions for the effective properties of a uniaxially fibre-reinforced composite. While in refs. [4, 9], we specialised the general theory for layered composites, in ref. [4] we focused on the influence of gradient effects on the remodelling variable. We remark that, as per the discussions in ref. [9], the study of homogenised systems undergoing remodelling effects requires dedicated numerical schemes capable of resolving the transfer of information from the internal structure (encoded in the cell problems and effective coefficients) to the macroscopic model and, vice versa, in a space and time-dependent way. To the best of our knowledge, there are no readily available computational schemes capable of solving coupled problems across scales as described in this work. A proposed numerical scheme to address such issues was provided in ref. [9], which, for complex microstructural settings, would be computationally expensive. Even in the benchmark scenario described in this work, it would be challenging without an analytical solution. Hence, an additional goal of our work is to provide analytical results that can serve as benchmark tests for numerical schemes for cases with simplified geometries and favourable symmetry properties. In fact, when more complex microstructures are considered, a good way of proceeding would be to employ validated numerical schemes. The analytical procedure is based on the theory of harmonic functions and complex potentials [57, 62, 69, 74] and, in the framework of this work, permits obtaining closed-form expressions for the effective coefficients of uniaxially fibre-reinforced composites parametrised by space and time through the remodelling tensor. It is noteworthy that in the linearised viscoelastic framework discussed in refs. [66, 75], closed-form expressions have been derived for the effective coefficients of unidirectional fibre-reinforced materials. These are parametrised by the complex variable dual to time arising from the Laplace–Carson transform. In the present work, however, the time- and space-dependent effective coefficients are driven by the evolution law for the inelastic distortions. Finally, we mention that another difference with respect to ref. [9] pertains to our constitutive selection in the characterisation of the progression of inelastic distortions.

The manuscript is structured as follows. Sections 2 and 3 present the kinematic description and the governing equations of the mathematical problem, respectively. Following this, in Section 4, we examine the system dissipation inequality to derive the constitutive relationships that dictate the behaviour of the composite constituents. This includes recasting the equation for the evolution of the remodelling tensor and the additional constraints it needs to fulfill. In Section 5, our focus is on conducting the AH of the momentum balance and the evolution law for the inelastic distortions. In doing this, we derive the cell and homogenised problems that result from our analysis. Next, we illustrate the general theory with the example of a uniaxially-aligned fibre-reinforced composite in Section 6. In particular, as detailed in Section 7, by introducing additional assumptions, we are able to solve the specified cell problems using complex variable methods. Subsequently, in Section 8, we derive the expressions for the effective coefficients appearing in the homogenised problem. Finally, in Section 9, we present and discuss some results by highlighting the impact of the remodelling tensor on the overall characterisation of the composite under study.

2 | KINEMATICS

Let $\mathcal{B}$ represent the reference configuration of a body consisting of two (continuum) solid constituents with reference placements $\mathcal{B}_{\mathrm{m}}$ and $\mathcal{B}_{\mathrm{f}}$ where the labels m and f are used to distinguish the matrix $\mathcal{B}_{\mathrm{m}}$ from the disjoint union of N inclusions $\mathcal{B}_{\mathrm{f}} = \sqcup_{\alpha=1}^{N} \mathcal{B}_{\mathrm{f}\alpha}$. In this setting, we can identify $\mathcal{B}$ as the heterogeneous body given by $\mathcal{B} = \bar{\mathcal{B}}_{\mathrm{m}} \sqcup \bar{\mathcal{B}}_{\mathrm{f}}$, where the bar notation is used to denote the topological closure operator, and, under the assumption of geometrically identical inclusions, we identify with Γ the interface separating $\mathcal{B}_{\mathrm{m}}$ and $\mathcal{B}_{\mathrm{f}\alpha}$ for each $\alpha = 1, 2, \dots, N$. Furthermore, the unit normal to Γ taken from the inclusion $\mathcal{B}_{\mathrm{f}\alpha}$ to the matrix $\mathcal{B}_{\mathrm{m}}$ is identified by $\boldsymbol{N}$, that is $\boldsymbol{N} := \boldsymbol{N}_{\mathrm{f}} = -\boldsymbol{N}_{\mathrm{m}}$ with $\boldsymbol{N}_{\mathrm{m}}$ being the unit normal to Γ taken from the matrix $\mathcal{B}_{\mathrm{m}}$ to the inclusion $\mathcal{B}_{\mathrm{f}\alpha}$.

By introducing the motion $\chi : \mathcal{B} \times [t_0, t_{\mathrm{f}}[\to \mathcal{S}$, where $\mathcal{S}$ is the three-dimensional Euclidean space and $[t_0, t_{\mathrm{f}}[$ is an interval of $\mathbb{R}^+$, the current configuration of $\mathcal{B}$ is $\mathcal{B}_t := \chi(\mathcal{B}, t)$. If $\chi_{\mathrm{m}} := \chi|_{\mathcal{B}_{\mathrm{m}}}$ and $\chi_{\mathrm{f}} := \chi|_{\mathcal{B}_{\mathrm{f}}}$ denote the restrictions of χ to $\mathcal{B}_{\mathrm{m}}$ and $\mathcal{B}_{\mathrm{f}}$, respectively, the current configurations of $\mathcal{B}_{\mathrm{m}}$ and $\mathcal{B}_{\mathrm{f}}$ are specified with $\mathcal{B}_{\mathrm{m}t} := \chi_{\mathrm{m}}(\mathcal{B}_{\mathrm{m}}, t)$ and $\mathcal{B}_{\mathrm{f}t} := \chi_{\mathrm{f}}(\mathcal{B}_{\mathrm{f}}, t)$. In particular, the interface between $\mathcal{B}_{\mathrm{m}t}$ and $\mathcal{B}_{\mathrm{f}t}$ is denoted with $\Gamma_t := \chi(\Gamma, t)$. Furthermore, we denote by $\boldsymbol{F}_\eta$, with $\eta = \mathrm{m}, \mathrm{f}$, the deformation gradient tensor associated with the motion $\chi_\eta : \mathcal{B}_\eta \times [t_0, t_{\mathrm{f}}] \to \mathcal{S}$ which, by adhering to a Cartesian framework, can be written in terms of the displacement field $\boldsymbol{u}_\eta$. That is,

$$\boldsymbol{F}_\eta(X, t) = \boldsymbol{I} + \mathrm{Grad}\, \boldsymbol{u}_\eta(X, t), \tag{1}$$

where $X \in \mathcal{B}_\eta$ and $\boldsymbol{I}$ denotes the second-order identity tensor.

As previously discussed in the introduction, our focus lies in describing the effective characteristics of the heterogeneous medium by capturing the presence of inelastic phenomena occurring within the internal structure of its constituents. As a means to achieve this description, we introduce the Bilby-Kröner-Lee (BKL) decomposition of each constituent's deformation gradient tensor (see, e.g., refs. [1, 10–12]) as

$$\boldsymbol{F}_\eta(X, t) = \boldsymbol{F}_{\mathrm{e}\eta}(X, t)\boldsymbol{K}_\eta(X, t). \tag{2}$$

The tensor $\boldsymbol{F}_{\mathrm{e}\eta}$ is regarded as a purely elastic contribution to the visible deformation, while the tensor $\boldsymbol{K}_\eta$ is associated to the distortions of inelastic nature occurring at the internal structure of each constituent. In the context of biomechanics, the generally non integrable tensor $\boldsymbol{K}_\eta$ is also called the *remodelling tensor* (see, e.g., refs. [1, 2, 10, 21, 76]) and generates variations in the constituent's mechanical properties. The BKL-decomposition (2) is used to reflect the structural evolution that each constituent undergoes, starting from the reference configuration, through $\boldsymbol{K}_\eta$, and introduces a stress-free state, usually called the *natural state*, which we denote by $\mathcal{N}_{\eta t}$. Due to the inelastic nature of remodelling for which the removal of external loads does not lead to a recovering of the original configuration of the constituent because of the presence of residual stresses, the stress-free characterisation of the natural state is achieved by a virtual tearing process leading to completely relaxed pieces [10, 13, 21, 77, 78]. In this context, for each pair (X, t) in $\mathcal{B}_\eta \times [t_0, t_\mathrm{f}[$, the remodelling tensor maps vectors of the tangent space $T_X\mathcal{B}_\eta$ into vectors of $\mathcal{N}_{\eta t}(X)$, while $\boldsymbol{F}_{\mathrm{e}\eta}$ maps vectors of $\mathcal{N}_{\eta t}(X)$ into vectors of the tangent space $T_{\chi_\eta(X,t)}\mathcal{S}$.

The determinants of the tensors $\boldsymbol{F}_\eta$, $\boldsymbol{F}_{\mathrm{e}\eta}$ and $\boldsymbol{K}_\eta$ are denoted by J_η, $J_{\mathrm{e}\eta}$ and $J_{\boldsymbol{K}\eta}$, respectively, and they are assumed to be strictly positive. In particular, focusing on inelastic processes that do not play a role in promoting growth and are instead related to remodelling, we establish the requirement that $J_{\boldsymbol{K}\eta}$ must equal 1 for all $X \in \mathcal{B}$ and $t \in [t_0, t_\mathrm{f}[$ [13, 79], that is, $\boldsymbol{K}_\eta$ is assumed to be isochoric. Therefore, by noticing that $\dot{J}_{\boldsymbol{K}\eta} = J_{\boldsymbol{K}\eta}\mathrm{tr}(\boldsymbol{K}_\eta^{-1}\dot{\boldsymbol{K}}_\eta) = \boldsymbol{K}_\eta^{-\mathrm{T}} : \dot{\boldsymbol{K}}_\eta$, where the superimposed dot denotes the partial derivative with respect to time, the isochoricity constraint on $\boldsymbol{K}_\eta$ can be rephrased as

$$\mathrm{tr}(\boldsymbol{K}_\eta^{-1}\dot{\boldsymbol{K}}_\eta) = \boldsymbol{K}_\eta^{-\mathrm{T}} : \dot{\boldsymbol{K}}_\eta = 0, \tag{3}$$

which implies that $\boldsymbol{K}_\eta^{-1}\dot{\boldsymbol{K}}_\eta$ is a deviatoric second-order tensor. We remark that although in this work we concentrate our efforts on describing remodelling processes within each constituent, the methodology can be generalised (even though not in a straightforward manner) so that growth-induced inelastic distortions are accounted for. A first step towards this path requires re-conceiving the constraint (3) to be rephrased in the form $\boldsymbol{K}_\eta^{-\mathrm{T}} : \dot{\boldsymbol{K}}_\eta = R_{\mathrm{g}\eta}$ [13, 18, 79], where $R_{\mathrm{g}\eta}$, referred to as growth law in ref. [18], is a scalar field accounting for the activation, progression, and deactivation of growth in each constituent and that can be found from constitutive considerations (see, for instance, refs. [18, 19, 27]).

3 | GOVERNING EQUATIONS

For the purposes of this work and similarly to refs. [14, 18], we consider that, for $\eta = \mathrm{m}, \mathrm{f}$, the kinematic descriptors of our model are χ_η and $\boldsymbol{K}_\eta$, which describe, respectively, the visible deformation of each constituent and the evolution of their internal structure. Within this context, we set our work in a purely mechanical framework where the material response of both constituents is influenced by the production of inelastic distortions. In particular, by considering a theory of grade one in the deformation, as is the case for elastic materials, and of grade zero in $\boldsymbol{K}_\eta$, we formulate the PVW associated to each constituent as [18]

$$\int_{\mathcal{B}_\eta} \boldsymbol{P}_\eta : \mathrm{Grad}\delta\chi_\eta \,\mathrm{d}V + \int_{\mathcal{B}_\eta} (\alpha_\eta \boldsymbol{I} + \boldsymbol{Y}_\eta) : \boldsymbol{K}_\eta^{-1}\delta\boldsymbol{K}_\eta \,\mathrm{d}V = \int_{\partial_\mathrm{N}\mathcal{B}_\eta} \boldsymbol{\tau}_\eta \cdot \delta\chi_\eta \,\mathrm{d}A + \int_{\mathcal{B}_\eta} \boldsymbol{Z}_\eta : \boldsymbol{K}_\eta^{-1}\delta\boldsymbol{K}_\eta \,\mathrm{d}V, \tag{4}$$

where $\delta\chi_\eta$ and $\delta\boldsymbol{K}_\eta$ refer to the virtual variations of χ_η and $\boldsymbol{K}_\eta$, respectively. In Equation (4), $\boldsymbol{P}_\eta$ denotes the first Piola–Kirchhoff stress tensor of the constituent η and $\boldsymbol{\tau}_\eta$ is a boundary contact force on the Neumann portion of the boundary of $\mathcal{B}_\eta$, namely $\partial_\mathrm{N}\mathcal{B}_\eta$. Moreover, α_η is the Lagrange multiplier associated with the constraint of isochoric inelastic distortions. The second-order tensors $\boldsymbol{Y}_\eta$ and $\boldsymbol{Z}_\eta$ in the PVW define two generalised forces of internal and external nature, respectively, and dual to $\boldsymbol{K}_\eta^{-1}\delta\boldsymbol{K}_\eta$ [14, 18]. While $\boldsymbol{Y}_\eta$ can be derived from constitutive principles (refer to Section 4), the external non-conventional force $\boldsymbol{Z}_\eta$ can be assigned according to the specific phenomenon under investigation and represents manifestations of various low-scale phenomena. For instance, when it comes to growth, Di Carlo & Quiligotti [80] ascribe it to non-mechanical processes, such as biochemical reactions occurring within living tissues (see, for instance, refs. [18, 27]). As per remodelling processes, a rather standard assumption is to take a null $\boldsymbol{Z}_\eta$ (see refs. [81, 82] and references therein), which as will be clear in the subsequent sections, still has a non-trivial relevance. Otherwise, a possibility in describing $\boldsymbol{Z}_\eta$ is to rely on a target value of stress and, thus, on the notion of homeostatic stress (see refs. [15, 19]).

Due to the specific geometrical setting of the constituents in $\mathcal{B}$, the boundary of $\mathcal{B}_\mathrm{m}$ is given by $\partial\mathcal{B}_\mathrm{m} = \partial^\mathrm{ext}\mathcal{B}_\mathrm{m} \sqcup \Gamma$, where $\partial^\mathrm{ext}\mathcal{B}_\mathrm{m}$ denotes the external boundary of $\mathcal{B}_\mathrm{m}$, that is, the boundary not in contact with $\mathcal{B}_\mathrm{f}$, whereas $\partial\mathcal{B}_\mathrm{f} = \Gamma$. In this setting, for $\eta = \mathrm{m}$, Equation (4) can be equivalently rewritten as

$$-\int_{\mathcal{B}_\mathrm{m}} \mathrm{Div}(\boldsymbol{P}_\mathrm{m}) \cdot \delta\chi_\mathrm{m}\, \mathrm{d}V - \int_\Gamma (\boldsymbol{P}_\mathrm{m}\boldsymbol{N}) \cdot \delta\chi_\mathrm{m}\, \mathrm{d}A + \int_{\partial_\mathrm{N}^\mathrm{ext}\mathcal{B}_\mathrm{m}} (\boldsymbol{P}_\mathrm{m}\boldsymbol{N}^\mathrm{ext}) \cdot \delta\chi_\mathrm{m}\, \mathrm{d}A + \int_{\mathcal{B}_\mathrm{m}} (\alpha_\mathrm{m}\boldsymbol{I} + \boldsymbol{Y}_\mathrm{m}) : \boldsymbol{K}_\mathrm{m}^{-1}\delta\boldsymbol{K}_\mathrm{m}\, \mathrm{d}V$$
$$= -\int_\Gamma \boldsymbol{\tau}_\mathrm{m} \cdot \delta\chi_\mathrm{m}\, \mathrm{d}A + \int_{\partial_\mathrm{N}^\mathrm{ext}\mathcal{B}_\mathrm{m}} \boldsymbol{\tau}_\mathrm{m}^\mathrm{ext} \cdot \delta\chi_\mathrm{m}\, \mathrm{d}A + \int_{\mathcal{B}_\mathrm{m}} \boldsymbol{Z}_\mathrm{m} : \boldsymbol{K}_\mathrm{m}^{-1}\delta\boldsymbol{K}_\mathrm{m}\, \mathrm{d}V, \tag{5}$$

where the minus sign in the first addend of the right-hand side of Equation (5) is a consequence of Cauchy's lemma (see, e.g. ref. [83] and references therein), while for $\eta = \mathrm{f}$,

$$-\int_{\mathcal{B}_\mathrm{f}} \mathrm{Div}(\boldsymbol{P}_\mathrm{f}) \cdot \delta\chi_\mathrm{f}\, \mathrm{d}V + \int_\Gamma (\boldsymbol{P}_\mathrm{f}\boldsymbol{N}) \cdot \delta\chi_\mathrm{f}\, \mathrm{d}A + \int_{\mathcal{B}_\mathrm{f}} (\alpha_\mathrm{f}\boldsymbol{I} + \boldsymbol{Y}_\mathrm{f}) : \boldsymbol{K}_\mathrm{f}^{-1}\delta\boldsymbol{K}_\mathrm{f}\, \mathrm{d}V = \int_\Gamma \boldsymbol{\tau}_\mathrm{f} \cdot \delta\chi_\mathrm{f}\, \mathrm{d}A + \int_{\mathcal{B}_\mathrm{f}} \boldsymbol{Z}_\mathrm{f} : \boldsymbol{K}_\mathrm{f}^{-1}\delta\boldsymbol{K}_\mathrm{f}\, \mathrm{d}V, \tag{6}$$

where $\boldsymbol{N}^\mathrm{ext}$ is the unit normal to $\partial^\mathrm{ext}\mathcal{B}_\mathrm{m}$ which has been decomposed as $\partial^\mathrm{ext}\mathcal{B}_\mathrm{m} = \partial_\mathrm{N}^\mathrm{ext}\mathcal{B}_\mathrm{m} \sqcup \partial_\mathrm{D}^\mathrm{ext}\mathcal{B}_\mathrm{m}$ with $\partial_\mathrm{D}^\mathrm{ext}\mathcal{B}_\mathrm{m}$ being the portion of $\partial^\mathrm{ext}\mathcal{B}_\mathrm{m}$ where Dirichlet boundary conditions are prescribed. We notice that Equations (5)–(6) hold true for arbitrary $\delta\chi_\mathrm{m}$ vanishing on $\partial_\mathrm{D}^\mathrm{ext}\mathcal{B}_\mathrm{m}$. Furthermore, under the assumption of no jump in the displacement field at the interface Γ, we can write $\chi(\Gamma, t) := \chi_\mathrm{m}(\Gamma, t) = \chi_\mathrm{f}(\Gamma, t)$, so that after adding together (5)–(6) over $\eta = \mathrm{m}, \mathrm{f}$, we obtain

$$-\sum_{\eta\in\{\mathrm{m},\mathrm{f}\}} \int_{\mathcal{B}_\eta} \mathrm{Div}(\boldsymbol{P}_\eta) \cdot \delta\chi_\eta\, \mathrm{d}V + \sum_{\eta\in\{\mathrm{m},\mathrm{f}\}} \int_{\mathcal{B}_\eta} \left\{\alpha_\eta\boldsymbol{I} + \boldsymbol{Y}_\eta - \boldsymbol{Z}_\eta\right\} : \boldsymbol{K}_\eta^{-1}\delta\boldsymbol{K}_\eta\, \mathrm{d}V$$
$$= -\sum_{\eta\in\{\mathrm{m},\mathrm{f}\}} \int_\Gamma \{[\![\boldsymbol{\tau}]\!] - [\![\boldsymbol{P}\boldsymbol{N}]\!]\} \cdot \delta\chi\, \mathrm{d}A + \int_{\partial_\mathrm{N}^\mathrm{ext}\mathcal{B}_\mathrm{m}} \left\{\boldsymbol{\tau}_\mathrm{m}^\mathrm{ext} - \boldsymbol{P}_\mathrm{m}\boldsymbol{N}^\mathrm{ext}\right\} \cdot \delta\chi_\mathrm{m}\, \mathrm{d}A. \tag{7}$$

In Equation (7), the jump of a quantity ϕ is defined as $[\![\phi]\!] := \phi_\mathrm{f} - \phi_\mathrm{m}$. Hence, considering that Equation (7) needs to hold for arbitrary choices of $\delta\chi_\mathrm{m}$ in $\mathcal{B}_\mathrm{m}$, on Γ and on $\partial_\mathrm{N}^\mathrm{ext}\mathcal{B}_\mathrm{m}$, and of $\delta\chi_\mathrm{f}$ in $\mathcal{B}_\mathrm{f}$ and on Γ, and assuming no jump in the tractions at the interface Γ, namely $[\![\boldsymbol{\tau}]\!] = 0$, the governing equations for the composite material undergoing structural transformations are

$$\mathrm{Div}(\boldsymbol{P}_\eta) = \boldsymbol{0}, \quad \text{in } \mathcal{B}_\eta, \tag{8a}$$
$$\mathrm{dev}(\boldsymbol{Y}_\eta) = \mathrm{dev}(\boldsymbol{Z}_\eta), \quad \text{in } \mathcal{B}_\eta, \tag{8b}$$
$$\alpha_\eta + \mathrm{sph}(\boldsymbol{Y}_\eta) = \mathrm{sph}(\boldsymbol{Z}_\eta), \quad \text{in } \mathcal{B}_\eta, \tag{8c}$$
$$\boldsymbol{K}_\eta^{-\mathrm{T}} : \dot{\boldsymbol{K}}_\eta = 0, \quad \text{in } \mathcal{B}_\eta, \tag{8d}$$

with the following interface and boundary conditions

$$[\![\boldsymbol{u}]\!] = \boldsymbol{0}, \quad \text{on } \Gamma, \tag{9a}$$
$$[\![\boldsymbol{P}\boldsymbol{N}]\!] = \boldsymbol{0}, \quad \text{on } \Gamma, \tag{9b}$$
$$\boldsymbol{\tau}_\mathrm{m}^\mathrm{ext} = \boldsymbol{P}_\mathrm{m}\boldsymbol{N}^\mathrm{ext}, \quad \text{on } \partial_\mathrm{N}^\mathrm{ext}\mathcal{B}_\mathrm{m}. \tag{9c}$$

In Equations (8b) and (8c), for a second-order tensor $\boldsymbol{A}$, $\mathrm{dev}(\boldsymbol{A}) := \boldsymbol{A} - \frac{1}{3}\mathrm{tr}(\boldsymbol{A})\boldsymbol{I}$ denotes the deviatoric part of $\boldsymbol{A}$ and $\mathrm{sph}(\boldsymbol{A}) := \frac{1}{3}\mathrm{tr}(\boldsymbol{A})\boldsymbol{I}$ its spherical part. Furthermore, we remark that Equations (8b) and (8c) are the result of separating the deviatoric and spherical parts of the balance of non-conventional forces $\alpha_\eta\boldsymbol{I} + \boldsymbol{Y}_\eta = \boldsymbol{Z}_\eta$. The investigation of this equation, which forms the basis for deriving the evolution law for the remodelling tensor, has attracted considerable attention in previous literature (see, for instance, refs. [14, 78, 81, 84, 85]), and generalisations have been achieved by including both $\boldsymbol{K}_\eta$ and its gradient, $\mathrm{Grad}\,\boldsymbol{K}_\eta$, in the set of kinematic descriptors, thereby conducting to a first-order theory of remodelling [17].

It is worth noticing that Equations (8a), (8b) and (8d) are enough to find the three components of the motion χ_η and the nine components of the tensor $\boldsymbol{K}_\eta$ (since the Lagrange multiplier α_η can be found upon assigning $\boldsymbol{Z}_\eta$ and constitutively determining $\boldsymbol{Y}_\eta$), provided that appropriate constitutive considerations are made for $\boldsymbol{P}_\eta$ and $\boldsymbol{Y}_\eta$ and that a Dirichlet boundary condition for χ_{m} on $\partial_{\mathrm{D}}^{\mathrm{ext}}\mathcal{B}_{\mathrm{m}}$ and initial conditions for χ_η and $\boldsymbol{K}_\eta$ are assigned. In this work, the AH technique is applied in regions sufficiently distant from the outer boundary of the medium under study so that the analysis in the following sections remains unaffected by the selection of the boundary condition.

4 | CONSTITUTIVE CONSIDERATIONS

In this section, we turn our attention to the dissipation inequality to find constitutive relationships dictating the behaviour of the composite constituents. We start by defining the dissipation density D_η per unit volume of $\mathcal{B}_\eta$ so that dissipation inequality in its global form reads [16, 19, 81]

$$0 \leq \int_{\mathcal{R}_\eta} D_\eta := -\overline{\int_{\mathcal{R}_\eta} \Psi_\eta}^{\boldsymbol{\cdot}} + \int_{\partial\mathcal{R}_\eta} (\boldsymbol{P}_\eta \boldsymbol{N}_\eta) \cdot \boldsymbol{v}_\eta + \int_{\mathcal{R}_\eta} \boldsymbol{Z}_\eta : (\boldsymbol{K}_\eta^{-1}\dot{\boldsymbol{K}}_\eta), \tag{10}$$

where Ψ_η denotes the Helmholtz free energy density per unit volume of $\mathcal{R}_\eta$ (a time-independent open subregion of $\mathcal{B}_\eta$) and $\boldsymbol{v}_\eta := \dot{\chi}_\eta$ is the velocity field of the η-th constituent. We notice that, in writing (10), we have neglected, as specified in Equation (8a), inertial terms and body forces. Furthermore, as indicated on the right-hand side of Equation (4), the last two integrals of (10) define the dissipation in terms of the external power to $\mathcal{R}_\eta$. In particular, by incorporating the contribution of the internal non-conventional force $\boldsymbol{Y}_\eta$ into the dissipation inequality through the balance of non-conventional forces, Equation (10) can be rewritten as

$$0 \leq \int_{\mathcal{R}_\eta} D_\eta = -\overline{\int_{\mathcal{R}_\eta} \Psi_\eta}^{\boldsymbol{\cdot}} + \int_{\partial\mathcal{R}_\eta} (\boldsymbol{P}_\eta \boldsymbol{N}_\eta) \cdot \boldsymbol{v}_\eta + \int_{\mathcal{R}_\eta} (\alpha_\eta \boldsymbol{I} + \boldsymbol{Y}_\eta) : (\boldsymbol{K}_\eta^{-1}\dot{\boldsymbol{K}}_\eta). \tag{11}$$

Thus, noticing that $(\boldsymbol{P}_\eta \boldsymbol{N}_\eta) \cdot \boldsymbol{v}_\eta = ((\boldsymbol{P}_\eta)^{\mathrm{T}}\boldsymbol{v}_\eta) \cdot \boldsymbol{N}_\eta$, the use of Gauss Theorem and the balance of linear momentum (8a) in the surface integral in (11) results, after localisation, in the following local form of the dissipation inequality

$$0 \leq D_\eta = -\dot{\Psi}_\eta + \boldsymbol{P}_\eta : \dot{\boldsymbol{F}}_\eta + \boldsymbol{Y}_\eta : (\boldsymbol{K}_\eta^{-1}\dot{\boldsymbol{K}}_\eta), \tag{12}$$

where we have also used the fact that $\boldsymbol{I} : (\boldsymbol{K}_\eta^{-1}\dot{\boldsymbol{K}}_\eta) = \mathrm{tr}(\boldsymbol{K}_\eta^{-1}\dot{\boldsymbol{K}}_\eta) = 0$. Furthermore, if $\Psi_{\nu\eta} = \hat{\Psi}_{\nu\eta}\circ(\boldsymbol{F}_\eta\boldsymbol{K}_\eta^{-1})$, with $\circ$ being the composition operator, is the Helmholtz free energy density per unit volume of the natural state, we can write $\Psi_\eta = \hat{\Psi}_\eta\circ(\boldsymbol{F}_\eta, \boldsymbol{K}_\eta) = J_{\boldsymbol{K}\eta}[\hat{\Psi}_{\nu\eta}\circ(\boldsymbol{F}_\eta\boldsymbol{K}_\eta^{-1})]$. Hence, by recalling that $\boldsymbol{K}_\eta$ is isochoric, that is, $J_{\boldsymbol{K}\eta} = 1$, the chain rule allows expressing the local dissipation inequality in the equivalent form

$$0 \leq D_\eta = \left\{ \boldsymbol{P}_\eta - \left(\frac{\partial \hat{\Psi}_{\nu\eta}}{\partial \boldsymbol{F}_\eta \boldsymbol{K}_\eta^{-1}} \circ (\boldsymbol{F}_\eta \boldsymbol{K}_\eta^{-1}) \right) \boldsymbol{K}_\eta^{-\mathrm{T}} \right\} : \dot{\boldsymbol{F}}_\eta + \left\{ \boldsymbol{Y}_\eta + \boldsymbol{F}_\eta^{\mathrm{T}} \left(\frac{\partial \hat{\Psi}_{\nu\eta}}{\partial \boldsymbol{F}_\eta \boldsymbol{K}_\eta^{-1}} \circ (\boldsymbol{F}_\eta \boldsymbol{K}_\eta^{-1}) \right) \boldsymbol{K}_\eta^{-\mathrm{T}} \right\} : \boldsymbol{K}_\eta^{-1}\dot{\boldsymbol{K}}_\eta. \tag{13}$$

Adopting the Coleman & Noll procedure [86], the first Piola-Kirchhoff stress tensor is then identified as

$$\boldsymbol{P}_\eta = \left(\frac{\partial \hat{\Psi}_{\nu\eta}}{\partial \boldsymbol{F}_\eta \boldsymbol{K}_\eta^{-1}} \circ (\boldsymbol{F}_\eta \boldsymbol{K}_\eta^{-1}) \right) \boldsymbol{K}_\eta^{-\mathrm{T}}, \tag{14}$$

while, we recognise Mandel stress tensor in the second addend of (13), namely

$$\boldsymbol{\Sigma}_\eta := \boldsymbol{F}_\eta^{\mathrm{T}} \boldsymbol{P}_\eta = \boldsymbol{F}_\eta^{\mathrm{T}} \left(\frac{\partial \hat{\Psi}_{\nu\eta}}{\partial \boldsymbol{F}_\eta \boldsymbol{K}_\eta^{-1}} \circ (\boldsymbol{F}_\eta \boldsymbol{K}_\eta^{-1}) \right) \boldsymbol{K}_\eta^{-\mathrm{T}}. \tag{15}$$

Thus, by additively decomposing $\boldsymbol{Y}_\eta$ as

$$\boldsymbol{Y}_\eta = \boldsymbol{Y}_{\eta\mathrm{d}} + \boldsymbol{Y}_{\eta\mathrm{en}}, \tag{16}$$

where $\boldsymbol{Y}_{\eta\mathrm{d}}$ denotes the dissipative part of $\boldsymbol{Y}_\eta$ and $\boldsymbol{Y}_{\eta\mathrm{en}} := -\boldsymbol{\Sigma}_\eta$ its non-dissipative part, the dissipation inequality reduces to [19]

$$0 \leq D_\eta = \boldsymbol{Y}_{\eta\mathrm{d}} : \boldsymbol{K}_\eta^{-1}\dot{\boldsymbol{K}}_\eta. \tag{17}$$

According to Equation (17), suitable forms of $\boldsymbol{Y}_{\eta\mathrm{d}}$ must adhere to the constraints imposed by this reduced dissipation inequality. In Section 4.2, we present a form that satisfies this constraint, ensuring that $\boldsymbol{Y}_\eta = \boldsymbol{Y}_{\eta\mathrm{d}} + \boldsymbol{Y}_{\eta\mathrm{en}}$ is fully determined from constitutive principles and, thus, as anticipated in previous sections, defined irrespective of whether $\boldsymbol{Z}_\eta$ is assumed to be null.

4.1 | First Piola–Kirchhoff stress tensor

As in refs. [4, 9], and given our motivations for providing analytical expressions for the homogenised mechanical properties of the composite, we proceed with the selection of the De Saint–Venant energy density

$$\hat{\Psi}_{\nu\eta}\circ(\boldsymbol{F}_\eta\boldsymbol{K}_\eta^{-1}) = \check{\Psi}_{\nu\eta}\circ(\boldsymbol{E}_{\mathrm{e}\eta}) = \frac{1}{2}\boldsymbol{E}_{\mathrm{e}\eta} : \mathscr{C}_{\nu\eta} : \boldsymbol{E}_{\mathrm{e}\eta}, \tag{18}$$

where $\boldsymbol{E}_{\mathrm{e}\eta} = \frac{1}{2}[(\boldsymbol{F}_{\mathrm{e}\eta})^{\mathrm{T}}\boldsymbol{F}_{\mathrm{e}\eta} - \boldsymbol{I}]$ is the elastic Green–Lagrange strain tensor and $\mathscr{C}_{\nu\eta}$ is the fourth-order elasticity tensor of the η-constituent referred to the natural state. We assume that $\mathscr{C}_{\nu\eta}$ is positive definite and possesses major and minor symmetries.

Substituting (18) into the expressions for the first Piola-Kirchhoff stress tensor and the Mandel stress tensor, namely (14) and (15), we obtain

$$\boldsymbol{P}_\eta = \left(\frac{\partial\check{\Psi}_{\nu\eta}}{\partial\boldsymbol{E}_{\mathrm{e}\eta}}\circ(\boldsymbol{E}_{\mathrm{e}\eta}) : \frac{\partial\boldsymbol{E}_{\mathrm{e}\eta}}{\partial\boldsymbol{F}_\eta\boldsymbol{K}_\eta^{-1}}\right)\boldsymbol{K}_\eta^{-\mathrm{T}} = \left\{\left(\mathscr{C}_{\nu\eta} : \boldsymbol{E}_{\mathrm{e}\eta}\right) : \frac{1}{2}\left((\boldsymbol{F}_{\mathrm{e}\eta})^{\mathrm{T}}\underline{\otimes}\boldsymbol{I} + \boldsymbol{I}\overline{\otimes}(\boldsymbol{F}_{\mathrm{e}\eta})^{\mathrm{T}}\right)\right\}\boldsymbol{K}_\eta^{-\mathrm{T}} = \boldsymbol{F}_\eta\{\mathscr{C}_\eta : (\boldsymbol{E}_\eta - \boldsymbol{E}_{\boldsymbol{K}_\eta})\}, \tag{19a}$$

$$\boldsymbol{\Sigma}_\eta = \boldsymbol{F}_\eta^{\mathrm{T}}\boldsymbol{P}_\eta = \boldsymbol{C}_\eta\{\mathscr{C}_\eta : (\boldsymbol{E}_\eta - \boldsymbol{E}_{\boldsymbol{K}_\eta})\}, \tag{19b}$$

where $\boldsymbol{C}_\eta := (\boldsymbol{F}_\eta)^{\mathrm{T}}\boldsymbol{F}_\eta$ is the right Cauchy–Green deformation tensor associated with the η-th constituent, $\boldsymbol{E}_\eta := \frac{1}{2}(\boldsymbol{C}_\eta - \boldsymbol{I})$ is the Green–Lagrange strain tensor and $\boldsymbol{E}_{\boldsymbol{K}_\eta} := \frac{1}{2}[(\boldsymbol{K}_\eta)^{\mathrm{T}}\boldsymbol{K}_\eta - \boldsymbol{I}]$. Furthermore, $\mathscr{C}_\eta$ denotes the fourth-order elasticity tensor pulled-back to the reference configuration and is defined as

$$\mathscr{C}_\eta := \left((\boldsymbol{K}_\eta)^{-1}\underline{\otimes}(\boldsymbol{K}_\eta)^{-1}\right) : \mathscr{C}_{\nu\eta} : \left((\boldsymbol{K}_\eta)^{-\mathrm{T}}\underline{\otimes}(\boldsymbol{K}_\eta)^{-\mathrm{T}}\right), \tag{20}$$

where for two second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, $[\boldsymbol{A}\underline{\otimes}\boldsymbol{B}]_{PQRS} = A_{PR}B_{QS}$ and $[\boldsymbol{A}\overline{\otimes}\boldsymbol{B}]_{PQRS} = A_{PS}B_{QR}$. According to Equation (20), the dependence of $\mathscr{C}_\eta$ on $\boldsymbol{K}_\eta$ illustrates how the structural evolution of the constituent affects its material properties in the reference configuration. Moreover, just like $\mathscr{C}_{\nu\eta}$, the fourth-order tensor $\mathscr{C}_\eta$ also exhibits major and minor symmetries, namely

$$[\mathscr{C}_\eta]_{ABCD} = [\mathscr{C}_\eta]_{BACD} = [\mathscr{C}_\eta]_{ABDC} = [\mathscr{C}_\eta]_{CDAB}. \tag{21}$$

Motivated by our scope of providing a procedure as analytical as possible, we consider that the constituents undergo infinitesimal deformations while concurrently experiencing re-organisation within their internal structure. Within this setting, our modelling approach is therefore intended for describing remodelling processes in media undergoing small deformations. However, although we opt to work under the assumption of infinitesimal elastic deformations, we retain the non-linear traits of the solid constituents through the remodelling tensor $\boldsymbol{K}_\eta$. This choice, despite its apparent simplicity, remains pertinent across diverse contexts, notably exemplified in biological scenarios, such as in bone structures [5, 87]. Thus, by considering only the linear terms in Grad $\boldsymbol{u}_\eta$, the linearised versions of the right Cauchy–Green deformation tensor and the Green–Lagrange strain tensor are

$$\boldsymbol{C}_\eta^{\mathrm{lin}} := \boldsymbol{I} + 2\boldsymbol{E}_\eta^{\mathrm{lin}}, \tag{22a}$$

$$\boldsymbol{E}_\eta^{\mathrm{lin}} := \mathrm{sym}(\mathrm{Grad}\,\boldsymbol{u}_\eta), \tag{22b}$$

where sym($\boldsymbol{A}$) denotes the symmetric part of the second-order tensor $\boldsymbol{A}$. Consequently, by exploiting the symmetry properties of $\mathscr{C}_\eta$ given in Equation (21), the first Piola–Kirchhoff stress tensor and the Mandel stress tensor can be rephrased as

$$\boldsymbol{P}_{\eta}^{\text{lin}} := \mathscr{C}_{\eta} : \text{Grad}\,\boldsymbol{u}_{\eta} - (\boldsymbol{I} + \text{Grad}\,\boldsymbol{u}_{\eta})(\mathscr{C}_{\eta} : \boldsymbol{E}_{\boldsymbol{K}_{\eta}}) = \mathscr{G}_{\eta} : \text{Grad}\,\boldsymbol{u}_{\eta} - \mathscr{C}_{\eta} : \boldsymbol{E}_{\boldsymbol{K}_{\eta}}, \tag{23a}$$

$$\boldsymbol{\Sigma}_{\eta}^{\text{lin}} := \mathscr{C}_{\eta} : \text{Grad}\,\boldsymbol{u}_{\eta} - [\boldsymbol{I} + 2\text{sym}(\text{Grad}\,\boldsymbol{u}_{\eta})](\mathscr{C}_{\eta} : \boldsymbol{E}_{\boldsymbol{K}_{\eta}}) = \left[\mathscr{G}_{\eta} - \boldsymbol{I}\overline{\otimes}(\mathscr{C}_{\eta} : \boldsymbol{E}_{\boldsymbol{K}_{\eta}})\right] : \text{Grad}\,\boldsymbol{u}_{\eta} - \mathscr{C}_{\eta} : \boldsymbol{E}_{\boldsymbol{K}_{\eta}}, \tag{23b}$$

where the fourth-order tensor $\mathscr{G}_{\eta}$ appearing in Equations (23a) and (23b) is defined as

$$\mathscr{G}_{\eta} := \mathscr{C}_{\eta} - \boldsymbol{I}\underline{\otimes}\left(\mathscr{C}_{\eta} : \boldsymbol{E}_{\boldsymbol{K}_{\eta}}\right), \tag{24}$$

and enjoys major symmetry, that is, $[\mathscr{G}_{\eta}]_{ABCD} = [\mathscr{G}_{\eta}]_{CDAB}$.

4.2 | Evolution law for $\boldsymbol{K}_{\eta}$

Following [18, 19], we recast the equation for the evolution of $\boldsymbol{K}_{\eta}$ from the balance of work-conjugated forces specified in Section 3 and the compliance of the dissipation inequality (17). To this end, we start by defining the dissipative part of $\boldsymbol{Y}_{\eta}$ as [18, 19]

$$\boldsymbol{Y}_{\eta\text{d}} := \mathscr{A}_{\eta}\circ(\boldsymbol{F}_{\eta}, \boldsymbol{K}_{\eta}) : (\boldsymbol{K}_{\eta})^{-1}\dot{\boldsymbol{K}}_{\eta}, \tag{25}$$

where $\mathscr{A}_{\eta}$ is assumed to be, for each $\eta = \text{m}, \text{f}$, a positive semi-definite fourth-order tensor. In particular, as in refs. [18, 25, 88], we use the following representation of $\mathscr{A}_{\eta}$

$$\mathscr{A}_{\eta}\circ(\boldsymbol{F}_{\eta}, \boldsymbol{K}_{\eta}) = \tfrac{1}{3}J_{\boldsymbol{K}_{\eta}}\mathfrak{a}_{\nu\eta}\boldsymbol{I}\otimes\boldsymbol{I} + J_{\boldsymbol{K}_{\eta}}\mathfrak{b}_{\nu\eta}\left(\boldsymbol{C}_{\eta}\underline{\otimes}(\boldsymbol{C}_{\eta})^{-1} + \boldsymbol{I}\overline{\otimes}\boldsymbol{I}\right) + J_{\boldsymbol{K}_{\eta}}\mathfrak{c}_{\nu\eta}\left(\boldsymbol{C}_{\eta}\underline{\otimes}(\boldsymbol{C}_{\eta})^{-1} - \boldsymbol{I}\overline{\otimes}\boldsymbol{I}\right), \tag{26}$$

where $\mathfrak{a}_{\nu\eta}$, $\mathfrak{b}_{\nu\eta}$ and $\mathfrak{c}_{\nu\eta}$ are material parameters with physical units of stress per time, such that $\mathfrak{a}_{\nu\eta} + 2\mathfrak{b}_{\nu\eta} \geq 0$, $\mathfrak{b}_{\nu\eta} \geq 0$ and $\mathfrak{c}_{\nu\eta} \geq 0$. Therefore, by recalling that $\boldsymbol{Y}_{\eta} = \boldsymbol{Y}_{\eta\text{d}} - \boldsymbol{\Sigma}_{\eta}$ (refer to Equation (16)) and using the isochoricity property of $\boldsymbol{K}_{\eta}$, the balance equation (8b) becomes

$$\mathfrak{b}_{\nu\eta}\text{dev}\left(\boldsymbol{C}_{\eta}\left((\boldsymbol{K}_{\eta})^{-1}\dot{\boldsymbol{K}}_{\eta}\right)(\boldsymbol{C}_{\eta})^{-1} + \left((\boldsymbol{K}_{\eta})^{-1}\dot{\boldsymbol{K}}_{\eta}\right)^{\text{T}}\right) + \mathfrak{c}_{\nu\eta}\text{dev}\left(\boldsymbol{C}_{\eta}\left((\boldsymbol{K}_{\eta})^{-1}\dot{\boldsymbol{K}}_{\eta}\right)(\boldsymbol{C}_{\eta})^{-1} - \left((\boldsymbol{K}_{\eta})^{-1}\dot{\boldsymbol{K}}_{\eta}\right)^{\text{T}}\right) = \text{dev}(\boldsymbol{\Sigma}_{\eta} + \boldsymbol{Z}_{\eta}), \tag{27}$$

which represents the governing equation for the evolution of the remodelling tensor $\boldsymbol{K}_{\eta}$. Furthermore, by multiplying Equation (27) to the left by $(\boldsymbol{C}_{\eta})^{-1}$, we can rephrase it as

$$2\mathfrak{b}_{\nu\eta}\text{sym}\left(\left((\boldsymbol{K}_{\eta})^{-1}\dot{\boldsymbol{K}}_{\eta}\right)(\boldsymbol{C}_{\eta})^{-1}\right) + 2\mathfrak{c}_{\nu\eta}\text{skew}\left(\left((\boldsymbol{K}_{\eta})^{-1}\dot{\boldsymbol{K}}_{\eta}\right)(\boldsymbol{C}_{\eta})^{-1}\right) = (\boldsymbol{C}_{\eta})^{-1}\text{dev}(\boldsymbol{\Sigma}_{\eta} + \boldsymbol{Z}_{\eta}). \tag{28}$$

Hence, in virtue of the symmetry of $(\boldsymbol{C}_{\eta})^{-1}\boldsymbol{\Sigma}_{\eta}$ and, thus, of $(\boldsymbol{C}_{\eta})^{-1}\text{dev}(\boldsymbol{\Sigma}_{\eta})$ [25, 89], and accounting for a form of $\boldsymbol{Z}_{\eta}$ such that $(\boldsymbol{C}_{\eta})^{-1}\text{dev}(\boldsymbol{Z}_{\eta})$ is also symmetric, Equation (28) splits into the system [18, 19]

$$2\mathfrak{b}_{\nu\eta}\text{sym}\left(\left((\boldsymbol{K}_{\eta})^{-1}\dot{\boldsymbol{K}}_{\eta}\right)(\boldsymbol{C}_{\eta})^{-1}\right) = (\boldsymbol{C}_{\eta})^{-1}\text{dev}(\boldsymbol{\Sigma}_{\eta} + \boldsymbol{Z}_{\eta}), \tag{29a}$$

$$2\mathfrak{c}_{\nu\eta}\text{skew}\left(\left((\boldsymbol{K}_{\eta})^{-1}\dot{\boldsymbol{K}}_{\eta}\right)(\boldsymbol{C}_{\eta})^{-1}\right) = \boldsymbol{O}, \tag{29b}$$

with $\boldsymbol{O}$ being the second-order null tensor and skew$(\boldsymbol{A})$ the skew-symmetric part of the second-order tensor $\boldsymbol{A}$. Within this framework, Equation (29a) represents the evolution law for $\boldsymbol{K}_{\eta}$, while Equation (29b) is an additional kinematic constraint. We remark that the evolution law defined in Equation (29a) relies on the computation of the right Cauchy-Green deformation tensor, $\boldsymbol{C}_{\eta}$, and the Mandel stress tensor, $\boldsymbol{\Sigma}_{\eta}$, and on the specification of the material parameters $\mathfrak{b}_{\nu\eta}$ and the external generalised force $\boldsymbol{Z}_{\eta}$. For instance, $\mathfrak{b}_{\nu\eta}$ can be specified as the multiplication of the initial yield stress of the constituent and a characteristic time scale of the inelastic distortions (see, e.g., refs. [4, 17]). Moreover, as previously mentioned, $\boldsymbol{Z}_{\eta}$ has been used to represent interactions originating from lower scales in biological tissues, such as chemical ones (see, e.g., refs. [14, 19]). When considered non-zero, it is typically defined on a phenomenological basis tailored to the problem under study. However, explicit and physically robust definitions of $\boldsymbol{Z}_{\eta}$ remain scarce in the literature and further research in this direction is therefore necessary. Within the context of remodelling, one possible approach is to encapsulate within $\boldsymbol{Z}_\eta$ the concept of homeostatic stress (see refs. [19, 90] for further discussions). We further notice that it is possible to derive simpler evolution laws compared to the one written in this section. For instance, by defining $\boldsymbol{Y}_{\eta\mathrm{d}}$ in Equation (25) as $\boldsymbol{Y}_{\eta\mathrm{d}} := \mathscr{K}_\eta : (\boldsymbol{K}_\eta)^{-1}\dot{\boldsymbol{K}}_\eta$ where $\mathscr{K}_\eta$ is a constant, symmetric, positive-definite fourth-order tensor, we recover the evolution equation presented in ref. [90] in the context of growth. We also remark that the evolution law for $\boldsymbol{K}_\eta$ in Equation (29a) can become more complex, for example, by incorporating spatial derivatives of the remodelling tensor $\boldsymbol{K}_\eta$ to capture the effects associated with the accumulation of geometrical dislocations (see, refs. [4, 17] and references therein for further details).

Considering the infinitesimal deformation regime in which we are operating, Equations (29a) and (29b) take the form

$$2\mathfrak{b}_{\nu\eta}\mathrm{sym}\left(\left((\boldsymbol{K}_\eta)^{-1}\dot{\boldsymbol{K}}_\eta\right)(\boldsymbol{C}_\eta^{\mathrm{lin}})^{-1}\right) = \mathrm{dev}\left(\boldsymbol{\Sigma}_\eta^{\mathrm{lin}}\right) + 2\mathrm{sym}(\mathrm{Grad}\,\boldsymbol{u}_\eta)\,\mathrm{dev}\left(\mathscr{C}_\eta : \boldsymbol{E}_{\boldsymbol{K}_\eta}\right) + [(\boldsymbol{C}_\eta)^{-1}\mathrm{dev}(\boldsymbol{Z}_\eta)]^{\mathrm{lin}}, \tag{30a}$$

$$2\mathfrak{c}_{\nu\eta}\mathrm{skew}\left(\left((\boldsymbol{K}_\eta)^{-1}\dot{\boldsymbol{K}}_\eta\right)(\boldsymbol{C}_\eta^{\mathrm{lin}})^{-1}\right) = \boldsymbol{O}, \tag{30b}$$

where $(\boldsymbol{C}_\eta^{\mathrm{lin}})^{-1} = \boldsymbol{I} - 2\mathrm{sym}(\mathrm{Grad}\,\boldsymbol{u}_\eta)$, $\boldsymbol{\Sigma}_\eta^{\mathrm{lin}}$ is given in (23b) and the linearised version of $(\boldsymbol{C}_\eta)^{-1}\mathrm{dev}(\boldsymbol{Z}_\eta)$ is denoted by $[(\boldsymbol{C}_\eta)^{-1}\mathrm{dev}(\boldsymbol{Z}_\eta)]^{\mathrm{lin}}$. Generally, $\boldsymbol{Z}_\eta$ can be viewed as a function involving the deformation $\boldsymbol{F}_\eta$, the inelastic distortions $\boldsymbol{K}_\eta$, and a collection of physical quantities of diverse nature, such as electric potential and concentrations of chemical agents [18]. Here, we will focus on the case $\boldsymbol{Z}_\eta$ does not feature the observable deformation so that $[(\boldsymbol{C}_\eta)^{-1}\mathrm{dev}(\boldsymbol{Z}_\eta)]^{\mathrm{lin}} = [(\boldsymbol{C}_\eta^{\mathrm{lin}})^{-1}]\mathrm{dev}(\boldsymbol{Z}_\eta)$.

4.3 | Summary of equations

Summarising the above results, the equations governing the evolution of the composite constituents can be written as

$$\mathrm{Div}\left(\mathscr{G}_\eta : \mathrm{Grad}\,\boldsymbol{u}_\eta\right) - \mathrm{Div}\left(\mathscr{C}_\eta : \boldsymbol{E}_{\boldsymbol{K}_\eta}\right) = \boldsymbol{0}, \quad \text{in } \mathscr{B}_\eta, \tag{31a}$$

$$\begin{aligned}2\mathfrak{b}_{\nu\eta}\mathrm{sym}\left(\left((\boldsymbol{K}_\eta)^{-1}\dot{\boldsymbol{K}}_\eta\right)\left[\boldsymbol{I} - 2\mathrm{sym}(\mathrm{Grad}\,\boldsymbol{u}_\eta)\right]\right) &= \mathrm{dev}\left([\mathscr{G}_\eta - \boldsymbol{I}\overline{\otimes}(\mathscr{C}_\eta : \boldsymbol{E}_{\boldsymbol{K}_\eta})] : \mathrm{Grad}\,\boldsymbol{u}_\eta\right) - \mathrm{dev}(\mathscr{C}_\eta : \boldsymbol{E}_{\boldsymbol{K}_\eta})\\ &+ [\boldsymbol{I}\underline{\overline{\otimes}}\mathrm{dev}(\mathscr{C}_\eta : \boldsymbol{E}_{\boldsymbol{K}_\eta} - \boldsymbol{Z}_\eta)] : \mathrm{Grad}\,\boldsymbol{u}_\eta + \mathrm{dev}(\boldsymbol{Z}_\eta), \quad \text{in } \mathscr{B}_\eta,\end{aligned} \tag{31b}$$

where, according to (31b), $\mathrm{dev}(\boldsymbol{Z}_\eta)$ needs to be symmetric and $\boldsymbol{A}\underline{\overline{\otimes}}\boldsymbol{B} := \boldsymbol{A}\overline{\otimes}\boldsymbol{B} + \boldsymbol{A}\underline{\otimes}\boldsymbol{B}$, with $\boldsymbol{A}$ and $\boldsymbol{B}$ being second-order tensors. The interface conditions complementing Equations (31a) and (31b) are

$$\boldsymbol{u}_{\mathrm{m}} = \boldsymbol{u}_{\mathrm{f}}, \quad \text{on } \Gamma, \tag{32a}$$

$$\left(\mathscr{G}_{\mathrm{m}} : \mathrm{Grad}\,\boldsymbol{u}_{\mathrm{m}} - \mathscr{C}_{\mathrm{m}} : \boldsymbol{E}_{\boldsymbol{K}_{\mathrm{m}}}\right)\boldsymbol{N} = \left(\mathscr{G}_{\mathrm{f}} : \mathrm{Grad}\,\boldsymbol{u}_{\mathrm{f}} - \mathscr{C}_{\mathrm{f}} : \boldsymbol{E}_{\boldsymbol{K}_{\mathrm{f}}}\right)\boldsymbol{N}, \quad \text{on } \Gamma. \tag{32b}$$

Moreover, the tensor $\boldsymbol{K}_\eta$ is subject to the following constraints

$$(\boldsymbol{K}_\eta)^{-\mathrm{T}} : \dot{\boldsymbol{K}}_\eta = 0, \quad \text{in } \mathscr{B}_\eta, \tag{33a}$$

$$2\mathfrak{c}_{\nu\eta}\mathrm{skew}\left(\left((\boldsymbol{K}_\eta)^{-1}\dot{\boldsymbol{K}}_\eta\right)\left[\boldsymbol{I} - 2\mathrm{sym}(\mathrm{Grad}\,\boldsymbol{u}_\eta)\right]\right) = \boldsymbol{O}, \quad \text{in } \mathscr{B}_\eta. \tag{33b}$$

We remark that Equations (31a)–(33b) still require the specification of initial and boundary conditions at $\partial_{\mathrm{D}}^{\mathrm{ext}}\mathscr{B}_{\mathrm{m}}$. The flexibility arising from employing the AH technique permits circumventing the need to formulate them at this stage as we are considering regions far from the outer boundary of the medium. For situations requiring homogenisation near the outer heterogeneous boundaries, additional insights are available in refs. [42, 91, 92].

5 | AH SCHEME

In this section, we focus on performing the asymptotic expansion of the momentum balance (31a) and the remodelling law (31b) together with the interface conditions (32a) and (32b) and the kinematic constraints (33a)–(33b).

5.1 | Multi-scale mathematical model

As required for the AH and as a first step in providing a multi-scale formulation of the model equations provided in Section 4.3, we identify two well-separated length scales L and l, which are associated, respectively, with the composite as a whole and its internal structure. Furthermore, we introduce the smallness parameter

$$0 < \varepsilon := \frac{l}{L} \ll 1, \tag{34}$$

alongside two formally independent dimensionless coordinates: $\tilde{X} := L^{-1}X$, referred to as the *slow* or *macroscopic* variable, and $\tilde{Y} := l^{-1}X = \varepsilon^{-1}\tilde{X}$, denoting the *fast* or *microscopic* variable. This characterisation yields that a given quantity $\Phi(X,t)$ can be written in a multi-scale fashion as $\Phi^{\varepsilon}(\tilde{X},\tilde{Y},t)$ [93, 94] and, consequently, by means of the chain rule,

$$\frac{\partial \Phi(X,t)}{\partial X_A} = \frac{1}{L}\left(\frac{\partial \Phi^{\varepsilon}(\tilde{X},\tilde{Y},t)}{\partial \tilde{X}_A} + \varepsilon^{-1}\frac{\partial \Phi^{\varepsilon}(\tilde{X},\tilde{Y},t)}{\partial \tilde{Y}_A}\right). \tag{35}$$

Hence, dropping the tilde notation, we can rephrase the system of equations governing the composite material in its multi-scale formulation as follows:

$$\frac{1}{L^2}\big(\mathrm{Div}_X + \varepsilon^{-1}\mathrm{Div}_Y\big)\big(\mathscr{G}^{\varepsilon}_{\eta} : (\mathrm{Grad}_X + \varepsilon^{-1}\mathrm{Grad}_Y)\,\boldsymbol{u}^{\varepsilon}_{\eta}\big) - \frac{1}{L}\big(\mathrm{Div}_X + \varepsilon^{-1}\mathrm{Div}_Y\big)(\mathscr{C}^{\varepsilon}_{\eta} : \boldsymbol{E}^{\varepsilon}_{\boldsymbol{K}_{\eta}}) = \boldsymbol{0}, \tag{36a}$$

$$\begin{aligned}
&2\mathfrak{b}^{\varepsilon}_{\nu\eta}\mathrm{sym}\Big(\big((\boldsymbol{K}^{\varepsilon}_{\eta})^{-1}\dot{\boldsymbol{K}}^{\varepsilon}_{\eta}\big)\Big[\boldsymbol{I} - \frac{2}{L}\mathrm{sym}\big(\mathrm{Grad}_X\boldsymbol{u}^{\varepsilon}_{\eta} + \varepsilon^{-1}\mathrm{Grad}_Y\boldsymbol{u}^{\varepsilon}_{\eta}\big)\Big]\Big)\\
&= \frac{1}{L}\mathrm{dev}\Big([\mathscr{G}^{\varepsilon}_{\eta} - \boldsymbol{I}\overline{\otimes}(\mathscr{C}^{\varepsilon}_{\eta} : \boldsymbol{E}^{\varepsilon}_{\boldsymbol{K}_{\eta}})] : (\mathrm{Grad}_X + \varepsilon^{-1}\mathrm{Grad}_Y)\,\boldsymbol{u}^{\varepsilon}_{\eta}\Big)\\
&\quad - \mathrm{dev}(\mathscr{C}^{\varepsilon}_{\eta} : \boldsymbol{E}^{\varepsilon}_{\boldsymbol{K}_{\eta}}) + \frac{1}{L}[\boldsymbol{I}\underline{\overline{\otimes}}\mathrm{dev}(\mathscr{C}^{\varepsilon}_{\eta} : \boldsymbol{E}^{\varepsilon}_{\boldsymbol{K}_{\eta}} - \boldsymbol{Z}^{\varepsilon}_{\eta})] : (\mathrm{Grad}_X + \varepsilon^{-1}\mathrm{Grad}_Y)\,\boldsymbol{u}^{\varepsilon}_{\eta} + \mathrm{dev}(\boldsymbol{Z}^{\varepsilon}_{\eta}),
\end{aligned} \tag{36b}$$

with interface conditions on Γ given by

$$\boldsymbol{u}^{\varepsilon}_{\mathrm{m}} = \boldsymbol{u}^{\varepsilon}_{\mathrm{f}}, \tag{37a}$$

$$\frac{1}{L}\Big(\mathscr{G}^{\varepsilon}_{\mathrm{m}} : \big(\mathrm{Grad}_X + \varepsilon^{-1}\mathrm{Grad}_Y\big)\boldsymbol{u}^{\varepsilon}_{\mathrm{m}} - L\mathscr{C}^{\varepsilon}_{\mathrm{m}} : \boldsymbol{E}^{\varepsilon}_{\boldsymbol{K}_{\mathrm{m}}}\Big)\boldsymbol{N} = \frac{1}{L}\Big(\mathscr{G}^{\varepsilon}_{\mathrm{f}} : \big(\mathrm{Grad}_X + \varepsilon^{-1}\mathrm{Grad}_Y\big)\boldsymbol{u}^{\varepsilon}_{\mathrm{f}} - L\mathscr{C}^{\varepsilon}_{\mathrm{f}} : \boldsymbol{E}^{\varepsilon}_{\boldsymbol{K}_{\mathrm{f}}}\Big)\boldsymbol{N}, \tag{37b}$$

and the following kinematic constraints for $\boldsymbol{K}^{\varepsilon}_{\eta}$

$$(\boldsymbol{K}^{\varepsilon}_{\eta})^{-\mathrm{T}} : \dot{\boldsymbol{K}}^{\varepsilon}_{\eta} = 0, \tag{38a}$$

$$2\mathfrak{c}^{\varepsilon}_{\nu\eta}\mathrm{skew}\Big(\big((\boldsymbol{K}^{\varepsilon}_{\eta})^{-1}\dot{\boldsymbol{K}}^{\varepsilon}_{\eta}\big)\Big[\boldsymbol{I} - \frac{2}{L}\mathrm{sym}\big(\mathrm{Grad}_X\boldsymbol{u}^{\varepsilon}_{\eta} + \varepsilon^{-1}\mathrm{Grad}_Y\boldsymbol{u}^{\varepsilon}_{\eta}\big)\Big]\Big) = \boldsymbol{O}. \tag{38b}$$

5.2 | Additional considerations

Before going further, we make some considerations on the topology of the composite material as well as on its main descriptors and intrinsic properties. In this regard, we start by identifying, the *elementary cell*, which we denote by $\mathscr{Y} = \mathscr{Y}_{\mathrm{m}} \sqcup \mathscr{Y}_{\mathrm{f}}$, and restrict our analysis to the case in which $\mathscr{Y}$ does not vary at different spatial locations. In other words, we invoke the notion of *macroscopic uniformity* [46, 47, 95] and confine our model to scenarios where the elementary cell can be selected independently of the macroscopic variable X. This assumption, notwithstanding, can be relaxed to underscore the possibility that the unit cell can change across the domain of definition and further discussions and consequences of this selection can be found, for instance, at refs. [45, 96, 97]). A fundamental consequence of the macroscopic uniformity is that for a given quantity $\Phi^{\varepsilon}_{\eta}(X,Y,t)$,

$$\frac{\partial}{\partial X_A}\int_{\mathscr{Y}_{\eta}} \Phi^{\varepsilon}_{\eta}(X,Y,t)\,\mathrm{d}V(Y) = \int_{\mathscr{Y}_{\eta}} \frac{\partial \Phi^{\varepsilon}_{\eta}}{\partial X_A}(X,Y,t)\,\mathrm{d}V(Y). \tag{39}$$

In addition, we also require that the descriptors and attributes of the composite medium are *locally periodic*. That is, for a quantity $\Phi^{\varepsilon}_{\eta}(X, Y, t)$, we call for $\Phi^{\varepsilon}_{\eta}(X, Y_{\mathrm{b}}, t) = \Phi^{\varepsilon}_{\eta}(X, Y_{\mathrm{b}} + \boldsymbol{e}_A, t)$, where $Y_{\mathrm{b}} \in \partial\mathscr{Y} \setminus (\partial\mathscr{Y} \cap \Gamma)$ and $\{\boldsymbol{e}_A\}_{A=1,2,3}$ is a basis of the three-dimensional real vector space associated with the local Cartesian frame of the elementary cell [4, 46]. For instance, if the reference cell is a cuboid, we define, for each $A = 1, 2, 3$, $\boldsymbol{e}_A = \alpha_A \boldsymbol{i}_A$ with $\alpha_A \in \mathbb{R}$ and $\{\boldsymbol{i}_A\}_{A=1,2,3}$ is the standard basis. We remark that, as it was for the macroscopic uniformity assumption, the assumption of local periodicity is a technical consideration that simplifies the mathematical treatment of multidimensional problems (see, ref. [96]). Local periodicity can be replaced with assumptions of local boundedness and regularity. For instance, in one-dimensional problems, local boundedness and regularity assumptions are sufficient to derive the homogenized problem and the effective coefficient, which is expressed as a harmonic mean [96, 98]. For multidimensional diffusion problems, local boundedness assumptions have been considered in the derivation of the equation of poroelasticity [95].

We also introduce the *average* of a generic field $\Phi(X, Y, t) := \Phi^{\varepsilon}_{\mathrm{m}}(X, Y, t)\Upsilon_{\mathrm{m}}(Y) + \Phi^{\varepsilon}_{\mathrm{f}}(X, Y, t)\Upsilon_{\mathrm{f}}(Y)$ (be it a scalar-, vector-, or tensor-valued) over the cell $\mathscr{Y} = \mathscr{Y}_{\mathrm{m}} \sqcup \mathscr{Y}_{\mathrm{f}}$ as

$$\langle \Phi \rangle(X, t) := \frac{1}{|\mathscr{Y}|} \int_{\mathscr{Y}} \Phi(X, Y, t) \mathrm{d}V(Y) = \frac{1}{|\mathscr{Y}|} \sum_{\eta \in \{\mathrm{m}, \mathrm{f}\}} \int_{\mathscr{Y}_{\eta}} \Phi^{\varepsilon}_{\eta}(X, Y, t) \mathrm{d}V(Y) = \langle \Phi^{\varepsilon}_{\mathrm{m}} \rangle_{\mathrm{m}}(X, t) + \langle \Phi^{\varepsilon}_{\mathrm{f}} \rangle_{\mathrm{f}}(X, t), \tag{40}$$

where $\Upsilon_{\eta}(Y)$ is the indicator function of $\mathscr{Y}_{\eta}$, that is $\Upsilon_{\eta}(Y) = 1$ if $Y \in \mathscr{Y}_{\eta}$, otherwise it is zero. In Equation (40), we also introduced the notation

$$\langle \Phi^{\varepsilon}_{\eta} \rangle_{\eta}(X, t) := \frac{1}{|\mathscr{Y}|} \int_{\mathscr{Y}_{\eta}} \Phi^{\varepsilon}_{\eta}(X, Y, t) \mathrm{d}V(Y). \tag{41}$$

5.3 | Formal expansions of main fields

Before performing the multiscale analysis of the momentum balance (36a) and the remodelling law (36b) together with the interface conditions (37a)–(37b) and the kinematic constraints (38a)–(38b), we prescribe formal expansions for the displacement $\boldsymbol{u}^{\varepsilon}_{\eta}$ and remodelling tensor $\boldsymbol{K}^{\varepsilon}_{\eta}$, namely

$$\boldsymbol{u}^{\varepsilon}_{\eta}(X, Y, t) = \sum_{k=0}^{+\infty} \varepsilon^k \boldsymbol{u}^{(k)}_{\eta}(X, Y, t), \tag{42a}$$

$$\boldsymbol{K}^{\varepsilon}_{\eta}(X, Y, t) = \sum_{k=0}^{+\infty} \varepsilon^k \boldsymbol{K}^{(k)}_{\eta}(X, Y, t). \tag{42b}$$

Thus, focusing on the addends appearing in the equilibrium Equation (36a) and the evolution equation for $\boldsymbol{K}_{\eta}$ given by (36b), we can write

$$(\boldsymbol{K}^{\varepsilon}_{\eta})^{-1} = (\boldsymbol{K}^{(0)}_{\eta})^{-1} + \varepsilon \boldsymbol{Q}^{(1)}_{\eta} + o(\varepsilon), \tag{43a}$$

$$\boldsymbol{E}^{\varepsilon}_{\boldsymbol{K}_{\eta}} = \boldsymbol{E}^{(0)}_{\boldsymbol{K}_{\eta}} + \varepsilon \boldsymbol{E}^{(1)}_{\boldsymbol{K}_{\eta}} + o(\varepsilon), \tag{43b}$$

where

$$(\boldsymbol{K}^{(0)}_{\eta})^{-1} = (\boldsymbol{K}^{(0)}_{\eta})^2 - I_1(\boldsymbol{K}^{(0)}_{\eta})\boldsymbol{K}^{(0)} + I_2(\boldsymbol{K}^{(0)}_{\eta})\boldsymbol{I}, \tag{44a}$$

$$\boldsymbol{Q}^{(1)}_{\eta} := \boldsymbol{K}^{(0)}_{\eta}\boldsymbol{K}^{(1)}_{\eta} + \boldsymbol{K}^{(1)}_{\eta}\boldsymbol{K}^{(0)}_{\eta} - \mathrm{tr}(\boldsymbol{K}^{(0)}_{\eta})\boldsymbol{K}^{(1)}_{\eta} - \mathrm{tr}(\boldsymbol{K}^{(1)}_{\eta})\boldsymbol{K}^{(0)}_{\eta} + \left[\mathrm{tr}(\boldsymbol{K}^{(0)}_{\eta})\mathrm{tr}(\boldsymbol{K}^{(1)}_{\eta}) - \mathrm{tr}(\boldsymbol{K}^{(0)}_{\eta}\boldsymbol{K}^{(1)}_{\eta})\right]\boldsymbol{I}, \tag{44b}$$

$$\boldsymbol{E}^{(0)}_{\boldsymbol{K}_{\eta}} := \tfrac{1}{2}[(\boldsymbol{K}^{(0)}_{\eta})^{\mathrm{T}}\boldsymbol{K}^{(0)}_{\eta} - \boldsymbol{I}], \tag{44c}$$

$$\boldsymbol{E}^{(1)}_{\boldsymbol{K}_{\eta}} := \tfrac{1}{2}[(\boldsymbol{K}^{(0)}_{\eta})^{\mathrm{T}}\boldsymbol{K}^{(1)}_{\eta} + (\boldsymbol{K}^{(1)}_{\eta})^{\mathrm{T}}\boldsymbol{K}^{(0)}_{\eta}]. \tag{44d}$$

We notice that while Equations (44b)–(44d) are definitions, Equation (44a) is an identity. That is the right-hand side of (44a) is the result of computing the inverse of $\boldsymbol{K}^{(0)}_{\eta}$ using the Cayley-Hamilton theorem. Moreover, $I_1(\boldsymbol{K}^{(0)}_{\eta}) := \mathrm{tr}(\boldsymbol{K}^{(0)}_{\eta})$ and $I_2(\boldsymbol{K}^{(0)}_{\eta}) := \frac{1}{2}\left[(\mathrm{tr}(\boldsymbol{K}^{(0)}_{\eta}))^2 - \mathrm{tr}((\boldsymbol{K}^{(0)}_{\eta})^2)\right]$ denote the first and second principal invariants of $\boldsymbol{K}^{(0)}_{\eta}$. In particular, in writing (44a) we have used the fact that $\det(\boldsymbol{K}^{(0)}_{\eta}) = 1$ which results from the constraint of isochoric inelastic distortions.

Specifically, since [4]

$$\det(\boldsymbol{K}_\eta^\varepsilon) = \det(\boldsymbol{K}_\eta^{(0)})\det\Big(\boldsymbol{I} + \varepsilon(\boldsymbol{K}_\eta^{(0)})^{-1}\boldsymbol{K}_\eta^{(1)} + o(\varepsilon)\Big) = \det(\boldsymbol{K}_\eta^{(0)}) + \varepsilon \det(\boldsymbol{K}_\eta^{(0)})\mathrm{tr}((\boldsymbol{K}_\eta^{(0)})^{-1}\boldsymbol{K}_\eta^{(1)}) + o(\varepsilon), \tag{45}$$

we can translate the constraint of isochoric inelastic distortions, namely $\det(\boldsymbol{K}_\eta^\varepsilon) = 1$, in terms of the addends in the expansion of $\boldsymbol{K}_\eta^\varepsilon$, which yields

$$\det(\boldsymbol{K}_\eta^{(0)}) = 1 \quad \text{and} \quad \mathrm{tr}((\boldsymbol{K}_\eta^{(0)})^{-1}\boldsymbol{K}_\eta^{(1)}) = 0. \tag{46}$$

Now, in view of (44a), the expansion of the elasticity tensor pull-backed to the reference configuration is given by

$$\mathscr{C}_\eta^\varepsilon = \mathscr{C}_\eta^{(0)} + \varepsilon\mathscr{C}_\eta^{(1)} + o(\varepsilon), \tag{47}$$

where

$$\mathscr{C}_\eta^{(0)} := ((\boldsymbol{K}_\eta^{(0)})^{-1}\underline{\otimes}(\boldsymbol{K}_\eta^{(0)})^{-1}) : \mathscr{C}_{\nu\eta}^\varepsilon : ((\boldsymbol{K}_\eta^{(0)})^{-\mathrm{T}}\underline{\otimes}(\boldsymbol{K}_\eta^{(0)})^{-\mathrm{T}}), \tag{48a}$$

$$\begin{aligned}\mathscr{C}_\eta^{(1)} :=\ & ((\boldsymbol{K}_\eta^{(0)})^{-1}\underline{\otimes}(\boldsymbol{K}_\eta^{(0)})^{-1}) : \mathscr{C}_{\nu\eta}^\varepsilon : ((\boldsymbol{K}_\eta^{(0)})^{-\mathrm{T}}\underline{\otimes}(\boldsymbol{Q}_\eta^{(1)})^{\mathrm{T}} + (\boldsymbol{Q}_\eta^{(1)})^{\mathrm{T}}\underline{\otimes}(\boldsymbol{K}_\eta^{(0)})^{-\mathrm{T}})\\ & + ((\boldsymbol{K}_\eta^{(0)})^{-1}\underline{\otimes}\boldsymbol{Q}_\eta^{(1)} + \boldsymbol{Q}_\eta^{(1)}\underline{\otimes}(\boldsymbol{K}_\eta^{(0)})^{-1}) : \mathscr{C}_{\nu\eta}^\varepsilon : ((\boldsymbol{K}_\eta^{(0)})^{-\mathrm{T}}\underline{\otimes}(\boldsymbol{K}_\eta^{(0)})^{-\mathrm{T}}).\end{aligned} \tag{48b}$$

Thus, $\mathscr{G}_\eta^\varepsilon$ can be written in the form

$$\mathscr{G}_\eta^\varepsilon = \mathscr{G}_\eta^{(0)} + \varepsilon\mathscr{G}_\eta^{(1)} + o(\varepsilon), \tag{49}$$

with

$$\mathscr{G}_\eta^{(0)} := \mathscr{C}_\eta^{(0)} - \boldsymbol{I}\underline{\otimes}(\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}_\eta}^{(0)}), \tag{50a}$$

$$\mathscr{G}_\eta^{(1)} := \mathscr{C}_\eta^{(1)} - \boldsymbol{I}\underline{\otimes}(\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}_\eta}^{(1)} + \mathscr{C}_\eta^{(1)} : \boldsymbol{E}_{\boldsymbol{K}_\eta}^{(0)}). \tag{50b}$$

Finally, although we still have to provide an expression for the external non-conventional force $\boldsymbol{Z}_\eta^\varepsilon$, for the time being, we accept that such a formula can be formally written as

$$\boldsymbol{Z}_\eta^\varepsilon(X, Y, t) = \textstyle\sum_{k=0}^{+\infty} \varepsilon^k \boldsymbol{Z}_\eta^{(k)}(X, Y, t), \tag{51}$$

with $\boldsymbol{Z}_\eta^{(k)}$ being locally-periodic second-order tensors that need to be identified. The subsequent phase of the asymptotic process involves substituting the above expansions into the model outlined in the previous section. This will lead to a system of differential equations obtained by gathering the coefficients of the same powers of the smallness parameter ε. Solving these equations will enable the description of the medium's evolution by encapsulating the features of its active internal structure within specific coefficients.

5.4 | The first cell problem

The first cell problem that emerges from the balance of linear momentum is

$$\frac{1}{L^2}\mathrm{Div}_Y\Big(\mathscr{G}_\eta^{(0)} : \mathrm{Grad}_Y\, \boldsymbol{u}_\eta^{(0)}\Big) = \boldsymbol{0}, \tag{52a}$$

$$\boldsymbol{u}_\mathrm{m}^{(0)} = \boldsymbol{u}_\mathrm{f}^{(0)}, \tag{52b}$$

$$\frac{1}{L}\Big(\mathscr{G}_\mathrm{m}^{(0)} : \mathrm{Grad}_Y \boldsymbol{u}_\mathrm{m}^{(0)}\Big)\boldsymbol{N} = \frac{1}{L}\Big(\mathscr{G}_\mathrm{f}^{(0)} : \mathrm{Grad}_Y \boldsymbol{u}_\mathrm{f}^{(0)}\Big)\boldsymbol{N}, \tag{52c}$$

where we recall that Γ denotes the interface separating the two constituents in $\mathscr{Y}$, namely $\Gamma = \mathscr{Y}_\mathrm{m} \cap \mathscr{Y}_\mathrm{f}$, and $\boldsymbol{N}$ denotes the unit normal from $\mathscr{Y}_\mathrm{f}$ to $\mathscr{Y}_\mathrm{m}$.

Remark 1. Regarding the above cell problem, we comment on the existence of a non-trivial locally-periodic solution. This requires, using the Fredholm alternative [99], to impose uniformly elliptic criteria on the differential operator. Proofs are available in the linear elastic case (see, for instance, refs. [39, 99]), that is when $\boldsymbol{K}_\eta \equiv \boldsymbol{I}$. However, the applicability of such a result to the above problem is less straightforward due to $\mathscr{G}_\eta^{(0)}$ being given by the difference between the elasticity tensor $\mathscr{C}_\eta^{(0)}$ and the fourth-order tensor $\boldsymbol{I}\underline{\otimes}(\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}_\eta}^{(0)})$. Hence, for the time being, we assume that the uniformly elliptic criteria holds. This implies that $\boldsymbol{u}_\eta^{(0)}$ must be (almost everywhere) constant with respect to Y (see, e.g., refs. [4, 100]) and, because of the no-jump conditions, we can write $\boldsymbol{u}_\eta^{(0)}(X, Y, t) = \boldsymbol{u}^{(0)}(X, t)$. It is worth noticing that, in the case of remodelling multilayered composite structures, explicit uniformly elliptic conditions were derived in ref. [4].

According to Remark 1, $\boldsymbol{u}_\eta^{(0)}(X, Y, t)$ is independent of the fast variable Y and, thus, within this framework that accounts for a zero-grade theory in $\boldsymbol{K}_\eta$, the evolution law for the inelastic distortions will not provide further cell problems accompanying the ones stemming from the balance of linear momentum as these will become an identity. Had there been a consideration of a first-grade theory in $\boldsymbol{K}_\eta$ as studied in ref. [4], it would have led to the emergence of an additional cell problem complementing Equation (52a).

5.5 | The local problems

Due to $\boldsymbol{u}_\eta^{(0)}$ being independent of Y, the second cell problem related to the balance of linear momentum is

$$\frac{1}{L^2}\mathrm{Div}_Y\left(\mathscr{G}_\eta^{(0)} : \mathrm{Grad}_Y \boldsymbol{u}_\eta^{(1)}\right) = -\frac{1}{L^2}\mathrm{Div}_Y\left(\mathscr{G}_\eta^{(0)} : \mathrm{Grad}_X \boldsymbol{u}^{(0)} - L\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}_\eta}^{(0)}\right), \tag{53a}$$

$$\boldsymbol{u}_{\mathrm{m}}^{(1)} = \boldsymbol{u}_{\mathrm{f}}^{(1)}, \tag{53b}$$

$$\frac{1}{L}\left(\mathscr{G}_{\mathrm{m}}^{(0)} : \mathrm{Grad}_Y \boldsymbol{u}_{\mathrm{m}}^{(1)} - \mathscr{G}_{\mathrm{f}}^{(0)} : \mathrm{Grad}_Y \boldsymbol{u}_{\mathrm{f}}^{(1)}\right)\boldsymbol{N} = -\frac{1}{L}\left\{\left(\mathscr{G}_{\mathrm{m}}^{(0)} - \mathscr{G}_{\mathrm{f}}^{(0)}\right) : \mathrm{Grad}_X \boldsymbol{u}^{(0)} - L\left(\mathscr{C}_{\mathrm{m}}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}_{\mathrm{m}}}^{(0)} - \mathscr{C}_{\mathrm{f}}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}_{\mathrm{f}}}^{(0)}\right)\right\}\boldsymbol{N}, \tag{53c}$$

while the cell problem resulting from the evolution equation for the inelastic distortions becomes a trivial identity. In particular, owing to the linear nature of (53a) in $\boldsymbol{u}_\eta^{(1)}$, we represent $\boldsymbol{u}_\eta^{(1)}$ by the *ansatz* [101]

$$\boldsymbol{u}_\eta^{(1)}(X, Y, t) = \boldsymbol{\xi}_\eta(X, Y, t) : \mathrm{Grad}_X \boldsymbol{u}^{(0)}(X, t) + \boldsymbol{\omega}_\eta(X, Y, t), \tag{54}$$

where $\boldsymbol{\xi}_\eta$ and $\boldsymbol{\omega}_\eta$ are locally periodic third-order tensor and vector fields, respectively. Equation (54) implies that

$$\mathrm{Grad}_Y \boldsymbol{u}_\eta^{(1)} = \mathrm{TGrad}_Y \boldsymbol{\xi}_\eta : \mathrm{Grad}_X \boldsymbol{u}^{(0)} + \mathrm{Grad}_Y \boldsymbol{\omega}_\eta, \tag{55}$$

where $\mathrm{TGrad}_Y \boldsymbol{\xi}_\eta$ is the fourth-order tensor given by

$$\mathrm{TGrad}_Y \boldsymbol{\xi}_\eta := [\mathrm{TGrad}_Y \boldsymbol{\xi}_\eta]_{ABCD}\, \boldsymbol{e}_A \otimes \boldsymbol{e}_B \otimes \boldsymbol{e}_C \otimes \boldsymbol{e}_D = \frac{\partial [\boldsymbol{\xi}_\eta]_{ACD}}{\partial Y_B} \boldsymbol{e}_A \otimes \boldsymbol{e}_B \otimes \boldsymbol{e}_C \otimes \boldsymbol{e}_D. \tag{56}$$

In particular, we require that the third-order tensor field $\boldsymbol{\xi}_\eta$ is the solution of the cell problem

$$\frac{1}{L^2}\mathrm{Div}_Y\left(\mathscr{G}_\eta^{(0)} : \mathrm{TGrad}_Y \boldsymbol{\xi}_\eta\right) = -\frac{1}{L^2}\mathrm{Div}_Y\left(\mathscr{G}_\eta^{(0)}\right), \quad \text{in } \mathscr{Y}_\eta, \tag{57a}$$

$$\boldsymbol{\xi}_{\mathrm{m}} = \boldsymbol{\xi}_{\mathrm{f}}, \quad \text{on } \Gamma, \tag{57b}$$

$$\frac{1}{L}\left(\mathscr{G}_{\mathrm{m}}^{(0)} : \mathrm{TGrad}_Y \boldsymbol{\xi}_{\mathrm{m}} - \mathscr{G}_{\mathrm{f}}^{(0)} : \mathrm{TGrad}_Y \boldsymbol{\xi}_{\mathrm{f}}\right)\boldsymbol{N} = -\frac{1}{L}\left(\mathscr{G}_{\mathrm{m}}^{(0)} - \mathscr{G}_{\mathrm{f}}^{(0)}\right)\boldsymbol{N}, \quad \text{on } \Gamma, \tag{57c}$$

while $\boldsymbol{\omega}_\eta$ is the solution of the additional cell problem

$$\frac{1}{L^2}\mathrm{Div}_Y\left(\mathcal{G}_\eta^{(0)} : \mathrm{Grad}_Y\boldsymbol{\omega}_\eta\right) = \frac{1}{L}\mathrm{Div}_Y\left(\mathcal{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}\right), \quad \text{in } \mathscr{Y}_\eta, \tag{58a}$$

$$\boldsymbol{\omega}_\mathrm{m} = \boldsymbol{\omega}_\mathrm{f}, \quad \text{on } \Gamma, \tag{58b}$$

$$\frac{1}{L}\left(\mathcal{G}_\mathrm{m}^{(0)} : \mathrm{Grad}_Y\boldsymbol{\omega}_\mathrm{m} - \mathcal{G}_\mathrm{f}^{(0)} : \mathrm{Grad}_Y\boldsymbol{\omega}_\mathrm{f}\right)\boldsymbol{N} = \left(\mathcal{C}_\mathrm{m}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}_\mathrm{m}}^{(0)} - \mathcal{C}_\mathrm{f}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}_\mathrm{f}}^{(0)}\right)\boldsymbol{N}, \quad \text{on } \Gamma. \tag{58c}$$

As per our previous discussions, provided the differential operators in the above equations enjoy uniformly elliptic properties, the existence of a solution to the problems for $\boldsymbol{\xi}_\eta$ and $\boldsymbol{\omega}_\eta$ is guaranteed if the average of the respective right-hand sides is zero. The latter is guaranteed when $\mathcal{C}_{\nu\eta}$ and $\boldsymbol{K}_\eta^{(0)}$ are assumed to be Y-constant for each $\eta = \mathrm{m}, \mathrm{f}$. A consideration we will be adopting in the upcoming sections. Furthermore, for uniqueness purposes, among all locally-periodic solutions of (57a)–(57c) and (58a)–(58c), we will choose those whose average is zero.

Finally, we notice that the two-cell problems concerning the auxiliary variables $\boldsymbol{\xi}_\eta$ and $\boldsymbol{\omega}_\eta$ no longer explicitly involve the leading-order displacement $\boldsymbol{u}^{(0)}$. However, as it will be clear in the following section, their connection to the homogenised problem will persist explicitly through the evolution of $\boldsymbol{K}_\eta^{(0)}$ and implicitly on $\boldsymbol{u}^{(0)}$. Notably, in the scenario where inelastic distortions are absent, that is, when $\boldsymbol{K}_\eta = \boldsymbol{I}$, the cell problem (58a)–(58c) implies that $\boldsymbol{\omega}_\eta = \boldsymbol{0}$, while the cell problem for $\boldsymbol{\xi}_\eta$ reduces to the classical one for the linear elastic case [91, 92].

5.6 | The homogenised balance laws

Upon taking the average of the equation that results from equating the coefficients of (36a) at the order ε^2 and introducing the *effective coefficients* (refer to ref. [9] for further details)

$$\mathcal{G}_\mathrm{eff} := \left\langle \mathcal{G}^{(0)} + \mathcal{G}^{(0)} : \mathrm{TGrad}_Y\boldsymbol{\xi}\right\rangle, \tag{59a}$$

$$\boldsymbol{D}_\mathrm{eff} := \left\langle \mathcal{G}^{(0)} : \mathrm{Grad}_Y\boldsymbol{\omega} - \mathcal{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}\right\rangle, \tag{59b}$$

the homogenised balance of linear momentum reads [101]

$$\mathrm{Div}_X\left(\mathcal{G}_\mathrm{eff} : \mathrm{Grad}_X\boldsymbol{u}^{(0)} + \boldsymbol{D}_\mathrm{eff}\right) = \boldsymbol{0}, \tag{60}$$

where with respect to the original balance of linear momentum (31a), the contribution of the elastic and inelastic properties at the lower scales, alongside the geometrical features of the internal structure, are encoded in $\mathcal{G}_\mathrm{eff}$ and $\boldsymbol{D}_\mathrm{eff}$ through $\boldsymbol{\xi}_\eta$ and $\boldsymbol{\omega}_\eta$.

Remark 2. We remark that although in Section 4.2 we focused on recasting the evolution law for the remodelling tensor and set specific constitutive considerations, the results summarised in the previous sections are independent of the particular evolution law chosen. Specifically, provided we work within a theory of grade zero in $\boldsymbol{K}_\eta$, ensuring no spatial derivatives of $\boldsymbol{K}_\eta$ are present in the evolution law, and the external generalised force $\boldsymbol{Z}_\eta$ is properly selected, the results summarised in Sections 5.4–5.5, including Equations (59a)–(60), will still hold. We also notice that while incorporating first-order spatial derivatives of the remodelling tensor $\boldsymbol{K}_\eta$ into the evolution law (29a) to account for the accumulation of dislocations [17] would introduce an additional cell problem associated with the homogenised evolution law for the remodelling tensor $\boldsymbol{K}_\eta$ (see Equation (97) in ref. [4]). Adopting the same constitutive assumptions for the elastic contributions implies that the form of the homogenised balance of linear momentum for the composite as a whole remains unchanged (compare Equation (60) and Equation (106) in [4]). This also applies to the effective coefficients $\mathcal{G}_\mathrm{eff}$ and $\boldsymbol{D}_\mathrm{eff}$ given in Equations (103a) and (103b) in ref. [4], albeit with different notation.

Now, by introducing the second-order tensor

$$\boldsymbol{\Pi}_\eta(X, Y, t) := [\boldsymbol{I}\underline{\otimes}\boldsymbol{I} + \mathrm{TGrad}_Y\boldsymbol{\xi}_\eta(X, Y, t)] : \mathrm{Grad}_X\boldsymbol{u}^{(0)}(X, t) + \mathrm{Grad}_Y\boldsymbol{\omega}_\eta(X, Y, t), \tag{61}$$

which accounts for both the solutions of the homogenised Equation (60) and of the cell problems (57a) and (58a). The homogenised equation for the evolution of inelastic distortions which results from equating in powers of ε Equation (36b), averaging using Equation (41) and adding together the resulting expressions, is given by

$$2\mathrm{sym}\left(\left\langle \mathfrak{b}_{\nu}^{\varepsilon}\left((\boldsymbol{K}^{(0)})^{-1}\dot{\boldsymbol{K}}^{(0)}\right)\left[\boldsymbol{I}-\frac{2}{L}\mathrm{sym}(\boldsymbol{\Pi})\right]\right\rangle\right)=\frac{1}{L}\mathrm{dev}\left(\left\langle\left(\mathscr{G}^{(0)}-\boldsymbol{I}\overline{\otimes}(\mathscr{C}^{(0)}:\boldsymbol{E}_{\boldsymbol{K}}^{(0)})\right):\boldsymbol{\Pi}\right\rangle\right)-\mathrm{dev}\left(\left\langle\mathscr{C}^{(0)}:\boldsymbol{E}_{\boldsymbol{K}}^{(0)}\right\rangle\right)$$
$$+\frac{1}{L}\left\langle\left[\boldsymbol{I}\underline{\overline{\otimes}}\mathrm{dev}(\mathscr{C}^{(0)}:\boldsymbol{E}_{\boldsymbol{K}}^{(0)}-\boldsymbol{Z}^{(0)})\right]:\boldsymbol{\Pi}\right\rangle+\mathrm{dev}\left(\left\langle\boldsymbol{Z}^{(0)}\right\rangle\right). \tag{62}$$

Furthermore, using the expansion of $\boldsymbol{K}_{\eta}^{\varepsilon}$ in (45) and the *ansatz* (54), the kinematic constraints for $\boldsymbol{K}_{\eta}^{(0)}$ are

$$(\boldsymbol{K}_{\eta}^{(0)})^{-\mathrm{T}}:\dot{\boldsymbol{K}}_{\eta}^{(0)}=0, \tag{63a}$$

$$2\mathfrak{c}_{\nu\eta}^{\varepsilon}\mathrm{skew}\left(\left((\boldsymbol{K}_{\eta}^{(0)})^{-1}\dot{\boldsymbol{K}}_{\eta}^{(0)}\right)\left[\boldsymbol{I}-\frac{2}{L}\mathrm{sym}(\boldsymbol{\Pi}_{\eta})\right]\right)=\boldsymbol{O}. \tag{63b}$$

6 | DESCRIPTION OF A BENCHMARK PROBLEM

In this section, we consider the case in which the composite under study, described by the systems of equations provided in the previous sections, possesses a fibre-reinforced structure. Specifically, we consider that the uniaxially cylindrical fibres are oriented in the direction specified by $\boldsymbol{i}_3$ and corresponding to the X_3-axis, where $\{\boldsymbol{i}_k\}_{k=1}^{3}$ is the set of standard Cartesian basis vectors. In this framework, we represent the cross-section of the microstructure as a matrix with evenly distributed circular inclusions and, especially, we choose the unit cell $\mathscr{Y}$ to be characterised by a square with a single inclusion identified with $\mathscr{Y}_{\mathrm{f}}$, so that the portion of the cell representing the matrix is $\mathscr{Y}_{\mathrm{m}}=\mathscr{Y}\setminus\mathscr{Y}_{\mathrm{f}}$. For the unit cell of the composite, we take a Cartesian reference frame with axes collinear with those of the global frame, and an associated system of Cartesian, microscopic coordinates (Y_1, Y_2, Y_3), with the Y_k-axis parallel to the X_k-axis and $k=1,2,3$.

Furthermore, we establish the condition where the first and second components of displacement are set to zero. Meanwhile, the third component, namely $[\boldsymbol{u}_{\eta}]_3$, is assumed to be a function solely dependent on X_1 and X_2. Based on these considerations, from Equations (42a) and (54), up to the first order, we have that

$$[\boldsymbol{u}_{\eta}^{\varepsilon}]_3=[\boldsymbol{u}^{(0)}]_3+\varepsilon\left([\boldsymbol{\xi}_{\eta}]_{33D}\frac{\partial[\boldsymbol{u}^{(0)}]_3}{\partial X_D}+[\boldsymbol{\omega}_{\eta}]_3\right). \tag{64}$$

Hence, $[\boldsymbol{\xi}_{\eta}]_{331}$, $[\boldsymbol{\xi}_{\eta}]_{332}$ and $[\boldsymbol{\omega}_{\eta}]_3$ will constitute the unknowns of our cell problems, while $[\boldsymbol{u}^{(0)}]_3$ will be the unknown of the homogenised balance of linear momentum. We observe that the constraints imposed on the displacement can be relaxed. Nevertheless, adopting such an assumption would necessitate addressing the in-plane problems associated with the cell problems–an aspect, although achievable using the methods that will be presented in the following sections, not within the scope of our current work.

6.1 | Material properties

In addition to the internal arrangement hypothesis, we consider that all the material properties vary in space only along the directions orthogonal to the Y_3-axis. Within this framework, we prescribe the elasticity tensor of the η-constituent in the natural state, $\mathscr{C}_{\nu\eta}$, to be independent of the macroscale variable so that, with abuse of notation,

$$\mathscr{C}_{\nu\eta}^{\varepsilon}(X,Y)=\mathscr{C}_{\nu\eta}^{\varepsilon}(Y)\equiv\mathscr{C}_{\nu\eta}^{\varepsilon}(Y_1,Y_2). \tag{65}$$

Besides, under the hypothesis of isotropic constituents and piece-wise constant elastic parameters,

$$\mathscr{C}_{\nu\eta}^{\varepsilon}(Y_1,Y_2)=\begin{cases}3\kappa_{\mathrm{m}}\mathscr{K}+2\mu_{\mathrm{m}}\mathscr{M}, & \text{in } \mathscr{Y}_{\mathrm{m}},\\ 3\kappa_{\mathrm{f}}\mathscr{K}+2\mu_{\mathrm{f}}\mathscr{M}, & \text{in } \mathscr{Y}_{\mathrm{f}},\end{cases} \tag{66}$$

where κ_{η} is the bulk modulus of the η-th constituent and μ_{η} is the shear modulus. Furthermore, $\mathscr{K}:=\frac{1}{3}(\boldsymbol{I}\otimes\boldsymbol{I})$ and $\mathscr{M}:=\mathscr{I}-\mathscr{K}$, with $\mathscr{I}:=\frac{1}{2}\left(\boldsymbol{I}\underline{\otimes}\boldsymbol{I}+\boldsymbol{I}\overline{\otimes}\boldsymbol{I}\right)$, are, respectively, the fourth-order tensors that extract the spherical and the deviatoric part of a symmetric second-order tensor. As specified in Equation (65), the elasticity tensor of each constituent referred to the natural state can depend, in general, on X and Y to characterise the inhomogeneous and multi-scale nature of the composite medium under consideration. However, for simplicity, as specified in Equation (66), we adopt the more standard assumption that the elasticity tensor of each constituent in the natural state is constant and that the only non-zero components are

$$[\mathscr{C}^{\varepsilon}_{\nu\eta}]_{1111} = [\mathscr{C}^{\varepsilon}_{\nu\eta}]_{2222} = [\mathscr{C}^{\varepsilon}_{\nu\eta}]_{3333} = \lambda_\eta + 2\mu_\eta, \tag{67a}$$

$$[\mathscr{C}^{\varepsilon}_{\nu\eta}]_{1122} = [\mathscr{C}^{\varepsilon}_{\nu\eta}]_{1133} = [\mathscr{C}^{\varepsilon}_{\nu\eta}]_{2233} = \lambda_\eta, \tag{67b}$$

$$[\mathscr{C}^{\varepsilon}_{\nu\eta}]_{1313} = [\mathscr{C}^{\varepsilon}_{\nu\eta}]_{2323} = [\mathscr{C}^{\varepsilon}_{\nu\eta}]_{1212} = \frac{1}{2}\big([\mathscr{C}^{\varepsilon}_{\nu\eta}]_{1111} - [\mathscr{C}^{\varepsilon}_{\nu\eta}]_{1122}\big) = \mu_\eta, \tag{67c}$$

where we have used the expression $\kappa_\eta := \lambda_\eta + \frac{2}{3}\mu_\eta$ with λ_η being Lamé's parameter.

6.2 | Remodelling tensor and constraints

We assume that the remodelling tensor $\boldsymbol{K}^{(0)}_\eta$ is diagonal with components $[\boldsymbol{K}^{(0)}_\eta]_{11} = [\boldsymbol{K}^{(0)}_\eta]_{22} = 1/\sqrt{p_\eta}$ and $[\boldsymbol{K}^{(0)}_\eta]_{33} = p_\eta$ where p_η is a scalar field, referred to as *remodelling parameter*. We remark that, for solvability issues, this specific characterisation of $\boldsymbol{K}^{(0)}_\eta$ requires that $\boldsymbol{Z}^{(0)}_\eta$ complies with additional conditions. These will be specified below. Furthermore, this form of the remodelling tensor complies, from the outset, with (63a), that is,

$$(\boldsymbol{K}^{(0)}_\eta)^{-\mathrm{T}} : \dot{\boldsymbol{K}}^{(0)}_\eta = \Big[(\boldsymbol{K}^{(0)}_\eta)^{-\mathrm{T}}\Big]_{AB}\Big[\dot{\boldsymbol{K}}^{(0)}_\eta\Big]_{AB} = -\frac{1}{2}\frac{\dot{p}_\eta}{p_\eta} - \frac{1}{2}\frac{\dot{p}_\eta}{p_\eta} + \frac{\dot{p}_\eta}{p_\eta} = 0. \tag{68}$$

On the other hand, by considering the assumptions made on the displacement vector, the only non-null components of the second order tensor $\boldsymbol{\Pi}_\eta$, introduced in Equation (61), are

$$[\boldsymbol{\Pi}_\eta]_{31} = \frac{\partial[\boldsymbol{u}^{(0)}]_3}{\partial X_1} + \frac{\partial}{\partial Y_1}\left([\boldsymbol{\xi}_\eta]_{33D}\frac{\partial[\boldsymbol{u}^{(0)}]_3}{\partial X_D} + [\boldsymbol{\omega}_\eta]_3\right), \tag{69a}$$

$$[\boldsymbol{\Pi}_\eta]_{32} = \frac{\partial[\boldsymbol{u}^{(0)}]_3}{\partial X_2} + \frac{\partial}{\partial Y_2}\left([\boldsymbol{\xi}_\eta]_{33D}\frac{\partial[\boldsymbol{u}^{(0)}]_3}{\partial X_D} + [\boldsymbol{\omega}_\eta]_3\right). \tag{69b}$$

However, as per the second constraint on $\boldsymbol{K}^{(0)}_\eta$ (see Equation (63b)) and provided $\mathfrak{c}^{\varepsilon}_{\nu\eta}$ is not zero, we have that

$$\frac{3}{2L}\frac{\dot{p}_\eta}{p_\eta}[\boldsymbol{\Pi}_\eta]_{31} = 0 \quad \text{and} \quad \frac{3}{2L}\frac{\dot{p}_\eta}{p_\eta}[\boldsymbol{\Pi}_\eta]_{32} = 0. \tag{70}$$

Therefore, since $\dot{p}_\eta = 0$ would imply no evolution in the inelastic distortions, we require that

$$[\boldsymbol{\Pi}_\eta]_{31} = 0 \quad \text{and} \quad [\boldsymbol{\Pi}_\eta]_{32} = 0. \tag{71}$$

In other words, $\boldsymbol{\Pi}_\eta$ must become the null tensor.

6.3 | Elasticity tensor and homogenised evolution law for the inelastic distortions

As per our previous discussions and due to our efforts in providing analytical expressions for the effective coefficients, we restrict our analysis to the case in which the leading order term in the expansion of $\boldsymbol{K}^{\varepsilon}_\eta$ depends only on the macroscale variables X_1 and X_2, and on t, namely, we set

$$p_\eta(X, Y, t) \equiv p(X_1, X_2, t). \tag{72}$$

Regarding Equation (72), would a first-order gradient theory on the tensor of inelastic distortions have been assumed as in ref. [4], the independence of p_η and, thus, of $\boldsymbol{K}_\eta^{(0)}$, on the microscopic variable Y is a consequence and not an assumption. Thus, together with the results in the previous section, the homogenised evolution equation (62) reduces to

$$2\mathrm{sym}\Big(\big\langle \mathfrak{b}_\nu^\varepsilon\big((\boldsymbol{K}^{(0)})^{-1}\dot{\boldsymbol{K}}^{(0)}\big)\big\rangle\Big) = -\,\mathrm{dev}\Big(\Big\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}\Big\rangle\Big) + \mathrm{dev}\big(\big\langle \boldsymbol{Z}^{(0)}\big\rangle\big). \tag{73}$$

In particular, we notice that the only non-zero components of the elasticity tensor pulled back to the reference configuration are

$$[\mathscr{C}_\eta^{(0)}]_{1111} = [\mathscr{C}_\eta^{(0)}]_{2222} = (\lambda_\eta + 2\mu_\eta)p^2, \quad [\mathscr{C}_\eta^{(0)}]_{3333} = (\lambda_\eta + 2\mu_\eta)p^{-4}, \tag{74a}$$

$$[\mathscr{C}_\eta^{(0)}]_{1122} = \lambda_\eta p^2, \quad [\mathscr{C}_\eta^{(0)}]_{1313} = [\mathscr{C}_\eta^{(0)}]_{2323} = \mu_\eta p^{-1}, \tag{74b}$$

$$[\mathscr{C}_\eta^{(0)}]_{1133} = [\mathscr{C}_\eta^{(0)}]_{2233} = \lambda_\eta p^{-1}, \quad [\mathscr{C}_\eta^{(0)}]_{1212} = \mu_\eta p^2. \tag{74c}$$

As per Equations (74a)–(74c), $\mathscr{C}_\eta^{(0)}$ keeps track of the microscopic changes through $\mathscr{C}_{\nu\eta}^{\varepsilon}$ and the macroscopic contributions through $\boldsymbol{K}_\eta^{(0)}$. Thus, in this scenario, the inhomogeneous nature of $\mathscr{C}_\eta^{(0)}$ is inherited from the production of inelastic distortions. Moreover, with respect to the notation introduced in Equation (50a), we can deduce that the only non-zero components of the tensor $\mathscr{G}_\eta^{(0)}$ are

$$[\mathscr{G}_\eta^{(0)}]_{1111} \equiv [\mathscr{G}_\eta^{(0)}]_{2222} = [\mathscr{C}_\eta^{(0)}]_{1111} - \sum_{J=1}^{3} [\mathscr{C}_\eta^{(0)}]_{11JJ} [\boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{JJ}, \tag{75a}$$

$$[\mathscr{G}_\eta^{(0)}]_{1122} = [\mathscr{C}_\eta^{(0)}]_{1122}, \tag{75b}$$

$$[\mathscr{G}_\eta^{(0)}]_{1133} \equiv [\mathscr{G}_\eta^{(0)}]_{2233} = [\mathscr{C}_\eta^{(0)}]_{1133}, \tag{75c}$$

$$[\mathscr{G}_\eta^{(0)}]_{3333} = [\mathscr{C}_\eta^{(0)}]_{3333} - \sum_{J=1}^{3} [\mathscr{C}_\eta^{(0)}]_{33JJ} [\boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{JJ}, \tag{75d}$$

$$[\mathscr{G}_\eta^{(0)}]_{1313} \equiv [\mathscr{G}_\eta^{(0)}]_{2323} = [\mathscr{C}_\eta^{(0)}]_{1313} - [\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33} = [\mathscr{C}_\eta^{(0)}]_{1313} - \sum_{J=1}^{3} [\mathscr{C}_\eta^{(0)}]_{33JJ} [\boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{JJ}, \tag{75e}$$

$$[\mathscr{G}_\eta^{(0)}]_{1212} = [\mathscr{C}_\eta^{(0)}]_{1212} - \sum_{J=1}^{3} [\mathscr{C}_\eta^{(0)}]_{22JJ} [\boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{JJ}, \tag{75f}$$

with

$$[\boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{11} = [\boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{22} = \frac{1}{2}\left(\frac{1}{p} - 1\right) \quad \text{and} \quad [\boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33} = \frac{1}{2}(p^2 - 1). \tag{76}$$

We notice that, by assuming that $\mathfrak{b}_{\nu\eta}^{\varepsilon}$ is the same in both constituents, that is, $\mathfrak{b}_{\nu\eta}^{\varepsilon} = \mathfrak{b}_\nu$, Equation (73) can be rephrased as

$$2\mathfrak{b}_\nu(\boldsymbol{K}^{(0)})^{-1}\dot{\boldsymbol{K}}^{(0)} = -\,\mathrm{dev}\Big(\Big\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}\Big\rangle\Big) + \mathrm{dev}\big(\big\langle \boldsymbol{Z}^{(0)}\big\rangle\big). \tag{77}$$

Equivalently, since

$$[\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}\rangle]_{11} = \frac{1}{|\mathscr{Y}|}\sum_{\eta=\mathrm{m,f}}\left\{\sum_{J=1}^{3}[\mathscr{C}_\eta^{(0)}]_{11JJ}[\boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{JJ}\right\}|\mathscr{Y}_\eta| = \frac{1}{|\mathscr{Y}|}\sum_{\eta=\mathrm{m,f}}\left\{(\lambda_\eta + \mu_\eta)p(1-p) + \frac{1}{2}\lambda_\eta\frac{p^2-1}{p}\right\}|\mathscr{Y}_\eta|, \tag{78a}$$

$$[\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}\rangle]_{22} = \frac{1}{|\mathscr{Y}|}\sum_{\eta=\mathrm{m,f}}\left\{\sum_{J=1}^{3}[\mathscr{C}_\eta^{(0)}]_{22JJ}[\boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{JJ}\right\}|\mathscr{Y}_\eta| = \frac{1}{|\mathscr{Y}|}\sum_{\eta=\mathrm{m,f}}\left\{(\lambda_\eta + \mu_\eta)p(1-p) + \frac{1}{2}\lambda_\eta\frac{p^2-1}{p}\right\}|\mathscr{Y}_\eta|, \tag{78b}$$

$$[\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)} \rangle]_{33} = \frac{1}{|\mathscr{Y}|} \sum_{\eta=\mathrm{m,f}} \left\{ \sum_{J=1}^{3} [\mathscr{C}_{\eta}^{(0)}]_{33JJ} [\boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{JJ} \right\} |\mathscr{Y}_{\eta}| = \frac{1}{|\mathscr{Y}|} \sum_{\eta=\mathrm{m,f}} \left\{ \lambda_{\eta} \frac{1-p}{p^2} + \frac{1}{2}(\lambda_{\eta} + 2\mu_{\eta}) \frac{p^2-1}{p^4} \right\} |\mathscr{Y}_{\eta}|, \quad (78c)$$

which yields that $[\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)} \rangle]_{11} = [\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)} \rangle]_{22}$, and the equation for the evolution of the inelastic distortions reduces to

$$-\mathfrak{b}_{\nu} \frac{\dot{p}}{p} = -\frac{1}{3}\Big([\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)} \rangle]_{11} - [\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)} \rangle]_{33} \Big) + [\mathrm{dev}(\langle \boldsymbol{Z}^{(0)} \rangle)]_{11}, \quad (79)$$

provided that $[\mathrm{dev}\,(\langle \boldsymbol{Z}^{(0)} \rangle)]_{11} = [\mathrm{dev}\,(\langle \boldsymbol{Z}^{(0)} \rangle)]_{22}$ and $[\mathrm{dev}\,(\langle \boldsymbol{Z}^{(0)} \rangle)]_{33} = -2[\mathrm{dev}\,(\langle \boldsymbol{Z}^{(0)} \rangle)]_{11}$.

7 | SOLUTION OF THE CELL PROBLEMS

Even though we are dealing with inelastic distortions that make the cell problems (57a)–(57c) and (58a)–(58c) to depend on the macroscopic variable and the solution of the homogenised problems, the considerations made in the previous sections make them possible to be solved by adapting the schemes available in, for instance, [57, 62, 66, 75, 102, 103] which are based in complex variable methods [57, 69, 74].

7.1 | Solution of the cell problem for $[\boldsymbol{\xi}_{\eta}]_{33D}$

Upon the above considerations, the cell problem associated with $[\boldsymbol{\xi}_{\eta}]_{33D}$, with $D = 1, 2$, reduces to

$$\frac{1}{L^2} \sum_{B=1}^{2} \left\{ [\mathscr{G}_{\eta}^{(0)}]_{3B3B} \frac{\partial}{\partial Y_B} \left(\frac{\partial [\boldsymbol{\xi}_{\eta}]_{33D}}{\partial Y_B} \right) \right\} = 0, \quad \text{in } \mathscr{Y}_{\eta}, \quad (80a)$$

$$[\boldsymbol{\xi}_{\mathrm{m}}]_{33D} = [\boldsymbol{\xi}_{\mathrm{f}}]_{33D}, \quad \text{on } \Gamma, \quad (80b)$$

$$\frac{1}{L} \sum_{B=1}^{2} \left\{ [\mathscr{G}_{\mathrm{m}}^{(0)}]_{3B3B} \frac{\partial [\boldsymbol{\xi}_{\mathrm{m}}]_{33D}}{\partial Y_B} \right\} N_B - \frac{1}{L} \sum_{B=1}^{2} \left\{ [\mathscr{G}_{\mathrm{f}}^{(0)}]_{3B3B} \frac{\partial [\boldsymbol{\xi}_{\mathrm{f}}]_{33D}}{\partial Y_B} \right\} N_B = \frac{1}{L} \left\{ [\mathscr{G}_{\mathrm{f}}^{(0)}]_{3B3D} - [\mathscr{G}_{\mathrm{m}}^{(0)}]_{3B3D} \right\} N_B, \quad \text{on } \Gamma. \quad (80c)$$

Regarding Remark 1, we notice that since $[\mathscr{G}_{\eta}^{(0)}]_{3131}$ and $[\mathscr{G}_{\eta}^{(0)}]_{3232}$ are equal and Y-independent for each $\eta = \mathrm{m}, \mathrm{f}$, the existence of a solution of Equation (80a) is guaranteed for each X and t, provided they are not zero, as the differential operator becomes a Laplacian. To find the solution, we take inspiration from [57, 62, 66, 103] and express $[\boldsymbol{\xi}_{\eta}]_{331}$ and $[\boldsymbol{\xi}_{\eta}]_{332}$ as

$$[\boldsymbol{\xi}_{\eta}(X, Y, t)]_{331} = \Re\{\varphi_{\eta}^{1}(X, Z, t)\}, \quad (81a)$$

$$[\boldsymbol{\xi}_{\eta}(X, Y, t)]_{332} = \Im\{\varphi_{\eta}^{2}(X, Z, t)\}, \quad (81b)$$

where φ_{η}^{1} and φ_{η}^{2} are complex potentials, $\Re\{\cdot\}$ and $\Im\{\cdot\}$ extract the real and imaginary parts of the quantities they are applied to, and $Z = Y_1 + iY_2$ with i being the imaginary unit. Specifically, we assume that for $D = 1, 2$, the complex potentials are defined as

$$\varphi_{\mathrm{m}}^{D}(X, Z, t) := a_0^D(X, t) Z + \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{+\infty} a_{\ell}^{D}(X, t) Z^{\ell} \frac{\zeta^{(\ell-1)}(Z)}{(\ell-1)!} = \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{+\infty} \left\{ a_{\ell}^{D}(X, t) Z^{-\ell} - A_{\ell}^{D}(X, t) Z^{\ell} \right\}, \quad (82a)$$

$$\varphi_{\mathrm{f}}^{D}(X, Z, t) := \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{+\infty} c_{\ell}^{D}(X, t) Z^{\ell}. \quad (82b)$$

In Equations (82a) and (82b), ζ denotes the quasi-periodic Weierstrass zeta function of periods w_1 and w_2 and $\zeta^{(k)}$ its k-th order derivative. The far right-hand side expression in Equation (82a) is obtained by calculating the Laurent series of

$\zeta^{(k)}(Z)$ about $Z = 0$ (see, for instance, refs. [68, 104] for further details), where

$$A_\ell^D(X,t) := \sum_{\substack{m=1\\ m \text{ odd}}}^{+\infty} m\Lambda_{\ell m}^D\, a_m^D(X,t), \quad \text{with} \quad \Lambda_{\ell m}^D := \begin{cases} \dfrac{(\ell+m-1)!}{\ell!m!} S_{\ell+m}, & \text{if } \ell, m > 1, \\ (-1)^{D+1}\pi, & \text{if } \ell = m = 1. \end{cases} \tag{83}$$

In Equation (83), $S_{\ell+m} := \sum_{w\in L\setminus\{0\}} w^{-(\ell+m)}$ denotes the so-called reticular sums where the lattice L is the set of all complex numbers of the form $w = rw_1 + sw_2$ with w_1 and w_2 being linearly independent and $r, s \in \mathbb{Z}$. We observe that the value of $\Lambda_{m\ell}^D$ for $m = \ell = 1$ stems from our choice of the unit cell shape to be a square, where $w_1 = 1$ and $w_2 = i$ [53, 62].

We remark that the complex coefficients a_0^D, a_ℓ^D and c_ℓ^D are parameterised by space and time. Indeed, in this work, by allowing the coefficients of the expansions (82a) and (82b) to keep track of X and t, we will be able to circumvent the computational complexities involved in determining the effective properties of a composite undergoing remodelling of its constituents which implies, as discussed in ref. [9], addressing from a numerical point of view the interactions across different scales of the cell and homogenised problems. Moreover, the solution of the cell problem (80a)–(80c) requires finding the complex coefficients a_0^D, a_ℓ^D and c_ℓ^D that satisfy the interface conditions (80b) and (80c). This approach relies on the insight that both, the real and imaginary components of an analytic function satisfy Equation (80a). Hence, we proceed in two steps which are briefly described below.

7.1.1 | Step 1 - Substitution into the first interface condition

As a first step, we substitute (82a) and (82b) into the interface condition (80b). Hence, we have that

$$\text{For } D = 1, \quad \Re\left\{\sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{+\infty} \left[a_\ell^1(X,t)Z^{-\ell} - A_\ell^1(X,t)Z^\ell\right]\right\} = \Re\left\{\sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{+\infty} c_\ell^1(X,t)Z^\ell\right\}, \quad \text{on } \Gamma, \tag{84a}$$

$$\text{For } D = 2, \quad \Im\left\{\sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{+\infty} \left[a_\ell^2(X,t)Z^{-\ell} - A_\ell^2(X,t)Z^\ell\right]\right\} = \Im\left\{\sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{+\infty} c_\ell^2(X,t)Z^\ell\right\}, \quad \text{on } \Gamma. \tag{84b}$$

In the specific scenario where the interface is a circle centred at the origin with a radius $R > 0$, we can write $Z = Re^{i\Theta}$ with $\Theta \in [0, 2\pi]$. Thus, by noticing that for a complex function ψ, $\Re\{\psi\} = (\psi + \overline{\psi})/2$ and $\Im\{\psi\} = (\psi - \overline{\psi})/(2i)$ where $\overline{\psi}$ is the complex conjugate of ψ, for each $\ell = 1, 3, \ldots$, Equations (84a) and (84b) can be rephrased as

$$(-1)^{D+1}\overline{a_\ell^D(X,t)}R^{-\ell} - A_\ell^D(X,t)R^\ell = c_\ell^D(X,t)R^\ell, \tag{85}$$

with $D = 1, 2$.

7.1.2 | Step 2 - Substitution into the second interface condition

Before substituting (82a) and (82b) into the interface condition (80c). We notice that the components of the unit normal vector $\boldsymbol{N}$ can be written in the form

$$N_1 = \frac{1}{R}\frac{\mathrm{d}Y_2}{\mathrm{d}\Theta} \quad \text{and} \quad N_2 = -\frac{1}{R}\frac{\mathrm{d}Y_1}{\mathrm{d}\Theta}. \tag{86}$$

Thus, exploiting the sparse characteristics of $\mathscr{G}_\eta^{(0)}$ given in Equations (75a)–(75f), the interface condition (80c) can be reformulated as follows

$$\begin{aligned}(-1)^{D+1}\frac{1}{L}[\mathscr{G}_\mathrm{m}^{(0)}]_{3131}\left(\varphi_\mathrm{m}^1 - (-1)^{D+1}\overline{\varphi_\mathrm{m}^1}\right) - (-1)^{D+1}\frac{1}{L}[\mathscr{G}_\mathrm{f}^{(0)}]_{3131}\left(\varphi_\mathrm{f}^1 - (-1)^{D+1}\overline{\varphi_\mathrm{f}^1}\right) \\ = -\frac{1}{L}(-1)^{D+1}\left([\mathscr{G}_\mathrm{m}^{(0)}]_{3131} - [\mathscr{G}_\mathrm{f}^{(0)}]_{3131}\right)\left(Z - (-1)^{D+1}\overline{Z}\right),\end{aligned} \tag{87}$$

where we have used Cauchy–Riemann equations for the real and imaginary parts of the complex potential φ_η^D, with $\eta = \mathrm{m}, \mathrm{f}$ and $D = 1, 2$, and integrated over $\Theta \in [0, 2\pi[$. Thus, using the series expansion in (82a) and (82b), Equation (87) yields

$$\frac{1}{L}\overline{a_\ell^D(X,t)}R^{-\ell} + (-1)^{D+1}\frac{1}{L}\chi(X,t)A_\ell^D(X,t)R^\ell = \frac{1}{L}(-1)^{D+1}\chi(X,t)R^\ell\delta_{\ell 1}. \tag{88}$$

which results from using Equation (85) and equating the coefficients of the complex exponential $e^{i\Theta\ell}$ for each $\ell = 1, 3, 5, \ldots$. Furthermore, the quantity $\chi(X,t)$ is defined as

$$\chi(X,t) := \frac{[\mathscr{G}_\mathrm{m}^{(0)}(X,t)]_{3131} - [\mathscr{G}_\mathrm{f}^{(0)}(X,t)]_{3131}}{[\mathscr{G}_\mathrm{m}^{(0)}(X,t)]_{3131} + [\mathscr{G}_\mathrm{f}^{(0)}(X,t)]_{3131}}, \tag{89}$$

which encodes the elastic properties of the composite constituents as well as the evolution of the remodelling parameter. In particular, by recalling the definition of A_ℓ^D introduced in (83) and defining $b_\ell^D := a_\ell^D R^{-\ell}\sqrt{\ell}$, we can rewrite Equation (88) in the equivalent form

$$\frac{1}{L}\overline{b_\ell^D(X,t)} + (-1)^{D+1}\frac{1}{L}\chi(X,t)\left\{\sum_{\substack{m=1\\ m\ \mathrm{odd}}}^{+\infty}\sqrt{\ell m}\,\Lambda_{\ell m}^D R^{\ell+m} b_m^D(X,t)\right\} = \frac{1}{L}(-1)^{D+1}\chi(X,t)\sqrt{\ell}R^\ell\delta_{\ell 1}. \tag{90}$$

Hence, since $b_\ell^D = \mathfrak{R}\{b_\ell^D\} + i\mathfrak{I}\{b_\ell^D\}$ and $\Lambda_{\ell m}^D = \mathfrak{R}\{\Lambda_{\ell m}^D\} + i\mathfrak{I}\{\Lambda_{\ell m}^D\}$, the infinite linear system represented in Equations (90) in the unknown b_ℓ^D, can be conveniently rewritten as

$$\left(\frac{1}{L}\begin{bmatrix}\mathfrak{I} & \mathfrak{O}\\ \mathfrak{O} & \mathfrak{I}\end{bmatrix} + (-1)^{D+1}\frac{1}{L}\chi(X,t)\begin{bmatrix}\mathfrak{R}\{\mathfrak{W}^D\} & -\mathfrak{I}\{\mathfrak{W}^D\}\\ -\mathfrak{I}\{\mathfrak{W}^D\} & -\mathfrak{R}\{\mathfrak{W}^D\}\end{bmatrix}\right)\begin{bmatrix}\mathfrak{R}\{\boldsymbol{b}^D(X,t)\}\\ \mathfrak{I}\{\boldsymbol{b}^D(X,t)\}\end{bmatrix} = \frac{1}{L}\boldsymbol{V}^D(X,t), \tag{91}$$

where $\boldsymbol{b}^D$ denotes the column vector $\boldsymbol{b}^D = (b_1^D, b_3^D, \ldots)^\mathrm{T}$, $\mathfrak{I}$ and $\mathfrak{O}$ denote, respectively, infinite identity and zero matrices, $\mathfrak{W}$ is the infinite matrix with components $[\mathfrak{W}^D]_{\ell m} = \sqrt{\ell m}\,\Lambda_{\ell m}^D R^{\ell+m}$, and $\boldsymbol{V}^D = ((-1)^{D+1}\chi R, 0, \ldots)^\mathrm{T}$. To find the solution $\boldsymbol{b}^D$, we truncate the system (91) for some $\ell = 1, 3, \ldots$ and take $m = 1, 3, \ldots, \ell$. As it will be clear in the following sections, the computation of the effective coefficients $[\mathscr{C}_\mathrm{eff}]_{3B3D}$ will rely only on the knowledge of b_1^D, that is, the first component of the vector $\boldsymbol{b}^D$, and, thus, of a_1^D.

Remark 3 (No inelastic distortions). We notice that in the case $\boldsymbol{K}_\eta \equiv \boldsymbol{I}$, that is, upon considering a composite made of two linear elastic and isotropic constituents with no evolution of their internal structure, the system of Equations (91) reduces to

$$\left(\frac{1}{L}\begin{bmatrix}\mathfrak{I} & \mathfrak{O}\\ \mathfrak{O} & \mathfrak{I}\end{bmatrix} + (-1)^{D+1}\frac{1}{L}\chi\begin{bmatrix}\mathfrak{R}\{\mathfrak{W}^D\} & -\mathfrak{I}\{\mathfrak{W}^D\}\\ -\mathfrak{I}\{\mathfrak{W}^D\} & -\mathfrak{R}\{\mathfrak{W}^D\}\end{bmatrix}\right)\begin{bmatrix}\mathfrak{R}\{\boldsymbol{b}^D\}\\ \mathfrak{I}\{\boldsymbol{b}^D\}\end{bmatrix} = \frac{1}{L}\boldsymbol{V}^D, \tag{92}$$

with χ rephrased as

$$\chi = \frac{[\mathscr{C}_\mathrm{m}^{(0)}]_{3131} - [\mathscr{C}_\mathrm{f}^{(0)}]_{3131}}{[\mathscr{C}_\mathrm{m}^{(0)}]_{3131} + [\mathscr{C}_\mathrm{f}^{(0)}]_{3131}}, \tag{93}$$

since $\mathscr{G}_\eta^{(0)} = \mathscr{C}_\eta^{(0)} - \boldsymbol{I}\underline{\otimes}(\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}) = \mathscr{C}_\eta^{(0)}$. In this simplified scenario, we notice that the dependency on X and t in χ and a_ℓ^D is lost and χ represents the relative difference in shear moduli between the matrix and the circular inclusion. Equations (92) and (93) coincide with those in ref. [105] for a two-phase composite elastic material.

7.2 | Solution of the cell problem for $[\omega_\eta]_3$

The cell problem associated with $[\boldsymbol{\omega}_\eta]_3$ is

$$\frac{1}{L^2}\sum_{B=1}^{2}\left\{[\mathcal{G}_\eta^{(0)}]_{3B3B}\frac{\partial}{\partial Y_B}\left(\frac{\partial[\boldsymbol{\omega}_\eta]_3}{\partial Y_B}\right)\right\} = 0, \quad \text{in } \mathcal{Y}_\eta, \tag{94a}$$

$$[\boldsymbol{\omega}_{\mathrm{m}}]_3 = [\boldsymbol{\omega}_{\mathrm{f}}]_3, \quad \text{on } \Gamma, \tag{94b}$$

$$\frac{1}{L}\sum_{B=1}^{2}\left\{[\mathcal{G}_{\mathrm{m}}^{(0)}]_{3B3B}\frac{\partial[\boldsymbol{\omega}_{\mathrm{m}}]_3}{\partial Y_B}\right\}N_B - \frac{1}{L}\sum_{B=1}^{2}\left\{[\mathcal{G}_{\mathrm{f}}^{(0)}]_{3B3B}\frac{\partial[\boldsymbol{\omega}_{\mathrm{f}}]_3}{\partial Y_B}\right\}N_B$$
$$= -\left\{\sum_{J=1}^{3}\left([\mathcal{C}_{\mathrm{f}}^{(0)}]_{3BJJ}[\boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{JJ} - [\mathcal{C}_{\mathrm{m}}^{(0)}]_{3BJJ}[\boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{JJ}\right)\right\}N_B, \quad \text{on } \Gamma, \tag{94c}$$

where we notice that, because of the sparse characterisitcs of $\mathcal{C}_\eta^{(0)}$, the right-hand side of Equation (94c) is equal to zero. This result implies that we are in the presence of a partial differential equation for $[\boldsymbol{\omega}_\eta]_3$ with no jump conditions on the interface Γ so that by the uniqueness of the solution of the problem specified in (94a)–(94c), we obtain $[\boldsymbol{\omega}_\eta]_3 = 0$, that is $[\boldsymbol{\omega}_\eta]_3$ is the trivial solution, as we are assuming that the average of $\boldsymbol{\omega}_\eta$ is zero. This result is analogous, to what we found in ref. [4], for which the solutions to the cell problems in the direction of no changes in material properties are the trivial solution.

8 | HOMOGENISED GOVERNING EQUATIONS AND EFFECTIVE COEFFICIENTS

8.1 | The homogenised balance of linear momentum and related effective coefficients

The considerations made so far, together with the previous results, permit considering the following equation to find $[\boldsymbol{u}^{(0)}]_3$

$$\frac{\partial}{\partial X_B}\left\{[\mathcal{G}_{\mathrm{eff}}(X,t)]_{3B3D}\frac{\partial[\boldsymbol{u}^{(0)}(X,t)]_3}{\partial X_D}\right\} = 0, \tag{95}$$

where, with abuse of notation, $X = (X_1, X_2)$ and $B, D = 1, 2$. Furthermore, we notice that $[\boldsymbol{D}_{\mathrm{eff}}(X,t)]_{3B} = 0$.

Now, together with the necessity of prescribing appropriate boundary and initial conditions to solve (95), we also need to know the effective coefficients $[\mathcal{G}_{\mathrm{eff}}(X,t)]_{3B3D}$, which are given by the expression

$$[\mathcal{G}_{\mathrm{eff}}]_{3B3D} = \left\langle [\mathcal{G}^{(0)}]_{3B3D} + [\mathcal{G}^{(0)}]_{3B3Q}\frac{\partial \xi_{33D}}{\partial Y_Q}\right\rangle, \tag{96}$$

with $B, D, Q = 1, 2$. These can be found through the solution of the cell problems associated with $[\boldsymbol{\xi}_\eta]_{331}$ and $[\boldsymbol{\xi}_\eta]_{332}$. Specifically, because of the local periodicity property of $\boldsymbol{\xi}_\eta$ and considering the orthogonality properties of $\sin(\Theta\ell)$ and $\cos(\Theta\ell)$ with respect to the inner product $\langle f(\Theta), g(\Theta)\rangle = \int_0^{2\pi} f(\Theta)g(\Theta)\mathrm{d}\Theta$, namely,

$$\langle \sin(\Theta\ell), \cos(\Theta k)\rangle = 0, \quad \text{for all } \ell, k \in \mathbb{Z}, \tag{97a}$$

$$\langle \sin(\Theta\ell), \sin(\Theta k)\rangle = \langle \cos(\Theta\ell), \cos(\Theta k)\rangle = 0, \quad \text{for all } \ell, k \in \mathbb{Z} \text{ with } \ell \neq k, \tag{97b}$$

$$\langle \sin(\Theta\ell), \sin(\Theta k)\rangle = \langle \cos(\Theta\ell), \cos(\Theta k)\rangle = \pi, \quad \text{for all } \ell, k \in \mathbb{Z}\setminus\{0\} \text{ with } \ell = k, \tag{97c}$$

the effective coefficients specified in (96) are expressed as

$$[\mathcal{G}_{\mathrm{eff}}]_{3131} = \langle [\mathcal{G}^{(0)}]_{3131}\rangle + \left\langle [\mathcal{G}^{(0)}]_{3131}\frac{\partial \xi_{331}}{\partial Y_1}\right\rangle = \frac{1}{|\mathcal{Y}|}[\mathcal{G}_{\mathrm{m}}^{(0)}]_{3131}\left(1 - 2\pi\Re\{a_1^1\}\right), \tag{98a}$$

$$[\mathcal{G}_{\mathrm{eff}}]_{3231} = \left\langle [\mathcal{G}^{(0)}]_{3232}\frac{\partial \xi_{331}}{\partial Y_2}\right\rangle = -\frac{2\pi}{|\mathcal{Y}|}[\mathcal{G}_{\mathrm{m}}^{(0)}]_{3131}\Im\{a_1^1\}, \tag{98b}$$

$$[\mathscr{G}_{\text{eff}}]_{3132} = \left\langle [\mathscr{G}^{(0)}]_{3131} \frac{\partial \xi_{332}}{\partial Y_1} \right\rangle = -\frac{2\pi}{|\mathscr{Y}|} [\mathscr{G}_{\text{m}}^{(0)}]_{3131} \mathfrak{I}\{a_1^2\}, \tag{98c}$$

$$[\mathscr{G}_{\text{eff}}]_{3232} = \langle [\mathscr{G}^{(0)}]_{3232} \rangle + \left\langle [\mathscr{G}^{(0)}]_{3232} \frac{\partial \xi_{332}}{\partial Y_2} \right\rangle = \frac{1}{|\mathscr{Y}|} [\mathscr{G}_{\text{m}}^{(0)}]_{3131} (1 + 2\pi \mathfrak{R}\{a_1^2\}). \tag{98d}$$

In the deduction of the above results, we have also taken into account formulae (85) and (88). We remark that if no inelastic distortions are considered, the above formulae coincide with those in the elastic case, with the only difference being that there would not be a dependency on X and t, as this is given by $\boldsymbol{K}_\eta^{(0)}$, and that $\mathscr{G}_{\text{m}}^{(0)}$ would reduce to $\mathscr{C}_{\text{m}}^{(0)}$.

Remark 4 (First-order approximation formula). We notice that, up to the first order, namely for $\ell = m = 1$, the system of Equations (91) reduces to

$$\left(\frac{1}{L} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-1)^{D+1} \frac{1}{L} \chi(X,t) \begin{bmatrix} \mathfrak{R}\{\Lambda_{11}^D\} R^2 & -\mathfrak{I}\{\Lambda_{11}^D\} R^2 \\ -\mathfrak{I}\{\Lambda_{11}^D\} R^2 & -\mathfrak{R}\{\Lambda_{11}^D\} R^2 \end{bmatrix} \right) \begin{bmatrix} \mathfrak{R}\{b_1^D(X,t)\} \\ \mathfrak{I}\{b_1^D(X,t)\} \end{bmatrix} = \frac{1}{L} \begin{bmatrix} (-1)^{D+1} R \chi(X,t) \\ 0 \end{bmatrix}, \tag{99}$$

with $\Lambda_{11}^D = (-1)^{D+1}\pi$. Therefore, by recalling that $b_\ell^D = a_\ell^D R^{-\ell} \sqrt{\ell}$, the solution of the linear system (99) is

$$\mathfrak{R}\{a_1^D(X,t)\} = (-1)^{D+1} \frac{\chi(X,t) R^2}{1 + \chi(X,t)\pi R^2} \quad \text{and} \quad \mathfrak{I}\{a_1^D(X,t)\} = 0, \tag{100}$$

and, thus, the effective coefficients $[\mathcal{G}_{\text{eff}}]_{3231}$ and $[\mathcal{G}_{\text{eff}}]_{3132}$ are null, while

$$[\mathscr{G}_{\text{eff}}(X,t)]_{3131} = \frac{1}{|\mathscr{Y}|} [\mathscr{G}_{\text{m}}^{(0)}(X,t)]_{3131} \left(\frac{1 - \chi(X,t)|\mathscr{Y}_{\text{f}}|}{1 + \chi(X,t)|\mathscr{Y}_{\text{f}}|} \right), \tag{101a}$$

$$[\mathscr{G}_{\text{eff}}(X,t)]_{3232} = \frac{1}{|\mathscr{Y}|} [\mathscr{G}_{\text{m}}^{(0)}(X,t)]_{3131} \left(\frac{1 - \chi(X,t)|\mathscr{Y}_{\text{f}}|}{1 + \chi(X,t)|\mathscr{Y}_{\text{f}}|} \right), \tag{101b}$$

with $|\mathscr{Y}_{\text{f}}| = \pi R^2$. Hence, a_1^D can be written in closed form and depending on the constituents' elastic properties through $\mathscr{C}_{\nu\eta}$, on the remodelling tensor $\boldsymbol{K}_\eta^{(0)}$ and on the geometrical features of the composite's microstructure. This means that we only need to find the remodelling parameter p as per Equation (79) to be able to compute the effective coefficients.

Additionally, from Equation (101a), the effective coefficient $[\mathscr{G}_{\text{eff}}]_{3131}$ (and since $[\mathscr{G}_{\text{eff}}]_{3131} = [\mathscr{G}_{\text{eff}}]_{3232}$, $[\mathscr{G}_{\text{eff}}]_{3232}$ as well), can be rewritten as

$$[\mathscr{G}_{\text{eff}}]_{3131} = \frac{1}{|\mathscr{Y}|} \left(\frac{1 - \chi|\mathscr{Y}_{\text{f}}|}{1 + \chi|\mathscr{Y}_{\text{f}}|} \right) \left[\frac{(1-p)^2(1+2p)\lambda_{\text{m}} - 2(1-p)p^2\mu_{\text{m}} + 2\mu_{\text{m}}}{2p^4} \right], \tag{102}$$

where we have used Equations (74b), (75f) and (76) to write (102), while χ is given in (89). Delving deeper into the expression in (102), we observe that two specific values of the remodelling parameter p possess relevant physical significance. Specifically, for $p = 1$, that is, when inelastic distortions are absent, we have that

$$[\mathscr{G}_{\text{eff}}]_{3131} = \frac{\mu_{\text{f}}(1 + |\mathscr{Y}_{\text{f}}|) + \mu_{\text{m}}|\mathscr{Y}_{\text{m}}|}{\mu_{\text{m}}(1 + |\mathscr{Y}_{\text{f}}|) + \mu_{\text{f}}|\mathscr{Y}_{\text{m}}|} \mu_{\text{m}}. \tag{103}$$

That is, when $p = 1$, the effective coefficient $[\mathscr{G}_{\text{eff}}]_{3131}$ is multiplicatively decomposed as a dimensionless weight incorporating the contributions of the shear moduli and adjusted by the area of the inclusion and the matrix, and the matrix shear modulus μ_{m}. Hence, $[\mathscr{G}_{\text{eff}}]_{3131}$ expresses the effective shear modulus of the composite material. Furthermore, when $p = 2$,

$$[\mathscr{G}_{\text{eff}}]_{3131} = \frac{5}{36} \frac{\mu_{\text{f}}(1 + |\mathscr{Y}_{\text{f}}|) + \mu_{\text{m}}|\mathscr{Y}_{\text{m}}|}{\mu_{\text{m}}(1 + |\mathscr{Y}_{\text{f}}|) + \mu_{\text{f}}|\mathscr{Y}_{\text{m}}|} (\lambda_{\text{m}} + 2\mu_{\text{m}}). \tag{104}$$

In this scenario, $[\mathscr{G}_{\text{eff}}]_{3131}$ is expressed by the multiplication of the same dimensionless quantity introduced in (103), multiplied by $5/36$, with the matrix P-wave modulus $\lambda_{\text{m}} + 2\mu_{\text{m}}$. Hence, within the first-order approximation formula given

TABLE 1 Values of the model parameters used in the numerical simulations.

Model parameters	Value Matrix (η = m)	Fibre (η = f)	Reference
λ_η	1 Pa	2 Pa	[9]
μ_η	0.1 Pa	0.06 Pa	[9]
$\mathfrak{b}_{\nu\eta}$	1 Pa · s	1 Pa · s	—
R	—	0.2	—

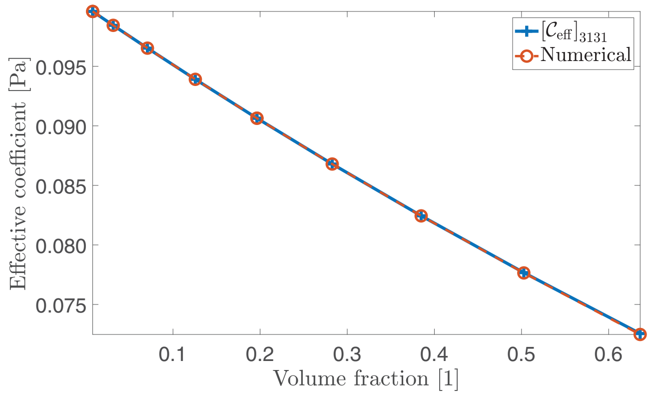

FIGURE 1 Comparison between $[\mathscr{C}_{\text{eff}}]_{3131}$ given in Equation (105) and those obtained with FE for different volumetric fractions of the fibre.

in (101a) for $[\mathscr{G}_{\text{eff}}(X,t)]_{3131}$, if the remodelling parameter evolves to take values within the interval [1,2], then p acts as a transition parameter, modulating the composite material's mechanical response from shear to compressional behaviour.

9 | RESULTS AND FURTHER DISCUSSIONS

In this section, we focus on the computation of the effective coefficient $[\mathscr{G}_{\text{eff}}(X,t)]_{3131} = [\mathscr{G}_{\text{eff}}(X,t)]_{3232}$ considering the formula specified in (101a) to highlight the fundamental role of the remodelling tensor in guiding the overall characterisation of the composite under study. In Table 1, we report the selection of parameters used in our calculations.

9.1 | Case I: No remodelling constituents

As previously discussed, the lack of readily available numerical schemes for problems like the one described in this work makes the comparison with computational results challenging. Nevertheless, to strengthen the validity of our approach, we first consider a scenario where inelastic distortions are absent and compare our results for the effective coefficients with those obtained from a finite element (FE) formulation implemented in COMSOL Multiphysics.

As anticipated in Remark 4, using the first-order approximation, if we set $p = 1$, the effective coefficient reduces to the effective elastic one, namely,

$$[\mathscr{G}_{\text{eff}}]_{3131} = [\mathscr{C}_{\text{eff}}]_{3131} = \frac{1}{|\mathscr{Y}|}[\mathscr{C}^{(0)}_{\text{m}}]_{3131}\left(\frac{1-\chi|\mathscr{Y}_{\text{f}}|}{1+\chi|\mathscr{Y}_{\text{f}}|}\right). \tag{105}$$

In particular, as shown in Figure 1, for the relevant parameter values in Table 1, the effective elastic coefficient $[\mathscr{C}_{\text{eff}}]_{3131}$ matches the numerically computed values for various volumetric fractions of the fibre inclusion. Moreover, we observe that this agreement is achieved even with the first-order approximation of $[\mathscr{C}_{\text{eff}}]_{3131}$.

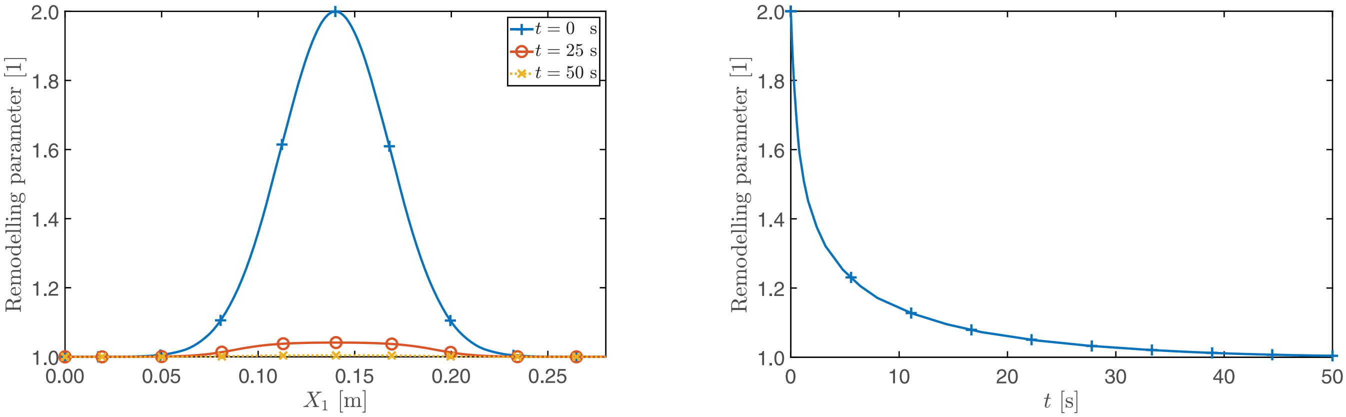

FIGURE 2 (Panel on the left) Spatial distribution of the remodelling parameter p at three distinct time steps with $X_2 = L/2$ and $X_1 \in [0, L]$. (Panel on the right) Time evolution of p at the centre of the domain $(L/2, L/2) \in \mathcal{B}_{\mathrm{h}}$.

9.2 | Case II: Remodelling constituents

In this section, we concentrate on computing $[\mathscr{G}_{\mathrm{eff}}(X, t)]_{3131}$, specifically for the scenario where the evolution of p is governed by Equation (79). As per Equation (101a), the computation of the effective coefficient does not rely on the need of solving the homogenised problem for $[\boldsymbol{u}^{(0)}]_3$, but on the knowledge of

$$\chi(X, t) = \frac{[\mathscr{G}_{\mathrm{m}}^{(0)}(X,t)]_{3131} - [\mathscr{G}_{\mathrm{f}}^{(0)}(X,t)]_{3131}}{[\mathscr{G}_{\mathrm{m}}^{(0)}(X,t)]}_{3131} + [\mathscr{G}_{\mathrm{f}}^{(0)}(X, t)]_{3131} \quad \text{and} \quad [\mathscr{G}_{\eta}^{(0)}]_{3131} = [\mathscr{C}_{\eta}^{(0)}]_{3131} - \sum_{J=1}^{3} [\mathscr{C}_{\eta}^{(0)}]_{33JJ} [\boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{JJ}. \quad (106)$$

That is, to compute $[\mathscr{G}_{\mathrm{eff}}(X, t)]_{3131}$, we need to find the remodelling parameter p through the evolution law (refer to Section 6)

$$\frac{\dot{p}}{p} = \frac{1}{\mathfrak{b}_{\nu}} \left(\frac{2}{3} \sigma_{\mathrm{T}} - [\mathrm{dev}\left(\langle \boldsymbol{Z}^{(0)} \rangle\right)]_{11} \right), \quad (107)$$

where $\sigma_{\mathrm{T}} := \frac{1}{2}([\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)} \rangle]_{11} - [\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)} \rangle]_{33})$.

We note that if $\boldsymbol{Z}_{\eta}$ is assumed to be the null tensor, as discussed in refs. [14, 19, 81], then no additional interactions, such as chemical ones in the context of growth [18, 19, 27], are considered in the evolution of p. In this case, we let the microstructural features of the composite have a leading role in influencing the progression of the inelastic distortions. These are given by the sole contribution of the Mandel stress tensor, represented by the final term in Equation (23b) which in (107) appears as the first term on the right-hand side, and the geometrical setting of the microstructure. On the contrary, for a non-null $\boldsymbol{Z}_{\eta}$, the evolution law for p would be influenced not only by intrinsic microstructural features, but also by the imbalance between these features and additional interactions encoded in $[\mathrm{dev}\left(\langle \boldsymbol{Z}^{(0)} \rangle\right)]_{11}$. Notably, if the right-hand side of (107) is zero, that is $[\mathrm{dev}\left(\langle \boldsymbol{Z}^{(0)} \rangle\right)]_{11} = 2\sigma_{\mathrm{T}}/3$, the evolution of p will be halted. Given the nature of this work and the lack of motivated expressions for $[\mathrm{dev}\left(\langle \boldsymbol{Z}^{(0)} \rangle\right)]_{11}$, we will assume, for the time being, that $\boldsymbol{Z}_{\eta} = \boldsymbol{O}$.

To solve Equation (107), we take the initial condition for p to be represented by a two-dimensional Gaussian function centred at the middle of the domain $\mathcal{B}_{\mathrm{h}} = [0, L] \times [0, L]$ with $L = 0.28$ m. Specifically,

$$p(X_1, X_2, 0) = 1 + \beta \exp\left(-\frac{(X_1 - L/2)2}{2(\tau_1)2} - \frac{(X_2 - L/2)2}{2(\tau_2)2} \right), \quad (108)$$

where $\beta \in \mathbb{R}$, $\tau_1 \neq 0$ and $\tau_2 \neq 0$ are constants. With this selection, our numerical experiment describes a situation in which the inelastic distortions are concentrated at the centre of $\mathcal{B}_{\mathrm{h}}$. In particular, we assume $\beta = 1$ and $\tau_1 = \tau_2 = L/10$ so that, according to Remark 4, p take values in the interval [1,2]. Particularly, due to the nonlinear nature of Equation (107), we utilised COMSOL Multiphysics to obtain its numerical solution.

In Figure 2, we report the distribution of the remodelling parameter p at three different times. These correspond to $t = 0$ s, $t = 25$ s, and $t = 50$ s. In particular, because of the symmetrical nature of the solution, we report in Figure 2 (panel on the left) the spatial distribution of p at $X_2 = L/2$ and with $X_1 \in [0, L]$. Furthermore, in Figure 2 (panel on the right),

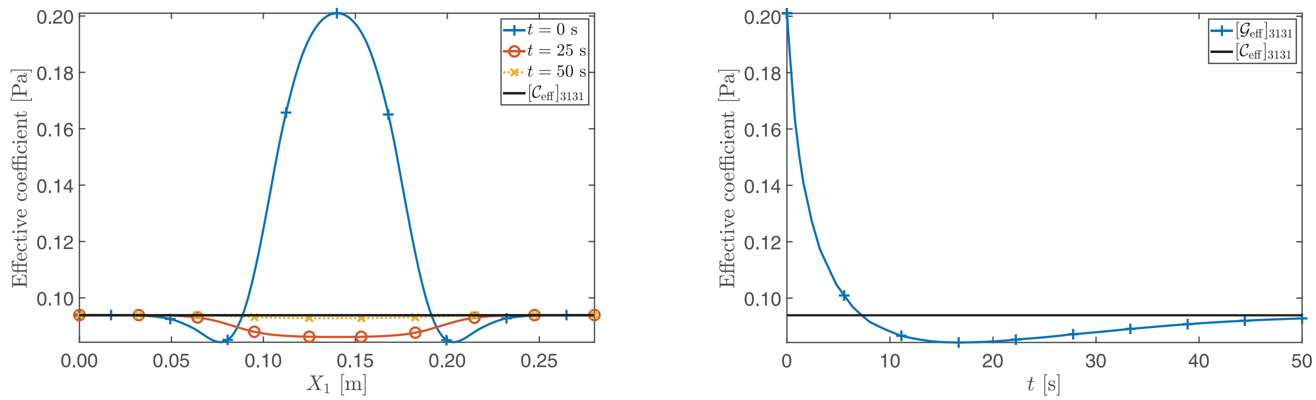

FIGURE 3 (Panel on the left) Distribution of the effective coefficient $[\mathcal{G}_{\text{eff}}]_{3131}$ at three distinct time steps with $X_2 = L/2$ and $X_1 \in [0, L]$. (Panel on the right) Time evolution of $[\mathcal{G}_{\text{eff}}]_{3131}$ at the centre of the domain $(L/2, L/2) \in \mathcal{B}_\text{h}$.

TABLE 2 L^2-, L^∞- and H^1-norm errors at different time steps.

	$t = 0$ s	$t = 25$ s	$t = 50$ s
$\|e(X_1, X_2, t)\|_{L^2(\mathcal{B}_\text{h})}$	$7.0095 \cdot 10^{-16}$	$7.8805 \cdot 10^{-16}$	$7.9368 \cdot 10^{-16}$
$\|e(X_1, X_2, t)\|_{L^\infty(\mathcal{B}_\text{h})}$	$2.8152 \cdot 10^{-8}$	$2.8152 \cdot 10^{-8}$	$2.8152 \cdot 10^{-8}$
$\|e(X_1, X_2, t)\|_{H^1(\mathcal{B}_\text{h})}$	$6.2159 \cdot 10^{-15}$	$7.9894 \cdot 10^{-16}$	$7.9372 \cdot 10^{-16}$

we depict the time evolution of p at the centre of the domain $(L/2, L/2)$. Notably, we observe that the inelastic distortions, which in this case are stress-driven as delineated in Equation (107), tend to dissipate over time and stabilise around the value $p = 1$ as time progresses. This behaviour aligns with the trend of σ_T, which (although not shown here) decreases at each time-step.

The impact of inelastic distortions on the effective coefficient $[\mathcal{G}_{\text{eff}}]_{3131}$ is particularly significant. That is because the effects of $\boldsymbol{K}_\eta^{(0)}$ are normalised by χ (see Equation (106)), the evolution of p significantly influences the distribution and progression of the effective coefficient via $[\mathcal{G}_\text{m}^{(0)}]_{3131}$ and $[\mathcal{G}_\text{f}^{(0)}]_{3131}$. Notably, attention to Figure 3 reveals that both the spatial and time distribution of $[\mathcal{G}_{\text{eff}}]_{3131}$ have a non-monotonic behaviour. In particular, focusing on the interval $[0, L/2]$ of Figure 3 (panel on the left), we notice that at initial times, $[\mathcal{G}_{\text{eff}}]_{3131}$ starts decreasing at around $X = 0.05$ m, while it increases at approximately $X = 0.07$ m. This tendency is driven by the growing prevalence of inelastic distortions near the centre of the considered domain (refer to Figure 2). Moreover, as time progresses and according to the reduction of inelastic distortions, the effective coefficient $[\mathcal{G}_{\text{eff}}]_{3131}$ loses its non-monotonic trend and converges to the effective elastic one $[\mathcal{C}_{\text{eff}}]_{3131} = 0.093908$ Pa. Considering the evolution of p, this corresponds to the portion of the domain where the remodelling parameter is close to one, indicating a situation of no occurrence of inelastic distortions. Consequently, as verified in Figure 3 (panel on the left), the overall mechanical properties of the composite do not change in this region. It is worth noticing that the convergence of $[\mathcal{G}_{\text{eff}}]_{3131}$ to the elastic effective coefficient $[\mathcal{C}_{\text{eff}}]_{3131}$ (i.e., for $p = 1$) is not reached in a monotonic way. Indeed, as per Figure 3 (panel on the right), $[\mathcal{G}_{\text{eff}}]_{3131}$ initially decreases by going below $[\mathcal{C}_{\text{eff}}]_{3131}$ and then, it increases over time to reach the elastic effective coefficient. These considerations underscore the potential impact of the evolution and distribution of inelastic distortions in, for instance, biomimetic applications where it could be desirable to accommodate the mechanical properties of a specimen according to the changes in its internal structure.

Additionally, we note that while our current discussion has focused on the scenario where the linear system (91) is considered up to order $\ell = m = 1$, the above considerations still hold, for the selected parameter values, regardless of whether higher orders of truncation are considered. This is reflected in Figure 4 where the effective coefficient's time evolution at the point $(L/2, L/2)$ is reported for the truncation orders $\ell = m = 1$ and $\ell = m = 3$. In this case, both curves overlap demonstrating the rapid convergence of the semi-analytical solution for a low truncation order.

Furthermore, in Table 2, we report, for three different simulation times, the L^2-, L^∞- and H^1-norm errors defined as [106]

$$\|e(X_1, X_2, t)\|_{L^2(\mathcal{B}_\text{h})} := \frac{1}{|\mathcal{B}_\text{h}|} \int_{\mathcal{B}_\text{h}} [e(X_1, X_2, t)]^2 \mathrm{d}A, \tag{109a}$$

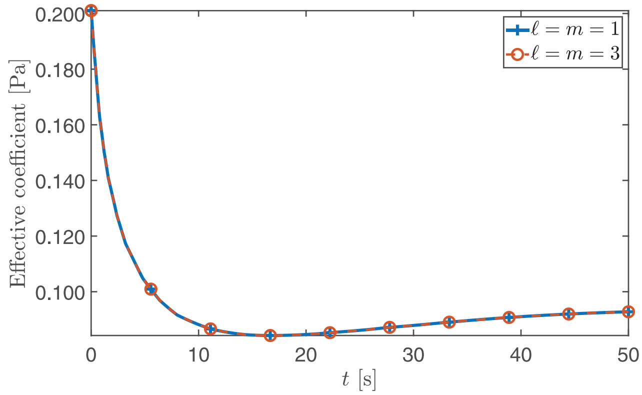

FIGURE 4 Time evolution of $[\mathcal{G}_{\text{eff}}]_{3131}$ at the point of $(L/2, L/2)$ in $\mathcal{B}_\text{h}$, for truncation orders $\ell = m = 1$ and $\ell = m = 3$.

$$\|e(X_1, X_2, t)\|_{L^\infty(\mathcal{B}_\text{h})} := \sup_{X_1, X_2 \in \mathcal{B}_\text{h}} \{e(X_1, X_2, t)\}, \tag{109b}$$

$$\|e(X_1, X_2, t)\|_{H^1(\mathcal{B}_\text{h})} := \frac{1}{|\mathcal{B}_\text{h}|} \int_{\mathcal{B}_\text{h}} [e(X_1, X_2, t)]2\text{d}A + \int_{\mathcal{B}_\text{h}} \|\text{Grad}e(X_1, X_2, t)\|^2 \text{d}A, \tag{109c}$$

with the error being given by

$$e(X_1, X_2, t) := \left|\{[\mathcal{G}_{\text{eff}}]_{3131}(X_1, X_2, t)\}_1 - \{[\mathcal{G}_{\text{eff}}]_{3131}(X_1, X_2, t)\}_3\right| \tag{110}$$

and where the subscripts “1” and “3” in Equation (110) refer to the truncation order in the linear system (91). The data presented in the table indicates that the error is negligible, highlighting the accuracy and reliability of the results for the first-order approximation.

10 | CONCLUSIONS

In this work, we focus on the computation of the effective properties of composite materials undergoing remodelling of their internal structure. Our motivation stems from the need to characterise the global properties of biological tissues that can alter their mechanical properties in response to permanent transformations, interpreted, in our formulation, through the concept of inelastic processes and quantified through the BKL decomposition of the deformation gradient tensor.

The mathematical model is set in a purely mechanical framework involving the balance of linear momentum and the balance of generalised forces. Notably, within the setting of this work, which relies on a theory of grade zero in the remodelling tensor, the main equations summarised in Sections 5.4–5.5, including the expressions for the effective coefficients (59a)–(59b) and the homogenised balance of linear momentum (60), will still hold regardless of the choice of remodelling law under certain considerations. These equations arise from a multi-scale framework using the AH technique, which entails identifying homogenised properties and equations informed by the solutions of local problems. The solutions to these local problems are therefore of fundamental importance. In this work, we opted for a benchmark scenario that leverages concepts and techniques from the theory of harmonic functions and complex potentials to pursue exact solutions. In doing so, we derived closed-form expressions for the relevant effective coefficients, notable for their dependence on the evolution of the remodelling tensor. Among the considerations adopted in the study of the benchmark problem, we leverage the infinitesimal deformation regime and the geometrical features of the internal structure of the composite material which we assume to be described as a uniaxially fibre-reinforced composite. Nevertheless, while these selections offer inherent mathematical advantages and can serve as good approximations for biological structures like bones in certain scenarios, we acknowledge their limitations in capturing the full range of nonlinear behaviours exhibited by biological tissues.

To underscore the influence of the remodelling tensor on the composite's overall properties, we examine the case in which the generalised force $\boldsymbol{Z}_\eta$ is zero. Thus, relying on the effects of the intrinsic microstructural features of the composite on its overall characterisation. A notable result is given by the influence of the remodelling tensor on the mechanical properties of the homogenised composite. This impact is highlighted by the space and time-dependent changes in the effective coefficients. In particular, within a first-order analytical formula for uniaxially aligned fibre-reinforced media, the evolution of inelastic processes makes the effective coefficient $[\mathscr{G}_{\text{eff}}]_{3131}$ to transition from a compressional to a shear behaviour. These considerations showcase the potential of the modelling approach in circumstances advantageous for biomimetic applications. Still, our work would benefit from additional considerations. For instance, we have yet to address the complete three-dimensional unit cell, which would involve solving the in-plane problems associated with $\boldsymbol{\xi}_\eta$ and $\boldsymbol{\omega}_\eta$. One must also address the question of whether solutions exist, which hinges on demonstrating the strong ellipticity of the differential operators defining the cell problems. Further avenues also include identifying suitable evolutionary laws that correspond closely with the specific phenomena under investigation or employing computational frameworks adept at accommodating intricate geometries.

Although in this work we do not compute the overall deformation of the composite, it is possible to predict it. Indeed, referring to the homogenised balance of linear momentum given in (95), to compute $[\boldsymbol{u}^{(0)}]_3$ we would only need to impose appropriate boundary and initial conditions since the effective coefficients appearing in it can be found independently through Equations (98a)–(98d). This is feasible because the homogenised evolution law for the inelastic distortions (79) decouples from the balance of linear momentum. However, since we aimed to show the potentialities of the approach, which is based on the analytical computation of the effective coefficients, we prefer to include these considerations in further investigations where a specific real-world problem is tackled. Nevertheless, we mention that the computation of the overall deformation of multi-scale and heterogeneous media undergoing remodelling has been investigated in refs. [4, 9] in which we took advantage of a simpler geometrical setting. Furthermore, if the constraints on the displacement vector discussed in Section 6 are relaxed, while still maintaining the microscopic depiction of the fibre-reinforced composite, a fully three-dimensional analysis can be conducted. This would involve solving the corresponding in-plane and out-of-plane problems derived from the cell problems, thereby fully addressing the deformation of the composite. Finally, we remark that, in a more general context, that is, if different considerations to those specified in Section 6 are made, both the homogenised balance of linear momentum (60) and the homogenised remodelling law (62) would be coupled. In this scenario, further computational efforts would be required to address the overall deformation of the composite.

As per some fields of applicability of our work, we mention the study of the electro-chemo-mechanics of nervous tissue. Nervous tissue, in particular, is susceptible to inelastic processes both during the progression of neurological diseases and as a natural consequence of ageing throughout one's lifespan. For instance, neurodegenerative diseases such as multiple sclerosis [107], Alzheimer's disease [108], and demyelination [109], generate changes in the properties of nervous tissue through the remodelling of its microstructure. Thus, by coupling the electrophysiological and inelastic effects, we could provide valuable insights into how inelastic processes such as axonal damage or ageing could affect the neurological function of the brain. For instance, axonal degeneration, in the form of morphological or composition changes, results in impaired communication between neurons which leads to neurological deficits and affected memory.

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

DATA AVAILABILITY STATEMENT

All data supporting this study are provided in full in the 'Results' section of this paper.

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